Navigating Transitions Along Continuums

PMENA 2012

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of the North American Chapter of the International Group for the Psychology of Mathematics Education

November 1 – 4, 2012

Kalamazoo, Michigan

Editors:

Laura R. Van Zoest
Jane-Jane Lo
James L. Kratky

Western Michigan University

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The major goals of the International Group and the North American Chapter are:

- To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
- To promote and stimulate interdisciplinary research in the aforesaid area, with the cooperation of psychologists, mathematicians, and mathematics teachers;
- To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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Preface

These Proceedings are a written record of the research presented at the 34th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA 2012) held in Kalamazoo, Michigan, November 1–4, 2012.

The theme of the conference, *Navigating Transitions Along Continuums*, focuses on an important set of opportunities for research to be useful in improving mathematics teaching and learning. Plenary speakers consider transitions across four continuum groupings: (1) student learning of mathematics (Jere Confrey); (2) professional learning, ranging from preservice mathematics teachers through teacher leaders (Deborah Ball and Suzanne Wilson); (3) school mathematics articulation, from topic to topic within grade levels as well as across grade bands (Amanda Jansen, Janie Schielack, Cathy Seeley, and Jack Smith); and (4) innovation to support mathematics learning, from the smallest of scale to the largest (Jo Boaler).

The Proceedings include papers from 2 plenary talks, 69 research reports, 124 brief research reports, 111 posters, and 10 working groups. The plenary and working group papers are the first and last chapters, respectively. Papers from the research reports, brief research reports, and posters are organized into chapters by topics. Each paper is indexed by authors and keywords. Underlined author indicates presenting author.

We would like to thank Hope Smith for her dedication to the technical details of putting together a high-quality document and James Kratky for his skill in making it easy to navigate. We are pleased to present these Proceedings as an important resource for the mathematics education community.

Laura Van Zoest & Jane-Jane Lo
Conference Co-Chairs

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Van Zoest, L. R., Lo, J.-J., & Kratky, J. L. (Eds.). (2012). *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Kalamazoo, MI: Western Michigan University.

Chapter 1

Plenary Papers

Van Zoest, L. R., Lo, J.-J., & Kratky, J. L. (Eds.). (2012). *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Kalamazoo, MI: Western Michigan University

ARTICULATING A LEARNING SCIENCES FOUNDATION FOR LEARNING TRAJECTORIES IN THE CCSS-M

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The paper describes the history of how learning trajectories (LTs) were associated with the Common Core State Standards for Mathematics (CCSS-M) and discusses the degree to which the two correspond faithfully. It reports on a website, www.turnonccmath.com, which organizes the K–8 standards into 18 LTs describing the development of big ideas over time, informed by empirical studies of learners. The paper illustrates how descriptors for each LT identify: (1) conceptual principles, (2) strategies representations, and misconceptions, (3) meaningful distinctions and multiple models, (4) coherent structure, and (5) bridging standards. The design principles for the website are illustrated describing how the CCSS-M are related to a learning trajectory on division and multiplication.

Keywords: Standards, Cognition, Teacher Knowledge, Learning Trajectories

The Common Core State Standards for Mathematics (CCSSO, 2010) have been represented as "fewer, clearer, and higher," reflecting the view that revised standards should be: (1) focused, (2) rigorous and applicable, and (3) coherent. They offer "a more coherent progression of learning" described as "… clearly articulat[ing] how knowledge builds from year to year. Each standard extends previous learning while avoiding repetition and large leaps in instruction" (Hunt Institute, 2012, p. 8). Despite this intent, the progressions themselves are not immediately accessible to readers, so other documents are needed to articulate and display these relationships in different formats. Our research group has done this as a set of posters (www.wirelessgeneration.com/posters) and as a website (www.turnonccmath.com). After reviewing the history of how learning trajectories became foundational in the writing of the CCSS-M, I describe the elements of a learning trajectory analysis of the CCSS-M as a means to support implementation of standards and conduct of related professional development. The advantages of researchers working together, to create resources on learning trajectories built on empirical study are discussed, along with a warning of the likely costs of failing to do so.

History of Learning Progressions in the CCSS-M

In the summer of 2009, a meeting was held at the Friday Institute for Educational Innovation in North Carolina where researchers on learning trajectories hosted the writers of the Common Core State Standards (CCSS) and other leaders from the Council of Chief State School Officers (CCSSO).[2] The proposed standards were to be based on scientific evidence. While the College- and Career-Ready Standards (U.S. Department of Education, 2010) could be sufficiently justified with evidence of international benchmarking and studies of the needs and expectations of colleges and entry-level careers, the grade-level standards required a basis in the research on student learning. A number of learning sciences and mathematics education researchers gave presentations (including M. Battista, D. Clements, J. Confrey, G. Kader, and R. Lehrer) on learning trajectories (also called "learning progressions"). After the conference, many of the attendees were invited to participate on the CCSS-M writing teams. The use of these teams during the Standards development was perceived by many as more sporadic than systematic—and the teams were only one voice among many (including state departments, mathematics faculty, and teachers) in influencing the development of the Standards. However, their ideas contributed significantly to the final document. In sum, the CCSS-M incorporated a foundation in learning trajectories that can propel the country forward now, and be strengthened over time. In the period since the publication of the CCSS-M, at least three groups have engaged in efforts to delineate the trajectories in more detail (Confrey et al., 2011; Hess & Kearns, 2010; McCallum, 2011).

www.turnonccmath.com

Once the CCSS-M was validated and widely adopted, and in response to the need expressed in the field for urgent assistance, the DELTA research group at North Carolina State University (NCSU) decided to connect the Standards more directly with associated research on learning trajectories. Many state leaders had reported that teachers perceived little change from their old or current state standards to the new CCSS-M, and expected that "crosswalks" would provide a sufficient basis to support the transition to the CCSS-M and the related curriculum and assessment. In this scenario, teachers would only change the way they teach new topics at the grain size of the individual grade levels and otherwise continue teaching by making small adjustments to their lesson plans. A close reading of the CCSS-M document, my understanding of the CCSS-M from experience on the National Validation Committee, and our group's close comparison of the CCSS-M to previous state standards, however, told a different story. There are major changes in when and where mathematical topics are emphasized, namely the intensity of content treatment at earlier grades and major shifts in several topics that will radically change teacher preparation and professional development. The "higher" and "fewer" aspects of the CCSS-M mean, also, that there is much less room for repetition of content at each grade.

We found learning trajectories useful in supporting implementation, because they focus attention on gradual and systematic student learning over time, a form of "genetic epistemology" (Piaget, 1970). The idea behind explicitly mapping learning trajectories onto the CCSS-M is to help teachers and students build consistently stronger understandings of big ideas by revising and modifying prior views in light of new conditions and challenges. Rather than emphasize a standard-by-standard view of implementation of new or revised content, learning trajectories support "vertical teaming" by teachers. This allows an exciting chance for teachers to discuss and plan their instruction based on how student learning progresses. An added strength of a learning trajectories approach is that it emphasizes why each teacher, at each grade level along the way, has a critical role to play in each student's mathematical development.

Our effort to build a website that synthesizes the relevant research and to lay out a manageable number of learning trajectories for the CCSS-M began as a result of a meeting of the *Measurement Mini-Center*.[3] Many of the group's participants had conducted pioneering work on learning trajectories, and each has his or her preferences about how to characterize, emphasize or order underlying proficiencies and concepts. Concerned that the interpretation of the CCSS-M should be better and more publicly informed by "learning sciences research," my research team drafted a synthetic trajectory built around the CCSS-M, drawing from these scholars' work, and brought it to the meeting for discussion. The Mini-Center's response to the effort was positive and constructively critical—the group reviewed the proposed trajectory, offered valuable suggestions and distinctions, and labored until an acceptable synthesis was negotiated. This specific trajectory as finalized is represented on the turnonccmath.com site (Confrey et al., 2011) and is described in more detail in a 2012 PME-NA paper (Lee, Nguyen, & Confrey, 2012).

Buoyed by this experience and stimulated by requests from the field, our NCSU team decided to undertake a full learning trajectories analysis of the K–8 Standards. Using a hexagon map of the CCSS-M (designed by Jere Confrey and ©Wireless Generation) to display the Standards and learning trajectories visually, I dissected the CCSS-M into 18 learning trajectories. Over a concentrated period of six months, the research team undertook writing, revising, and interlinking *descriptors*, which are text-based descriptions of standards in terms of students' movement from more naïve to more sophisticated ideas for each of the trajectories. Our working assumptions were that the web-based environment would: (1) provide the opportunity for continuous incremental improvements in the descriptors that would serve the needs of the field for rapid access to the associated learning trajectories for the Standards, and (2) permit us to gradually strengthen the site based on feedback and review. In the next sections, the hexagon map is introduced along with an explanation of the framework used to analyze the trajectories and unpack them into descriptors.

Turnonccmath: by Grade

The website http://www.turnonccmath.com displays a "hexagon map" of the CCSS-M. In designing this map, decisions to use a predictable and consistent method to assign standards to hexagons were largely pragmatic. Standards in the CCSS-M are of many different grain-sizes, which added considerable challenge to the effort in mapping them to hexagons. Standards were assigned to individual hexagons using the following scheme: (1) If a Standard has no subparts, the hexagon represents the entire standard. However, multipart Standards were too dense to be summarized in a single hexagon. Therefore, (2) for any Standard with subparts (e.g., a, b, c, etc.), each subpart was assigned its own hexagon. The map can be displayed in three views: by grade levels, by LT with the LTs labeled, and by LTs without labels. The topics within the standards generally proceed from less complex (lower left) to more complex (upper right).

The hexagons for the different grade levels occur in bands that are more or less orthogonal to the progression of the topics. In the grade level display, the lower left ends of any relevant learning trajectory contain the earliest grade-level standards, beginning (if applicable) with kindergarten standards, followed by first through eighth grade Standards built on top and to the right, and coded such that a hexagon's background color represents its grade level. The text color in each hexagon represents the content strand; for example in K–8, blue text corresponds to Number and Operations; red text corresponds to Measurement and Data, and black text corresponds to Geometry. In terms of the relative positions of different main content strands and learning trajectories, I chose to put Number and Operation-related standards on the bottom with Measurement-related standards on top of those, diagonally, and then Geometry-related standards above measurement. At the very top is a peninsula where the very thin learning trajectory for Elementary Data (Statistics) and Modeling is placed. This trajectory comprises K–5 standards in the Measurement and Data cluster that address how to build and interpret data representations. Having opposed the writers' decision to reduce the treatment of statistical reasoning in the CCSS-M at the elementary level, I left space to expand these standards in future revisions.

From the grade-level display, one can discern certain patterns. For instance, one can see that third grade is almost entirely comprised of standards on number and measurement, with only one standard in geometry. In contrast, one can see that in sixth grade, there are three distinct clusters of topics: (1) statistics, (2) ratio and proportion, and (3) equations and expressions.

The Relationship Between the Learning Trajectories and the CCSS-M

The purpose of a learning trajectory is to describe and synthesize what is known about how students reason over time. The term *Learning Trajectory* (LT) has varied meanings in mathematics education. Simon (1995) first defined the term hypothetical learning trajectory (HLT) to be "The learning goals, the learning activities, and the thinking and learning in which students might engage" (p. 133). We define it as, "a researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e., activities, tasks, tools, and forms of interaction), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time" (Confrey, 2008; Confrey, Maloney, Nguyen, Mojica, & Myers, 2009, p. 2-346). We view a learning trajectory as a path through a conceptual corridor in which there are predictable obstacles and landmarks and thus a student's particular path is an issue of expected probabilities and likelihoods: LTs permit one to specify at an appropriate and actionable level of detail what ideas students need to know during the development and evolution of a given concept over time.

Learning trajectories provide a way to create coherence within the CCSS-M by drawing attention to how knowledge develops over time. If teachers try to implement the CCSS-M standard-by-standard, they will be unlikely to leverage the underlying structure of the standards and support gradual transformations in student reasoning. When we have worked with teachers in unpacking our learning trajectories, they have commented on the value of creating a "story" which illustrates how the ideas are likely to evolve in the minds of students when they are provided appropriate curriculum tasks, instruction, and opportunities for discourse. Therefore, our goal is to provide this type of support to teachers by providing them efficient and coordinated access to related research. In the end, the success of the CCSS-M rests on its potential to support alignment, including curriculum, assessment (formative *and* summative), and professional development, at a level not previously possible. But to achieve the deep and lasting change envisioned by the Common Core State Standards Initiative and the mathematics education community, the knowledge of learning trajectories must be made clear, accessible, compact, and well-integrated within the CCSS-M.

The relationship between the learning trajectories and the Standards is complex. To a degree, the CCSS-M were built on the foundation of learning trajectories. But it would not be accurate to say that there is an isomorphic relationship between the CCSS-M and the learning trajectories. In fact, acknowledging this, the Standards' writers call the progressions in the standards, "standards progressions" (Common Core Writing Team, 2011). The reasons include:

1. Different researchers have differing views of learning trajectories, even within strands;
2. Not all topic areas have been studied as learning trajectories; and
3. The writers took suggestions from mathematicians who conflated learning trajectories with logical progressions created by "thought experiments," independent of empirical verification.

This outcome is to be expected in a document resulting from negotiations and differences of opinion among disciplinary scholars, researchers and practitioners; moreover, it creates the possibility now to systematically test, compare, and refine those trajectories in light of students' work. Also, in order to construct "fewer" and "clearer" standards, the learning trajectories in the CCSS-M are of necessity *abridged*; that is, they do not and could not contain a full treatment of all the big ideas contained in the research literature. To address this in our analysis, we added "bridging standards" as needed. These statements are similar in structure to the CCSS-M standards, but represent topics that would be required in a more fully articulated (i.e., unabridged) learning trajectory. Because of the dual nature of standards as both assessment targets and targets of understanding, bridging standards can permit one to describe standards that need to be addressed in preparation for a later standard but which will not be assessed directly at that specific time. Finally, even after debate and review, there are a few standards that were poorly constructed, inconsistent, or unadvisable, based on mathematics education or learning sciences literature; a bridging standard may be added to improve the coherence of the trajectory overall.

Standards, by themselves, can serve as a skeleton for learning trajectories, but they need to be interpreted and made unabridged to serve this purpose. Moreover, the interpretation must make explicit the connections to the research base and provide a more complete articulation of how the ideas in a trajectory evolve in light of students' documented behaviors, emergent relations and properties, and generalizations (Confrey, Maloney, Wilson, & Nguyen, 2010). To this end, and so that there would not be too many LTs to manage, we decided to create a mapping such that every standard would belong to exactly one LT, each targeting a key "big idea" or set of related big ideas. The CCSS-M document itself does not suggest an instructional sequence or rigid ordering of the Standards beyond specifying grade level, as the authors have stated: "These Standards do not dictate curriculum or teaching methods" (CCSSO, 2010, p. 3). Therefore, we reorganized standards within a trajectory if this would show the student learning development more clearly (while keeping the grade level position of standards and topics). Thus, sequencing within grade was malleable; we adjusted it to fit the learning trajectories structure (hence the numbering of the standards can be "out of order" within a grade). We also assisted readers in seeing the internal structure of and the relations among the learning trajectories by (a) creating sections to reveal underlying development, (b) providing structural overviews, and (c) cross-referencing and referencing forward and backward within a LT.

Turnonccmath: by Learning Trajectories

The hexagon map of the CCSS-M, with learning trajectories labeled, is shown at www.turnonccmath.com (Figure 1). The two-dimensional structure of the map lends itself to parallel structures among some learning trajectories, in some cases, to represent close relationships between various big ideas. One of these is the fundamental role played by (1) *counting*, (2) *equipartitioning*, (3) *addition and subtraction*, and (4) *place value and decimals* in developing an early sense of number and operations. These four learning trajectories are situated at the lower left portion of the map. *Counting* is directly tied into *addition and subtraction* and develops in tandem with *place value and decimals*. *Equipartitioning* leads directly to supporting the development of (5) *division and multiplication*, and subsequent rational number reasoning, with contributions from *addition and subtraction*. (6) *Fractions* are most closely related to equipartitioning and division and multiplication, with (7) *ratio and proportion and percents* being most closely tied to division and multiplication and fractions in topic and grade-level development within the CCSS-M. (8) *Rational and irrational numbers* link to *ratio and proportion and percents.*

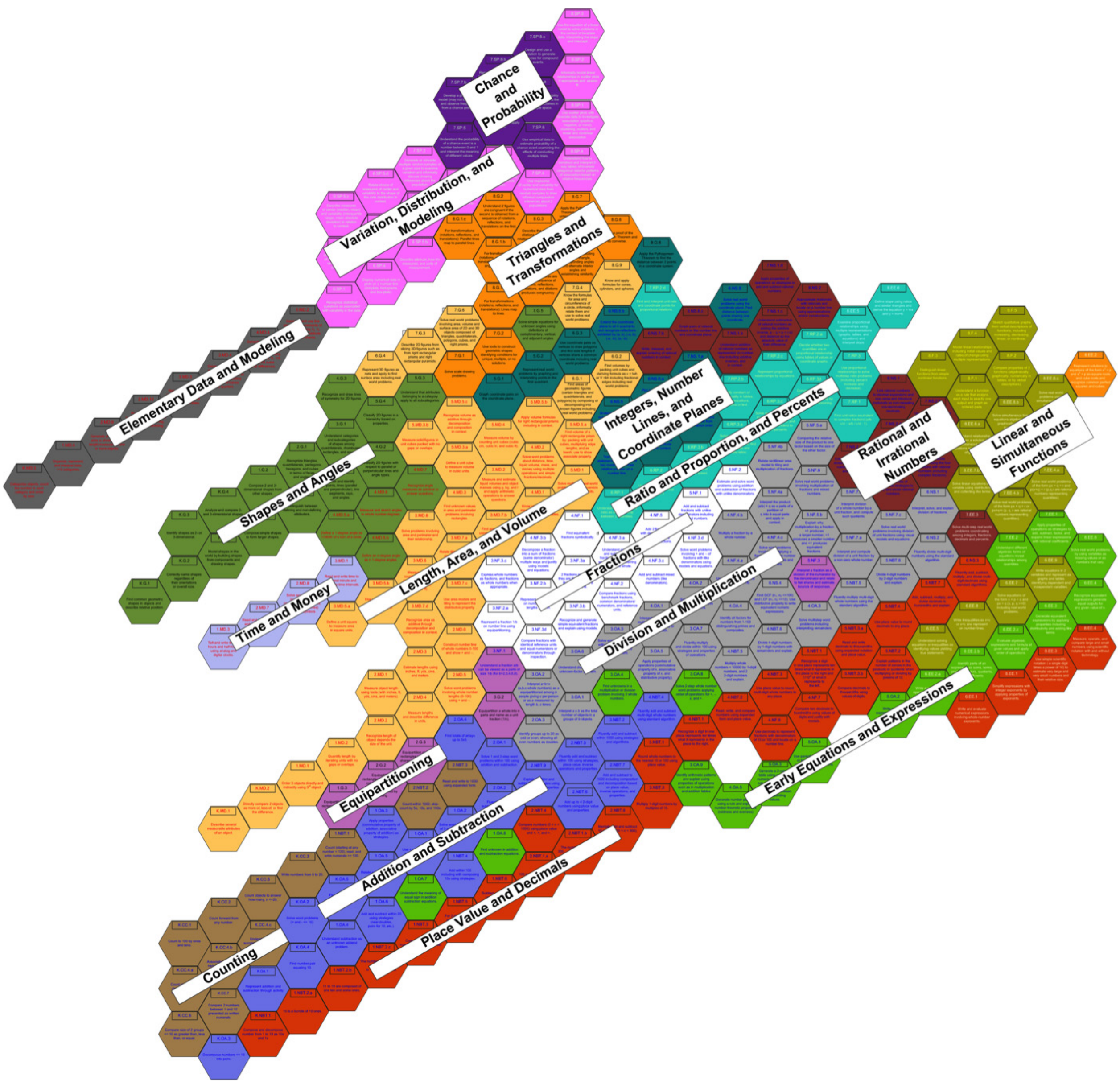

Figure 1. Hexagon Map of K–8 Common Core State Standards for Mathematics with individual learning trajectories color-coded and labeled

The learning trajectory for (9) *length, area and volume* is situated next to equipartitioning in recognition of their close relationship in early reasoning about shapes and measurement, and because they cover a considerable amount of conceptual development in spatial, measurement, and geometrical reasoning. This forms a large anchor LT. (10) *Time and money*, a small early-grades set of topics, is tucked into the left side of the map. The measurement cluster has close links to (11) *shapes and angles*, which carries into (12) *triangles and transformation* as students progress into the middle grades. *Integers, number lines and coordinate planes* (13), a mostly 6th-grade set of topics, are placed close by to support further development of other middle-grades trajectories linking geometry and number systems. The cluster of learning trajectories that comprise data, statistics, and probability—(14) *elementary data and modeling,* (15) *variation, distribution and modeling,* and (16) *chance and probability*—are located along the top of the map, as they were most closely related to each other. The limitations of the two-dimensional space on which the map was constructed prevented us from linking them more closely to measurement and ratio reasoning.

Further upwards and to the right are the more complex topics of (17) *early equations and expressions,* which are built on the four operations and which link to (18) *linear and simultaneous functions* to create a foundation for algebra in the 9–12 Standards.

A Framework for Unpacking Learning Trajectories

When one hovers the cursor over a hexagon on the hexagon map of www.turnonccmath.com, the full Standard is presented verbatim in a box in the bottom left corner. If one clicks on a hexagon or learning trajectory, a new window with the descriptors for the selected learning trajectory appears. The descriptors are organized as follows: A "Structural Overview" is presented at the beginning of each LT, identifying the sections of the LT and showing its development across the relevant grades. Sections are then used to create a sub-organization of the learning trajectory. In addition, a framework of five elements was created to systematize the unpacking of the each trajectory:

1. *Conceptual principles*: These are a list of underlying cognitive principles, identified by researchers, which support the overall development of the ideas.
2. *Strategies, representations, and misconceptions:* When students encounter new tasks that are presented as a cognitive challenge, they invent strategies and representations as they solve them, demonstrating their ways of thinking and, often, revealing related misconceptions that need to be addressed instructionally. Because misconceptions typically have a kernel of "right thinking" (Confrey, 1990), these thoughts must be elicited and then refined into alternative conceptions or valid intermediate steps on paths to more sophisticated thinking.
3. *Meaningful distinctions and multiple models:* All educators recognize the value of prior knowledge and the importance of identifying clear targets for learning. A major challenge, however, lies in identifying and evaluating intermediate states of proficiency and understanding their role in moving students forward in their thinking. To describe these intermediate states, teacher and researchers must recognize or invent meaningful distinctions; vocabulary terms for these tend to exhibit properties that are both cognitive and mathematical, such as partitive vs. quotative division, which later simply collapse to "division." We refer to these as "meaningful distinctions." In addition, for "big ideas"—also described as a learning trajectory's "domain goal of understanding"—there are often multiple earlier models that correspond to the different schemes that govern recognition of situations in the real world. These big ideas are typically captured as a "generalization" that, while "encapsulating" their meanings in the minds of experts, hides or loses the details of the distinctions and models, so students should be afforded sufficient opportunity to explore the distinctions and models *before* they move to the generalization, in order to understand its many referents and applications.
4. *Coherent structure:* In a learning trajectory, a pattern often emerges in how a topic is developed; commonly, that pattern is repeated as the students expand it at later grades and apply it to increasingly complex cases, representations, tools, choices of numeric values, or spatial

dimensions. For example, students' understanding of area is expanded as the lengths of the sides take on fractional values. Understanding such structure, and considering which parts of it remain invariant and which change under these expansions, is a characteristic of mathematical reasoning.

5. *Bridging standards*: Moving from "abridged" learning trajectories represented in the CCSS-M to more fully-articulated, "unabridged" standards requires the addition of "bridging standards" that might not have represented major intellectual targets within the CCSS-M but which may nonetheless be necessary to support a successful progression of learning for students. Based on our structural analysis, we sometimes found gaps or inconsistencies in the Standards. In these cases we also added bridging standards. The bridging standards are identified by their use of a capital letter (A, B, C, …) at the end of the standard number, and the use of brown font. Each bridging standard includes an explanation for its addition to the descriptors document.

A question can be raised about the relationship of the eight mathematical practices to our learning trajectories analysis of the CCSS-M. We do not address the practices directly in the analysis, although the practices are critical elements of the curricular instantiations of the CCSS-M. First of all, we emphasize that a learning trajectories analysis is not a curricular analysis, although one can conduct analysis of curricula using the learning trajectory construct (Nguyen & Confrey, in press) by considering the learning trajectory as a boundary object (Confrey & Maloney, in press; Star & Griesemer, 1989). Furthermore, as students progress along a learning trajectory, they will employ the various mathematical practices, such as applying repeated reasoning, and using precision, articulating arguments, or building or critiquing new modeling.

An Example: The Division and Multiplication LT

Data on large-scale assessment show weakness in U.S. student knowledge and understanding of division and multiplication (NAEP, 2009). Furthermore, division and multiplication are topics around which there is considerable research. Fischbein et al. (1985) introduced the idea of primitive schemes for division and multiplication, claiming two for division (partitive and quotative) and only one for multiplication. Partitive division was linked to schemes based on dealing (usually to obtain the size of a share or group) while quotative division, later commonly referred to as "measurement division" (Simon, 1993), was linked to repeated subtraction or addition, in an iterative manner.

Elaborating further on how children learn multiplication, many researchers (Kamii, 1985; Steffe & Cobb, 1998) describe a process of accumulating equal-sized groups by describing how children learn to coordinate the process of differentiating the roles of numbering the groups and naming the group size. In doing so, they derive multiplicative structures from additive ones. They describe a gradual process of skip counting, double counting, and eventual description as a product, *ab*, comprised of a number of groups, *a*, of a particular size *b*. Because multiplication then is comprised of two elements, group size and number of groups, these researchers tend to follow Fischbein et al. (1985), in recognizing the two types of division, one focused on finding the size of the group (partitive) and the other the number of groups (quotative).

Other researchers categorize word problem types in multiplication or division (e.g., equal groups, rates, comparison, Cartesian products, scaling, etc. undertaken by scholars such as Nesher (1980, 1988, 1992), and Carpenter, Fennema, and Romberg (1993). These scholars have a tendency to associate multiplication with a certain set of problems and each type of division with other sets of problems. For example, equal groups problems are associated with multiplication, fair sharing problems are associated with partitive division, and measurement problems (e.g., How many 3 inch ribbons are there in a ribbon that is 36 inches long?) with quotative division. It is preferable, in our opinion, to distinguish among the questions asked (e.g., the size of a group or fair share and the number of groups or the number of shares) and to associate these questions, and not problem categorizations, with the processes students use to solve a problem. One advantage is that this leaves open the possibility of students using other approaches (e.g., co-splitting (Corley, Confrey, & Nguyen, 2012), or the use of arrays or area models models (Battista, Clements, Arnoff, Battista, & Borrow, 1998; Outhred & Mitchelmore, 2000). Researchers who rely on categorization schemes (CGI, others) tend to focus on these as applications of operations rather than to go further to use them to define the underlying cognitive schemes (Carpenter & Fennema, 1992).

A contrasting trend in research was introduced by Vergnaud in his work on multiplicative conceptual fields (MCF) (Vergnaud, 1983, 1988), when he articulated the relations among ratio and proportion and multiplication and division. The MCF, he argued, consisted "of all situations that can be analyzed as simple or multiple proportion problems and for which one usually needs to multiply or divide" (Vergnaud, 1988, p. 141). He connected the many parts of the MCF to a four part relationship (visually, a two-by-two arrangement) among quantities in which movement horizontally was described as a functional, demonstrating a direct variation relationship between two quantities (i.e., $f(x) = ax$) and vertical movement was referred to as an "*isomorphism of measures.*"

In a related vein, in 1988, I articulated my splitting conjecture (Confrey, 1988), arguing that multiplication and division could be linked to ratio and proportion as derived from an early application of an operation I labeled *splitting,* and subsequently also labeled *equipartitioning*. In a three-year teaching experiment of children in 3rd–5th grade, I demonstrated the advantages to student learning of co-defining multiplication, division, and ratio (Confrey & Scarano, 1995) and showed the effects of teaching fractions as expressing a particular subset of ratio relations.

Data suggest that, contrary to most textbook sequencing, equipartitioning and partitive division are understood at an early age (Bell, Fischbein, & Greer, 1984; Confrey et al., 2009; Confrey & Scarano, 1995). Moreover, approaching division and multiplication through early experience with ratio has been supported by research on protoratio (Noelting, 1980a; Noelting, 1980b; Resnick & Singer, 1993), on splitting (Confrey, 1988; Confrey & Scarano, 1995), and on distribution (Streefland, 1984, 1991).

Schwartz (1988) distinguished between referent-transforming and referent-preserving operations, suggesting that additive structures are referent-preserving (preserves the referent unit, e g., 4 apples plus 3 apples equals 7 apples) while multiplicative ones are referent-transforming (does not preserve the referent unit, e.g., 20 coins shared among (divided) 5 people results in 4 coins per person). He also introduced the distinction between extensive quantities (magnitude) and intensive quantities (indirectly measured as composed from other quantities). However, I argue that multiplication can also be referent-preserving when only the particular unit changes (e.g., in the case of measurement conversion, the use of groups, or scaling).

This second set of approaches deemphasize the role of addition and subtraction in the construction of division and multiplication. Instead I view division and multiplication as related operations describing the same situations in reverse. The two operations are interlocked in a four-part relationship that can be described by ratio relations. For example, in the "division problem" 20 coins shared among 5 people results in 4 coins per person, the ratio relationship is 20 coins : 5 people :: 4 coins : 1 person. Multiplication can be used to describe the movement from 4 coins to 20 coins and 1 person to 5 people and division can be used to describe the reverse movement. Because they rely on ratios, this treatment of division and multiplication is necessarily related to the use of two distinct quantities: the case of referent-preserving division and multiplication is cast as the reduced case where groups, unit-changes, or a scalar are introduced. These approaches also tend to support the extension of the operations to non-whole numbers, and more intuitively anticipate the operator construct of rational numbers (Behr, Harel, Post, & Lesh, 1994), which I locate in this trajectory.

Both generalized approaches recognize the use of division and multiplication in area measurement and find ways to incorporate it. In the first approach through counting and additive structures, arrays can be viewed as a transitional tool. If the groups are lined up in columns and placed side by side, then the resulting array can be viewed as representing both the number of groups (rows) and the size of the groups (columns). Proceeding from the discrete case to the continuous case can still support a definition of the multiplication operation in terms of the number of groups and their sizes. The integrated approach also uses area problems but does so through the application of scaling operations from the single unit on the lengths of the sides of a rectangle, and subsequently on the area of the resulting rectangular figure.

In deciding how to approach the learning trajectory, I sought ways to:

1) combine the strengths of both models, while emphasizing importance of multiplicative structures;
2) build from what the children already knew from the related learning trajectories of equipartitioning, length, area and volume, and addition and subtraction;
3) ensure the approaches were sensitive to the variety of situations connected to division and multiplication; and
4) anticipate how sufficient the models would be as the numeric values in the problems changed from whole numbers to non-whole rational numbers.

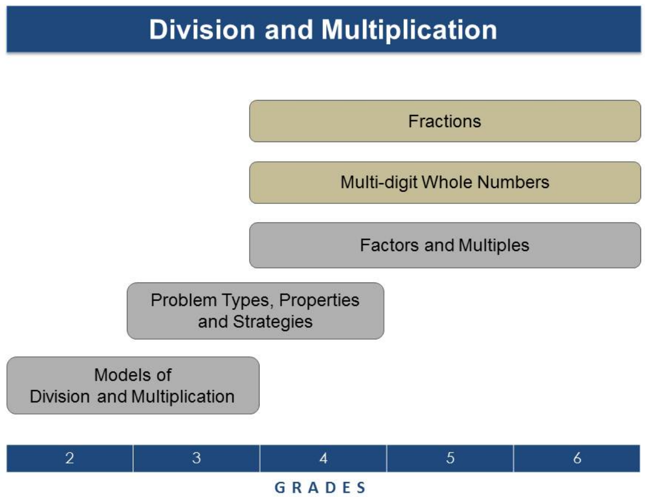

Figure 2. Structural Overview diagram for Division and Multiplication learning trajectory

Framework for Learning Trajectories, Applied to the Division and Multiplication LT

The Structural Overview of the learning trajectory is shown above (Figure 2) whereby one can see that the LT stretches from second through sixth grade. Students develop three models and then apply them to a variety of problem types. As they become fluent in the number facts, they learn about factors and multiples and then extend their knowledge to more complex cases. In the following sections, a window into the structure of the division and multiplication learning trajectory (DMLT) is provided using the five-element framework described previously.

The Target of the Learning Trajectory for Division and Multiplication

Learning trajectories always incorporate assumptions about what students have experienced and know, and what the target of that learning should be at the upper end of the trajectory. The primary target of the DMLT is for students to understand the relationships captured in the equation: $ac/bd \div a/b = c/d$. As explained below, these relationships can be understood either as they reside in a ratio box or in relation to two-dimensional area relations (which can later be extended to higher dimensions).

Ratio boxes relate two quantities such that the relationship is preserved across multiplicative changes to both quantities. All but elementary uses of the ratio box for fair sharing explicitly show the preservation of the ratio across multiplicative changes by using two pairs of "arrows," one which shows the multiplicative or divisional operation that relates the two sets of numbers vertically and showing the other relationship horizontally (Confrey, 1995). Noelting refers to these as, respectively, "between" and "within ratio relations (Noelting, 1980a, 1980b). Characteristic of a ratio box is that the pairs of opposite arrows are identical.

The DMLT can be summarized as an evolving sequence of types of ratio boxes and area models. Those ratio boxes start with a "fair sharing box," and proceed to a division/multiplication box (D/M box) to complete the DMLT. In the ratio and proportion and percents LT, the boxes evolve into a fully developed ratio box. Figure 3, below, illustrates the fully developed ratio box. Given any three values students find a fourth unknown value of the proportion, and describe the relationships represented by the operator arrows, either as shown here as multiplication, or its inverse, division (not shown).

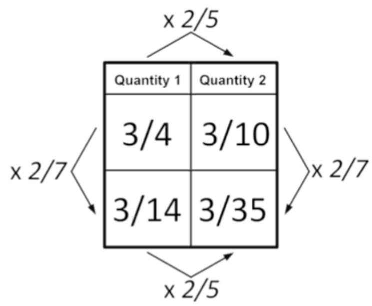

Figure 3: A ratio box solution, with multiplication shown

The DMLT begins from a "reduced ratio box" known as a fair-sharing box in the equipartitioning LT (EQLT). Second-graders can fill in the column headers and the two rows when sharing, for example when fair-sharing 12 coins among 3 people, they fill in 12 and 3 in the top row, and 4 and 1 in the bottom row (Figure 4a). Also based on the EQLT, they express the sizes of upper row numbers relative to lower row numbers as "*b times as many*." At this young age and lacking any formal introduction to multiplication or division, children are not expected to use the arrow notation. For the EQLT, the final target goal can be expressed in a ratio box (Figure 4b) corresponding to Standard 5.NF.3 ("Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem").

Number of coins	Number of persons
12	3
4	1

Figure 4a: Fair share box for equipartitioning a collection of 12 coins

Number of coins	Number of persons
a	b
a/b	1

Figure 4b: Generalized fair share box for equipartitioning collections

Building from the fair sharing box, the first target for the DMLT is a slightly more sophisticated reduced ratio box called a "division/multiplication box" (D/M box). The D/M box (Figure 5a) also has a 1 in the lower right corner because in the four-part relations for MCF, for division and multiplication, one cell is equal to 1. For example, in the problem "at a tire shop, six cars are getting their 4 tires changed. How many tires are needed?," the final D/M box would have two columns—one for the number of tires and one for the number of cars—and show 24 tires associated with six cars and 4 tires with *one* car. The number facts, $6 \times 4 = 24$, $24 \div 6 = 4$, and $24 \div 4 = 6$, do not show the *one*. At first, the use of the D/M box can be constrained to whole numbers only. The D/M box differs in two respects from the fair-sharing box. Firstly, it is not restricted to fair-share situations, and secondly, as students learn to work with division and multiplication operations symbolically, they add arrows to define the relationships (operators) explicitly. The associated area model, can also initially use whole numbers (Figure 5b).[4]

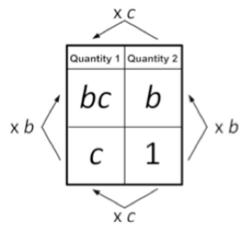

Figure 5a: D/M box adapted for whole-number multiplication

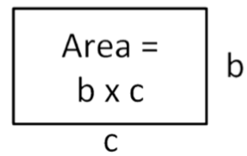

Figure 5b: Whole-number multiplication model for the area of a rectangle

In order to understand the D/M box and the rectangle area model, students describe and work with all three related equations of $a \times b = ab$, $ab \div$ b = a and $ab \div$ a = b. These intermediate goals are presented here in symbolic form for brevity, for the benefit of experts; students, however, are expected to understand where they come from, explain and represent them, relate them to prior and related knowledge with justifications, and apply them to solve a rich variety of problems. In addition to correctly producing their answers, students are expected to be able to move about flexibly and fluently in multiplicative space using factors, including primes and multiples, and recognize, discover, and use the relevant properties and practices.

The final target for the DMLT is a D/M box showing division, multiplication, and a rectangular area model (Figures 6a, b, c) where the non-one values in the cells can be any rational numbers. The DMLT can be understood now as poised between (a) equipartitioning, and (b) ratio and percent. As will also be shown, it draws on elements of other LTs on the length, area and volume, addition and subtraction, and place value and decimals.

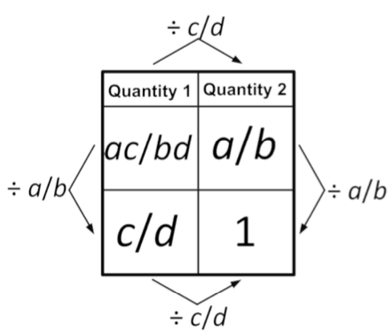

Figure 6a. D/M box (division)

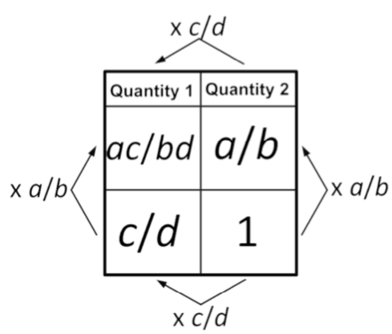

Figure 6b. D/M box (multiplication)

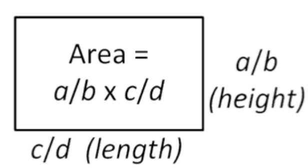

Figure 6c. Area model

Also, later in the length, area, and volume LT, the product can include more than two dimensions (essential for the associative property), so that one can explain volume as $v = l \times w \times h$, or as v = area $\times\, h$, and one can increase dimensionality as required for modeling multiplication in higher dimensions that lack obvious spatial analogues. This set of related learning trajectories: equipartitioning, division and multiplication, ratio and proportion and percents, and length, area and volume, together with similarity (within the triangles and transformations LT), comprise the majority of the content that resides in the multiplicative structures.

It is important to understand as fully as possible the target or domain goal understanding for a learning trajectory, because while it often cannot be directly taught, it must be reached as the product of a careful series of transformations based on empirical study of student learning. By delineating it carefully, one can distinguish intermediate states that are productive from ones which will limit students' chances of obtaining a full and nuanced perspective.

Distinctions and Models

A synthesis of the literature yields three fundamental *models* for the joint operations of division/multiplication, each of which generate both division and multiplication contexts. These are (a) referent-transforming, (b) referent-preserving, and (c) referent-composing models. These three models are necessary to sufficiently link division and multiplication to its related trajectories, from equipartitioning and addition/subtraction to ratio and proportions and percent, fractions, chance and probability, and length, area, and volume, and to support mathematical modeling. The three models are described below:

a) *Referent-Transforming.* Division/multiplication in these models involves changes in the attributes or referents connected with the quantities, or action on a quantity of one attribute or referent by a quantity of another attribute or referent. For instance, in fair sharing, coins are shared among people to produce coins per person (Figure 7). Rate problems also fit in this category. In relation to the D/M Box, the student sees $6 \times 3 = 18$ as shifting from 6 people to 18 coins by means of a multiplication by 3 coins per person, which transforms the referent using an intensive quantity as an operator. There are two associated division problems for fair sharing $18 \div 6 = 3$ and $18 \div 3 = 6$, each of which is referent-transforming. Students are likely to solve the first one partitively and the second quotatively.

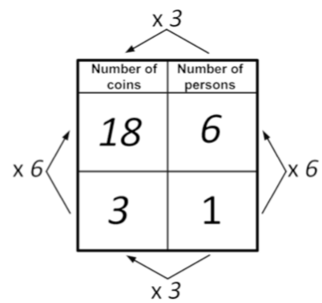

Figure 7. D/M box used to model referent-transforming multiplication

b) *Referent-Preserving*. Division/multiplication in these models involves a multiplicative comparison of two amounts of a single quantity. This can be accomplished using a new unit, a composite unit such as a group or a scale, or by using one amount to measure another while the referent or attribute is maintained. For example, if one is told that the distance from New York to Kansas City is six times the distance from New York to Baltimore (approximately 200 miles), the D/M box would look like Figure 8a:

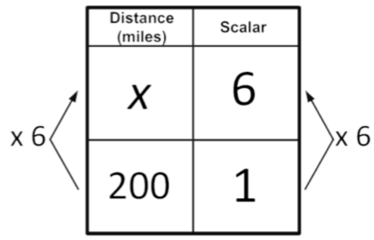

Figure 8a: Referent-preserving multiplication problem modeled with a D/M box

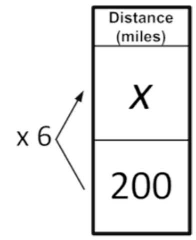

Figure 8b: The same D/M box with arrow indicating the scalar

The scale in the right-hand column is, by most accounts, unit-less, but the right column is used to establish the vertical arrow, or the "within" or referent-preserving relation, "multiply by 6." Thus to solve this problem, one maps miles to miles, multiplying by the dimensionless scalar 6, to get 1200 miles. Because the left-hand column with the scalar multiplication is sufficient to solve the problem, a two-by-one display of this relationship is sufficient as shown in Figure 8b. Likewise we suggest that problems involving groups and measurement conversions can and probably should be treated as referent-preserving because only the unit and not the referent changes.

We note that because the D/M box always has a 1 in one cell, collapsing it to a 2×1 box or a 1×2 box is always possible because the operator arrows will "carry" the information from the non-one cell as illustrated in figure 8b. These collapsed views permit one to assert a single model for division/multiplication; a drawback of this curtailment, if done too early, conceals some of the richness of the relational reasoning.

c) *Referent-Composing.* Division/multiplication in these problems involves the creation of a new referent or attribute not previously associated with the other quantities. For example, the division/multiplication associated with area produces square inches from side lengths in inches. In Cartesian products, a number of shirts and a number of pants produce a number of outfits, and so on. Volume as a product of three length measures or as a product of length and area, and higher dimensions also fit in this category. Arrays can form a transitional representation linking referent-preserving and referent-creating, such that the product can be computed by multiplication of the number of dots in each of the two sides, but the product remains a number of dots so no new referent is composed. The row and column structure, while geometrically extending in two dimensions (length and width) still produces a product that is a total number of dots.

These three models of division and multiplication can be summarized as shown in Table 1 along with examples of problem contexts associated with each model.

Table 1. Three Models of Division/Multiplication, Along with Common Contexts for Each

Model 1: Referent-Transforming	Model 2: Referent-Preserving	Model 3: Referent-Composing
Fair Sharing	Unit Conversion	Arrays
Rate	Scaling	Area
Equal-sized Groups		Cartesian Product

Note the placement of the equal-sized groups context, in which one reasons with the number of groups, the size of the group; the resulting product is placed in both models 1 and 2. A problem such as "a bookshelf has four shelves with six books on each, how many books are there?" can be viewed as referent-transforming (number of books per shelf × number of shelves = number of books) or as referent-preserving (4 groups of 6 books).

As a result of this analysis, the team recognized that the transition to division and multiplication needed to be broadened and strengthened. We analyzed the experiences of children that would support these varied models, especially in the earlier trajectories of equipartitioning and addition/subtraction. The expectation for the DMLT was that students would encounter the models as simple whole-number cases until they built up their repertoire, became fluent and flexible in their knowledge of the associated facts, and explored the properties. As the numbers became larger, the algorithms would be developed. Though not fully developed in this paper, students' introduction to non-whole quantities in the LT division and multiplication involves reconceptualizing meanings based on their understanding of relational naming (describing 12 shared among 4 as 1/4 of the collection) and reassembly from EQLT. Over time, students generalize across the various number types, models and applications as division/multiplication more abstractly. However, by avoiding overgeneralizing and simplifying to one single model, students should remain flexible in selecting appropriate models for division and multiplication in modeling activities.

Bridging Standards

From EQLT, children enter third grade with experience in fair sharing, relational naming, and composition of splits, all of which can support their movement to division/multiplication. Composition of splits refers to children splitting a split (such as a rectangle into two parts vertically and three parts horizontally) and learning to predict six (2 × 3) instead of five (2 + 3) resulting parts. The addition and subtraction LT also links to DMLT through a standard on the array structure and repeated addition. The length, area, and volume LT also contributes to students' conceptions of division and multiplication, and the relevant commutative and distributive properties with such activities as finding a common unit for area measurement and composing and decomposing rectangular areas. Nonetheless, a set of bridging standards were needed—first, to make the necessary connections to these earlier learning trajectories, and secondly, to interpret the meaning of the standards in light of our targets and distinctions.

There are four Standards in CCSS-M that specifically carry the weight of introducing division and multiplication:

- 3.OA.1: "Interpret products of whole numbers";
- 3.OA.2: "Interpret whole-number quotients of whole numbers";
- 3.OA.3: "Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem"; and
- 3.OA.6: "Understand division as an unknown-factor problem."

(Note: 3.OA.4 is placed in elementary algebra because it involves solving for an unknown in any position in $a \times b = c$; 3.OA.5 (concerned with properties) and 3.OA.7 (concerned with fluency) are placed in the next section of the DMLT.)

While these four standards are sufficient to support the distinctions offered above, they are awkward to interpret standard by standard: three of them are required to introduce and link multiplication and division (3.OA.1, 2,and 6), and the examples mentioned along with the first two in the CCSS-M document seem to imply that a problem type is linked to an operation (groups to multiplication and fair sharing to division). Furthermore, 3.OA.3 seems to imply that the problem situations are used to apply the operations rather than that the operations are developed to model the situations. This bias seems to be pervasive in the K–8 Standards.

However, what appears to be awkwardness in the Standards can be addressed because the examples therein are not intended to limit the cases but only to illustrate them. Therefore in our interpretations, we explain the three cases of multiplication (referent-transforming, referent-preserving and referent-composing), then treat division similarly, using Standard 3.OA.6 to link the operations. While the model remains referent-transforming, the observed processes for the division problems may appear as partitive or quotative.

Standard 3.OA.3 provides an opportunity to summarize the entire framework with descriptions of the overall D/M box for whole numbers and the area model. In preparation for Standard 3.OA.3, three bridging standards were required for the model for referent-composing D/M. The bridging standard 3.OA.F ("Students reason with arrays using multiplicative relationships") was added to provide students opportunities to work multiplicatively with arrays. This was necessary because the standard authors had restricted the approach to arrays in second grade to repeated addition (2.OA.4: "Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends"). This constraint ruled out other approaches such as by decomposing and composing arrays into other equivalent arrangements (for instance, rearranging a 6 × 4 array as a 12 × 2 or a 24 × 1), or using skip counting.

Building on a bridging standard from the EQLT (2.G.C: "Equipartition a rectangle using vertical and horizontal cuts and predict the resulting number of parts."), another bridging standard, 3.OA.D ("Students learn to code composition of splits as multiplication and can state the associated division problem"), supports students in coding compositions of splits as multiplication and division. From the length, area and volume LT, the standard 3.MD.7.b ("Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning"), links to the emerging DMLT. To complete the idea of referent composition then for both area and for pairing of attributes to create Cartesian products, bridging standard 3.OA.B was added, stating "Relate multiplication and division problems to rectangular area (e.g., 3 inches × 4 inches = 12 square inches) and Cartesian products (e.g., 3 pants × 2 shirts = 6 possible outfits)."

With this set of three bridging standards carefully linked to the four CCSS-M Standards, third grade students who accomplish the related content should be able to apply all three models to situations to produce both division and multiplication problems and solve for unknowns in all of the three positions of the problem in standard 3.OA.3. Well-prepared with three models, students can be carefully introduced to the cases in which non-whole numbers are involved, topics that are discussed more fully on the website. As argued previously, this approach is also powerful because it builds explicitly from prior learning trajectories and anticipates later ones.

Strategies, Representations, and Misconceptions

The previous section on distinctions and models supports students in creating a rich variety of representations for multiplication and division (groups, tree diagrams, measures, scaled drawings, and Cartesian products shown as two dimensional cross products). A second important area of development involves how children learn their "multiplication and division facts." Confrey and Scarano (1995) had demonstrated that children are not given adequate support to "move in multiplicative space." Most teachers assume that multiplication should be introduced separately from division and that learning number facts should proceed in the same order as addition facts, from small to large numbers. Instead, the LT research shows how many forms of interrelationships among and between multiplication facts can be fostered by teaching children rich strategies that build on early understanding of numbers. For example, instead of teaching multiplication facts in the order of the counting numbers (i.e., ×1, ×2, ×3, etc.), Confrey showed that a sequence of double (×2), double-double (×4), double-double-double (×8), then multiplying by 10 and then by 5 (×10 ÷2), then tripling (×3), multiplying by 6, (triple-double, or ×3 ×2), and by 9 (triple-triple), and then, finally, by 7, is more readily understood by students, and makes more sense to them. (The related division facts are practiced simultaneously with multiplication facts in this sequence.) Instead of viewing multiplication facts as simply a list of things to be memorized, students begin to get a foundation of the multiplicative relationships among numbers—what I have previously called "moving around in multiplicative space" (Confrey, 1995).

Two misconceptions are addressed in the DMLT. An early standard in the LT regards the idea of "evenness" (as contrasted with "oddness"), and the descriptors carefully articulate two approaches, (1) fair sharing by two, and (2) pairing up. In addition, the descriptors warn that students use the term "even" to describe when a collection can be fairly or evenly shared, for example, in the sentence, "It came out even." The descriptors discuss how the term "even" therefore can be used simultaneously by students in two conflicting ways, (1) to describe when a factor divides evenly—then the result is even (so that six shared among two is three which is "even" or fair), and (2) to describe that when a number is "even," i.e., is divisible by two. The two meanings must be distinguished by students, so they avoid or resolve a "misconception." This is a prime example in which we wrote into the descriptors an important distinction that we believe many teachers would not readily recognize and discuss with their students.

The second, more widely recognized, misconception is "multiplication makes bigger and division makes smaller" (MMBDMS) (Greer, 1992). The CCSS-M address this misconception directly in 5.NF.5.b ("Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1").

In the DMLT, the misconception is addressed in relation to each of the three models. In the unit transforming model, the descriptors illustrate that any two numbers can be related in an equation, such as rate × time = distance, so that 30 mph can be multiplied by a half hour to produce 15 (i.e., fewer) miles. Students also learn to interpret division of two quantities, in the form a/b and c/d, as a ratio of fractions or ratios ($3/4 \div 1/2 = 3/2$). This example demonstrates that division can result in a larger quantity than the quantity one begins with. In referent-preserving situations, division by n is shown to be equivalent to multiplication by $1/n$, with students learning to predict the effects of multiplication by a/b as a composition of multiplication and division, just as was done originally in Dienes's work on operators (e.g., stretchers and shrinkers) (Dienes, 1967). Finally, for contexts using the area model, students learn that area measured in square units can be of a smaller magnitude than the magnitudes of either of the sides.

Conceptual Principles

The development of conceptual principles in the DMLT can revolve first around the ideas of factors and multiples. Overreliance on multiplication as exclusively derived from repeated addition leaves students insensitive to the distinctions between additive and multiplicative reasoning. As noted above most students are not given enough experience moving in multiplicative space. In the descriptors, we also offer the view that students should be challenged to find multiple ways using only multiplication and division to move among numbers, such as between 15 and 24 (dividing by 5 and multiplying by 8). I called these types of problems "daisy chains" in earlier work (Confrey & Scarano, 1995). This encourages students to work with common factors. In addition, it helps students to develop knowledge of the principles of multiplication by 1 (identity), multiplication by zero, the commutative property of multiplication, the associative property of multiplication, and, later, multiplicative inverses. It can also lead to students recognizing rational number multiplication and division. In the DMLT, we also treat distributivity very carefully and explicitly, as it is the means by which the additive structures are linked to the multiplicative structures.

Coherent Structure

The coherence of the DMLT's structure can now be summarized. The LT builds from the prior LTs of (a) equipartitioning; (b) length, area, and volume; and (c) addition and subtraction to establish the three models applied to whole numbers. The interrelationships among the ideas of factors and the patterns in the multiplicative table are used to support the evolution of the properties and draw connections to multiplicative vs. additive comparison. Then at the upper end of the LT, two types of extensions occur: the application of the problems to multidigit algorithms using the distributive property, and the inclusion of fractions and ratios as operators. These extensions are carefully constructed in the context of the three underlying models. The extensions to fractional operators are also connected to the learning trajectory on length, area and volume where the MMBDMS misconception can be most readily remediated.

Overall the LT is designed to set up the movement to ratio reasoning through connections to the two Standards on tables of values, 4.MD.1 on conversions and 5.OA.3 on tables of values. Finally, students are prepared for the culmination of equipartitioning in the fifth grade standard (5.NF.3: "Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem"). The target goal of the LT is reached in a set of Standards that include 6.NS.1 ("Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem"), and 7.RP.1 ("Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units").

With this example of how an LT is related to the standards, one can see that the process of linking an LT to standards requires careful and synthetic applications of empirical research literature. The overall framework for multiplication and division is thin in the early grades and tends to overemphasize a relationship to additive structures, resulting in an underdeveloped framework for multiplicative structures. We have attempted to articulate a stronger framework for a stronger multiplicative structures approach by adding a few key bridging standards within the learning trajectory which link to equipartitioning and help to explain how multiple models of division and multiplication can be supported in classroom instruction. The authors of the CCSS-M left room for such interpretations by avoiding the mistake of *defining* multiplication as repeated addition (which had been included in early drafts of the CCSS-M). The learning trajectory also makes the case for both strong distinction among the strategies, and strong relationships among the models, strategies, and associated properties.

Implications for Researchers and Professional Developers

The www.turnonccmath.com website was visited more than 7000 times between its release in April 2012 and late May 2012. The primary visitors have been state and district personnel and teachers looking for a means to make sense of and make instructional interpretations from the CCSS-M. Some found the website on their own while others have found it as a result of presentations and mailings. We are currently in the process of improving the site in several ways. We are adding in the relevant references to research that we were unable to do in the first round due to the pressures of time and the focus on creating coherence and consistency in the descriptors; as one can imagine, this has been hard work. We are also preparing to undertake an expert review process, similar to the process we conducted for vetting the LT on length, area and volume with the researchers from the Measurement Mini-Center.

We are also committed to working with districts and states using the LTs and their descriptors as a basis for professional development. These efforts include both pre-service and in-service teachers. We have worked with Colorado, West Virginia, North Carolina, and Washington, and have received requests from other states. In this work, it becomes clear that the foundation of knowledge in the unpacking is not on its own sufficient to support professional development; the examples in this paper make it clear that the written descriptors by themselves can serve as an important part of efforts to help teachers understand the mathematical knowledge embedded in the trajectories and to translate them into robust learning trajectory-based classroom practice.

There are numerous opportunities for college and university faculty and state and district mathematics coordinators to use these materials to support professional development. We have engaged in creating digital presentations to show, in a more visual and story-based way, how the LTs are linked to the standards. One could imagine building webinars and course materials to provide hands-on experiences for teachers with these ideas as well, assuming sufficient available resources. Some of the teams developing the original LTs have already created related professional development materials that can be used in creating a nationwide application of this work.

Perhaps even more relevant to the PME-NA audience is the potential professional value of the website to the research community. To some degree, the influence of learning trajectories/progressions on the CCSS-M was mitigated by ambiguity, dispute, or lack of synthesis by the research community. While this is not surprising in a field as young as ours, its maturation depends on our willingness to undertake synthesis, and suggests it would be wise to engage in more of this kind of activity. While researchers may wish to "do their own thing" or await some other body to interpret and synthesize the development of the Standards, it would improve our professional reputation as a field if we were to take up this challenge ourselves.

It is often reported that in medicine, prior to the famous Flexner report (Flexner, 1910), physicians received education in general basic science and then apprenticed to a working physician until they were ready to establish their own practice. If that mentor was a strong and knowledgeable role model, the apprentice was likely to emerge as a well-qualified and very competent physician as well. If not, another "quack" might be added to the rolls. After the Flexner report, the medical field stepped up to create a practitioner-informed practice-oriented knowledge base for "clinical training" of physicians and to standardize medical education. In some ways, we are in a similar predicament in mathematics education research. Someone studying in a strong program, or apprenticing with a strong faculty member, tends to move into teacher education well prepared. Study in a less rigorous program and navigating the literature without any guidance leaves one tasked with "inventing" a deep understanding of the literature: the job is highly inefficient, at best, and likely to leave a student poorly prepared to take up highly informed work or to make insightful contributions. Synthesis work is challenging, sometimes grueling, and yet remarkably satisfying. The www.turnonccmath.com website is meant to serve as one contribution to increasing the accessibility, completeness, and consistency of the interpretation of the significant portion of the research base in mathematics education on student learning.

Our research group has been the beneficiary of one of the REESE synthesis grants to bring together a literature on rational number reasoning that consists of some 600 articles. This experience has led us to this synthesis of the LTs work with the CCSS-M. It may be the case that the idea of LTs will fade, just as so many movements in mathematics education do (e.g., metacognition, problem solving, differentiated instruction, active mathematics teaching, and individualized instruction; the list is, sadly, quite long). Many valuable lessons resided in those movements, and for the field to become robust for guiding the conduct of practice, it must create a means for its empirical work to accrue progressively and be refined over time. Such a means would help reduce the frequency with which we see the same studies conducted (e.g., students mistaking the visual path of a function's representation for the behavior of the function has been studied too many times to count), and help to define a cutting edge field where scholars can aim to make progress. All of these suggestions fulfill the vision of the conference organizers for this PME-NA annual meeting to discuss transitions. The bulk of this paper addressed how to create supports for teachers as they transition to the CCSS-M, but the discussions herein also address transitions for professional developers and researchers in the everyday conduct and sharing of our practices.

Endnotes

[1] The author was a member of the National Validation Committee for the Common Core State Standards.

[2] This meeting was jointly hosted by the DELTA research group (directors Confrey and Maloney) and the Consortium for Policy Research in Education (co-sponsors F. Mosher, P. Daro, and T. Corcoran).

[3] The Mini-Center comprises faculty and senior researchers (J. Smith, organizer, J. Confrey, J. Barrett, R. Lehrer, M. Battista, D. Clements, B. Dougherty, D. Heck) and associated postdoctoral researchers and graduate students.

[4] One can also use the D/M box (Figure 5a) to apply to area, if one starts with a unit square and views b as stretching b into a strip of b units, for example, as a strip along the top of Figure 5b. Then if c represents a c × 1 strip vertically along the left edge, then stretching it by b produces bc; and the ratios are preserved. This model seems too abstract and so we prefer to introduce the area model separately.

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SCHOOL MATHEMATICS ARTICULATION: CONCEPTUALIZING THE NATURE OF STUDENTS' TRANSITIONS (AND TEACHERS' PARTICIPATION IN THEM), K–16

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Students experience a variety of challenges as they move from one level in school to the next. In this session, we consider and discuss two central questions related to students' progressions through their mathematical experience, particularly at transitions roughly characterized as elementary to middle school, middle school to high school, and high school to post-secondary: What are the key dimensions/aspects of such transitions? What kinds of system-level responses address students' issues with transitions, particularly when they are problematic? We discuss research and practice related to students' challenges and the nature of system-level responses to various aspects of these school mathematics transitions, including mathematical content, curriculum, students' dispositions, classroom teaching practices, and school structures. Characteristics of selected strategies and programs are discussed and questions for further research are presented.

If students' motivation to learn mathematics, attitude toward mathematics, and interest in mathematics tends to decline as students progress through levels of education (Middleton & Spanias, 1999), then it is worthwhile to look more closely at how students experience school mathematics over time. Additionally, concerns have been expressed about the shortage of qualified workers for careers in mathematics, science, engineering, and technology (National Science Board [NSB], 2006) and the mathematical demands of informed civic engagement. Although we recognize that there are many fulfilling professional paths outside of STEM fields, it is important to consider how the accumulation of students' school mathematics experiences over time could inform their career choices and their relationships with the discipline of mathematics. If students choose not to engage further in mathematics beyond their required school experience, ideally they would make this choice because they prefer another option, not because their school experiences have taught them that mathematics is an intellectual and practical activity to avoid.

This paper addresses school mathematics articulation in terms of students' experiences as they move through school – from kindergarten through college. The study of students' progressions through levels of education provides insights about what we know and don't know about being a mathematics learner at various points in time in students' lives. We summarize research findings from some select studies to describe some transition issues as students move from (a) elementary school to middle school, (b) middle school to high school, and (c) high school to post-secondary experiences. To provide some conceptual clarity for the study of students' progressions through levels of education, we ask, "What are some of the aspects and dimensions of students' transition experiences as they move through their schooling?" To address this question, we will discuss various conceptualizations of students' transition experiences across school settings, such as: factors in school mathematics settings that can change over time, student-level factors that could indicate variations in their "transition" experiences, and conceptual lenses for viewing these factors (person-environment fit, what counts as a "mathematical transition," boundary-crossing, and rite of passage). We follow this discussion with the question, "What kinds of system-level responses address transition issues?" In response, we describe a few promising system-level responses to describe possible efforts to support students as they progress through mathematics programs over time. Finally, at the end of the paper, we explore promising possibilities for future research on students' transitions.

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Focus on Students' Experiences as They Move through School Settings

One of our premises in this paper is that students' experiences as learners of mathematics as they progress through school are important to understand and support. In addition to learning mathematics content, students are *becoming* mathematics learners as they move through school settings:

> As they [students] are compelled to sit in a mathematics classroom for a significant period of their school life, they come to learn how to participate in that context – they learn when to respond, when to resist, how to appear busy but avoid work. They learn how to cope with the embarrassment, the joy, the cajoling. They learn how the actions in the classroom have meaning and how some of the actions of teachers, texts and students take on substantially different meanings for themselves and others. They learn how to *be* a mathematics student. They develop a sense of who they are as learners within this context, a context which may be very different from other subjects within the school context and beyond the school context. (Boaler, William, & Zevenbergen, 2000, p. 3)

In this manner, we foreground the study of students' identities, as some have argued that "…learning and a sense of identity are inseparable: They are the same phenomenon." (Lave & Wenger, 1991, p. 115). Although our perspective does not equate learning and identity development, we highly value identity development as a significant outcome of students' school mathematics experiences, in addition to learning academic content. As students move through school settings, students develop their beliefs and practices as learners of mathematics and develop affiliations (or not) with the subject matter. Understanding students' experiences as a process of identity development is a way of conceptualizing learning. According to Wenger (1998), learning occurs through participating in communities of practice. Participating involves not only thinking and acting, but also developing increasingly central membership within communities. From this perspective, learning "changes who we are by changing our ability to participate, to belong, to negotiate meaning" (Wenger, 1998, p. 226).

What students learn—the ways students come to participate, come to view themselves, and come to view mathematics—is situated within opportunities to participate, and opportunities to participate are likely to vary as students move across school settings. School settings can be considered to be "facilitating contexts" (Grootenboer & Zevenbergen, 2008, p. 245) in which students have opportunities to develop relationships with mathematics. We recognize that opportunities to participate in school experiences may change as students move from one classroom to another. However, in this paper we focus on changes that can occur between grade bands—moving from elementary school to middle school, middle school to high school, and high school to other post-secondary experiences. An assumption in work on students' school transitions is that there are often more differences in mathematics teaching and learning between school buildings than within them and that these differences have implications for students' experiences.

We acknowledge that structures of school settings vary within the United States and also can differ between the U.S. school system and those of other North American nations. For instance, within the U.S. structure of elementary and middle schools, there are various configurations. Students may attend schools that include kindergarten through eighth grade on the same campus. Another structure involves schools constructed by grades K–5 on one campus and grades 6–8 on another campus. Still other configurations include grades K–6 on one campus and grades 7–9 on another (with high school starting at grade 10 rather than grade 9). For the purposes of this paper, we consider "elementary school" to encompass kindergarten through fifth grade, "middle school" to address grades six through eight, and "high school" to include grades nine through twelve. These demarcations follow the grade bands described in the *Principles and Standards for School Mathematics* (NCTM, 2000).

There may be an embedded assumption in work on school articulation and transitions across school settings that students remain in a particular school setting or school district for an extended period of time. We recognize that students may be mobile even during a particular school year. According to recent data (US GAO, 2010), 11.5 percent of K–8 schools have high rates of student mobility, such that more than 10 percent of students left by the end of the school year. These schools also had higher percentages of students who were low-income, English language learners, and received special education. However, we believe that the conceptual lenses and the transition issues affecting students discussed in this paper could be applied to some degree to students moving into new schools or new classrooms at other points in time.

Experiencing transitions involves navigating change, such as changes in approaches to mathematics teaching and curriculum between school buildings. In our effort to understand school mathematics articulation (or lack of it) over time, we do not believe that it is entirely possible or necessarily desirable to eliminate the changes that students experience. A perfect alignment of experiences over time is not possible or even ideal. Rather, we hope to consider the kinds of changes students might experience, whether students are aware of these changes and how they might respond, how to support students in navigating changes between school settings over time, and whether the changes that students encounter are purposefully created or occurring haphazardly.

The four authors of this paper collaborated because of our different experiences with scholarship around students' school mathematics transitions. Amanda and Jack worked together, with a number of other scholars, on the Mathematical Transitions Project [MTP]. Funded by the National Science Foundation (Jack Smith, Principal Investigator), our work in MTP investigated students' experiences as they moved from middle school to high school and high school to college when the mathematics curriculum materials shifted from either reform to traditional or traditional to reform (cf., Smith & Star, 2007; Star & Smith, 2006; Star, Smith, & Jansen, 2008; Jansen, Herbel-Eisenmann, & Smith, in press, 2012). Cathy and Janie were involved with writing a series of articles for *Teaching Children Mathematics* (Schielack & Seeley, 2010), *Mathematics Teaching in the Middle School* (Brown & Seeley, 2010), and *Mathematics Teacher* (Hull & Seeley, 2010) about students' experiences as they move from one level of education to another. This paper is an opportunity to synthesize and share what we've learned and to encourage mathematics educators to do more to consider how to understand and support developing students in the context of moving across school settings.

Conceptualizing Students' School Mathematics Transitions

Research on students' school mathematics transitions can be conducted from a range of perspectives. There can be a focus upon (1) the *internal* experiences of students, (2) the success (or not) of particular students in "moving along" as judged by *external* standards (grades, course-taking, etc.), (3) the success (or not) of institutions in supporting *aggregate* student success over time, (4) the effects of curriculum or teaching practice as they *correlate* to students' experience (internal) and/or success (external). To explore some of these foci, we share a representation that highlights some of the main factors in school mathematics settings that could either change or remain consistent over time and student-level dimensions that could indicate variations in their "transition" experiences (Figure 1).

We wanted this figure to display four temporal stages in time or grade bands (elementary years, middle school years, high school years, and post-secondary years of college and/or career). Also, we wanted to highlight a few factors in the school settings as well as student-level dimensions (both internal and external). The central column (large arrow) lists important student-level dimensions, such as learning, achievement, dispositions, patterns of working, and identity/direction. These dimensions could be relevant for students at any point in time. The ovals represent factors in school settings that may influence students' at any point in time and that could influence or shape any of the student-level dimensions. Figure 1 highlights some of the complexities of students' transitions across school mathematics programs, as changes along any of these factors in school settings or changes in any of the student-level dimensions could be significant to students in their experiences with school mathematics.

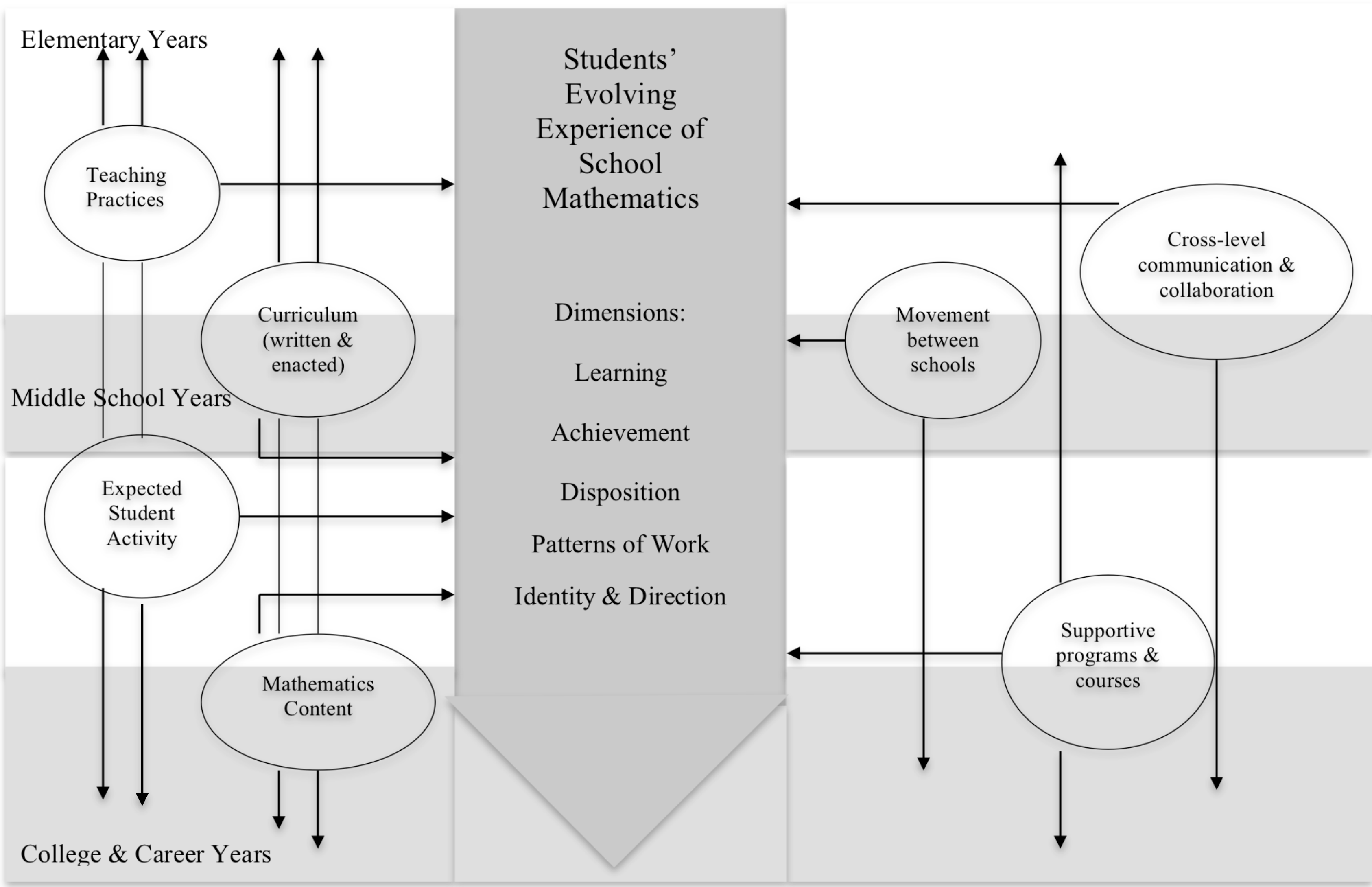

Figure 1: Visualizing students' transitions and the factors that shape them

Aspects and Dimensions of Issues in Students' Transitions

We wrestled with the order through which to present students' experiences over time. In the spirit of backwards design (Wiggins & McTighe, 2005), we considered starting our discussion of transition issues with post-secondary experiences (college and careers) and working back to earlier stages of schooling. This approach would have affording reflecting upon where students could land and working backward to support their successful journeys. Alternatively, we could begin with elementary school and move forward, because this timeline aligns with how students experience the accumulation of school mathematics experiences. After much discussion, we chose the latter option, as (for one reason) it is such a familiar frame for the issues that we consider.

Below, we share findings from a few select research studies. These studies provide insight on some conceptual lenses that have been used to understand the nature of students' transition experiences as they move across school settings: stage- or person-environment fit, boundary-crossing, and rite of passage. We present each of these conceptual lenses as we discuss the grade band of students or level of transition that the researchers studied. However, we do not mean to suggest that the conceptual lens should be used only with this grade band. Rather, we believe that these conceptual lenses could provide insight at any transition period, so lenses used previously to understand the transition from elementary school to middle school could also be useful for studying other transitions, such as the transition from high school to college (or to other post-secondary experiences). Other conceptual lenses that are useful for understanding students' experiences will be presented later in the paper.

The projects we discuss also highlight transition issues that students might experience as they move through school over time. We selected these studies because they highlighted ways to see interactions between dimensions of students' school mathematics transitions and system level responses, as many other studies report outcomes rather than offering explanations about why students might experience these outcomes.

Elementary School to Middle School

Schielack and Seeley (2010) previously summarized some of the issues that students often experience when moving from elementary into the middle grades mathematics. They described a student-level dimension: decreases in achievement in mathematics over time. Prior research suggests that, in general, students experience significant declines in academic achievement as they move from elementary school to middle school (Alspaugh 1998). They also highlighted factors in school settings, such as surface and substantive differences in curriculum materials, the variance in instructional approaches between settings, changes expectations for students' work, and increased difficulty in content. Some of the surface level features in the curriculum materials include change in color schemes, word density, font size, frequency of word problems or computational items in exercise sets. More substantively, the curriculum materials generally differ with respect to the types of representations used. For instance, elementary school mathematics textbook representations may be large and spacious (e.g., use of area models to represent fractions), and middle school mathematics textbook representations often using more symbolic and compact representations (e.g., linear models or number lines to represent fractions). Instructional approaches often vary in the degree to which teachers enact direct instruction and position students as receivers of knowledge or whether teachers encourage open exploration in which teachers act as facilitators. Schielack and Seeley (2010) acknowledge that these differences in curriculum or instruction could be reversed at the different grade levels. Changes in expectations for students' work in middle school include increases in the amount of independent work, including homework.

One conceptual lens that researchers have used to view students' experiences as they move from elementary school to middle school is the *stage-environment fit* perspective, which focuses on the degree to which students' developmental needs are met by the structures and practices of schooling. Foundational research on adolescent development (outside of mathematics education) has conceptualized students' experiences in the context of educational change in terms of fit between students' developmental needs and the school environment. This line of research addresses changes in adolescents' motivation over time as they move between school buildings. Declines in student motivation have been explained as a lack of a stage-environment fit (or a mismatch) (Eccles et al., 1993, following Erikson [1968]). "[R]esearch has found that academic declines in interest and self-concept are a function of the mismatch between the school environment and the adolescent" (Zarrett & Eccles, 2006, p. 17). A stage-environment fit is the quality of the match between the developmental needs of adolescents and the nature of the learning and social opportunities afforded to them. From this perspective, declines in students' motivation are not conceived as student deficits but as results of misalignment between students' needs at their stage of development and the learning and social opportunities afforded to them in their school experiences. Alternatively, students' motivation could remain strong or improve if there exists a fit between the student's needs and his or her experience in school.

Studies of the transition from elementary school to middle school reveal some examples of stage-environment mismatches (e.g., Eccles et al., 1993; Roeser, Eccles, & Sameroff, 2000; Wigfield, Eccles, & Roderiguez, 1998). The move toward ability grouping in the transition to middle school emphasizes social comparison at a time of heightened self-focus for adolescents. If teacher control increases and students' choices decrease in middle school, this conflicts with adolescents' increasing needs for autonomy. If teachers become more distant in their interactions with students in middle school, this may conflict with an adolescents' need to foster stronger relationships with adults outside the home.

Transition studies often provide insight at more of a top level, such as why students would continue to engage or not in school generally. However, there are some noteworthy exceptions. Roeser, Eccles, and Sameroff (2000) found that when middle school students perceived their school's curriculum to be meaningful, relevant to their lives, and supportive of their autonomy, they also expressed higher academic competence and higher academic value. Additionally, Midgley, Feldlaufer, and Eccles (1989) found that, during their transition into middle school, students who perceived lower degrees of support from their new mathematics teachers also reported lower intrinsic values for mathematics. More troubling, these findings were stronger for lower achieving students.

Looking back at our representation in Figure 1, these studies highlight particular factors in school settings and student-level dimensions. A move toward ability grouping indicates an example of change in the structure of the mathematics program that students experience. A more distant teacher-student relationship, reduced teacher support, and an increase in teacher control represent changes in instructional practices. An example of curricular factors was the degree to which the curriculum was perceived to be meaningful and relevant. These studies varied in terms of whether the factors in the school setting were perceived by students (self-reported) or observed by researchers. Student-level dimensions described were their reflections upon their identities (heightened attention to self, need for relationships with adults outside of the home) and disposition (need for autonomy, sense of competence, high value for academics).

Middle School to High School

As students move from middle school to high school, some of the factors in school settings and student-level dimensions occur again for students, and additional factors and student-level dimensions are incorporated for others. Brown and Seeley (2010) describe a range of factors in school settings and student-level dimensions that often change as students move from middle school into high school. Regarding school factors, they identified potentially insufficient alignment of mathematics instruction and curriculum materials across grade bands, specific issues with mathematics content in high school (e.g., mandatory Algebra I), and problems that could occur if high school teachers construct students as being "unmotivated" rather than trying to understand what they can do to motivate students. They also describe student-level dimensions, particularly decreases in achievement that seem to occur if students experience lack of alignment in curricular or instructional approaches (differences in the degree to which the mathematics programs are problem-centered and evoke sense-making or focus on teacher-directed procedural instruction). As we review some of the prior research on students' transitions into high school mathematics programs, we describe three conceptual lenses: person-environment fit, defining the nature of a mathematical transition, and boundary crossing.

Person-Environment fit. The person-environment fit conceptual lens is a variation on stage-environment fit. Studies of stage-environment fit have been conducted in the context of the transition from middle school to high school (e.g., Barber & Olsen, 2004; Isakson & Jarvis, 1999), not only in the context of the transition from middle school to elementary school. A transition into high school typically involves similar changes as those that occur during a transition from elementary to middle school (e.g., decreased autonomy in the classroom, decreased support from teachers). A transition into high school may be less disruptive than a transition to middle school, since the first transition tends to have more impact (Barber & Olsen, 2004). However, for students who attended a school structured in grades K-8 prior to high school, the transition into high school may be their first move between buildings. Students' transitions into high school have been associated with declines in their academic performance (Barber & Olsen, 2004; Isakson & Jarvis, 1999; Rice, 2001). For high school students, one important developmental challenge of middle adolescence is to become more self-reliant and self-governing (Kimmel & Weiner, 1995; Powell, Ferrar, & Cohen, 1985). In contrast to early adolescents, middle adolescents face increased future-related pressures as they begin to prepare for their lives beyond K–12 education. This perspective draws attention to student-level dimensions, such as achievement and identity (becoming more self-reliant) as well as direction (future-related pressures).

A fit or a mismatch with one's environment may not necessarily be developmental, so we believe that *person-environment fit* (Hunt, 1975) can be a more appropriate term for understanding students' experiences than stage-environment fit. Stage-environment fit addresses the fit between the school setting and a student's developmental needs, but there may be other aspects of the person that can fit or not with the environment. For example, a fit or mismatch may be due to the alignment (or lack thereof) between students' individual epistemological beliefs about the nature of knowing and the approach to mathematics instruction in their classrooms, and these beliefs may not be tied to students' development. Epistemological beliefs have been shown to vary by gender (Gilligan, 1982) and by curricular contexts (Star & Hoffmann, 2005). Boaler (1997) explained high school females' experiences in different mathematics programs in terms of the fit or mismatch between their academic contexts and women's ways of knowing. She drew on the work of Gilligan (1982), who described differences between "separate" and "connected" knowing. Separate thinkers prefer to work with subjects that are characterized by logic, rigor, absolute truth and rationality; and connected thinkers prefer to use intuition, creativity, personal processes and experience. The young women in Boaler's study expressed a preference for learning mathematics through a more open, problem-solving approach that supported their autonomous sense making in mathematics, or an approach more aligned with connected knowing, and they expressed dislike for a more closed, teacher-led approach that was more aligned with separate knowing. Epistemological beliefs represent a student-level dimension (disposition). School-level factors described in this work include teaching practices, enacted curriculum, and expected student activity.

Looking across conceptual lenses that address "fit" (or mismatch) with environment, a common approach is to characterize the degree of overlap between students' perceived needs and preferences and what the environment affords. Both students and school settings change over time. An implicit assumption with this research is that two settings should adjusted to become more aligned (and aligned in ways that also support developing learners). The alternative perspective is that change and challenge is essential to healthy development so the focus should be on how students adjust to those changes and challenges But what frames are available to examine adjustment to changing mathematical contexts for learning? Two perspectives, the concept of a mathematical transition and the lens of boundary crossing, provide alternative ways to view moving into new school settings other than fit with environment.

What counts as a "mathematical transition"? Something to consider in research on educational change is whether changes are noticed by students and how students respond when noticing particular changes. The Mathematical Transitions Project [MTP] team (cf., Smith & Star, 2007; Star, Smith, & Jansen, 2008; Jansen, Herbel-Eisenmann, & Smith, in press, 2012) did assume that changes that adults observed in curriculum and instruction when students moved from one building to another would necessarily be important for students. Our starting point in characterizing what counts as a mathematical transition was to understand the transition experience from students' perspectives (either when moving into a mathematics program that was reform-oriented from a more traditional program or when moving into a mathematics program that was more traditional from a reform-oriented program). The term, "reform-oriented," in this discussion simply signals the possibility of significant changes in curricular content and instructional practice.

We proposed that students experienced a *mathematical transition* if data indicated a significant change along two or more (out of four) dimensions. These dimensions were chosen to capture students' cognitive, affective, and behavioral experiences and included: (a) whether a student reported a significant number of *differences* between their middle school and high school programs, (b) self-reported changes in a student's *disposition* toward mathematics, (c) significant changes in mathematics *achievement*, and (d) self-reported changes in a student's *approach to learning* mathematics. We defined a "transition type" as any combination of significant changes in two or more of these dimensions. (For more on concepts and methods in MTP, see Smith and Star [2007].) Note that factors in school settings were captured from students' perceptions, with respect to the differences that students self-reported. Student-level dimensions that we investigated included their achievement (in terms of course grades over time and overall GPAs), dispositions (self-efficacy, attitude toward mathematics, reports of career goals), and patterns of work (approach to learning). We collected our data over two and a half years at two high school sites and two universities and followed approximately 25 students at each site.

Two of the MTP sites captured students' experiences as they moved from middle school to high school, and results indicated that two-thirds of our focal participants did not experience significant changes in achievement and less than 20% of our high school students changed their learning approaches (Smith & Star, 2007). (When high school students' achievement did change, in approximately three-fourths of these cases, achievement fell.) However, we do not mean to suggest that such lack of change when moving into high school is representative or typical, as we only worked with about 25 students at each high school. Rather, these data suggest that students could have a mathematical transition when moving into high school that does not primarily focus upon changes in achievement or learning approach as the most relevant student-level dimensions.

When students experienced "mathematical transitions" at the two MTP high school sites, they noticed significant differences between their middle school and high school mathematics programs and changed their dispositions toward mathematics. Patterns in students' disposition changes at each site appeared to vary with respect to curricular shifts (Smith & Star, 2007). When the dispositions towards mathematics of students who moved into a high school with a reform-oriented mathematics program (in which the teachers used Core Plus Mathematics Project [CPMP] [Hirsch, Coxford, Fey, & Schoen, 2005]) changed, they became more positive. In contrast to their prior experiences in a more traditional middle school, these high school students reported liking CPMP's focus away from repeated practice on very similar problems, increase in story problems, more group work, and a focus on understanding and sense making. Students who moved into a more traditional high school mathematics program from a reform-oriented middle school mathematics program (in which the teachers used the Connected Mathematics Project [CMP] [Lappan, Fey, Friel, Fitzgerald, & Phillips, 1995]) experienced a range of disposition changes. Among the students whose dispositions changed, there was a mix of both more positive and more negative dispositions toward mathematics in high school, with slightly more students developing slightly more negative dispositions. Students whose dispositions changed reacted to similar factors in school settings (e.g., more distant teacher-student relationships in high school, more challenging mathematics content in high school, more word problems in reform mathematics programs), but some students preferred the middle school and others preferred the high school mathematics program.

Learning during boundary crossing. Rather than assuming that school settings could potentially become more aligned or assuming that change in mathematics programs over time is inherently problematic, the conceptual lens of learning during boundary crossing could be used for understanding how students experience educational change. "Boundary crossing" refers to a person's interactions and transactions across different settings (Akkerman & Bakker, 2011). Jansen, Herbel-Eisenmann, and Smith (in press, 2012) drew upon the concept of boundary crossing to examine two cases of MTP students' transitions into the high school site in which students moved from a middle school that used CMP into a high school with a more traditional mathematics program. Following Akkerman and Bakker (2011), a "boundary" was seen as "a sociocultural difference leading to a discontinuity in action or interaction"

(p. 134). A discontinuity could involve changes along any of the factors in school settings, as presented in Figure 1. Such changes could lead to students' experiences of adjusting their roles in each setting. Following Jackson (2011), "setting" was a distinct physical space, and we considered that different physicals spaces (i.e., school buildings) could typically "enclose" different school practices. Rather than viewing boundary crossing experiences as barriers to learning, they can be perceived as potential resources for learning.

Jansen, Herbel-Eisenmann, and Smith (in press, 2012) presented cases of two students that exemplified two learning processes that could occur during boundary crossing in the process of transitioning out of a reform mathematics program into high school. Drawing on Akkerman and Bakker's (2011) characterizations of learning mechanisms during boundary crossing, our cases illustrated two processes of making sense of practices in multiple contexts: identification and reflection.

> Where identification represents a focus on a renewed sense of practices and a reconstruction of current identity or identities, reflection results in an expanded set of perspectives and thus a new construction of identity that informs future practice. (Akkerman & Bakker, 2011, p. 146)

These conceptualizations highlight that learning during boundary crossing can involve reifying one's current identity (identification) or constructing a new identity through expanding one's perspectives about practices in both settings (reflection). In this work, we present analytic tools for identifying students' boundary crossing experiences and describe the nature of learning that appeared to occur during those experiences.

The case of Bethany (identification) illustrated a student who had a strong preference for her CMP experience in middle school and fought to retain the aspects of that experience that she preferred, even when her high school experience did not provide clear opportunities to do so. For instance, she valued that her middle school mathematics teachers explicitly encouraged her to develop and share her own solution methods, expressed frustration that her high school mathematics teachers were "teaching one way… I've never done the same way as the teacher," and experienced conflicts with her mathematics teachers when they took off points for solutions that were correct yet did not align with their taught procedure or when they would not listen to her ideas for how to solve the problem. Through making sense of her boundary-crossing experience, she appeared to solidify her identity as a learner and doer of mathematics.

Ethan's case (reflection) demonstrated a student who, through his continual use of metaphors, expanded his perspective about school mathematics through experiencing two different mathematics programs. One of these metaphors included running water over an ice cube tray. He observed that the middle school mathematics teachers filled the ice cube trays slowly and carefully while the high school mathematics teachers ran the water quickly over the trays, which represented the degree to which teachers monitored student understanding in each setting. He reported liking his high school mathematics program slightly more than his middle school mathematics program, because he appreciated what he perceived to be an increase in autonomy and challenge. Learning occurred for Ethan through reflection because he constructed new understandings of the differences between the two settings and developed a new sense of identity (that he called "ambidextrous") such that he believed that he could be successful in either type of setting.

High School to Post-Secondary School (College and/or Careers)

We recognize that ideally all students should have a diversity of learning and work options beyond high school; not every student should be expected to attend a four-year university. This diversity of potential post-secondary experiences adds additional challenges to studying the transition out of high school. Even college-bound students who have been historically successful in mathematics may not be successful in college mathematics (Smith & Star, 2007). It is important to understand the range of factors at play in students' experience with the transition to post-secondary experiences. Hull and Seeley (2010) note some factors in students' experiences that appear to be lacking: adults might have lower expectations for students' post-secondary goals than students have for themselves and students are often lacking information about post-secondary experiences, including what colleges require for entrance or placement in particular mathematics courses.

Although we acknowledge the range of post-secondary experiences, the prior research that we describe below primarily focuses upon students' transition from high school to college mathematics programs. We describe two conceptual lenses, including the MTP conception of a mathematical transition and the concept of "rite of passage." Additionally, we describe other transition issues that students might experience when moving into college or a work career.

What counts as a "mathematical transition"? There were two university sites in the MTP project; one included students who experienced CPMP or another reform-oriented curriculum in high school and entered a university with a more traditional calculus program and the other site included students who experienced a more traditional high school mathematics program and moved into a university with a reform-oriented calculus program.

At both university sites, the changes that students experienced appeared to be more similar than different, which suggests that their transitions had more to do with moving into college than shifts in their curriculum materials (Smith & Star, 2007). More than four-fifths of the students at both university sites experienced drops in their achievement. When students' dispositions toward mathematics changed at either university site, their dispositions became more negative. About half of the students at each university site changed their approaches to learning mathematics. Students who moved into the university that had a more traditional mathematics program than they had experienced in high school changed their intellectual participation by struggling to attend class when the dominant activity was lecture presentation in college. Students may have struggled to attend class due to being able to choose whether or not to attend class in college or because they had a preference for their high school mathematics courses that were not lecture-oriented. However, at both sites, the general pattern in their learning approach changes was to read mathematics textbooks more carefully and extensively in college, to complete more homework (even when voluntary), study more for tests, and to seek more help from institutional resources or peers (but not from teachers) in college. Most participating students at both sites reported significant differences between high school and college. Some of these differences were more about the move into college generally, such as the new and more difficult mathematics content that they observed, and other differences were more closely aligned with the shift in curricular programs, such as the increase or decrease in contextual story problems, the increase or decrease of fixed procedures available to solve the problems, and the increase or decrease of the expectations to explain solutions in writing or verbally. Although most students at both universities experienced change on our dimensions, these changes seemed to be more about moving into college generally than the shifts in curricular programs.

Rite of passage. Clark and Lovric (2008, 2009) addressed a need for a theoretical model to understand the high school to university transition in mathematics by adapting the concept of "rite of passage." This concept affords an understanding of both the nature of the transition experience and suggests possibilities for supporting students as they move into college. Below, we will describe how the rite of passage concept provides insights on the nature of the transition experience, and later in the paper we will revisit the concept to consider how to support students' transitions into college.

Rite of passage is a concept from anthropology that describes how people experience a crisis, according to Clark and Lovric (2008, 2009). In such a crisis, routines are interrupted, changed, and distorted (discontinuities in experience). In rites of passage, young people re-establish balance and bring back more regular routines. There are three phases associated with a rite of passage: separation (distancing one's self from a high school mathematics experience and beginning to anticipate the tertiary experience), liminal or transitional phase, and the incorporation phase. The process involves cognitive conflict and culture shock. This rite of passage is marked by a physical separation from family and former homes; combined with the large scale of university settings and programs, shock and stress may be inevitable. The success of moving through a rite of passage depends at least in part upon the assistance offered to the individual undergoing the experience.

Clark and Lovric (2008, 2009) relate the experience of moving from secondary mathematics programs into tertiary programs. They describe the discontinuities in terms of college faculty perceiving a lack of preparation in students' technical or procedural facility and analytic skills and deficiencies in students' fundamental notions about the nature of mathematics (particularly a lack of understanding about the role of proof in mathematics). Additionally, citing Tall (1991), they note that college students struggle with building the cognitive apparatus needed to handle advanced mathematics. A student who completes this rite of passage becomes able to think in more productive ways that are aligned with the new environment. From the perspective of rite of passage, the transition into college will likely involve significant discontinuities. Rather than trying to remove the discontinuities, the goal is to think about how to support students with successfully navigating the discontinuities. (Clark and Lovric's [2008, 2009] suggestions, as informed by the rite of passage concept, will be explored later in this paper.)

System-Level Responses to Support Students as They Progress Through School Experiences

Given the range of discontinuities students might experience in school-related factors as they move through grade bands over time (and associated or co-occurring responses at student-level dimensions), it would be useful to explore some recommendations for supporting students with their transitions through mathematics programs over time. We do so with a caveat: many of these system-level responses have not been thoroughly examined empirically. Some of the recommendations address minimizing the discontinuities between factors in school settings at the transitions between grade bands. Other recommendations take discontinuities between grade bands as a given and focus on how to help students navigate them. More research is needed to understand the conditions under which these system level responses are more and less effective for supporting students in their transitions across grade bands.

Regarding efforts to minimize discontinuities, a consistent recommendation has been for teachers to communicate across grade bands about mathematics teaching and learning (Brown & Seeley, 2010; Hull & Seeley, 2010; Schielack & Seeley, 2010). Teachers at the earlier temporal stages of transitions can develop awareness of what their students will experience in the future and prepare them. Teachers at the later temporal stages can learn more about what their future students will have experienced and what they might be capable of doing or understanding about mathematics. The specific recommendations about how to go about this sort of communication vary slightly. It has been suggested that elementary and middle school mathematics teachers can visit each other's classrooms and have comparative discussions about assignments and students' work (Schielack & Seeley, 2010). Middle school and high school teachers could engage in cross-site collaboration to improve alignment in instructional practices and collaboratively study mathematical goals and expectations (Brown & Seeley, 2010). College faculty and high school teachers could collaborate to develop a shared understanding about what students need to know, develop tasks that exemplify these expectations, and establish exemplars of student work that reflect the depth of knowledge that should be promoted (Hull & Seeley, 2010). To engage in such cross-site collaboration, teachers would need support (in terms of time, structure, and guidance) and shared motivation for working toward better alignment across grade bands. Where the latter may exist in some, perhaps many communities, the support resources typically do not.

Given potential challenges associated with reducing unproductive discontinuities, system-level supports designed to support students with navigating transitions seem more pragmatic and promising. Teachers could create a support network with other teachers, counselors, administrators, and parents to provide students with an early vision about what being "good in mathematics" could mean for students' futures (Schielack & Seeley, 2010). Middle school teachers could work to create classroom cultures that actively engage students such that they support students' cognitive, emotional, and social development (Brown & Seeley, 2010). These efforts could usefully promote the ideas that mathematical competence is malleable rather than fixed and that being good at mathematics involves effort rather than solving problems quickly, and provide every student with a sense of belonging in the mathematics classroom. High school teachers could promote high expectations for every student, build strong relationships with students to reinforce high expectations, and know about (and communicate with students about) what students need to do to prepare for college mathematics courses and mathematics placement exams (Hull & Seeley, 2010).

To prepare for success in tertiary education, high school students should receive clear messages about the importance of taking mathematics for all four years in high school and how effort matters for mathematics learning, support for learning to read mathematics textbooks for understanding, and encouragement to form study groups among peers.

Seeley and colleagues have made some specific content recommendations to support students with navigating transitions. For the middle grade bands, they advocate promoting proportional reasoning to support success in high school mathematics (Brown & Seeley, 2010). In high school, they advocate increasing mathematical expectations for students, but rather than advocating that every student take calculus, they recommend that some students take a fourth year mathematics course consisting of statistics, probability, data analysis, and modeling (Hull & Seeley, 2010). These "new" areas of mathematical content are promising focal areas given the nature and demand of many fields of work, before and after college.

Rite of passage and implications for supporting the college transition. Considering the rite of passage conceptual lens, Clark and Lovric (2008, 2009) made some specific recommendations to assist students as they navigate their transition into their college mathematics programs. Rather than change college mathematics courses to be more like high school courses, the rite of passage perspective suggests that it is more appropriate to focus upon making expectations more transparent to students. This would mean telling high school students more directly, accurately, and in detail about their future work in university or college mathematics classrooms.

Regarding mathematics placement tests, Clark and Lovric (2009) suggest that recognizing that a rite of passage involves the whole student, an effective mathematics placement test would incorporate more than mathematics content. Beyond testing mathematics background knowledge and skills, placement tests could capture the whole individual. This would include measuring students' attitudes toward learning mathematics, their motivation, and their preferences for learning and social engagement in the classrooms, and designing appropriate mathematical learning experiences according to the outcomes.

These authors note that a rite of passage takes time and should not (and cannot) be accelerated.

> Rather than trying to "ease the transition" or "make it smoother," it [a successful transition program] needs to acknowledge that the transition [to college] will be painful, difficult, and—perhaps most importantly—that it will take time. Students undergoing transition need to know that all discomfort, pain, stress, even severe anxiety—in the end—will be proven worthwhile. Confusion and uncertainty are integral parts of everyone's learning process (Clark & Lovric, 2009, p. 764).

Realistic expectations for the length of time it will take for students is important, as short orientation sessions about how to be a more effective note-taker or how to manage time are not enough to help students (Clark & Lovric, 2008). We should not expect that short, one-shot workshops are enough to support students with a transition to college.

Additionally, a rite of passage suggests that individuals who engage in the process should take some responsibility for it (Clark & Lovric, 2008). To ease the process of students taking responsibility, groups of students can be brought together to support each other as they navigate the transitions together (Clark & Lovric, 2009). Students who have already successfully transitioned into college can serve as mentors to first-year students. It is not inappropriate, from this perspective, to expect first-year college students to accept at least some responsibility for taking initiative to negotiate the transition. Too much help may serve to disempower students.

Promising Possibilities for Future Research

Given the complexity of students' experiences in school over time, we are hesitant to prescribe specific questions for future research. However, we would like to suggest an issue to consider and a promising theme for researchers to pursue if they are interested in better understanding students' transitions in school mathematics. An issue to consider is which processes and outcomes to investigate when conducting research on students' transitions. Additionally, we believe that a promising path to pursue would be to further document the effects of interventions designed to support students with their school mathematics transitions over time.

We recommend that researchers build upon and extend this line of research through examining factors other than achievement and course-taking. Outcomes such as achievement and performance measures have dominated prior research (cf., Barber & Olsen, 2004; Hill & Parker, 2006; Isakson & Jarvis, 1999; Post, Medhanie, Harwell, Norman, Dupuis, Muchlinski, Andersen, & Monson, 2010; Rice, 2001). However, dispositional factors may be as potent for mathematics learning as any set of factors, particularly as a mediating variable between instruction and performance or understanding. Accounting for these mediating variables could enhance research on transitions by providing explanations or insights about why students' performance or achievement outcomes or course-taking patterns occur.

Future studies about transitions between school settings should be more closely situated in relation to shifts in curriculum or instructional practices of specific subject matter. Relating studies of transition in relation to specific subject matter can provide insights for how the teaching of particular content can support or constrain the degree to which students will continue to engage or not with that content.

There is a need to continue to report the effects of promising interventions that support productive outcomes. Certainly this paper did not exhaustively explore all of the research that has been conducted on productive interventions, but there is a need for more research that uncovers conditions that lead to students experiencing school mathematics transitions in productive ways. We recognize the challenge in this sort of work. There is a severe difficulty of relating change in any variable to the effect of one factor represented in the intervention.

Conclusions

In this paper, we examined questions about aspects and dimensions of students' transitioning through educational settings over time along with concerns for system-level responses to support students as they move through these transitions. We highlighted conceptual lenses to help understand students' experiences over time during transition points in school mathematics programs as well as issues that students may experience due to factors in school settings and student-level dimensions. We advocate for attention, through both research and practice, to students' socio-emotional well-being and developing identities as they navigate changes in their mathematics programs over time. Understanding the nature of changes that students experience at transition points across their school experiences can be helpful for those who are invested in supporting students' mathematics learning and development.

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Chapter 2

Curriculum and Related Factors

Research Reports

Brief Research Reports

Poster Presentations

A MODEL FOR DESIGNING COGNITION-AND-INSTRUCTION-BASED GOAL TRAJECTORIES FOR RESEARCH IN K–6 MATH CURRICULA

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This paper describes a model for building cognition-and-instruction-based goal trajectories (GT) in the context of a study that examines the validity of curriculum-embedded assessments. The model consists of six design processes and two constraints. The GT is constructed from curriculum-specified learning goals as well as developmental progressions and learning trajectories derived from empirical research. The GT is designed to inform both the selection of assessment activities for data collection and the interpretation of empirical results. Two primary results of the design process are presented: (a) a goal trajectory for promoting algebraic understanding and (b) the relationships between the trajectory and features of the Common Core State Standards. Implications of the design model for building GTs that can be used to assess student reasoning are discussed.

Keywords: Assessment and Evaluation; Curriculum Analysis; Learning Trajectories

Introduction

Learning trajectories are constructs designed to approximate variability and change in student knowledge states over time. They are domain-specific and therefore relate to understanding and reasoning in a particular domain such as algebra, geometry, place value, and rational number (e.g., Clements & Battista, 2000; Clements, Wilson, & Sarama, 2004; Confrey & Maloney, 2010; Daro, Mosher, & Corcoran, 2011; Fuson, 1998; Griffin, 2009; Simon, 1995; Simon & Tzur, 2004). With optimal design, learning trajectories can be used to support formative assessment processes that include connecting observed student performances to domain-referenced (e.g., "student *x* is distance *y* from expected 'expert' performance levels") and individual-referenced (e.g., "student *x* is distance *y* from expected student *x* performance levels given what the teacher understands about the knowledge states of student *x*") ways of acting (Cowie & Bell, 1999). Thus, if a trajectory reveals a diagnostic range of student understanding that a teacher or student is likely to encounter it may provide a basis for instructional responses that promote learning.

Most learning trajectories are designed to directly inform learning and instruction. Indeed, the *goal trajectory* (GT) concept described here is based upon the well-established idea of the learning trajectory, but the GT serves a different purpose which is to make the otherwise implicit models of learning progressions in a math curriculum explicit, an express priority for researchers interested in tracing student knowledge states in the context of a math curriculum. The present paper describes a model for building a GT and explicates its utility for evaluating the variation and growth of mathematical understanding and reasoning in the contexts of particular curriculum-embedded assessments in K–6 math curricula. The research is situated in the context of a larger study designed to address some of the most pressing problems of classroom assessment practice, and is aimed at strengthening the linkages among assessment design, instruction, and student learning.

The current notion of the GT incorporates elements of the developmental progressions that partially compose typical learning trajectory constructs (e.g., Fuson, 1997; Griffin, 2009). Elsewhere, cognition-and-instruction-based design methods have been designed for "forward engineering" a mathematics curriculum (e.g., Clements & Battista, 2000). By contrast our GT serves a purpose of principled retrospective evaluation that is focused on the embedded assessments in an existing mathematic curriculum.

Thus, the current approach to formulating a goal trajectory will be most useful to researchers and practitioners that work in situations where an instructional sequence is present (i.e., in a "scope and sequence") but where a developmental progression—as defined by empirically and theoretically grounded models of learning—is implicit. The approach consists of six primary design processes: (a) Define the design product, (b) Specify the purpose of the product, (c) Identify the features of the design product, (d) Evaluate, (e) Update, and (f) Classify the features. To illustrate these design processes, we focus on the goals that comprise the algebraic reasoning strand of the standards-driven curriculum, *Everyday Mathematics* (EM; Bell et al., 2007).

Method: Six Design Processes and Model Constraints

Our curriculum-and-instruction-based model for building a goal trajectory has six design processes and two design constraints (see Figure 1). The processes are cognitive activities that are either expressed by an individual or distributed across several people and media.

a. Define Goal Trajectory as the Design Product

The first process, *Define the Design Target*, refers to activities in which the researcher or evaluator articulates what will be designed. In the present case we sought to design a GT that modeled variability and growth in knowledge states in and over time for fifth-graders learning how to reason algebraically in the context of a specific math curriculum. We wanted the GT to be a cognitive model with cognitive units at a level of specificity described by the curriculum. Additionally, we wanted the GT to have properties such that it could be used to estimate variability and growth through its different "levels."

b. Specify the Purpose of the Goal Trajectory

The second process, *Specify the Purpose*, refers to activities in which the researcher or evaluator specifies the aim of the design product. It addresses the question, "Why do we need or desire to design such a product (i.e., the GT)?" In the present case, the purpose of designing a GT that models variability and growth in student knowledge states in and across time was to help us (a) select curriculum-embedded activities, and (b) interpret student performance on the selected assessments. The GT is an important tool in our investigation of the cognitive, instructional, and inferential validity of curriculum-embedded assessments. Thus, in the current situation the purpose was pragmatic. However, in other cases the design product can have empirical, pragmatic, and/or theoretical considerations.

c. Identify the Features of the Goal Trajectory

The third process, *Identify the Features of the Goal Trajectory*, operationalizes the elements of the design product. In the present situation the features were cognitive units and properties of the GT. As mentioned earlier we were concerned with preserving the level of cognitive specificity described by the curriculum. In the context of *Everyday Mathematics* (EM), the cognitive units were tied to the learning goals such as *Use patterns to find basic facts* and *Use rules to complete function tables/machines*. The learning goals comprised the Patterns, Functions, and Algebra (PFA) learning strand in the Grade 5 EM curriculum. Another feature was the ordinal property of the GT. Our intent was to design a GT with ordinal levels that could approximate variation in student performance and growth in cognitive complexity over time.

d. Evaluate Process Outcomes

As shown in Figure 1, the fourth process in the model, *Evaluate*, serves at least two functions. One is to evaluate the agreement between the purpose of the design product (i.e., process b) and its features throughout progress in the design cycle. For example, given the purpose of the design (see section *b. Specify the Purpose*), selecting cognitive units at the larger grain sizes of learning strands (e.g., measurement, number, or geometry) or content threads (e.g., patterns and functions, algebraic notation and solving number sentences, or properties of the arithmetic operations) would not have given the GT the necessary power to model cognitive *variability* in or among students. At those levels the GT would only describe two knowledge states: haves and have-nots. Therefore, it was critical to evaluate each feature of the GT with this constraint in mind.

A second function of the *Evaluate* process is to assess the extent that the design features and the method for assigning them into meaningful levels of the GT is viable given the model's design constraints which are explained below. The dashed circular path indicates that (a) the outcomes of two related processes are cross-evaluated (e.g., outcomes of processes *c* and *f*) and (b) the decision to move forward with the design depends on the balance of that cross-evaluation; if the balance is positive (i.e., consistent with the scope of the model) then advance, if negative (i.e., inconsistent with the scope of the model) then the model needs to be updated (process *e*).

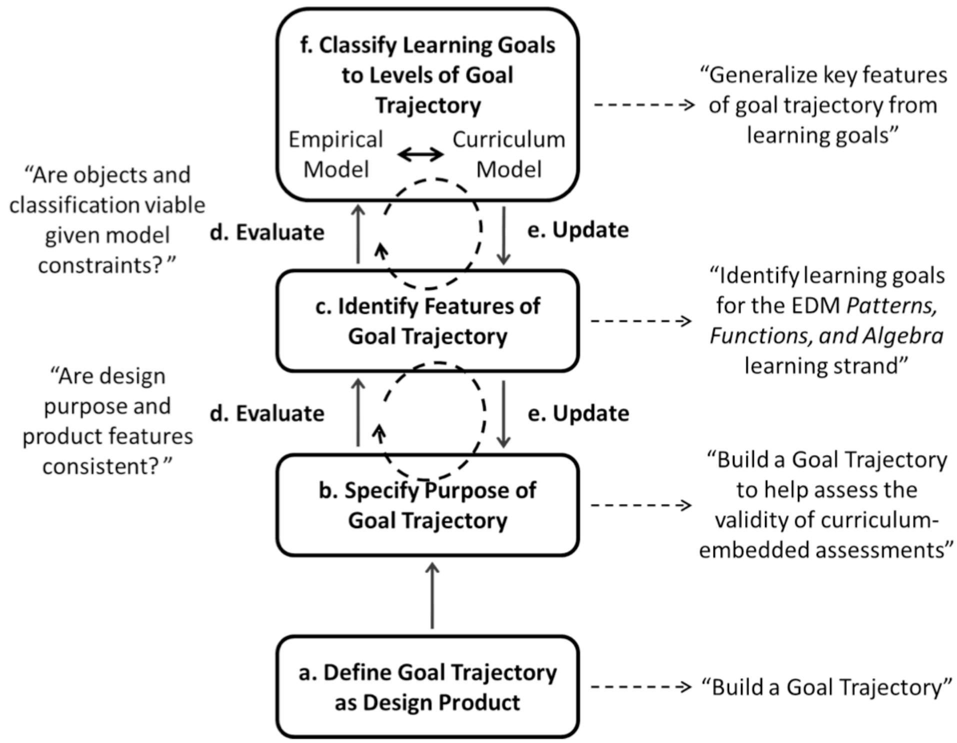

Figure 1. Model of goal trajectory design processes with examples

e. Update

The fifth process, *Update*, serves to make process outcomes consistent with the model or make the model consistent with process outcomes. If an evaluation of two process outcomes reveals an inconsistency (e.g., a learning goal defined as a feature of the GT does not "fit" into a level of the GT), then one or both of those outcomes will need to be updated. In this example a decision may be made to modify a trajectory level, a decision may be made to expand the trajectory by adding a level; or a decision may be made to modify the learning goal. If the two evaluated outcomes are related to processes *c* and *f*, then it may also be necessary to evaluate the outcome of process *b*. This particular chain of evaluations may support a decision to update the purpose of the design (e.g., the GT is useful for selecting embedded assessments but not for interpreting student performance). The cyclic iterations between *Evaluate* and *Update* processes can be one, few, or many in the actual design cycle. Indeed, the model is referred to as a design "cycle" because it is not linear in a strict sense. It is important that researchers or evaluators engaged in the design cycle keep careful records of the model's development from initial conception to final design. In our project we write reports that trace the nature of the design cycle as it unfolds.

Once the learning goals were identified in the curriculum and extracted, we met with the curriculum developers to evaluate (a) the extent that our search for PFA learning goals was exhaustive, (b) our understanding of the curriculum layout, and (c) the degree that the level of learning goal information we decided to use at that point in our design would enable us to build the desired GT. Indeed, our in-depth curriculum analysis revealed several layers of learning goal information. In its early stages, our GT referenced information from all of these layers. However, based on discussions with the curriculum developers we updated the model to include only a single source of learning goal information, the Grade-

Level Goals Chart. Our rationale for this decision was that the Grade-Level Goals Chart highlights the units in which the Grade 5 PFA learning goals are introduced. Using the Grade-Levels Goal chart as our point of reference we were able to “see” the concepts and skills that encompassed the Grade 5 curriculum over time. This satisfied a demand of our model (i.e., build a GT whose levels express ordinal relations) and we were ready to enact the sixth design process.

f. Classify Features into Levels of the Goal Trajectory

The sixth design process in our model for building a goal trajectory is *Classify*. To classify means to abstract a smaller set of cognitive constructs from the learning goals that approximated the major forms of reasoning in the trajectory. The Grade-Level Goals Chart yielded 38 PFA learning goals across the 12 units of the Grade 5 curriculum. The goals were organized into seven general levels of reasoning that were scheduled to be introduced in the PFA trajectory. In effect, the *Classify* process “collapses” all related learning goals across task demand (e.g., recall vs. recognition) and external representation format (e.g., base-10 blocks vs. arrays) resulting in a general set of learning goals and a manageable GT. Notice how Figure 1 indicates that the *Classify* process is constrained by two sources of information: (a) prior research in developmental psychology, cognitive psychology, and mathematics education on the development of and variability in algebraic reasoning (i.e., the “Empirical Model”), and (b) the instructional sequence of key concepts and skills as outlined by the curriculum (i.e., the “Curriculum Model”). As depicted in Figure 1, the resulting learning trajectory was subjected to an Evaluate-Update Cycle before final approval.

Learning Goal Trajectory for Understanding Patterns, Functions and Algebra

The result of the design processes in the current case is the Patterns, Functions, and Algebra (PFA) goal trajectory shown in Table 1. The design processes revealed that the general PFA goal trajectory for acquiring algebraic thinking was implicitly characterized by EM as growth from none or very little understanding of patterns, to identifying and using patterns, to formalizing patterns as a means for solving problems, to generalizing rules from patterns and sequences, to formalizing rules in notational, graphical, and tabular formats, to finally being able to reason with and about variables. The organization of the trajectory was consistent with a growing body of research in cognitive science and mathematics education which suggested that algebra acquisition could be defined by cognitive growth along a multi-path continuum of reasoning with patterns and sequences, generalizing rules from patterns and sequences, representing functions among rules, patterns, and sequences, and formalizing variables to think about functions (Carraher & Schliemann, 1992; Kaput & Blanton, 1999; Moss & McNabb, 2011; Smith & Thompson, 2007; Warren & Cooper, 2008).

Table 1: Goal Trajectory for Understanding Patterns, Functions and Algebra

	Level of Understanding	Examples
6	**Abstract Algebraic Functions** (Represent functions using words, algebraic notation, tables and graphs; represent patterns and rules using algebraic notations; translate from one representation to another; use representations to solve problems involving functions)	• Use a variable to represent unknown quantities to solve problems • Represent an algorithm as a general pattern with variables • Solve linear equations with one unknown and multiple operations using trial-and-error or equivalent equation strategies • Solve problems involving functions using representations; including translating from one representation to another
5	**Algebraic Functions** (Represent functions using words, symbols, tables and graphs; use those representations to solve problems)	• Represent functions using algebraic notations • Use representations of function(s) in tables and graphs to solve problems • Use patterns, tables and graphs to define relationships between volumes of 3D solids or between radius and area; • Represent rates with formulas, tables and graphs
4	**Function Rules** (Describe and/or write rules for functions involving the four basic arithmetic operations; use rules to solve problems)	• Identify and use patterns in graph coordinates to match graphs with situations • Use patterns to identify the relationship between numerators and denominators; use patterns to identify relationships between fractions and decimals • Generate rule for comparing, ordering fractions • Describe the patterns in an area model • Use rules to complete function tables/machines • Use words and symbols to extend patterns/ to describe the operations of Addition, Subtraction, Multiplication and/or Division and/or create/use rules to solve problems
3	**Numeric Pattern Rules** (Use words or symbols to create and/or describe rules for numeric patterns; use rules to extend patterns and solve problems)	• Use words and/or symbols and/or arithmetic notation and extend patterns to describe geometric rules • Use and describe patterns to find sums • Describe number patterns related to exponents and/or use them to solve problems
2	**Numeric Patterns** (Identify, use, expand, describe, or create numeric patterns)	• Complete number sequences • Use patterns to find basic facts • Describe and extend patterns among facts and their extension • Identify and/or use patterns in skip counting • Count in Equal Intervals
1	**No Understanding of Patterns**	• Not able to complete number sequences or count in equal intervals

Relationships Between the PFA Goal Trajectory and the Common Core State Standards

In addition to being consistent with empirical models of growth in algebraic reasoning, the trajectory also aligned with the mathematical content domains and practices outlined by the Common Core State Standards (CCSS) in several interesting ways. First, the Grade 5 EM trajectory for understanding patterns, functions, and algebra embodies two Grade 5 CCSS content domains: *Operations and Algebraic Thinking*

(OA) which focus on writing and interpreting numerical expressions and analyzing patterns and relationships and *Number and Operations in Base 10*. Second, the Grade 5 goal trajectory relates to these CCSS content domains across Grade 2, Grade 3, and Grade 4 but the mathematical foci (i.e., "clusters") vary among the grades. For example, whereas the CCSS Grade 5 OA domain has two relevant clusters that focus on (a) writing and interpreting numerical expressions, and (b) analyzing patterns and relationships, the CCSS Grade 3 OA domain has four clusters that emphasize (a) representing and solving multiplication and division problems, (b) understanding properties of multiplication and the relationship between multiplication and division, (c) multiplying and dividing using strategies (e.g., $8 \times 4 = 32$ therefore $32 \div 4 = 8$) and properties of operations, and (d) solving for unknown quantities that involve the four operations in addition to identifying and explaining arithmetic patterns. Aspects of the goal trajectory also map onto features of the Grade 6 CCSS content domain, *Expressions and Equations*, which includes clusters that focus on (a) applying and extending what is understood about arithmetic to algebraic expressions, (b) reasoning about and solving one-variable equations and inequalities, and (c) representing and analyzing the relationships between dependent and independent variables.

Besides aligning with the CCSS mathematical *content domains*, we also found the goal trajectory to be well-aligned with the CCSS mathematical *practices*; that is, the various habits of mind that mathematics instructors are expected foster in their students such as constructing viable arguments and reasoning with others, modeling with mathematics, using appropriate tools strategically, and attending to precision. There are various mathematical practices that map onto particular levels of the goal trajectory. For instance, take *Use a variable to represent unknown quantities to solve problems*, taken from the sixth level of understanding in the goal trajectory, *Abstract Algebraic Functions* (Table 1). The level of understanding relates to the CCSS mathematical practice that indicates variables are used to solve problems because they can help *make sense* of quantities and relationships. This mathematical practice implies that variables have greater utility than as simple tools for identifying or recalling answers. A second example of the alignment between the trajectory and the mathematical practices described by the CCSS can be found if one looks at *Complete number sentences* in the *Numeric Patterns* level of understanding in the goal trajectory. The latter is related to the CCSS mathematical practice that promotes the capacity to seek and use structure to describe and extend facts and patterns. The implication is that engaging students in practices that give them opportunities to identify the structure of number sequences should lead to efficient pattern identification strategies that can be applied across different task situations.

Discussion

A six-process model for building curriculum-and-instruction-based goal trajectories for cognitive research and instructional assessment was proposed. We instantiated the processes of the model in the context of our work with the Patterns, Functions, and Algebra learning strand in the Grade 5 *Everyday Mathematics* curriculum. The design processes yielded a unique representation of the goal information that was already represented—albeit, "hidden"—in the organization of the curriculum. Interestingly, the representation that we constructed as the PFA goal trajectory was quite different from the representation of that information as presented by the curriculum.

Re-Presentations of Curriculum-Embedded Goal Structures

Cognitive psychologists have reliably shown that different representations of equivalent information can vary in the way that they preserve information, and this in turn can yield differential affordances for accessing and utilizing the same information (e.g., Larkin & Simon, 1987; Palmer, 1978; Zhang & Norman, 1994). An evaluation of the model proposed in this paper indicates that the benefits of the constructed GT are the result of the aforementioned *representational effect* (Nickerson, 1988; Zhang, 1997). Indeed, the GT affords fresh and important insights into student understanding that expand upon what is available from the *Everyday Mathematics* curriculum materials, while also remaining faithful to the curriculum by basing the GT on the curricular learning goals and instructional materials. For one, the goal trajectory allows us to predict and account for a wider range of student performance on an activity than what is usually estimated by the curriculum, because the curriculum-based representation is typically limited to dichotomous evaluations of student performance such that student performance either reflects evidence of goal acquisition or it does not. A second benefit of the PFA goal trajectory is that it makes it possible to interpret student performance in terms of the cognitive constructs that are relevant to a particular domain in the contexts of the curriculum and scientific progress. Thus, the goal trajectory affords greater diagnostic information about student performance relative to the learning and acquisition of algebraic thinking.

Investigating Curriculum-CCSS Goal Alignment

Although the CCSS are based on notions of a learning trajectory or progression, their explicit description of one is limited to expectations of mathematical content domains and practices *across* not *within* grades. By comparing our constructed GT to the CCSS it became clear that for a teacher at a particular grade the CCSS was not intended to represent the expected understandings and reasoning patterns of students "well below or well above grade-level expectations," nor was it meant to account for variation contributed by English language learners or children with special needs. We propose that GTs help to illuminate—within the context of a particular mathematics curriculum—the potential for multiple levels of knowledge and reasoning that may be observed as students complete a given activity.

Mapping the CCSS Operations and Algebra content domain onto the GT of an elementary grades math curriculum revealed interesting relationships between each level of the goal trajectory and the CCSS. In particular, as the GT levels progressed, the number of shared relations between each level and the standards increased. Whereas the earlier levels of the trajectory shared a one-to-one relationship with the CCSS standards, the advanced levels of the trajectory shared a one-to-many relationship with the standards in which a single level of the GT was linked to multiple goals in the CCSS. Finally, in support of the CCSS's position about the breadth of mathematical practices, our analysis indicated that the CCSS mathematics practices were differentially instantiated at each GT level of understanding. The extent that these patterns will emerge with other GTs (e.g., Number and Numeration) and the empirical validity of the GT levels is currently being investigated.

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LONGITUDINALLY INVESTIGATING THE IMPACT OF CURRICULA AND CLASSROOM EMPHASES ON EQUITY IN ALGEBRA LEARNING[1]

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This paper explores how curriculum and classroom conceptual and procedural emphases affect the learning of algebra for students of color. Using data from a longitudinal study of the Connected Mathematics Program (CMP), we apply cross-sectional HLM to lend explanatory power to the longitudinal analysis afforded by Growth Curve Modeling that we have reported elsewhere. Overall, we find that the achievement gaps tend to decrease over the course of the middle grades. However, differences in open-ended problem solving and equation solving performance persist for African American students. Classroom conceptual and procedural emphases also appear to differentially influence the performance of Hispanic and African American students, depending on the aspect of algebra learning that is being measured.

Keywords: Curriculum Analysis; Equity and Diversity; Algebra and Algebraic Thinking

Classrooms in the United States are becoming increasingly ethnically diverse. However, disparities in the mathematics achievement of different ethnic groups remain a persistent challenge (Lubienski & Crockett, 2007). Since teaching and learning are cultural activities, students with different ethnic and cultural backgrounds may respond differently to the same curriculum. Given the development and implementation of curricula based on the *Standards* documents developed by the National Council of Teachers of Mathematics (NCTM, 1989, 2000), a key question about curriculum reform is: How does the use of a *Standards*-based curriculum like CMP impact the learning of students of color as compared to White students?

In our project, Longitudinal Investigation of the Effect of Curriculum on Algebra Learning (LieCal), we used a longitudinal design to examine the similarities and differences between a *Standards*-based curriculum called the Connected Mathematics Program (CMP), and more traditional curricula (non-CMP). We investigated not only the ways and circumstances under which the CMP and non-CMP curricula affected student achievement gains, but also the characteristics of these reform and traditional curricula that hindered or contributed to the gains. One aspect of the LieCal analysis was an examination of potentially differential effects of curriculum and procedural and conceptual emphases in the classroom on the achievement of students of color. In this paper, we present results from a cross-sectional analysis of student growth within each grade level. This analysis allows us to add depth to our previous analysis using Growth Curve Modeling by probing effects that are significant at individual grades but which were not uncovered in our longitudinal analysis.

Background

Algebra readiness has been characterized as the most important "gatekeeper" in school mathematics (Pelavin & Kane, 1990). In particular, success in algebra and geometry has been shown to help narrow the disparity between minority and non-minority participation in post-secondary opportunities (Loveless, 2008). Research shows that completion of an Algebra II course correlates significantly with success in college and with earnings from employment. The National Mathematics Advisory Panel (2008) found that students who complete Algebra II are more than twice as likely to graduate from college as students with less mathematical preparation. Furthermore, the African-American and Hispanic students who complete

Algebra II cut the gap between their college graduation rate and that of the general student population in half. However, success in high school algebra is dependent upon mathematics experiences in the middle grades. In fact, middle school is a critical turning point for students' development of algebraic thinking (College Board, 2000).

In a *Standards*-based curriculum like CMP, the focus is on conceptual understanding and problem solving rather than on procedural knowledge. Students are expected to learn algorithms and master basic skills as they engage in explorations of worthwhile problems. However, a persistent concern about Standards-based curricula is that the development of students' higher-order thinking skills comes at the expense of fluency in computational procedures and symbolic manipulation. In addition, it is not clear whether this potential trade-off might play out differently for students from different ethnic backgrounds. Some reports have suggested that Hispanic and African American students using the CMP curriculum may in fact show greater achievement gains than students from other backgrounds (Rivette, Grant, Ludema, & Rickard, 2003). Still, research is needed to assess whether and how the use of a Standards-based curriculum such as CMP can improve the mathematics achievement of all students while helping to close achievement gaps (Lubienski & Gutiérrez, 2008; Schoenfeld, 2002).

Since the effectiveness of a curriculum depends critically on how it is implemented by teachers in real classrooms, studies of the effectiveness of Standards-based curricula must examine how teachers use the curricula (Kilpatrick, 2003; NRC, 2004; Wilson & Floden, 2001). The data gathered must be analyzed in appropriate ways to control for variations in classroom instruction and the learning environment. In order to determine the effects of curriculum on learning, it is essential to examine the classroom experiences of the teachers and students who are using the different curricula. In this paper, we take features of classroom instruction into consideration when we examine the impact of curricula on students' learning of algebra. In particular, we examine the extent to which teachers emphasize concepts and procedures in the classroom. As was reported by Moyer, Cai, Nie, and Wang (2011), CMP teachers placed more emphasis on conceptual understanding whereas non-CMP teachers placed more emphasis on procedural knowledge.

Our previous longitudinal analyses of the LieCal data using Growth Curve Modeling showed that over the three middle school years, African American students experienced greater gain in symbol manipulation when they used a traditional curriculum. The use of either the CMP or a non-CMP curriculum improved the mathematics achievement of all students, including students of color. The use of CMP contributed to significantly higher problem-solving growth for all ethnic groups (Cai, Wang, Moyer, Wang, & Nie, 2011). In this paper, we take a cross-sectional approach and examine the achievement of students of color in each grade level while controlling for the conceptual and procedural emphases in classroom instruction.

Method

Sample

The LieCal project was conducted in 14 middle schools of an urban school district serving a diverse student population. When the project began, 27 of the 51 middle schools in the district had adopted the CMP curriculum, and the remaining 24 had adopted more traditional curricula. Seven schools were randomly selected from the 27 schools that had adopted the CMP curriculum. After the seven CMP schools were selected, seven non-CMP schools were chosen based on comparable demographics. In sixth grade, 695 CMP students in 25 classes and 589 non-CMP students in 22 classes participated in the study. We followed these 1,284 students as they progressed from grades 6 to 8. Approximately 85% of the participants were minority students: 64% African American, 16% Hispanic, 4% Asian, and 1% Native American. Male and female students were almost evenly distributed.

Assessing Students' Learning

Learning algebra involves honing procedural skills with computation and equation-solving, fostering a deep understanding of fundamental algebraic concepts and the connections between them, and developing the ability to use algebra to solve problems. Thus, to assess students' learning of algebra, it is important to consider their conceptual understanding, their symbol manipulation skills, and their ability to solve problems. We used state test scores in mathematics and reading as measures of prior achievement. We used LieCal-developed multiple-choice and open-ended assessment tests as dependent measures of procedural knowledge and conceptual understanding in algebra, respectively. The two LieCal-developed tests were administered four times, once as a baseline in the fall of 2005, and again each spring (2006, 2007, and 2008).

We used multiple-choice items to assess whether students had learned the basic knowledge required to perform competently in introductory algebra. Each version of the multiple-choice test was comprised of 32 questions that assessed five mathematics components (Mayer, 1987): translation, integration, planning, computation (or execution), and equation solving. For this paper, we report on the results from the translation, computation, and equation-solving components of the multiple-choice tasks. In addition, the open-ended tasks were designed to assess students' conceptual understanding and problem-solving skills. These tasks were adopted from various projects including Balanced Assessment, the QUASAR Project (Lane et al., 1995), and a cross-national study (Cai, 2000). Since only a small number of open-ended tasks can be administered in a testing period, and since grading students' responses to such items is labor-intensive, we distributed the non-baseline tasks over three forms (five items in each form) and used a matrix sampling design to administer them. Examples of the items and tasks used in the LieCal assessments can be found in Cai et al. (2011).

The multiple-choice items then were scored electronically, either right or wrong. The open-ended tasks were scored by middle school mathematics teachers, who were trained using previously developed holistic scoring rubrics. Two teachers scored each response. On average, perfect agreement between each pair of raters was nearly 80%, and agreement within one point difference out of 6 points (on average) was over 95% across tasks. Differences in scoring were arbitrated through discussion. The two-parameter Item Response Theory (IRT) model was used to scale student assessment data on both multiple-choice tasks and open-ended tasks (Hambleton, Swaminathan, & Rogers, 1991; Lord, 1980).

Conceptual and Procedural Emphases as Classroom-level Variables

Mathematical proficiency includes both conceptual and procedural aspects (NRC, 2001), and teachers can shape instruction in ways that emphasize either or both aspects. We used conceptual and procedural emphases as classroom variables when examining the impact of curriculum on students' learning. To do so, we estimated the levels of conceptual and procedural emphases in the CMP and non-CMP classrooms using data from 620 lesson observations of the LieCal teachers, which we conducted while the students were in grades 6, 7, and 8. Each class was observed four times, during two consecutive lessons in the fall and two in the spring. Further details about the observations are documented in Moyer et al. (2011). One component of the observation was a set of 21 items using a 5-point Likert scale to rate the nature of instruction for each lesson. Of the 21 items, four were designed to assess the extent to which a teacher's lesson had a conceptual emphasis. For example, observers rated a lesson's conceptual emphasis using the following item: "The teacher's questioning strategies were likely to enhance the development of student conceptual understanding/problem solving." Another four items were designed to determine the extent to which a teacher's lesson had a procedural emphasis. For example, observers rated a lesson's procedural emphasis using this item: "Students had opportunities to learn procedures (by teacher demonstration, class discussion, or some other means) before they practiced them." Factor analysis of the LieCal observation data confirmed that the four procedural-emphasis items loaded on a single factor, as did the four conceptual-emphasis items. Since students changed their classrooms and teachers as they moved from grade 6 to grade 7 and from grade 7 to grade 8, each student could have a different value each year for three years, but all students in the same classroom at each grade had the same value.

Quantitative Data Analysis

To examine student growth within each school year while controlling for multiple factors such as gender, ethnicity, and classroom conceptual and procedural emphases, we used hierarchical linear modeling (HLM). After unconditional models were fitted, two sets of conditional cross-sectional HLM analyses were conducted. The first set of models was composed of cross-sectional hierarchical linear models that included student-level variables and a curriculum variable. These models used four different student achievement measures: open-ended, translation, computation, and equation solving. Each HLM model used data from one of the four dependent achievement measures in one of three middle grades, together with an independent prior achievement measure, namely the results of the state mathematics testing in the fall of the corresponding year. So, each model examined a single type of learning within a specific grade level. Since we had four achievement measures at each of three grade levels, there were 12 cross-sectional models in this first group.

The next set of models built on the first group of models by adding two classroom-level variables: the conceptual emphasis of the classroom and the procedural emphasis of the classroom. These cross-sectional HLM models were of the following form:

Level-1 Model

$$Y_{ij} = p_{0j} + p_{1j}(\text{Prior Achievement}_{ijk} - \overline{X}_1) + p_{2j}(\text{Gender}_{ijk} - \overline{X}_2) + p_{3j}(\text{African American}_{ijk} - \overline{X}_3) + p_{4j}(\text{Hispanic}_{ijk} - \overline{X}_4) + p_{5j}(\text{Other Ethnicity}_{ijk} - \overline{X}_5) + r_{ijk}$$

Level-2 Model

$$p_{0j} = b_{00} + b_{01}\text{CMP} + b_{02}\text{Conceptual Emphasis}_j + b_{03}\text{Procedural Emphasis}_j + r_{0j}$$

Interactions between conceptual emphasis, procedural emphasis, and curricula were tested, but found to be not significant.

Results

We present the results of our analysis in two parts. First, we report on the cross-sectional HLM models that included student-level and curriculum variables. Then, we examine the impact of including the classroom-level conceptual and procedural emphasis variables in the models.

Student-Level and Curriculum Cross-sectional HLM Models

Table 1 shows the standardized results from an examination of the performance of African-American and Hispanic students relative to White students, when controlling for prior achievement, gender, and curriculum (but not conceptual and procedural classroom emphases).

Table 1: Effect of Ethnicity on Mathematics Achievement

	Grade 6		Grade 7		Grade 8	
	African American	Hispanic	African American	Hispanic	African American	Hispanic
Open-ended	-0.50***	-0.21*	-0.26**	-0.22*	-0.28***	-0.13*
Translation	-0.24**	--	--	--	--	--
Computation	-0.37***	-0.22*	--	--	--	--
Equation solving	-0.35**	-0.23*	-0.24**	-0.22*	--	--

* $p < .05$. ** $p < .01$. *** $p < .001$.

In the sixth grade, an achievement gap was seen between African American students and White students on all four student achievement measures, and between Hispanic students and White students on the open-ended, computation, and equation solving measures. The gaps on the open-ended and equation solving measures remained in the seventh grade for both groups. However, performance on the computation and translation measures had equalized across the groups. In the eighth grade, the only gap that remained was on the open-ended items. The overall trend was a gradual decline or elimination of the achievement gap among the ethnic groups.

To better understand if using the CMP curriculum would reduce achievement gaps, we conducted separate parallel analyses for CMP and non-CMP students. The results are shown in Table 2. In the analysis of the combined CMP and non-CMP student sample, achievement gaps for the translation and computation measures occurred only in the 6th grade: White students outperformed African American students on both measures, and White students outperformed Hispanic students on computation. However, in the analyses of the separate student samples, we found that although all three of these gaps appeared for the non-CMP students, the only achievement gap for the 6th grade CMP students was in computation. In grades 7 and 8, the performance parity on computation and translation items observed in the combined sample of students was mostly preserved in the separate analyses, except for the appearance of a gap between CMP 8th grade African American students and White students on computation items.

Mirroring the results from the combined sample, White students outperformed African American students on open-ended items across all three grades regardless of curriculum. For students using CMP, White students also outperformed Hispanic students on these items in Grades 7 and 8. For non-CMP Hispanic students, however, there were no parallel achievement gaps. For the equation solving items in the combined analysis, White students outperformed African-American and Hispanic students in grades 6 and 7, with no achievement gap in grade 8. These gaps were attributable to the CMP students; there were no achievement gaps found for equation solving items among the non-CMP students. For CMP students, White students outperformed African American students in all three grades, and White students outperformed Hispanic students in grades 7 and 8.

Table 2: Effect of Ethnicity on Mathematics Achievement for CMP / Non-CMP Students

	Grade 6		Grade 7		Grade 8	
	African American	Hispanic	African American	Hispanic	African American	Hispanic
Open-ended	-0.40** / -0.91***	-- / --	-0.27** / -0.23*	-0.31**/ --	-0.36** / -0.26**	-0.28**/ --
Translation	-- / -0.37*	-- / --	-- / --	-- / --	-- / --	-- / --
Computation	-0.43** / -0.27**	-- / -0.35**	-- / --	-- / --	-0.22* / --	-- / --
Equation solving	-0.35* / --	--/ --	-0.44** / --	-0.45**/ --	-0.23**/ --	-0.22*/ --

* $p < .05$. ** $p < .01$. *** $p < .001$.

Student-Level, Classroom-Level and Curriculum HLM Models

We built on the results of Table 1 with the addition of the conceptual emphasis and procedural emphasis classroom-level variables. Our goal in adding these variables to the analysis was to begin to probe the complexity that underlies conclusions we might otherwise draw from one-dimensional comparisons of students in different ethnic groups. With respect to the analysis of the combined CMP and non-CMP students, however, the addition of the classroom-level variables did not greatly perturb the results save for the disappearance of the gap in Hispanic students' performance on open-ended tasks in the 8th grade.

We again conducted parallel analyses for the CMP and non-CMP students, this time including the conceptual and procedural emphasis classroom-level variables. The results are presented in Table 3. For the CMP students, two achievement gaps were no longer statistically significant with the addition of the classroom variables: 8th grade African American students on computation items, and 8th grade Hispanic students on equation solving items. For non-CMP students, the performance gap of 6th grade African American students on translation and computation items ceased to be significant.

Table 3: Effect of Ethnicity on Mathematics Achievement for CMP/Non-CMP Students Controlling for Conceptual and Procedural Emphases

	Grade 6		Grade 7		Grade 8	
	African American	Hispanic	African American	Hispanic	African American	Hispanic
Open-ended	-0.40** / -0.90***	--/ -0.33*	-0.28** / -0.27*	-0.30**/ --	-0.36** / -0.23**	-0.29**/ --
Translation	-- / --	-- / --	-- / --	-- / --	-- / --	-- / --
Computation	-0.37** / --	-- / -0.33**	-- / --	-- / --	-- / --	-- / --
Equation solving	-0.35*/ --	-- / --	-0.45** / --	-0.44**/ --	-0.21** / --	-- / --

* $p < .05$. ** $p < .01$. *** $p < .001$.

For the combined student groups, the performance of Hispanic students in the 8th grade was not significantly different from 8th grade White students for all four achievement measures. Similarly, the performances of 8th grade African American and White students were not significantly different except on the open-ended items; there was no achievement gap between African American and White students in the 8th grade on translation, computation, and equation solving items. When analyzed as separate groups, the CMP and non-CMP students of color generally showed achievement gaps on open-ended items compared to White students using the same curriculum. Within the CMP student group, there were also achievement gaps for African American students on equation solving items.

Discussion

In examining how Standards-based curricula such as CMP affect the mathematics learning of students of color, it is important to use nuanced analyses to look beyond one-dimensional comparisons (Lubienski, 2008). The longitudinal growth curve analysis of the LieCal data provided mixed conclusions regarding the use of the CMP curriculum with students of color (Cai et al., 2011). Though, over the course of the middle grades, African American and Hispanic students had growth rates similar to students not in their ethnic groups on the open-ended, translation, and equation solving measures, African American students had a smaller growth rate on the computation measure. The cross-sectional HLM analysis in this paper provides detail not captured in the longitudinal analysis.

Overall, the results of the cross-sectional analysis show a trend of decreasing gaps in achievement. Whereas Hispanic and African American students score significantly lower than White students on most or all of the measures at the end of 6th grade, by the end of 8th grade, only the open-ended measure still reflects a gap. Moreover, when classroom conceptual and procedural emphasis is taken into account, the only difference that remains at the end of 8th grade is in African-American students' performance on the open-ended tasks. Despite the slower growth rate in African American students' performance on computation tasks that was identified in the longitudinal analysis, the effect seems to be largely limited to the 6th grade.

When the cross-sectional analysis is limited to the CMP students, the open-ended measure reflects a persistent gap between White students and students of color. Similarly, for African American students in the CMP group, equation solving remains an area of challenge throughout the middle grades. Even when classroom conceptual and procedural emphasis variables are included, these gaps remain. Indeed, these performance gaps in the CMP analysis do not decrease with grade level, as many of the other performance gaps do. Thus, despite the fact that the longitudinal analysis showed comparable growth curves for White students and students of color on the open-ended measure, the cross-sectional analysis suggests that there may be opportunities within the CMP curriculum for developing open-ended problem-solving skills that are being differentially accessed by students of different ethnic backgrounds.

It is interesting to note how the influence of classroom emphasis variables played out differently for different student groups. For example, the profile of Hispanic CMP students' equation solving performance was somewhat different from the African American students'. For Hispanics, the negative CMP effect was limited to the 7th grade. Classroom conceptual and procedural emphases, not curriculum, appear to account for Hispanic student performance differences in the 8th grade. Moreover, the reverse appears to be the case with respect to the translation and computation measures in the 6th grade. When controlling for classroom emphasis, there was no longer an achievement gap for African American students. This difference in the effects of classroom emphasis on Hispanic and African American students merits exploration.

In conclusion, the longitudinal and cross-sectional analyses continue to paint a mixed picture of the effects of the CMP curriculum for students of color. By grade 8, most performance differences on the measures in this study were no longer significant. Though African-American students' computation skills appeared to grow more slowly across grades 6 through 8, the effect of this difference seems to have been primarily limited to grade 6. However, the persistent gaps between African American students and White students on the open-ended and equation solving measures, even when classroom emphases are taken into account, invite further investigation.

Endnote

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UNPACKING THE COMMON CORE STATE STANDARDS FOR MATHEMATICS: THE CASE OF LENGTH, AREA AND VOLUME

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Adoption of the Common Core State Standards present challenges to school districts, school administrators, and teachers. To assist in this endeavor, we present our work on unpacking the CCSS-M for the length, area, and volume Learning Trajectory (LT). The overarching theme of "genetic epistemology" and a five-characteristic framework guided our work of unpacking the Standards. As a result, we added "Bridging Standards" to mediate students' progression through the length, area, and volume LT to provide a coherent structure through this trajectory. The implications of our work were discussed.

Keywords: Standards; Learning Trajectories; Curriculum

Objective

The Common Core State Standards for Mathematics (CCSS-M) (CCSSI, 2010) are a major revamping of existing and past state standards. Adoption of the CCSS-M presents many challenges for school administrators and teachers. In particular, the learning trajectories that ostensibly undergird the Standards are not readily accessible to readers because they are *abridged* within the standards and do not contain a full treatment of the research base (Confrey, 2012). Hence, there are gaps between standards reducing their cohesiveness. Finally, the Standards authors state: "These Standards do not dictate curriculum or teaching methods" (CCSSI, 2010, p. 3) which is commendable, however, this implies that teachers need resources and support to understand the gradual evolution of the "big ideas" within the Standards.

Our research group has unpacked the grade K–8 standards for the CCSS-M (http://www.turnonccmath.com) (Confrey et al., 2011) by mapping each of the K–8 standards onto 17 LTs (Confrey, 2012). For each trajectory, we unpacked the Standards, or parts of a Standard if it had a few parts (e.g., 3.MD.7 had three parts: 3.MD.7.a, 3.MD.7.b, and 3.MD.7.c), into descriptors to include a careful discussion of the full learning trajectory. The descriptors include: (1) Conceptual principles; (2) Misconceptions, strategies, and representations; (3) Introduction of meaningful distinctions about mathematical concepts and multiple models of situations; (4) A coherent Structure of Development underlying the LT; and (5) Bridging Standards. Other groups who are unpacking the standards tend to elaborate on the mathematical content in each Standard (e.g., McCallum, Black, Umland, & Whitesides, 2010) or make comparisons between existing standards and the CCSS-M (e.g., North Carolina Department of Public Instruction, 2011). Though important, these approaches do not always give perspective on how students' mathematical ideas advanced under instruction. In this paper, we present our work on unpacking the K–5 CCSS-M Standards for the *length, area, and volume* LT. Drawing upon the literature, we created an initial draft to reveal a coherent structure for this LT.[1]

Literature Review

Learning Trajectories

The term *learning trajectories (LT)*, has different meanings among researchers in mathematics education. Simon (1995) first defined a *hypothetical learning trajectory* (HLT) to be, "The learning goals, the learning activities, and the thinking and learning in which students might engage" (p. 133). Our research group defines a learning trajectory to be,

> a researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e., activities, tasks, tools, and forms of interaction), in order

> to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time. (Confrey, Maloney, Nguyen, Mojica, & Myers, 2009)

We view LTs as expected probabilities of students' progresses in their development of mathematical knowledge in terms of sequence and likelihood. LTs permit one to specify at an appropriate and actionable level of detail what ideas students need to know during the development and evolution of a given concept over time. This definition allowed us to unpack and sequence of the CCSS-M Standards guided by the research literature on spatial measurement.

Learning Trajectories for Length, Area, and Volume

Synthesizing the literature in length and area measurement (Nguyen, 2010), we found three different viewpoints on measurement: (1) those who have built LTs for length and area using an external iterating unit (Barrett, Clements, Klanderman, Pennisi, & Polaki, 2006; Battista, 2007; Battista, Clements, Arnoff, Battista, & Borrow, 1998; Clements & Sarama, 2009; Outhred & Mitchelmore, 2000); (2) those who have investigated the use of common units as measure (Lehrer et al., 1998; Nguyen, 2010); and (3) those who have built an entire numeration system based on measurement (Dougherty & Venenciano, 2007). By approaching measurement as a "systematic process to compare two or more quantities" (Confrey, 2011), we expanded the meaning of measurement beyond association a number of units with a given quantity to include building number-unit relationships using units that are internal of and external to the object being measured. In our work, we treated students' learning of the concepts and skills of length, area and volume as progressions through a single LT instead of separate LT for the above reason.

Development of length trajectories. Sarama and Clements (2009) have proposed a LT for length measurement based on a mixed method analysis and synthesis from other studies (e.g., Hiebert, 1981; Lehrer, 2003; Piaget, Inhelder, & Szeminska, 1960; Stephan, Cobb, & Gravemeijer, 2003). Their LT identified five areas of how students build concept and skills through instructional experiences: (1) alignment of endpoints to compare lengths (Piaget et al., 1960); (2) comparing the lengths of two objects using a third object and transitive reasoning (Hiebert, 1981); (3) finding the lengths of an object by "tiling" or "iterating" smaller identical objects as length units and associating higher counts with longer objects (Hiebert, 1981; Lehrer, 2003; Stephan et al., 2003); (4) understanding that length measure requires equal-length units (Ellis, Siegler, & Van Voorhis, 2000); and (5) using rulers and length measures to investigate real-world phenomenon (Lehrer, 2003; Stephan et al., 2003).

The evolution of students' concepts and skills on length measurement is described in terms of students' developmental progressions and their action schemes (see Sarama & Clements, 2009, pp. 289–291 for details). Seven levels were identified in the LT: (1) Pre-length quantity recognizer; (2) Length quantity recognizer; (3) Length direct comparer; (4) Indirect length comparer; (5) End-to-end length measurer; (6) Length unit relater and repeater; and (7) Length measurer. Sarama & Clements' (2009) work and its supporting corpus of studies provided the research input needed to unpack the Length Standards (Sarama, Clements, Barrett, Van Dine, & McDonel, 2011).

Development of area trajectories. Researchers have documented that to have a deep understanding on area, students must first understand the idea of systematic coverage (no overlaps or gaps) by a square unit (Outhred & Mitchelmore, 2000). They learn to align the units into an array of rows and columns, relating rows and columns to the lengths of the sides, and finally to calculate area from the number of units of length and width (Battista et al., 1998). Other aspects of a more complete learning trajectory for area would include developing student understanding about measuring with a square unit versus with a ruler, linking to lattice point arrays, the impact of different sized units on the magnitude of the area, linking area and perimeter, and extending to triangles or circles. Finally, it would include student understanding of the calculation of fractional area with an anticipation that the product of two numbers produce an area that is smaller than an area of either one of the linear dimensions by 1 unit (e.g. ½ in. x $^3/_4$ in. = $^3/_8$ sq. in. is less than ½ in by 1 in. or $^3/_4$ in by 1 in.).

Nguyen (2010) documented that students could construct common units to compare areas when asked to compare two or more areas without the provision of an external unit. Through equipartitioning (Confrey et al., 2009) of the two areas into smaller areas, students created a same-sized area unit embedded in the original areas to be used as the basis of comparison. He also demonstrated that students eventually generalized that if two areas are equal, they must be measured by the same-sized unit the same number of times. As a result, his students were able to correctly predict the effects of changing the unit size on the measure of an area. Others have investigated a number of these ideas (Simon & Blume, 1994), but work remains to synthesize these findings into a unified description linked to student behaviors.

Development of volume trajectories. Battista and Clements (1996) showed five levels of student behaviors when working volume tasks. At Level A, students only begin to conceptualize a set of cubes that forms a rectangular array. At Level B, students have conceptualized the cubes, but do not utilize the inherent layer structure of a 3-dimensional cube. At Level C, cube faces are used, however, either all of the face cubes are counted or outside the cubes. At level D, students use the volume formula and count a row of face cubes to calculate volume. Lastly, level E is reserved for outliers. Students who were not yet at Level A were generally unable to find out how many cubes there were in a 3-dimensional box, since seeing a mental array picture is only the beginning step to Level A understanding. To such students, the L × W × H formula means very little. Those who applied the formula tended to ignore the three-factor product that results from volume measurement. Multiplication was also not the only operation relied on to calculate volume. Addition, skip counting, and repeated addition were also used.

Battista (1999) followed with a teaching experiment to see if fifth graders could enumerate cubes. All six students in the study were able to structure and enumerate 3D cube arrays. However, their use of layering did not immediately lead to its use in subsequent predictions. Battista (2007) currently claims seven levels of sophistication in students' uses of cubic arrays to construct volume, ranging from organization or location of units in arrays, to introducing composite units, emergent array structures, and spatial structuring and enumeration.

Curry and Outhred's (2005) work distinguishes "packing volume" with cubes and "filling volume" with liquid or sand. While investigating students' understanding of the relationship between length, area, and volume, they discovered that student scores on packing volume tasks were highly correlated with scores on length. In these tasks, students were asked to pack an area with a unit box. They performed much better on tasks involving filling volume with water or sand. The authors conjectured that a filling procedure and length iteration were related processes. This literature informed our consideration of the contents to be included in the descriptors.

Unpacking the Length, Area, and Volume Trajectory

An overarching theme of our work is to consider the "genetic epistemology" (Piaget, et al., 1960) of how instruction refines students' informal mathematical idea successively and develop more complex ideas, as informed by research from a cognitive and instructional standpoint. The adoption of the genetic epistemology approach motivated a five-characteristic framework for unpacking the mathematical content of the Standards into the descriptors. First, the descriptors provide an explicit breakdown of complex mathematical ideas into its *conceptual principles*. For example, the descriptor for standard 1.MD.2 spells out the principles of using a length unit to measure. Second, the descriptors address the *misconceptions, strategies, and representations* that students may encounter as their informal ideas evolve into complex mathematical ideas. For example, the descriptor for standard 2.MD.1 addresses the misconception in using a ruler, where students may misinterpret the number of tick marks spanned by an object as its length. Third, the descriptors identify *meaningful distinctions* about a mathematical concept. These distinctions lead to *multiple models* of problems and support students' generalizations. For example, the descriptor for standard 3.MD.5.b makes three distinctions about the idea of "an area of n square units" as: (1) iterating an area unit $n - 1$ times, (2) "n times as big" as an area unit, and (3) a sweep of a line segment over a distance. Fourth, the organization of the descriptors of a LT reflected a genetically *coherent structure* of development through which students develop "big ideas." For example, the descriptors of this LT are organized to highlight the genetic sequence in which students develop length, area, and volume by: (1) Defining the attribute, (2) Direct comparison, (3) Indirect comparison, (4) Measuring using a unit with no gaps or overlaps, and (5) Compensatory and Additive principles. Fifth, we introduce “Bridging Standards,” additional mathematical knowledge that mediates students’ progression from prior concepts in earlier Standards to more sophisticated and formal concepts in later standards. These Bridging Standards and their descriptors provide a complete genetic epistemological account of a LT. For example, qualitative comparison of area and volume were added as Bridging Standards, since this mathematical knowledge was instrumental to the coherent structure underlying students’ development of measurement, but was not included in the CCSS-M.

We approach the task of unpacking the CCSS-M by describing students’ development in terms of the characteristics mentioned above. Our unpacking proceeded in the following manner. First, we sequenced the relevant Standards in a way that generally reflects research findings about how students progressively learn the ideas. A set of sequenced Standards can be regarded as an *abridged* LT. Second, based on the abridged LT, we built an unabridged version where we incorporated research findings to bridge the instructional gaps between and within the standards of a LT. For length, area, and volume, we synthesized different research findings in the domain of spatial measurement into a unified description of how students’ mathematical knowledge evolved as they encounter activities, tasks, tools, and forms of interaction. Third, we added Bridging Standards when we felt the research suggested mediating ideas that were necessary to be learned before progressing to the next standard in the LT.

We drafted the text of the unpacked LTs in the format of a two-column table, in which the left column showed the standards and its codes as sequenced in the LT and the right column showed the descriptor of the standard. We used Confrey’s (2010) hexagons map to represent how the LTs develop over time and to depict how they are relate to each other visually. The length, area, and volume LT was organized into six sections: (1) Attributes, Measuring Length and Capacity by Direct Comparison; (2) Length measurement using units and tools; (3) Area and Perimeter; (4) Volume Measurement; (5) Conversion; and (6) Area and Volume of Geometrical Shapes and Solids. The move to subdivide the entire LT into sections does not signify some disconnect between the contents of the descriptors but rather permit us to focus on unpacking the more intertwined connections among some Standards. In fact, cross-references between the Standards were often made when drafting the descriptors.

Report of the Unpacking of Length, Area and Volume Standards

We wrote 50 descriptors in the length, area, and volume LT (36 from CCSS-M and 14 Bridging Standards). Below we present a summary of the mathematical knowledge that we have unpacked, according to the five-characteristic framework. The most updated edition of the descriptors can be accessed online (http://www.turnonccmath.com).

Conceptual Principles of Length, Area, and Volume

In the descriptors, we unpacked a list of conceptual principles to be mastered by students across length, area, and volume. They are: the Conservation Principle, the Compensatory Principle, the Principle of Unit Placement, the Principle of Unit Conversion, and the Additive Principle. The Conservation Principle states that the length (or area or volume) of an object remains unchanged under any rigid transformation. The Compensatory Principle states that there is an inverse relationship between the size of the unit (length, area, or volume) used for measurement and the measure (count of the units). The Principle of Unit Placement states that the units used to measure the length (or area or volume) of an objects must be placed without gaps or overlaps and along a path aligned with the object's length (or arrays in the case of area and volume). The Principle of Unit Conversion states that smaller units can be composed to form larger units and that larger units can be regrouped into smaller units. The Additive Principle states that the joining of two lengths (areas or volumes) are sums of the lengths (areas or volumes). From the LT perspective, these principles are foundational to students' development across length, area, and volume. This does not imply that they are taught directly, but rather that the students’ understanding of them evolves gradually through the course of activities and tasks.

Misconceptions, Strategies and Representations

We identified a number of misconceptions informed by NAEP results. These concerned students'' use of rulers and their understanding about area and perimeter. For example, when measuring the length of an object, many students do not check if the object aligns with the zero mark. They also tend to treat tick marks on the ruler as the length of the object instead of the interval between the tick marks. In area and perimeter, students tend to measure the perimeter of a rectangle using square tiles around the corner and believe that increasing the perimeter of a rectangle always increase its area.

We described length as being represented on a number line by equally spaced intervals from 0 as a useful representation of addition and subtraction. Addition of two numbers ($a + b$) could be thought of as combining a length of a units with another length of b units. Subtraction of two numbers, $a – b$ can be thought of as comparing the difference between two line segments or taking away b units from a line segment of a units. For strategies, we also highlighted various ways in which students can directly compare two lengths, two areas and two volumes. Because length, area, and volume have different spatial properties, the strategies of direct comparison varied. For example, straight lengths can always be directly compared, while some areas may overlap and need decomposition to compare. Likewise, the capacity of two containers can be directly compared if poured into cylinders with the same base, whereas volumes of solids will require a systematic means of decomposition.

Distinctions and Models

While the Standards did not introduce any distinctions between volume of a solid and the volume of a container, we use “capacity” to refer to the latter in the descriptors. We also make distinctions among concepts of area and volume which were not explicit in the Standards. For example, the area of a rectangle can be viewed as composed of smaller square units versus the sweeping of a length over a distance. Likewise, we distinguished between volume as the packing of space-filling units versus the sweeping of an rectangular area over a height.

We also distinguished the area formula of rectangles involving fractions from whole-number lengths and introduced four models of fractional multiplication of lengths based on equipartitioning of areas in the descriptors: (1) a whole number and a unit fraction; (2) two unit fractions; (3) two proper fractions but not unit fractions; and (4) one or two mixed numbers (or improper fractions). This is consistent with the sequence in the standards for fractions for multiplication, which is developed fully in the division and multiplication LT. Likewise, in the unpacking of the volume formula of a rectangular prism, we introduced different models of Volume = L × W × H related to the associative property. Coordinating across learning trajectories and providing multiple models supports future development in these topics.

Coherent Structure

As Smith and Gonulates (2011) reported, the Standard’s treatment of length measurement is the most complete in alignment with the research literature as students are expected to distinguish length as a measureable attribute (K.MD.1), directly compare two objects based on length (K.MD.2), order three objects based on length (1.MD.1), iterate a length unit to express the length of an object as a whole number of those length units (1.MD.2), use tools to measure the length of objects (2.MD.1), and measure the length of an object using different length units (2.MD.2).

However, for area measurement, the Standards writers presented an abridged version of this sequence where students immediately iterate a unit square to cover a rectilinear area and call this measure n unit squares (3.MD.5.a and 3.MD.5.b), then learn to measure area by counting unit squares (3.MD.6), and finally find the area of a rectangle by multiplying the length by the width (3.MD.7.a). They then include a standard for students to understand that areas are additive (3.MD.7.d), a Standard that was missing in the Length content. Similarly, for volume measurement, the sequence first started with students’ measurement, estimation of volume and one-step volume problems of involving any of the four operations (3.MD.2). Due to the abridged treatment, the structure underlying students' development in Area and Volume was incomplete.

To ameliorate these issues, we identified from the length, area, and volume contents a template of key ideas found in students' development of spatial measurements. We then applied this template across length, area, and volume Standards in our unpacking. As a result, a coherent structure of the LT descriptors emerged across length, area, and volume, which showed how students' concepts and skills of Length and Area and Volume become more sophisticated under instructions over time: (1) Describe and recognize the measureable attribute; (2) Direct comparison of two objects; (3) Indirect comparison of two objects; (4) Comparison of three or more objects; (5) Define what is meant by *n* units; (6) Express the attribute as a whole number of the units. (7) Measure the attribute twice using different units (compensatory principle); (8) Measure to determine how much bigger or smaller; and (9) Recognize the attribute as additive. Describing students' development of mathematical knowledge within such a coherent structure leveraged on the relevant research in providing teacher readers a sense of an overall developmental progression of students' knowledge as well as the interconnectedness between different Standards when unpacked.

Addition of "Bridging Standards"

As a result of our undertaking of "generic epistemology" account of students' learning, we introduced a total of 14 Bridging Standards unpacked with descriptors based on the coherent structure ands. Five were associated to the conceptual principles of length, including the missing additive principle; five were associated to the conceptual principles of area; three were associated to the volume concepts; and the last one connected the surface area with the volume of the cylinder. The last Bridging Standard was added based on a suggestion from a district curriculum coordinator who noted its absence. When read as parts of the trajectory, these descriptors filled in the knowledge gaps between some Standards and provided a coherent structure for students' development of length, area, and volume.

Discussion

The length, area, and volume Standards in the CCSS-M provide an example of why carefully unpacking the Standards is important. We detailed a trajectory of weaving the relevant Standards together in our unpacking in place of a piece-wise Standard-by-Standard elaboration. Next, we discuss the implication of our work for State Standards and Curricula.

Cross-walk between CCSS-M, State Standards and Curriculum

Comparing the CCSS-M and existing State Standards provides a quick and pragmatic way of evaluating the amount of re-alignment needed for curricular and assessment purposes. However, this approach is insufficient in itself to prepare teachers for implementation. For example, how should matched State Standards be re-ordered to maintain a coherent learning path? Do unmatched State Standards matter to students' learning? A minimalist approach might do more harm in this case. Unpacked using a LTs perspective, the descriptors provide educational practitioners access to a research basis in making educated decisions. For example, the coherent structure of moving from "Defining attributes" to "Comparison" in the length, area, and volume LT provide grounds for including addition of areas as a grade-level objective in the CCSS-M. Similarly, an LT analysis supports a means to conduct content analyses of proposed curricula and CCSS-M. The five characteristics of our unpacked descriptors provided teachers with curricular "landmarks" in anticipation of identifying and filling in instructional gaps in curricula.

Endnote

[1] We thank the participants of the 2011 Measurement Mini-Center Conference (Rich Lehrer, Doug Clements, Jeff Barrett, Jack Smith, Mike Battista and others) for reviewing an earlier draft of these descriptors. This process of peer-review enriched our work with the current views of the research community.

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ANALYSIS OF TWO-VARIABLE FUNCTION GRAPHING ACTIVITIES

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This is a study about the didactical organization of a research based group of activities designed using APOS theory to help university students make constructions, needed to understand and graph two-variable functions, but found to be lacking in previous studies. The model of the "moments of study" of the Anthropological Theory of Didactics is applied to analyze the activities in terms of their institutional viability.

Keywords: Advanced Mathematical Thinking; Geometry and Geometrical and Spatial Thinking

Introduction

Functions of two variables are of great importance in applied mathematics and engineering. However, despite their importance, there are few publications that take advantage of their particularities in order to study their teaching and learning. The first published article we found that explicitly treats functions of two variables is by Yerushalmy (1997). In it she insisted on the importance of the interplay between different representations to generalize key aspects of these functions and to identify changes in what seemed to be fixed properties of each type of function or representation. Kabael (2009) studied the effect that using the "function machine" might have on student understanding of functions of two variables, and concluded that it had a positive impact in their learning. In other work, Montiel, Wilhelmi, Vidakovic, and Elstak (2009) considered student understanding of the relationship between rectangular, cylindrical, and spherical coordinates in a multivariable calculus course. They found that the focus on conversion among representation registers and on individual processes of objectification, conceptualization and meaning contributes to a coherent view of mathematical knowledge. Martínez-Planell and Trigueros (2009) investigated formal aspects of students' understanding of functions of two variables and identified many specific difficulties students have in the transition from one variable to two-variable functions. Using APOS theory, they related these difficulties to specific coordinations that students need to construct among the set, one variable function, and R^3 schemata. In a study about geometric aspects of two variable functions, Trigueros and Martínez-Planell (2010) concluded that students' understanding can be related to the structure of their schema for R^3 and to their flexibility in the use of different representations. These authors gave evidence that the understanding of graphs of functions of two variables is not easy for students and in particular, that intersecting surfaces with planes, and predicting the result of this intersection, plays a fundamental role in understanding graphs of two variable functions and was particularly difficult for students. More concretely, students showed difficulty intersecting fundamental planes (that is, planes of the form $x = c$, $y = c$, or $z = c$ where c is a constant) with surfaces given in different representational formats. Hence they had difficulty with transversal sections, contour curves, and projections. Finally, Trigueros and Martínez-Planell (2011) used the moments of study of the Anthropological Theory of Didactics to analyze the didactical organization of a widely used calculus textbook (Stewart, 2006) and showed that its organization was neither effective in fostering the needed constructions nor viable from the praxeological point of view. This stressed the need to supplement traditional calculus textbooks with activities which cover those aspects found to be lacking in most textbooks. Our research questions in the present study are:

- Does a set of activities designed using a genetic decomposition of functions of two variables help students interiorize actions found to be necessary for a process conception of function of two variables?
- Is the didactical organization of the activity sets conducive to their functioning well at the institutional level?

Theoretical Framework

Since APOS is a well known theory we will only briefly discuss the notion of a genetic decomposition which is important to the content of this paper. For more information on APOS the reader may refer to the brief discussion in Trigueros and Martínez-Planell (2010), or more extensive treatments in Asiala et al (1996), and Dubinsky (1991, 1994).

In APOS Theory, the study of student understanding of a particular concept starts with a "genetic decomposition." This is a hypothesis advanced by the researcher and based on his/ her knowledge, experience, and any available previous data of the actions, processes, and objects that must be constructed in order to attain the desired conceptual understanding. A genetic decomposition is not unique, as different researchers might propose different decompositions. However, it is important that the decomposition be contrasted with data obtained from student interviews to ascertain the constructions actually being made by students. Typically, research data results in revisions of the genetic decomposition as researchers discover unforeseen constructions made by students, or constructions that are assumed to be readily made by students but which are not. The resulting revised genetic decomposition can be used in research and also in the design of activities that may be incorporated into the instructional cycle and that may help students make the desired constructions.

The initial genetic decomposition for function of two variables is given in Trigueros and Martínez-Planell (2010). In order to accommodate results of that study, the genetic decomposition was refined to include the following paragraph on the construction of the schema for R^3, which is important in the present study: The Cartesian plane, real numbers, and the intuitive notion of space schemata must be coordinated in order to construct the Cartesian space of dimension three, R^3, through the action of assigning real numbers to points in R^2, and the actions of representing the results of those actions as 3-tuples, in a table or as points in space. These actions are interiorized into processes that make it possible to consider different sets or subsets, in particular fundamental planes, in each representation register. These processes can be encapsulated into objects on which further treatment actions or processes can be performed. These treatment actions or processes include intersecting fundamental planes with other surfaces to form transversal sections, contour curves and projections, and processes of conversion of those sets and subsets among representations in a schema which evolves and that can be thematized as a schema for three-dimensional space, R^3.

A set of activities was prepared to help students make the constructions suggested by the revised genetic decomposition. We considered it important to analyze and discuss its organization and effectiveness. The *moments of study* of the Anthropological Theory of Didactics (ATD) was used as a tool for the epistemological analysis of the group of activities. In ATD the mathematical activity and the activity of studying mathematics are considered parts of human activity in social institutions (Chevallard, 1997; Bosch & Chevallard, 1999). This theory considers that any human activity can be explained in terms of a system of *praxeologies*, or sets of practices which in the case of mathematical activity constitute the structure of what are called *mathematical organizations* (MO). Mathematical organizations always arise as response to a question or a set of questions. In a specific institution, one or several techniques are introduced to solve a task or a set of tasks. Tasks and the associated techniques, together form what is called the *practical block* of a praxeology. The existence of a technique inside an institution is justified by a technology, where the term "technology" is used in the sense of a discourse or explanation (*logos*) of a technique (*technè*). The technology is justified by a theory. A theory can also be a source of production of new tasks and techniques. Technology and theory constitute the *technological-theoretical block* of a praxeology. Thus a praxeology is a four-tuple ($T/\tau/\theta/\Theta$) (tasks, techniques, technologies, theories), consisting of a practical block, (T/τ), the tasks and techniques, and a theoretical block, (θ/Θ), made up of the technological and theoretical discourse that explains and justifies the techniques used for the proposed tasks.

Within an educational institution a mathematical praxeology is constructed by a didactic process or a process of study of a MO. This process is described or organized by a model of six moments of study (Chevallard, 2007) which are: *first encounter* with the praxeology, *exploratory moment* to work with tasks so that techniques suitable for the tasks can emerge and be elaborated, *the technical work moment* to use and improve techniques, the *technological-theoretical moment* where the technological and theoretical discourse takes place, the *institutionalization moment* where the key elements of a praxeology are identified, leaving behind those that only serve a pedagogical purpose, and *evaluation moment* where student learning is assessed and the value of the praxeology is examined. It is important to clarify that the order of the moments is not fixed. It depends on the didactical organization in a given institution, but independently of the order it can be expected that there will be instances where the class will be involved in activities proper to each of the “moments”.

In a recent article, Trigueros, Bosch, and Gascón (2011) discussed the elements of APOS and ATD theories that may be used to expand the theoretical basis of each of these theories without violating their respective basic tenets. They observed that the model of the moments of study may be used in APOS theory to examine instruction based on activities designed in accordance to APOS.

Method

In view of the results obtained in Trigueros and Martínez-Planell (2010, 2011), four activity sets were designed to help students make those constructions found to be needed to understand functions of two variables. The activity sets dealt with (a) fundamental planes and surfaces, (b) cylinders, (c) graphs of functions, (d) contour maps and graphs of functions. All activity sets stress the use of sections in graphical analysis. For example, in a problem of the first activity set students are given the set $S=\{(x,y,z): z=x^2+(2+y)^3x+y^2\}$ and are asked to draw on a Cartesian plane its intersection with the plane $y=-2$; represent physically the intersection in space (using the manipulative in McGee, 2009); draw in three-dimensional space the resulting intersection curve making sure it is placed in its corresponding plane; and give three points in the intersection. This is to be done right after students are introduced to three-dimensional space, after they have constructed fundamental planes as processes, and before functions of two variables are defined. It aims to have students act on their process of fundamental plane thus helping the encapsulation of fundamental planes into objects. In another problem students are asked to represent physically in space the set $\{(x,y,z): z=xy^2, y=0\}$ and draw it in three-dimensional space. This is a variation of the algebraic representation of the previous example.

After designing the activity sets, they were analyzed in terms of the genetic decomposition and revised until the researchers agreed they covered those constructions predicted by the genetic decomposition. Then, the moments of study of the ATD were used to analyze their didactic organization in two different institutions. For example, the problems presented above are designed to be part of *the moment of the first encounter*, where students meet an important idea needed to construct their R^3 schema.

Activities were classroom tested and revised in two consecutive semesters. After class testing the activities, a set of interviews was undertaken to evaluate them. This produced new observations leading to further improvements on the activity sets. Fifteen students were chosen and interviewed after they had just finished an undergraduate multivariable calculus course. Of the 15 students, 9 had used the activity sets and 6 had not. Each of these two groups of students had equal number of above average, average, and below average students, as judged by their professors. Each interview lasted approximately 45 minutes. They were transcribed and analyzed independently by the two researchers. The conclusions were negotiated.

The interview questions relevant to this study are reproduced below:

1. Draw or represent in three-dimensional space the set of points in space that satisfy the equation $y=2$ and that are also in the graph of the function $f(x,y)=x^2+y^2$.
2. What can you say of the intersection of the plane $x=0$ with the graph of the function $f(x,y)=x\sin(y)$? Represent the intersection in three-dimensional space.
3. Students were to choose the graph of $f(x,y)=\sin(xy)$ among six given surfaces.

Results

APOS and Activity Sets

Results suggest that most students who used the activities had an interiorized process of intersecting planes with surfaces. Orlando, who used the activities, obtained a correct graph:

Orlando: I believe this is a cone … it would be… a circle, may I draw it?
Interviewer: Yes, of course
Orlando: … then this is a parabola on the zx plane that is 4 units up… [even though he says "zx" plane he draws and represents it physically correctly in the plane $y = 2$].

Note that even though initially he gave an incorrect answer, Orlando decided the issue by using sections, as practiced repeatedly in the activity sets, thus obtaining the correct graph. The most common student mistakes on the first question were: acting on the familiarity of "$x^2 + y^2$" conclude that the graph was a cylinder (without using sections) and then trying to obtain the intersection geometrically from that graph; and obtaining the correct formula $z = x^2 + 4$ but being convinced this is a parabola on the *xz* plane, not placing it correctly in space. Students not using the activities were more prone to commit these errors as they had less practice intersecting fundamental planes with surfaces and placing the resulting curve in space. Valerie, a student who did not use the activities, seems not to have interiorized the use of sections as a process:

Valerie: … $x^2 + y^2$ would be, a circle … this is harder than I thought … if I draw it … in the *xy* plane, it would be a circle in the *xy* plane, then, if $y = 2$ … it doesn't give the radius…

Question 2 revealed students' difficulties with free variables. Most students did not realize that after substituting $x = 0$ into $z = x\sin(y)$, the variable *y* can take any value, so that the desired intersection is the *y* axis. For example, Jackeline, who had used the activity sets in her class, was able to respond correctly; however it seems she avoids dealing with the free variable by using other sections to visualize the graph of the surface:

Jackeline: … would have the sine function, then as *x* increases the amplitude is going to increase … so this would be a line [under questioning she specifies it is the *y* axis]

On the other hand Victor, also troubled by the free variable, but who did not use the activity sets, does not evidence a process of using sections:

Víctor: $x = 0$, this is confusing … the entire function $x\sin(y)$ becomes 0 … therefore this would be a plane like this and a plane like this… the intersection consists of two planes

In Question 3, the pattern observed in previous questions continued with students who used the activity sets in class showing more of a tendency to use sections and thus performing better.

Activity Sets and the Moments of Study

According to ATD, a balance of the moments of study is needed for materials to help student learning in an institution. As mentioned before, activities that show the importance and usefulness of intersecting fundamental planes with surfaces can be considered as pertaining to the *moment of the first encounter*. The analysis of the activities showed that a large part of them are related to the *moment of task exploration*. This is no wonder, given that in APOS theory reflection on actions so that they may be interiorized into processes, and applying actions to processes so that they may be encapsulated into objects is of fundamental importance, The activity sets start by exploring a wide range of types of tasks aimed at giving students the opportunity to start building a schema for R^3 in which fundamental planes, intersections of fundamental planes with subsets of R^3, free variables, and quadratic surfaces, in different representational formats will be understood. Task exploration continues in the second activity set with cylinders, that is, surfaces in three-dimensional space described with only two variables. This gives the opportunity to have students reflect on how to plot the graph of $z = x^2$ in three dimensions by initially exploring a point by point representation. In our previous studies we had conjectured that the action of point by point representation may be interiorized into the process of drawing graphs by sections, and this construction was included in the refined genetic decomposition. The last set of interviews showed clearly that this construction is necessary and how a lack of interiorization can act as an obstacle for the coordination of important processes needed to learn the particularities of functions of two variables. The interiorization of actions such as graphing $\{(x, y, 0): y = |x|\}$, $\{(x, y, 1): y = |x|\}$, and $\{(x, y, 2): y = |x|\}$, help students reflect on what is happening as *z* takes on different values, and can be interiorized when they are asked to draw the graph of $y = |x|$ in three-dimensional space. Coordination of different processes and reflection on them leads students to develop a method for drawing cylinders in three dimensions. Later, in the third and fourth activity sets, tasks explicitly involving the use of those methods for functions of two variables gives students the opportunity to start by point by point construction actions and quickly move on to generalize the constructed processes for other functions like $f(x, y) = x^2 + y$. They are also asked to verify their graph by giving values to *z* and showing the resulting curves as part of the surface drawn previously, an activity which may be considered as part of the *moment of evaluation* as are problems in which students compare their graphs of surfaces to contour diagrams they draw. Other tasks that are explored include darkening the curves where specific fundamental planes intersect a given graph of a surface; matching a given set of formulas to a given set of graphs of surfaces with justifications given in terms of transversal sections, which can be considered as *technological- theoretical moment*. The fourth and final activity set reviews transformations in the context of graphing functions of two variables. The variety of activities in the *moment of task exploration* stresses the use and geometric significance of transversal sections making the *technical work moment* explicit. Many of the problems are broken down into parts to guide students in a step by step construction and reflection on the graphing process. This is in accordance with the didactical approach of APOS theory and is intended to complement traditional textbooks, which (Trigueros & Martinez-Planell, 2011) tend to overlook students' difficulties using transversal sections and contours to graph two-variable functions.

The *technical work moment* is present in the activities as the number and variety of problems enables an increasing number of students to construct a process of graphing functions of two variables with understanding. Activity sets allow students to build a schema for R^3 with the necessary coordinations to sustain ensuing graphing activities. Although traditional books present techniques for graphing functions of two variables, the number and variety of problems directly exploring the use of fundamental planes is limited.

As discussed in Chevallard (2007), the *technological-theoretical moment* is closely interrelated with each of the other moments of study. This is also the case in this topic. The technology of using traces or cross-sections to draw the graph of a two-variable function is introduced in the moment of first encounter and developed with multiple opportunities to do task explorations using the activity sets. Even though the activity sets do not include an explicit discussion of the theory, they include opportunities to discuss and justify the methods used by students; also throughout the activity sets it becomes clear that substituting a number for a variable in an equation with three variables corresponds to intersecting a fundamental plane with the graph of the equation. This being the "*technology*" (in the sense of explanatory discourse) used for graphing functions of two variables, the consistent use of this idea aids the construction of cross-sections, projections, and contours, otherwise found to be difficult for students. Many textbooks typically do not explicitly emphasize the role of fundamental planes in graphing activities and hence students seem to come out of these courses without a clear notion of this "*technology*."

The *moment of institutionalization* is present when the activity sets are formally included in the course syllabus, but more importantly when fundamental planes are explicitly used as an important justification technology throughout the course, for example, when explaining partial derivatives, tangent planes, differential, directional derivatives, iterated integration, and drawing solids whose volumes or mass is to be computed with a double or triple integral. The idea of analyzing a function of two variables by using knowledge of functions of one variable is pervasive in the course, so that opportunities abound during class discussion for building upon the knowledge of fundamental planes constructed early on in the course, and to institutionalize the processes and objects constructed. Ideas in the activity sets that serve only a pedagogical purpose are not institutionalized; for example, the action of plotting individual points in a graph of a function is quickly interiorized into a process of graphing by sections. The *moment of evaluation* is abundantly available as activity sets present an opportunity for students to auto-evaluate and discuss their work. It is also present if activities are collected and corrected to evaluate students, used in group activities, or used as the basis of test items, or when activity sets themselves are evaluated by studies, such as this one, comparing student performance.

Conclusion

Results suggest that the activity sets help students interiorize actions described in the genetic decomposition of function of two variables into processes, and encapsulate processes into objects and thus, when used effectively, have the potential to improve students' understanding of graphs of functions and their performance in graphing activities. This can only improve as activity sets are iteratively used and discussed in class, refined on the basis of classroom observations, and further studied in depth with successively improved interview instruments, as has been shown in this study. For example, this study uncovered the need to target activities early on that explore the use of free variables, the convenience of using surfaces with graphs that are unlikely to be memorized by students, and the need that some students have of doing a point by point sketch of a graph before they are able to effectively use sections. Some work remains to be done to complete the sets of activities to explore other aspects of the construction of the concept of functions of two variables, such as recognizing domain and range, and working with restricted domains, but so far, interview results show that they improve students' understanding.

The activity sets shows the presence of all the moments required in the study of the graph of these functions. In comparison with traditional texts and courses, the moment of first encounter is clearly present in the activity sets, while the moment of task exploration offers a wide range of activities and opportunities to interiorize actions into processes or to encapsulate processes into objects. The moment of work on the techniques presents the challenge of balancing the number of activities in each set that can realistically be used in class, or be assigned to students. The moment of institutionalization is present when the praxeology developed in the activities is built upon throughout the rest of the course. Finally, the moment of evaluation is present when evaluating student individual and group performance in the activities, including in similar test items, and most importantly, when evaluating the effectiveness of the activity sets per se.

APOS and semiotic representation theories are cognitive theories of learning and as such are limited in their capacity to describe and predict the effects on learning of social and institutional constraints. However, we have shown a situation where one of the models of the ATD can be useful in analyzing the design of activities that result from a cognitive analysis of a learning situation. Constructions and coordinations found to be missing in studies of students' construction of graphs of two-variable functions can be addressed with activities specifically designed to foster those constructions, in a pedagogical organization that takes the different moments of study into account.

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EXPLORING U.S., TAIWAN, AND MAINLAND CHINA TEXTBOOKS' TREATMENT OF LINEAR EQUATIONS

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Although Chinese students outperform U.S. students according to international mathematics studies, it is suggested U.S. students are equally or even more successful than Chinese students in sense-making and open-ended problems (Cai, 1995, 2000). We hypothesize different treatments on learning to solve problems in Chinese and U.S. curricula may contribute to the difference in performance. We explored what and how meaning, strategy, and procedure were introduced in curriculum. Two U.S. textbooks and one textbook from each of China and Taiwan were analyzed. The initial results show that the Chinese textbooks focus on efficient problem-solving strategies, and the opportunity for students to develop equation transformation skills. The U.S. textbooks place significant emphasis on understanding components of equations and the rationale of equation transformation by presenting procedural steps in a detailed way.

Keywords: Algebra and Algebraic Thinking; Curriculum Analysis; Problem Solving

Introduction

International mathematics studies in the past decade show Chinese students outperform U.S. students (TIMSS 2003, 2007) for both fourth and eighth grades. However, studies also show U.S. students are equally (Cai, 1995; Cai & Silver, 1995) or even more successful (Cai, 2000) than Chinese students in sense-making and open-ended problems such as story problems or problems that can not be solve by simply applying a formal or standard algorithm. We hypothesize textbooks' different treatments on problem solving contribute to the difference in performance because curriculum has been identified as one of the main factors that affect teachers' teaching and students' learning (McCrory, Francis, & Young, 2008). The purpose of the study is to explore different treatments, if there are, between textbooks for what textbooks say (e.g., concepts, strategies, procedure) and how textbooks say (e.g., statement, work example, question, group activity) about solving one-variable linear equations.

Theoretical Framework

Problem solving has been regarded as the heart of mathematics (Schoenfeld, 1992). Polya identified four parts of problem solving which are to understand the problem, make a plan, carry out the plan, and look back at the completed solution. To elaborate, problem-solving process includes subject matter knowledge (e.g., definitions, properties) for understanding a problem, heuristics or strategies to make a plan, the application of definitions or properties for implementing a plan, and the justification of solution procedure according to the plan and definitions or properties. To explore what textbooks can say about solving problems (of one-variable linear equations), based on Polya's idea described above, we identify three categories of knowledge, which are meaning, heuristic and strategy, and procedure. Meaning contains definitions, properties, or rules such as the definition of term or equation. Heuristic and strategy refer to problem-solving plan in general such as isolating the variable or combining like terms. Procedure refers to the application of definitions, properties, or strategies such as combining $2x$ and $3x$ when solving $2x + 3x = 10$ or dividing both sides of the equation $4x = 16$ by 4.

How can textbooks say about problem solving? Textbooks can convey knowledge in the following six forms: statement, work example, question, problem, demonstration, and group activity (Smith et al., 2008). To illustrate, problem-solving strategies could be conveyed in a statement, a question, a work example, or a group activity. Knowledge could also be conveyed in an open or explicit way (Alro & Skovsmore, 1998;

Christiansen, 1997; Silver, 1997; Voigt, 1994). For example, problem-solving steps could be presented in detail (explicit way) or with details skipped (open way).

Methods

This study asks the following question: How do U.S., Taiwan, and China textbooks treat the meaning, strategy, and procedure for solving one-variable linear equations? Particularly, what do the textbooks say about the meaning, strategy, and procedure, how are the meaning, strategy, and procedure presented quantitatively, and how are they introduced qualitatively? We selected one 7th grade mathematics textbook from each of China and Taiwan based on its large market share, and we selected two "Algebra I" textbooks from the U.S. that hold different philosophy of design (i.e., traditional and integrated) and have large market share. We analyzed chapters about solving one-variable linear equations. A coding scheme was constructed based on the framework including 24 meaning items, 17 strategy items, and 25 procedure items. The three authors were paired into two teams (the U.S. team and the Chinese team). The U.S. team achieved 88% inter-rater reliability from 10% coding materials. The Chinese team resolved all of the discrepancies instead. The following table (Table 1) illustrates our coding. The left part of the table is to-be-coded text, and the right part contains codes and descriptions.

Table 1: Coding Example

<table>
<tr><td colspan="2">Text to be coded</td><td>Code</td><td>Code Description</td></tr>
<tr><td colspan="2">The goal of solving an equation is to isolate the variable p.</td><td>S-S06</td><td rowspan="6"><u>The letter before hyphen</u>
S: Statement; W: Work example
<u>The code after hyphen</u>
S06: Strategy, isolate the variable
S10: Strategy, Combine like terms
S15: Strategy, Unitize coefficient
P02: Procedure, Multi/Divide variable terms
P03: Procedure, Add/Subtract constant terms
P04: Procedure, Multi/Divide constant terms
P05: Procedure, Application of add/subtract from both sides
P06: Procedure, Application of multi/divide both sides</td></tr>
<tr><td>Example 1:
(1/6)p + 42 = 21</td><td>
Original equation</td><td></td></tr>
<tr><td>(1/6)p + 42 – 42 = 21 – 42</td><td>Subtract 42 from each side</td><td>W-P05</td></tr>
<tr><td>(1/6) = –21</td><td>Combine like terms</td><td>W-P03 (2)
W-S10</td></tr>
<tr><td>6(1/6)p = –21*6</td><td>Unitize the coefficient of p</td><td>W-P06
W-S15</td></tr>
<tr><td>p = –126</td><td></td><td>W-P02
W-P04</td></tr>
</table>

Analysis and Results

The current results are based on 40% to-be-coded pages of the two U.S. textbooks (Teacher Edition) and 40% to-be-coded pages of the two Chinese textbooks (Student Edition). The distribution of meaning, strategy and procedure as well as the top meaning, strategy, and procedure items are compared between the U.S. and Chinese textbooks.

The distribution of meaning, strategy and procedure are similar between U.S. Integrated and Taiwan textbooks where strategy is about 25% and procedure is about 70% of all coded knowledge items (i.e., meaning, strategy, and procedure items). On the other hand, U.S. Traditional and China's textbooks have similar distribution of meaning (about 5%), strategy (about 50%), and procedure (about 45%, See Table 2).

Table 2: The Distribution of Meaning, Strategy, and Procedure

Percentage/ Frequency	U.S. Int.	U.S. Trad.	Taiwan	China
Meaning	1.8/6	5.4/9	5.5/10	5/7
Strategy	26.7/87	50.6/85	26.5/48	51.8/72
Procedure	71.5/233	44/74	68/123	43.2/60

The top three meaning items from each of the four textbooks show that the Chinese textbooks focus on the meaning of equations and meaning of solution of equations (M08 and M12 respectively). However, the U.S. textbooks focus on the meaning of components of equations (e.g., M06: Term, M07: Like Terms) and the meaning of operations on an equation (e.g., M09: Equivalent Equations, M14: Addition Property of Equations, M15: Multiplication Property of Equations, See Table 3).

Table 3: The Top Five Meaning Items

Meaning Item / Percentage	U.S. Int.	U.S. Trad.	Taiwan	China
1st	M06/33.3	M09/22.2	M08/20	M12/28.6
2nd	M07/33.3	M14/22.2	M12/20	M08/14.3
3rd	M11/16.7	M15/22.2	M15/20	M10/14.3

The top six strategies from each of the four textbooks show that the Chinese textbooks put much more emphasis on story problems than the U.S. textbooks. The Chinese textbooks have more than 57% but the U.S. textbooks have less than 48% strategies about solving story problems (S01, S02, and S03, see Table 3). If we take away items about solving story problems (the shaded cells in Table 3) from the top six strategies, the left three strategies show the U.S. textbooks have S07 (Undo/Inverse Operations) and S08 (Manipulative) that have not been seen in the Chinese textbooks, and the Chinese textbooks have S14 (Move Terms) and S17 (Eliminate Parentheses) that have not been seen in the U.S. textbooks (See Table 4).

Table 4: The Top Six Strategy Items

Strategy Item/ Percentage	U.S. Int.	U.S. Trad.	Taiwan	China
1st	S01/17.2	S02/21.2	**S01/35.4**	**S02/25**
2nd	S07/16.1	S08/17.6	**S02/18.8**	**S01/18.1**
3rd	S08/16.2	S03/15.3	**S03/14.6**	**S03/13.9**
4rd	S03/14.9	S09/12.9	S09/10.4	S10/12.5
5th	S02/12.6	S01/10.6	S10/8.33	S15/11.1
6th	S13/9.2	S07/5.88	S17/4.17	S14/6.94

The top four procedure items from each of the four textbooks show that the Chinese textbooks have significant skip in procedure (P22), but the U.S. textbooks tend to make procedure explicit (P03-P06: the application of addition and multiplication property of equations, see Table 5).

Table 5: The Top Five Procedure Items

Procedure Item/ Percentage	U.S. Int.	U.S. Trad.	Taiwan	China
1st	P03/27.9	P04/29.7	P03/24.4	P01/23.3
2nd	P04/18.9	P03/20.3	P04/20.3	P22/18.3
3rd	P05/12.9	P05/16.2	P22/13	P03/15
4rd	P06/8.58	P06/13.5	P21/9.76	P23/11.7

Discussion

According to the results, the Chinese textbooks apparently have different treatments on solving one-variable linear equations compared to the U.S. textbooks in the following three phases. First, the Chinese textbooks focus on the meaning of equations and solution of equations. However, the U.S. textbooks place emphasis on the components of an equation or the rationale of operations on an equation. Second, for the top six strategies, the Chinese textbooks have significant problem-solving strategies that focus on efficiency (e.g., Move, eliminate) and that have not been seen in the U.S. textbooks, and the U.S. textbooks have significant problem-solving strategies that focus on understanding (e.g., undo, manipulative) and that have not been seen in the Chinese textbooks. Third, the Chinese textbooks treat procedure in a quite open way (with procedural steps skipped), but the U.S. textbooks treat procedure in a quite thorough way (with procedural steps presented in detail). In brief, we find the Chinese textbooks focus on the task of solving equations by providing efficient problem-solving strategies, and the opportunity for students to figure out missing procedural steps in a solution procedure. The U.S. textbooks place significant emphasis on the understanding of components of an equation and rationale of operations on an equation. Problem-solving procedure is also presented in a detailed way to help student understand the reasoning in a solution procedure. Knowing whether written curricula place different emphasis on the meaning, strategy, and procedure of problem solving where students have demonstrated strength and weakness, as in this case for solving linear equations, can support revision and improvement of those materials, and efforts to improve the enacted curriculum as well.

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DIMINISHING DEMANDS: SECONDARY TEACHERS' MODIFICATIONS TO TASKS FOR ENGLISH LANGUAGE LEARNERS

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English language learners (ELLs) are the fastest growing segment of U.S. students. Many teachers who have little or no training with regard to effective teaching strategies for ELL students now face the challenge of transitioning existing curriculum materials for use with ELL students. In this qualitative study, I examined three high school teachers' modifications to mathematical tasks for their ELL students and the resulting impact these modifications had on the tasks' cognitive demand. The primary data sources for this study include interviews, observations, classroom artifacts, and surveys. The findings suggest that teachers made modifications to both the tasks' content and the instructional formats used for the tasks. These modifications frequently resulted in lowered cognitive demand. Implications include suggestions for classroom practice and mathematics educators.

Keywords: Curriculum; Equity and Diversity; High School Education

Finding strategies to improve the educational outcomes of English language learners (ELLs) is imperative as they are the fastest growing segment of U.S. students (Wolf, Herman, & Dietel, 2010). Though the majority of teachers now have at least one ELL student in their classroom, only a third of teachers have received training to effectively teach ELL students (Ballantyne, Sanderman, & Levy, 2008). The mismatch between training and the realities of teaching has left many teachers to their own devices as they seek out or create curriculum materials for their ELL students. The purpose of this study is to examine modifications teachers make to mathematics tasks as they attempt to create a better alignment between their curriculum materials and their ELL students.

Perspectives

The demographics of U.S. students are rapidly changing. ELLs comprise approximately 11% of the students in U.S. public schools. This percentage represents a 51% increase in the decade since the 1997–1998 school year (National Center for English Language Acquisition, 2011). This rapid increase in the number of ELL students has resulted in many states that previously had small ELL populations experiencing large increases in ELLs. With these dramatic increases comes a new set of challenges to many school districts. Many teachers with no experience or training related to teaching ELL students now have several ELL students in their classroom. Additionally, recent results from standardized tests have revealed this quickly growing segment of students continues to reside on the lower end of the achievement gap in mathematics (Fry, 2008). These situations highlight the growing need to train teachers to teach effectively ELL students in both sheltered and mainstream classrooms. Of particular importance is the selection and use of appropriate curriculum materials for these students.

The selection of tasks is an important part of a teacher's practice and student learning. Kloosterman and Walcott's (2010) examination of NAEP results concluded there exists a "positive relationship between what is taught and what is learned" (p. 101). This implies the types of problems enacted impact the type of learning that occurs. Due to the important role tasks play in the mathematics classroom, I have focused my study on mathematical tasks. I have adopted Stein and Smith's (1998) definition of a mathematical task as a portion of the classroom centered on the development of a mathematical concept. Although several studies have examined teachers' use of mathematical tasks (e.g., Stein, Smith, Henningsen, & Silver, 2009), a review of the literature uncovered no studies specifically examining teachers' selection and use of tasks with secondary ELL students. In this study I examined three high school teachers' modifications to mathematical tasks. The following research questions related to this purpose:

1. How do teachers modify mathematics tasks for their ELL students?
2. In what ways do modifications teachers make to tasks impact the cognitive demand?

Methods

I employed a qualitative, multiple case study methodology. The participants—Ms. Thomas, Ms. Hunter, and Mr. Dubois—were secondary mathematics teachers who taught a ninth grade, mathematics class comprised entirely of ELLs, a so called sheltered mathematics course. I purposefully selected these teachers because of their role as sheltered mathematics teachers. Each of the teachers was in their sixth year of teaching.

This study is part of a larger study in which I examined teachers' selection and enactment of mathematics tasks for ELL students. The primary data sources for the present study are surveys, interviews, observations, and classroom artifacts. I observed each teacher's sheltered mathematics course daily for two weeks. Each observation was video recorded and partially transcribed. I conducted daily interviews with the teachers prior to observing their teaching and then conducted two extended interviews after the two weeks of observation, each of which I transcribed verbatim. The classroom artifacts included the tasks presented to the students.

I analyzed the data using the constant comparison method decoupled from grounded theory. This involved many rounds of inductive coding. I first analyzed each teacher individually and identified emerging themes using analytic memos. I then collapsed these themes into codes as I analyzed each of the different data sources for each teacher. I then performed a cross case analysis looking across the three teachers to identify those codes that were relevant to all the teachers. I consulted with my major professor in developing and verifying the codes. In the following section I discuss findings related to the teachers' task modifications.

Findings

Each of the teachers discussed the need to modify the content of tasks for their ELL students. I use the term content to refer to the features of the task including the written presentation, the mathematical values included in the task, and the task's visual presentation. Throughout my discussions with the teachers, each stated the need to modify the language of tasks for their ELL students. The teachers made statements such as "[I had to] cut out a lot of words" and "simplify" the tasks for ELLs. When asked if they modified tasks for all of their classes, the teachers stated that they did on occasion, but in general did not have as great a need for these modifications in their non-sheltered classes.

In addition to simplifying the language, the teachers discussed the need to simplify the mathematical content for their ELL students. Ms. Thomas discussed the need to lower the difficulty of tasks on several occasions during my time in her classroom. Though Ms. Hunter and Mr. Dubois did not explicitly state the need to simplify the content of tasks for their ELL students, they did describe modifications to tasks that simplified the mathematics. The simplification of content also extended to the teachers' presentations of mathematical content and their avoidance of mathematical proofs. Related to simplifying the mathematical content, the teachers discussed their desire to modify tasks so that they had only one solution or one solution path. Each of the teachers noted he or she thought presenting students with a multitude of ways to solve a particular problem created unnecessary confusion. Therefore, they preferred to set up tasks with a particular solution method in order to preempt possible student confusion.

Beyond discussion related to simplifying the mathematics and the presentation of the task, Ms. Thomas also stated that her task modifications for sheltered students often included visual representations. Ms. Hunter and Mr. Dubois did not directly discuss making modifications of this type with their sheltered students; however, during the interviews they did express approval of tasks that included visual representations.

In addition to modifications to the tasks' content, the teachers modified the instructional format they used for the tasks they selected for their sheltered course. I use the term instructional format to refer to the arrangement of students, time allowed for a task, and the resources with which the teachers provided students during the teachers' explanation of the task set up. The teachers often stated that the instructional formats they chose for their sheltered students served as a modification to their typical routine used with non-sheltered students.

Each of the teachers discussed the arrangement of students as a modification to the tasks they used, though the arrangements differed among the teachers. Ms. Thomas discussed her use of small groups within her sheltered course, a practice she avoided with her non-sheltered students. Similarly, Mr. Dubois often assigned problems and then encouraged students to work with and help one another. Ms. Hunter preferred direct instruction, often stating that her sheltered students did not value cooperative learning and got off task too easily.

The teachers often provided students with time limitations as they set up the tasks. For example, before a task that required students to rotate between stations, Ms. Thomas told students they would have five minutes at each station. The time constraints set up by the teachers seemed to try to focus student activity on mathematics and eliminate off task behavior.

In terms of resources provided during task set up, the teachers encouraged their students to draw on graphics, vocabulary aides, and manipulatives as they worked on tasks. Because the scope of this study did not include an in-depth examination of the teachers' non-sheltered courses, I cannot claim the teachers used these resources exclusively when setting up tasks for their ELLs. Although, in some instances, the teachers did explicitly state this was the case.

None of the task modifications resulted in an increase in cognitive demand. Of the modifications I have described, several did not result in a change in the cognitive demand. These modifications instead contributed to the maintenance of cognitive demand. These modifications included the use of visual representations, the time constraints, and the inclusion of resources.

The teachers included visual representations to supplement the written tasks in an attempt to connect the representations with other mathematical ideas in the task. The teachers did not explicitly connect the visual representations to the intended task outcomes; more typically, the representations were included to help students visualize concepts. The lack of intent to connect representations to the task or include reasoning about the representations as part of the outcome prevented the representations from increasing the demand. The time constraints the teachers placed on the tasks helped to prevent the tasks from devolving into non-mathematical activity. Though time in itself cannot raise the cognitive demand, Stein et al. (2009) cited time as a task feature that can aid in the maintenance of cognitive demand. The provision of resources during the task set up did not impact the cognitive demand prior to implementation. For the most part, the teachers suggested to students that they could use calculators, visual aids, textbooks, etc., but did not explicitly discuss how they should use them in conjunction with the task. Therefore, the inclusion of these resources did not work to raise or lower the cognitive demand.

The majority of tasks selected by the teachers were already low in cognitive demand. Therefore, modifications that lowered cognitive demand often resulted in memorization level or non-mathematical tasks. In general, the teachers thought text heavy problems obfuscated the mathematics for their ELL students. The teachers' decisions to simplify the language of tasks often resulted in lowered cognitive demand. On a number of occasions, the context described in a task would require students to interpret the situation and tie their numerical responses to the situation. The elimination of this connection lowered the cognitive demand, a phenomena of which the teachers were not aware.

The teachers' modifications to the mathematical content were the only modifications that had the intentional outcome of a lowered cognitive demand. The teachers' avoidance of proof in their sheltered courses led to lowered expectations in terms of students' justification of answers. In addition to avoiding proof, the teachers' reluctance to embrace multiple solution paths or tasks with multiple solutions lowered the cognitive demand. Stein et al. (2009) discussed the inclusion of multiple solutions and solution paths as a feature of high cognitive demand tasks. The teachers' decisions to avoid difficult mathematics so as not to confuse students resulted in the students experiencing mathematics through low cognitive demand tasks. The persistent use of low cognitive demand tasks is in opposition to the task literature that suggests a variety of tasks is important for student learning (Stein et al., 2009).

Discussion

Each of the teachers in this study cared about their students and modified tasks in ways they thought would improve their ELL students' learning outcomes. Providing teachers with more training on how to modify tasks for ELL students while maintaining cognitive demand is an important step towards improving the learning outcomes for ELLs. Teachers must approach the simplification of language in tasks with extreme care so as not to lessen the cognitive demand. Teachers should also avoid conflating language and mathematical abilities as they modify tasks. This may help to avoid modifications that unnecessarily simplify the mathematical content of tasks. Similarly, teachers should carefully consider the instructional format of lessons to support students without lowering the cognitive demand.

Knowing how to modify curriculum materials in ways that maintain the mathematical rigor is important for students to build mathematical understanding. This research may allow curriculum developers to understand the challenges teachers encounter when selecting curriculum materials for ELLs. This understanding can lead to improvement in curriculum materials that support teachers of ELLs. Finally, teacher educators can build on the findings of this research to develop strategies to better prepare teachers for this rapidly increasing population of students.

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ANALYZING THE DIAGRAMMATIC REGISTER IN GEOMETRY TEXTBOOKS: TOWARD A SEMIOTIC ARCHITECTURE

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Geometry diagrams are multisemiotic texts that encode meaning across a range of communication systems. We propose a scheme for analyzing how geometric diagrams function as resources for mathematical communication in terms of four semiotic systems: type, position, prominence, and attributes. The semiotic architecture we propose draws on research in systemic functional linguistics (Halliday, 2004; O'Halloran, 2005); the architecture suggests a way of analyzing how geometry diagrams function as mathematical texts.

Keywords: Geometry; Classroom Discourse; Curriculum Analysis

Introduction

Mathematical communication employs various semiotic systems to make meaning, in particular language, symbols, and visuals (Lemke, 2003; O'Halloran, 2005). Duval (2006) uses *register* to refer to these semiotic systems. Building on that use of register, in their study of the types of translation tasks that students might be assigned in the geometry class, Weiss and Herbst (2008) argue that the diagrams of high school geometry comprise a distinct mathematical register—the *diagrammatic register*. The symbols of this register are "…pictures of (idealized) 'real' things…"—e.g., circles, points, parallel lines—together with the system of "markup signs"—e.g., congruence, perpendicularity, and parallelism markings—that encode the properties of those objects or permit references to those objects (Weiss & Herbst, 2008, p. 19).

The system of markup signs for geometric properties in diagrams and the norms that govern how diagrams represent specific geometric relations are products of the 20th century. The diagrams of Euclid and Descartes, as well as those in early 20th century plane geometry textbooks—particularly those from what Herbst (2002) called the Era of the Text and the Era of the Originals—were collections of strokes (for lines, line segments, and circles) and letters (for points). The diagrammatic register in these early textbooks of plane geometry lacked many of the features that one would expect from the diagrams in mainstream textbooks from the later 20th and 21st centuries. Figure 1 illustrates some of these differences.

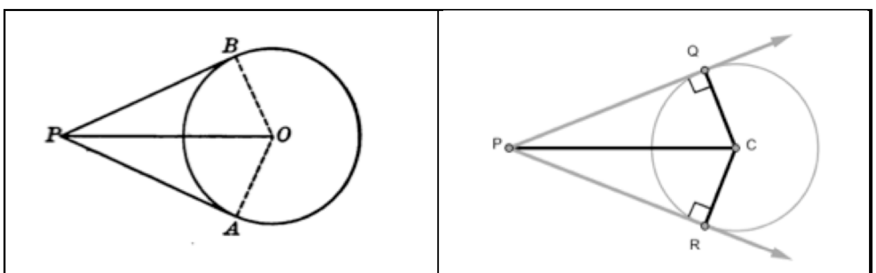

Diagram from Wentworth's *Plane Geometry* (1913, p. 107)	Replication of diagram from Merrill *Geometry* (1990, p. 366)

Figure 1: Comparison of two geometry diagrams

Figure 1 shows two different diagrams from two different textbooks (Wentworth, 1913, and Foster et al., 1990, respectively) that accompany the statement of the same theorem: that the external tangents drawn from a point to a circle entail segments ($\overline{PB}$, $\overline{PA}$ and $\overline{PQ}$, $\overline{PR}$, respectively) that are congruent. The diagrams in Figure 1 have commonalities. For example, each figure shows the segments one might use to prove the tangent segments theorem, and each figure labels the points one would expect to use in the proof

(P, B, O, A and P, Q, C, R). Yet the diagrams in Figure 1 are also clearly different: the later diagram marks <PQC and <PRC as right angles and uses different colors (the lighter lines are blue, the darker lines are red) for different strokes, while the earlier diagram uses different styles of lines ($\overline{BO}$, $\overline{BA}$ are dashed) and uniform choices for the thickness of strokes. The observable differences between these diagrams suggest that readers of diagrams need to be able to interpret and integrate different semiotic systems as they interact with diagrams. The semiotic architecture presented below aims to characterize these and other systems used in the visual display of geometric diagrams.

Semiotic Systems in Geometric Diagrams: Type, Position, Prominence, Attributes

We propose four semiotic systems to describe the range of variation in geometry diagrams. These systems are referred to as the *type*, *position*, *prominence*, and *attributes* systems. Our use of "system" concords with its use in functional grammar: systems contain the paradigmatic ordering of a language (a "what-could-go-instead-of-what relation," Halliday, 2004, p. 23). The systems we identify inventory the choices that are available when creating a geometry diagram. We identified these systems by analyzing the diagrams in 30 geometry textbooks published by mainstream publishers (including Merrill, Ginn and Company, McGraw-Hill, Glencoe, and World Book) that span the 20th century—from 1899 to 2004. The systems we elaborate below capture the possible variations in how geometry diagrams function as representations (note: visual representations of these systems are in a longer version of this paper, available at: http://hdl.handle.net/2027.42/91288)

The Type System

The Type system categorizes the different *parts* of a geometric diagram according to their visual qualities. In any diagram, there could be parts that represent geometric objects (e.g., dots, strokes, regions) and parts that represent geometric (and potentially other mathematical) properties of objects (e.g., hash marks, arrows, small arcs). The parts of a diagram can be differentiated analogously to the way that *free* and *bound* morphemes are differentiated in linguistics (Engelhardt, 2002, p. 24). The free parts are those that can appear on their own (e.g., dots, strokes), while the bound parts are those that can only appear with others (e.g., hash marks on strokes, arrows on strokes).

The divisions in the Type system are visual, not geometric. Keeping visual properties distinct from the geometric properties allows one to study how different geometric properties are represented as visual parts in diagrams. Thus, for example, lines, segments, and rays are all examples of {strokes: straight} and are visually of the same kind. The geometric differences between lines, segments, and rays are encoded through the use of different Attributes (see below).

The Position System

While the Type system provides a scheme for identifying the possible participants in any statements a diagram can make (e.g., points A, B, and C lie on line *l*), the Position system captures how those different participants relate to each other spatiographically—where parts are located relative to one another and how those parts are oriented relative to the frame of reference of the page (Laborde, 2004). Categories in the Position system include *distance* (visual space between parts), *orientation* (heading of the part relative to a set of reference axes), and *connection* (the links between parts, such as strokes that share a dot), with subcategories that depend on a chosen frame of reference (e.g., radial, rectangular).

The Prominence System

Prominence refers to the visual prominence of a part in the display (O'Halloran, 2005, p. 136). There are emphasis and difference subsystems. Emphasis communicates the visual emphasis of a part, through choices for weight (strokes), gauge (dots), transparency (regions), and style (letters and symbols). Difference communicates the visual difference of a part with respect to other like parts, through choices for color (all parts), pattern (regions), fill (dots), and style (stroke). The interaction of these different systems is evident in Figure 1, where circle C (right frame) is given less emphasis relative to strokes $\overline{PQ}$,

$\overline{PR}$ and $\overline{PC}$, by virtue of its lighter weight, yet linked to $\overline{PQ}$ and $\overline{PR}$ —while being set apart from $\overline{PC}$ — through choices in color.

The Attributes System

The principal system that communicates the geometric properties of a diagram is the Attributes system. Attributes can be *relational* or *existential*; like the word "attribute," "relational," and "existential" are chosen to draw an analogy to functional linguistics. In this case, it is the distinction between relational and existential processes (Halliday, 2004). Relational processes serve to "characterize and identify" (Halliday, 2004 p. 210), while existential processes are those "…by which phenomena of all kinds are simply recognized 'to be'" (Halliday, 2004 p. 171). Similarly, the relational attributes of parts are those diacritical markings, measures, and labels that serve to identify and classify relations that hold among specific parts. These markings are resources in the diagram that encode geometric properties. Thus, for the modal viewer, a marked right angle is *right*, regardless of what it might actually look like (and conversely: an unmarked angle that looks right might not be).

Complementing the relational attributes are the existential attributes. Like their linguistic cousins, existential attributes are so named because they actually stipulate the existence of a part in a diagram. Consider, for example, points D, E, C, B, and A in Figure 5, a diagram in Wells and Hart's *Plane Geometry* (1915). In this diagram, the presence of the letters 'D', 'E', 'C', 'B', and 'A' positioned at the ends of the straight strokes mark the existence of points on their ends.

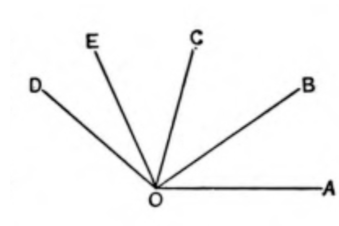

Figure 2: Diagram from Wells and Hart's *Plane Geometry* (1915, p. 19)

Arrows serve as existential attributes when they are applied to the ends of straight strokes, as the means of stipulating that a given straight stroke is a line (two arrows) or a ray (one arrow). The right frame of Figure 1 (see above) has examples of these attributes as they are applied to the stroke from P-R and the stroke from P-Q, thereby bringing into existence ray $\overrightarrow{PQ}$ and ray $\overrightarrow{PR}$ (as opposed to bringing into existence a segment or a line). Apart from the relational and existential attributes that apply to single parts, there are also attributes such as captions or arrows (transformational) that apply to the entire diagram or to several parts.

Summary

Geometry teachers have been concerned with how to teach students to communicate with geometric diagrams for more than 100 years (Baker, 1902). The evolution of the diagrammatic register in 20th century geometry textbooks speaks to this concern, and the semiotic architecture we have proposed in this report is one means through which this evolution can be analyzed. Studying the development of the diagrammatic register in 20th century textbooks will shine a light on how the multiple, ambiguous, and sometimes conflicting roles that diagrams play in student mathematical reasoning are semiotically managed. The work reported here is a step in this direction.

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TRANSITIONING TO THE COMMON CORE STANDARDS FOR MATHEMATICS

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In this study, we investigated various aspects of the transition from state-specific mathematics standards to CCSSM, including a comparison of particular content strands of state standards compared to CCSSM and states' plans for implementation of CCSSM. With regard to content, findings indicated shifts in (1) grade levels at which fluency with mathematical topics is expected, (2) the amount of time spent learning topics, (3) focus on particular content, and (4) the way in which certain aspects of mathematics are addressed. Findings related to the implementation of CCSSM across states indicate variation in modifications made to CCSSM and the rate at which CCSSM is being implemented.

Keywords: Curriculum Analysis; Standards

Purpose

The 2010 publication of the *Common Core State Standards for Mathematics* (CCSSI, 2010) and its subsequent adoption will undoubtedly shift the focus and nature of mathematics education in the U.S. Because of the recent adoption and assessments in 2014–2015, state departments of education across the U.S. are transitioning in different ways and with varying levels of urgency. In this paper, we report findings for the following research questions: (1) To what extent have specific mathematics content in past state standards changed in their placement (grade level) and/or focus in the *CCSSM*? (2) What are the initial actions taken by states as they transition from existing state standards to *CCSSM*?

Perspective

Curriculum standards have dictated the content taught at particular grade levels, and due to the high stakes associated with the mandated assessments, the learning standards have strongly influenced what students have an opportunity to learn (Weiss, Pasley, Smith, Banilower, & Heck, 2003). Researchers have documented considerable variation across state mathematics standards, including the relative emphasis on particular topics, the grade level(s) at which specific content is addressed, and the types of expectations present in the standards (Reys, 2006; Smith, 2011).

CCSSM aims to move K–12 mathematics "toward greater focus and coherence" (CCSSI, 2010, p. 3) and outlines the mathematical expectations for K–12 students across the United States. It reflects the knowledge and skills needed to prepare students for success in both post-secondary education and their future careers (CCSSI, 2010). A survey in 35 common core adoption states, conducted by the Center for Educational Policy (CEP), found that the majority of state representatives believe *CCSSM* is more rigorous than previous mathematics state standards (Koeber & Stark-Renter, 2012). State representatives have begun taking steps to prepare districts for the implementation of *CCSSM* by developing timelines for implementations and crosswalks that compare state standards to *CCSSM*. The results from the CEP survey also indicate that most states do not expect to fully implement *CCSSM* until the 2014–15 school year (Koeber & Stark-Renter, 2012); however; states have begun to take initial steps toward the transition to implement this milestone in curriculum governance in the U.S. These include states (1) augmenting *CCSSM* to address district needs, and (2) developing a timeline and implementation plan for transitioning state standards to *CCSSM* (Reys et al., under review).

Modes of Inquiry

Data Sources and Analysis

The content standards analysis, conducted to answer the first research question, used two data sets: (1) learning expectations from state standards used in earlier state standards analyses (hereafter called "State Standards") (Reys, 2006; Smith, 2011) and (2) learning expectations from the *CCSSM* for the same content/strands as previous analyses. For the second research question, to capture the preliminary actions taken by state departments of education, all *CCSSM*-related documents available on each state department of education website were collected and summarized based on the actions state representatives were taking or planning to take relative to the implementation of *CCSSM*.

In State Standards (Reys, 2006; Smith, 2011), researchers used varying methods to capture themes across state standards as well as variation across states. For example, some teams analyzed the grade placement of particular topics to determine the number of grade levels that students spent learning a specific topic and at what grade level students were expected to demonstrate fluency. Other teams used an existing framework, such as the van Hiele levels of geometric thinking to analyze the descriptive geometry GLEs. Additionally, teams used different foci in examining the standards, ranging from examining specific mathematical topics within a content strand (e.g., fraction computation) to studying entire content strands (e.g., measurement). For all analyses, standards were tagged with state and grade level identifiers for analysis purposes. The same data collection and analyses process were employed in the comparative analysis of state standards and *CCSSM.*

Data analysis for states' initial actions with implementation of *CCSSM* consisted of two processes. First, the most recent state standards were searched to determine if additional standards or changes were made in comparison to *CCSSM*. All standards that were either modified or added to a state's curriculum document were collected for analysis. Second, state timeline documents were analyzed to study trends among state's *CCSSM* implementation schedules.

Findings

Comparison between State Standards and *CCSSM*

Analyses of K–8 state mathematics standards, conducted prior to the release of *CCSSM* under the auspices of the Center for the Study of Mathematics Curriculum (Reys, 2006; Smith, 2011), provided the lens through which changes in K–8 mathematics expectations as outlined in *CCSSM* are identified. Although our analyses revealed similarities between the mathematics described in State Standards and the *CCSSM*, the differences between the two were considerable and will likely be the focus of discussion as states transition to the *CCSSM*. These differences fall into four categories of shifts: (1) a shift in grade levels at which fluency is expected, (2) an expansion or contraction in the amount of time students will spend learning particular topics, (3) a change in overall focus on particular mathematical content at specific grade levels, and (4) a shift from including certain aspects of mathematics in individual standards to addressing them in more general terms in *Standards for Mathematical Practices*.

One key finding from State Standards (Reys, 2006; Smith, 2011) was that states vary considerably in the grade levels at which they expect fluency with particular topics. Therefore, it is inevitable that *CCSSM* will cause adjustments to learning expectations in order for many states to transition from the individual states' standards. Discrepancies in grade placement of standards are prominent when examining fraction computation and mastery of basic facts. For example, 40 of the 42 states examined in State Standards placed fluency with multiplying fractions at a later grade level than *CCSSM*. In contrast, mastery of basic facts is expected at an earlier grade level in *CCSSM* than was found in State Standards.

Differences also exist in the amount of time students will spend learning particular topics in State Standards and in *CCSSM*. For example, whole number computation for addition and subtraction is generally taught over a period of three years in State Standards; however, the development of this topic in *CCSSM* spans five years from the initial exposure of adding and subtracting whole numbers until fluency is expected. Conversely, while topics pertaining to probability were found across all grades K–8 in State

Standards, *CCSSM* confines coverage of probability to grade 7. Many probability topics found only in grade 7 of *CCSSM* are developed across multiple grade levels in State Standards, beginning in some states as early as grade 3. The overall focus on some mathematical topics in State Standards has also shifted in *CCSSM*. For example the emphasis on relationships between operations (e.g., multiplication as repeated addition) and mathematical properties (e.g., distributive property) increased three-fold in *CCSSM* compared to State Standards.

Finally, there was a shift from including certain aspects of mathematics in individual standards to addressing them in more general terms in the *Standards for Mathematical Practices (SMP)*, overarching statements that are included at the beginning of each grade level. For example, calculator and/or technology use was found in at least one standard at all grades K–8 in State Standards, with the overall number of standards increasing across grade levels. However, *CCSSM* does not mention technology and/or calculators within the individual standards until grades 7 and 8. Instead, *CCSSM* addresses the use of technology within the *SMP*, including the expectation that students are able to “use technological tools to explore and deepen their understanding of concepts” (CCSSI, 2010, p. 7). Likewise, reasoning for verification expectations (e.g., predicting, conjecturing, hypothesizing, justifying, drawing conclusions), common in State Standards are absent in *CCSSM*. However, reasoning abstractly and quantitatively, construct viable arguments and critique the reasoning of others and making sense of problems and persevering in solving them are addressed globally in *SMP* (CCSSI, 2010).

State Modifications of *CCSSM*

While the intention of *CCSSM* is common standards across the United States, states are granted permission to make some adjustments to *CCSSM* in order to better meet the needs of their local districts: “While states will not be considered to have adopted the common core if any individual standard is left out, states are allowed to augment the standards with an additional 15% of content that a state feels is imperative” (Achieve, 2010). As of February 2012, 35 of the 45 adoption states have not added any additional standards or changed the language of the standards (Reys et al., under review). Seven states (AL, AZ, CA, CO, IA, MA, NY) have added additional standards. California was the only state to move standards from one grade level to another grade level. Three states (AL, CA, and CO) have added or changed the wording of standards. Two states (MD and ND) have made changes to the format and/or annotated *CCSSM*. North Dakota added an “annotations” column with examples, definitions and comments in the state’s *CCSSM* document to help district administrators and teachers understand the standards and provide guidance in interpreting them (Reys et al., under review).

States’ Development of Transition Timeline

A number of states have developed implementation timelines, describing their plans and deadlines for transitioning from the current state standards and assessments to *CCSSM*. In order to transition to *CCSSM* most states developed “crosswalk documents.” These documents compare the current state standards to the *CCSSM*. The purpose of the crosswalk document is to assist teachers in understanding the shifts in learning expectations and more important, the necessary changes in instructional emphasis. In addition to the crosswalk documents, some states created “bridging documents” that address timelines for transitioning from current standards to *CCSSM*. The transition timelines address the timeframe for when teachers are expected to implement *CCSSM*. Some states also include plans for professional development.

Discussion

CCSSM is the latest educational reform measure in the U.S. designed to elevate student achievement in an understanding of mathematics. The transition from a system of state standards to the adoption and implementation of *CCSSM* will inevitably lead to several changes in K–8 mathematics. These changes have implications for multiple mathematics education stakeholders (e.g., curriculum developers, mathematics teachers). Although change is hard, the hope is that *CCSSM* will challenge the field to refocus our efforts on helping students be prepared for careers and college readiness. The shift of particular topics as well as the introduction of new content and the deletion of other topics will necessitate a transition period as teachers alter their instruction to accommodate *CCSSM*.

Although the transition to *CCSSM* may be seen as difficult, most states have already begun the implementation process. This fast action by states to create crosswalks and prepare professional development for their teachers provides some evidence that *CCSSM* is important and states are ready to make a difference in children's lives. However, with the quick implementation also brings obstacles (e.g., new curriculum, high school course sequencing). Providing teachers with curriculum that is aligned to the goals and standards in *CCSSM* may be the single obstacle that could cause this initiative to fail. Therefore, it is important to continue to monitor the situation especially as schools begin to transition to and implement *CCSSM* in more grade levels.

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STUDENT ACTIVITIES IN FOUR CURRICULA: THE CASE OF ANGLE LESSONS

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This study examines student activities presented in written lessons on angle in four elementary mathematics curriculum programs, based on the assumption that student activities reveal the content students learn and the way students learn the content. In doing so, we analyzed the content span of the activities, relationships among activities, and the relationship between activities and other components of the lesson. The four programs share some commonalities and yet exhibited stark differences in the nature of student activities. This study suggests the significance of examining student activities in relation to individual and overall lessons in the written curriculum.

Keywords: Curriculum; Curriculum Analysis; Instructional Activities and Practice; Geometry and Geometrical and Spatial Thinking

This paper examines mathematical activities in four different elementary mathematics curriculum programs, especially those from lessons to teach angle. Activities not only provide the context in which students learn mathematics, but also embed the mathematics that students explore. Types of activities and their nature provoke certain kinds of thinking and learning, and shape students' learning experiences. In this study, we particularly focus on student activities used to develop the concept of angle in each program, in terms of their content span, the relationship among the activities, and their relationship to other components of the lesson, in order to infer what students are expected to do and learn. We examine student activities, not teacher activities, to account for the kinds of activities in which students are expected to engage in various curriculum programs. In this study, a curriculum program refers to written curriculum materials for day-to-day teaching, not one-day resources or supplemental materials. This may include, but is not limited to, a textbook, a student book, and teacher's guide.

Theoretical Perspectives

Student activities that are assigned in curriculum programs are crucial to understanding what students are expected to do and learn in the mathematics classroom (Li, 2000; Sternberg, 1996). Student activities can be understood as promoting authentic learning (D'Ambrosio, 1987; Smith & Stein, 1998; Stein, Grover, & Henningsen, 1996), as opposed to teacher demonstration and practice problems. Student activities genuinely drive and generate students' actions and performance that shape student learning (Bloom, 1956; Gronlund, 1978; Reed & Bergemann, 2001), by generating students' struggle or perturbation (Piaget, 1975) or motivating to fulfill goals of the activities (Leont'ev, 1981). Moreover, student activities can prompt students to think in diverse and sophisticated ways (Bloom, 1956; Gronlund, 1978; Reed & Bergemann, 2001). As such, examining student activities is crucial to infer what students are learning and thinking in the classroom. In particular, we examine student activities in a set of lessons on angle, in terms of the content span and the relationships among activities and the relationship between activities and other components of the lesson, in order to account for the nature of the activities, i.e., what students are expected to do and learn in each of the programs.

According to Leont'ev (1981), activities are processes. An activity requires a set of actions to accomplish the goal of the activity. Leont'ev emphasizes interrelatedness and situatedness of activities. Student activities can be considered in the same sense. A student activity is a process of learning while students perform a series of actions to reach the goal of the activity. A student activity is necessarily context-based. Whether it is concrete, real world, abstract, or imaginary, a student activity is bounded by the lesson and the previous activities and experiences. In this study, we analyze student activities as they are presented in the written curriculum materials. This constitutes the context of the study as well as the constraints of the study.

The span of the content in student activities tells the overall mathematical goals that each curriculum program envisions. Therefore, examining the content span helps one to imagine what students may be experiencing and learning in the classroom. It also helps one to see how mathematical content builds up as lessons move forward and how learning progresses. In fact, knowledge developed through prior activities serves as a resource to develop a new understanding in later activities (Kajander & Lovric, 2009). In this perspective, the correlation between prior and later activities or content is crucial. Moreover, the relationships between activities and other components of the lesson illuminate the role and significance of activities within the lesson, which helps explain the nature of student activities. These relationships influence students' trajectory of learning and thinking (Steffe, 2011). For example, an activity followed by teacher demonstration inevitably constrains students' thinking and action.

Methods

Four curriculum programs chosen for the study are *Investigations in Number, Space, and Data* (INV hereafter), Scott Foresman–Addison Wesley *Mathematics* (SFAW hereafter), *Math Trailblazers* (MTB hereafter), and the Korean elementary mathematics program, *Mathematics* (KMath hereafter). KMath is chosen because Korean students outperformed those in other countries in a number of international comparison studies (Mullis, Martin, Gonzalez, & Chrostowski, 2004; Schmidt, Blömeke, & Tatto, 2011), and yet we know little about what Korean students learn and what kind of curriculum programs they use. KMath is based on the National Curriculum of Korea revised in 2007 and is the only program available at the elementary level in the country. The three American programs represent a range of elementary mathematics programs, from reform-oriented and research-based to commercially developed. Reflecting reform needs in mathematics teaching and learning, INV and MTB were developed with funding from the National Science Foundation, and yet their approaches are slightly different: INV emphasizes student strategies and genuine investigation of mathematical ideas, whereas MTB integrates science and language arts with mathematics and covers advanced rigorous mathematics. SFAW is one of the programs commercially developed, and yet there is an attempt to incorporate research findings and reform recommendations in the program.

For the analysis, we collected curriculum materials/resources for both teachers and students that were needed for day-to-day teaching and learning, such as teacher guides and student books. These materials provided the details of the mathematical content and context for each lesson and student activities. First, we identified lessons exploring the concept of angle in each of the four programs. Next, we extracted student activities from each lesson, along with the mathematical content embedded in them. In determining what to consider as student activities, we relied on each program as they designated a certain portion of the lesson as activity. All four programs included at least one section for "activity" in each lesson.

The overall analysis focus was given to the features of the activities used to develop the concept of angle in each program. We created detailed descriptions of activities along with specific actions and content embedded. We also examined a general structure of the lessons in each program, in relation to the location and role of the activities in each lesson. In our subsequent analysis, we paid particular attention to what kind of actions students were expected to do in those activities, how each of these activities was connected to other components of the lesson, how they were related to each other, especially how later activities were built on previous activities, and how those activities were organized as a whole to develop the concept of angle. These helped characterize the nature of activities to teach the concept of angle as well as their scope and sequence. Finally, common features and differences in various aspects were compared among the four curriculum programs.

Results and Discussion

The four programs share some commonalities in terms of content covered, and yet they exhibited stark differences in the nature of student activities. In all four programs, right angles are introduced in grade 3 in the context of exploring polygons (e.g., triangles). While INV, MTB, and KMath have lessons on angle in grade 4, SFAW explores angle across grades 3, 4, and 5, one lesson in each grade. KMath devotes one entire chapter to the concept of angle (8 lessons) in grade 4, and all American programs address angle in the unit/chapter of geometry. Angles are further explored in polygons in grade 5 in INV, MTB, and KMath.

Student activities in the four programs address the concept of angle in quite distinct ways and promote different kinds of student actions and thinking. SFAW activities include minimal content; INV activities promotes students' thinking about the relationship among angles; MTB and KMath activities are the most diverse in terms of content embedded in them. The four curriculum programs, ranked from least to greatest in the extent to which student activities play a role in the lessons, are SFAW, KMath, MTB, and INV. It is even possible to teach a lesson without a student activitie in SFAW. KMath and MTB activities involve frequent teacher interventions toward lesson goals. In contrast, INV lessons are organized by student activities and discussions around them. Students explore mathematical ideas during activities and share what they found or did in the whole group discussion. INV lessons cannot be completed without activities, and the role of activities in a lesson is crucial.

MTB activities share some common aspects with KMath and INV. On the one hand, like KMath activities, some MTB activities are short and small-scale, involving teacher intervention along the way of exploration. On the other hand, as with INV activities, some MTB activities require extensive inquiry about the concept and mathematical relationships. In general, MTB activities progress along with discussions and teacher intervention when appropriate, whereas INV lessons designate certain time for discussion before or after activities, usually the beginning or the end of the lesson, in which students publicize and formalize what they found during activities.

KMath activities move from enactive to iconic, and to symbolic representations (Bruner, 1960) and from concrete to abstract fairly quickly. Concrete and enactive approaches are used at the beginning to introduce the ideas, and once reaching the abstract and symbolic level, concrete and enactive activities are rarely used. This is quite different from American programs analyzed. Moreover, KMath activities are highly structured and constrain students' learning experiences toward specific lesson goals. Following the steps, one by one, leads students to reach the desired outcomes in the activities. KMath activities also promote precision and accuracy very early on in the lessons, whereas activities in the American programs in general do not emphasize precision to that extent.

Commercially developed programs have a significant market share, which indicates their substantial influence on classroom practice. In the current reform era, these programs try to incorporate many reform efforts. For example, SFAW includes lessons on relationships among multiplication facts (e.g., $5 \times 9 = 5 \times 5 + 5 \times 4$) and uses games in some lessons. However, it is evident that activities in SFAW lessons on angle have a limited potential. This may be due to the topic chosen. Activities used in lessons on other topics may be different. This also suggests the importance of examining other topics as well, preferably involving a number of lessons and activities. Despite this possible explanation, the fact that there are only three angle lessons, one in each grade, and the fact that their activities have little connection among them indicate the need for improvement.

This study illustrates the significance of examining student activities in the curriculum. As activity theory posits, it is important to analyze a set of actions involved in each of these activities in order to investigate the characteristics of activities even further. An in-depth analysis of activities in relation to their specific actions is a follow-up study needed. In addition, examining actual student activities in the classroom in relation to those in the curriculum is an important study to conduct, since activities presented in the written curriculum are only "envisioned" activities, not actual student activities.

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EFFECT OF PROFESSIONAL DEVELOPMENT ON TEACHERS' IMPLEMENTATION OF A REFORM ORIENTED CURRICULUM

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Transitioning to reform mathematics curricula presents a difficult challenge for many teachers. Often professional development is targeted to help teachers implement curricular materials and can include many different components. This paper presents results from a quantitative study of teachers' implementation of an integrated mathematics curriculum based on their varying levels of participation in a targeted professional development. Results show that participation in workshops increased teachers' textbook implementation. Also, absent workshop participation, instructional coaches did not increase teachers' implementation. These results have important implications for the design of professional development and for researchers conducting curricular evaluations and studies of teaching effectiveness.

Keywords: Curriculum; Professional Development; Curricular Implementation

Implementing reform mathematics curricula represents a challenging transition for many teachers (Ziebarth, 2003), especially for those whose perceptions of mathematics education are grounded in traditional views of teaching mathematics. Although many view the textbook as the most important catalyst for changing what occurs in mathematics classrooms, the adoption of the curriculum alone will not likely transform teachers' instructional practices (Arbaugh, Lannin, Jones, & Park-Rogers, 2006; M. S. Wilson & Lloyd, 2000). Teachers typically use the same instructional practices used by their teachers (Ball, 1988; Tyack & Cuban, 1995), and in order for them to change their instructional practices to reform instruction they need ongoing and sustainable support (Ball & Cohen, 1999; Loucks-Horsley, Love, Stiles, Mundry, & Hewson, 2003; Putnam & Borko, 1997; S. M. Wilson & Berne, 1999). The NRC (2004) contends teachers need adequate professional development (PD) *before implementing* new curricular materials, continued support *while implementing*, and time for reflection *during and after implementation* (p. 46). PD designed to assist teachers before, during, and after implementing reform curriculum has been shown to be effective (Krupa & Confrey, 2010); however, teachers still face difficulties when implementing curriculum for the first time (Krupa, 2011).

Objectives

While it is apparent that ongoing support can help teachers change their instructional practices, it is yet to be determined *how* this type of support impacts implementation of curricular materials. The purpose of this paper is to report quantitative findings on the impact different components of a PD model have on teachers' implementation of the reform mathematics textbook, *Core-Plus Mathematics* (Coxford et al., 2001). This research is an important first step towards determining: the extent to which textbooks are used for instruction and the significance different components of PD have on textbook implementation. Specifically, the research question to be addressed is: *Are there quantitative differences in teachers' implementation of Core-Plus based on varying levels of participation in a specialized professional development?*

Theoretical Perspectives

In its report about curricular effectiveness, the NRC (2004) noted the importance of documenting the faithfulness of implementation and recommended researchers document the "implementation fidelity" of the curricular materials. Implementation fidelity measures the extent to which textbook materials are used for instruction, which is important for documenting the opportunity to learn students are given, but are not indicative of the quality of teaching (McNaught, Tarr, & Grouws, 2008; National Research Council, 2004).

The Comparing Options in Secondary Mathematics: Investigating Curriculum (COSMIC) research team was designed to evaluate high school students' mathematics learning from different curricular programs (COSMIC, 2005). They have provided methodological approaches and instruments to document and measure implementation fidelity. They have created indices for *opportunity to learn* (OTL), *extent of textbook implementation* (ETI), and *textbook content taught* (TCT).

The COSMIC team measured the OTL, ETI, and TCT through Table of Contents Records (TOC-logs), which were self reported by the teachers and customized for the textbook they were using (McNaught et al., 2008). For each lesson of the textbook, teachers indicated if they taught the content (a) primarily from the textbook, (b) primarily from the textbook with some supplementation, (c) primarily from an alternative source, or (d) not at all. The OTL index measured the percentage of textbook content that was taught, either solely from the textbook or through supplemental materials. The ETI index weighted the options in the TOC-logs, giving the content taught primarily from the textbook the a weight of one, content with some supplementation a weight of two-thirds, content mostly from alternative sources a weight one-third, and content not taught a weight of zero. The weights were then summed and divided by the total number of lessons contained in the textbook. This measured the degree to which the textbook was used directly to teach the content. Similarly, the TCT index used the same weighted sum but divided by the total number of lessons taught through any means. This is a measure of how the textbook was used to teach content in the textbook and ignores the topics students were not taught. Each of these three indices was measured at the course level.

Methods

Context and Sample

The North Carolina Integrated Mathematics Project (NCIM) was developed to create and support a community of teachers using the reform oriented *Core-Plus* integrated curriculum materials, particularly in high needs schools. Spread throughout rural parts of the state, the seven partner schools in the NCIM project were identified as low-performing based on North Carolina accountability measures. To prepare teachers to implement *Core-Plus*, in order to strengthen STEM education at these schools, the project directors and evaluation team designed four components for the NCIM PD: (1) a summer workshop providing in-depth education on use of curricular materials (one or two weeks), (2) a web-based environment supporting information exchange, (3) two face-to-face follow-up conferences, and (4) instructional coaches who visited each site monthly. The context of the NCIM project, non-field test sites and ones with a high percentage of minority students, supports research in areas that have not been well researched. For more information about the PD components (see Krupa & Confrey, 2010, 2012).

The sample included groups of teachers with various NCIM PD experiences. Group A teachers participated in all facets of the PD (n = 7), Group B teachers participated in the workshop only (n = 6), Group D teachers were not involved in any aspect of the NCIM PD (n = 6), and Group F were classified as NCIM project teachers but only had an instructional coach and were not part of the summer workshops (n = 2).

Data Sources and Analysis

Each teacher completed a TOC-log for each unique course they taught during the 2009–2010 year. The OTL, TCT, and ETI indices were computed using the COSMIC approach (McNaught et al., 2008) for the 41 logs completed by this sample of teachers ($n_A = 17$, $n_B = 11$, $n_D = 8$, $n_F = 5$). Due to the sample sizes, to determine quantitative differences in teachers' implementation across PD exposure, an ANOVA for unbalanced data was used, followed by Scheffe's Test to determine differences among specific groups (Hollander & Wolfe, 1999). The TCT measures were not normally distributed and the non-parametric distribution free tests, Kruskal-Wallis and Dunn's Test (Dunn, 1964) were used to determine differences in TCT among groups.

Results

Teacher Implementation Indices Disaggregated by NCIM Participation

Opportunity to learn. The OTL index across all *Core-Plus* teachers indicates that on average just over half of the content in the textbook was covered (52.30), though there was considerable variation among teacher's OTL indices (13.93), ranging from 27.69 to 81.71 (Table 1). Teachers who participated in the project workshops have higher OTL indices than non-workshop participants. Scheffe's post hoc test found significant differences in the mean OTL between Groups B and F and Groups A and F ($\alpha = 0.05$). These data suggest the importance of workshop attendance on textbook implementation. The two teachers who were provided with an instructional coach, absent workshop attendance, had significantly lower textbook OTL.

Table 1: Mean (and Standard Deviations) of the Implementation Indices

	OTL	ETI	TCT
Group A (All NCIM components)	56.19 (14.31)	51.21 (15.13)	90.65 (8.32)
Group B (Workshops only)	58.79 (9.90)	57.69 (8.44)	98.48 (3.83)
Group D (No NCIM exposure)	46.67 (9.74)	33.85 (14.90)	70.58 (17.99)
Group F (Coaches only)	33.80 (6.30)	26.18 (6.54)	77.98 (17.14)
Entire Sample	52.30 (13.93)	46.51 (16.73)	87.29 (14.94)

Extent of textbook implementation. Recall the ETI index is a weighted measure describing the degree that the textbook, rather than other materials, was used to teach the content. There was a significant difference in the mean ETI across teachers in all four groups ($F = 10.31$, $p < 0.0001$). Scheffe's post hoc analysis determined differences in Groups B and D, B and F, A and D, and A and F ($\alpha = 0.05$). These data indicated that participation in the workshop significantly increased teachers' ETI indices. Teachers involved in the NCIM PD supplemented the textbook less frequently and rarely used alternative sources. Group D teachers used alternative sources more frequently than others groups.

Textbook content taught. Recall that the TCT index restricted the ETI to consider only the *Core-Plus* content that was taught. The nonparametric Kruskal-Wallis test showed differences in location for the groups ($\chi^2 = 17.02$, $p = 0.0007$). To determine which groups had significantly different TCT indices, Dunn's nonparametric post hoc test for multiple comparisons was utilized and showed differences in the TCT indices for Groups B and D and Groups B and F ($\alpha = 0.05$). When Group B teachers taught content in the textbook, they were directly using the textbook for their instruction instead of supplements. Group D teachers used alternative sources for instruction more frequently than teachers who took part facets of the NCIM PD. Groups with teachers attending the summer workshops rarely used alternative sources for instruction and utilized the textbook as the primary resource in their instruction.

Significance

The TOC-logs provided evidence of the variance in teachers' implementation of textbook content among teachers with varying levels of NCIM PD experience. It was clear that teachers who participated in the NCIM summer workshops utilized the textbook for teaching content in the *Core-Plus* curriculum more frequently than teachers who did not attend the workshops. Next steps in this research will be to report qualitative findings from classroom observations and teacher interviews to understand *how* different components of the PD model shaped teachers' textbook implementation. As teachers navigate the transition between different curricula and standards, it is imperative researchers understand how PD offerings effect instruction so that high-quality, targeted PD can be designed and implemented to meet teachers needs.

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EDUCATIVE SUPPORTS FOR TEACHERS IN MIDDLE SCHOOL MATHEMATICS CURRICULUM MATERIALS

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In this paper I describe opportunities for teacher learning present in four middle school curricular series in the areas of introduction to variable and geometric transformations. I focus on one part of my analysis, the description of the opportunities present for developing Subject Matter Knowledge, Pedagogical Content Knowledge (for Topics and Practices), and Curricular Knowledge. My results indicated that opportunities for teachers' development of Pedagogical Content Knowledge for Practices or Curricular Knowledge were most prevalent, whereas Subject Matter Knowledge was the least prevalent. In particular, opportunities lacked rationale guidance, or guidance that enables teachers to develop an understanding of why particular mathematical or pedagogical approaches might be appropriate.

Keywords: Curriculum Analysis; Mathematical Knowledge for Teaching; Teacher Knowledge; Middle School Education

There have been many efforts to reform mathematics teaching, but for pedagogical change to be realized there is the need for substantial teacher learning (Remillard, 2000). Educative curriculum materials, or materials for Grades K–12 that are "intended to promote teacher learning in addition to students' learning" (Davis & Krajcik, 2005, p. 3), are a potential source for opportunities for teacher learning. Ball and Cohen (1996) advocated for such materials because curriculum materials are used on a daily basis, affording them a "uniquely intimate connection to teaching" (p. 6).

Focus of Study

The focus of my study was to describe opportunities for teacher learning embedded in written middle school mathematics curriculum materials. In particular, I examined the opportunities for teacher learning by investigating the *content* of the teachers' guides and how this content was *expressed*. In this paper, I describe the results related to my analysis of the *content*.

Theoretical and Analytical Framework

The teacher plays an active role in designing and enacting the curriculum in their classroom. Furthermore, the curriculum is a guide not only for students, but for teachers. Dewey (1902) argued that "its primary indication, is for the teacher, not for the child. It says to the teacher: Such and such are the capacities, the fulfillments, in truth and beauty and behavior, open to these children" (p. 39). Curriculum materials continue to be guides for teachers and research indicates that teachers do learn from using materials (see Males, 2011, for a detailed review). Although empirical work has rarely investigated the features of written materials and how these features promote learning, research on teachers' use of materials and what and how they learn from using materials indicates that they may play a role in this learning.

Towards a Framework for Investigating the Content Supports in Curriculum Materials

To analyze content supports I adapted a framework from Beyer, Delgado, Davis, and Krajcik (2009) that allowed me to describe the types of *knowledge* and *guidance* available in the teachers' guides. Due to space limitations I do not include the entire coding scheme in this paper, but instead include the four knowledge domains with some explanatory text in Figure 1.

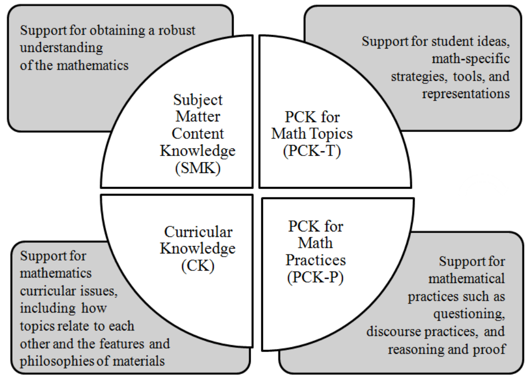

Figure 1: Four domains of knowledge

Knowledge. To develop expertise in teaching mathematics one must have many types of knowledge and be able to integrate these in ways that help one productively promote students' learning of mathematics. In essence, teachers require a specialized type of knowledge of their discipline, knowledge that allows them to teach, not just know their subject matter (Shulman, 1986). First, *Subject Matter Content Knowledge* involves having an understanding of subject matter that goes beyond the "mere subject matter major" (Shulman, 1986, p. 9). Teachers must be able to understand that something is so and also why something is so. Second, *Pedagogical Content Knowledge* can be described as subject matter knowledge for teaching. This includes knowing the most useful forms of representing content in ways that allow for its comprehensibility by others, knowing when and how students may excel or struggle, and knowing strategies for working with students' ideas. Finally, *Curricular Knowledge* is knowledge about the range of programs for the teaching of subject matter, the instructional materials available, and the knowledge related to making decisions about the fruitfulness of using particular materials in particular situations.

Guidance. Unlike materials that merely described what to teach, educative curricula go beyond this and provide opportunities for teacher learning through two types of support: *Enactment Guidance* and *Rationale Guidance* (Beyer et al., 2009). *Enactment Guidance* includes more than just knowing *what* to teach, but also knowing *how* to teach it. For example, this might include a sample of a class discussion in which the teacher asks specific questions to elicit students' justification for their reasoning or to evaluate the reasoning of their classmates. Such examples provide support for how teachers might pose questions in related contexts to elicit similar student responses. *Rationale Guidance* enables teachers to know why particular mathematical or pedagogical approaches might be appropriate. Supports such as this allow teachers to make sense of their curriculum materials and develop what Drake and Sherin (2009) call "curriculum vision," or a sense of where the curriculum materials are going and an understanding of the "particular kinds of learning and teaching practices described in the curriculum materials" (p. 324). An example might include a discussion of why having students create multiple representations for a particular situation is important by describing how the facility between representations will help students develop a stronger concept of linearity.

Method

Sample and Procedures

I mindfully choose four series with large market share in the United States and varied design principles. I included curriculum materials that are categorized as "Standards-based" (Senk & Thompson, 2003), and those that were not. I purposefully chose multiple curricular series within the Standards-based category because we know little about the differences between curricula in the same category. I chose the

Connected Mathematics Project 2 (CMP), *Math Connects* (Glencoe), *Mathematics in Context* (MiC), and *Transition Mathematics* (UCSMP).

Since the structure and features were repeated throughout the texts I chose to analyze units related to the introduction to variable and geometric transformations because these topics were addressed heavily in standards documents and research indicates that these topics are typically problematic for students or teachers (Clements, 2003; Kieran, 2007).

On each page of each unit I examined each sentence and assigned one or more content codes, if applicable, and also coded the location of the content support (i.e., *Unit*, *Section*, *Lesson*). Sentences were coded for multiple supports if it was warranted. I entered all codes into a spreadsheet for ease of calculating frequencies and percentages across all units and curricula and used relative frequencies on summaries sheets to explore themes. In addition, I had a second coder code a random sample of 10% of the corpus, stratified by unit. Percent agreement was calculated at the sentence level and an agreement of at least 85% was reached for each unit.

Results and Discussion

I present my clearest and most significant findings here. For more details, see Males (2011).

Types of Guidance

For all curricula and units, content supports more often provided *Enactment Guidance*. *Rationale Guidance*, or guidance that helps supports teachers in developing a sense of why particular mathematical or pedagogical approaches might be appropriate, accounted for no more than 6% of support in any unit. CMP and Glencoe were consistent in their distribution across the two units. MiC and UCSMP included a higher percentage of *Rationale Guidance* in their variable unit than in their transformations units, however this difference was modest.

Knowledge Addressed

Figure 2 shows the percentages of support for the four types of knowledge for both the variable and transformations units in each curriculum.

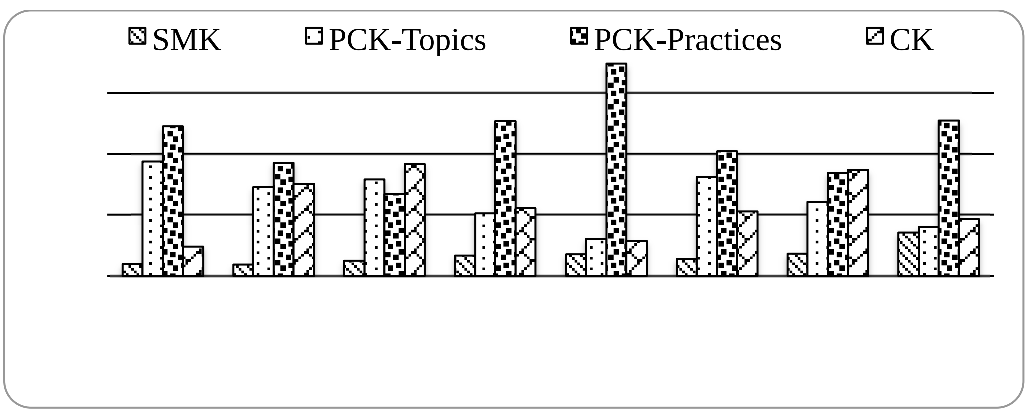

Figure 2: Percentages of content supports by unit and curriculum

The most prevalent content supports in three out of four curricula addressed *Pedagogical Content Knowledge for Practices*, accounting for over 37% of the support in CMP, Glencoe, and UCSMP. These supports included those designed to help teachers engage students in mathematical practices such as questioning, reasoning and proving, and using terminology. MiC, on the other hand, split its attention more evenly between *Curricular Knowledge* and *PCK for Practices* or *Topics.* Supports included those related to developing an understanding of the curricular features and storyline. Of the 31 individual content supports in my framework, 15 were infrequent or unobserved across at least three of the curricular series.

Location of Content Supports

Although most supports were located at the *Lesson* level, a substantial amount of support, particularly for CMP and MiC, was located at the *Unit* or *Section* level, accounting for 28–53% of their support. These results are important because the location of educative supports may impact whether teachers use them. Schneider and Krajcik (2002) found that teachers learned from support located at the lesson level, rather than support located in other sections of the textbook.

Opportunities and Implications

Although each curriculum provided access to some content supports, this access might not be sufficient. Of the 31 content supports, 15 were infrequent or unobserved across at least three of the curricular series and the supports that were present did not often provide *Rationale Guidance*. The lack of this type of guidance may diminish the ways in which teachers engage with and learn from the support. When curriculum authors discuss their rationale they open up a space in which teachers can engage with them around the underlying principles on which the curriculum is designed and. Generally, this space was not provided. In order for teachers to be able to learn from materials, authors need to speak *to* rather than *through* teachers (Remillard, 2000). For this to be realized more attention is needed on content supports, guidance, and where this is located.

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MEETING A NEW STANDARD: USING SAXON MATHEMATICS FOR GRADE 8

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As Common Core State Standards are adopted, assessment in the United States will soon change as a result, and school districts will turn to use textbooks that best align with these standards. This study examines range of knowledge and depth of knowledge, outlined in Webb's criteria for alignment (1997), to compare Saxon's Course 3 textbook with the Common Core State Standards for seventh and eighth grades. By analyzing the alignment between these sources, judgment can be made about whether the curriculum set forth in Course 3 aligns with the mathematics students are now expected to know how to do in the eighth grade. Results of alignment will inform teachers in seventh and eighth grades as they make instructional decisions. Results show 31% alignment of Course 3 material with the eighth-grade standards.

Keywords: Curriculum Analysis; Middle School Education; Policy Matters

Introduction

As states adopt the Common Core State Standards (CCSS), much attention has been focused on comparing previous state documents to the new CCSS with regard to curriculum and assessment (Cobb & Jackson, 2011; Porter, McMaken, Hwang, & Yang, 2011). Textbooks will become outdated with changes to the CCSS, and studies that compare existing textbook curricula to the CCSS will help teachers and school administrators in their transition to CCSS compliance. Kilpatrick (2011) acknowledges the critical role of teachers in implementing curriculum changes by understanding the intended curriculum and better implementing it in the classroom.

Through comparing Range of Knowledge (ROK)—the span of mathematical topics, and DOK—the complexity of knowledge required to meet the objectives, included in Saxon's *Course 3* with outlined objectives in the 7th and 8th grades CCSS (Common Core State Standards Initiative, 2010), we will provide information regarding the presence of a coherent and challenging written curriculum for 8th grade. In particular, the examination of older textbooks such as *Course 3* gives a way of evaluating programs from school districts that are unable to purchase new textbooks after CCSS implementation. Results will provide information to teachers regarding usable curriculum for students in middle school mathematics. With the eventual goal of studying student achievement data from CCSS assessments, these results can form a basis for future work examining teacher instruction, the second important component in judging a program's effectiveness (NCTM, 1995; Tarr, Chavez, Reys, & Reys, 2006). The Saxon text was chosen because of its unique lesson design and incremental sequencing (Hake, 2007).

This study will address the following research question:

How closely do the mathematical topics and depth of knowledge of the content in *Saxon Math: Course 3* and in the Common Core State Standards for grades 7 and 8 align?

Theoretical Framework

The Webb model was designed to analyze the alignment of state assessments and content standards and uses a combination of "qualitative expert judgments" and "quantified coding and analysis" of standards and assessments (CCSSO, 2010). The model is extended in this study to analyze the alignment between objectives in the CCSS and instructional content (as shown in the Saxon Course 3 textbook). By utilizing Webb's framework, this study will judge the alignment of ROK and DOK of the CCSS to Saxon's Course 3.

Range of Knowledge

One way to judge alignment between standards and assessments (or in this case standards and curriculum) is to examine whether both address a similar span of knowledge within content strands. Webb's ROK criterion is met if the full range of each major concept appears in both documents (Webb, 1997, 2007). This study matches textbook topics with CCSS objectives.

Depth of Knowledge

One factor to examine when judging alignment of standards, curriculum, and assessment is the alignment according to complexity of knowledge. Webb's framework gives a four-point hierarchy system. The first, *recall and reproduction*, requires the learner to recall information (e.g., a fact, definition, term or simple procedure). The second level, *skills and concepts/basic reasoning*, requires the learner to provide a non-habitual response that requires some thinking. The third level, *strategic thinking/complex reasoning*, involves higher-order thinking skills, reasoning, explaining, and using evidence. The fourth level, *extended thinking/reasoning*, requires critical thinking, planning, reasoning, and explanation over a long period of time, which must signify some higher-order thinking over the long period of time.

Method

Data Sources

The CCSS Initiative was coordinated by the National Governors Association Center for Best Practices (NGA Center) and the Council of Chief State School Officers (CCSSO). These standards will provide consistent and appropriate benchmarks for students nationwide and enhance global competitiveness (CCSS Initiative, 2010a). Work is underway to develop assessments for field testing in 2013-2014 that are aligned to these standards.

Saxon Math: Course 3 includes distributed instruction, practice, and assessments and contains 132 lessons that are not organized by chapters like traditional textbooks. This text is a part of the middle school series of Saxon's Courses 1–3, for students in sixth through eighth grades. The organizing principle in this text is mathematical thinking, with skills, concepts, and problem solving all connected by consistent mathematical language (Hake, 2007).

Procedures and Instruments

Webb's criteria were used to judge the alignment of ROK and DOK of those topics in Saxon's Course 3 to that of the CCSS for grades 7 and 8.

To analyze ROK, two raters with middle school educational experience and extensive mathematics knowledge matched mathematical topics as stated in lesson titles and subtitles with CCSS objectives. We matched key words from CCSS objectives with the same key words in lesson titles and subtitles. Examples of key words used include *represent proportional relationships*, *compute unit rates*, and *area and circumference of a circle*. If no lessons aligned with an objective by key words alone, raters looked through the instructional material and example problems in each lesson to find instances of alignment with the objective.

For those lessons that matched a CCSS objective, DOK was analyzed. We first coded the CCSS objectives in terms of the DOK levels as described in the previous section. Next, the level of instructional content and examples in the textbook were coded according to DOK. Content in the textbook and in the CCSS was analyzed using key words, verbs, and objects.

Thirty-eight CCSS objectives for grade 7 and 32 CCSS objectives for grade 8 were transferred to a spreadsheet for analysis. There are 132 lessons in the Course 3 textbook including investigations (application or exploration activities). All lessons that matched with an objective for either grade 7 or 8 were recorded, as well as the DOK level of each objective and each lesson corresponding to the objective.

Results

Raters had perfect agreement for number of lessons matching CCSS in grade 8 and differed slightly for three strands in grade 7 (Ratio/Proportion and Number Systems differed by 2 lessons, and Expressions/Equations differed by 4 lessons). The discrepancies resulted from disagreement about standards for *rational numbers*. Raters disagreed about lessons using whole numbers or integers as a match to standards referring to rational numbers. All results were based from the matches agreed upon by both raters. Results of alignment showed 41 of 132 lessons matching the CCSS standards for grade 8 and 51 matching the standards for grade 7 (see Figures 1 and 2). Looking closer at the CCSS objectives for each grade, it was found that 5 of 32 eighth-grade CCSS objectives (in Geometry, Functions, and Expressions/Equations) failed to match with any Course 3 lesson. For seventh grade, 2 of 38 CCSS objectives (both in Geometry) failed to match. Combined, 7 of 70 total CCSS objectives failed to match with the textbook, which still shows acceptable alignment with regard to ROK according to Webb's criterion (Webb, 1997).

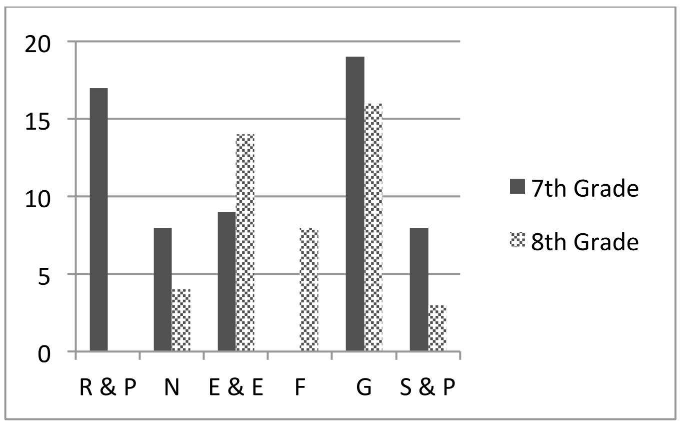

Figure 1: Number of aligned lessons for grades 7 and 8

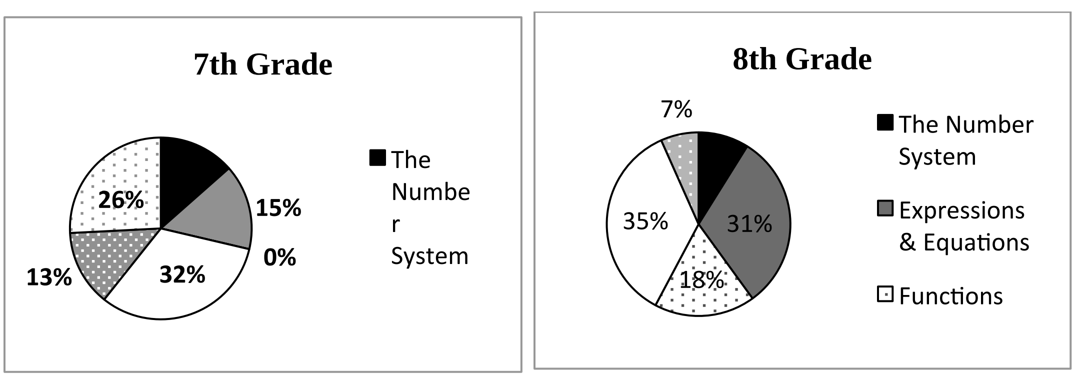

Figure 2. Proportion of aligned lessons in each domain for grades 7 and 8

An examination of DOK shows that over half of lessons were matched at the appropriate DOK level given by the CCSS objectives as Webb's criterion requires at least 50% of matches to be at or above the DOK level given by each standard (Webb, 1997). For example, if a CCSS standard is coded as level 2, the corresponding lesson objective must be coded as level 2 or above. Thirty-one of 38 seventh-grade CCSS objectives, and 22 of 32 eighth-grade objectives were found to be in alignment for both ROK and DOK.

To provide a completely CCSS-aligned, challenging written curriculum, teachers need to supplement from other sources to address these objectives. Other Saxon textbooks, such as *Algebra 1* and *Geometry*, contain lessons that provide a match to these CCSS objectives.

Discussion

As the CCSS become an important source for a consistent national framework for grades K–12 curriculum, assessment will be aligned to the standards. Future studies regarding instructional curriculum will be needed to ensure alignment between expectations, curriculum, and assessment, and these studies will provide assurance of equity among students nationwide.

This study provides a comparison of *Saxon Math: Course 3* to the CCSS for grades 7 and 8 in terms of ROK and DOK. Results of this study are important for school officials who make program decisions regarding curriculum and for teachers who must implement the school-provided textbook. Furthermore, results of this study provide important background for alignment studies of enacted curricula and of assessment relating to the CCSS, which will ultimately give ways to examine middle school achievement.

This study revealed a 69% match of DOK levels to the eighth-grade CCSS objectives, but future study is needed to examine teacher instruction and use of the textbook in the classroom. The matches between lessons and CCSS objectives may be much different than study of enacted curriculum. An alignment study between enacted curricula and CCSS objectives could provide more information about how learning is aligned to these objectives.

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INTENTIONALLY INTEGRATING STEM: A PROPOSED FRAMEWORK

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The goals of this study were to articulate a framework for the development of integrated STEM projects for middle school students in which mathematics is meaningfully represented. Analysis of teachers' mappings of processes central to each of the STEM fields was used to develop the proposed Integrated STEM Process Framework. Here we present the framework and provide an example of a project that was designed through its use.

Keywords: Informal Education; Modeling; Problem Solving

Introduction

STEM has become a very loaded buzzword in education, especially as it relates to policy and as a result of that, funding. STEM stands for *science, technology, engineering, mathematics* and is interpreted by some as an "and" statement and others as an "or" statement (i.e., the former pointing to the integration of the disciplines and the later to the disciplines as independent, but important). NCTM president Michael Shaughnessy's recent editorial (February 2, 2012) noted both the strength of STEM and the problem with STEM. Its strength being an advocacy for investing resources in these disciplines to remain "globally competitive and scientifically and technologically innovative" which is critically important for the field of mathematics education. Yet at the same time he notes that because of this advocacy position STEM has taken on a "generalist" meaning (e.g., STEM programs, STEM schools, and STEM curricula) in which mathematics often takes a back seat and there is real concern of important mathematics content being lost. This is very troublesome when you consider the role that mathematics plays in all scientific, technological and engineering fields.

STEM-based understandings and experiences that prepare learners beyond the classroom are of imminent need, as today's STEM students are tomorrow's leaders in science, technology, engineering, mathematics and education (Prabhu, 2009). National standards for mathematics, science and technology education all highlight the importance of preparing students for college and careers through the integration of science, technology, engineering and mathematics concepts. The *Principals and Standards for School Mathematics* (NCTM, 2000) noted, "The need to understand and be able to use mathematics in everyday life and in the work place has never been greater" (p. 4), emphasizing the importance of students being able to recognize and apply mathematics to science and engineering. This has been emphasized more recently in the Common Core State Standards for Mathematics (CCSSI, 2010) through the articulation of mathematical practice standards that state, "students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace" (p. 7). Such calls point to the importance of integrative STEM experiences for all students; however, as noted by Shaughnessy, this cannot be done at the expense of important mathematics content.

Integrative STEM education signifies the intentional integration of science and mathematics with the processes, content and procedure of technology and engineering education (Sanders & Wells 2010). Though there is an obvious need for opportunities for students to participate in integrative STEM experiences, designing such experiences for classroom use is not easy. Last year a STEM project team brought together prospective science, mathematics, and technology teachers to design a *purposefully* integrated STEM project for middle school students. The term "purposefully" is used because of the central intent aimed at assuring that each discipline was incorporated to into the project in a *meaningful* way. For example, the aim was not for the effort to turn into a science project in which students used minimal mathematics. Nor was the intention for the project to become a "real world" math problem simply using a superficial science setting. What was needed was a framework that would provide both structure and a common language to guide the work. In this article, a framework for the development of purposefully integrated STEM projects is proposed. The framework includes guidelines for identifying a context and the actual design of the project. Finally, we conclude with an example of a purposefully integrated STEM project that was designed using this framework and suggestions for future work.

Proposed Integrated STEM Framework

Science, Technology, Engineering and Mathematics Processes

At the outset of our project there was an understanding that in order to design a project for students that was truly integrated the teacher team would have to work collaboratively to understand the content and common pedagogical practices of each discipline. The science teachers were pushing for the students to be expected to use the scientific method. The technology/engineering teachers were thinking in terms of engineering design principals (NASA, 2010). Finally, the mathematics teachers wanted to make sure that mathematical concepts were not relegated to just being a "tool" within STEM work, but that students were engaged in meaningful mathematical processes such as modeling. The search began for this common language by comparing and contrasting existing processes. Immediately, the cyclical nature of each process became evident and the team began to map them to one another.

The mapping was promising, so the next step was to meet with a group of approximately 30 middle school science, technology/engineering, and mathematics teachers at a statewide conference (many of whom taught at STEM schools). The idea of finding a common language that would capture the processes of each of the STEM disciplines was presented. The participants were provided a copy of NASA's engineering design process (the process with which we knew they were least familiar). Then they were asked to identify how the processes of scientific inquiry and a models and modeling prospective of mathematical problem solving (Lesh & Doerr, 2003) related to the engineering design process. Finally, they noted phrases that captured the intent of each phase.

Social Relevance as a Context

One of the first choices that must be made when designing an integrated STEM project is the context in which it will be set. STEM oriented connections that engage learners based on interest and direct relevance form effective educational efforts (Tate et al., 2007). As such, while the context needs to be related to content standards it should also have social relevance. This can be achieved through the investigation of real problems facing practitioners and researchers in STEM fields. It is recommended that teams of STEM teachers partner with local experts (e.g., informal educators, businesses, researchers) to identify contexts that are both personally and socially relevant to students to design a STEM project within. These emphases on context along with the mappings above were used to inform the development of what we refer to as the Integrated STEM Process Framework (Figure 1).

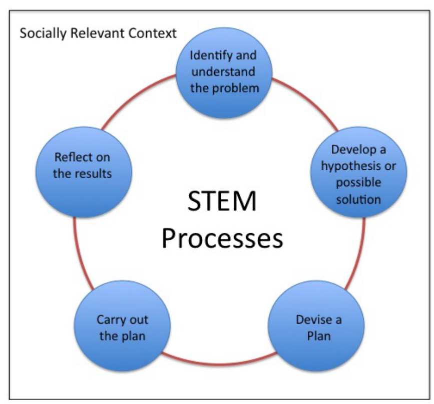

Figure 1: Integrated STEM process model

Proposed Integrated STEM Process Framework

We propose that the Integrated STEM Processes provide a framework for thinking about the development of purposefully integrated STEM projects. Notice that the processes are situated within a socially relevant context. The processes are represented as cyclical, consistent with mathematical modeling/problem solving processes, the scientific method, and engineering design processes. Language that was common among all three existing processes was used, while attempting to reduce the processes to a concise quantity of steps that still captured the essence of each. The intent is that teachers use this framework to inform project designs. In doing so, the goal is that projects should be designed such that students have opportunities to move through this cycle *at least* once within the context of each of the STEM disciplines. This will ensure that each of science, technology/engineering, and mathematics is being incorporated in a thoughtful and meaningful manner. An example of an integrated STEM project designed for middle school students utilizing this framework follows.

An Example: Inquiry on the Neuse River

A team of five prospective science, technology/engineering, and mathematics teachers chose the context of estuarine ecology for this project since it is a component of the state standard course of study for middle school science and recent research that shows that there is a need to be concerned about the effect of estuarine ecology on our nation's drinking water and fish supply. In addition, this was a particularly relevant context for local middle school students given the project team's proximity to an estuary. As such, a partnership was formed with the NCSU Center for Applied Aquatics Ecology (CAAE) in order to offer the project team and prospective teachers CAAE researcher expertise as support in this endeavor.

The prospective teachers began by meeting with the CAAE researchers to discuss their work in a local estuary. During this conversation the researchers noted the problems they had with their very expensive instrumentation, notably keeping the instruments free from the attachment of harmful barnacles. The barnacle issue ended up being the impetus for the design of their integrated STEM project—to design a way for the scientists at the CAAE to protect their instrumentation, used for water sampling, from destructive barnacles. The project was piloted with a small group of students in an out of school setting.

The project was designed to be integrative but with particular goals set within each discipline. Within the context of science, students would learn about estuaries and barnacles. For example, since barnacles live in salt water it is important that students understand that the salinity in an estuary can vary dramatically, depending on depth and the direction of wind currents as well as water temperature. Most importantly, they would then take what they have learned and design an experiment to determine whether or not a protective covering is effective. Within the context of technology/engineering, students would design and construct a protective covering for the water-sampling instrument. Finally, with respect to mathematics students would naturally be drawing on their knowledge of measurement and data analysis when designing and constructing their protective coverings and when designing and carrying out their scientific investigation. At this point in the design process all of the STEM disciplines were represented in the project. When the teachers compared the project to the proposed framework they felt as if mathematics was used as a tool within the design of the protective covering and the scientific experiment. However, they did not feel as if students would have gone through the STEM framework cycle with mathematics. To rectify this the students were also asked to compare the effectiveness of each of the protective coverings, which required them to also draw on their understanding of area (including composition of area), surface area and percents. The prospective mathematics teachers piloted this project with a small group of students in two half-day meetings outside of school.

Reflections and Future Work

Our goal at the outset of this project was to delineate a framework to help teams of teachers from STEM fields find a common language and goal for designing integrated STEM projects. Again, for us that meant a project in which all four STEM disciplines are at the forefront. While this was a preliminary investigation, we feel confident that the framework we have proposed will be helpful for teams of teachers attempting to do this kind of work. Teachers were instrumental in the development of this framework. The project team received overwhelmingly positive feedback from the STEM teachers that participated in the framework workshop that suggested that they were themselves in search of a framework to guide their project development. Furthermore, teachers from each of the disciplines saw their project goals illustrated in the framework. Even so, the mathematics incorporated into the project was somewhat "forced." By that we mean, analysis of the data collected during the experiment was the most obvious mathematics to include, but in order to be sure that mathematics was represented as a process and not solely a tool the teachers added additional prompts to compare the effectiveness of the designs. Further work needs to be done to see if this framework does in fact help to guide teachers toward more meaningful incorporation of mathematical concepts through the models and modeling perspective of problem solving incorporated with the other STEM processes.

The example provided was designed and piloted by a group of preservice teachers in an out of school setting. It is unclear how such a project would—or even could—fit within a school setting. Future research should focus on how integrated STEM projects, that seem to more naturally fit in informal settings (where the disciplines are not split into classes), might be incorporated into schools without losing important mathematics instructional time. While some have expressed understandable concern about the generalization of STEM, we propose that through the use of the Integrated STEM Process framework, integrated STEM projects can be designed in such a way that important mathematical concepts are addressed meaningfully.

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A MISSED OPPORTUNITY: AN EXAMINATION OF MULTIPLICATION AND DIVISION IN A CANADIAN ELEMENTARY MATHEMATICS CURRICULUM

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We present preliminary results from our analysis of multiplication and division tasks included in the teachers' manuals of the Nelson curriculum series. Our analysis of tasks from 14 manuals for grades 1 through 6 focused on (a) the relative proportion of tasks that require problem solving, (b) the ways in which the tasks were presented, (c) the relative frequency of Partitive and Measurement division problems, and (d) the relative frequency of different multiplication and division problem types (Carpenter et al., 1999). The results demonstrated an emphasis on the development of students' conceptual understanding of the operations in the context of word problems. In addition, we observed a greater emphasis on understanding multiplication as repeated addition, suggesting that opportunities to develop multiplicative understanding may be limited. We conclude with additional analyses we are currently conducting.

Keywords: Curriculum Analysis; Elementary School Education; Problem Solving

According to Doyle (1983), "Tasks influence learners by directing their attention to particular aspects of content and by specifying ways of processing information" (p. 161). The potential, however, for mathematical tasks to positively impact students' level of mathematical understanding largely depends on the quality of the task itself (Osana et al., 2006). Given the important role of mathematical tasks in student learning, the types of tasks presented in elementary curricula warrants attention. Accordingly, the primary goal of the present study was to analyze an elementary curriculum series used extensively in Canada (*Nelson Mathematics*; Kestell & Small, 2004) with a specific focus on multiplication and division. More precisely, we focused our analysis on the *context* and *content* (i.e., the problem type) of multiplication and division activities as outlined in the teachers' manuals of the curriculum series.

Theoretical Framework and Objectives

Our decision to focus on these two features of mathematical tasks is supported by the literature on making mathematics "problematic" (Hiebert et al., 1996) and the developmental research on multiplication and division (e.g., Carpenter et al., 1993). Research has shown that tasks situated in a problem solving context[1] promote a more meaningful understanding of mathematics content because it involves applying knowledge versus simply acquiring it (Hiebert et al., 1996). Thus, it is important to consider the *contexts* that are used with students to introduce and explore concepts of multiplication and division.

In addition to the task context, the *task content*, or problem type, should align with students' mathematical development. Multiplication, for instance, is often introduced to children using an additive model in which it is conceptualized as repeated addition. Park and Nunes (2001), however, demonstrated that children's concept of multiplication originates in a schema of correspondences, not addition; therefore, Park and Nunes conclude that instruction should emphasize multiplicative reasoning rather than repeated addition. To develop an understanding of division, it is important to address two models of division: partitive and measurement. Indeed, although children's initial understanding of division is rooted in the action of sharing (Correa et al., 1998) an understanding of the concept of division also involves understanding the relationships between the dividend, divisor, and quotient (Correa et al., 1998). Developing this relational view requires a multifaceted view of division, which includes conceptualizing it as the inverse of multiplication (Greer, 1992).

This paper presents preliminary results from our analysis of multiplication and division as it is treated in the *Nelson* curriculum. Our first objective was to examine the **contexts** in which these two operations are used. More specifically, we examined the relative proportion of tasks in the Nelson series that require problem solving, as well as the ways in which the tasks were presented (i.e., word problems, equations). The next two objectives addressed task **content**: (a) to examine the relative frequency of both *models* of division (Partitive and Measurement), and (b) to examine the relative frequency of different multiplication and division problem *types* (Carpenter et al., 1999). We are currently analyzing the data to describe the ways in which frequency patterns change over the grade levels, the results of which will not be reported here.

The teachers' manuals that accompany the curriculum contain highly scripted descriptions of classroom lessons and activities. Thus, while the manuals clearly describe the intended mathematics curriculum, they also give some indication of what teachers are actually doing in their classrooms. Accordingly, our results paint a picture of the types of activities experienced by a large number of Canadian children in the area of multiplication and division, which can serve to provide an important context for examining their mathematical performance locally and internationally.

Method

The Nelson Mathematics Series

The Nelson Mathematics (NM) series is K–8 mathematics curriculum that is in use in the provinces of Ontario and in several English language school boards in Québec. The NM series was designed for use in the context of major curricular reforms in both provinces (e.g., *Quebec Education Program*; 2005). With respect to mathematics, the core of these reform initiatives corresponds to key principles in the NCTM (2000), such as problem solving and communication. The teacher's manual, called the *Teacher's Resource,* includes 14 color chapter booklets with accompanying resource materials for teachers' use in the classroom.

Coding and Analysis

Data sources. To examine the presentation of multiplication and division in the NM series, we analyzed all the printed material in the Teacher's Resource manuals at each grade level 1 through 6. We coded 14 manuals at each grade level (one manual for each chapter in the student text), for a total of 84 manuals. Each manual is further divided into lessons. For each lesson, the teacher is provided with information about the lesson's goals and required materials for implementing it, and is also provided with scripts and activities on how to introduce the concepts related to the lesson, problems for students to work on in class, and assignments to foster reflection on what was learned.

Task selection. Each lesson consisted of a series of tasks. We defined "task" as any activity assigned by the teacher to the students. Our first round of coding involved classifying the tasks as Multiplication, Division, or Other. Only Multiplication and Division tasks were included in the analysis. Multiplication tasks included those in which (a) the multiplication symbol was used, (b) the student was specifically told to use multiplication, or (c) the structure implied multiplication in a word problem context. A similar classification was used for division tasks.

Coding rubric. All Multiplication and Division tasks were then further classified as Problem Solving (PS) and Non-Problem Solving (NPS). A PS task involved solving for an unknown quantity. This included word problems such as, "Each basket has 5 apples. There are 6 baskets, how many apples are there altogether?" An NPS task is one where there is no unknown to find. In general, the goal of these tasks is to model a mathematical relationship (e.g., use these blocks to show the different ways you can represent $3 \times 2 = 6$).

PS tasks were further subdivided into those that situate models of multiplication and divisions in word problem contexts. In these tasks, the operation is not specified, so the student needs to rely on his or her conceptual understanding of the problem structure to solve it. We called these tasks PS-Not Specified (or PS-NS). There were also PS tasks that were coded as Specified (PS-S), and these were tasks in which the operation was either specified symbolically (i.e., calculate: 7×3, $48\div6$) or in the context of a word problem (e.g., “0.3 of the 400 students in the school are going to Montreal. Multiply to find how many students are going on the trip.”)

The PS-NS tasks were further classified according to problem type (i.e., Grouping/Partitioning, Rate, Price, and Multiplicative Comparison; see Carpenter et al., 1999). Grouping/Partitioning problems describe scenarios that involve collections of discrete objects that are grouped or partitioned into equal parts. An example of a Partitive Division Grouping/Partitioning problem is, “*Robert has 15 stamps that he would like to give to 3 of his friends. He would like to give each friend the same number of stamps. How many stamps does each friend get?”* In contrast, the other problem types (Rate and Price problems) often involve continuous quantities, such as those related to distance and weight. Rate problems, for instance, describe one quantity in relationship to another (e.g., 3 miles per hour), and often the quantities used are continuous. Multiplicative Comparison problems are unique in that a relationship of two quantities is described. That is, the size of one of the quantities (i.e., the referee) is based on how many times bigger or smaller it is compared to another quantity (i.e., the referent).

Results

Context of Multiplication and Division Tasks

Multiplication and division tasks only began to appear in the second grade. In addition, the use of Problem Solving (PS) tasks increased steadily between grades 2 and 6. The ratio of PS tasks to Non-Problem Solving (NPS) tasks started at 1.24 in grade 2 and ended at 2.9 in grade 6. A further analysis indicated that all the PS tasks at the second-grade level were of the PS-NS variety, meaning that all the activities on multiplication and division in grade 2 were couched in word problem contexts. We also found that although the frequency of PS-S problems increased with grade level, the frequency of PS-NS problems was always greater, suggesting that there is an emphasis on the development of students’ conceptual understanding of the operations in the context of word problems across all grade levels.

Content of Tasks: Problem Types

To analyze task content, we further coded (1) the PS tasks involving division according to the model represented (Partitive or Measurement), and (2) all the PS tasks according to problem type. We found that the frequency of Measurement Division problems compared to Partitive Division problems differed across grade levels. In particular, 66.7% of the division problems in Grades 2 and 5 were Partitive; in the other grades, this percentage was lower (48.15% in third grade; 41.67% in fourth grade; and 43.48% in sixth grade).

In general, the frequencies of problem types other than Grouping/Partitioning were relatively low. More specifically, for each grade level, the Grouping/Partitioning problems were the most frequent and Multiplicative Comparison problems were the least frequent. The frequency of Grouping/Partitioning problems was proportionally highest in Grade 2 and Grade 3, representing 86% and 81%, respectively, of the total number of tasks at each level. The high frequency of Grouping/Partitioning problems continues in Grade 4, 5, and 6, accounting for 64%, 72%, and 61%, respectively, of the total. The frequency of Rate and Price problems did increase as the grade level increased. The frequency of the Multiplicative Comparison problems, on the other hand, declined after Grade 4 and did not appear at all in Grades 2 and 3.

Conclusion

The results addressing the relative proportion of problem-solving tasks and their presentation suggest that the NM series introduces multiplication and division with a focus on conceptual understanding. As students experience with multiplication and division progresses, the frequency of modeling tasks decrease to promote more experience with problem solving. While the high frequency of problem-solving tasks suggests that the NM series engages students in reform-orientated mathematical reasoning and facilitates conceptual understanding, the results from the task content analysis demonstrate that the task context results are somewhat misleading. Indeed, the paucity of multiplicative comparison, rate, and price problems, compared to grouping/partitioning problems, demonstrates an emphasis on understanding multiplication as repeated addition. From a developmental perspective, these results suggest the orientation in the NM series toward repeated addition may hinder the development of children's multiplicative reasoning (Park & Nunes, 2001). Currently, we are conducting further analyses to determine whether there is a significant difference in the frequency of Partitive Division and Measurement Division problems and examining changes in these frequencies over the grade levels.

Endnote

[1] Our conceptualization of "problem solving" involves engaging students in the process of determining an unknown quantity. While a more traditional definition associates problem solving with tasks that do not provide an obvious solution method (Hiebert et al., 1996), our definition was broader and even included tasks where the solution method was made explicit.

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CONVINCE ME: AN INVESTIGATION OF ARGUMENTATION IN A MATHEMATICS COURSE FOR IN-SERVICE TEACHERS

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This study investigates the forms of argumentation a mathematics professor intends for in-service teachers to learn and the forms addressed in the course. The teachers are enrolled in a graduate level mathematics course intended for practicing teachers. Additionally, teacher's perceptions of mathematical argumentation and the forms they employ in course activities will be explored. Both case study and grounded theory approaches will be used to guide the data collection and analysis. Interviews with the professor and teachers will be conducted, along with observations of the mathematics course and of teachers' classrooms.

Keywords: Mathematics; Argumentation; Proof; Teacher Education–Inservice/Professional Development

Introduction and Literature Review

Mathematical argumentation is part of curriculums, for example in the teaching of two-column proofs in geometry. Proof, as Hanna (2000) claims is "prominent" in curriculum, but it is not the only form of argumentation in mathematics. Researchers Pedemonte and Reid (2011) claim that further research is needed in the area of abduction, which can be thought of the process of developing a certain form of argument. Models developed in other disciplines, such as Toulmin's model of argumentation, have been used by researchers to study argumentation (e.g., Giannakoulias, Mastorides, Potari, & Zachariades, 2010; Krummheuer, 2007; Pedemont & Reid, 2011). Both proof and argumentation are explicitly addressed in the NCTM Reasoning and Proof Process Standard (2000). Not only that, but they acknowledge in their recommendations that there are multiple forms of argumentation. The NCTM states that students should have the ability to:

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof.

For teachers, the ability to develop well-formed mathematical arguments is important for a few reasons. First, it supports curriculum and mathematics standards; teachers are expected to teach and ask students to engage in mathematical argumentation. They must respond to students' mathematical claims or explanations. Walshaw and Anthony (2008) claim, "effective pedagogy is inclusive and demands careful attention to students' articulation of ideas" (p. 527). Thus, teachers' experiences in mathematical argumentation are an important part of pedagogy. Further, Krummheuer (2007) who assumes that student mathematical learning is predicated on engagement in argumentation practices considers mathematical argumentation as an everyday activity in the mathematics classroom. Kennedy (2009) discusses the importance of mathematical argumentations stating "The ideal mathematical inquiry proceeds through a form of argumentation…" (p. 73). If teachers are not familiar with developing and evaluating mathematical arguments for themselves, it is unlikely they would feel comfortable asking their students to do so and assessing students based on their arguments. Additionally, considering Kennedy's remarks, deficiency in knowledge of mathematical argumentation could hinder efforts to incorporate inquiry practices in the classroom. Mathematical argumentation of teachers is an indicator of their mathematical content knowledge. Developing a valid mathematical argument, such as a proof, can show one can connect mathematical ideas together to verify, discover, explain, and achieve other purposes as listed by Hanna (2000).

Lastly, mathematical arguing is a way for teachers to "do" mathematics. Teacher education programs around the country vary in their structure, but in general programs have modest requirements regarding proof-based mathematics courses or courses which might have a variety of argumentation types, yet mathematical argumentation is a teaching expectation. Giannakoulisa, Mastorides, Potari, and Zachariades (2010) advise that refutation is a form of argumentation that needs greater emphasis in teacher education.

The focus of the research is to study the mathematical argumentation of a professor and in-service teachers enrolled in a master's program. The research pursued here is motivated by the following research questions: What forms and in what ways does the professor intend for teachers to learn mathematical argumentation? What are teachers' perceptions of mathematical argumentation in the course and what forms do they use? How do in-service teachers employ mathematical argumentation in their instruction?

Methodology

The research study's subjects will include one professor from the mathematics department at a university in the Rocky Mountain region and in-service middle and high school mathematics teachers enrolled in a master's program designed for practicing teachers. The professor is the instructor of an algebra course offered in the program, which the in-service teachers (henceforth called students) are enrolled for Spring 2012. This is the second time this professor has taught the course in the program. The format of the course is online and all course meetings are offered through the synchronous software package called Elluminate. This course was chosen for several reasons. First, it is an advanced level mathematics course and so forms of argumentation such as proof and counterexamples are likely to be encountered. Second, the researcher's previous experience working with the program has helped to gain a sense there exists a wide range of mathematical backgrounds and varying degrees of experiences with forms of mathematical argumentation. Thus, this might provide more opportunities to see the variety of ways students argue in a course they take. Third, the instructor of the course expressed enthusiasm at the prospect of conducting this research study in his course.

Data will consist of field notes from observing course meetings, which are audio and video recorded with Elluminate software, the recordings, interviews with the professor and selected students, written work collected from students, and observations of classroom visits to see the students teach in their own classrooms. Interviews throughout the semester with the course instructor will be based on observations of the course and written work produced by in-service teachers. Questions posed during interviews with the professor will seek to draw out information regarding ways the instructor plans to address argumentation, how he intends to engage students in argumentation, and forms of argumentation he perceives students employing. Potential students for interviews will be chosen based on the forms of argumentation they may have employed, questioned, or in the way in which they responded to a given mathematical argument. Interviews with students will be focused on their perceptions of mathematical arguments, forms they have used in the course, and exploring why they chose to argue a certain way.

Both techniques from grounded theory and case study approaches will be employed. Because a small number of students are expected to be selected (possibly two or three) to participate in interviews and observations of their teaching practice, this satisfies one of Merriam's criteria that case study is an appropriate approach when the phenomenon is "intrinsically bounded" (2009, p. 41). The professor and each student will be considered as separate cases. The interactions between the professor and students will provide valuable data concerning the teaching, use, and development of mathematical arguments. As asserted by Grbich (2007), in such cases when "interactions between persons or among individuals and specific environments" is under investigation, grounded theory is a suitable approach (p. 70). Also, because little is known how teachers develop arguments in mathematics, it is another reason why grounded theory is an appropriate approach (Grbich, 2007). While Toulmin's model of argumentation has been used by numerous researchers (e.g., Giannakoulias, Mastorides, Potari, & Zachariades, 2010; Krummheuer, 2007; Pedemont & Reid, 2011), it seems better suited to analyzing single episodes or instances of argumentation than to analyze the process of constructing and argument and documenting someone's forms of argumentation over a period of time to see if any changes occur.

Data collection began in the spring semester of 2012 and will continue through May 2012 and so I will continue to collect data after submission of the proposal. Observations of the course meetings and opinions from the professor will be taken into account for inviting particular students to participate in interviews and classroom observations. Open and in-vivo coding will be conducted for initial coding stages of the interview data of the professor and students, which will be partially or fully transcribed, depending on what is determined to be pertinent to the study. As Merriam (2009) states, "triangulation remains a principle strategy to ensure for validity and reliability" and so this is one strategy I will employ in analyzing data (p. 216). Also, member checks, and efforts to establish researcher reflexivity (e.g., Cho & Trent, 2006; Merriam, 2009) will be used.

Preliminary Findings and Discussion

My data set will consist of what I have gathered at this time and data I will continue to collect throughout the semester. Data I currently have consists of field notes of course observations and links to recorded class sessions. Preliminary analysis of observations indicates there is a variety of ways the professor engages students in argumentation practices. Also, students appear to be diverse with respect to arguing mathematically, providing arguments of various forms, from giving an example to respond to the professor's prompt for an argument to citing a theorem.

One of the goals of this research is to develop a scheme for examining argumentation in mathematics. Ideas or elements from Toulmin may be used in the formation of this scheme, as it is a well-established tool for analyzing mathematical arguments. Based on what I observe happening in the course I may use the data to modify Toulmin's model of argumentation so that it helps follow a person's use of argumentation over time. By developing a scheme for analyzing mathematical argumentation it is hoped that a better understanding of how teachers develop mathematical arguments and of forms they is achieved.

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REASONING-AND-PROVING TASKS IN A GEOMETRY AND MEASUREMENT TEXTBOOK FOR PROSPECTIVE ELEMENTARY TEACHERS

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Although elementary teachers are expected to engage their students in the process of reasoning-and-proving in everyday mathematics learning, many prospective teachers have had limited experiences with this process. College mathematics courses for prospective teachers and the mathematics textbooks chosen for these courses can play an important role in prospective teachers' opportunities to learn about reasoning-and-proving as undergraduate students. In this article, we examine the opportunities in a geometry and measurement textbook for prospective teachers to engage in reasoning-and-proving. The findings have implications for how instructors might choose to implement a textbook in ways that support the development of rich conceptions of reasoning-and-proving with prospective teachers.

Keywords: Reasoning and Proof; Curriculum Analysis; Teacher Education–Preservice

Introduction

Reasoning-and-proving (RP) is fundamental to the work of doing authentic mathematics, both in mathematics and classroom. Descriptions of mathematicians' practice and K-12 standards documents alike note that the proving process involves exploration of patterns, which can lead to the generation of conjectures, and can then be tested and revised or proven informally or formally (Lakatos, 1976; NCTM, 2009). The hyphenated term *reasoning-and-proving* (Stylianides, 2008) denotes the range of activities including investigating patterns, formulating conjectures, generating arguments, evaluating others' arguments, and communicating mathematical knowledge, which are "frequently involved in the process of making sense of and establishing mathematical knowledge" (Stylianides, 2009, p. 259).

Proof in K–12 classrooms is often restricted to verifying given statements using a two-column format (Herbst, 2002). This emphasis represents only one aspect of the RP processes, and contributes to the pervasive difficulties and limited views of proof held by K–12 students and their teachers (Balacheff, 1988; Martin & Harel, 1989). The full range of RP processes can be accessible at the elementary level, and as such, elementary teachers should be equipped to teach RP in meaningful ways. It is crucial, therefore, to help transition prospective teachers of elementary grades (PTEs) from conceptions of proof as empirical arguments, towards understandings of the RP as a process in which one engages to make meaning in mathematics.

Mathematics for elementary teachers courses are the primary site for supporting this development of PTEs' knowledge for teaching. Geometry and Measurement is a common content slice for these courses, with explicit attention to the work of proof (Cannata & McCrory, 2007; McCrory, Siedel, & Stylianides, 2008). Analyses of popular math for elementary teachers texts suggest that RP opportunities are sparse (McCrory et al., 2008), but those analyses used broad approaches involving key word searches of the table of contents and indices of texts. In this study, we use Stylianides's (2009) analytic framework to examine the treatment of RP in a textbook used in teaching PTEs Geometry and Measurement to characterize the opportunities for PTEs to learn about RP. Specifically, we analyzed the Geometry and Measurement chapters of a popular text used in teaching mathematics content courses in the United States, a text designed to help PTEs to "*explain why* mathematics works the way it does" (Beckmann, 2008, italics added, p. xix), make sense of mathematics, and carry those abilities into their future classroom.

Analytical Framework

The unit of analysis used in considering the text was the mathematical instructional task (Henningsen & Stein, 1997), as tasks are a key determinant of students' opportunities to learn (NCTM, 2000). A mathematical task is a set of questions or text segment oriented that develops a particular idea. Analyzing the textbook at the task level, we identified opportunities to create or evaluate conjectures and/or provide mathematical justification within each task.

Table 1: Reasoning-and-Proving Framework (adapted from Stylianides, 2009)

	Reasoning-and-Proving					
	Mathematical Generalizations			*Mathematical Arguments*		
	Investigate Mathematical Relations	*Conjecture*	*Evaluate Claim*	*Evaluate Argument*	*Provide a Non-proof Argument*	*Provide a Proof*
Components & Subcomponents of RP	• Provided Examples • Generate Examples	• Make a Conjecture	• Evaluate a Mathematical Claim	• Evaluate a Mathematical Argument	• Empirical Argument • Rationale	• Proof

To characterize the RP opportunities afforded by tasks in the text (Beckmann, 2008), we applied a modified version of Stylianides's (2009) analytic framework. This framework identifies a task as related to RP if it provides opportunities for students to create or evaluate mathematical generalizations or mathematical arguments (Table 1).

Method

In the Geometry and Measurement chapters of Beckmann's (2008) textbook, we first grouped questions together into tasks and identified all tasks with RP opportunities. The textbook contained 115 tasks related to these four chapters. We coded the questions within tasks using the analytical framework (Table 1). The definitions for each subcategory (detailed further in Stylianides, 2009) guided the coding of each question, and a single question could be coded with multiple categories, when applicable. If a question did not fit the criteria for any categories of the framework, then it was coded as not RP-related. Two trained raters conferred to refine the descriptions and procedure for applying this framework; double-coding 31% of questions, reaching agreement on 81% of the codes, and resolving any disagreements through discussion.

Results

One goal of this analysis was to examine the extent to which the textbook provided opportunities for PTEs to engage in RP; understanding the ways in which RPTs were distributed informed this purpose. Of the 115 tasks, about 57% contained at least one question related to RP, and these RPTs were distributed across the four chapters. Figure 1 shows the ways in which RPT and nonRPT distribute across sections of these four chapters. There were a number of sections in these chapters for which RPTs were prevalent, accounting for at least 50% of the tasks in sixteen of the 26 sections in the textbook. Moreover, the later subsections in some chapters (8-10) tended to have more RP tasks than earlier sections or there was an even spread among subsections (11). It could be problematic if the chapter frontloaded RPTs in the beginning subsections. The sections of the textbook with fewer RPTs were also spaced out so that PTEs could arguably have opportunities to encounter RPTs at various points throughout a course using this textbook.

We also aimed to identify the nature of RP opportunities afforded in the text by investigating the types of RP processes elicited in RPTs. Figure 2 shows the ways in which the tasks were distributed with respect to the six types of RP processes. The most common category was *providing nonproof arguments* in which 47 of the 66 RPTs (71%) contained at least one question prompting students to provide mathematical explanations, not explicitly proofs. Two categories especially relevant to teaching, evaluating claims and arguments, accounted for the fewest tasks.

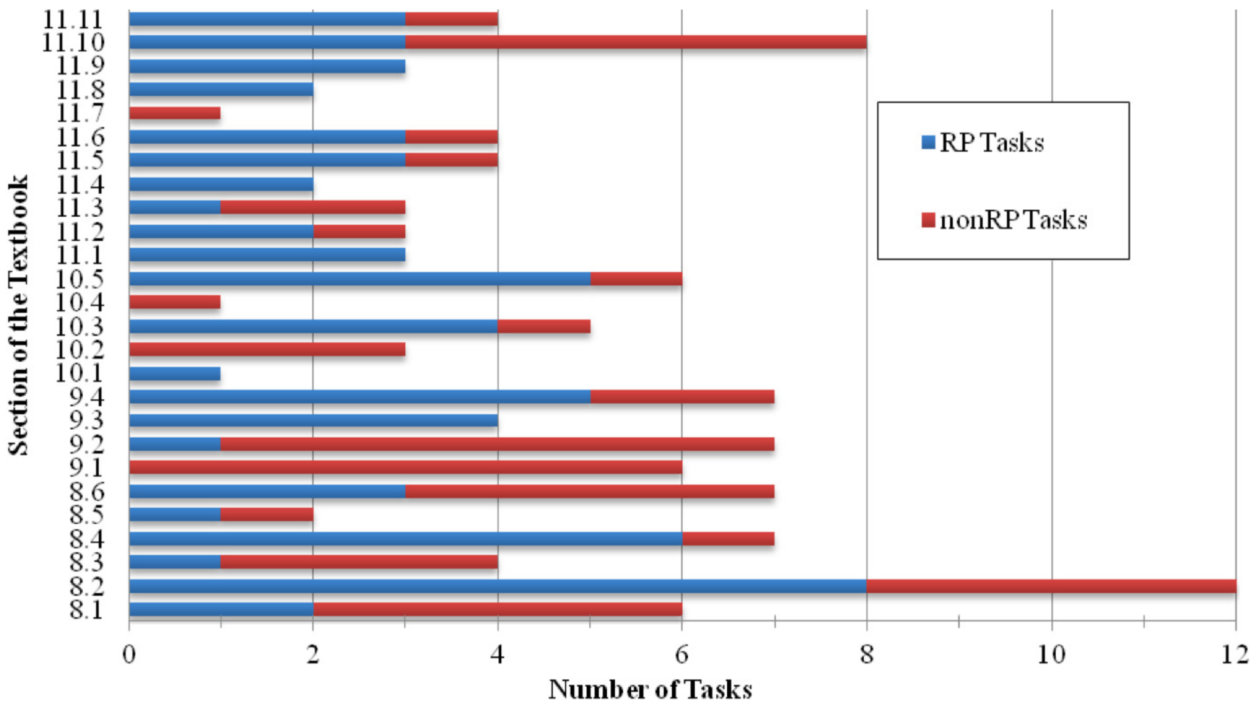

Figure 1: Breakdown of RPTs and nonRPTs in the textbook

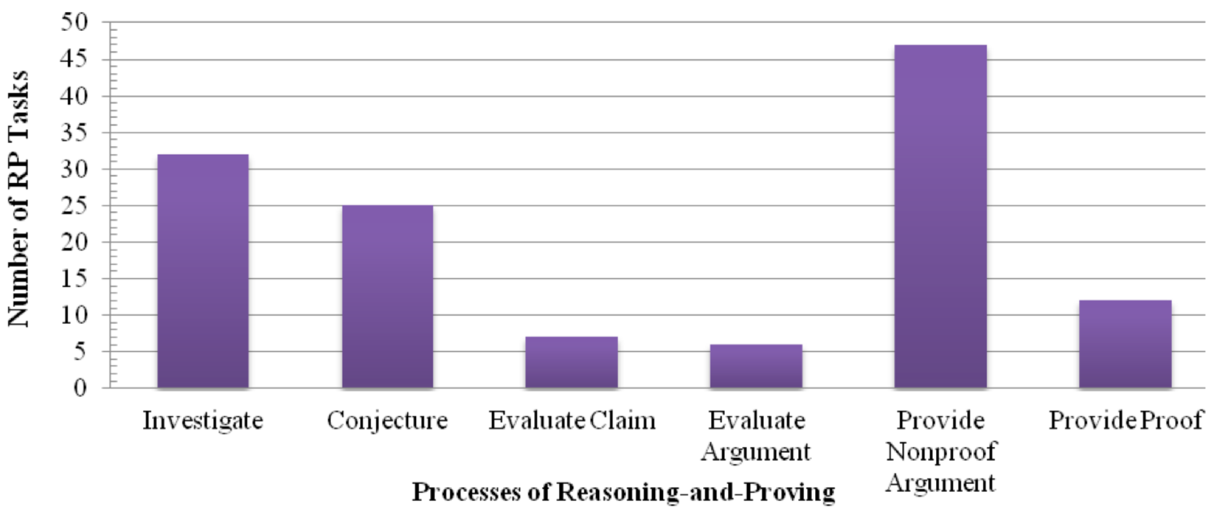

Figure 2: Number of Tasks Addressing RP Processes

Discussion

This analysis indicated that Beckmann's (2008) textbook provided a range of opportunities for PTEs to engage in RP. Throughout the geometry and measurement chapters, students investigate patterns, generate conjectures, and justify mathematical claims with rationales or with proofs. There were also opportunities to engage in many RP processes within one task, meaning they could potentially experience the process of generating and refining arguments. This analysis provided a more detailed view of RP opportunities in this textbook than previous research (McCrory et al., 2008) by identifying PTEs' opportunities to engage in specific RP processes.

As was particularly apparent in Figure 2, however, a majority of these RP opportunities were about generating nonproof arguments. This prevalence of nonproof arguments suggests that the textbook would likely be implemented to provide opportunities for PTEs to generate informal arguments for why mathematical statements were valid. Such a focus could provide helpful opportunities for PTEs to practice communicating their mathematical reasoning more clearly, which is important to demonstrate their understanding of mathematical idea and to prepare them to explain mathematical ideas to students in the future. Since many prospective and practicing teachers maintain the misconception that empirical arguments are proofs (Knuth, 2002; Steele, 2006), however, it is also important to help PTEs transition from this way of thinking to a more robust view of the role of proof in mathematics. Although the inclusion of rationales, explanations, and empirical arguments could potentially help PTEs articulate how they are thinking about the mathematical ideas, without some contextualization and thoughtful implementation on the part of the college instructor, PTEs may walk away from a course using this textbook with that conception maintained or reinforced. An instructor, on the one hand, could modify a task by pressing PTEs to provide a proof even though the task did not explicitly call for one. An instructor, on the other hand, could contextualize the fact that the argument PTEs generated was not a proof, facilitating class discussions about features of proofs and nonproofs. Modifying and contextualizing tasks from the textbook is an important aspect of instructors' teaching practice that needs to happen to broaden PTEs' experiences with and conceptions of RP. Supporting college instructors in their teaching of RP and studying the way in which RP tasks from this textbook were enacted are important avenues of further study.

Endnote

[1]In the fall of 2012, the first author will be a faculty member at Bowling Green State University in the Department of Mathematics and Statistics.

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A CROSS-NATIONAL STANDARDS ANALYSIS: QUADRATIC EQUATIONS AND FUNCTIONS

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Keywords: Standards; Curriculum Analysis; Curriculum; Algebra and Algebraic Thinking

The newly released national curriculum standards, *Common Core State Standards (CCSS),* have aroused wide interests in the field of education. As claimed by the founders, the standards emphasize the correlation with the real world, attempt to reflect the knowledge and skills that students need for success in college and careers, and eventually help them to compete successfully in the global economy (Common Core State Standards Initiative, 2011). Undoubtedly, the new standards have pushed all K–12 teachers, students, researchers, policy makers, and even parents into a crucial transition phase. People are eager to learn the features of the new standards and much concerned about how they can make a smooth transition while adapting the *CCSS* into their daily work.

As a group of international researchers studying mathematics education in the United States (authors are from Caribbean, China, Turkey, and U.S.), we decide to investigate the *Common Core State Standards for Mathematics (CCSSM)* through an international lens. We designed a cross-national comparative study to investigate the similarities and differences among curriculum standards of the four countries with different education systems, specifically, how one of the most conceptually challenging topics, quadratic equations and functions (e.g., Vaiyavutjamai, Ellerton, & Clements, 2005), is introduced in different countries. Comparing to the previous studies (e.g., Reys, Dingman, Nevels, & Teuscher, 2007; Reys, 2006), a more comprehensive theoretical framework was created, which is the three dimension comparison of characteristics of standards: content, mathematical reasoning, and cognitive level. The results show that all the standards introduce students to the foundational concepts of quadratic functions, however, with various procedural and conceptual expectations. Our ultimate goal of doing the cross-national comparison is not to simply rank nations, but to provide a basis for considering current practice and possible alternatives and to help teachers improve their students' learning of mathematics.

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UNDERSTANDING HOW AN ELEMENTARY TEACHER RECOGNIZES AND USES CURRICULUM FEATURES

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Teachers' use of curriculum materials is in great variation (Remillard, 2005). So is the way in which teachers read curriculum materials (Sherin & Drake, 2009). The relationship between teachers' reading and use of curriculum materials has not been articulated, and little research examined how teachers recognize and use curriculum features (CFs). The purpose of this case study is to understand how a fifth-grade teacher, Caroline, recognizes and uses curriculum features as she teaches mathematics using the *Investigations in Number, Data, and Space* curriculum.

I observed two lessons that Caroline taught using *Investigations*. Each lesson lasted one hour twenty-five minutes. Data collected include her written plans for the two lessons, associated notes, classroom videotapes, pre- and post-interviews, and the curriculum materials used. Caroline's lesson plans, associated notes, and interviews helped identify CFs she had recognized and planned to use during instruction. The videos enabled me to identify which CFs recognized during planning were actually used in her teaching and how they were used.

Caroline recognized and used CFs such as key representations and models, instructional approaches and mathematical tasks, as well as support and guidance provided for teachers. While reading/skimming the *Investigations* lessons, Caroline recognized the significance of certain CFs, by using her knowledge of CFs and benefits they offer, which she gained from the professional development (PD) she participated in. Such recognition led her to evaluate those CFs' suitability and appropriateness for her classroom and plan how to use them during instruction. For example, when reading "teacher notes," Caroline recognized the different methods students might use and the importance of discussing these methods during the lesson. Also, Caroline recognized the importance of "dialogue boxes" because of the guidance they provided, such as the type of questions to ask and the kinds of responses students might give.

Caroline's use of CFs was evident during instruction. For example, Caroline made connections among the multiple strategies students generated based on guidance from the dialogue boxes, leading to productive classroom discussions. These connections fostered students' understanding of the different strategies as well as their efficiency. When using the CFs that she recognized, she made some adjustments based on students' needs. Therefore Caroline's recognition and use of CFs was influenced by what her students knew and were able to do. Using Forbes and Davis's (2010) framework, I classified her adaptations as distributed improvisation. However, some CFs, such as "ongoing assessment," which elaborated kinds of students' thinking to look for, were neither recognized nor used explicitly. These findings suggest that PD programs should consider in-depth explorations of CFs to build teachers' capacity in recognizing and using them in productive ways.

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THE INFLUENCE OF VISUAL REPRESENTATIONS ON LEARNING FROM MATHEMATICS LESSONS

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The inclusion of information that is interesting, but irrelevant to the lesson, has been found to distract learners and diminish comprehension in a phenomenon referred to as the *seductive details effect* (Lehman, Schraw, McCrudden, & Hartley, 2007). Math textbooks often contain images, either decorative (i.e., for aesthetic purposes only) or contextual (i.e., related to the background of the lesson) that are irrelevant to the mathematical concepts being taught.

There is empirical evidence that decorative images have a negative influence on learning (Levin, Anglin, & Carney, 1987), likely because of the seductive details effect. In contrast, contextual images have been shown to help with aspects of reading comprehension for some populations (cf. Pike, Barnes, & Barron, 2010), although the effects of contextual images on learning from math lessons have not been explored. It is unknown whether contextual images would distract from mathematics learning or if they would benefit mathematics learning through assistance with reading comprehension. The purpose of this study is to examine the influence of contextual and decorative images on learning from a mathematics lesson. Eye-tracking methodology was used to determine if the inclusion of these images, which are mathematically irrelevant, caused diminished visual attention to the lesson text and graphs, which are mathematically relevant.

Forty-one undergraduate students participated by reading four mathematics lessons on functions. The data indicated that there was little visual attention to either decorative or contextual images. Including decorative or contextual images did not influence visual attention towards math relevant information in the lesson (i.e., the graph and lesson text). Therefore, it can be inferred that the students tended to ignore the images in the lessons. There were no differences in written recalls of lessons or answers to questions across image conditions. Compared to the lesson text, little visual attention was directed towards the graphs, which were mathematically relevant visual representations. This is unfortunate because graphs can assist in mathematics learning (Shah, Mayer, & Hegarty, 1999). An important direction for future research may be to develop methods to direct learner attention towards graphs.

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ACTIVITY THEORY: THE THEORETICAL FRAMEWORK THAT GUIDES THE ACTIVITY OF GENERALIZATION FROM KINDERGARTEN TO COLLEGE

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In this poster I present the results of a research project in which activity theory informed the analysis of the design of some curricula materials. Through examples, I will present to what extent the curriculum tasks and accompanying teacher material included in specific lessons in the teacher's edition for the third grade textbooks create the potential for teachers to mediate students engagement in describing, extending, and making generalizations. The same theoretical analysis may inform curricula designers and educators about their integral role in creating the learning environment, the goals, and the development of a generalization activity.

Keywords: Activity Theory; Generalization; Curriculum Analysis; Advanced Mathematical Thinking

The practice of generalization is a powerful process that should be present in mathematical learning from kindergarten to college. In order to be able to investigate how curricula from elementary, middle, high school and undergraduate courses create contexts in which students may perform different forms of generalization, we need a theoretical framework. Activity theory provides basic principles that allow us to understand generalization as an activity that is socially and historically developed through tools and artifacts mediations, internalization of social knowledge, and that is transformed through learning and development. I propose that the generalization process in mathematics to be considered an activity system. I will present the means of the generalization activity using Leontiev's activity theory interweaved with Rubinshtein's description of the generalization process. The theoretical definition of the activity of generalization will be used to critique examples of task from textbooks designed to target generalization activities. Moreover, this theoretical approach of the process of generalization definition may bring a new perspective on how to organize mathematics instruction in its transition from elementary school level to high school and college levels.

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ARTICULATING HOW GEOMETRIC THINKING RELATED TO CIRCLES IS DEVELOPED IN HIGH SCHOOL MATHEMATICS CURRICULA

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Secondary school mathematics topics examining circles are found in standards, assessments, and college entrance examinations. Typically, circle-related topics are found at the end of high school textbooks (Donoghue, 2003). The implementation of curriculum using these textbooks often leads to limited-to-no coverage of circle topics and missed opportunities for student learning. Senk (1989) suggests that geometry courses alone cannot transition students from low to high levels of geometric thinking in one high school course. The Common Core State Standards for Mathematics (CCSSI, 2010) articulates that circle topics are important for college and career readiness by advocating for the understanding and application of theorems about circles. In CCSSM, all students are also expected to find arc lengths and areas of sectors of circles. Based on these findings, it is clear that curricula across high school grade levels must develop student thinking in a clear and deliberate manner, the ultimate goal being to transition students to higher levels of geometric thinking, including attention to mathematics topics related to circles.

This poster focuses on understanding the nature of geometric thinking related to circles found in three different high school curriculum programs. The researchers identified two research goals:

1. What levels of geometric thinking are required for the treatment of circles found in secondary school mathematics?
2. How do high school curriculum materials develop students' geometric thinking concerning circles?

In our analysis, a framework using van Hiele levels of geometric thinking (Fuys, Geddes, & Tischler, 1988) provided a lens for describing the development of curriculum tasks focused on addressing the CCSSM domain of circles. Using this framework, the researchers classified related tasks found in the Core-Plus Mathematics Project, Kendall Hunt Discovering Series (formerly Key Curriculum Press), and University of Chicago School Mathematics Project.

For the indicated curriculum projects, the analysis focuses on the nature of curricular efforts to develop the levels of student thinking related to circles. The results inform classroom practice across different courses when supporting students as they transition from lower to higher levels of geometric thinking. In addition, the results of this study inform curriculum developers as they strive to create a focused and coherent school mathematics curriculum.

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IN PURSUIT OF A COHERENT CURRICULUM – WILL COMMON CORE STATE STANDARDS DO IT?

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Recently, Common Core State Standards (CCSS) have been adopted by more than forty states in the United States. One of the goals of CCSS is to have consistent and coherence standards because of criticism that U.S. curricula are "a mile wide and an inch deep" (Schmidt, Wang, & Mcknight, 2005) and that there is little consensus regarding when and how certain mathematical topics should be introduced and developed across the K–12 mathematics curriculum (National Council of Teachers of Mathematics Research Committee [NCTM], 2011). Curriculum coherence was found in countries that performed well on the Trends in International Mathematics and Science Study (TIMSS) while American curriculum does not have such coherence (Schmidt et al., 2005). Does CCSS show a similar pattern as top performing countries? How did CCSS and other state standards introduce mathematical topics prior to or without adoption of CCSS? This study attempts to answer these questions.

In previous studies, researchers analyzed TIMSS curriculum frameworks, textbooks, and standards from different states (Schmidt et al., 2005; Valverde & Schmidt, 2000). We will use a similar approach. First, we will use a method called "General Topic Trace Mapping" (GTTM) where experts from different countries are asked to identify all grade levels that certain topics are covered. The result shows a map reflecting the grade-level coverage of each topic for each country (Schmidt, et al., 2005). The results of this mapping will be compared in regards to similarities and dissimilarities to Schmidt et al.'s (2005) comparison of top performing countries. This will partially answer coherence of CCSS compared to other countries. Second, we will choose four states—two who have adopted CCSS (New York and California) and two who have not (Virginia and Texas)—and compare the state mathematics standards before New York and California adopted CCSS (their 2005 and 2007 standards, respectively) and the state standards from Texas and Virginia. By comparing these documents to CCSS and curricula of other countries, we will determine how coherent these standards are prior to or without adoption of CCSS. This would give us better ideas on whether adopting CCSS will bring us more coherent mathematics curriculum or not. Analysis will be conducted and data will be ready for PME-NA 2012.

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EQUITY IN MATHEMATICS TEXTBOOKS: A REPORT ON PROGRESS

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Analyzing textbooks can provide insight into the way society views certain groups and individuals. The images portrayed in textbooks have the ability to influence students' beliefs about self, ethnicity, social class, or sex, and hence produce what is known as "stereotype threat." Good, Woodzicka, and Wingfield (2010) define stereotype threat as "a phenomenon by which individuals, fearful of confirming a negative stereotype about their group, display decreased performance on a task relevant to the negative stereotype" (p. 135). Research examining the effects of stereotype threat suggests that images producing stereotype threat can have a negative impact on student achievement (Good et al., 2010). Zeldin and Pajares (2000) assert that "individuals' beliefs about their competencies in a given domain affect the choices they make, the effort they put forth, their inclinations to persist at certain tasks, and their resiliency in the face of failure" (p. 216). In order to avoid stereotype threat, it is critical that textbooks represent a variety of individuals doing mathematics. Several studies in the 1980's and 1990's (e.g., Heintz, 1987; Allen & Ingulsrud, 1998) documented the lack of equity in mathematics textbooks with respect to gender, though few studies have been conducted recently. Research with a focus on minority representation in mathematics textbooks is even more sparse.

This study examines equity in mathematics textbooks with a focus on race and ethnicity. We analyzed three middle school mathematics textbooks series commonly used in the United States. Middle school textbooks were chosen because children in the early adolescent years are highly susceptible to outside influence and are beginning to find their personal identities (Baker & Leary, 1995). The series selected were the most recent editions of Pearson's *Connected Mathematics 2*, Saxon Publishers' *Saxon Math*, and Holt McDougal's *Mathematics*. These books were chosen to provide a range of both traditional and reform-based textbooks. Every image in each of the textbook series was examined with respect to race and ethnicity. Our categories were Asian, black, Hispanic, Middle Eastern, Native American, and white. Persons for whom we could not determine ethnicity were classified as unknown. Our goal was to compare the representation of ethnic groups in the textbooks to that of the U.S. population. We also examined how these groups are being portrayed with a focus on activities and careers by analyzing trends found in the photos. Activities were classified using modified versions of classification schemes used by Heintz (1987) and Allen and Ingulsrud (1998). Analysis is ongoing and results will provide a detailed update on the progress made by mathematics textbooks with regards to equity.

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GENDER EQUITY IN MIDDLE SCHOOL MATHEMATICS TEXTBOOKS

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Beginning in the early 1970s, a flood of research in textbook analyses produced data demonstrating gender bias in mathematics textbooks (e.g., Winifred, 1973). Most word problems and pictures depicted men in prominent roles, while women were placed in passive roles or stereotypical roles, such as sewing and cooking (Garcia, 1990; Winifred, 1973). As a result of these findings, research from the 1990s showed significant improvement in balancing gender representation throughout the pictures and word problems in mathematics textbooks; however overall equality had not been reached in terms of numbers or in the types of roles portrayed by men and women in the texts (Clarkson, 1993). Some scholars noticed that one response to earlier studies has been to remove people and therefore gender from the texts leading to a depersonalization of mathematics (Garcia, 1990; Parker, 1999). Little research has been done in this area since the early 1990s. This led us to the following research questions: Are there an equal number of males and females in recent mathematics textbooks? Are males and females still portrayed stereotypically? Are textbooks removing people altogether, depersonalizing mathematics, to avoid the situation?

We chose to analyze textbooks used in middle school mathematics classes specifically because between the ages of nine and thirteen, children are beginning to define themselves and are more receptive to social influences (Baker, 1995). We selected popular textbooks based on varying approaches to teaching mathematics. Thus, we chose three series ranging from more traditional to more reform: *Saxon Math, Holt McDougal Mathematics,* and *Connected Mathematics 2*. We examined every image in each of the 6th, 7th and 8th grade textbooks, comparing the number of images that had people, animals, and objects. The pictures that included people were further examined to determine the gender of each person, which careers and roles were being portrayed by each gender, and which famous people were depicted. We categorized roles using an adaptation of classifications by Heintz (1987) and Allen and Ingulsrud (1998). In this poster, we provide our results regarding the stated research questions.

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INSTANCES OF MISCOMMUNICATION BETWEEN CURRICULUM AND TEACHER

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An important goal for developers of mathematics curricula is to clearly communicate with the teacher the key mathematical ideas and the ways that those key ideas can be the basis for effective lessons. In the language of Gehrke, Knapp, and Sirotnik (1992), developers create a “formal curricula” filled with their intentions about the key ideas and how to teach them; there also exists an "intended curricula" of the teacher who plans what to use and how. This poster will highlight instances of miscommunication of key ideas between curricula and teachers.

This poster uses data gathered as a part of a project investigating teacher curriculum use. Within this project, 6 teachers were observed using two different curricula. The example chosen for this proposal is from Juliet (pseudonym), a third grade teacher using the 2nd edition of *Investigations in Number, Data, and Space* to teach multiplication and division stories. After observations of Juliet’s classroom were conducted, an interview was given that centered on how she read the curriculum and that influences the decisions she made while planning. The example below details an activity whose purpose is to help students realize how you can determine whether a particular story requires multiplication or division. Students are expected to work through stories by acting them out, using drawings, or cubes. Student thinking should be focused on looking for numbers of groups and numbers in each group. The following is an excerpt from the teacher guide description of the activity, “Highlight for students that this problem identifies the number of groups and the number of items that are in each group. Because they need to find how many there are altogether, this is a multiplication problem” (Unit 5, Lesson 4.2). Juliet said how she interpreted this passage, and its impact on her planning and teaching.

> This is kind of the basis for everything that you observed. The keywords for recognizing what multiplication and division are, and how to pull those out of the story problem and use them to the advantage for the kids. So, yeah. This right here (pointing to the passage above) was the meat and bones of what this pack of lessons was about.

Juliet took from the curriculum passage above that the “meat and bones” of the activity was the identification of keywords. These keywords are the basis for deciding between multiplication or division. The curricula, however, focuses on identifying groups and items within groups. There is only this one sentence that hints towards a keyword approach. The observers and Juliet both found these lessons difficult for students. The researchers attribute this to the keyword approach, which represented a miscommunication between the text and teacher.

The disparity that exists between the formal curricula and the intended/enacted curricula illustrates the need for teacher learning. Certainly, careful and precise writing on behalf of the developers is necessary, but the examples provided in this poster session facilitate discussion on teacher knowledge and capacity needed to design and enact lessons. This can also inform teacher education and professional development programs.

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AN ACADEMIC PERSPECTIVE OF MATHEMATICAL LITERACY FOR ENGINEERING STUDENTS

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Keywords: Standards; Post-Secondary Education; Curriculum

The need to reform undergraduate Science, Technology, Engineering, and Mathematics (STEM) programs has been prominent in recent years (Ferrini-Mundy & Gu̇çler, 2009). The Programme for International Student Achievement (PISA, 2003) defines mathematical literacy as "an individual's capacity to identify and understand the role that mathematics play in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen" (p. 24). Mathematics is seen as a language with which STEM students must gain fluency. Two subtleties are overlooked when using this metaphor: (1) Translation implies that one moves between two different languages when, throughout history, mathematical formalism has been used to articulate natural phenomena in both science and in every day life. (2) Literacy in mathematics entails both the dexterity and the resourcefulness to recognize and employ mathematical principles and structures. As history has shown, reforms in STEM education do not succeed through instructional modifications alone. Instructional models must be grounded in a deep reconceptualization of the skills and knowledge bases necessary for productive functioning within various disciplines, and this perspective change must precede the development of curricula (Niss & Hojgaard, 2011). Our research was directed towards development of such conceptualization by examining the relationship between mathematical competencies and mathematical literacy. In doing so, we collected and analyzed data sources from two investigations (1) the STEM professors' perceptions about the essential mathematical concepts necessary for first year engineering students, and (2) review of reports on an interdisciplinary task force whose aim is to define indicators of mathematical literacy for engineering students to be used in the creation of a first year mathematics course for engineering students.

Data sources consisted of audio-recorded interviews designed specifically to document participants' responses to key issues identified by the literature including: mathematics as it applies to their discipline and what they consider as mathematical competence pertaining to their own specific needs. We utilized the competencies put forth by the Danish KOM project to tag and describe mathematical content mentioned in interviews (Niss & Højgaard, 2011). Common and distinguishing patterns of skills and goals identified by different participants were identified using discourse analysis.

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ALGEBRA I PREPAREDNESS AND THE MEASURE UP PROGRAM: IMPLICATIONS FROM A THEORETICAL MODEL

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Keywords: Algebra and Algebraic Thinking; Assessment and Evaluation; Curriculum Analysis

Here we present a model based on theoretical implications from studies of El’konin-Davydov mathematics implementation (i.e., Davydov, 2008; Slovin & Venenciano, 2008; Dougherty & Slovin, 2004; Morris & Sloutsky, 1995) and developed through structural equation modeling. The Measure Up (MU) project developed elementary curricula grounded in concepts of measurement and quantitative reasoning. MU experience, prior mathematics achievement, age, and logical reasoning capability were used as predictors of algebra preparedness. Logical reasoning was simultaneously used as a mediating variable.

The sample consisted of 129 fifth or sixth graders, 40 MU and 89 non-MU students, from a research laboratory school. Test items measuring logical reasoning were identified using exploratory and confirmatory factor analyses. Fit indices χ^2 (5) = 7.20, p = 0.21, CFI = .97, RMSEA = 0.06, WRMR = 0.57, and a composite reliability (Raykov, 2007) of .76 confirmed the conceptual relatedness of the items. Similarly, items measuring algebra preparedness were identified using factor analyses and resulted in a model, χ^2 (5) = 5.25, p = .39, CFI = 1.0, RMSEA = 0.01, WRMR = .45, with composite reliability of .89.

The development of the final SEM model [χ^2(30) = 39.4, p = .12, CFI = .94, TLI = .93, RMSEA = .05, WRMR = .96] implied that algebra preparedness was strongly mediated by logical reasoning capabilities, to the extent that effects from prior achievement could only be observed through logical reasoning. MU experience was the only variable that made a significant, direct contribution to algebra preparedness. Age and prior mathematics achievement were found to be positive, significant, indirect contributors to algebra preparedness. The path coefficient from the MU experience to algebra preparedness was positive, supporting earlier findings that MU experience leads to greater algebra preparedness. The path from prior achievement to algebra preparedness was negative and not significant, suggesting that only a particular aspect of prior achievement contributed to a preparedness for work with variables.

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COGNITIVE DEMAND OF MAIN NAEP MATHEMATICS ITEMS

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Keywords: NAEP; Cognition; Assessment and Evaluation

To understand the connections between student achievement and teacher demographic information as measured by the National Assessment of Educational Progress (NAEP), we analyzed the cognitive demand of fourth- and eighth-grade NAEP mathematics assessment items. Using the item classification frameworks developed by Stein and Smith (1998) and Webb (1997), we analyzed and aligned released NAEP items based on the cognitive processes necessary to engage in the task. This work served as precursor to a larger study in which we are building clusters of items that will form achievement score sub-scales.

To investigate these connections, we initially sought to determine which items placed higher demands on students. NAEP items are classified by complexity level as defined by the NAEP assessment framework. The NAEP complexity level classification was designed as an indication of the level of demand that a particular item places on a student. We considered using the existing NAEP complexity level as a basis to build clusters of items, but were concerned about the wide range of types of student thinking necessary to engage in tasks classified at the same complexity level. Consider the following two task descriptions as an example of a situation that highlights this discrepancy in the complexity level classification. Both of these items were administered as part of the 2009 fourth-grade Main NAEP mathematics assessment. In the first item students were asked what number should be placed into the blank to make the following number sentence true: $\_\_ - 8 = 21$. In the second item, students were asked to use several provided shapes (four parallelograms and two triangles) to cover a composite figure. Both items were classified by NAEP as low complexity items. When considering using the NAEP classification for our research purpose, we became troubled because of the difference in the types of thinking in which students would need to engage in order to attempt these tasks. For some students the solution of the first task may be immediately obvious (recall or mental manipulation); for others, perhaps employing a simple procedure would determine the solution. In our opinion, the problem-solving path is clearly defined with no pre-planning necessary. However, the second item requires students to make a plan for solving the problem, even if the plan employed is trial and error. The answer is not immediately obvious, and the problem-solving pathway is defined by the child, not by the item prompt. This example supports the statement in the most recent mathematics framework that “Mathematical complexity deals with *what* the students are asked to do in a task. It does not take into account *how* they might undertake it” (NAGB, 2010).

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Chapter 3

Mathematical Processes

Research Reports

Brief Research Reports

Poster Presentations

MIDDLE SCHOOL STUDENTS' EXAMPLE USE IN CONJECTURE EXPLORATION AND JUSTIFICATION

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Although students' difficulties in developing and understanding proofs in mathematics is well documented, less is known about how students' example use may support their proof practices, particularly at the middle school level. Research on example use suggests that strategic thinking with examples could play an important role in exploring conjectures and developing appropriate justifications. This paper introduces a framework of middle-school students' example exploration, distinguishing between the types of examples students use and the uses examples play in making sense of and proving conjectures. Drawing from clinical interviews with 20 students, we present thirteen categories of example types and seven categories of uses, followed by a discussion of each set of categories and their connections to one another.

Keywords: Middle School Education; Problem Solving; Reasoning and Proof

Objectives: The Importance of Supporting Proof in School Mathematics

Proof in school mathematics has received increased attention over the past decade, with researchers arguing that it must be a central part of the education of all students at all grade levels (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002). Both the *Common Core State Standards for Mathematics* (Common Core State Standards Initiative, 2010) and the *Principles and Standards for School Mathematics* (Naitonal Council of Teachers of Mathematics, 2000) argue that a central hallmark of mathematical understanding is the ability to prove, and that the mathematics education of students from pre-kindergarten through grade 12 should enable all students to develop and evaluate mathematical conjectures, arguments, and proofs. Middle school in particular is a critical time for students to develop the ability to reason deductively, resulting in recommendations for curricular and pedagogical changes emphasizing proof in beginning algebra classes (Epp, 1998; Marrades & Gutierrez, 2000).

These recommendations pose serious challenges, however, given that many students struggle to recognize, understand, and produce deductive arguments (e.g., Chazan, 1993; Harel & Sowder, 1998). Researchers have posited that a critical source underlying students' struggles to understand proof is their treatment of examples. On the one hand, students tend to engage in example-based proofs, pointing to a few successful examples as justification that a mathematical statement is true (e.g., Healy & Hoyles, 2000; Porteous, 1990). On the other hand, deliberate exploration of examples is not explicitly supported as a strategy to foster deductive reasoning; students have few opportunities to strategically analyze examples in order to make sense of a mathematical statement or to gain insight into the development of its proof.

We suggest that providing students with opportunities to carefully analyze examples may contribute to their abilities to develop and make sense of conjectures and their proofs. Studies of mathematicians suggest that the process of experimenting with examples is a critical aspect of proof development (Epstein & Levy, 1995). Although scholars have noted a number of potential roles of example use, little research has focused on characterizing these roles with regard to facilitating students' learning to prove. In fact, very little is known about how middle school students think with examples, whether their example use can facilitate deeper mathematical understanding, or whether and how examples can support students' attempts to develop proofs.

This paper presents the results of a study aimed at identifying the roles of middle school students' example use. We introduce a framework that distinguishes between the *types* of examples students use and the *uses* examples play in making sense of and proving conjectures. Our findings indicate that students made use of a variety of example types and used examples in different ways in order to check a conjecture's correctness, convince themselves and others that it held true, better understand a conjecture, and develop justifications to support their statements.

Theoretical Background

One common model of students' mathematical reasoning is that their understanding of mathematical justification is "likely to proceed from inductive toward deductive and toward greater generality" (Simon & Blume, 1996, p. 9). [For this discussion, inductive refers to generalizing from examples, and is not to be confused with mathematical induction, a valid method of proof.] This expected progression is reflected in various mathematical reasoning hierarchies (Balacheff, 1988; van Dormolen, 1977; Waring, 2000) as well as in many curricular programs (e.g., Lappan et al., 2002). However, not only do students find this transition difficult to navigate, studies also suggest that their development may not be as straightforward as the induction-to-deduction model; in fact, students may follow a "zig-zag path" (Polya, 1954) between example exploration, conjecture, proof, and back again (e.g., Ellis, 2007).

One approach to helping students navigate the transition to deductive reasoning involves emphasizing the limitations of examples as proof, thus helping students recognize the need for deductive arguments. It has, however, proven difficult to help teachers leverage this technique in order to successfully foster their students' proof abilities (Bieda, 2011). In addition, this approach positions example-based reasoning strategies as stumbling blocks to overcome. We suggest an alternative stance by positioning strategic thinking with examples as an important object of study in its own right. From this perspective, reasoning with examples is viewed as a potential foundation for the development and understanding of conjectures and proofs.

The Roles of Examples

Examples play a critical role in mathematical practice, and the time spent analyzing particular examples can provide not only a deeper understanding of a conjecture, but also insight into the development of its proof (Epstein & Levy, 1995). The role examples play in the work of middle and high school students, however, is less well understood. Although research has demonstrated students' overwhelming reliance on examples as a means of verification and justification, less is known about how students think strategically with examples.

Research on students' thinking does suggest that examples can have different potential roles and uses. For instance, Buchbinder and Zaslavsky (2009, 2011) introduced four different types of examples (confirming, non-confirming, contradicting, and irrelevant) and examined their status in determining the validity of mathematical statements. Other studies have identified different example types as well, including start-up examples, boundary examples, crucial experiments, reference examples, model examples, counterexamples, and generic examples (Alcock & Inglis, 2008; Balacheff, 1988; Michener, 1978; Watson & Mason, 2001). Studies examining the role of examples in understanding conjectures have found that analyzing structural similarities across examples can support proof development (Pedemonte & Buchbinder, 2011).

This body of research suggests that example use plays an important role in understanding conjectures and potentially supporting the development of valid proofs. However, there remains much to be learned about what types of examples students exploit, particularly at the middle school levels, and how they use them when developing and exploring conjectures. In this study we accordingly characterize the roles and strategic uses of examples in terms of a more comprehensive framework for developing, exploring, and proving conjectures.

Methods

Participants and Instrument

Participants were 20 middle-school students (12 sixth-graders, 6 seventh-graders, and 2 eighth-graders), each who participated in a semi-structured 1-hour interview. Eleven students were female and 9 students were male. Seventeen students were in general 6th, 7th, or 8th-grade mathematics courses using the Connected Mathematics curriculum, while 2 students were in algebra and 1 student was in geometry.

The interview instrument presented students with seven conjectures (see Table 1 for sample conjectures). The interviewer asked the participants to examine the conjectures, develop examples to test them, and then, when they could, provide a justification. The conjectures addressed ideas in number theory and geometry that were accessible to a middle-school population, and every conjecture except Conjecture 6 was true. Fifteen out of the 20 participants viewed only the first four conjectures; the remaining 5 participants had extra time to view all seven conjectures, resulting in 95 total responses to code. After the students worked with examples for each of the conjectures, they were asked why they chose the examples they did.

Table 1: Sample Interview Conjectures

Conjecture 1	Eric thinks this property is true for every whole number. First, pick any whole number. Second, add this number to the number before it and the number after it. Your answer will always equal 3 times the number you started with.
Conjecture 4	Bob thinks this property is true for every parallelogram. The angles inside any parallelogram add up to 360 degrees.
Conjecture 6	Kathryn thinks this property is true for every whole number. First, pick any whole number. Second, multiply this number by 2. Your answer will always be divisible by 4.

Data Analysis

Coding began by identifying each of the examples students produced for each conjecture. We then developed emergent codes to identify example types and uses. *Types* refer to the different characteristics of examples students used, and *uses* refer to the roles that the examples played in students' investigations. The research group discussed the codes and clarified uncertainties as emergent codes solidified. Codes to determine example types depended on the participant's discussion of the example, rather than on a determination based only on the example itself. For instance, the same number, 1, could be considered "common" from one student's point of view or a "boundary case" from another student's point of view. Furthermore, the same example could be coded in multiple ways based on the participant's explanation. Three different researchers on the project team coded portions of the conjecture responses so that each conjecture response was ultimately coded independently by at least two different team members.

Results and Discussion

We found 13 categories of example types (Table 2) and seven categories of example uses (Table 3). Each table introduces the category name with the number of instances in which the example type or use occurred in the data set, its definition, and a representative example to illustrate each example type and use. We discuss the example types first, and then present example uses.

Table 2: Types of Examples

Example Type	Definition	Data Example
Dissimilar Set (41)	An inclusion of examples that are all different from one another.	"I tried to pick both prime numbers…composite numbers and then odd numbers and even numbers. So just have a variety of different kinds of numbers."
Generic (28)	An example with unimportant specific characteristics. These examples demonstrate a general idea about the conjecture.	[Uses 50 to explain why Conjecture 1 works]: "So this one [51] is 1 more, and this one [49] is 1 less. So if you take this one [the 1 from 51] over to that one [49] it turns into 50. And then that one goes into 50, and 50."
Common (24)	An example described as typical or one that many would think of.	[After choosing 10 and 15 as example cases]: "I picked a more uncommon number [15] and a more common number [10], and they both worked out."
First Thought Of (22)	Example is the first that came to mind; no evidence of thoughtful decision about example selection.	"I just…kinda what popped into my head."
Unusual (21)	An example that does not occur often and may have odd or strange characteristics.	[To explain using 1,028 as an example]: "Well, the 1028's kind of like a little stand-out number because it's – it's large."
Random (12)	Example is arbitrarily chosen, with the "randomness" intentional to highlight the likelihood of the conjecture being true.	This code refers to what students consider random, rather than true mathematical randomness: "For a problem like this you want to pick random numbers. Not selected numbers."
Conjecture Breaking (11)	Example chosen to disprove the conjecture; counterexample.	"I wanted to do numbers that were hard for it…it was less likely for them to be divisible by 3, I think."
Easy (11)	Examples that are easy to operate on or compute with.	[*Prompt: So why did you try it out with 15?*] "Because it's an easy number to use."
Known Case (9)	Student picks an example in which properties or features pertaining to the conjecture are already known.	"A rectangle is a parallelogram, so that is four 90 degree angles, which is 360."
Boundary Case (5)	An extreme example or a special case example, such as the identity.	[Explaining the use of 0]: "You kind of have to try it with every not possibility like, not like 3, you know, to, you know, 100, but kind of like get down to the origin of the number. Like 1 and then, you know, 0."
Similar Set (3)	Deliberate inclusion of examples similar to one another.	[*Prompt: Are these two (triangles) different or similar from each other?*] "Similar, because they have same numbers in two of the sides and different numbers in this side."
Progression (2)	First one type of example is chosen, then student deliberately switches to a different type, and may continue switching to new types.	"You would first test a typical number to just see, like, okay in general was this going to be true. And then if that – if he was true on that, then you say, okay, then I would test a more unusual number to just, like, to test his property."
Favorite (2)	Example represents a favorite number or shape.	"The only reason I picked 6 is because that's my lucky number."

Unsurprisingly, we found evidence that students chose example types that were not always deliberate or thoughtful: For instance, the categories *first thought of*, *favorite number*, and *easy* represented example types that were not necessarily connected to the content of the conjecture at hand. These example types also did not typically support the development of deductive arguments. However, these categories only represented 18% of all of the example types. When examining the participants' discussion of their examples, we also found many cases of deliberate and thoughtful example choices, which we discuss below.

By far the most prevalent type of examples was the *dissimilar set* type; many participants indicated a belief that choosing a variety of examples was a more reliable method of testing a conjecture. For instance, one student explained the importance of choosing a dissimilar set:

> You should find numbers that maybe aren't as alike to test just so you have all kinds of differences covered. Like when you're maybe testing students for a survey, you want to have as many different students and maybe different race, different families, and everything. Just a bigger background so maybe you'll get more accurate information.

The two students who used a *progression* of example types demonstrated similar reasoning by first picking common numbers, then deliberately shifting to less common numbers.

The inclusion of a dissimilar set often resulted in a discussion about the importance of picking both *common* and *unusual* examples. Some students indicated that unusual examples are more convincing than other types of examples. For instance, one student, Eva, tested a conjecture that every even number added to half of itself would be divisible by 3. She explained that she deliberately tested numbers that could only be divided by 2, such as 10. Eva tested those unusual numbers because "it was less likely for them to be divisible by 3, I think." It is worth noting that in this case, a number such as 10 was unusual in Eva's eyes in relationship to the conjecture, even though 10 might not be an unusual number for her in general. Unusual examples and boundary case examples both played an important role: they could lead to *conjecture breaking* example types, and they were particularly convincing because if a conjecture held for an unusual or extreme example, it may be more likely to be true overall.

There were some example types that were more strongly connected to proof development, such as *known cases* and *generic* examples. Although deductive arguments were not solely developed through these example types, known case examples and generic examples helped students reason through the structure of the conjectures. For instance, Rodrigo examined Conjecture 4 (Table 1), and in order to better understand the conjecture, he began with a known case example, the rectangle. Rodrigo knew that the conjecture held true for a rectangle: "A rectangle is a parallelogram, so that is four 90-degree angles, which is 360." He then took the rectangle and adjusted it to think about how a new example would work with the conjecture (Figure 2):

Figure 2: Rodrigo's adjustment of the "known case" rectangle into a new parallelogram

After examining the new example, he said, "Oh yes it would work for every one – this doesn't really matter any more." Developing a generic example, Rodrigo explained further: "That's just a random drawing (Figure 3). It is a rectangle in disguise because you cut this off and you put this, over here, ta da! And it becomes a rectangle. And rectangles, well, see, equals 360."

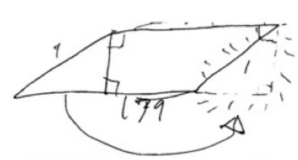

Figure 3: Rodrigo's generic example

By "random," Rodrigo indicated that the particular nature of his example was not important because it illustrated a more general point; hence, this example type was coded as *generic* rather than *random*. Across the participant group, it is notable that 15% of the example types were generic in nature; it was the most prevalent code, second only to *dissimilar set*. In general, the types of examples students chose in order to foster their understanding of the conjectures suggest that middle school students can and do engage in deliberate and strategic example choices.

Example Uses

Students also demonstrated a variety of example uses. Each of the seven categories in Table 3 includes a frequency, a definition, and a representative data example.

Table 3: Example Uses

Example Use	Definition	Data Example
Check (69)	Student selects examples to test whether the conjecture holds.	"Just, you know, test, just to see if it actually does work or not."
Support a General Argument (28)	Student uses a generic example to describe a more general phenomenon in support of a deductive proof.	"When you're taking half of it, then that number is, because it ends up being thirds. So it's always going to be true because if you do…514. That's always going to be 1208, which means that it's broken up into thirds, so no matter what it's going to be divisible by 3."
Convince (25)	After checking the conjecture, student tries additional examples in order to convince oneself or others that the conjecture must be true.	"You can't be sure if you only test one number because one number, because in almost every case there is exceptions to the stuff if it's not true."
Understand (21)	Student uses an example to make sense of the conjecture; may lead to insights that support deductive proof.	"Let's try 5…okay. Those are the two that I needed. Now I kind of know the logic behind it."
Asked (19)	Student was asked to choose and example; the only evidence that a student produces an example is because s/he was explicitly asked to do so.	S: "I'm totally convinced it's true." I: "You don't even need to – do you need to test out any examples?" (Student shakes head.) I: "Okay. Let's say that you didn't know it was true. Are there any kinds of rectangles you would want to test it out on?"
Support Empirical Proof (9)	Student offers examples as a justification of the truth of a conjecture.	I: "Say that you wanted to show that this was always true." S: "I would use these examples, and probably a few more."
Disprove (6)	Student tests an example in an attempt to disprove the conjecture.	S: "Any whole number? Oh, I thought you just meant even numbers. I wouldn't think that's true then." (Tries 9 to disprove). "Nine times 2 equals 18. 18 divided by 4 equals question mark."

The most prevalent use of examples was in *checking the correctness* of a conjecture; 39% of the example use instances occurred when students used examples to test conjectures. Part of this prevalence may be due to the fact that students were encouraged to test examples during the interview. Checking correctness occurred with many different example types, ranging from the first number thought of to unusual examples to dissimilar sets. Among the other example uses, there were some connections between how students used examples and the types of examples they employed. The strongest connection was between generic examples and the *support a general argument* use. This link is unsurprising because the purpose of a generic example is to illustrate a broader point. Similarly, using examples to *disprove* a conjecture typically relied on conjecture breaking example types, but also occasionally made use of boundary cases or unusual examples.

Another set of links emerged when students used examples to convince and understand. The example types that students viewed as more convincing, such as dissimilar sets, unusual examples, and boundary cases, were often the ones they employed when continuing to further check examples after an initial test. For instance, Alyssa tested Conjecture 1 with the number 4, and found that it worked. She then explained that she was not convinced: "I think I need to try it a few more times to make sure." She indicated that she should try different numbers, such as both even and odd numbers, in order to really be sure the conjecture would work. While testing a dissimilar set of examples in order to further convince herself that the conjecture held true, Alyssa began to use the examples to understand the structure of the conjecture. Through this process, she was then able to produce a general argument, using the initial example, 4, as a generic example: “When you add the number before and the number after, those two numbers will equal twice the first number I guess. Because, like, for 4 + 3 + 5, if you drop one off the 5…then 3 would kind of be 4. So it’d be 4 + 4 + 4. Which would be, like, 12, or 4 times 3.” Alyssa’s general argument was not unusual amongst the 20 participants; we coded students’ justifications as part of a larger study and found that after exploring examples, students who attempted justifications were able to produce deductive arguments or valid counterexamples a little over half the time.

It is worth noting that in 19 responses, students did not see a need to produce an example at all; this typically occurred because the student already believed the conjecture to be true, and therefore not in need of testing. For example, Andre was asked to consider the conjecture that for any triangle, the sum of the length of any two sides are greater than the length of the third side. Andre did not see a need to test this property because “That’s a property already proven by the, you know, the community.” This finding is in contrast to previous results suggesting that students want to test conjectures even when presented with their proofs (e.g., Chazan, 1993).

Conclusion

This study presented a framework of the example types and example uses middle school students employed when making sense of, exploring, and attempting to prove conjectures. Our findings support earlier studies suggesting that students’ uses of examples can play an important role in exploring and understanding conjectures, as well as in potentially supporting the development of valid proofs (Alcock & Inglis, 2008; Buchbinder & Zaslavsky, 2011; Pedemonte & Buchbinder, 2011). Moreover, our study suggests that distinguishing between example types and example uses may be an important component in better understanding students’ thinking with examples; this distinction can also provide a potential structure for more in-depth analysis of how example type may be linked to example use in future studies.

One compelling finding was that many of the students who explored conjectures with multiple examples were able to produce deductive arguments, valid counterexamples, or general arguments that relied on generic examples. These results run counter to the many studies demonstrating K–16 students’ difficulties in producing valid mathematical proofs (e.g., Chazan, 1993; Healy & Hoyles, 2000). The fact that the students were able to produce valid arguments after in-depth example exploration provides initial evidence that strategic and thoughtful use of examples can indeed support the development of mathematically appropriate proofs, even at the middle school level. This suggests the importance of continuing to study the roles examples can play in supporting middle-school students’ learning to prove.

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TWO FORMS OF REASONING ABOUT AMOUNTS OF CHANGE IN COVARYING QUANTITIES

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This paper addresses how secondary students might reason about amounts of change in covarying quantities. Two empirically based forms of covariational reasoning are distinguished. The first form—reasoning about quantities as varying simultaneously and independently—supports tandem comparison of amounts of change. The second form—coordination of change in one quantity with change in a related quantity—supports coordinated comparison of amounts of change. By expanding the mental actions of Carlson et al.'s (2002) covariation framework, these forms of reasoning provide finer grained distinctions in the "Quantitative Coordination" level of covariational reasoning. Distinctions made between these forms of reasoning might help to explain how students could begin from informal reasoning to transition to more formal reasoning about average and instantaneous rate of change.

Keywords: Algebra and Algebraic Thinking; Reasoning and Proof; High School Education

A student reasoning covariationally would be mentally "coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002, p. 354). By conducting fine-grained investigations with secondary students, researchers have articulated the nature of relationships that students might make between covarying quantities (Johnson, 2012; Saldahna & Thompson, 1998). These articulations provide landmarks within a continuum of reasoning about covarying quantities.

This paper draws on two empirically based forms of secondary students' reasoning about amounts of change in covarying quantities to expand the mental actions of Carlson et al.'s (2002) covariation framework. These forms of reasoning make finer grained distinctions in the "Quantitative Coordination" level of covariational reasoning. Distinctions made between these forms of reasoning might provide insight into how students could begin from informal reasoning to transition to more formal reasoning about average and instantaneous rate of change.

A Brief Overview of the Covariation Framework (Carlson et al., 2002)

Consideration of undergraduate and beginning graduate students' responses to tasks involving recognizing and characterizing how changes in one variable affected change in another variable (Carlson, 1998) led to the development of a covariation framework. The covariation framework (Carlson et al., 2002) provides a continuum of mental actions supporting five levels of covariational reasoning, with each level increasing in sophistication: Coordination, Direction, Quantitative Coordination, Average Rate and Instantaneous Rate. Researchers infer underlying mental actions from certain behaviors associated with each level of covariational reasoning. Classifying a student as reasoning covariationally at a particular level means that the student is able to perform mental actions supporting not only that level, but also all preceding levels of covariational reasoning (Carlson et al., 2002).

For the purposes of this paper, I focus on the Quantitative Coordination (QC) and Average Rate (AR) levels. The QC level supports the mental action of coordinating an amount of change in one quantity with the change in another quantity (Carlson et al., 2002). For example, a student who related amounts of change in volume to changes in height would provide evidence of reasoning at the QR level. The AR level supports the mental action of coordinating an average rate of change in one quantity with uniform change in another quantity (Carlson et al., 2002). For example, a student who related the rate of change in volume with respect to height to uniform changes in height would provide evidence of reasoning at the AR level. In a study of college calculus students, Carlson et al. (2002) found that even after students took a course focusing on rate and varying rate, students consistently applied covariational reasoning at the QC level, but not at the AR level. Further explication of the QC level of covariational reasoning might help to account for variation in the students' reasoning and suggest whether or not students' reasoning might advance to levels of Average (AR) and Instantaneous Rate (IR).

A key distinction between the QC and AR levels is the consideration of an amount of change (QC) versus the consideration of a rate of change (AR). In this paper, I provide two distinct forms of QC level reasoning that seem to support the addition of finer-grained mental actions to the covariation framework. These additional mental actions further explicate what it could mean to coordinate an amount of change in one quantity with change in another quantity.

Two Forms of Reasoning about Amounts of Change in Covarying Quantities

In this section I articulate both forms of reasoning, providing empirical support for each. I draw on three secondary students' (Austin, Jacob, and Hannah—names are pseudonyms) work on a task relating the typical high temperature of a city to the day of the year (see Fig. 1). Austin and Jacob were 11th graders enrolled in a Precalculus course and Hannah was a 10th grader enrolled in a Geometry course. The task required students to investigate how the typical high temperature varied as the day of the year varied. Each student worked on the task during an individual clinical interview (Clement, 2000), for which I served as the interviewer.

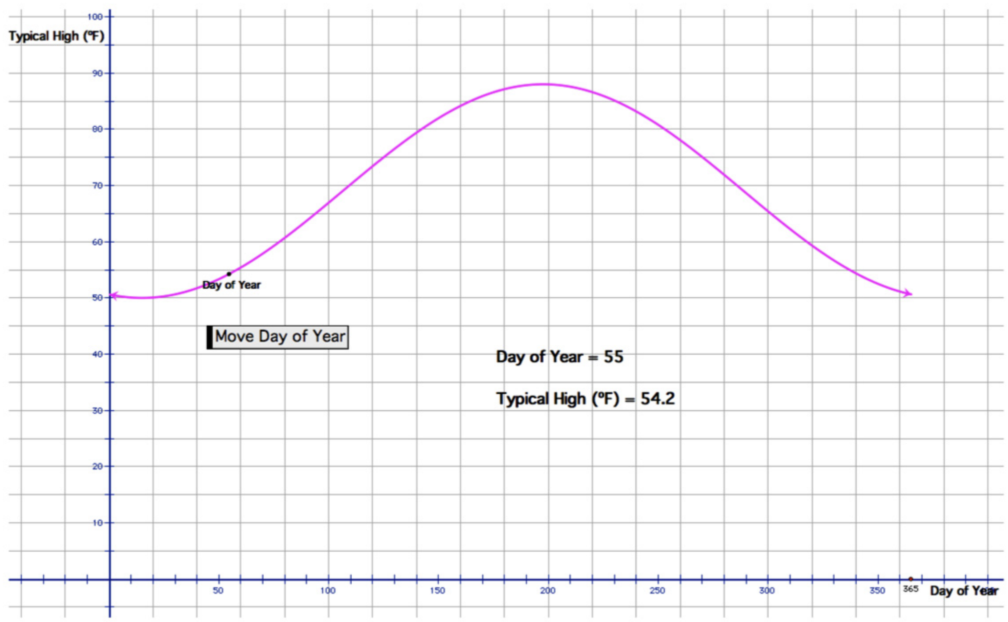

Figure 1: Dynamic Cartesian graph

The task incorporated a dynamic Cartesian graph (see Fig. 1) created using Geometer's Sketchpad Software (Jackiw, 2001). A student interacting with the graph could click and drag on the active point or press one of the animation buttons. As the day of the year changed, the corresponding typical high temperatures changed accordingly. As part of this task, I asked each student to use the graph to make a prediction about how the typical high temperature would continue to increase or decrease as the day of the year changed. Because the interviews were semi-structured, the actual prompt varied from student to student based on his or her individual work.

I employ an actor-oriented perspective (Lobato, 2003) when investigating students' reasoning about covarying quantities. By quantity, I mean an individual's conception of a "quality of an object in such a way that this conception entails the quality's measurability" (Thompson, 1994, p. 184). For example, a student could conceive of area as a quantity measuring an amount of flat surface being covered. By covarying quantities, I mean quantities that are changing together. For example, as a square is being enlarged, its side length and area are varying together. Drawing on students' explanation, written work, and gestures, I make claims about the mental actions involved in students' reasoning.

Changing Simultaneously and Independently

In the excerpt that follows, Austin used amounts of change in temperature and days to make claims about how the decreasing temperature is changing as the day of the year varied. When Austin used the word slope, he was referring to an association of an amount of days with an amount of degrees.

Interviewer: And when it decreases, if you had to describe for me, as it's going along, how is it decreasing as it's going along?

Austin: It just starts, like it's kind of rounded, or it's going more days for the temperature. It's kind of staying hot for a while and then once it starts to get close to say two hundred forty, two hundred thirty days, then it starts to decrease pretty much at that same constant rate as the other side as it increased.

…

Interviewer: And so, when you talk to me about decreasing, can you tell me what's decreasing?

Austin: The temperature is decreasing with the amount of days you go on from that top two hundred days.

Interviewer: So in the top here, how is that temperature decreasing?

Austin: From day two hundred to my line there *[longest horizontal segment shown in Fig. 2]*, it's close to about two hundred fifty, so in fifty days it's decreasing about seven degrees, which isn't that much. I'll write that down. It's fifty degrees in seven days there. *[Writes* $\frac{\text{50 degrees}}{\text{7 days}}$*]*

…

Interviewer: So suppose I were to ask you to consider the interval between day two hundred and day two twenty. How do you think that change would compare to this fifty days and seven degrees?

Austin: I'd say it'd be, it would change a little less because there's more or, there's less of a slope in those twenty days compared to that section there.

Interviewer: Can you show me? You can use the card *[Austin had been using a note card as a straightedge]*, or just show me what you mean by less.

Austin: You could just say like if I drew a line here, *[Draws in the upper left set of horizontal and vertical segments shown in Fig. 2]* it's changing a little, a lot less than compared to that. *[Draws in the lower left set of horizontal and vertical segments shown in Fig. 2.]*

Interviewer: And how does that affect the, how does that relate to the changing temperature?

Austin: It's just going to have a steeper slope, which means the more days, or the least, the lesser amount of days, compared; it takes for the temperature to drop a certain amount.

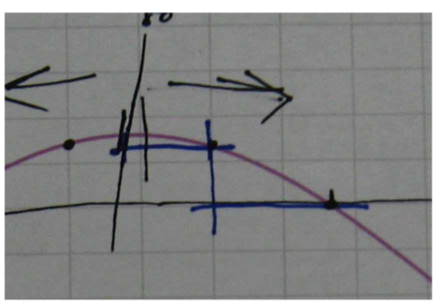

Figure 2: Line segments Austin drew to represent the changing amounts of temperature and days

To determine how the temperature might continue to decrease, Austin specified an interval of days and then compared the amount of change in temperature to the amount of change in days. He determined particular numeric amounts of change because he could compare the lengths of horizontal and vertical segments. With either specifying or not specifying numerical amounts, he used an interval, determined amounts of change in each quantity, and compared those amounts of change in the interval. Although not included in this excerpt, he did use division to compare the amounts of change in temperature and days. However, even when he used division, he interpreted the result as an amount of days per one degree of change in temperature, thereby preserving both individual quantities. Using Carlson et al.'s (2002) covariation framework, Austin was reasoning at the QC level, because he related amounts of change in covarying quantities.

Austin's reasoning shared similarities with Jacob's reasoning. In the excerpt that follows, Jacob determined an average rate of change in temperature per day for a five-day interval. He chose other five-day intervals that he predicted might have the same average rate of change, and then calculated the average rate of change on those intervals to make comparisons.

Interviewer: So if you were to determine an average change per day between days one ninety and one ninety-five, how would you figure that out, between one ninety and one ninety-five?

Jacob: Okay, well I'd take um, minus one ninety and I'll just do one ninety-five. Day one ninety-five has the high of eighty-seven point eight-nine, nine eight, (87.98) and one-ninety is a change of eighty-seven. Er, it doesn't have a change it has a temperature of eighty-seven point eight four (87.84). So to find change, ninety-eight minus eighty-four is point one four (0.14) that is for five days worth. So I would take point one four (0.14) divided by five to find the change in days, like per day so it changes point zero eight, two eight per day (0.028).

…

Interviewer: Are there any other time periods on the graph when you might expect an increase of point zero two eight (.028) degrees per day?

Jacob: Uh huh. Whenever, I'll go back to the beginning. Um, I'd say maybe somewhere around here. We'll say, we'll make it nice and make it forty. We'll try this.

Interviewer: Can you tell me why you picked this day?

Jacob: I just thought it looked like it wasn't moving up much.

Interviewer: And can you tell me how you determine if something looks like it's not moving up much?

Jacob: Um, yeah, it moves over a lot more than it moves up, so it means that it is not getting that much hotter as the days go on. But since it's curved inwards instead of outwards, I don't know if that is going to affect it, but I'm just going guess and write it down. Day, I would write day for the rest and the high was fifty-one point seven four (51.74). Day thirty-five, fifty-one point one (51.1), point six four (.64) difference for five days that is a lot bigger than this, point six four (.64) divided by five is somewhere around, yeah, point one two eight (.128) so that is a lot bigger I was wrong then, I'll go five more days, I hope so, I will be right this time.

Interviewer: Why are you moving left?

Jacob: Because if I went right it's getting greater, the intervals between each five days is getting bigger, because earlier I forget where I said it, yeah, here, it is moving up by about six point two degrees (6.2) every twenty days. … Six point two (6.2) divided by twenty, about point three one (.31), and up here it is just point zero one five (.015), so I don't, I don't see what's the point of even trying to go up because I know it is just going to get greater. So I will try, what day is this, thirty, fifty point, fifty point six one (50.61)… Fifty-one point one four (51.14) minus fifty point six one (50.61), point five three (.53). I don't know what that was—and so that's for five days so divide that by five so per day it changes point one zero six (0.106), that's still not even close. Let's go all the way back to the beginning day, it starts at day one and day six, I'll make another chart. How many do I have now, five? Yeah. Day one, day six, we have fifty point five three (50.53). Day six, fifty point two two (50.22), difference of, I am just going to use the calculator because I know what I want to say, point three one (.31) divided by five, point zero six two (.062). So I was wrong, we are probably not going to have a change like this. But that is kind of close, I guess, but that is as close as it is going to get. It just gets bigger and bigger as it is going, until it gets up to the top.

To determine how the average rate of change in temperature given a five-day interval might compare to 0.028 degrees per day, Jacob calculated the average rate of change in different five-day intervals. As indicated by his comment about being "curved inwards instead of outwards," he identified curvature as a physical attribute of the graph. He could use the shape of the graph to make some informed choices about where to begin his calculations. However, he was not able to use curvature to make sense of the varying average rate of change in temperature per day because his focus was on the results of his calculations. When his calculations did not support his hypotheses, he assumed that it was not possible to have another interval with the same average rate of change in degrees per day. Using Carlson et al.'s (2002) covariation framework, Jacob was reasoning at the AR level, because he considered the rate of change of temperature with respect to time for equal amounts of time (five-day intervals).

Together, Jacob and Austin's responses provide empirical support for reasoning about covarying quantities as changing simultaneously and independently (see also Johnson, in press). This way of reasoning involves the simultaneous varying of quantities such that both are changing in tandem. Using this form of reasoning, a student could compare amounts of change in one quantity with amounts of change in another quantity in uniform or nonuniform intervals. A student could also use this way of reasoning to compare average rates of change in one quantity with respect to another quantity in uniform or nonuniform intervals. When using this form of reasoning, a student begins by forming intervals. In doing so, a student can compare amounts of change (or average rates of change) across intervals. Comparing variation in amounts of change in an interval would not be the student's goal. Instead, the student's goal is to find an amount of change (or average rate of change) in an interval, making varying change in the interval irrelevant.

Changing with Respect to Another Quantity

In the excerpt that follows, Hannah attended to variation in the intensities of increases and decreases in typical high temperature with respect to changes in amounts of days. Her reasoning stands in contrast to Austin's and Jacob's because she did not work from calculations to make claims about changes in the typical high temperature. Instead, she used descriptors such as "increases are increasing," "steady increase," and "increase its decrease" to indicate the variation in the intensity of an increase or decrease.

Interviewer: And so if you were to take a look over the whole year and talk to me about when the temperature, the typical high is changing the most or the least?

Hannah: The typical high changing the least would be like at the peak *[Makes a circling motion around the maximum of the graph shown in Fig. 1]* like near the one hundred ninety-seventh day, but like the least, or the most change would be around right here *[Motions to the part of the graph near day 60]*, like where the steady increase is going *[Slides her finger along the graph until about day 120]*, and like same on the other side, like around in there. *[Motions to the part of the graph near day 300.]* The peak is more like the least change.

Interviewer: And if you also had to talk about a range of days, and you talked about increasing increases,

Hannah: Mhmm.

Interviewer: When do you think, does it seem like those increases are increasing?

Hannah: Um, it looks like the increases are increasing right here *[Motions to the part of the graph between days 60 and 120.]* and then like the increases decreasing would be up closer to the point *[Referring to the active point which is on day 197]*.

Interviewer: When does it seem like the change happens from increasing increases to decreasing increases?

Hannah: It seems like it really changes before the steady increase. It's where the increase increases and after the steady thing is where it starts to change to decreasing the increase.

Interviewer: And what about the decreases?

Hannah: The decreases is pretty much the same, like as the increases, except this is where *[Points to the part of the graph to the right of the maximum]* it starts to decrease its increase, or decrease its decrease, or no, increase its decrease, so that the other side towards the end *[Points to the right* *most portion of the graph]* would be where it's, the smaller decreases come.

Interviewer: Could you explain to me increase its decrease, just to make sure I'm understanding how you are thinking about these things?

Hannah: Like for, on the decrease side, around, like right after the point *[the maximum]*, like where the highest high is. Right after that the decrease is larger than what's after it. So the decrease starts off bigger and then as it goes on the decrease gets smaller. And then it goes into that steady one and then eventually the steady one goes smaller.

To determine how the intensity in the increases and decreases might vary, Hannah drew on the curvature of the graph to make claims regarding the intensity of the change. Hannah's work extends beyond noticing a physical attribute of the graph, because she could use an attribute (curvature) to make claims about variation in the increases and decreases in amounts of temperature. Using Carlson et al.'s (2002) covariation framework, Hannah was reasoning at the QC level, because she related amounts of change in covarying quantities. Although she attended to variation in the intensity of increases and decreases, she provided no evidence of considering average rate of change in temperature with respect to change in days.

Hannah's response provides empirical support for reasoning about covarying quantities such that one quantity changes with respect to changes in another quantity (see also Johnson, 2012). Using this way of reasoning, a student could vary one quantity (using uniform or nonuniform increments) and investigate how another quantity is changing with respect to that variation. Unlike a student reasoning about covarying quantities as changing simultaneously and independently (e.g., Austin & Jacob), a student reasoning about covarying quantities such that one quantity changes with respect to changes in another quantity (e.g., Hannah) does not necessarily form intervals to determine and compare amounts of change.

Expanding Carlson et al.'s (2002) Covariation Framework

Reasoning about covarying quantities such that one quantity changes with respect to changes in another quantity supports students' consideration of variation in intensity of quantity indicating a relationship between varying quantities. At the heart of this way of reasoning is the coordination of the covarying quantities such that one quantity is changing with respect to another quantity. In contrast, reasoning about covarying quantities as changing simultaneously and independently supports students' linearization of nonlinear situations, but does not support students' consideration of variation in intensity of a rate of change in a single interval. As evidenced by Jacob's work, reasoning about covarying quantities as changing simultaneously and independently could support covariational reasoning at the AR level. However, it seems unlikely that a student's mental actions would support reasoning about instantaneous rate of change.

I propose that the Carlson et al.'s (2002) covariation framework be expanded to account for students' reasoning about covarying quantities as changing simultaneously and independently (e.g., Austin & Jacob) or about covarying quantities such that one quantity changes with respect to changes in another quantity (e.g., Hannah). Using the current framework, Hannah and Austin were both reasoning at the same level (QC). However, these students were coordinating amounts of change in covarying quantities in very different ways. Making distinctions between the ways in which students coordinate amounts of change in covarying quantities can create two paths to the subsequent levels of Average (AR) and Instantaneous Rate (IR). Table 1 indicates two distinctions (Type 1 and Type 2) in the QC level of the covariation framework.

Table 1: Expanding the Covariation Framework

Level of Covariational Reasoning	**Mental Action**	**Behaviors**
Quantitative Coordination: *Existing*	"Coordinating each amount of change of one variable with changes in the other variable" (Carlson et al., 2002, p. 357)	• "Plotting points/constructing secant lines" • "Verbalizing awareness of the variable amounts of change of the output while considering changes in the input" (Carlson et al., 2002, p. 357)
Quantitative Coordination *Expansion: Type 1*	Coordinating amounts of change in quantities such that the quantities are varying simultaneously and independently	• Specifying intervals (uniform or nonuniform), determining amounts of change in those intervals, and comparing those amounts of change • Using amounts of change to make claims about covarying quantities
Quantitative Coordination *Expansion: Type 2*	Coordinating amounts of change in quantities such that change in one quantity depends on change in another quantity	• Allowing one quantity to change with respect to another quantity • Describing variation in the intensity of change in covarying quantities

By making these distinctions in the QC levels, students' transitioning to more advanced levels of covariational reasoning might be more closely examined. Students engaging in QC Type 1 covariational reasoning seem likely to advance differently to the levels of AR and IR than would students engaging in QC Type 2 covariational reasoning. For example, Jacob reasoned in a way consistent with QC Type 1 and provided evidence of reasoning at the AR level. To extend to the IR level of covariational reasoning, a student could begin by shrinking the interval on which average rate of change is being determined. In Jacob's work on the task, he was able to shrink the interval when prompted. However, his goal was not to shrink the interval because his focus was comparing average rates of change in different intervals. In contrast, it made sense for Hannah to consider smaller intervals because for her the change in temperature was dependent on the change in the day of the year. Future research might investigate how students using these different types of QC covariational reasoning advance to AR and IR levels of covariational reasoning.

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A FRAMEWORK FOR MATHEMATICIANS' EXAMPLE-RELATED ACTIVITY WHEN EXPLORING AND PROVING MATHEMATICAL CONJECTURES

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Examples play a critical role in mathematical practice, particularly in the exploration of conjectures and in the subsequent development of proofs. Although proof has been an object of extensive study, the role that examples play in the process of exploring and proving conjectures has not received the same attention. In this paper, we present a framework that characterizes ways in which mathematicians utilize examples when investigating conjectures and developing proofs. The data consist of 133 mathematicians' responses to two open-ended survey questions. The framework offers categories for the types of examples, uses of examples, and example strategies that mathematicians discussed in reference to their work with conjectures. In addition to presenting the framework, we also discuss potential educational implications of the results.

Keywords: Advanced Mathematical Thinking, Reasoning and Proof

Introduction

A perennial concern in mathematics education is that students fail to understand the nature of evidence and justification in mathematics (Kloosterman & Lester, 2004). Mathematics education scholars have suggested that students' struggles with understanding the nature of evidence and justification may be due, in large part, to their views concerning the role of examples; in particular, students tend to be overly reliant on examples and often infer that a (universal) mathematical statement is true on the basis of checking a number of examples that satisfy the statement (e.g., Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009; Porteous, 1990). One approach designed to help students overcome their overreliance on examples is to help them understand the limitations of examples as a means of justification and thus appreciate the need for a proof (e.g., Harel & Sowder, 1998; Stylianides & Stylianides, 2009; Zaslavsky, Nickerson, Stylianides, Kidron, & Winicki, in press). Although such an approach may indeed help students understand the limitations of example-based reasoning as well as appreciate the need for proof, it characterizes example-based reasoning strategies as obstacles to overcome. Given the essential role examples play in the exploration of conjectures and in subsequent proof attempts, we suggest that example-based reasoning strategies should not be positioned only as barriers. The field may benefit from a greater understanding of the ways in which those who are adept at proof, such as mathematicians, leverage examples in order to support their thinking and activity.

Although the role of examples in learning mathematics has received attention in the literature (cf., Bills & Watson, 2008), considerably less attention has been directed toward the specific roles examples play in exploring and proving conjectures. In this paper, we present a framework that serves to characterize the variety of ways in which examples arise in mathematicians' exploration of conjectures and development of proofs. In particular, the framework characterizes the *types* of examples mathematicians may choose, the ways in which they may *use* examples, and their described *strategies* for utilizing the examples. We also discuss potential implications of this work for the teaching and learning of proof in school mathematics.

The Interplay Between Example-Based Reasoning and Proof

Epstein and Levy (1995) contend that "Most mathematicians spend a lot of time thinking about and analyzing particular examples," and they go on to note that "It is probably the case that most significant advances in mathematics have arisen from experimentation with examples" (p. 6). Clearly, examples play a critical role not only in mathematicians' development of and exploration of conjectures, but also in their subsequent development of proofs of those conjectures. Indeed, there is often a back-and-forth interplay between mathematicians' example-based reasoning activities and their deductive reasoning activities (e.g., Alcock & Inglis, 2008). Several mathematics education researchers have accordingly examined various aspects of the interplay between example-based reasoning activities and deductive reasoning activities among both mathematicians and mathematics students (e.g., Buchbinder & Zaslavsky, 2009; Iannone, Inglis, Mejia-Ramos, Simpson, & Weber, 2011; Knuth, Choppin, & Bieda, 2009).

Antonini (2006), for example, asked advanced mathematics graduate students to generate examples with specific mathematical properties. The results of this work led to an initial categorization of the students' strategies for producing examples. Antonini hypothesized that the categorized strategies may play an important role in the production of conjectures and proofs. Building upon this research, Iannone et al. (2011) used Antonini's framework to categorize the strategies undergraduate mathematics students used to generate examples during their attempts to produce proofs of conjectures. They were surprised to find that example generation did not seem to have a positive effect on proof production tasks, and they called for more research to be done in studying examples. Finally, Buchbinder and Zaslvsky (2009) also provided a framework for categorizing high school mathematics students' uses of examples when evaluating the validity of mathematical statements; they specifically classified examples as confirming, non-confirming, contradicting, and irrelevant. Although the preceding studies did not focus explicitly on the role examples play in exploring conjectures and developing proofs (or counterexamples), the research underscores the nature of the interplay between example-based reasoning activities and deductive reasoning activities, and it thus serves to inform the research presented in this paper.

Methods

Participants consisted of 133 mathematicians who responded to an online survey sent to the mathematics departments at 27 U.S. universities. The focus of this paper is on these experts' responses to the following open-ended prompt: *If you sometimes use examples when exploring a new mathematical conjecture, how do you choose the specific examples you select in order to test or explore the conjecture? What explicit strategies or example characteristics, if any, do you use or consider?* Approximately 58% of the experts were completing PhDs in mathematics, 30% had PhDs in mathematics (in a variety of different mathematical areas), and 12% had advanced degrees in other STEM-related fields; 67% were male. The data consist of these mathematicians' self-reported responses about their work with examples; while we acknowledge limitations to such data, they led us to an initial framework for experts' example-related activity.

Members of the research team independently examined the expert responses to the questions with the intent of identifying the various types of examples the mathematicians reported. During this initial coding of the data, however, it became clear that *types* of examples alone did not sufficiently capture the richness of the responses; in particular, we also coded the data with respect to the mathematicians' *uses* of examples and their *strategies* for using examples. After codes for types, uses, and strategies emerged, two members of the research team re-coded all of the responses; any discrepancies were resolved through discussion with the entire research team.

It is important to note that a particular response often could be coded in multiple ways simultaneously, both within a category (e.g., receiving multiple example-type codes), and across categories (e.g., coded as a particular example-type and as a particular use of examples). For instance, the response "I first do examples that are easiest to test. If those are consistent with the conjecture, I try more general examples, focusing on those for which the conjecture might fail." received the following codes (defined in the following section): ***Types****: Easy to Compute, General/Generic, Counterexample/*

Conjecture Breaking; ***Uses****: Check, Break the Conjecture;* ***Strategies****: Multi-Stage Example Exploration: Increasing in Generality.* The total frequencies thus do not necessarily sum to 133, the total number of respondents to the prompt.

Results

In the three sub-sections that follow, we present the components of the framework that characterize the mathematicians' example-related activities when exploring and proving conjectures. Given the page limitations of the conference proceedings, we do not go into great detail about the framework; however, we do provide representative verbatim data excerpts to illustrate the various framework categories (italics in the excerpts indicate researchers' rationale for the respective codes). We examine the results further in the Discussion section.

Types of Examples

Mathematicians described a variety of types of examples that they use when exploring conjectures (Table 1). *Simplicity* was the most frequent type of example discussed by the experts, and they also often considered counterexamples and complex examples in their work.

Table 1: Types of Examples

Type (Frequency)	Definition	Representative Data Excerpt
Simplicity (72)	Expert appeals to an easy, simple or basic example. Includes "trivial" and "small."	*Easy* ones! Start with toy cases and slowly build up the complexity.
Counterexample /Conjecture Breaking (36)	Expert picks an example that might disprove the conjecture. The expert might explicitly say "a counterexample," but this can also be inferred.	Likewise, I might also check *an example for which I believe the conjecture is most likely to fail* (sort of like a stress test).
Complex (36)	Expert picks a complex example in order to test whether the conjecture holds for tricky ones; synonyms include "non-nice," "non-trivial," or "interesting."	I try to find examples that include all of the (foreseen) barriers to a proof. *The "hardest" examples in the sense of what I'm trying to prove.*
Easy to Compute (32)	Expert chooses an example that is easy to manipulate. The difference between this code and "Simple" is that the expert says something about computing or working the example out.	I usually use appropriate low-level examples. For example, those that may be *easy to compute* and/or for which it is reasonable to check the conjecture.
Properties (26)	Expert takes into account some specific mathematical property – he or she might reference a "property" or "features," or might mention particular properties.	… For number-based conjectures, *I choose 0, numbers close to 0 (both positive and negative), very large and very small numbers, for examples, both integers and non-integers.*
General/Generic (22)	Expert states that he or she uses general or generic examples, or describes examples that are viewed as representative of a general class of cases or otherwise lack special properties.	Try the *most general example* which is still practical to test.
Boundary Case (19)	Expert picks an extreme example or number, or a "special" case, such as the identity.	…I will next try to test some strange or pathological examples, *to really push the boundaries of what might be possible in this situation*.
Familiar/Known case (18)	Expert chooses an example with which he or she is familiar, or in which properties related to the conjecture are already known.	Use examples *I'm familiar with* and see if everything still holds.

Type (Frequency)	Definition	Representative Data Excerpt
Unusual Examples (13)	Expert picks an unusual number, which would be described as something that does not come up often. "Rare," "obscure," "strange," and "weird" are also synonyms.	Next, I try something slightly more *obscure*.
Random (10)	Expert describes the example as randomly chosen; this includes genuine mathematical randomness, such as cases in which examples are chosen with a random number generator.	Try *"random"* examples in cases where that makes sense.
Exhaustive (9)	Expert looks for "all" of the examples in an exhaustive manner. This can be by testing all possible examples or by using a computer.	If it is difficult to find examples, write a computer program to find *all examples with specific characteristics*.
Common (9)	Expert describes the example as typical, common, or one many would choose.	Ones that are not special, *ones that I judge to be typical*.
Dissimilar Set (9)	Expert indicates that he or she purposely selects a variety of different types of examples.	If there are too many, I would try to select examples with *widely different properties*.

Uses of Examples

Table 2 highlights the ways in which the mathematicians discussed how they use examples as they examine conjectures; these ranged from using an example to check whether a conjecture is true to carefully selecting examples that might provide insight into how to prove the conjecture.

Table 2: Uses of Examples

Use (Frequency)	Definition	Representative Data Excerpt
Check (41)	Expert selects examples to make a judgment about the correctness of a conjecture; "test," "verify," and "check" are all synonyms.	I would start with an example that is easy to *check*. If that example works out and agrees with the conjecture I would *check* a less trivial example then.
Break the Conjecture (35)	Expert tries examples to break the conjecture; this can include specifically looking for a counterexample.	Then once I'm more comfortable with it, *try it with some example I regard as less likely to verify the conjecture and keep looking for a counter-example*.
Make Sense of the Situation (16)	Expert uses an example to deepen his or her understanding of why the conjecture might be true or false, or to gain mathematical insight.	First test for the most simple cases, *also to understand the conjecture a little better*. …
Proof Insight (8)	Expert indicates that his or her production of examples (or counterexamples) might have a direct bearing on understanding how to prove the conjecture.	Eventually, if I figure out the conjecture is true for all the examples tested, the search for a counter-example *should have given me some insight in how to prove it*.
Generalize (5)	Expert mentions using the example to generalize or to allow the expert to work in a more general situation.	… Hopefully it's obvious why what you're looking for is true in the easiest case. *You can then see if that reason generalizes.*
Understand Statement of the Conjecture (3)	Expert uses an example to better understand the statement of the conjecture.	I use simple examples first, *so I understand what the conjecture says* and then build up to more complicated ones.

Example Strategies

Table 3 displays the strategies that mathematicians said they employ when using examples to explore and/or prove conjectures. In order to be coded as a strategy, a mathematician's response had to explicitly describe a systematic approach for how he or she used examples when exploring a conjecture; simply listing one or two actions one would take or examples one would try did not constitute a strategy unless the actions or examples were explicitly connected. The *Multi-Stage Example Exploration* strategy occurred when mathematicians indicated a progression in their choices of examples; typically they described starting with simple examples, and moved toward more complex, more general, or more extreme examples. The *Property Analysis* strategy involved an examination of particular properties of the chosen examples, and insights about these properties provided further insight into the conjecture or the proof. In the *Analysis of Related Proof Activities* strategy, mathematicians described engaging in proof activities that subsequently affected how they chose and used examples. In these cases, the mathematicians' choices and uses of examples were explicitly linked to attempts to prove or to disprove the conjecture. Finally, the *Systematic Variation* strategy occurred when mathematicians suggested that they would start with an example, but would then carefully modify it in some way to further their progress in exploring or proving a conjecture.

Table 3: Example Strategies

Strategy (Frequency)	Subcategory (Frequency)	Definition	Representative Example
Multi-Stage Example Exploration (62)	*Increasing in Complexity (29)*	Expert begins with simple or easy examples and shifts to more complex or complicated examples.	*I use simple examples first*, so I understand what the conjecture says *and then build up to more complicated ones.*
	Increasing in Extremity (17)	Expert begins with simple or typical examples and builds to boundary cases, special cases, or conjecture-breaking cases.	*First look for the easiest examples. Build sophistication, and look for extremal cases.*
	Increasing in Generality (16)	Expert begins with simple or special examples and shifts to more general or generic examples.	Following Polya's suggestions, *looking at simplest examples first*. *Then try to choose a few representative/generic/random examples.*
Property Analysis (12)	*Supporting or Non-Supporting Examples (11)*	Expert attempts to determine the properties of examples that either support the conjecture in particular ways, or to determine the properties of examples that do not support the conjecture.	I try *to find specific properties* of my guess examples that prevent them from doing what I want them to do. *Sometimes this allows a slow building up of properties that can eventually say something useful about the conjecture.*
	Test Cases (1)	Expert analyzes special test cases depending on the critical properties of the examples related to the conjecture.	Also, depending on what I think will be important in proving (or disproving) the conjecture, *I will include some special test cases.*

Strategy (Frequency)	Subcategory (Frequency)	Definition	Representative Example
Analysis of Related Proof Activities (11)	*Generalize From Attempts to Prove (6)*	Expert proves related conjectures or lemmas and attempts to generalize to the conjecture at hand. Or, proves the conjecture for specific examples and then attempts to build to a general proof. Or, attempts to generalize from examples related to unsuccessful proof attempts.	Based…*on an unsuccessful proof of the conjecture* (the point where I got stuck may give me suggestions of where to look for negative examples).
	Generalize From Attempts to Disprove (5)	Expert attempts to generalize from examples that disprove related conjectures. Or, attempts to generalize from properties of examples that failed to disprove the conjecture.	I also *try to guess counterexamples*. This guessing typically fails, and if it does, *I try to find specific properties of my guess examples that prevent them from doing what I want them to do.*
Systematic Variation (10)	*Known Case Adjustment (7)*	Expert takes a known case and makes small adjustments to the example's properties, inputs, or characteristics.	Take an example *I've already done and perturb it a bit.*
	Multiple Property Variation (3)	Expert varies multiple properties or characteristics simultaneously or independently in a systematic fashion.	If there are *multiple properties, I often try to vary them independently* to see if I can discover their individual effects.

Discussion

The three-part framework highlights the complex roles examples play in the work of mathematicians, identifying multiple types of examples, uses of examples, and example-based reasoning strategies that mathematicians take into account as they engage in exploring and proving conjectures. What may not necessarily be evident in the presentation of the framework, however, is the ways in which the mathematicians' extensive domain knowledge plays a critical role in conjecture-related activity. Through our analysis, we identified four ways in which mathematicians' domain expertise appears to influence both their example choices and the ways in which they think strategically with examples when making sense of conjectures.

First, mathematicians noted that their approaches to exploring and proving a conjecture were dependent on whether or not they thought the conjecture was true. Their initial instincts about a conjecture's likelihood to be valid influenced the types of examples they chose and the strategies they employed. For example, one expert said, "If I am not sure whether the conjecture is true, I start by considering an example for which I believe it will be true … If I think it is not true, I choose the simplest example for which I believe the conjecture will fail. If I am fairly certain it is true, I usually try to consider the most general case." Second, mathematicians indicated that for many conjectures, constructing an example was not necessarily trivial (or even possible) in certain mathematical domains. One expert said, "In my work … the invariants I work with are incredibly hard to compute," and another noted that, "In my mathematical experience, the trick is to FIND examples" (expert's emphasis). Third, mathematicians implied that prior experience and intuition often played a part in their choice of examples. They described ways in which they capitalize on their prior experiences with a given area in order to access examples that would be most relevant to particular types of conjectures. One expert described choosing examples, "Based on the intuition, on the experience (examples already present in the literature may give a feeling)."

Finally, the mathematicians exhibited a meta-awareness in which they were able to see their example-related activity in terms of a broader context of their mathematical activity. In other words, the mathematicians showed intentionality about their work with examples—they were aware of what their examples could do for them and were often explicitly deliberate about their example choices. In the following response, the expert displays a clear strategy and cognizance when it comes to choosing examples when exploring a conjecture: “First test for the most simple cases, also to understand the conjecture a little better. Then once I'm more comfortable with it, try it with some example I regard as less likely to verify the conjecture and keep looking for a counter-example. Eventually, if I figure out the conjecture is true for all the examples tested, the search for a counter-example should have given me some insight in how to prove it.”

The role that examples play in the work of mathematicians stands in contrast to the role examples typically play in the work of mathematics students. For instance, some studies suggest that experts’ meta-awareness of examples described above differs from students’ example use. Knuth et al. (2011) suggest that middle school students may have difficulty considering the characteristics of their examples in the way mathematicians do. Additionally, while other studies have shown that, like mathematicians, students make use of similar types of examples (such as *Simple, Common*, or *Unusual*) (Cooper et al., 2011), the ways in which students and mathematicians appear to use examples may differ. Using examples to check a conjecture’s accuracy and then as a justification of its truth is common in student populations (Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009; Porteous, 1990), whereas mathematicians indicated that they rely on examples not only to check conjectures, but also to better understand them and to gain insights into their proofs.

The domain expertise that the mathematicians possess also clearly contributes to the differences between the roles examples play in the work of mathematicians and in the work of students. In the latter case, for example, the majority of conjectures with which students are charged with proving are true (and students often know this in advance as well); example uses such as *Break a Conjecture* are thus rarely employed. Students are also often unable to build upon familiar examples and on their intuition due to their relatively limited mathematical experience. This may be one reason why students seldom demonstrate the meta-awareness that mathematicians demonstrate. Our results point to the power of intentional example exploration in supporting one’s understanding of conjectures and their proofs. By better understanding mathematicians’ strategies when thinking with examples, we can uncover and elaborate ways to more effectively support students’ example exploration and subsequent proof development.

Concluding Remarks

Our findings suggest some implications for the teaching and learning of proof in school mathematics. Mathematicians’ practices of engaging in systematic, multi-stage example exploration suggest that students may benefit from learning how to vary their example use and to assess the relative merits of different types of examples when exploring conjectures. Teaching practices that encourage exploration of multiple example types, and that require students to clarify and justify their use of examples, could support a greater understanding of the conjectures. Also, during a classroom discussion, comparing different sets of examples across groups of students could highlight the ways in which some types and uses of examples may be more beneficial than others in supporting both understanding and proof development. In this sense, a stronger understanding of the strategies mathematicians employ as they use examples to develop, explore, and prove conjectures may ultimately inform the design of instructional practices and curricula that effectively foster students’ abilities to prove. Mathematicians clearly possess an awareness of the powerful role examples can play in exploring, understanding, and proving conjectures, as well as the ability to implement example-related activity in meaningful ways. Thus, in order for students to develop such awareness and ability, it is important to help them learn to think critically about the types of examples, uses of examples, and associated strategies they can employ as they engage in exploring and proving conjectures. Building on both our findings and on others’ prior work, the exploration of students’ example use could inform a new approach to conjecture development and proof, one that highlights the power of strategic example-based reasoning and activity.

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TRANSITION FROM DERIVATIVE AT A POINT TO DERIVATIVE AS A FUNCTION

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This paper explores how textbooks address two central concepts in differential calculus, derivative at a point and derivative function, make the transition from one concept to the other, and establish connections between them. We analyzed how the three most widely used calculus textbooks present these two aspects of the derivative, focusing on visual means and word use in the books. In contrast to their thorough discussion on the limit process for the derivative at a point, the books make a quick transition to the derivative function by "letting a point ***a*** *vary" and changing "f '(a) to f '(x)." Then, they graph f '(x) using several values of the derivative at a point. In addition, the books often use the term "derivative" without specifying which of the two concepts is meant, and are inconsistent in the use of letters, so that it is unclear whether a letter (a or x_1) denotes an arbitrary but fixed number or a variable.*

Keywords: Advanced Mathematical Thinking; Post-Secondary Education

Introduction

With a gradual growth in research in teaching and learning calculus, there have been several studies about students' thinking about the derivative. Most studies have reported students' conceptualizations about the derivative (e.g., Tall, 1987; Thompson, 1994), and their notations (e.g., Hahkioniemi, 2005; Zandieh, 2000) by addressing several mathematical aspects. This study focuses on two aspects: the derivative at a point as a specific value, and the derivative function as a function. Other researchers have emphasized these aspects (e.g., Oehrtman, Carlson & Thompson, 2008), but few studies have been done especially about the derivative as a function, nor about the transition and connection between derivative at a point, and the derivative function.

Motivation of this study came from Park's (2011) study about calculus instructors' and students' discourses on the derivative. The results showed that instructors addressed some aspects of the derivative implicitly in class using the word "derivative" without stating whether it was "derivative at a point" or "derivative function," and how these two concepts are related. During the interviews, students also used the word "derivative" without specifying and often to support incorrect notions such as "derivative as tangent line." From these results, we started wondering how to help students realize the relation and difference between derivative at a point and derivative function, make a transition from one to the other and build connections between them. As a first step, we decided to explore how widely-used calculus textbooks address the derivative as a point-specific concept and as a function. Specifically, we address the following questions:

1. How textbooks for Calculus I address the derivative at a point?
2. How textbooks for Calculus I address the derivative of a function?
3. Whether and how textbooks for Calculus I make a transition/connection between the derivative at a point and the derivative of a function?

This study is important for several reasons. First, it focuses on a central, but not yet sufficiently analyzed, relation between two main concepts of differential calculus, derivative at a point and derivative as a function. By studying students' opportunities to establish such relation through the material presented in the textbooks, if the analysis shows gaps or inadequacies in the presentations, we will be able to suggest ways instructors may complement how the books presented the idea. The textbooks analyzed in this study, which are used by over 70% college calculus instructors, share many similarities in their approaches to derivative. Second, exploring the relation between derivative at a point and derivative function is important because it offers calculus students an opportunity to revisit central aspects of function, namely a relation between thinking about function pointwise and across an interval. Though the concept of function is fundamental to understand calculus concepts, many students who received A's still have incomplete conceptions of function after their second calculus course (Oehrtman et al., 2008).

Theoretical Background

Function at a Point and Function on an Interval

There is a rich body of research on how students understand function, which also has provided several conceptualizations of functions. The studies, which address developmental stages of understanding functions, have made a clear distinction about function at a point and function on an interval (e.g., Breidenbach, Dubinsky, Hawks, & Nichols, 1992). Most studies describe the first stage of understanding function as being able to generate an output value of a function when an input value is given. A person at this stage would think of function as a value for a given input. Monk (1994) called this view of function "*pointwise understanding,*" and Dubinsky and McDonald (2002) called it "Action." The next stage is described as being able to see dynamics of a function. Monk (1994) called this stage "*across-time understanding,*" and described it as an ability to see the patterns in change of a function resulting from patterns in input variables. Dubinsky and McDonald (2002) called it "Process." Breidenbach et al. (1992) found that a transition from the first to the second stages is not natural, and some calculus students are at the first stage, and thus they have trouble seeing calculus concepts dynamically.

Derivative at a Point and Derivative as a Function

Existing studies on students' thinking about the derivative can be divided regarding the two views of functions. Studies about the derivative as a point-specific value showed that students' thinking about the limit is related to their thinking of local linearity (Hahkioniemi, 2005) and tangent line (Tall, 1987). Studies about the derivative as a function that mainly address co-variation showed the importance of what is varying in a function. Oehrtman et al. (2008) compared the rate of change of the volume of a sphere with respect to its radius (its surface area) and the rate of change of the volume of a cube with respect to its side (not its surface area). Thompson (1994) related the rate of change to students' thinking of the derivative.

However, few studies have been done about the relation between these two types of understanding of the derivative. Monk (1994) addressed these two types based on students' written answers on four survey problems, but did not give much detail about whether and how students related these two concepts. Park (2011) interviewed 12 calculus students and found that using one word "derivative" for both "the derivative function" and "the derivative at a point" was related to their conception of the derivative as a tangent line. The students were changing what the word "derivative" refers to in various contexts and used it as a mixed notion of a point-specific concept but a function, which the tangent line represents. They also used this idea to justify an incorrect statement, "a function increases if the derivative increases." Analysis of their class lessons about the derivative showed that the instructors were not explicitly addressing the derivative at a point as a number, and the derivative function as a function. In this current study, whether and how the calculus textbooks relate these two mathematical aspects will be explored.

Words and Visual Mediators

This study is based on the communicational approach to cognition (Sfard, 2008), which views mathematics as a discourse characterized by four features: word use, visual mediators, routines, and endorsed narratives. This study focuses on the first two features. A *word* in mathematical discourse can be used differently in a different context. For example, the word "derivative" is used as the derivative at a point and the derivative function (e.g., "is the derivative positive here?"). Quantifiers (e.g., one & any) play an important role to determine if "derivative" is a point-specific value or a function. *Visual mediators* refer to visual means of communication. This paper focuses on various notations of the derivative and letters for a point and variable. For example, if the derivative at a point is denoted as $f'(a)$, and the derivative function as $y = f'(x)$, "a" is used as a number, and "x" is used as a variable. The derivative at a point can be visually mediated by the slope of the tangent line and the derivative function by its graph.

Method

Based on Bressoud's (2011) study, we chose three textbooks that are widely used by Calculus I course instructors in the United States: one edition by Stewart (43%), Hughes-Hallett et al. (19%), Thomas et al. (9%). In each book, we explored the sections about the rate of change, and the derivative. We developed an analytical tool using an existing framework (Park, 2011). Though there were slight differences in each book, we identified five phases: (a) rate of change, (b) the derivative at a point, (c) transition, (d) the derivative function, and (e) connection. The first phase addresses the rate of change without using the word "derivative." The derivative at a point is defined in the second phase. In the third phase, the books make a transition to the derivative function, and define it in the fourth phase. Last, they connect back to the derivative at a point graphically. We examined book descriptions through their visual mediators and word use (Table 1). We focused on whether key terms—slope, rate of change, and derivative—were used as static or dynamic based on whether it is defined at a point, multiple points, or on intervals with a variable. Because books have limitations showing dynamics, we carefully looked at the descriptions for the figures including quantifiers and letters.

Table 1: Analysis Table

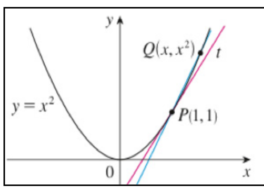
Stage	Visual Mediator			Word Use			
	Table	Graph	Symbolic Notations	Key Term	Static		Dynamic
					A point	Multiple points	

Results

In this section, we focus on the most-widely used book (Stewart, 2010) with the details of how we used the key words and visual mediators to reach our conclusions. The analysis of other books is addressed in the Discussion.

Velocity and Slope of Tangent

Stewart's (2010) *Calculus* addresses the slope and velocity in the chapter of Limit without using the word *derivative*. First, it shows how to obtain the slope of tangent line to a curve $y=x^2$ at $P(1, 1)$ using the point $Q(x, x^2)$ approaching P (Figure 1).



x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

$m_{PQ} = \frac{x^2-1}{x-1}$ …for the point $Q(1.5, 2.25)$… $m_{PQ} = \frac{2.25-1}{1.5-1} = 2.5.$
The table…shows the values of m_{PQ} for several values of x close to 1. The closer Q is to P, the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2.

Figure 1: Graph of $y = x^2$ and values of slope of secant lines (p. 45)

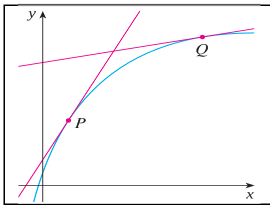
Using the same method, the book calculates the velocity of a ball after 5 seconds as *49 m/s* (the distance: $s(t) = 4.9t^2$), and relates its velocity at $t = a$ to the slope of tangent to the curve, $s(t)$. Here, the book addresses the slope of the tangent line to a curve and the velocity as point-specific concepts, at $x = 1$, $t = 5$, and $x = a$. The book used a as if a were a number rather than an arbitrary value or multiple values without stating that a could be any point. The dynamic aspect of the concepts was only addressed in the limit process finding the slope of tangent from secant lines. Thus, this section addressed the velocity and slope of a tangent line at a *specific* (single) point.

Derivative at a Point

The book calls the "special type of limit" in the slope and velocity "a *derivative*," and uses the word with phrases, "of a function," "at *a*," or an equation in this section. It rewrites the slope of the tangent line of $y = f(x)$ at $x = a$ as $m=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, and defines "the *derivative* of a function *f* at a point *a*" as the same limit (p. 107). The letter *a* is used only for a point until the book calculates "the derivative of a function $f(x) = x^2-8x+9$ at the number *a*" as "$f'(a) = 2a-8$" and finds the slope at (3, -6) as "$f'(3) = 2(3)-8=-2$." Though this calculation implies that *3* is one value of *a,* the book does not explicitly state it or write $a = 3$ (p. 107).

Rate of Change

The book defines the instantaneous rate of change of $y = f(x)$ with respect to *x* at $x = x_1$ as $\lim_{x_2\to x_1}\frac{f(x_2)-f(x_1)}{x_2-x_1}$, interprets it as "the derivative $f'(x_1)$" and changes it to "the derivative $f'(a)$." It then gives two interpretations, "the slope of tangent line to a curve when $x = a$" and "the instantaneous rate of change…at $x = a$," and makes a connection between them (Figure 2, p. 108).



[On] the curve, $y = f(x)$…the instantaneous rate of change is the slope of the tangent to this curve at the point, where $x = a$. This means that when the derivative is large (and therefore the curve is steep, as at the point *P*), the *y*-values change rapidly. When the derivative is small, the curve is relatively flat (as at point *Q*). and the *y*-values change slowly.

Figure 2: Graphs of two tangent lines (p. 108)

For the cost of producing *x* yards of fabric, $C = f(x)$, the book explains "the derivative, $f'(x)$" as "the rate of change of the production cost with respect to the number of yards produced" in dollars/yard and asks to find or compare the meaning of $f'(1000) =9$, $f'(50)$, and $f'(500)$ (p. 109).

In this section, the book uses the word "derivative" three times. "The derivative, $f'(x_1)$" indicates that it is defined at a "fixed point x_1" (p. 109). In Figure 3, "derivative" is used to describe the function behavior as in "when the derivative is large, the *y* value change rapidly" (p. 108). Because the book specified the point *P*, it is clear that the sentence is about the local function behavior near P, but it can be true anywhere on the interval if "the derivative" is used as a function. At the end, the book calls all rates of change of various functions at several points "derivatives." It uses the notation, $f'(x)$ for the first time. In the fabric problem, it interprets $f'(x)$ as if it were a point-specific value, but gives its units in general terms using the units of different quotients without making a connection to its interpretation. In the second problem, it interprets "$f'(1000) = 9$," as "when $x = 1000$, C is increasing 9 times as fast as *x*." Though $f'(1000)$ was used as a value of $f'(x)$ at $x = 1000$, the relation between notations, $f'(1000)$ and $f'(x)$, was not stated.

Transition from the Derivative at a Point to the Derivative of a Function

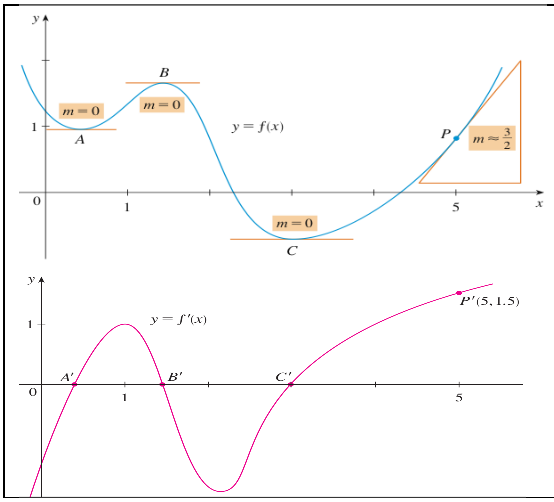
The book summarizes that all previous discussions were about "a fixed point," which confirms the word, "derivative," "*x*" in the fabric example and "*a*," in graphs as point-specific values. Then, in $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, the book changes the "point of view and let[s] the number *a* vary, … replace[s] *a* by a variable *x*, and…obtain[s] $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$" (p. 114). Here, the nature of "*a*" was specified as "vary[ing]" and connected to the "variable, *x*."

The Derivative Function

The book defines "$f'(x)$ as a new function" that assigns to "any number *x*…the number $f'(x)$," and connects it to "the slope of the tangent line to the graph of *f* at the point $(x, f(x))$" (p. 114). It also emphasizes that the variable *x* in $f(x)$ and $f'(x)$ are the same by comparing the domain of f', $\{x| f'(x)$ exists$\}$ that may be smaller than the domain of *f* (p. 114).

Connection from the Derivative Function to the Derivative at a Point

The book then graphs *f '(x)* using slopes of tangent lines to the curve, *f(x)* (Figure 3).



We can estimate the value of the derivative at any value of *x* by drawing the tangent at the point (*x*, *f(x))* and estimating its slope. For instance, for *x* = 5 we draw the tangent at *P* in [the] Figure and estimate its slope to be about 3/2 , so *f* '(5) ≈1.5. This allows us to plot the point *P' (*5, 1.5) on the graph of *f* ' directly beneath *P*. Repeating this procedure at several points, we get the graph shown in Figure 2(b). Notice that the tangents at *A*, *B*, and *C* are horizontal, so the derivative is 0 there and the graph of *f* ' crosses the *x*-axis at the points *A*', *B*', and *C*', directly beneath *A*, *B*, and *C*. Between *A* and *B* the tangents have positive slope, so *f* '(*x)* is positive there.

Figure 3: Graphs of a function and its derivative function (p. 115)

In Figure 3, the book makes a connection from the derivative function to the derivative at a point by stating the value of the derivative at "any" point of *x* using the slope at the point, finding the slope 1.5 at *x = 5*, and plotting (5, 1.5) for *f '(x)*. It again uses the point-wise approach to find the zeros for "the derivative." Then, it uses the interval-wise approach to determine whether *f '(x)* was positive or negative between these zeros. Here, the word, "derivative" first is used as *the derivative function* because it was defined "at any value." The second one in "the derivative is zero there and the graph of *f '* crosses the *x*-axis" is used as a point-specific value. To refer to the function that the second graph represents, the book consistently used the notation *f '(x)*. When it describes the sign of the "slope" of "tangents" on intervals, it used the singular "slope" instead of "slopes." Though "the slope" can be inferred as "the slope" as a function because the book was using "the slope" for several values, it would have been "the slopes of the tangents."

Summary

To address the concept of the derivative, Stewart (2010) (a) uses the velocity and slope at a point, (b) defines the derivative of a function at a point, (c) interprets it as the instantaneous rate of change, (d) makes a transition by letting point *a* vary and replacing it with variable *x*, (e) defines the derivative of a function, and (f) constructs the graph of *f '(x)* using the slope of tangents to *y = f(x)*. Table 2 shows key words and visual mediators used in each of these phases.

Table 3: Visual Mediator and Word Use in Stewart (2010)

Phase	Visual Mediator			Word Use			
	Table	Graph	Symbolic Notations	Key Terms	Static		Dynamic
					A point	Multiple points	
(a)	Fig. 1	Fig. 1	$\frac{x^2-1}{x-1}$	The slope	At the point (1, 1)		Limit. "Values close to 1"
			49m/s	The velocity	After 5 seconds		
			$\frac{4.9(a+h)^2-4.9a^2}{(a+h)-a}$	The velocity The slope	At time $t = a$ At point P		Limit. "As h approaches 0"
(b)			$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$	The derivative of a function f	At a point a		
			$f'(a) = 2a-8$	The derivative of $f(x) = x^2-8x+9$	At the number a		
			$f'(3) = 2(3)-8=-2$	The slope of the tangent line	At (3, -6)		
			$\lim_{x_2\to x_1}\frac{f(x_2)-f(x_1)}{x_2-x_1}$	Rate of change The derivative $f'(x_1)$	At $x = x_1$		
(c)		Fig. 2	$f'(a)$	Derivative		At the points P & Q	
			$f'(x)$	Derivative Rage of change of C	With respect to x		
			$f'(1000) = 9$,	The rate of increase	After 1000 yd		
			$f'(50)$, $f'(500)$	The rate of increase		After 50 & 500 yd	
(d)			$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$	Derivative	At a fixed point a		Let a vary
(e)			$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$	Derivative			[of] variable x
(f)			$f'(x)$	A new function			Any number x
		Fig. 3	f'	Derivative Graph of f'	At $x = 5$	Where f' crosses the x axis	Any value of x

As shown in Table 2, Stewart (2010) develops the concept of the derivative from a number to a function by showing graphical representations of the slope of a single tangent line, the slopes of multiple tangent lines, and the derivative of a function. It also uses the slope, the rate of change, the derivative at a specific point, multiple points, and any points by using numbers and letters with or without subscripts and changing what those key words represent. Graphical notations were consistent with such development: first a single tangent line, then multiple tangent lines, and graph of $f'(x)$ in turn (see Figures 1, 2, & 3). However, other visual mediators, letters for a point and variable were not consistent. Though the book mainly used a for a single point or multiple (discrete) points, and x for the variable, in the second and third examples in phase (b), "a" was used as if it were a variable, because 3 is substituted in a in the next step. The book does not mention that a can be any value. In a later section, it calls "a" "the fixed point." Then, in phase (c), it again uses "a" for multiple points, which seems to transfer what "a" represents from a single value to any values. However, in the next step, it uses "$f'(x)$" as if it were one value of the rate of change of a cost function and interprets $f'(1000)$, $f'(50)$, and $f'(500)$ without making a connection back to $f'(x)$ or mentioning they are the specific values of $f'(x)$.

The word "derivative" is also used inconsistently. First, it is used as *"the derivative* $f'(x_1)$*"* as the rate of change at a point, and again in "the derivative $f'(x)$ with respect to x" as if it were a point-specific concept (before the book defines the derivative function, $f'(x)$). Then, in Figure 3, the book uses the word "derivative" twice: one for the derivative of a function (at any points), and the other one for the derivative at points where f' crosses the x axis. The word is used without its referent–the derivative function or the derivative at a point–or notation–$f'(x)$ or $f'(a)$. The book relates these two concepts twice. First, it makes a transition from the derivative at a point to the derivative function by letting "a" vary and changing "a" to "x." Second, after defining the derivative of a function, it makes a connection back to the derivative at a point based on the slopes of several tangent lines to the original function at discrete points.

Discussions and Conclusions

As mentioned earlier, the textbooks address the concept of the derivative first as the velocity and slope without using the word "derivative," define the derivative of a function at a point, and then the derivative function. With slight differences in representations, the books we analyzed have some common characteristics in connecting the derivative at a point and the derivative of a function. First, the use of numbers and letters with or without subscripts is not consistent. For example, Thomas et al. (2010) uses $t = 1$ and $t = 3$ in a problem statement and $t_0 = 1$ and $t_0 = 3$ in its solution. It also uses a letter with a subscript x_2 for a value approaching a fixed value x_1. Second, the word "derivative" is not used explicitly; most times, it is used without its referent, the derivative at a point or the derivative function. Especially, when the word is used after defining both concepts, it is not clear whether "derivative" is used as a point-specific value or as a function. With this implicit use of the letters and key words, the derivative at a point as a value of the derivative function is also not consistently addressed. For example, all three books use notations $f'(x)$ and $f'(a)$, and substitute a number in x or a before mentioning that the concept of the slope or the rate of change can be considered at more than one (or any) point on an interval or defining the derivative function. To define the derivative function, they all change the view to let "a" or "x_0", which used to be a fixed value, "vary" and change it to "x." After the definition, they show the graphing process of the derivative function based on the slopes to the curve $y = f(x)$. In this process, the word "derivative" is also used implicitly, which is problematic because they are graphing "the derivative function" based on "the derivative" at discrete points. Hughes-Hallet et al. (2010) even draws the graphs of a function and its derivative function on one x-y plane, which does not show that they represent different values, such as distance and velocity.

Calculus I is a first college course, in which students practice abstract mathematical thinking and prepare for upper level mathematics courses. Mathematicians, including textbook authors, may think that students have mastered the concept of function before they start the course. However, many studies show that this is not necessarily true; calculus students do not always have complete understanding of function in secondary level, and thus have trouble seeing the derivative as a function (Park, 2011). Calculus books cannot and need not include all the explanations of a function, which should be addressed in the previous mathematics classes. However, inconsistent use of key words and visual notations supporting the concept of the derivative as a point-specific value and as a function may confuse calculus students who do not have a solid understanding of a function. The way the concepts of the derivative are built—(a) heavy discussion on the limit process in the derivative, obtaining the slope of the tangent from a sequence of secant lines; (b) a simple transition from the derivative at *a*, $f'(a)$, to the derivative function $f'(x)$; and (c) graphing $f'(x)$ based on several values of $f'(a)$– is not consistent with the way the concept of function was built before. Changing a view of seeing "*a*" as a fixed value to any values may not be *simple* to students and graphing $f'(x)$ after giving its definition may not be ideal. Constructing the derivative function based on the derivative at discrete points before defining the derivative function may remind students about how a function was constructed and thus help them guess what those values represent and how they change as *x* values change, and finally think about the derivative as a function before they see the formal definition.

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RELATIONSHIPS BETWEEN MATHEMATICAL PROOF AND DEFINITION

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This article discusses results from interviews with two undergraduate students in an introductory proofs course. The researcher assessed the participants' general proof schemes and built models of the participants' conception of probabilistic independence and mutual exclusivity. The participants were then tasked with asserting a relationship between independence and mutual exclusivity and trying to prove the asserted relationship. The results discuss possible interactions between students' conception of mathematical ideas and their approaches to proof.

Keywords: Proof Scheme; Definition; Concept Image; Probability

Mathematical proof and mathematical definition are two areas of research that have recently gained heightened attention from researchers (Edwards & Ward, 2004; Harel & Sowder, 1998; Ko, 2010; Vinner, 1991). These two areas, though generally studied separately, are intrinsically related (Edwards & Ward, 2004; Weber & Alcock, 2004). Little research, however, has explicitly explored relationships between proof and definition. The purpose of this research is to explore the connections and relationships between undergraduate students' proof schemes and their understanding and use of definition.

Theoretical Framework

A Framework for Discussing Proof

Harel and Sowder (1998, 2007) have provided a fundamental framework for research in students' conceptions of proof. This framework begins by defining proof as a two-part process: ascertaining and convincing. A student's notion "of what constitutes ascertaining and persuading" is called that student's proof scheme (Harel & Sowder, 1998, p. 244). There are varying levels of proof schemes described in the literature. Broadly, these proof schemes are External, Empirical, and Analytical — an External proof scheme includes validations by authority, ritual, or symbolism, an Empirical proof scheme is inductive or perceptual, and an Analytical proof scheme is more rigorous and logical. It is important to note that, at a given time, no person completely displays evidence of exactly one proof scheme. Because of this, a student's proof scheme is a generalization of the types of proof schemes evident through his or her work. For example, a student could exhibit both Empirical and Analytical proof schemes within a short period of time or even within a single proof; such a student could be said to have an "emerging Analytical" or "Empirical/Analytical" proof scheme.

Weber and Alcock (2004) also contribute a framework for discussing students' semantic and syntactic proof productions. This framework draws distinctions between students' use of instantiations in proof (syntactic) and the formal manipulation of logical mathematical statements (semantic). While this framework is constructed outside of Harel and Sowder's (1998) proof schemes, the two classification systems for students' mathematical tendencies seem as though they could work successfully together as neither excludes the other.

A Framework for Discussing Definition

Vinner (1991) distinguishes between the ideas of concept image and concept definition. He defines the concept image as a non-verbal entity such as a "visual representation of the concept… [or] a collection of impressions or experiences" (p. 68) which our mind associates with the concept. In contrast, the concept definition is the formal mathematical definition of a concept. These two ideas are not necessarily—and, one could argue, seldom—the same thing. Vinner uses the sentence, "my nice green car is parked in front of my house," as an example of concept image (p. 67). He argues that the reader or listener does not necessarily consider the definition of each word in the sentence, but that each word invokes a generic concept image in his or her mind, the collection of which allows the sentence to take form as a whole impression.

Probability as a Context

Manage and Scariano (2010) provided a useful context in which this research was conducted. The authors found that most of the students in their study thought that two events being independent implied that they were mutually exclusive and vice versa. Although one's initial reaction may be to conclude this exact relationship, after careful consideration of the two concepts one realizes the two terms have almost exactly opposite meanings. This "almost" is attributed to cases in which one or both of the events has zero probability. Otherwise (i.e., if two events have nonzero probability), independence implies that two events are not mutually exclusive and mutually exclusive events are not independent. When asked to prove this relationship between independence and mutual exclusivity, one must address his or her own conceptions of independence and mutual exclusivity, compare the two concepts, ascertain the relationship between the two, and try to convince others. So, this relationship between mutual exclusivity and independence will provide a context for exploring the use of definition in proof.

Methods

The researcher conducted semi-structured interviews with three undergraduate students—Alex, Betty, and Caroline—who were enrolled in an Introduction to Proofs course. All three students were mathematics majors in their third year and were chosen randomly from a group of volunteers. None of the students were compensated for their participation. Each participant completed three interviews, each lasting approximately one hour.

Each of the three interviews had its own unique goal: (Interview 1) to gauge the participants' general proof schemes, (Interview 2) to gain insight into the participants' concept definitions and concept images of specific mathematical terms, and (Interview 3) to observe and analyze the participants' use of definition and imagery while proving relationships about the mathematical terms discussed in the previous interview.

Data Collection

All interviews were recorded using both video and audio devices. The researcher kept notes throughout the interviews and all participant work was collected. The first interview was designed to gather a general understanding of each participant's proof scheme. The interview consisted of each participant assessing a "matrix of proofs," which is a 3-by-3 grid of mathematical proofs. Each row in the matrix contained three variations of proof of the same mathematical relationship, reflecting Harel and Sowder's (1998) three major proof schemes—Analytical, External, and Empirical. The participants were asked to assess each proof for mathematical and logical correctness. From these responses, the researcher determined the aspects of mathematical proof that the participants considered important and/or necessary or, conversely, unimportant and/or unnecessary. In turn, the researcher used the participants' responses and reasoning in order to form a notion of each participant's general proof schemes.

In the second interview, the researcher collected the participants' definitions of mutually exclusive events and independence. The researcher also asked the participants to consider several events in various sample spaces and determine whether pairs of events were mutually exclusive and/or independent. Participants were also invited to introduce their own events and sample spaces to elaborate points that came up during discussion. This was intended to provide the interviewer with insight not only into how the participants defined each of the two terms, but also how these terms were applied in various probabilistic contexts. The researcher could then distinguish between the participants' concept definitions (collected directly) and concept images (drawn from examples, phrasing, etc.).

The third interview was designed to provide a context wherein the participants could assert a mathematical relationship between independence and mutual exclusivity and then attempt to prove this relationship. The participants were asked to assert two main relationships: given that two events have nonzero probabilities in the same sample space, does independence imply mutual exclusivity and does mutual exclusivity imply independence? These questions were posed as two separate multiple-choice questions, as in Manage and Scariano (2010).

Data Analysis

The researcher analyzed each video and audio recording after each interview in order to determine the participants' proof schemes, identify major themes in the participants' reasoning, model participants' understanding of mutual exclusivity and independence, draw quotes from the dialogue to support such models, and develop individualized clarifying tasks for the subsequent interview. Throughout the video analysis, video clips were taken that supported or challenged working models of participant thinking. These videos were collectively re-analyzed in order to confirm or reevaluate a model. The researcher would then develop questions for the subsequent interview that would be used to help clarify conflicts within the model.

Results

For the sake of depth and limited length, we discuss only Alex and Betty here.

Alex

Throughout the first interview, Alex exhibited a predominantly Analytic proof scheme. Eventually he correctly supported all Deductive proofs and refuted all Empirical and External proofs, citing appropriate flaws in logic or reasoning. In a few instances, he showed signs of relying on a proof's form rather than content, signifying an occasional tendency toward a Ritual (External) proof scheme. Alex was also very pedantic about precise details, reflecting a skeptical point of view and checking for logical progression in each proof.

Alex displayed a deep understanding of examples and their use in proof. This was quickly evident in the first example in the matrix of proofs. This proof, applying an Inductive (Empirical) proof scheme, used an example of a large random number that exhibited the desired result. After reading the argument, Alex immediately said, "Yeah, this is bogus." He later refuted other inductive proofs very similar to the first. These examples highlight Alex's ability to refute proofs that inappropriately use examples.

Another interesting aspect of Alex's proof scheme is his emphasis on the axioms of real numbers and considering the space in which he was working. These qualities were evident in three instances. In the first two cases, Alex explicitly applied the axiom of the closure of integers under addition and multiplication. In the second case, Alex also invoked the associativity axiom for real numbers. In the third case, Alex suggested that the sum of the interior angles of a triangle might not be 180° in non-Euclidean space. While this could be a manifestation of the rigor required in his Introduction to Proofs course, it is evident from these examples that Alex had internalized a mindset that considers the system in which a proof is argued and its fundamental axioms. It should be noted that Alex's use of axioms in this interview reflects Harel and Sowder's (1998) Intuitive-Axiomatic proof scheme.

These qualities of Alex's proof scheme combine to support an initial emphasis on the form of a proof and then careful investigation of motivation and justification at each line of an argument. This emphasis was manifested in his pedantic discussions of the proof writer's justification, his use of axioms, and refutation of proofs by example. We see Alex used the form of a proof to make initial judgments, but his skepticism forced him to evaluate a proof based on its line-by-line merit. From this, we can conclude that Alex generally displays an Intuitive-Axiomatic (Analytical) proof scheme with tendencies toward a Ritual (External) proof scheme.

In the second interview Alex explored two sample spaces and discussed a few other examples that he used to help describe his understanding of mutual exclusivity and independence. As we will see, Alex displayed an extremely internalized and powerful conception of independence. Alex defined independence as, "[when] the outcome of one event does not affect the outcome of a subsequent event." This definition implies an emphasis on a sequence of events, where one of the events being considered must occur prior to the other.

With regard to mutual exclusivity, however, Alex was less certain of a formal definition—changing his definition twice throughout the interview until eventually declaring, "[Mutual exclusivity is when] performing an event or series of events causes a subsequent event to have zero probability of happening." Again, Alex uses the word "subsequent" in his definition, which implies that this relationship is defined over a period of time. It is important to note that Alex's initial definition of mutual exclusivity (consistent with the mathematical definition) was not defined over time, but rather instantaneously. It was not until he had considered examples in the two given sample spaces that he changed this definition to more closely resemble his definition of independence.

When prompted for an example of independent events, Alex gave two examples: a die and a coin. He stated that rolling a six on the first roll of a die does not affect rolling a six on the second roll of a die and gave an analogous explanation for the coin. These examples are consistent with his definition of independence, implying that the two events in consideration take place at separate times. His initial examples of mutually exclusive events exhibited what he described as "well-defined states" including raining versus not raining, sides of a die ("you can't roll both a 5 and a 1"), and a coin ("it's either heads or tails"). These examples support his original definition, which considers the two outcomes instantaneously in that it cannot both rain and not rain at the same time. Later in the interview, however (after changing his definition of mutual exclusivity), Alex described repeatedly drawing "any card" without replacement until all spades were exhausted. In this case, drawing "any card" and drawing a spade were mutually exclusive since drawing "any card" can eventually cause drawing a spade to have probability zero. This example seems much more convoluted than the first three examples, but supports Alex's newer definition of mutual exclusivity.

We can see that Alex's conception of independence was so strong that it not only influenced how he defined mutual exclusivity, but also caused him to reject three different examples and develop a new concept image for mutual exclusivity wherein one event must cause a subsequent event to be impossible. This new concept image was so strong that, when asked to reconcile this new definition with his original examples, Alex reneged on their mutual exclusivity (e.g., heads on a coin does not cause "not tails" later).

Equally intriguing is the fact that Alex independently asserted a corollary to his new definition of mutual exclusivity. In this corollary, Alex stated that if the two events are mutually exclusive, then they cannot be independent. This reflects the (almost) exact relationship outlined in Manage and Scariano (2010) and investigated in the third interview of this research. Alex used an explanation analogous to that described in Manage and Scariano (2010). He asserted that, since one event causes the second event to have zero probability, the first event changes the probability of the second event and therefore the two events are not independent. It should be noted, however, that Alex did not consider the case when the second event in the sequence already had zero probability.

In the third interview, Alex responded that if two events were mutually exclusive this implied that they were not independent. This claim was made using his final definition of mutual exclusivity. He directly referenced his own corollary from the second interview in which he made this exact assertion. Alex also claimed that if two events are independent then they are not mutually exclusive. He supports his answer choice by saying, "one event's not affecting the other event at all so, I mean, it's not going to cause it to have zero probability cause it's not changing the probability of the next event." As with the first question, this answer choice supports the relationship between the mathematical definitions of independence and mutual exclusivity for nonzero events.

Betty

Betty displayed a predominantly Analytic proof scheme with the exception that she accepted one proof based on its appearance and another proof based on its form. Betty correctly refuted the three examples of Inductive (Empirical) proofs, but accepted one Deductive proof because it "seem[ed] more mathematical." Her refutation of the inductive proofs shows her understanding of the importance of a general proof for all cases. Betty's acceptance of a proof based on its seeming mathematical qualities and acceptance of a false proof by the principle of mathematical induction, however, indicate a tendency toward External (Ritual) and Empirical (Perceptual) proof schemes.

Betty showed an insistence on understanding very specific aspects of a proof rather than drawing any assumptions about the proof's process with the exception of one case. She quickly accepted a proof by mathematical induction. Here, she was likely preoccupied with the form (or "look") of the proof, rather than its mathematical validity. This idea was supported when Betty stated that her class had recently discussed the principle of mathematical induction. When asked which of the first three proofs she preferred, Betty chose the last proof because the processes in the second proof were not obvious to her. This reflects a need to understand connections in a proof, even though this need was temporarily suspended in the case of mathematical induction. This need was also addressed later in the interview, when Betty described the process of verifying for herself relationships she felt she did not understand in class.

Throughout the rest of the first interview, Betty correctly refuted Empirical and External proofs and accepted Analytical proofs. She rejected the false proofs with little hesitance. At one point, Betty described combinations of negating the hypotheses statements of the Inductive proofs, showing a clear understanding of logical proof, counterexamples, and proof by contradiction. She also reflected an ability to identify false proof by example. These examples show a healthy skepticism of Authoritarian and Ritual proof, both of which are External proof schemes. Additionally, Betty's explanations in refuting Inductive proof schemes support an emphasis on proof for all cases.

When asked what it meant for two events in a sample space to be independent, Betty responded, "The intersection is zero. Is it? That's what I'm asking. I don't remember." Betty almost instantly changed this to, "Two events are independent if the probability of A occurring does not affect the probability of B occurring." Betty then described the independence of events A and B using the equation $P(A) = P(A|B)$. Neither of these representations necessarily implies a chronological distinction between events A and B (as was seen with Alex's use of the word "subsequent"). But, when prompted for an example of independent events, Betty described the act of picking a card from a deck of fifty-two cards and putting it back so that the probability of picking a second card is not affected. Similarly, when asked for an example of events not being independent, Betty provided the case of picking a card and not replacing it. These examples are consistent with a conception of independence in the context of a "with replacement" and "without replacement" conditioning event.

In contrast, Betty defined mutually exclusive events with the statement, "you can't have both at the same time." This definition explicitly states that the events can be compared instantaneously. Here, Betty gave the example that the choosing the queen of hearts and choosing the jack of diamonds are mutually exclusive because they cannot both occur when one card is drawn. We notice that this definition is consistent with the mathematical definition and that this example is consistent with Betty's definition. Betty did spend much more time defining mutually exclusive events compared to her definition of independence, but once she determined this definition, she held firm to its accuracy saying, "I'm sorted now." This reflects her need in first interview to prove relationships in order to understand them.

Betty's initial confusion of independent events as events that "don't happen at the same time" reflects the most common misconception in Manage and Scariano (2010). Although she quickly changed her mind about the definition of independence, this confusion was apparent in her use of mathematical notation to represent the ideas (discussed below). Also, when explaining her conditional notation of independence, Betty described two independent events as "completely separate," which one could argue is a descriptor more applicable to mutually exclusive events since their intersection is empty.

More than once, Betty wrote an equation involving probabilities saying, "That's just something I remember from probability." For instance, she initially used "$P(A \cap B)=0$" to represent independence and used the equation "$P(A \cap B)=P(A)*P(B)$" to define mutual exclusivity. These equations were quickly erased. The former, however, was eventually used to describe mutual exclusivity. For the latter, Betty admitted, "[I have] no idea where that came from or if that's even mutually exclusive. And I would not be able to come up with [it]." We notice that Betty's second description of independence, $P(A)=P(A|B)$, is true unless the probability of B is zero. In this case, the statement $P(A|B)$ makes no sense, although it could be adapted to say, "two events A and B are independent if both have nonzero probability, $P(A)=P(A|B)$, and $P(B)=P(B|A)$."

In this interview, we see that Betty's concept definitions, though initially inconsistent, are each strongly internalized when evaluating the independence and mutual exclusivity of specific events in specific sample spaces. This is evident because once Betty defined each term, she was "sorted" on how to verify them and seemed to develop quick checking schema in order to do this (e.g., "Can these happen at the same time?"). Her spoken reasoning for two events' independence and mutual exclusivity reflected these quick checks.

In response to each of the two questions in the third interview, Betty concluded that there was not enough information about the sample space and that two mutually exclusive events can be both independent and not independent. This led her to respond that there was not enough information about the sample space or the context of selecting events in the sample space to determine a relationship. She explained that in the previous interview she had seen mutually exclusive events that were both independent and not independent (a copy of her responses from the second interview was presented to her). She also explained that she saw independent events that were both mutually exclusive and not mutually exclusive in the second sample space. From this, Betty reasoned that more information was needed about both the sample space and the actions taken between the occurrence of the first event and second (e.g., replacement, non-replacement). Again, we see independence is affected by the context in which the events take place.

Betty's proof scheme showed that she is more inclined to want to verify mathematical relationships on her own. This was evident as she "sorted" herself about the definitions of independence and mutual exclusivity. During this process, Betty successfully reconciled her definitions of the terms with symbolic representations (about which she was admittedly unsure) that she had recalled from her statistics course. Betty used these definitions to investigate the sample spaces in the second interview, the results of which had a direct affect on her reasoning in the third interview. Because Betty's definition of independence relied so heavily on the sample space and whether replacement occurred, she had examples of all different combinations of independence and mutual exclusivity.

Conclusion

We see that proof schemes can be both restricted and enhanced by students' definitions of the mathematical ideas they consider. Though her reasoning was logically based on her previous experiences in the samples spaces, Betty's conception of independence and mutual exclusivity caused her to require more information about the sample spaces in question, in turn restricting her ability to draw conclusions between the two concepts. On the other hand, Alex's ability to adapt his concept image and concept definition of mutual exclusivity allowed him to logically conclude both directions of the relationship between mutual exclusivity and independence, however correct or incorrect his definition may have been.

In his proof, Alex claimed from his concept definition of mutual exclusivity that each mutually exclusive event would cause the other to have zero probability. This would make the two events "not independent" since his definition of independence necessitated each event to "not affect a subsequent event." Using similar reasoning, Alex concluded that independence implied "not mutual exclusivity." It should be noted however that, despite Alex's focus on "proof for every case" in the first interview, he failed to assert a relationship for the case when one or both events were given to have zero probabilities. The contrast between his assertions about proof and his actions in proving this relationship reflects the "pathological" nature of zero probability cases pointed out by Kelly and Zwiers (1986). Interestingly, this also points to a characteristic of his definitions that may have influenced his thought process: an event with zero probability cannot "happen first" and therefore can neither cause nor affect any other event, as the definitions require.

Betty's was unable to logically assert any certain relationship between the two concepts. This resulted from such strong concept images of independence and mutual exclusivity. More specifically, Betty's personal experiences in the sample spaces allowed her to provide counterexamples to any explicit relationship between the two concepts. Since specific characteristics of sample spaces affected two events' independence, she required information about a sample space in order to make inferences about the events in question. This prevented Betty from generalizing to all cases an explicit relationship between mutual exclusivity and independence, which her proof scheme required.

Recalling the Alex and Betty's general proof schemes (mostly Analytical with slight Empirical and External tendencies), we consider how these related to their use of definition. Alex's dynamic concept image and unsolicited production of the lemma for the definition of mutual exclusivity reflect an Analytical frame of mind that is also geared toward finding and asserting relationships between the two concepts. We see with Betty, however, that a mostly Analytical proof scheme alone is not sufficient to connect the relationships between mutual exclusivity and independence. This is because her conceptions of the two ideas were so powerful that she was comfortable using the four cases from her exploration to show that no relationship existed. From these two cases, we see that little inference can be made about how a student uses definition relative to Harel and Sowder's (1998) proof schemes.

But we can also consider these cases with respect to Weber and Alcock's (2004) semantic and syntactic proof productions. Because he produced it immediately after changing his concept definition of mutual exclusivity to more closely resemble his concept definition of independence, we see that Alex's lemma (and therefore responses in the third interview) was a direct result of his comparing the two concept definitions. A syntactic approach to the relationship was not fruitful, however, until he changed his definition. Conversely, Betty's use of previous instantiations (a semantic approach) prevented a definite relationship between the concepts from forming. It is unclear, though, whether Betty even thought her concept definitions might need to be changed. From this, we see some indication that a syntactic approach may play some role in aiding the adaptability of definition and that a semantic approach could be more restrictive.

From this research, we have seen how the adaptability of a student's concept image allows him or her to compare seemingly disparate concepts in new contexts. Here, the phrase "seemingly disparate" reflects the understanding of the concepts from the students' initial points of view. This action reflects Vinner's "interplay between definition and image," but is different in that the participants were not comparing a definition and image of a single mathematical concept, but rather two different but related images (1991, p. 70). This interplay is not addressed in his work, but yields a result similar to that of Vinner's interplay where an adaptation of image allows one to make sense of a perceived relationship. In this case, the adaptation of two images allowed a relationship to be perceived. Conversely, in Betty's case, rigidity restricted her perception of a relationship between independence and mutual exclusivity.

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HOW STUDENTS REASON DIFFERENTLY IN EVERYDAY AND MATHEMATICAL CONTEXTS: TYPICALITY AND EXAMPLE CHOICE IN JUSTIFICATION

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Middle school students bring with them to the classroom powerful, informal resources for reasoning about mathematical ideas. However, little research has examined how these resources can interact with or support skills of mathematical justification. Here, we explore how middle school students consider inductive strategies—the use of examples in proof—when confronted with conjectures. We discuss ways in which these students might reason about mathematical objects like numbers and shapes strategically as they test examples. We argue that critical to such strategic reasoning is flexible application of mathematical and everyday ways of knowing.

Keywords: Reasoning and Proof; Middle School Education

Students bring with them to the mathematics classroom powerful intuitive ways of reasoning based on their everyday experiences interacting with the world. An important goal in mathematics education has been to find ways to leverage these resources or "funds of knowledge" (Moll & Gonzalez, 1997) to support mathematics learning. However, this search has proven to be problematic. Students not only have trouble applying their "school math" knowledge to complex, situated real world problems (e.g., Masingila, Davidenko, & Prus-Wisniowska, 1996; Walkington, Sherman, & Petrosino, 2012), but they struggle to productively use knowledge from their everyday experiences in school-based tasks (e.g., Reusser & Stebler, 1997; Walkington, Nathan, Wolfgram, Alibali, & Srisurichan, in press). A situated perspective on learning acknowledges that the interplay between the practices valued in school and everyday activity is complex, and that these two sets of practices will not always overlap. However, recent work has uncovered ways for students' concrete, situated experiences to support the learning of mathematical formalisms (e.g., Walkington & Sherman, 2012).

One area in which the interaction between everyday experience and formal mathematical knowledge has not been well-examined is mathematical justification. Here, we define justification or proving as "the process employed by an individual to remove or create doubts about the truth of an observation," and emphasize that this process is often based on intuition, internal conviction, and necessity (Harel & Sowder, 1998, p. 243). The importance of the construction and evaluation of mathematical arguments is accentuated in both the Common Core and NCTM standards (CCSSI, 2010; NCTM, 2000). But how are children's intuitive ways of reasoning important when considering mathematical justification, which has been traditionally characterized as a formal and disembodied chain of axiomatic, deductive statements?

Recent research has revealed the *inductive* or *example-based* reasoning strategies that children use when considering mathematical conjectures (Knuth et al., 2011). For instance, when presented with the conjecture "the sum of any two even numbers is even," a student might test different even numbers, like 2 and 20. Some studies have suggested that this kind of reasoning might allow students develop more general arguments (Knuth et al., 2011). Here we will examine how students' everyday and mathematical knowledge interacts with their evaluation of example-based justifications. We argue that students must navigate along a learning continuum as they gain expertise with mathematical arguments, which ultimately leads to flexible and appropriate application of everyday and mathematical knowledge. Gaining an understanding of this continuum, of the ways in which students think about the nature of evidence in inductive justification, may help mathematics educators in better supporting students' learning to prove.

Theoretical Framework

Use of Example Objects in Justification

Many of the problems people face in life resist formal solution. There is no deductive proof for beliefs about friends, nor a valid algorithm for picking a spouse. Instead people must employ inductive reasoning strategies. Some of the most well studied inductive strategies in the cognitive science literature are example-based (see Feeney & Heit, 2007). One way to decide if a person will be a good friend is to compare them to others. But which others? Children and adults employ a number of strategies for selecting good examples in their everyday lives, strategies that often are in line with formal principles of inductive inference.

In mathematics, students also tend to use inductive reasoning when confronted with conjectures (Chazan, 1993; Knuth et al., 2011; Harel & Sowder, 1998). Such reasoning has sometimes been identified as problematic because students may use *only* examples, without moving towards more powerful general arguments. However, examples may still play a critical role in understanding conjectures and constructing more general justifications. For instance, mathematicians use examples as tools when confronted with conjectures (Alcock, 2004). Expert mathematicians (N = 133) indicated they use examples to verify and understand conjectures, generalize from examples to a proof, and seek counter-examples or try to "break" the conjecture (Lockwood et al., 2012). Examples play an important role in the development of proofs.

Typicality and Example Choice in Non-Mathematical and Mathematical Domains

In scientific domains, three principles of example selection (see Osherson et al., 1990) have been identified as useful when drawing conclusions about a class or type: *quantity*—more examples are better than fewer, *diversity*—a wide variety of examples are better than a set of very similar examples, and *typicality*—generic or "average" examples are better than special or "weird" examples. Thus in trying to decide whether all birds have hollow bones, one would want to check many birds, a diverse set of birds, and relatively typical birds. Here we focus on typicality—a typical example shares properties with many members of its class and has few distinctive properties. One challenge in developing accounts of example-based inference is identifying which features are used to compute typicality. In science, people have robust intuitions about features that are "biologically relevant." That cats and goldfish are both kept as pets does not seem relevant in determining their biological relatedness. However, untangling everyday notions of typicality from typicality based on properties of *mathematical* objects may be more difficult.

In previous work, we found it useful to distinguish two types of mathematical typicality (Williams et al., 2011). The *everyday typicality* of an object is how common it is in everyday life—i.e., how many experiences a person has with objects of that kind in their day-to-day activity. The *mathematical typicality* of an object is how typical it is when its mathematical properties are considered in relation to the properties of *all* objects of that type. The number "0" would be a mathematically *atypical* number because it has properties that no other integers share (e.g., additive identity). The number 322 might be a *typical* number in a mathematical context because it does not have many properties that make it distinct from the set of whole numbers. Middle school mathematics is an interesting site for exploring these two types of typicality, as many of the objects that are highly *atypical* mathematically (e.g., numbers like 0 or 1) are highly *typical* in everyday life. Students may struggle to reconcile these two different conceptions of typicality. But do typicality judgments really matter when considering mathematical justifications?

When the expert mathematicians (N = 133) were asked how they choose examples when exploring conjectures, many responses referenced the typicality of their examples. They reported choosing common examples with no special properties or generic or general examples, unusual, obscure, or "tricky" examples, examples with special properties, and examples that are boundary cases (Lockwood et al., 2012). These mathematicians seemed to have found ways to use typicality *strategically*—to allow typicality judgments to support and inform their exploration of mathematical conjectures. But what about middle school students? Do they consider typicality when exploring conjectures with examples, and if so, what type of typicality?

We presented middle school students (N = 20) with conjectures about numbers, and students reported purposefully varying the typicality of the examples they chose when testing conjectures. Students reported trying to test both typical and atypical numbers, or trying to test unusual numbers to see if the conjecture would hold (Cooper et al., 2011). Students' reports of what made a number typical varied—some were attuned to whether the number was prime or the relative size of the number, while others identified typical numbers based on their everyday experiences. Overall, it seemed that students were reasoning strategically about the typicality of their examples. In the present study, we implement a large-scale survey to assess how students use mathematical and everyday typicality when considering examples in justification.

Research Questions

Our research questions are: (1) How do middle school students use typicality *strategically* when considering examples? and (2) How are students' conceptions of mathematical typicality consistent or at odds with their everyday notions of typicality?

There are two dimensions along which middle school students might demonstrate using mathematical typicality strategically. First, students might realize that conjectures that hold for mathematically atypical objects (i.e., objects with mathematically special properties) may not hold for all objects. For instance, a conjecture holding for the number "1" may not be strong evidence that the conjecture would hold for all numbers, since 1 has special properties (e.g., multiplicative identity). However, this conception of mathematical typicality might be directly at odds with students' everyday notions of typicality, because although 1 is highly atypical in a mathematical context, it is highly typical in students' everyday life. Thus if typicality is used strategically, we might see a reversal. Students may recognize that a number like 1 is highly *atypical* in a mathematical context, despite being highly *typical* in an everyday context.

Second, students might use mathematical typicality strategically if they realize that superficial features of a mathematical object are not particularly important when considering whether conjectures that hold for that object will hold for most objects. Students might realize that when a parallelogram is in a non-standard orientation, this is unlikely to impact most mathematical conjectures in middle school mathematics. Similarly, a student might realize that the relative size of a number (e.g., 3 or 103) or the cultural significance of a number (e.g., 13) might not be particularly important. This strategic use of mathematical typicality may be at odds with everyday notions of typicality—in daily life, students are accustomed to seeing shapes in standard orientation and working with relatively small numbers, so objects that do not conform to these experiences might be considered atypical. Thus we argue that strategic use of typicality requires students to flexibly switch between their "everyday" and "mathematical" lenses.

Methods

A total of 475 middle school students (46% female) from a suburban middle school in a Midwestern state were included in the study. Students were distributed across grades 6 (144 students), 7 (160 students), and 8 (163 students), and mathematics classes used reform texts. The school demographics were 48% Caucasian, 21% African American, 14% Asian, 11% Hispanic, and 1% Native American, with 37% low income, and 10% English Language Leaners (ELL).

A survey was administered to all participants during their normal math classes. Each survey contained questions relating to two of four different domains: numbers, parallelograms, triangles, and birds (birds are omitted here). For each domain, students were presented with mathematical objects or items in that domain (e.g., a small equilateral triangle or the number "6") and asked to rate each item's typicality on a 1–7 scale in a mathematical context and in an everyday context. Figure 1 gives an example of the instructions students received on the survey (left) and actual survey items (right). Mathematical objects were selected by the researchers to either cover the space of possible mathematical properties in the domain (e.g., the parallelogram in Figure 1 is a rectangle; we also included squares, rhombi, etc.), or to be completely devoid of any property that would distinguish the object mathematically (e.g., a long, skinny rhomboid with no 90 degree angles). The order of the 9 items within each context and the 2 domains was randomized.

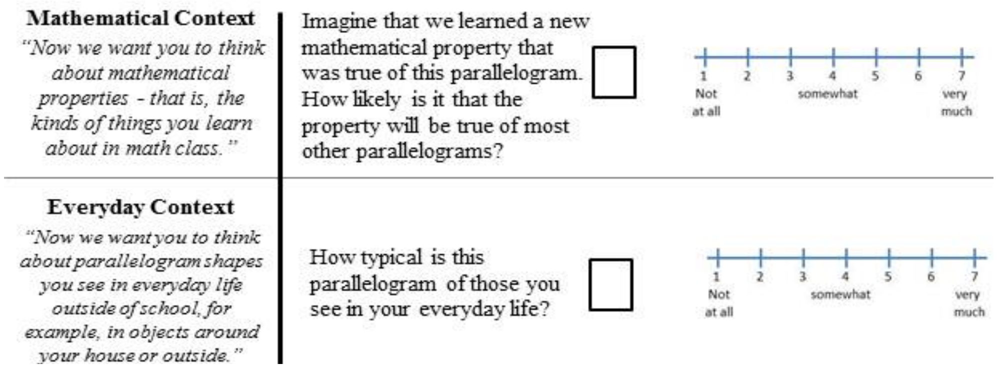

Figure 1: Example of questions on middle school survey

Table 1: Mathematical (italics) and Everyday (underline) Properties Entered into Model

Numbers	Parallelograms	Triangles
Prime or perfect square *Power of 2 or 10* *Multiple of 5 or 10* *Identity properties (i.e., 0 or 1)* Relative magnitude (small or large)	*Square, rectangle, or rhombus* Size (small, large, "skinny") Orientation (standard, non-standard, left-leaning)	*Isosceles, equilateral, scalene* *Obtuse, acute, right* Size (small, large, "skinny") Orientation (standard, non-standard)

The data were analyzed using hierarchical linear regression models (Snijders & Bosker, 1999) where repeated observations were nested within students nested within teachers. Three different models, one per domain, were fit to the data. Random effects included student, teacher, and which mathematical object (i.e., which specific number or shape) the question referenced. Fixed effects included context (Figure 1), the mathematical and everyday properties of the object (Table 1), and the interaction of these two terms. Properties that did not have significant main effects or interactions with context were removed. Fixed effects for gender and grade were not significant in any of the models. Mathematical and everyday properties of numbers, triangles, and parallelograms entered into the model are in Table 1. These properties were chosen based on the mathematical knowledge of a team of mathematicians, psychologists, and mathematics educators and former K–12 teachers, as well as based on previous results from our studies of inductive reasoning (Knuth et al., 2011; Cooper et al., 2011).

Results and Discussion

Number

As can be seen from Table 2, across mathematical and everyday contexts, students rated small numbers ($p < .001$), numbers ending in 5 ($p = .020$), and powers of 10 as being more typical ($p < .001$). This suggests two ways in which students might *not* be considering mathematical typicality strategically. First, students seemed to believe that conjectures that hold for mathematically-special numbers, like powers of 10 or multiples of 5, would be *more* likely to hold for other numbers. From a mathematical standpoint, properties that hold for these numbers may be *less* likely to hold for other numbers. Second, students rated that conjectures that held for small numbers were *more* likely to hold for other numbers. Here, students may have been considering a superficial or mathematically irrelevant feature when considering mathematical conjectures. In both cases, students' everyday notions of typicality, their familiarity encountering small numbers, multiples of 5, and powers of 10 in their lives, may have influenced their mathematical notions of typicality—whether it makes sense for properties that hold for these numbers to hold for most other numbers. We also see no evidence of the desired reversal for mathematical typicality that might evidence strategic thinking. Students did not indicate that numbers with special properties—like prime numbers—were atypical in a mathematical context.

Table 2: HLM Analysis of Students' Typicality Ratings for Number

	Estimate	**S.E.**	***t***	***p***	Sig.
(Intercept)	3.61	0.46	7.70	< .001	***
Mathematical Context	(ref.)				
Everyday Context	0.19	0.17	1.10	0.274	
Large	(ref.)				
Small	0.77	0.17	4.42	< .001	***
Ends with 5	0.68	0.27	2.55	0.020	*
Power of 10	1.32	0.25	5.39	< .001	***
Prime	-0.17	0.22	-0.76	0.445	
Identity (0 or 1)	-0.37	0.35	-1.05	0.298	***
Everyday Context: Small	0.59	0.09	6.67	< .001	***
Everyday Context: Prime	0.33	0.11	3.07	0.001	**
Everyday Context: Identity	0.55	0.16	3.44	< .001	***

* $p < .05$. ** $p < .01$. *** $p < .001$.

However, looking at the interaction terms in Table 2, we do see evidence that students are at times using mathematical typicality strategically. First, although students rated small numbers as typical regardless of the context, being small had a larger impact on typicality in an everyday context ($p < .001$). This suggests that students may realize that superficial characteristics, like relative size, are less important when considering a number mathematically. Second, students found both prime and the identity numbers more typical in an everyday context ($p = .001$ and $p < .001$). Thus although students expressed their familiarity with these numbers by giving them high everyday typicality ratings, this familiarity did not inflate mathematical typicality ratings.

Parallelograms

Across mathematical and everyday contexts, students rated squares as being more typical (Table 3; $p = .015$). This again suggests that students might not be considering mathematical typicality strategically—these ratings suggest that properties that hold for squares are more likely to hold for other parallelograms. Students' everyday familiarity with squares might be interfering with viewing a square as a mathematical object that has special properties (e.g., 90° angles). We again do not see evidence of the desired reversal—students do not rate mathematically special parallelograms (like squares) as being less typical in a mathematical context.

Table 3: HLM Analysis of Students' Typicality Ratings for Parallelograms

	Estimate	**S.E.**	***t***	***p***	Sig.
(Intercept)	2.88	0.61	4.7	< .001	***
Mathematical Context	(ref.)				
Everyday Context	0.39	0.27	1.41	0.165	
Standard Orientation	0.43	0.24	1.78	0.080	
Large	(ref.)				
Small	0.19	0.55	0.34	0.710	
Leans Left	-0.51	0.34	-1.52	0.134	
Square	0.88	0.34	2.58	0.015	*
Rectangle	0.63	0.32	1.99	0.055	
Rhombus	0.57	0.40	1.44	0.154	
Everyday Context: Standard Orientation	0.49	0.11	4.49	< .001	***
Everyday Context: Small	-1.04	0.25	-4.20	< .001	***
Everyday Context: Leans Left	0.52	0.15	3.48	< .001	***
Everyday Context: Square	0.55	0.15	3.64	< .001	***
Everyday Context: Rectangle	1.31	0.14	9.30	< .001	***
Everyday Context: Rhombus	0.56	0.18	3.14	0.001	**

* $p < .05$. ** $p < .01$. *** $p < .001$.

However, looking at the interaction terms, we see considerable evidence that students can use mathematical typicality strategically. Although students rated squares as being typical regardless of the context, squares were considered even more typical in an everyday context ($p < .001$). Similarly, students rated rectangles and rhombi as more typical in an everyday context ($p < .001$ and $p = .001$). Students seemed to recognize that although these shapes were common in their everyday lives, this consideration should not inflate their ratings when determining whether properties that hold for these shapes will hold for other shapes. Students also allowed superficial properties of parallelograms—like size and orientation—to influence their everyday typicality ratings ($p < .001$), but not their mathematical typicality ratings.

Triangles

Across mathematical and everyday contexts, students found equilateral, isosceles, and standard orientation triangles more typical (Table 4; $p = .037$, $p = .004$, $p < .001$, respectively) and skinny triangles less typical ($p = .002$). This suggests that students may not be using mathematical typicality strategically. Students expressed that conjectures that hold for special triangles like isosceles and equilateral triangles are more likely to hold in general, and that conjectures that hold for skinny or non-standard orientation triangles, superficial considerations, are less likely to hold in general. Here, again, students do not seem to be differentiating between everyday typicality (the commonness of equilateral and isosceles triangles in their everyday life, and the rarity of skinny and non-standard orientation triangles) and mathematical typicality (whether conjectures that hold for certain triangles are likely to hold for other triangles). We also see no evidence of the desired reversal—students did not rate mathematically special triangles as atypical in a mathematical context. However, looking at the interaction terms, students seem to sometimes reason strategically about mathematical typicality. Although students rated equilateral triangles as typical regardless of context, they were even more typical in an everyday context ($p = .002$). Further, right triangles were typical in an everyday context ($p = .004$), but students did not let everyday familiarity inflate ratings in a mathematical context. Students may realize that although these triangles are highly salient in their everyday experiences, this familiarity should not affect whether conjectures that hold for these triangles will hold for other triangles.

Table 4: HLM Analysis of Students' Typicality Ratings for Triangles

	Estimate	S.E.	*t*	*p*	Sig.
(Intercept)	4.98	0.41	12.16	< .001	***
Mathematical Context	(ref.)				
Everyday Context	0.15	0.20	0.75	0.457	
Skinny	-0.74	0.17	-4.28	0.002	**
Isosceles	0.63	0.17	3.64	0.004	**
Equilateral	0.75	0.33	2.28	0.037	*
Acute	(ref.)				
Obtuse	-0.049	0.19	-0.25	0.794	
Right	0.20	0.27	0.75	0.453	
Standard Orientation	0.59	0.18	3.19	0.009	**
Everyday Context: Equilateral	0.68	0.22	3.19	0.002	**
Everyday Context: Obtuse	-0.16	0.10	-1.56	0.124	
Everyday Context: Right	0.46	0.16	2.91	0.004	**

* $p < .05$. ** $p < .01$. *** $p < .001$.

Summary and Conclusions

We examined whether middle school students use typicality *strategically* when considering conjectures, and found mixed results. When numbers or shapes had special mathematical properties, students considered them *more* typical in a mathematical context. However, properties that hold for these special objects should be *less* likely to hold for other objects. In other cases, superficial characteristics impacted whether students thought that conjectures that held for an object would hold for other objects. Both behaviors suggest that students might be conflating everyday typicality with mathematical typicality. Despite these results, students did sometimes distinguish mathematical and everyday contexts; they appropriately recognized the relevance of mathematically special and surface-level properties in each domain. This suggests that students have important resources for using typicality strategically, and for differentiating how objects should be considered in the math classroom and everyday life. But are these behaviors really characteristic of mathematical expertise? We recently presented the survey to 339 mathematicians. Initial analyses suggest that mathematicians do use typicality strategically in the ways we predicted, and they recognize everyday and mathematical typicality as two distinct entities that are often in opposition. This stands in contrast to how middle school students considered typicality, as they had difficulty reconciling mathematical and everyday contexts.

Our results suggest that students must negotiate an important learning continuum regarding mathematical conjectures. Initially, students appear to have difficulty reconciling their mathematical experiences with numbers and shapes with their concrete, salient everyday experiences. However, expertise in mathematics is characterized by flexible application of formal mathematical knowledge and everyday experience, based on the features of the problem and the social context. Thus students should be encouraged to critically reflect on how mathematical objects like numbers and shapes are considered differently in the mathematics classroom when exploring conjectures, compared to interacting with these objects in day-to-day life. Our work suggests that mathematicians are able to move flexibly between each of these two viewpoints, and use both examples and typicality judgments as resources in their work. Strategic use of examples and considerations of typicality may thus be important in helping students think more critically about the nature of mathematical evidence and in moving students towards making important generalizations about why mathematical conjectures hold, both of which ultimately could support deductive reasoning and formal mathematical proof.

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INVISIBLE PROOF: THE ROLE OF GESTURES AND ACTION IN PROOF

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The role of reasoning and proof in mathematics is undeniably crucial, and yet research in mathematics education has repeatedly indicated that students struggle with proof production. Our research shows that proof activities can be illuminated by considering action and gesture as a modality for crucial aspects of mathematical communication. We share two examples that highlight the importance of gesture and action in supporting students' mathematical proof production. We conclude by discussing the implications of our work for already existing schemes for classifying proof.

Keywords: Geometry and Geometrical and Spatial Thinking; Reasoning and Proof; Instructional Activities and Practices

Research in mathematics education has consistently shown that students, as well as pre-service and in-service teachers, struggle with constructing, interpreting, and evaluating proofs (Knuth, 2002; Healy & Hoyles, 2000; Chazan, 1993). Such research is deeply concerning, as proof is "an essential component of doing, communicating, and recording mathematics" (Schoenfeld, 1994, p. 74). Yet these results are perhaps unsurprising, as only recently has mathematics education begun to emphasize proof in the curriculum (e.g., National Council of Teachers of Mathematics [NCTM], 1989, 2000). NCTM recommends that proof and reasoning be taught from prekindergarten to 12th grade. The reasoning and proof Process Standards most relevant to our research include: the investigation of mathematical conjectures; the development and evaluation of mathematical arguments and proofs; and the use of "various types of reasoning and methods of proof" (NCTM, 2000, p. 322). Building upon NCTM's work, the new but widely-adopted *Common Core State Standards for Mathematics* identify constructing viable arguments and critiquing the reasoning of others as critical skills for students to learn across grade levels, while the standards for high school geometry specifically call for students to construct mathematical proofs of theorems (Common Core State Standards Initiative, 2010).

As the field continues to struggle with teaching proof, it is worth considering alternative forms of support for students. Healy and Hoyles (2000) found that correct proof production is easier for students when they can engage in building narrative forms, rather than algebraic ones. Various authors have examined the role of digital geometry environments in supporting the development of proof (Chazan, 1993; Hoyles & Jones, 1998). Here, we take a novel approach by looking to theories of embodied cognition, examining alternative methods of supporting proof production and communication through the modalities of action and gesture. We place ourselves on PME-NA's learning continuum, as we are using an innovative method of examining ***student learning*** during proof production. This paper makes no claims as to the newness of action and gesture in supporting mathematical learning and communication—in fact, it is the ubiquitous yet overlooked role of action and gesture to which we wish to draw attention. Consequently, we contribute a new lens in order to reveal heretofore invisible proofs.

In this paper, we begin by briefly defining the practice of mathematical proof, and then explore relevant research on embodied cognition, action, simulated action, and gesture. We then share two excerpts of students using gesture and action to support proof from a recent study we conducted. In these excerpts and others, we find that the practice of proof is greatly enriched by considering both verbal and physical modalities. Finally, we share our future plans for this research, and examine the potential implications for teaching proof.

Theoretical Framework

Mathematical Proof and Justification

We conceptualize mathematical justification using Harel and Sowder's (1998) proof scheme, and our intended "mode of thought" (e.g., the modality of mathematical observation and reflection; p. 240) is body-based action and gesture. Harel and Sowder define *proving* as "the process employed by an individual to remove or create doubts about the truth of an observation" (p. 241). They further identify two subprocesses of proving: ascertaining (the proof activities an individual engages in when attempting to convince themselves); and persuading (the proof activities an individual engages in when attempting to convince others). As proof occurs in a social context where the argument must be communicated to an audience effectively and convincingly, we argue that each subprocess is essential when considering the learning of proof.

Harel and Sowder's (1998) proof scheme includes multiple categories and levels for classifying mathematical proofs. For our purposes, we focus on the *analytical>transformational* proof scheme, which involves "operations on objects and anticipations of the operations' results" (p. 259). In particular, when students are utilizing the analytical>transformational proof scheme, they are *transforming* a mathematical object or concept by varying some relationships purposefully in anticipation of certain results, observing the resulting changes, and deducing mathematical properties accordingly. Although this is a powerful and effective method of proving for students to learn, until now little research has examined how gesture and body-based action can play a role in supporting these dynamic transformations.

Gesture and Action

Theories of embodied cognition suggest that cognitive processes are not algorithms acting upon amodal mental systems, but rather they are bound up with the action and perception systems of the thinker (Barsalou, 1999; Barsalou, 2008; Glenberg & Robertson, 2000; Wilson, 2002). These action and perception systems, in turn, are not only guided by cognitive processes, but they also constitute and transform those processes. In other words, gestures and actions are not simply byproducts of cognition—they are coupled to cognitive processes (Shapiro, 2011) and they influence cognition. For example, gesture accompanied by speech may elaborate upon the thoughts possessed by the speaker (contributing additional information not contained by the speech acts), as well as feed back into processes that transform the speaker's cognition (Alibali & Kita, 2010; Goldin-Meadow & Beilock, 2010; Nathan & Johnson, 2012).

Gestures are a particular form of action that represent the world, rather than acting upon the world directly (Goldin-Meadow & Beilock, 2010). Furthermore, gestures are more than mere movements; as McNeill (1992) says, they "can never be fully explained in purely kinesthetic terms" (p. 105). Gesture is tied tightly to action, in our view, following Hostetter and Alibali's (2008) conceptualization of gestures as "manifestations of the simulated actions and perceptions that underlie thinking" (p. 508). Gestures are symbols that serve both to communicate and to affect the gesturer's cognition. Whether participants' gestures are produced as communicative or cognitive acts may appear to be a crucial distinction that we are in need of making. However, Hostetter and Alibali (2008) determine such a distinction to be somewhat false:

> … gestures are a natural by-product of the cognitive processes that underlie speaking, and it is difficult to consider the two separately because both are expressions of the same simulation…. [G]esture and speech may express different aspects of that simulation … but they derive from a single simulation; thus, they are part of the same system. (p. 508)

Consequently, we use verbal and gestural data side by side in our analyses for the purpose of triangulating on participants' cognition. By considering multiple modalities in this fashion, we are able to gain access to elements of ascertaining (convincing oneself) and persuading (convincing others) proof activities that would otherwise remain hidden in plain sight.

Gesture, Action, and Mathematics

Learners' gestures and actions have been found to support mathematics learning in many previous studies (e.g., Glenberg et al., 2007; Nathan, Kintsch, & Young, 1992; Alibali & Nathan, 2012; Alibali & Goldin-Meadow, 1993), and are "involved not only in processing old ideas, but also in creating new ones" (Goldin-Meadow, Cook, & Mitchell, 2009, p. 271). In our research, we build upon this prior work while venturing into new territory: the role of action and gesture in supporting proof production. In the following section, we discuss gesture and mathematics in a general fashion, and draw out some threads that are particularly relevant to the domain of proof.

In some cases, gesture or body-based action may allow students to manipulate conceptual objects in a fashion similar to dynamic geometry software. In these systems, students can build objects that maintain invariant relationships even as the object is manipulated and acted upon (e.g., when a single vertex is moved on a triangle, the connected sides will stretch to meet the new location of the vertex, always keeping a triangular shape) (Hoyles & Jones, 1998). These environments can support students in generating and verifying conjectures about the relationships contained within these objects. Similarly, the real-world context within which action and gesture are produced can give feedback about the legitimacy of the constructs evoked by the mathematical conjectures. Whereas with paper and pen impossible triangles can be constructed (e.g., a triangle where the hypotenuse is labeled as longer than the sum of the remaining two sides; see Table 1 for the relevant task), using one's body to construct the triangle can constrain a student from expressing such an impossibility.

One of the features of dynamic geometry software is that it affords the possibility of testing a large number of examples while maintaining the relevant invariant features. Gesture and simulated action may have a similar affordance. Hostetter and Alibali (2008) note that, "Because mental images retain the spatial, physical, and kinesthetic properties of the events they represent, they are dependent on the same relationships between perceptual and motor processes that are involved in interacting with physical objects in the environment" (p. 499). In other words, mental simulation, as evidenced through action and gesture, also allows such testing. We hypothesize that action and gesture can influence cognition in a way that is soundly based upon the physical experience of space within the world. In this way, action and gesture may support analytical>transformational proof production.

Given the potential influence of action, gesture, and simulated action upon cognition, we designed an experiment to examine the role of action and gesture upon mathematical proof production and communication. In the following section, we share our methodology and mathematical conjecture tasks.

Methods

The data reported in this paper were drawn from a larger study of the role of action and gesture in proof. Participants were 36 students (22 F; 14 M) at a large Midwestern university enrolled in a psychology course, and they received partial class credit for participating. The average age of our participants was 20 years old (15 freshmen; 9 sophomores; 7 juniors; 4 seniors; 1 part-time student).

In one-on-one interviews, participants were asked to justify a variety of mathematical theorems from number theory and geometry; here we report on the two conjectures shown in Table 1. Our research question was: *How are gestures and actions used in the ascertaining and persuading phases of proof?* In the larger study (not reported here), we encouraged participants to produce particular sorts of gestures for some conjectures. However the analysis here focuses only on cases in which students spontaneously gestured and used action when proving (i.e., their behavior was not manipulated).

Table 1: Conjectures

Conjecture 1: Gears	Conjecture 2: Triangle Inequality
An unknown number of gears are connected in a chain. You know which direction the first gear turns. How can you determine what direction the last gear will turn? Provide a justification for your answer.	*Mary came up with the following conjecture:"For any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side." Provide a conceptual justification as to why Mary's conjecture is true or false.*

We intentionally created a context in which participants would be likely to produce gestures while reasoning mathematically by setting up the discourse context (Hostetter & Alibali, 2008) in various ways. First, we emphasized the talk-aloud nature of the experiment to all of our participants, and we verbally prompted them to reason out loud if they fell silent. For some, we removed pen and paper; and for others we also removed the chair so that they (and the interviewer) had to remain standing. Furthermore, by designing the protocol to give no feedback to the participants, we emphasized *innovative* proof instead of *imitative*, and shifted the mode of thought to mental and physical by lessening the availability of pen and paper.

Each participant was videotaped, with the camera(s) capturing both small and large gestures. The data were analyzed in Transana, a qualitative analysis software program, and team members selected various gesture segments with the units of analysis based on each mathematical conjecture. In particular, we used *multimodal analysis* techniques (Alibali & Nathan, 2012; McNeill, 1992), and coded how each student used action and gesture during the ascertaining and persuading phases of their proof (Harel & Sowder, 1998).

Results

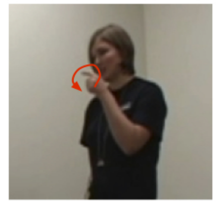
Our first example shows how simulated action (manifested through gesture) can illuminate the ascertaining phase of proof. The second example demonstrates the role of paired gesture and speech during the persuading phase. Our presentation we will provide additional examples.

Simulated Action Illuminates Ascertaining

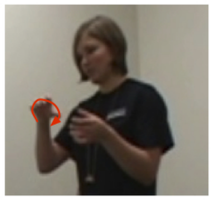
To illustrate the illumination of proof by simulated action, we share an excerpt from the Gears conjecture (Table 1), as the participant leverages her body as a tool for simulating the actions of the gears and identifying parity (shown in Figure 1). The excerpt contains both ascertaining and persuading phases of the proof, and is annotated as such.

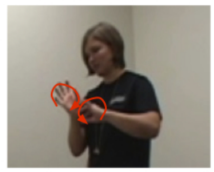
1 **Well, if you know which direction the first gear is turning, then you also know that the second gear must to be turning in the opposite direction,**

(Ascertaining) Uses her hands to replicate the turning of a gear



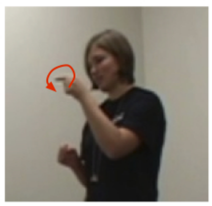
2 **Because one gear turning would push the other gear...**

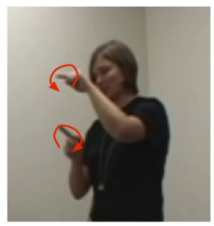




(Ascertaining) Uses both her hands to simulate the movement of two gears turning against each other

3 **See if one gear was turning this way, then the spokes on it would push...**

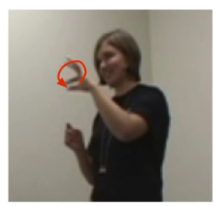


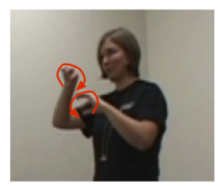


(Ascertaining) Uses her hands to simulate the spokes of one gear pushing the second gear

4 **If one gear was turning like that, then the spokes would push the other gear in the opposite direction, so you would know that every other gear from the first gear was turning the same direction as the first gear.**

(Persuading) Uses her hands to simulate the movement of two gears as she solves the conjecture.





5 **So all the odd gears would be turning in the original direction and all the even gears would be turning in the opposite direction. So if the last gear was odd it would be turning in the original direction and if the last gear was even it would be turning in the opposite of the original direction.**

Figure 4: Simulated action illuminates ascertaining

Paired Gesture and Speech Persuading

To illustrate the pairing of gesture and speech to support persuading in embodied proof, we examine an excerpt that occurs after a participant has solved the Triangle Inequality conjecture (Table 1) and has shifted into the persuading phase of the proof (Figure 2). The verbal element of the proof provides a specific example, as simultaneously the gestural components communicate the generalizability of the participant's proof.

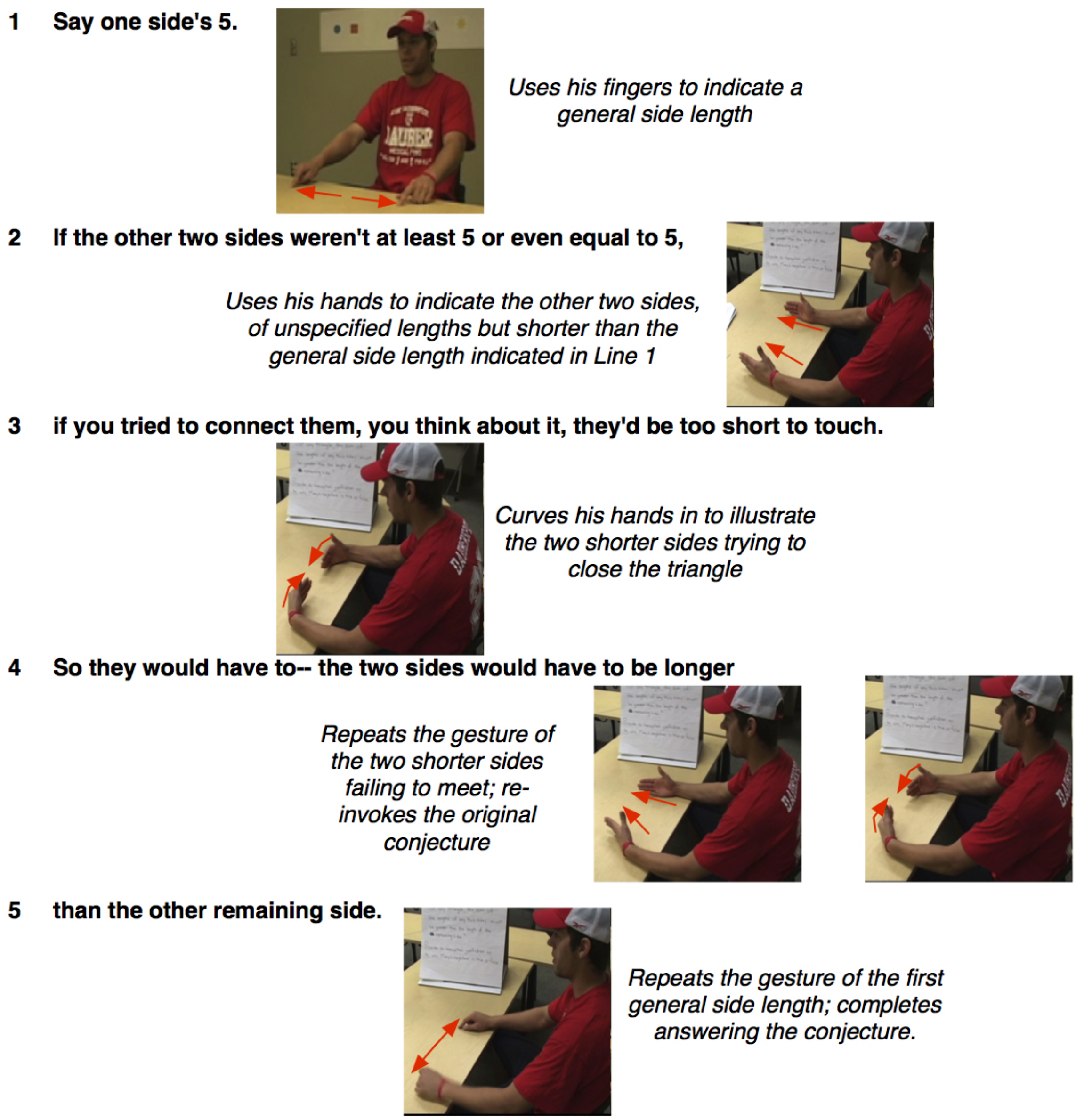


Figure 2: Paired gesture and speech persuading

In Figure 2, the verbal and gestural components are woven together to provide a complete proof. Attending only to the verbal proof elements would result in an incomplete empirical justification, as the participant would appear to be basing his entire argument upon testing a single (and incompletely described) triangle. However, in considering the gestures, we gain insight into the participant's full argument that goes beyond empirical, into analytical and even axiomatic proof schemes—the realms of a mathematically legitimate proof. It is through multimodal communication of gesture paired with speech that the student presents the most compelling and persuasive case for supporting his conjecture.

Discussion

These two examples highlight the multi-modal nature of proof, and show that understanding proof production can require attending to more than just students' verbal and written work. In Figure 1, the participant's embodied account reveals how she relies on an early instance to establish a conjecture about gears that is sufficiently general to support a deductive proof scheme. In Figure 2, the persuading phase offered might seem superficial (overly empirical) as a strictly verbal account. However, the participant's accompanying gestures reveal a corroborative proof scheme that is analytical in the sense that it relies, not on the particular lengths or topology, but their structural role. At the same time, it is transformational in how it utilizes actions to support the goal of portraying the impossibility of any triangle that rejects the premises.

These are not rare examples from our data, but rather they are characteristic of many other proof schemes we observed. Alongside our exploration of the different modalities of proof, we are examining various ways that gesture and action can support mathematical learning and, consequently, proof production. Although the data reported here come from proofs that participants spontaneously generated, interventions that manipulate action and gesture show promise for supporting analytical>transformational proof production (Walkington et al., 2012). As many students have difficulty producing traditional deductive proofs, preferring inductive empirical reasoning (Chazan, 1993; Healy & Hoyles, 2000; Hoyles & Healy, 1999, 2007), gestures and actions may provide an accessible bridge between the two. The potential of simple physical movements to support mathematical understanding is vast—and a crucial new area of study, given the importance of proof to the mathematical community and the general difficulty of engaging students in proof practices.

Conclusions

The implications of this emerging research on embodied cognition are profound for mathematics education in general, and the teaching of mathematical proof in particular. Action and gesture provide another modality for mathematics learning and expression, which may particularly support those students who struggle with the abstract notation traditionally used with proof. Extending the examination of proof production into gesture and action allows us to conceptualize a more complete model of cognition (Shapiro, 2011), and consequently allows us to design new activities that more coherently account for different strategies of proof production.

Our research provides a starting point for those examining mathematical proof through the modalities of action and gesture, and we continue to research the impact of action and gesture upon proof production. This work raises an important question: How does an embodied account influence earlier proof frameworks (e.g., Harel & Sowder, 1998; Healy & Hoyles, 2000)? Our next step is to answer exactly that question, and provide a multidimensional framework that incorporates a proof scheme with a spectrum of gesture, action, and proof.

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A DEVELOPMENTAL PERSPECTIVE INTO STUDENTS' CONTEXTUALIZATION OF PROBLEM SOLVING

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The purpose of this paper is to investigate students' contextualization of problem solving, not the problems. This study draws on the naturalistic paradigm and uses a developmental perspective to explore students' representations and metaphors used during problem solving. Students of comparable abilities employed similar representations, tended to use analogous metaphors during problem solving, and perceived solutions as outside of a problem's context.

Keywords: Linguistics; Problem Solving; Reasoning and Proof

Introduction

Problem solving is central to mathematics and instruction should give students daily experiences with it (Kilpatrick, Swafford, & Findell, 2001). Prior problem-solving experiences including teachers' scaffolding or suggestive language influence students' problem-solving behaviors and perceptions (Lesh & Zawojewski, 2007). The aim of this paper is to understand how students' contextualize problem solving. We illuminate relationships between students' problem-solving performance and experiential expressions via metaphors and representations employed during problem solving.

Related Literature

Embodied Cognition

The theoretical framework for this study stems from the embodied cognition perspective (Lakoff & Núñez, 2000). Students' problem solving is influenced by the cognitive network (i.e., beliefs and academic knowledge) and external relationships with the environment and other individuals (Lesh & Zawojewski, 2007). Prior experiences are difficult to communicate at times for teachers and students, but linguistic tools, such as metaphors, used by students can be rich with representational elements (Kövecses & Benczes, 2010). Metaphors denote one figure of speech as another figure of speech (Merriam-Webster, 2011). They embody experiences and are a means to support transfer through language, thought, and action.

Problem Solving and Representations

A problem is a developmentally appropriate challenge for which a problem solver has a goal but the means for achieving it are not immediately apparent (Schoenfeld, 2011). It requires making sense of the problem and the involved decisions to achieve the desired goal (Schoenfeld, 2011). When solving a problem, the existence of "a" solution or "the" solution is uncertain. Moreover, a pathway to such solutions is unclear (Schoenfeld, 2011). Research on students' problem solving indicates that prior experiences and knowledge, beliefs and dispositions, and culture play a huge role in how individuals approach problem solving (Lesh & Zawojewski, 2007).

Representations characterize a product or process (Goldin, 2002), or more specifically "an item that corresponds in an iconic sense to another item, an 'original' to which it refers" (von Glasersfeld, 1985, p. 2). Re-presentation characterizes a "conceptual construct that has no explicit reference to something else" (von Glasersfeld, 1985, p. 2). This distinction is critically linked to a contextualized understanding of mathematics (Goldin, 2002). Learners encode familiar contexts as internal representations such as beliefs, competencies, and expectations (Goldin, 2002). These internal representations are (a) based on everyday experiences, (b) shared by many, (c) extensively linked within one's cognition, (d) developed prior to learning mathematics in a context, and (e) supported by one's culture (Goldin, 2002). Thus, prior experiences greatly impact students' perceptions (i.e., representations) of problem solving (Schoenfeld, 2011).

Metaphors

As representations associate one item to an iconic other, the linguistic, cognitive counterpart is the conceptual metaphor. Current conceptual metaphor theory includes the literal component and conceptual component (Lakoff & Johnson, 2003). The literal component is the actual literal expression, while the conceptual metaphor is a mapping between two objects: the source and the target domain. The source domain is the experientially known domain and the related concept is the target domain. For example, "Your theoretical framework has a solid foundation" would involve the conceptual metaphor of "THEORIES ARE BUILDINGS." The target domain is theoretical framework and source domain is building. Variations of being (e.g., are and were) indicate unidirectional flow from the target to source domain. Conceptual metaphors can be classified in one of three hierarchical categories: structural, ontological, and orientational (Kövecses & Benczes, 2010; Lakoff & Johnson, 2003). Structural metaphors tend to describe a complex concept, such as time or understanding, in terms of a concrete experiential object, such as a limited resource (i.e., "DON'T WASTE MY TIME"). Ontological metaphors employ less structured target domains and necessitate a new defined reality to understand the shared experience. Personifications are regularly ontological. Orientational metaphors broadly conceptualize a specific direction inherent in human development. For example, the literal expression, "Things are looking up" demonstrates the conceptual metaphor of GOOD IS UP. Conceptual metaphors are used to map how individuals' cognitive domains are related to expression of their experiences (Lakoff & Johnson, 2003).

The relationship between the experiences of the teacher and student are vital to mathematics education. Teachers and students share an experiential set: solving mathematics problems. However, the student's and teacher's perspectives of what constitutes mathematical problems and/or solutions are complex in structure (Lakatos, 1976). Metaphors are culturally designed to articulate these implicit perspectives, and they have been found to encourage and incite cognition (Lakoff & Núñez, 2000).

Research Questions

The two research questions are: (1) How do middle and high school students' problem solving compare? (2) How do middle and high school students contextualize problem solving?

Method

Research Design

This study drew on a naturalistic paradigm and phenomenological inquiry to closely examine students' contextualization of problem solving (Short, 1991). Researchers employed a developmental perspective to explore students' problem solving.

Participants

Six participants for this qualitative study were representatively selected from investigations with larger samples. Data from sixth-, tenth-, and eleventh-grade students were collected during a think aloud conducted during two prior studies. Three middle and high school students from each study were selected. One sixth- and eleventh-grade pair (i.e., Theta and Kappa) performed above average compared to participants in the larger samples. A second pair had average performance (i.e., Beta and Lambda) and a third pair performed below average compared to peers (i.e., Gamma and Mu). Pairs two and three involved sixth- and tenth-grade students.

Data Collection

All participants completed a think aloud during a 40-minute period, which was video recorded. Sixth-grade participants completed four problems and high school participants responded to three problems. All participants were asked to solve developmentally appropriate problems using materials (e.g., manipulatives and markers) provided during the interview.

Data Analysis

Three analyses were conducted with videotapes and interview transcripts. First, students' responses were scored as correct or incorrect/no response by two mathematics educators. Correct responses had (a) solutions that answered the problem, and (b) representation(s) that supported the solution. Interrater agreement (IRA) was used for the first and second analyses and calculated using r_{wg}. Second, correct responses were coded using a representation coding protocol (Lesh & Doerr, 2003). Representation categories included symbolic, pictorial, tabular, verbal, concrete model, and mixed. IRA for these analyses was ideal, $r_{wg} = 1$. The third analysis was conducted by one researcher and intended to categorize students' conceptual metaphors used during the think aloud. The three conceptual metaphors were structural, ontological, and orientational (Kövecses & Benczes, 2010; Lakoff & Johnson, 2003).

Results

Participants with comparable performance tended to use similar representations. Theta and Kappa answered more problems than peers and also employed a variety of representations. Moreover, they did not immediately implement a symbolic approach like other participants. Gamma wrestled with symbolic expressions to explore one problem. Similarly, Mu read the problem and immediately combined numbers. Beta's attention focused on manipulating a concrete approach for one task, and then tried, albeit unsuccessfully, to employ symbolic representations with other problems.

Participants' metaphor use offered insight into their contextualization of problem solving. Theta and Kappa tended to use action verbs more often than their peers. For example, Kappa used "equals" more often than Lambda and Mu, who tended to use variations of "to be." As a whole, middle school participants employed metaphors far less than their high school counterparts. Kappa, Lambda, and Mu said "got" and variations of "to be" frequently whereas high school students' language was more complex in vocabulary and grammar structure. For example, Gamma stated that he was "going in the other direction" and "getting off track." These literal metaphors align with the structural conceptual metaphor of PROBLEM SOLVING IS A JOURNEY. Concomitantly, Theta had the literal metaphor, "my mind hit a wall" indicating the same conceptual metaphor as Gamma. Less successful students said "(verb) out" more often than their peers. Lambda frequently made comments like "figure out this problem," "take him [number] out," and "draw it [representation] out." These types of ontological metaphors indicated that students perceived the solution as outside of the problem's context. Thus, problem solving, as interpreted by students, can be characterized as working from within one context and outward to another where the solution lies.

Conclusion

The aim of this study was to examine students' representations, contextualizations, and metaphors of mathematical problem solving. A common theme emerged across grade levels: effective problem solvers tended to use nonsymbolic representations and more conceptual metaphors to support their problem solving. Students' contextualization suggests that problem solving is moving towards a solution, which is not readily associated with the task's context. Kappa and others' strategies often employed symbolic representations, which divorce mathematical symbols from their context. These results aligned with Santos-Trigo's (1996) findings that students perceived symbolic representations as more appropriate than others during problem solving, and students were reticent to explore nonsymbolic representational approaches. The perception of mathematics as abstract due to its highly symbolic nature may have encouraged students to disassociate the problem's context from the problem and solution. Thus, practical considerations are necessary to enhance learners' contextualization of problem solving.

This exploration also suggested a new model to draw on students' experiences. The student-described experiences with problem solving indicated that students perceived problems ontologically as containers.

Linguistically, students contextualized problem solving with the ontological conceptual metaphor of PROBLEMS ARE CONTAINERS. This result was surprisingly natural as Kövecses and Benczes (2010) argue, the experiential understanding of in and out is inherent with human existence. The ontological metaphor of container is powerful and intimately involved with our perception of the world. The container (i.e., problem) held all knowledge needed to “solve” the problem. Therefore, the action of “solving” the problem was to use the given knowledge to move one’s understanding from inside to outside the container.

This research led to a transition along a developmental continuum of students’ perceptions of problem solving via the compass of contextualization. The proposed model can support future investigations into enhancing students’ nonsymbolic representation use during problem solving and their problem-solving outcomes.

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TO PROVE OR DISPROVE: IN WHAT WAYS DO UNDERGRADUATE STUDENTS USE INTUITION TO DECIDE?

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In this study, students used their intuition to decide whether to prove or disprove mathematical assertions. The investigation involved individual task-based interviews with three undergraduate students in transition-to-proof courses and analysis to determine students' various uses of intuition to decide on the truth of a statement. Tall's three worlds of mathematics—embodied, symbolic, and formal—provided categorization for students' intuitions, and the intuitions were situated along three continua of mathematical thinking. The students' use of intuition and the categorization of their intuitions suggest that the students were at various points in the transition to advanced mathematical thinking. Thus, differences in students' intuitions may correspond to differences in students' competence in making valid decisions and translating their intuitions into counterexamples.

Keywords: Advanced Mathematical Thinking; Post-Secondary Education; Reasoning and Proof

"The move from elementary to advanced mathematical thinking involves a significant transition" (Tall, 1991, p. 20). Intuition is necessary for deciding on the truth value of a mathematical assertion and producing proofs and counterexamples (Fischbein, 1987; Hersh, 1997; Wilder, 1967). Undergraduate students, however, often lack intuitive understandings that support effective reasoning during proof productions (Harel & Sowder, 2007; Moore, 1994). Preliminary results from a study of how undergraduate students decided whether to prove or disprove a mathematical assertion will illustrate in what ways (a) they used their intuitions to decide, (b) they used their intuitions to support formal reasoning, and (c) their intuitions can be situated in Tall's (2004, 2008) *three worlds of mathematics*.

Theoretical Framework

Student intuitions can be categorized using Tall's (2004, 2008) three worlds of mathematics—the *embodied*, *symbolic*, and *formal* worlds—and situated along three continua of mathematical thinking. According to Tall, these worlds are distinct yet overlapping, allowing them to intertwine in various ways as individuals mature mathematically. Development of mathematical intuition begins with initial experiences with mathematics and is nurtured by further experience and mathematical knowledge (Fischbein, 1987; Wilder, 1967). Intuitions can be rooted in any of the three worlds and often span them, being influenced by experiences that overlap worlds.

The embodied world encompasses "perceptual representations of objects" and visuo-spatial reasoning (Tall, 2008, p. 7). Embodied intuitions may be based on vague mental representations that are acquired through repeated experiences with mathematics (Davis & Hersh, 1980; Fischbein, 1987). These mental representations may connect to prior knowledge that resides in any of the three worlds (Burton, 2004; Hadamard, 1945).

The symbolic world incorporates symbolic manipulations, including arithmetic and algebraic reasoning. The symbols in this world "allow us to switch effortlessly from processes to *do* mathematics to concepts to *think* about" (Tall, 2004, p. 285). Symbolic intuitions may develop from repeated and various experiences with computational mathematics that allow for generalizations and abstractions based on recognizing iterative reasoning.

The formal world comprises formally defined objects, deductive reasoning, and mathematical proof. Formal intuitions build on experiences with mathematical logic and assist with reasoning about formal mathematical statements and constructing mathematical proofs (Hadamard, 1945; Tall, 1991).

In this paper, intuition is situated along three continua of mathematical thinking based on the degree of students' use of (a) intuition to make sense of formal statements and decide on their truth value; (b) links among intuitions, proofs, and counterexamples; and (c) formal intuition. Intuitions of lower-division undergraduate students often reside in the embodied and symbolic worlds because of their limited experience in the formal world. However, in order to advance their mathematical thinking, they must develop formal intuitions and learn to use their embodied and symbolic intuitions to make sense of formal mathematical statements and build bridges to mathematical proofs. Thus, in order for students to develop the intuition of mathematicians, they must make "a fundamental transition in their thinking processes" (Tall, 1991, p. 7).

Method of Inquiry

The data in this paper come from a larger exploratory study that (a) investigated how students decide whether to prove or disprove mathematical assertions, (b) identified difficulties students have in making these decisions, and (c) examined connections between students' decision procedures and their success in constructing associated proofs and counterexamples.

Data Collection

The participants were four undergraduate students in transition-to-proof mathematics courses at two private liberal arts institutions in Ohio and West Virginia. I asked each instructor to choose two high-achieving students to participate, but this was not feasible at one institution. I conducted individual, semi-structured, task-based interviews with each student (Goldin, 2000). Each interview was audio-recorded and transcribed. Participants completed three prove-or-disprove tasks, including the following:

> *Injective-function task*: Let $f : A \to B$ be a function and suppose that $a_0 \in A$ and $b_0 \in B$ satisfy $(a_0, b_0) \in f$. Prove or disprove: If $(a, b) \in f$ and $a \neq a_0$, then $b \neq b_0$.

I instructed the students to think aloud during the tasks and to clarify or expand on their thinking as necessary. Upon completion of the tasks, I asked the students about difficulties they had with the tasks and general strategies they used for prove-or-disprove tasks.

This paper focuses on three students, called Ann, Dave, and Chris, whose intuitions represent increasing degrees of development of mathematical thinking. Each student reported using their *intuition*, *instinct*, or *gut feeling* when deciding on the truth value of the statements in the tasks. Dave and Chris are from one institution, and Ann is from the other.

Data Analysis

The main goal of the analysis was to determine the various ways that students were reasoning while deciding whether a mathematical assertion was true or false. Consequently, I performed a content analysis on the transcripts in order to detect themes in the data (Patton, 2002). The transcripts were coded and intuition emerged as a theme. I classified intuitions by reasoning type and situated them in the three worlds of mathematics. Intuitions categorized as *embodied* involved visuo-spatial reasoning, verbal phrases that invoke a visual image or action, and visual representations of objects. Intuitions categorized as *symbolic* involved symbolic manipulations, equations, operations, and properties of operations. Based on the above characterizations, I situated examples and informal definitions in either the embodied or symbolic worlds. Intuitions categorized as *formal* involved logical reasoning and formal definitions. Many intuitions contained combinations of these reasoning types and were situated in multiple worlds.

Results

I present below snapshots of three students, Ann, Dave, and Chris, whose intuitions resided in various combinations of the embodied, symbolic, and formal worlds and represented increasing degrees of development of mathematical thinking. The students used their intuitions in distinctive ways to determine the truth value of the injective-function task.

Ann

Ann's general intuition involved the use of numbers, equations, or graphs, but she had a difficult time getting started on the tasks. For the injective-function task, Ann said, "Well, when I think functions, I just always think of like a graph. That's just where my thought instantly goes. . . . I think like, everything like algebra, so I just gotta, hmm, I like equations." Ann's intuition on this task combines the visuo-spatial concept of graphs with a symbolic characterization of functions as equations, thus it is embodied-symbolic. When I asked Ann if she had a particular equation in mind, she replied, "Not really. That's what I'm trying to figure out." Ann was unable to use her embodied-symbolic intuition to determine a suitable function, graph, or equation to make sense of or decide on the truth or falsity of the injective-function task.

Dave

Dave's general intuition stemmed from "experience with problems" which helped him recognize tasks that are similar to ones he had seen before. On the injective-function task, Dave's intuition was in the formal-embodied world: "That sounds just like the definition of a function, one thing can't map to two, one element of the domain can't map to two elements of the codomain. So I guess I'm probably trying to prove that." However, he hesitated to begin a proof, and said "I'm just wondering if I'm confusing vertical line test with horizontal line test, and confusing not a function with not one-to-one." Thus, his formal intuition of the definition of a function was coupled with an embodied visual intuition related to the definitions of function and one-to-one. He reread the assertion, decided the definition was for one-to-one, and quickly produced and verified the counterexample $f(x) = x^2$. Dave's embodied-formal intuition was off target at first, but then led him to construct a correct counterexample.

Chris

Chris depended on his intuition to help him construct proofs and counterexamples: "I find it really difficult to start unless I have an idea. . . . I'm pretty sure it's this. Why do I think that? I think that because of this, this, and this, and eventually that kinda leads me through." For the injective-function task, Chris's embodied-formal intuition invoked a visual image of a function related to the definition of one-to-one. "Well my intuition so far is that it can be. It seems like if it were a function from **R** to **R**, then if it's just not a one-to-one function, or something, then it can double back on itself." He quickly wrote $f(x) = x^2$ and proved it was a counterexample by noting that different inputs result in the same output. Thus, he combined his visual embodied intuition with the formal definition of one-to-one, resulting in an embodied-formal intuition. This intuition was decisive in his ability to complete the injective-function task successfully.

Discussion

The ways in which Ann, Dave, and Chris used their intuitions and the categorizations of their intuitions suggest that the students were at various points on the three continua of mathematical thinking detailed above. Ann's embodied-symbolic intuition did not help her make sense of the assertion, decide on its truth value, or construct a proof or counterexample. Although Dave struggled to understand the assertion and decide on its truth value, his symbolic-formal intuition led him to construct a correct counterexample. Chris's embodied-formal intuition helped him successfully understand the assertion, decide on its falsity, and construct a counterexample.

Each student used intuitions from overlapping worlds when deciding on the truth of the task, but the similarities in the categorization of the students' intuitions may conceal differences that influenced their success. Dave and Chris linked their formal intuitions to the formal definition in the task, and Ann's lack of formal intuition limited her ability to make sense of the task. Thus, students using formal intuitions may be better equipped to make decisions and translate their intuitions into counterexamples than students using only embodied and symbolic intuitions.

However, it was the differences in the students' embodied intuitions that seemed to affect their success the most. Each student had a distinct embodied intuition that played a key role in their work on the task.

Ann's superficial correlation of graphs and functions was not helpful to her because she was unable to draw a graph corresponding to the function in the task. Dave's embodied version of the definitions of function and one-to-one as the vertical and horizontal line tests made it clear to him that he was confusing these definitions, thus leading the way to a counterexample. Chris's vague image of a function doubling back on itself and violating the definition of one-to-one provided the key idea for his counterexample to the task. Chris said that when he has an intuition of a definition related to functions, "I feel like I can kind of have a function in my mind, which, it's a very strange function that does whatever I want it to." Thus, differences in students' embodied intuitions may correspond to differences in students' competence in making valid decisions and translating their intuitions into counterexamples.

Although this paper explored the intuitions of only three students working on one prove-or-disprove task, it presents differences in their intuitions that may warrant consideration in future work. How can Ann's graphical intuition be refined so that it can help her relate to the formal aspects of the task? How did Dave come to think of definitions in both embodied and formal terms? And how can we teach students to invoke vague images in the way that Chris does?

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EXPLORING THE SECONDARY–POSTSECONDARY TRANSITION WITH TEACHERS: THE CASE OF SYMBOLIZING

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This research is interested in considering teachers' view in issues of transition between school levels. As a framework, ethnomethodology enables us to take teachers' "ways of doing" mathematics (symbolizing, using representation, proving, etc.) as key elements to those issues. Using this concept, we present data from a one-year collaborative research project involving secondary and postsecondary teachers. We analyzed data from a task the researcher and the teachers examined together in one of the meetings, highlighting teachers' ways of doing in relation with transitional issues around the use of symbolism to represent exponential functions.

Keywords: Secondary-Postsecondary Transition; Collaborative Research; Symbolism

Introduction

Several researches highlight difficulties due to new demands and new knowledge's organization when moving from secondary to postsecondary mathematics (see Robert, 1998). Researchers commonly address this issue from the perspective of advanced mathematics (e.g., Robert & Schwarzenberger, 1991), identifying potential problems, but without studying the transition as such. Others compare mathematical organizations of tasks at both levels (e.g., Bosch et al., 2004; Gueudet, 2004; Praslon, 2000; and Winslow, 2007). For instance, Artigue (2004) starts from secondary level to examine transitional issues by characterizing the culture of high school mathematics based on its curriculum. According to Hall (1959), it is, however, the implicit "ways of doing" that lead to greatest cultural differences. In classrooms, teachers do, and make their students do mathematics in a certain way. Transitions in mathematics education have yet to be examined from the perspective of those ways of doing. This project aims at exploring the transition from secondary to postsecondary mathematics with teachers through their ways of doing mathematics. The study anchors in our previous researches showing several students' difficulties with symbolism when transitioning from high school to college (Corriveau, 2007).

A Collaborative Research Taking Account of Teachers' Perspectives

Collaborative research is founded on the idea of researchers and teachers working together on issues directly related to the practitioners' practices (Desgagné et al., 2001). An idea is to conduct research "with" teachers rather than "on" or "about" them. In this project, we engaged the issue of transition with a group of six participants: 3 secondary schools teachers and 3 college teachers (12th to 14th year of study). They join a collaborative research focusing on issues around the transition from secondary to postsecondary mathematics.

There were regular meetings (7) between the researcher and teachers to work around the participants practices (their ways of doing mathematics). Except for the first, informational meeting (Nov. 2010), they were full "one-day" reunions (Jan. 2011 to Nov. 2011). Typically, the teachers alternatively worked in teams and as a group around a task submitted by the researcher. The task itself was designed around actions-situations teachers face in their daily practices, but also *breaching* from their ordinary (an ethnomethodological method used to reveal everyday hidden insights and/or "ways of doing"). All meetings were recorded and transcribed, and are now analysed using qualitative emergent analysis methods and softwares.

Conceptual Framework Underlying the Analysis

Ethnomethodology (Garfinkel, 1967) is a perspective enabling us to explore “ways of doing” mathematics by teachers. Ethnomethodology’s research interest is to study everyday methods people use to accomplish and constitute a socially organised activity (such as teaching math at a specific level). For the analysis, we are interested in what we call ***mathematical ethnomethods*** which are *ways of doing mathematics* that are mobilized by teachers in their daily working life. These ethnomethods are *indexed* to situations and certain *circumstances* which allow them to be shared among teachers of a given level. In this perspective, the actor (the teacher) is assumed to be knowledgeable, meaning that those ethnomethods also concerns teachers’ *rationale* for his/her ways of doing. Hence, actors in a social activity share and create meanings that show common ways of doing and saying, because they allow the participants to understand one another.

In a first phase of the analysis, we identify and describe ways of doing shared by teachers of a same level, the circumstances, indexicality, rationality of these ways of doing. This categorization (ways of doing, circumstances, indexicality and rationality) leads to a description of mathematical ethnomethods.

Results: Symbolizing an Exponential Function

We illustrate the analysis with an example about ways of symbolizing an exponential function at secondary and post-secondary levels. We gave teachers excerpts from textbooks showing different ways of symbolizing an exponential function (see Figure 1, for example). We ask them to comment what is done at the other level.

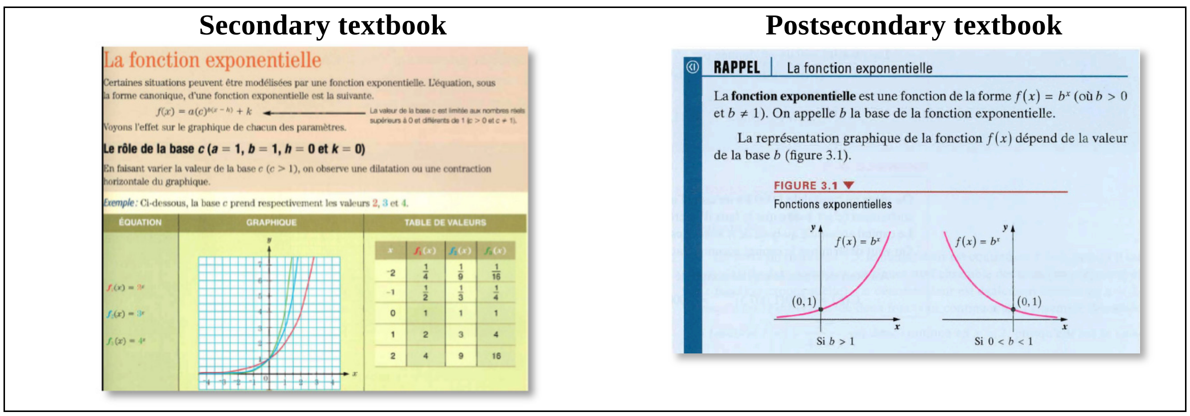

Figure 1: Example of symbolization of an exponential function

The choice of this situation is significant. It had been thought so the question immediately puts teachers in their practice (using their textbook, working on a function they both study and use with their students). This task also led to unfamiliarity: teachers were faced with ways of symbolizing presented in textbook from the other level (inspired by the concept of breaching we mentioned earlier). In the interaction between teachers, many ways of doing around symbolism emerged. The following fragment gives an example, involving Paul and Patricia, from postsecondary level, and Sean, Sam and Sandra, who work in secondary schools, reacting on the formula presented in a secondary textbook $f(x) = ac^{b(x-h)} + k$:

Paul:	Do you really present the exponential function this way? Really?
Patricia:	You don’t like it? This writing? It’s heavy, hey? I also find it really heavy.
Paul:	Ah very heavy.
Sean:	But we will eventually simplify this writing.
Paul:	Maybe but even if you do it, why is it written this way? Students will be afraid right away, they will be scared.
Sam:	It’s because of the 4 parameters...

Sandra: There is a basis to that...
Sean: Students are already pros [with parameters] when we study this function [exponential]. It's true, they are very comfortable working with the parameters.

From the conversation, we understand that the use of parameters (a, b, h, and k) as presented in the textbook ($f(x) = ac^{b(x-h)} + k$) is not a familiar symbolization for postsecondary teachers. Secondary teachers express familiarity with this symbolization and do not recognize the heaviness pointed out by Paul and Patricia. Sam explains to them the *circumstances* of the use of this particular symbolization and the ways of using (introducing) it:

Sam: I start with $f(x) = c^x$ and then, I present the four parameters one by one […]…
Patricia: You don't show this [$f(x) = ac^{b(x-h)} + k$] right away.
Sean: No…
Sam: No. We always start with the function $f(x) = c^x$, the basic function, and we see some properties in the graphic. Then, we transform it. We use parameters and draw the graphic and we ask "what is interesting here".

Sam explains he starts with something like $f(x) = c^x$, a basic function, and then, progressively adds complexity by introducing parameters (a, b, h and k) until it becomes $f(x) = ac^{b(x-h)} + k$. This progression in the symbolization is also associated with the graphic of the function: each parameter is related to an effect when compared to the basic function (a translation, a reflection and dilatation/shrinkage). In reaction, Patricia explains the way of symbolizing an exponential function at postsecondary level:

Patricia We use b^x in the review. We will work with this writing when we teach the derivative, when we demonstrate properties. But then, we can change it, we can play with it in applications. It doesn't have to be x. x can become $\sin(x)$, we can play with it, it can become $a^{\sin x}$.

At postsecondary level, teachers use the basic function $f(x) = b^x$ as a reminder of what is an exponential function, in definitions and theorems, and the teacher explain to students that x can become something more complex. Then, in applications, the complexity of the symbolization shows when the letter x becomes $sin(x)$, $x + \Delta x$, etc. This results in another way of using symbolism (to vary the meaning of the letter/variable): "we can play with x, with the symbolism" said a teacher from postsecondary level. When secondary teachers heard how symbolization is used in applications at the postsecondary level, they asked why not use something like $g(x) = a^{f(x)}$ as a basic symbolization rather than $f(x) = b^x$: it would be "more logic!" To this, postsecondary teachers explained that it is also important for them to graphically represent the function, which would not be possible with the first, more general, case.

From the symbolization point of view, it is interesting to notice that these ways of using symbols leads to different meaning in relation with generalization. In the way teachers from the secondary level use the symbols, the function $f(x) = c^x$ represents one specific type of functions, a function which can be represented in a graphic (*indexicality of the symbolism*). The symbolization $f(x) = ac^{b(x-h)} + k$ then represents every other case. This second symbolization is thus more general then the first one. Also, considering this notation as more general is linked with a certain way of using "letters." Letters are indexed on the one hand to numbers (variables and parameters), and on the other hand, to an effect in the graphic (parameters). Contrastingly, from the way teachers use symbols at postsecondary level, $f(x) = b^x$ is considered the most general function because b and x can become something more complex. Letters are not indexed to numbers but to more complex algebraic expressions. At the same time, when representing the graphic of the function $f(x) = b^x$, the symbols have other signification, closer to what they are used for by the secondary level teachers. Thinking in terms of transition, student then have to deal a meaning for the letters which is partially the same as in secondary school, but also have to abstract and expand the possibilities of these symbols.

Conclusion

By contrasting ways of doing at each level, we highlighted important aspects of teachers' ways of using symbolism to represent an exponential function. Teachers explained their ways of symbolizing but also pointed out *circumstances* (e.g., $f(x) = c^x$ at secondary level to represent a particular function that will become more complex, $f(x) = b^x$ at postsecondary level as a reminder, in definitions and theorems), and *rationales* (e.g., $f(x) = b^x$ is used rather than $g(x) = a^{f(x)}$ to be able to represent the graphic) for these ways of doing. And we can also see how symbolism is *indexed* to specific meanings (a symbol can relate to numbers and effects in the graphics at secondary level; a symbol can relate to a complex algebraic expression at postsecondary level). Such analysis take us to conceptualize symbolism as a *process*, in the way it is talked about and use by high school teachers, and/or as a *pregiven object*, from the perspective brought forth by the teachers at the postsecondary level.

Such analysis take us to conceptualize symbolism as a *process*, in the way it is talked about and use by high school teachers, and/or as a *function*, from the perspective brought forth by the teachers at the postsecondary level. We believe this shift can be an interesting tool for researchers to address secondary-postsecondary transition from the student's perspective.

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INVESTIGATING THE RELATIONSHIP BETWEEN TASK DIFFICULTY AND SOLUTION METHODS

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The present study sought to investigate the role of task difficulty in students' preference for visual or nonvisual methods as they solved calculus tasks. Data were collected from 498 high school students enrolled in Advanced Placement Calculus courses. Results indicate that the mode of representation and task difficulty were factors influencing the students' preference for visual or nonvisual methods. The students used more visual methods when presented with graphic calculus tasks and more nonvisual methods when presented with algebraic calculus tasks. As task difficulty increased, the number of visual methods increased significantly, and the number of nonvisual methods decreased significantly.

Keywords: Advanced Mathematical Thinking; High School Education; Calculus; Visual and Nonvisual Methods

Background

Research on differences in students' preference for solution methods has been of considerable interest to educators for many years (e.g., Krutetskii, 1976; Kozhevnikov, Hegarty, & Mayer, 2002; Lean & Clements, 1981). Most previous studies investigating differences in students' preferred methods have concentrated on gender and cognitive abilities and failed to take into account the role of task difficulty (e.g., Lowrie & Kay, 2001). Thus, the purpose of this study was to investigate the relationship between task difficulty and preference for visual and nonvisual methods.

Krutetskii (1976) identified types of mathematical giftedness based on students' preferences for or verbal-logical (or nonvisual) or visual thinking. Following the work of Krutetskii, Lean, and Clements (1981), Moses (1977), Suwarsono (1982), and Presmeg (1985) have recognized that individuals could be placed on a continuum (i.e., degree of mathematical visuality) according to their preference for visual processing. Visualizers are considered as learners who prefer to use visual-pictorial processes, and verbalizers as learners who prefer not to use visual-pictorial processes when there is a choice on a specific task.

Individual differences in solution methods have been reported by a number of researchers (e.g., Fennema & Tartre, 1985; Hegarty & Kozhevnikov, 1999; Lean & Clements, 1981). Battista (1990) and Bremigan (2005) showed that low achieving students used more visual solutions (i.e., drawing strategies without visualization) than nonvisual. Ben-Chaim, Lappan, and Houang (1989) investigated the effect of instruction on middle school students' preferences for visual and nonvisual methods. There were significant differences in preferences among students by grade level prior to the instruction. In the studies with high school students, Gallagher and De Lisi (1994) and Gallagher, De Lisi, Holst, McGillicuddy-De Lisi, Morely, and Cahalan (2000) identified strategy use and flexibility as factors contributing to differences in mathematical performances. In a study with sixth grade students, Lowrie and Kay (2001) found that task difficulty was related to preference for visual or nonvisual methods, and that the students used more visual methods as the task difficulty increased. From the review of literature, it appears that research findings regarding preferred solution methods have been inconclusive, and few studies investigated students' preference for visual or nonvisual methods in calculus. Therefore, this study sought to investigate the role of task complexity in students' preference for visual or nonvisual methods as they solved calculus tasks presented graphically and algebraically.

Methods

Participants

The participants were 498 high school students who were enrolled in Advanced Placement (AP) calculus courses at seven high schools in two districts in Florida in the United States at the time of the study. Of the 498 students, 290 were males and 208 were females. Approximately 59% of the sample were White, 18% were Hispanic, 15% were Asian, 5% were African American, and 1% were Multi Racial. The remaining 2% indicated "Other" as their ethnic group.

Materials and Procedure

Over a 2-year period, 498 students were asked to complete 20 calculus tasks. All students received standardized instructions and were tested in their intact classrooms. All participating students gave their informed consent and were debriefed at the end of the study.

The calculus tasks, which could be solved by visual or nonvisual methods, were administered in a packet. The calculus packet contained 14 graphic (7 derivative and 7 antiderivative) and 6 algebraic (3 derivative and 3 antiderivative) tasks and a corresponding questionnaire consisting of a visual and a nonvisual solution for each task. In each task, the students were presented with the graph or the equation of a function and were asked to draw a possible derivative or antiderivative graph. Upon completion of the tasks, the students were given the questionnaire and were asked to choose for each task a method of solution that most closely described how they solved the tasks.

First, the calculus tasks were divided into two groups as graphic and algebraic. Then, each of the two groups was divided into two subgroups. That is, eight graphic tasks (4 derivative and 4 antiderivative) were classified as relatively easy, and six graphic tasks (3 derivative and 3 antiderivative) as more difficult. Easy tasks require sketching derivative or antiderivative graphs of continuous functions (i.e., linear, quadratic, cubic, exponential, and trigonometric), whereas difficult tasks sketching derivative or antiderivative graphs of functions with an infinite discontinuity or a corner (e.g., $1/x$, $1/x^2$, and $|x - 1|$). Two algebraic tasks, which require sketching derivative or antiderivative graphs of quadratic or cubic functions, were classified as relatively easy. Four algebraic tasks, which require sketching derivative or antiderivative graphs of absolute and logarithmic functions, were classified as more difficult.

Students' responses for each task were categorized into four subgroups: (a) visual and correct, (b) nonvisual and correct, (c) visual and incorrect, or (d) nonvisual and incorrect. A score of 1 was given for each response category on each test (i.e., Easy Graphic, Difficult Graphic, Easy Algebraic, and Difficult Algebraic). It is important to note that few students chose both visual and nonvisual methods for a task on the questionnaire, and these responses were not included in the analysis.

Results

The mean number of responses for each of the response categories derived from easy and difficult tasks is reported in Tables 1 and 2. A paired-samples t-test was conducted to compare the total number of correct responses for easy and difficult tasks presented graphically and algebraically. As expected, there was a statistically significant decrease in the number of correct answers from Easy to Difficult tasks presented graphically, $t(1, 497) = 15.19$, $p < 0.001$. The eta squared statistic (0.32) indicated a large effect size. Similarly, on the algebraic tasks, the number of correct responses for easy tasks was significantly higher than difficult tasks, $t(1, 497) = 24.35$, $p < 0.001$. The eta squared statistic (0.54) indicated a large effect size. The results indicate a statistically significant difference in the task complexity between the tests.

Table 1: Means and Standard Deviations for Graphic Tasks

	Easy		*Difficult*	
	M	*SD*	*M*	*SD*
Response Category				
Visual and correct	0.30	0.28	0.31	0.33
Nonvisual and correct	0.24	0.21	0.06	0.13
Visual and incorrect	0.21	0.21	0.35	0.29
Nonvisual and incorrect	0.22	0.23	0.23	0.28

Table 2: Means and Standard Deviations for Algebraic Tasks

	Easy		*Difficult*	
	M	*SD*	*M*	*SD*
Response Category				
Visual and correct	0.09	0.21	0.10	0.17
Nonvisual and correct	0.48	0.36	0.08	0.16
Visual and incorrect	0.10	0.24	0.24	0.26
Nonvisual and incorrect	0.31	0.34	0.53	0.32

A paired *t*-test revealed significant differences in the number of visual (correct and incorrect) and nonvisual (correct and incorrect) between easy and difficult tasks. On the graphics tasks, the number of visual methods significantly increased, and the number of nonvisual methods significantly decreased as the task complexity increased. When the tasks were presented algebraically (see Table 2), the students used more nonvisual methods than visual on easy and difficult tasks. However, with the increasing level of task complexity, the number of nonvisual methods significantly decreased while the number of visual methods significantly increased.

Discussion

The present study examined the relationship between task difficulty and preference for visual and nonvisual methods. Although the mode of representation (i.e., graphic vs. algebraic) was not of primary interest to this study, the results indicate that the mode of representation influenced the students' preference for visual or nonvisual methods. That is, the students used more visual methods when presented with graphic calculus tasks and more nonvisual methods when presented with algebraic calculus tasks.

The results also suggest that task difficulty was related to preference for visual or nonvisual methods. This finding supports the conclusions of Lowrie and Kay (2001), who found that that task difficulty has an influence on the way students represent mathematics problems. As task difficulty increased, the number of visual methods (correct and incorrect) increased significantly, and the number of nonvisual methods (correct and incorrect) decreased significantly, suggesting that the students were likely to use visual methods for more difficult tasks. Visual solutions are image-based and involve determining the shape of derivative or antiderivative graphs based on visual estimates of slopes. On the other hand, nonvisual solutions are generally equations-based and involve translation to an equation, computing the derivative or integral of the equation, and then using this new equation to draw the graph. Although the design of the study does not enable this finding to be inferred from the data, the results support the hypothesis that students are likely to use visual methods in more complex tasks because increasing complexity makes it difficult for students to estimate the equation of a function and might influence their tendency to use visual methods.

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STUDENTS' WAYS OF THINKING ABOUT COMBINATORICS SOLUTION SETS: DELETION AND EQUIVALENCE CLASSES

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Students' ways of thinking about combinatorics solution sets, the sets of elements being counted, were investigated in a study that engaged fourteen undergraduates with no formal experience with combinatorics in individual task-based interviews. This paper focuses on two ways of thinking that emerged from the data analysis: Deletion and Equivalence Classes. Both involve creating a related combinatorics problem and finding a relationship between the solution set of the new problem and that of the original problem. The relationship is additive in Deletion and multiplicative in Equivalence Classes.

Keywords: High School Education, Post-Secondary Education, Problem Solving

According to Piaget and Inhelder (1975) children's combinatorial reasoning is a fundamental mathematical idea based in additive and multiplicative reasoning. Indeed, as Kavousian (2008) said "without much prior knowledge of mathematics, one can solve many creative, interesting, and challenging combinatorial problems" (p. 2). This indicates that students should be able to solve combinatorial problems by employing their additive and multiplicative reasoning. However, the research shows that students of all ages often struggle to solve combinatorial problems (English, 1991; Hadar & Hadass, 1981; Lockwood, 2011)

In order to address these difficulties, much of the prior research on combinatorics education has focused on students' actions, not their reasoning and understanding. Thus, it will be foundational to understand the stable patterns in reasoning that students apply in a variety of combinatorial situations. These coherent patterns in reasoning are known as *ways of thinking* (Harel, 2008). The research study described here aims to answer the following research question: What are students' ways of thinking about the set of elements being counted in combinatorics problems?

The set of objects being counted has been called the "solution set"(Lockwood, 2011). Framed in this language, the second research question investigates students' ways of thinking about the solution set of combinatorial problems.

Theoretical Perspectives

The philosophical perspective underlying this study is that "knowledge is not passively received either through the senses or by way of communication, but it is actively built by the cognizing subject"(von Glasersfeld, 1995, p. 51). This idea that mathematical knowledge is constructed as the learner engages actively in the tasks is central to this research. Harel (2008) contends that there are two different categories of mathematical knowledge: ways of understanding and ways of thinking. Humans' reasoning "involves numerous *mental acts* such as interpreting, conjecturing, inferring, proving, explaining, structuring, generalizing, applying, predicting, classifying, searching, and problem solving" (Harel, 2008, p. 3). *Ways of understanding* refers to the reasoning applied to a particular mathematical situation—the cognitive products of mental acts carried out by a person. *Ways of thinking*, then, refer to what governs one's ways of understanding—the cognitive characteristics of mental acts—and are always inferred from ways of understanding. Reasoning involved in ways of thinking does not apply to one particular situation, but to a multitude of situations. Ways of understanding and ways of thinking thus comprise mathematical knowledge (Harel, 2008).

Research Methodology

Data for this study come from a series of "individual exploratory teaching interviews" (Steffe & Thompson, 2000) conducted at a large southwestern university in the USA. Fourteen students from a second-semester Calculus for Engineers course participated in individual task-based interviews. Each student participated in 2 hour-long interviews with the researcher (the present author) in a two-week period in spring 2011. None of the students had formal experience with combinatorics. The purpose of these interviews was to catalogue students' ways of thinking the elements of solution sets. Each interview involved the researcher as the teaching agent, one of the students, a series of tasks, and a method of audio and video-recording the interview.

There were a few phases of retrospective analysis. The researcher discussed the data with two mathematics education researchers during the study. Content logs including summaries of the video for each task were created for each student following each interview and relevant portions of the video were transcribed as necessary. At the end of the study, the researcher used open coding (Strauss & Corbin, 1998) to identify and catalogue the ways of thinking in which each student engaged.

Results

Several different ways of thinking emerged from the data analysis. One category of ways of thinking was present as students determined the sizes of subsets of the solution set. Another, the *Odometer* category, involves holding an item constant and cycling through possible items for the remaining spots in order to generate all elements of the solution set (Halani, 2012). A third category comprised of *Deletion* and *Equivalence Classes* thinking is discussed here. Both of these ways of thinking involve creating a new, related combinatorics problems and then finding a relationship between the solution set of the new problem and that of the original problem. The relationship is additive in Deletion and multiplicative in Equivalence Classes.

Deletion

Consider the following task "Situation: A security code for a computer involves two letters. It is case insensitive, but the two letters must be different from each other. Question: How many possible security codes are there for this computer?" Sophie found the answer to be $2^{26} – 2$ and described her thought process below:

> *This time you need to take out any duplicates of the letters. So it can't be like A and A or anything else like that. Um, so let's see. [….] Like, 2 to the 26th [writes 2^{26}] and then you would end up minusing, well, yeah, you would end up subtracting [continues writing $2^{26} – 26$ because this [circles 2^{26}] would give you the total number of combinations, and this [circles 26] is the number of combinations that are invalid because they are using the same letter.*

Here, Sophie tried to count the number of security codes where repetition may be allowed, thereby creating a related problem and finding the cardinality of its solution set. Then, she identified those elements that are "invalid" because they use the same letter. She understood that she could "take out" these elements in order to find the cardinality of the solution set of the original problem. These remarks point to a way of thinking known as *Deletion* thinking.

Typically, while engaging the Deletion way of thinking, students will first consider a given task with solution set *A*. Then they will construct a related problem with a solution set *C*. This solution set will contain a subset *B* which has a bijective correspondence with *A*. By counting the total number of elements in *C* and the elements of *C* which are not in *B*, the students will find the cardinality of *B*. Since *B* corresponds to *A*, they will have found the cardinality of *A*, the solution set whose size they were trying to find. In essence, they are "deleting" the elements of the solution set *C* that are not in *B*.

The fact that Sophie incorrectly determined the number of security codes where repetition is allowed is not relevant to identifying her way of thinking. What is relevant is that even before she attempted to enumerate the elements of the solution sets, she outlined her thinking about the situation by saying "you would need to take out any duplicates of the letters." Again, one difference between a way of thinking and a way of understanding is that a way of thinking is present in a multitude of situations. Sophie's solution here is not enough to show that she is engaging in the Deletion way of thinking. However, her solutions to other tasks and the way she outlined her thinking suggest that she was in fact engaging in Deletion thinking.

Equivalence Classes

When attempting to determine the number of ways four people could sit around a circular table, Slang drew representations of different table arrangements as shown in Figure 1:

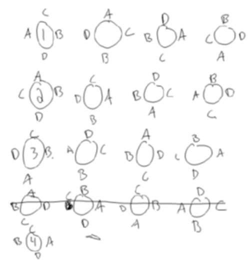

Figure 5: Slang's written work for the Table Problem with four people

From her partial representation and explanations, it is clear that Slang constructed a new problem in which rotations of the circle are considered as a different table arrangement. She then grouped the tables so that each row corresponded to a table arrangement she wanted to count. Once Slang had drawn three different rows, she realized that there would be a total of 24 different circles she would be drawing and four circles per row so there are 24/6 or 4 ways four people could sit around the table. For this reason, she only drew one representative of the fourth row.

In a similar manner to students engaging in Deletion thinking, Slang constructed a new problem and found a relationship the solution set of the new problem and the one whose cardinality she was trying to find. However, the relationship was additive in Deletion thinking but multiplicative in her case. This is a new way of thinking known as *Equivalence Classes*.

A student engaging in the Equivalence Classes way of thinking will first consider a given task with solution set *A* and then create a related problem with a solution set *S*, which can be partitioned into equivalence classes of the same size—each class corresponding to an element of *A*. After grouping elements into these equivalence classes, he or she would then quantify the size of each equivalence class and relate the size of the equivalence classes to the size of *S*, in order to find the size of set *A*. See Figure 2.

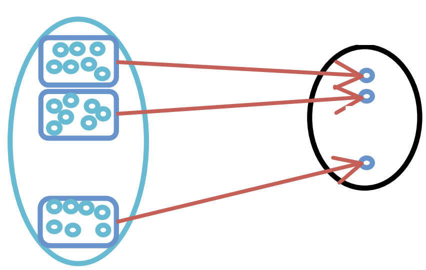

Figure 2: Equivalence classes

Discussion

In this study, students did not always engage in ways of thinking which are productive for solving certain tasks. In particular, most students were able to naturally engage in Deletion thinking, but struggled with Equivalence Classes. Some of these students could construct new questions, determine the size of the new solution set, construct equivalence classes and even quantify the size of the equivalence classes, yet they could not determine the multiplicative relationship between the new solution set and the original one. Thus, it seems as if some students need to be encouraged to develop certain ways of thinking.

This study which focused on understanding students' ways of thinking about the set of elements being counted and how that thinking expresses itself in their attempts to solve combinatorial problems can be foundational for future studies and for teaching practice. It can serve to assist teachers in implementing instructional interventions designed to help students develop productive ways of thinking about combinatorics and support curriculum developers in organizing tasks to build upon students' ways of thinking. In addition, this study could provide a framework for analyzing how the ways of thinking about combinatorics solution sets are distributed across various mathematical populations. This researcher hopes to conduct further studies to investigate how students develop their ways of thinking about the solution sets as they progress through a variety of combinatorial tasks and the instructor implements interventions designed to encourage particular ways of thinking, including Equivalence Classes thinking.

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STUDENTS' PERCEPTIONS OF THE ROLE OF THEORY AND EXAMPLES IN COLLEGE LEVEL MATHEMATICS[1]

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Previous research has thoroughly reported on college level students' inability to deal with non-routine problems and to justify in mathematical terms the ways in which they deal with routine problems. Researchers have conjectured that these inabilities are the result of the absence of theoretical content or its dissociation from tasks and corresponding techniques in the teaching approach to college mathematics. In this preliminary research report, we present empirical evidence that, at the college level, students perceive theory exclusively as a validatory discourse and don't recognize the value of generalized examples. Students' perceptions of the role of theory and examples in mathematics have to be addressed if we want them to genuinely engage in mathematical activity.

Keywords: Post-Secondary Education; Advanced Mathematical Thinking

Recent research has shown that, at the college level, mathematical theory is either completely absent or strongly dissociated from the teaching of procedures (techniques to find limits, calculate derivatives, integrate, solve systems of linear equations, etc.; e.g., Lithner, 2004; Sierpinska, 2004; Barbé et al., 2005; Hardy, 2009). The absence or dissociation of theoretical discourses or even mathematical intuition might leave students with no resources to deal with non-routine tasks (e.g., Selden et al., 1999; Lithner, 2000) and be a contributing factor to students behaving more as *institutional subjects* than as *mathematical subjects* in front of mathematical tasks (Hardy, 2010). Studies show that this absence or dissociation is typical of the teaching approach to college mathematics: Elementary and Intermediate Algebra, Calculus, and Linear Algebra (e.g., Raman, 2004; Sierpinska, 2004; Barbé et al., 2005; Boesen et al., 2010).

What drives this research project is the question "if we were to incorporate theory in our teaching approach to Calculus, how should we do it so that we provide students with discourses to justify in mathematical terms their approaches to problem solving and with tools and strategies to deal with non-routine tasks?" As a first step towards answering this question, we investigate students' perceptions and uses of theory and examples. Bills et al. (2006) point out that students may not perceive and use examples in the ways intended by educators; based on 30 subjects we present a preliminary report on students' perceptions and uses of theory and examples.

Perspective and Educational Context

We consider that knowing mathematics includes the ability of behaving mathematically and that the process of learning mathematics requires behaving as mathematical subjects. Our understanding of mathematical behavior includes mathematical reasoning, analytic and deductive thinking, mathematical approaches to problem solving (see Schoenfeld, 1987), and a preoccupation for theoretical consistency and validation. Furthermore, a subject behaves mathematically when he or she perceives theory not only as a discourse that justifies techniques but also as a discourse that connects and produces techniques to solve different types of tasks. Our use of the notions of *tasks*, *techniques* and *theory* is to be understood in terms of the notion of *praxeology* within the Anthropological Theory of the Didactic (Chevallard, 1999; we refer to *theory* meaning the theoretical block of a praxeology).

The educational context of this project is the teaching and learning of college level Calculus courses, with particular focus on the topic of limits of functions, as they are taught in several universities across Canada, the US and different locations in Europe (e.g., Lithner, 2000, 2004, 2010; Barbé et al., 2005). In many well described cases, the routinization of tasks and a prevalent institutional normative discourse result in certain stable institutional practices that define a teaching culture in the Calculus community. This

Van Zoest, L. R., Lo, J.-J., & Kratky, J. L. (Eds.). (2012). *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Kalamazoo, MI: Western Michigan University.

culture can be gleaned from course outlines, assessment instruments, and textbooks. One of its main characteristics is the dissociation of techniques-to-solve-a-given-task from a theoretical-(even-intutitive)-mathematical-valid-explanatory-discourse.

Methodology

Four teaching approaches to the topic limit of functions at infinity were designed and video-taped. Forty subjects were recruited among students at the university level who have recently passed a pre-Calculus course and were randomly assigned to one of four conditions. Condition 1 corresponds to a teaching approach that consists of a list of particular (numerical) examples of finding limits of functions; techniques are presented in terms of sociomathematical norms (Voigt, 1995) with complete absence of theoretical content. In condition 2, a list of theoretical results together with some generalized examples is given first. Then, a list of particular (numerical) examples is given. In this approach, theory is present but is dissociated from particular examples. Condition 3 corresponds to a teaching approach consisting exclusively of theoretical results; these results are derived from and/or illustrated by means of generalized examples, no numerical examples are shown. The fourth condition, which will be analyzed at a later stage, focuses on theory and mathematical intuition and clearly links them to tasks and techniques. Mechanisms of generalization of some techniques are also shown.

Subjects were met individually; they completed a pre-test, attended the video-taped lecture and engaged in a task-based interview.

Results and Analysis

In this preliminary report, we present results from subjects assigned to conditions 1, 2, and 3 corresponding to three types of problems that subjects deal with in the task-based interview. Problems of type 1 consist of finding limits at infinity of particular rational functions. In conditions 2 and 3, a technique for finding such limits is discussed in general terms, while particular examples are only given in conditions 1 and 2. Subjects in condition 3 were unable to deal with most problems of type 1 while subjects in conditions 1 and 2 were able to deal with those that strongly resembled examples given in the lectures; for instance, they were able to find limits at infinity of rational functions where the value of the limit is a non-zero constant (an example shown in both conditions 1 and 2) but were unable to do so when the value of the limit was zero or infinity. Problems of type 2 consist of finding limits of rational functions with generalized constants. Only two subjects (out of the 30) engaged in these problems. They did so by experimenting with concrete values of the constants and then generalizing the results to all possible values. Only one of these two subjects was able to justify his generalization. This subject showed an explicit interest in mathematical theory and strongly criticized condition 1 for the lack of it. The remaining 28 subjects displayed a hesitant behavior and eventually refused to deal with the problems. They justified their behavior on a self-declared inability to work with generalized constants (subjects refer to these as "letters"). Problems of type 3 consist of finding limits of functions that were not discussed in any of the three conditions. In particular, subjects are asked to find the limits of the product of an exponential function and cosine function—however, examples including exponential and trigonometric functions are given in all conditions 1 and 2. Only one subject engaged and succeeded in dealing with problems of type 3 (the theory-bound subject mentioned in the previous paragraph). The other 29 subjects argued that they did not know how to deal with these tasks; they blamed this on the lack of examples of that particular type.

Table 1: Three Types of Problems Posed to Subjects in the Task Based Interview

Type 1	Type 2	Type 3
Find limits of rational functions (e.g., $\lim_{x\to\infty}\frac{3x^2-x-2}{5x^2+4x+1}$, $\lim_{x\to\infty}\frac{x^2-4}{x+2}$)	Find limits of rational functions whose expressions contain generalized constants (e.g. $\lim_{x\to\infty}\frac{x^c}{x^4+1}$ where c is a non-negative integer)	Find limits of functions not discussed in any of the three conditions (e.g. $\lim_{x\to\infty} e^x\cos(x)$)

Subjects assigned to conditions 1 or 3 criticized the approaches for their lack of theory and examples respectively. When asked about the effect of adding theory to condition 1, subjects said that they "like to know why things are true"[2] but that theory "doesn't help in knowing how to deal with problems." When asked about the effect of adding (particular) examples to condition 3, subjects said that "without examples [we] don't know what to do" and that they are "unable to apply theory to an example without being shown how to do so." Certainly, these students do not recognize generalized examples as "examples on how to apply theory."

When dealing with problems in the task-based interview, subjects assigned to conditions 1 and 2 constantly reviewed the notes they had written during the corresponding lecture, in search of examples that *matched* the proposed problem. This is more significant when we consider subjects assigned to condition 2 who ignored their notes on theoretical explanations, generalized techniques, etc. For instance, in condition 2, they had been shown a technique for finding limits of rational functions in general terms and only one numerical example; when dealing with problems of type 1, subjects only referred to the numerical example and got stuck when the function they had to deal with did not have the exact same features as those of the example. When asked why they did not use their notes apart from the example, they argued that they "wouldn't know how to apply these to a concrete problem." These students, as students in condition 1, use particular examples as templates (instead of using generalized examples).

Subjects assigned to condition 3 did not refer to their notes when dealing with problems in the task-based interview. When asked, for example, whether they thought that a technique to find limits of rational functions had been explained in the lecture, most subjects said that they "didn't remember" or that they "didn't understand the explanation." In some cases, subjects said that they "didn't pay attention to explanations in the lecture as [they] know they are not important for solving problems." When the interviewer insisted they refer to the section of their notes where a technique to find limits of rational functions is explained, subjects stated that they were uncomfortable with "expressions that use letters rather than numbers," and that they have a hard time "recognizing what the letters mean" and understanding how to use techniques explained in general terms.

Discussion

Our study highlights features of the learning culture that college level (Calculus) students share: the ways they perceive mathematical theory and particular examples. In his paper of 2004, Lithner showed how particular examples in Calculus textbooks are used as templates to solve other problems. Our data shows that students not only use particular examples as templates but that at the time when they are (institutionally) ready to start a Calculus course, they need these particular examples; generalized ones don't count as they don't know how to use them. Moreover, they do not recognize generalized examples as a tool to solve problems; the existence of generalized constants in an expression seems to act as an obstacle for them to recognize the example as such. It seems as though students don't realize that particular examples hide the limitations of a technique, and are at loss when presented with non-routine problems. Furthermore, although students perceive theory as important because "it explains why things are true," they fail to see it as a generator of techniques for solving problems. We believe that these inabilities, although they are of different natures, might have a common source in certain institutional norms that define a predominant mathematics teaching and learning culture; a culture in which if and when theory is present, it is not necessary. Thus, it won't suffice to incorporate theory into the teaching approach; it is the very essence of mathematical theory and its relation to problem solving that has to be addressed if we want college level students to engage in mathematical practices as mathematical subjects.

Endnotes

[1] This ongoing project is funded by FQRSC, grant number: NP-145336.

[2] In this report, sentences between quotes paraphrase what subjects explained during the task-based interviews.

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SEEING THE GENERAL IN THE SPECIFIC: A CLOSER LOOK AT STUDENTS' EXAMPLE BASED REASONING

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In this paper, we explore distinctions among middle school students' example based mathematical justifications. This work has emerged naturally as part of a research and professional development project focused on middle school teachers' development in their understanding of mathematical justification and their ability to support their students in learning to justify. We share a selection of example based student justifications in order to illustrate a distribution ranging from pure empirical arguments to generic proofs. We observe that the most sophisticated of these generic example based arguments are but a step away from general proofs. This progression could provide a viable starting point for design research focused on developing instruction to support students in leveraging examples to produce general arguments.

Keywords: Reasoning and Proof; Middle School Education

It has been well documented that K–12 students tend to both produce and prefer empirical justifications (Chazan, 1993; Knuth, Choppin, Slaughter, & Sutherland, 2002). However, this research can suggest a dichotomy between example-based reasoning and rigorous proof. Justifications using generic examples (Mason & Pimm, 1984), which we will refer to as *generic proofs* (Selden & Selden, 2008) straddle this dichotomy. On the one hand, they involve the use of specific examples, while on the other hand, these examples are reasoned about generically (i.e., focus is on the attributes seen as common to all instances rather than attributes specific to the example being considered). Additionally, generic examples can typically be refined to be general proofs by introducing more general representations (including algebraic symbols). This suggests that it may be more productive to think in terms of a progression from pure empirical reasoning to reasoning about examples generically on the way to rigorous general proof.

The JAGUAR Project

The JAGUAR (Justification and Argumentation: Growing Understanding of Algebraic Reasoning) project is a research and professional development project focused on middle school teachers' development in terms of their understanding of mathematical justification and their ability to support their students in learning to justify. The project team worked with 12 teachers in two different states over the course of two years. The teachers were exceptional in the sense that (for most of the teachers) we had evidence that their practice was already developed to the point that their classrooms were rich in mathematical discourse. An important goal of the project was for the teachers to develop in their ability to support their students in constructing valid mathematical justifications.

Data Collection and Analysis

Early in the project, the research team met with the participating teachings in summer workshop. One of the questions that was considered (in collaboration with the participating teachers) was what forms of reasoning (Stylianides, 2007) were appropriate to expect and promote in a middle school mathematics classroom. Because of its prevalence in students' thinking, example-based reasoning was something that the project team and the teachers expected to see regularly in students' justifications. The project team worked with the teachers to design four lessons that both fit well into the teachers' existing curriculum and strongly featured justification. For each teacher, we videotaped the implementation of each of these lessons in two consecutive years. While the focus of our data collection was on the teachers, we were able to collect a large number of student justifications in the form of collected written assignments, poster presentations, and contributions to whole class discussions.

This brief research report is based on our ongoing analyses of the students' individual written justifications and poster presentations. A major aspect of this analysis is the categorization of the forms of reasoning employed by students in their justifications. The goal of this analysis is to construct a taxonomy of forms of reasoning that is relevant to middle school mathematics. Note that an argument can employ more than one form of reasoning, and we have found that some of our categories naturally overlap (e.g., pattern based reasoning can typically be considered to be a special case of example based reasoning). The focus here is on the distinctions we are seeing among students' example-based arguments.

Illustrative Results

Here we will present a progression of illustrative student justifications. This progression can be subdivided roughly into two levels. The first level consists of empirical justifications ranging from naïve empiricism to arguments that show evidence of awareness of the potential limitations of empirical reasoning. The second level consists of justifications in which examples are used generically to some extent. The most sophisticated justifications of this type are essentially general proofs although they contain residue of example based reasoning in their use of representations that are not fully general.

Level 1: Empirical use of examples in justifications. Not surprisingly, we identified many instances of students using empirical reasoning. For example, when asked whether a number trick (using the distributive property) worked for all numbers (after it had been verified for the numbers 1–10), one student responded as follows:

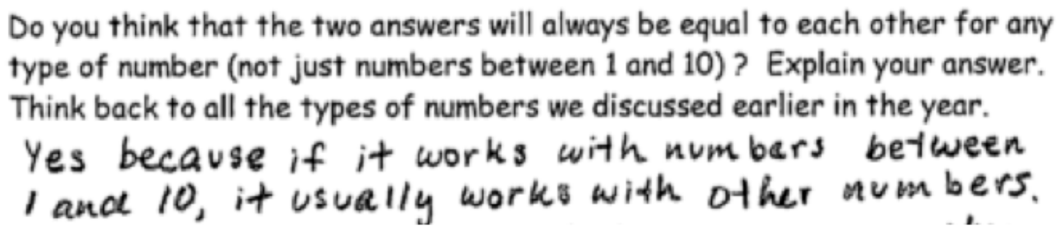
Do you think that the two answers will always be equal to each other for any type of number (not just numbers between 1 and 10)? Explain your answer. Think back to all the types of numbers we discussed earlier in the year.
Yes because if it works with numbers between 1 and 10, it usually works with other numbers.

Figure 1: Naïve empirical justification

While the justification in Figure 1 shows little awareness of the limitations of checking examples, the justification shown in Figure 2 (to a very similar task) describes a much less naïve empirical approach in which one carefully selects examples in order to verify a representative sample (apparently) in an attempt to decrease the likely hood of missing a disconfirming case.

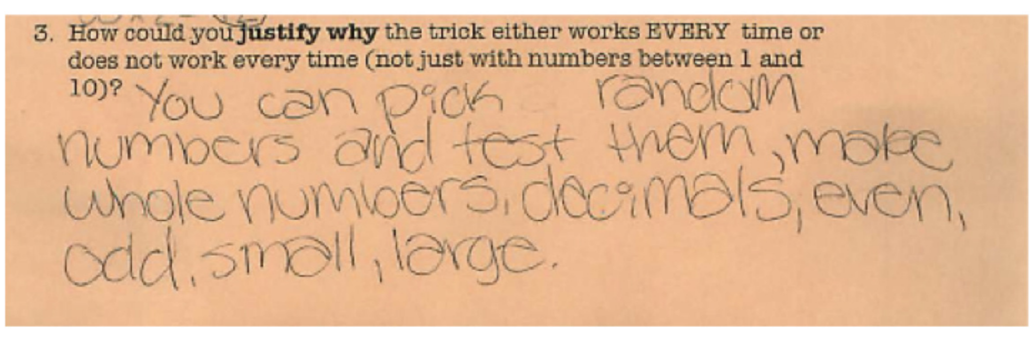
3. How could you **justify why** the trick either works EVERY time or does not work every time (not just with numbers between 1 and 10)? You can pick random numbers and test them, make whole numbers, decimals, even, odd, small, large.

Figure 2: Less naïve empirical justification

Level 2: Generic examples. While the example shown in Figure 2 suggests an increased awareness of the limitations of checking examples, it does not address *why* the number trick works and so does not really provide a starting point for developing a general argument. However, we found many examples in which students did reason generically about examples. Some of these only showed only a rudimentary analysis of why a statement was true for a specific example. In Figure 3 we see a justification of this type.

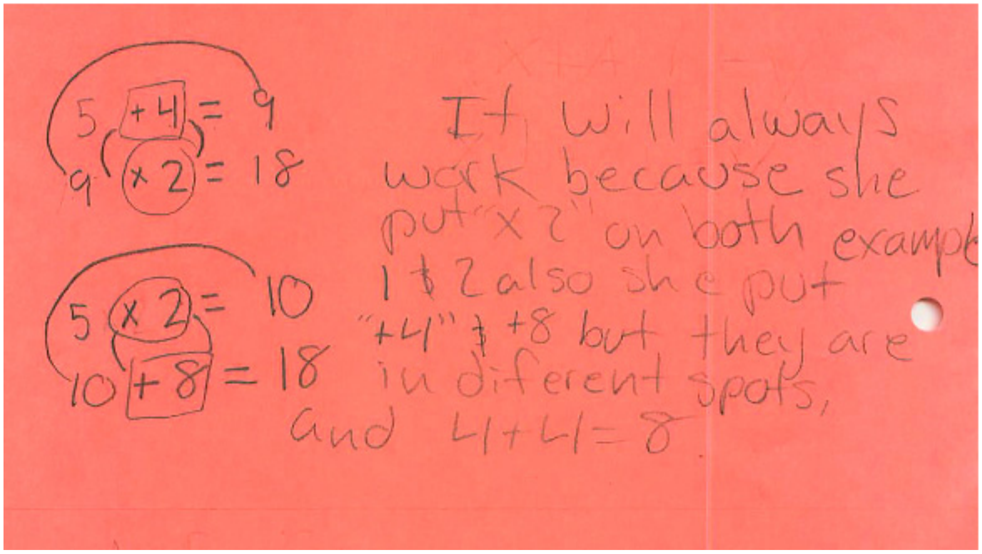

Figure 3: Slightly generic example

In the justification shown in Figure 3 we see the student analyzing the structure of the example in order to ascertain *why* the number trick works. The student has identified a type of equivalence between the two statements based on the doubling that occurs in each procedure and a compensation (doubling the four to get eight) for the fact that this doubling occurs at a different step in the two procedures. By way of contrast, the justification shown in Figure 4 (Justifying a formula for the perimeter of a chain of hexagons) fully explains why the formula works in general by carefully analyzing a specific example.

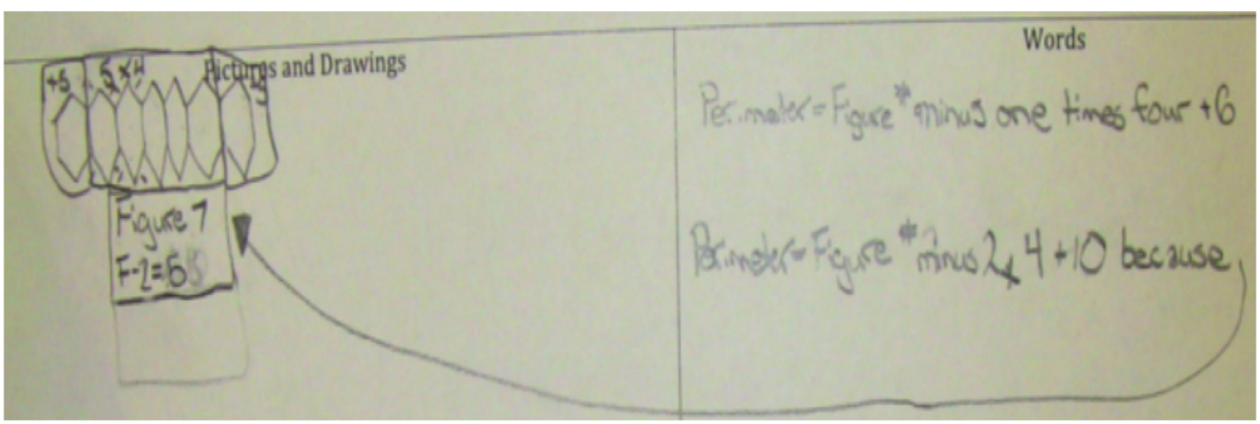

Figure 4: Generic proof

Note that a justification of the type shown in Figure 4 comes very close to being a general proof. The argument does not depend in any way on the specific nature of the example. One could make the figure more general using ellipses and make greater use of algebraic symbols. Then when accompanied by a verbal explanation, this argument would serve as a very explanatory general proof.

Discussion

The examples presented here suggest some milestones along a trajectory from pure empirical reasoning to generic proof. Further analysis will be needed to further flesh out this trajectory. Such a trajectory would be an important part of a taxonomy of forms of reasoning relevant to middle school mathematics and would provide a solid starting point for design research aimed at developing instructional approaches to supporting students in developing more powerful example based reasoning.

Acknowledgments

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INFERRING IMPULSIVE-ANALYTIC DISPOSITION FROM STUDENTS' ACTIONS IN SOLVING MATH PROBLEMS

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The research reported in this paper is part of a larger study designed to investigate the validity of the Likelihood-to-Act (LtA) survey—an Likert-scale instrument, currently under development and testing, for assessing students' impulsive-analytic disposition in mathematics. Transcripts and videos of 15 interviewees' responses to five problems, adapted from the LtA Survey, were analyzed in terms of (a) solution strategies, and (b) impulsive-analytic disposition. Two scores were derived from quantifying the codes that were assigned to the 75 problem-solving episodes. These scores were highly correlated to one another and were correlated to the LtA_Difference (impulsive minus analytic) score, obtained from the LtA Survey administered several weeks prior to the interviews. These results can be taken to support the validity of the LtA instrument.

Keywords: Cognition; Metacognition; Problem Solving

Lim, Morera, and Tchoshanov (2009) use the term *impulsive disposition* to refer to a tendency to proceed with an action that comes to mind without analyzing the problem situation and without considering the relevance of the anticipated action to the problem situation. Most mathematics educators would want their students to progress from impulsive disposition to *analytic disposition*—a tendency to study the problem situation prior to taking actions. Self-awareness of impulsive tendency is likely to propel students towards a more analytic tendency. One way to create such awareness is to have an instrument that can accurately measure one's impulsive-analytic disposition.

Lim et al. (2009) created an instrument, called the *Likelihood-to-Act* (LtA) survey, for gauging students' impulsive-analytic disposition. The LtA survey has been undergoing a few rounds of testing and revisions (Lim & Mendoza, 2010; Lim & Morera, 2011). The current version consists of 16 pairs of Likert items. Figure 1 below shows a pair of impulsive and analytic items where respondents indicate on a scale of 1 (extremely unlikely) to 6 (extremely likely) how likely they are to respond to the given mathematical situation in the described manner. Three measures can be derived from the LtA survey: (a) the impulsive subscale based on the 16 impulsive items, (b) the analytic subscale based on the 16 analytic items, and (c) the LtA_Difference score is the average difference between the impulsive and analytic scores.

3. $\frac{3}{4}+\frac{1}{10}+\frac{9}{10}$ When asked to find the answer for the above arithmetic expression without using a calculator, how likely are you to begin by finding the common denominator?	**1**	**2**	**3**	**4**	**5**	**6**
19. $\frac{3}{5}+\frac{11}{12}+\frac{1}{12}$ When asked to find the answer for the above arithmetic expression without using a calculator, how likely are you to begin by studying the fractions to see if you can predict the answer?	**1**	**2**	**3**	**4**	**5**	**6**

Figure 1

The objectives of this research are: (a) to investigate the validity of the LtA survey, and (b) to explore ways of inferring impulsive-analytic disposition from students' actions while solving math problems that were designed specifically to elicit impulsive solution strategies. In this research report, we seek to answer the following questions:

1. What solution strategies did participants use in solving those math problems?
2. To what extent does student impulsivity inferred from their problem-solving behaviors during an interview agree with the LtA score of impulsivity?

Theoretical Perspectives

Research on mathematical problem solving has been studied extensively. In particular, Schoenfeld (1992) identifies four categories of cognition that provide a framework for analyzing problem-solving behaviors: resources, heuristics, control, and beliefs. He reported that 60% of solution attempts were of the "read, make a decision quickly, and pursue that direction come hell or high water variety" (p. 356). This behavior is a consequence of one's lack of metacognitive control. The "reading and making a decision" part is suggestive of impulsive disposition.

Unlike typical problem-solving research that involve *problems* where a solution approach is not readily known to students, our study focuses on students' responses to familiar-looking tasks where they are likely to take actions spontaneously. We use tasks that are similar to those in Frederick's (2005) *cognitive reflection test*: "A bat and ball cost \$1.10 in total. The bat costs \$1.00 more than the ball. How much does the ball cost?" (p. 27). Students who have an impulsive tendency are likely to answer 10 cents, which is the difference between \$1.10 and \$1.

From a cognitive science perspective, certain impulsiveness is inevitable due to our cognitive structures. According to dual process theories, humans have two distinct cognitive systems of reasoning. "System 1 processes are rapid, parallel and automatic in nature: only their final product is posted in consciousness" whereas "System 2 thinking is slow and sequential in nature and makes use of the central working memory system" (Evans, 2006, p. 454). Mathematicians know when they can rely on System 1 and when they need to be cautious. Many math students, on the other hand, lack the control and tend to rely on the first idea that comes to mind. Helping them become aware of the need to be reflective and analytic is an important step.

Method

Among the 495 undergraduates who took the LtA survey, 90 volunteered and 16 took part in a 60-minute three-part interview. This paper focuses on the problem-solving part where participants solve five math problems and then explained their reasoning.

P1. Find the answer for $\frac{3}{4}+\frac{1}{10}+\frac{9}{10}$
P2. Solve for n: $90 + 1234n + 567 + 89n = n + 1234n + 567 + 89n$
P3. Five lampposts are spaced evenly along a street. The distance between the first lamppost and the last lamppost is 220 m apart. What is the distance between any two neighboring lampposts?
P4. Project P took 30 workers, each working 8 hours, to complete. Project Q took 20 workers, each working 3 hours, to complete. Which project was bigger in size?
P5. Paula is bicycling from home to school. At 8 o'clock she has already cycled 2.4 miles. What is her speed?

The first 12 interviews were conducted by the first author and the remaining four interviews by the second author. We analyzed 75 think-aloud problem-solving episodes (15 participants solving 5 problems each; the interviewee who spoke mainly in Spanish was excluded).

The 75 episodes were analyzed in two ways to generate two quantifiable scores. The first analysis involved data exploration and code creation, where the focus was on the solution strategies used by interviewees in solving the five problems. For each interviewee, we divided the number of instances of impulsive strategies by the total number of instances of strategies (both impulsive and analytic) to obtain the *Impulsive Strategy Percent*.

In the second analysis, the video of each episode was analyzed. Using a rubric, each episode was assigned one of five codes: "I+" (strong indication of impulsive disposition), "I-" (weak impulsive), "A+" (strong analytic), "A-" (weak analytic), and "U" (unsure). The authors and a graduate student met to go over the rubric and a training set of nine episodes. The inter-rater reliability for the remaining 66 episodes was 0.76. After discussing all discrepant cases, each code was then assigned a number (1 for "A+" and 5 for "I+"). The sum of scores for the five problems constitutes the *Impulsive Disposition Score*, which ranges from 5 to 25.

Results and Discussion

Table 1 below shows the list of solution-strategy codes and number of occurrences. Of the 102 instances (some interviewees used more than one strategy in a problem), 43 are impulsive with most of them occurring in the two non-contextualized problems (P1 and P2). The remaining 59 analytic instances occurred mainly in the three contextualized problems (P3, P4 and P5), partly because students have to read and understand the problem situation.

Table 1

Item	Strategy Code	Description	Instances
P1	CD	Uses a common denominator of 10, 20, or 40	14
	ADDLAST	Adds the last two fractions first	2
P2	COMBLT	Combines like terms	12
	CANCELTRM	Cancels any common terms found on both sides	6
P3	DIVIDE5	Divides the total distance by five	10
	DIVIDE4	Divides the total distance by four	6
	DRAWPIC	Draws a picture	12
	GUESS-N-CHK	Makes an educated guess and checks if it is correct	1
P4	RATIOCOMP	Creates and compares two ratios	4
	LOGIC	Uses direct reasoning to solve problem	4
	TOTALHOURS	Multiplies workers by hours to find total hours worked	10
P5	DIVIDE8	Divides the total distance by eight	1
	FORMULA	Uses speed = distance/time formula	1
	PROPORTION	Thinks of setting up a proportion	1
	DRAWPIC	Draws a picture	4
	MISSINFO	Believes problem is unsolvable; missing info	14

Table 2 below shows the codes we assigned to each episode as well as the Impulsive Disposition Score for each interviewee. "I+" and "I-" codes appeared mainly for P1 and P2 whereas "A+" and "A-" occurred mainly for P4 and P5.

Table 2

	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12	S13	S14	S15
Problem 1	I+	I+	I+	I+	A+	I-	I+	I+	I+	A+	I+	I+	I-	I+	I+
Problem 2	I+	A+	I+	I+	I+	I-	I-	I+	I+	A+	A-	A-	I-	A-	A+
Problem 3	I-	A-	A+	I+	I+	I-	I-	I+	I-	A+	A-	A+	A+	I+	A+
Problem 4	I-	A+	A+	A-	A+	A+	I+	A+	A+	A+	A+	A+	A+	A+	A-
Problem 5	A-	A+	A+	A+	A+	A+	A-	A+	A+	A+	A+	U	A+	A+	A+
Imp. Disp. Sc.	20	10	13	18	13	14	20	17	16	5	11	12	11	14	10

Table 3 below lists the three measures of impulsivity for each participant, with the top three highest scores in each row highlighted. These results seem to be consistent, except S13.

Table 3

Participant	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12	S13	S14	S15
Imp. Strategy %	62	22	33	50	40	50	71	60	50	0	33	38	33	50	43
Imp. Disp. Sc.	20	10	13	18	13	14	20	17	16	5	11	12	11	14	10
LtA_Difference	2.81	0.31	0.88	2.00	-0.56	1.13	0.44	0.81	0.44	-2.75	-0.31	0.44	1.50	0.25	0.81

Twelve out of the 15 interviewees have a positive value for LtA_Difference score; in other words, they were more likely to respond in the impulsive manner than in the analytic manner described in an LtA pair (see Figure 1). On the other hand, only three interviewees exhibited more impulsive solution strategies than analytic solution strategies, and only five interviewees have Impulsive Disposition scores that are greater than 15. These results suggest that participants are generally more analytic in an interview setting and more impulsive in a survey-taking setting.

The Pearson's correlation between Impulsive Strategy Percent and Impulsive Disposition Score is 0.927 with a *p*-value of 0.000. This high correlation can be attributed to the fact that these two scores are based on the same data source—the problem solving part of the interview. The correlation between Impulsive Strategy Percent and LtA_Difference is 0.670 with a *p*-value of 0.006. The correlation between Impulsive Disposition Score and LtA_Difference is 0.693 with a *p*-value of 0.004. These high correlations suggest that the analysis of the problem-solving part of the interviews supports the validity of the LtA_Difference score.

Concluding Remarks

There was a strong correlation between the Impulsive Strategy Percent and Impulsive Disposition Score, partly because the problems in the interview were designed specifically to elicit impulsive solution strategies. Nevertheless, it is possible for students to use an impulsive (or analytic) solution strategy but reason in an analytic (or impulsive) manner.

The two scores from the interview data are positively correlated with the score obtained from the LtA survey. These strong correlations suggest that the LtA survey is a valid instrument. However, not all LtA items are necessarily valid. The analyses of the other parts of the interview will inform the strengths and weaknesses of the LtA items; this information will be valuable for refining the LtA items.

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CONVERSATIONAL REPAIR AS A SCAFFOLDING STRATEGY TO PROMOTE MATHEMATICS EXPLANATIONS OF STUDENTS WITH LEARNING DISABILITIES

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The linguistic concept of conversational repair is proposed to scaffold students with learning disabilities/difficulties (LD) to better transit to the reform-based discourse-oriented mathematics classroom. With a multiple-baseline-across-participants design, participants were asked to self-explain their reasoning/thinking during word problem solving, while the experimenter gives repair requests as a scaffolding strategy to elicit explanation repair from them during intervention. This study attempts to examine the effectiveness of this strategy in improving these students' reasoning ability measured by the characteristics and quality of their self-explanations, and their problem solving ability measured by performance scores.

Keywords: Elementary School Education; Problem Solving; Reasoning and Proof; Standards (broadly defined)

Pursued Problem

Self-explanation can deepen understanding and improve learning outcome (Aleven & Koedinger, 2002; Neuman & Schwarz, 2000). It plays an especially important role in current reform-based mathematical classroom discourse where *all* students are expected to learn to be clear and convincing in expressing their own ideas and to listen to, understand, and make connections with others' ideas and hence sharpen their thoughts (National Council of Teachers of Mathematics, 2000).

Students with learning disabilities (LD) encounter difficulties articulating or explaining well their reasoning processes due to the mathematical and communicative difficulties they may have. As such, they may need to repair incorrect or inaccurate articulations while self-explaining. Thus the use of repair request techniques is proposed as an intervention to elicit conversational repair from students with LD to improve their self-explanations.

Classroom Discourse and Students with LD

The importance of communication in learning stems from Vygotsky's social development theory, which emphasizes that social interaction is crucial in shaping cognition (Kotsopoulos, 2010). There are increasing interests in the role classroom communication plays in teaching and learning within academic content areas (Hicks, 1995-1996). Current reform in mathematics education calls for reasoning and communication aspects of mathematics learning to develop student competencies. Explanation is an important component in mathematics classroom discourse. Self-explanation is to explain to oneself problem-solving process and reasoning behind the process. If a student is able to make good self-explanation, s/he can also make him/herself clear when explaining to others.

According to existing literature, though it is difficult for students with disabilities to participate in discourse-oriented learning (Baxter, Woodward, & Olson, 2001; Maccini & Gagnon, 2002), they could still learn the thinking behaviors such as asking questions, disagreeing, explaining, and suggesting solutions (Berry & Kim, 2008). They can also benefit from the discourse-oriented classroom as their normally achieving peers do (Berry & Kim, 2008; Kroesbergen & Van Luit, 2002, 2003; Woodward & Baxter, 1997) if teachers use effective instructional strategies (Baxter, Woodward, & Olson, 2001).

In a constructivist learning process, to help a student produce a satisfactory explanation of thinking or reasoning, a teacher should collaborate with the student through dialogues with scaffolding techniques whenever needed for the student to achieve that goal.

Research Questions

The research questions are: (a) What are the characteristics of self-explanation utterances of students with LD, pertinent to repair frequencies and types before, during, and after intervention? (b) Will conversational repair increase the quality of self-explanation of students with LD? (c) Will the students gradually need fewer and fewer repair requests as they go through the intervention? (d) How will conversational repair help students' performance on word problem solving tests?

Methodology/Research Design

A multiple baseline design across participants was used. The experimental design included two major conditions for each participant: baseline (including pre-intervention assessment and post-intervention assessment conditions) and intervention. Chronologically, the experiment included four phases: pre-test phase, intervention phase, post-test phase, and transfer phase. When a phase ended, the next phase began the following day.

Measures

Dependent measures included students' word problem solving performance and self-explanation quality in the criterion test and the transfer test. The criterion test and its alternate forms were used in the pre-test, the intervention, and the post-test phases. The criterion test comprised 10 one-step *equal group* (EG) word problems (i.e., number of items in each group × number of groups = total number of items) with the unknown's position in a problem systematically varied. The transfer test had the same format as the criterion test except that the 10 word problems were two-step EG word problems.

Scoring. The participants' quality of self-explanation and accuracy of problem solving were scored. For each test, the quality of self-explanations was calculated as the total points earned for self-explanation statements divided by the total statements produced. Accuracy of problem solving referred to the percentage of problems solved correctly in each test. It was calculated as the total points earned divided by the total possible points.

Participants and Setting

About five students with LD from 4th grade in a Midwest urban public elementary school in the United States were selected as participants. They were pulled out to a quiet conference room four times a week to conduct the experiment. The whole process of the experiment was videotaped.

Procedures

In the baseline condition, students were asked to solve 10 one-step equal group (EG) word problems in each session. They first wrote down all of the problem-solving processes, including math equation(s) in the space below the problems, and then explained the solving steps to the investigator.

In the intervention condition, students' explanations were scaffolded by the investigator's repair requests. During intervention, participants' initial statements were evaluated for the quality of self-explanation. Also, the number of repair requests needed from the investigator to complete a satisfactory explanation of a problem was counted. When the first participant showed a clear and stable increase in self-explanation scores, a second student was introduced to the intervention condition.

Treatment components. Repair request referred to the prompts from the investigator requesting the participants to repair their explanation. The repair requests in this study were designed based on Weiner (2005): (1) requests for general information, (2) requests for specification/clarification, (3) requests for revision, and (4) direct other-repair. The forms of the requests were based on and maximally followed the basic types defined by Schegloff, Jefferson, and Sacks (1977).

1. The request for general information **"It is a good start. Could you tell me more?"** is given after a participant is given enough time to read and think about the problem, and produces a problematic initial explanation (being roughly correct but unclear, or incorrect, or offering no response at all) that needs to be

improved. After the participant repairs (or does not repair) the explanation, the following intervention will address the repaired (or initial, in case the participant does not try again) explanation if it is still problematic.

2. If repaired explanation is incorrect or offers no response at all, which reflects incorrect reasoning, the investigator will follow up to provide a hint and then request a revision by saying **"This number means…and this number means…"** (*Tells the participant the meaning of the two known numbers*.) **"So do you want to revise your explanation?"** If the following repaired explanation is still incorrect or offers no response at all, the investigator will implement direct teaching (based on Xin et al., 2008 and Xin & Zhang, 2009), and the participant will be asked to repeat what the investigator has just said; if the following repaired explanation becomes roughly correct but not clear enough, the investigator will implement the actions in "requests for specification/clarification."

3. If repaired explanation is roughly correct but not clear enough, the investigator will request specification/clarification of the unclear parts by repeating repairable parts in the participant's response and adding *a **wh-*** question word, or, by saying **"And could you tell me more about why you did so?"** If the following repaired explanation is still roughly correct but not clear enough, the investigator will do direct other repair. That is, the investigator will model a full-scored explanation to the participant. Then the participant is asked to repeat what the investigator has just said.

4. If in any place in the process the explanation is worth full score, the participant will be moved to the next problem.

Results

Prior to this study, a pilot study (Xin et al., in preparation) has shown that in a constructivistic learning environment, students with LD were still passively involved in the teaching-learning interaction when the teacher applied little scaffolding techniques. Therefore, the results of this study will show how the participants' performance in self-explanation and problem solving change from the baseline to the intervention condition. The self-explanation scores, test scores, and frequencies of repair requests will be presented in a graph and a systematic visual comparison of within- and across-conditions will be conducted to see the levels, trends, and variability of the data.

Discussion

This study has theoretical significance. First, it directly addresses the call for mathematical communication/discourse by the math education reform. It helps improve not only the achievement levels, but also articulation of math thinking and reasoning. Secondly, it focuses on students with LD. In the call for inclusive classroom, quality participation of all students (including those with LD) in classroom discourse will be crucial to the success of the whole class. Third, current research on students' verbalization in the field of special education covers various topics. However, few studies specifically examine the characteristics and quality improvement of self-explanations of students with LD. Fourth, this study enriches the research on self-explanation by extending it to the field of special education. Fifth, this study incorporates conversational repair, a concept in pragmatics, into math education with students with LD or learning difficulties. This study's practical significances include first, it provides a clearer picture for teachers and practitioners on how students with LD explain their reasoning in problem solving process. Second, it offers a strategy (conversational repair) that teachers or practitioners could easily use in conversations to work with the student by providing scaffolding to improve their explanation. Third, the repair requests and responses can also occur between students with LD and their normally achieving peers. As such, it can facilitate communication and discussion between students with LD and their normal achieving peers, and improve group work.

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WHAT KINDS OF ARGUMENTS DO EIGHTH GRADERS PREFER: PRELIMINARY RESULTS FROM AN EXPLORATORY STUDY

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In this work, using Harel and Sowder's framework as a guiding lens for data collection and analysis, we examined the proof schemes used by 16 eighth graders when validating mathematical statements from number theory, algebra, geometry and probability and statistics. We also elicited the participants' views regarding soundness of alternative explanations offered to justify the same statements. Our goal was to see what features of arguments they viewed as convincing and ways in which they reconciled differences between their own validating schemes and those of others, if any existed.

Introduction

Students' difficulty with both writing deductive proofs and recognizing them as viable vehicles for verifying validity of mathematical conjectures has long been recognized (Balacheff, 1988; Chazan, 1993; Coe & Ruthven, 1994; Knuth et al., 2002). It has also been established that students usually maintain different perspectives than those of mathematicians about the meaning of proofs and how to go about proving mathematical statements (Herbst, 2006).

Harel and Sowder (1998) posited that an individual's scheme for proving statements consists of what the person considers as ascertaining and persuading. These schemes are not built in a vacuum and are impacted by students' personal experiences, vocabulary and thinking habits. Harel and Sowder (1998) summarized the students' proof schemes to be of three types: external conviction, empirical, and analytical. External conviction proof schemes include instances where students accept the validity of an argument by referring to external sources. As such, they rely on either the appearance of the argument (instead of its content) that could include symbolic manipulation, or rely on words of an external authority such as a textbook or a teacher. Empirical proof schemes, inductive or perceptual, include instances when a student relies on examples or mental images to remove doubt about the validity of an argument. While an inductive proof scheme draws heavily on examination of cases for convincing oneself, a perceptual proof scheme is grounded in more intuitively coordinated mental procedures without realizing the impact of specific transformations. Lastly, analytical proof schemes follow logical deduction when validating conjectures, relying on either transformational structures (operations on objects) or axiomatic modes of reasoning which include resting upon defined and undefined terms, postulates or previously proven conjectures.

There is consensus that in order for students to move towards relying on analytical schemes when attempting to validate mathematical statements, they need to recognize deductive arguments as convincing statements instead of merely mechanical procedures that need to be followed. Despite this, research has shown consistently that students' misconceptions about proofs impede their capacity to recognize this critical issue (Herbst, 2006; Healy & Hoyles, 2000; Harel & Sowder, 1998). Due to the vital role that proofs and proving process play in the discipline of mathematics, the need to conceptualize ways in which students might be assisted in their development in this area is equally as important as to exam their thinking during the transitional band of moving from informal to formal mathematics. The existing body of research on students' proof schemes has focused primarily on students enrolled in upper secondary school levels or undergraduate university courses. It is not quite clear what schemes younger students, particularly those in middle grade levels might use when confronted with mathematical tasks needing validation. The purpose of our research was to address this gap in the literature.

Purpose of the Study

The goal of the study was threefold. First, we intended to examine how a group of eighth graders attempted to validate mathematical statements from four different content areas. Second, we intended to investigate the type of arguments that were most likely to be accepted by the students. Third, we explored whether exposure to various types of validation arguments (e.g., external, empirical, and analytical) motivated a reconsideration of utility of each when confronted with similar situations.

Methodology

This report is a part of a much larger, longitudinal research project in which we study the development of problem solving and mathematical reasoning skills of a cohort of approximately 150 children as they progress from eighth to tenth grade in their respective schools. All children come from urban communities, attending over 30 different middle and high schools across the state of Ohio. The sample used for analysis in the current study consists of 16 eighth graders involved in the larger project. At the time of data collection approximately one half of the children were enrolled in a geometry course and the other half in an algebra course (Integrated Algebra or Algebra I). The sample consisted primarily of children from historically underrepresented groups in STEM areas (75% Black, 15% Latino).

Data Collection and Analysis

Harel and Sowder's (1998) framework guided the development of the data-collection instrument, *Survey of Reasoning*, as well as data analysis efforts. *Survey of Reasoning* consisted of four problems that focused on proving and disproving conjectures in number theory, algebra, geometry, and probability & statistics, respectively. Each problem consisted of several parts. The participants needed to first decide on the validity of the statement of the problem and support their conclusion using their own explanations; then four different types of arguments to justify the validity of the same problem were given and students were asked to decide which argument they preferred and why. Lastly, the participants were asked to judge whether each of the given arguments was convincing and whether it was mathematically complete. Students needed to provide narrative explanations to further support their claim in answering each of the questions. The content of the problems used in the survey will be shared during the presentation.

The choice of arguments offered in the survey was essential to the results we could obtain from the survey. In order to distinguish what kinds of arguments were most accessible to the participants, we deliberately designed arguments that represented different proof schemes as identified by Harel & Sowder's (1998) model.

In analyzing the data, the participants' responses were coded according to indicators associated with each of the three proof schemes. Prominent modes of reasoning used by the participants were catalogued and tallied across the sample and problem type. Qualitative analysis of responses included identifying particular lines of reasoning participants had used when verifying statements, issues they had considered when suggesting an argument as mathematically complete, as well as whether consistency existed between problems regarding what students considered most convincing and most mathematically complete.

Results

Table 1 illustrates the proof schemes that students used in responding to each of the problems on the survey (NA indicates that either participants didn't offer a response or their work was not understandable to the researchers). Consistent with the findings of previous research the participants in this study relied heavily on empirical evidence when verifying arguments. However, the dominance of the empirical scheme varied according to the mathematical context on which they worked.

Table 1: The Proof Scheme of the Arguments Created by Students in Each Problem

	Number Theory	Geometry	Probability	Algebra	Accumulative
External	0	2	1	0	3
Empirical	12	8	3	8	31
Analytical	1	2	5	4	12
NA	3	4	7	4	18

Table 2 illustrates the proof schemes the students preferred among the 4 arguments provided in each problem. These results again depict that students' explanations can vary depending on the context. As such, the results suggest possible transitional paths for production of proof schemes learners are inspired by understandable and informative arguments. Interestingly, even the same individuals who had expressed preference of their own arguments favored a different argument type when answering the same question in another problem.

Table 2: The Proof Scheme of the Arguments Preferred by Students in Each Problem

	Number Theory	Geometry	Probability	Algebra	Accumulative
Empirical	11	4	2	0	17
Analytical	2	10	14	16	42
Own	1	1	0	0	2
NA	2	1	0	0	3

The participants indicated an analytical argument as their favorite choice in 11 of 31 cases where the explanations created by themselves followed empirical approaches; while among the 12 cases where students offered analytical explanations, only 1 indicated preference for an empirical argument given to support the same problem. This suggests that students can and do recognize the advantages of analytical approach over empirical explanations. Once students were able to produce analytical explanations they were unlikely to prefer empirically based argument in the same context.

What students considered as *convincing argument* was not always consistent with what they recognized as mathematically complete. Several participants' choice of *mathematically complete* arguments in each of the four problems could have relied more on formal appearance of statements. In particular, in explaining their choice they often suggested that while the argument they had selected was least understandable to them they felt it "seemed more mathematical than others."

Perhaps the most important result concerned the potential learning that seemingly emerged from the review of optional arguments provided in each item. In conceptualizing ways in which children may be assisted in the development of their reasoning skills and the use of formal deductive arguments when verifying statements, we had conjectured that by increasing their exposure to a variety of argument types (drawing from multiple representations and visual models) we could increase their repertoire of approaches and hence influence resources from which they could draw when working on a mathematical task. This conjecture was indeed confirmed as a large number of students suggested that by studying the four arguments, they had not only gained a better understanding of the task but also methods of proving or disproving statements. This merits further and extensive study. Samples of the content of the survey and students' responses will be shared during the presentation at the conference.

Summary

Results of the study indicated that induction from empirical evidence was the most common reasoning scheme used by the participants as evidenced on their responses on the survey. However, since the participants exhibited the tendency to use empirical reasoning mode differently when verifying statements in different content areas, we believe it is premature to assert that empirical reasoning is the natural validation method used by students at this grade level in every situation. Results of our quantitative analysis demonstrated the external, empirical and analytical aspects of participants' reasoning when justifying various conjectures and shed light on contextual factors that may encourage each type of reasoning.

Our data indicated the existence of a gap between what students considered as *mathematically complete* and *convincing* arguments, and the latter were more preferred and more likely to motivate their reconsideration of how they could justify a statement. This suggests that by offering arguments that represent different proof schemes in different contexts, we might have a greater chance of transitioning students from informal to more formal modes of reasoning by providing them the access to a greater representational repertoire that they could use to build intuition and understanding.

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QUANTITATIVE REASONING ABOUT ABSOLUTE VALUE DURING GUIDED REINVENTION OF SEQUENCE CONVERGENCE

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We focus on two pairs of students from two different guided reinvention teaching experiments where students created rigorous formal sequence convergence definitions. Neither pair had been instructed on formal limit definitions, but both pairs were able to reinvent definitions consistent with formal theory. Obvious differences existed between how each pair reasoned about absolute value to denote the distance between individual sequence terms and the value of the limit. One pair quickly conceived of absolute value as a measurable attribute of a sequence graph (i.e., a quantity), while the other pair initially conceived of absolute value as a transformation. We detail the emergence of both pairs' quantitative reasoning about absolute value and shed light on why the second pair's adoption of absolute value as a quantity came much later.

Keywords: Advanced Mathematical Thinking; Design Experiments; Learning Progression; Learning Theory

Introduction

Some recent studies have adapted the guided reinvention heuristic to detail some of the challenges students face while attempting to articulate their emerging formal definitions of limit concepts (Swinyard, 2011; Oehrtman et al., 2011; Martin et al., 2011). In the case of sequence convergence, students constructed and refined their definition primarily through iterative cycles of attempting to capture and test salient relationships among quantities they identified in graphical examples. This paper begins to detail the role of such quantitative reasoning (e.g., Thompson, 1994) during guided reinvention of limit definitions by focusing on students' reasoning concerning absolute value while constructing formal sequence definitions.

Theoretical Perspective

A *guided reinvention* approach was chosen to support students in progressively constructing formal mathematics for themselves (e.g., Gravemeijer, 1998, 1999). Oehrtman et al. (2011) described student reinvention of sequence convergence as an iterative refinement process where students wrote a definition, evaluated their definition against examples and non-examples, acknowledged problems with their definition, discussed potential solutions, and attempted to incorporate solutions into a new definition, thereby initiating another iteration. *Problems* are identified by students typically due to conflicts between their concept image and their current definition. Students' explicit resolutions of problems lead to ideas that remain stable throughout remaining iterations. During the process, a mathematician may identify other *problematic issues* with a definition that the students have yet to identify as problems. When students persist in overlooking a problematic issue, facilitators can act as *conflict producers* by, for instance, asking students to interpret their definition applied to a particular example. After students wrestle with a problem for a significant time and have sufficient understanding of solution elements, but remain unable to come to a satisfactory resolution, facilitators might also act as *solution providers*.

Quantitative reasoning involves reasoning about quantities that includes the mental actions of conceiving of an object (e.g., a graph of the sequence $\{a_n\}$), attending to measurable attributes of the object (e.g., distances on a graph), constructing representations of measurements of these attributes (e.g., $|a_n - L|$,

N, or ε), and developing and reasoning about relationships between these constructed quantities (e.g., for all $n \geq N$, $| a_n - L | < \varepsilon$) (Moore, Carlson, & Oehrtman, 2009; Thompson, 1994). The research reported here focuses on the interaction between problems and quantitative reasoning about absolute value by asking the following in the context of guided reinvention: (a) How do students come to reason quantitatively about the role of absolute value? (b) What roles do problems play in students' quantitative reasoning about absolute value? and (c) What roles can facilitators play in helping students to reason quantitatively?

Method

Participating students had experience with sequences during second-semester calculus courses, but had not received instruction on formal limit definitions. Each teaching experiment was comprised of 90- to 120-minute sessions with the objective of generating a rigorous definition for *sequence convergence*. Megan and Belinda (pseudonyms) were selected from a nonstandard calculus class using Oehrtman's (2008) instructional framework where concepts involving limits are systematized around a theme of unknowns, approximations, errors, and bounds on errors. Over eight days, they participated in six sessions at a large, southwest, urban university. Joann and David (pseudonyms) were selected from a more traditional calculus class using Tan's *Calculus* (2009). Over one month, they participated in nine sessions at a small, southwest university.

Each pair initially created what they viewed as prototypical examples of sequences converging to 5 and not converging to 5. They wrote a definition which they iteratively revised until they felt it captured all their examples and excluded all their non-examples. Facilitators steered discourse to help students progress, but at the same time were diligent to preserve their "intellectual autonomy" (Gravemeijer, 1998, p. 279).

The research team created content logs containing time-stamped descriptions of the students' and facilitators' activities and theoretical notes about how progress was being made toward a formal definition. The timeline was coded for problem(s) being addressed and for students' reasoning about absolute value. Attempts were made to determine the interactions between problems and reasoning about absolute value and why particular reasoning emerged.

Results

For each pair of students, we focus only on their quantitative reasoning concerning absolute value in relation to the problems they were engaging as they iteratively refined their definition.

Megan & Belinda

Megan and Belinda first engaged the problem of "bad early behavior" of a convergent sequence with terms moving away from the limit. Their resolution was to include their notion of "at some point *n*" into a definition to identify a point at which terms became "closer and closer" to 5. Megan explained this notion by *pointing to* individual points on their graph (**Figure 6**), and noting how points get both further away and closer to 5. While she attached descriptions of closeness to relative locations of points, she did not indicate any explicit measurable attribute of the graph or mention any formula.

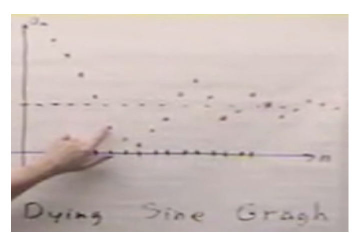

Figure 6: Pointing to individual points

Within 12 minutes of the experiment's start, Megan spontaneously rearticulated "closer and closer" in the context of errors. Both students highlighted vertical distances between points and the limit (7) and attached these vertical distances to the expression $| 5 - a_n |$, which they then included in their definition.

Afterwards, they returned to using dynamic language without reference to errors as they wrestled with another problem: "How close is close?" They made little progress during this time. However, within five minutes of facilitators guiding the students back to approximation language, they coordinated error with the emergence of error bounds and produced their next definition which contained an explicit relationship between both ("$| 5 - a_n | < .01$"). They viewed this relationship as resolving the issue of "How close is close?" provided that .01 is deemed as an "acceptable error range." Their use of error and its coordination with other quantities remained relatively stable throughout the remainder of the reinvention.

Figure 7: Highlighting error as vertical distance

Joann & David

Joann and David first discussed absolute value while engaging the problem of clarifying the meaning of "approaches" for a damped alternating graph. Initially equating "approaches" with "monotonically decreases," they voiced the following convergence test: Shift the graph down to zero, reflect the negative-valued terms over the *x*-axis, and if it "…converges to zero, it converges to 5." They expressed this in Definition 2 as "$| 5 - a_n |$ approaches or equals 0." Unlike Megan and Belinda, they took an hour to incorporate absolute value into their definition.

While testing Definition 2 against a damped oscillating sequence (as in Figure 3) where each subsequent term does not necessarily move closer to the limit, Joann and David concluded that their notion of "approaches" did not work. This led Joann to articulate ideas about "closeness" using her concept of "breaking decimal barriers" which evolved to a notion consistent with ε in the standard ε-N definition. They felt that they had resolved their problem but as they applied their definition to graphs they consistently interpreted absolute value as a transformation.

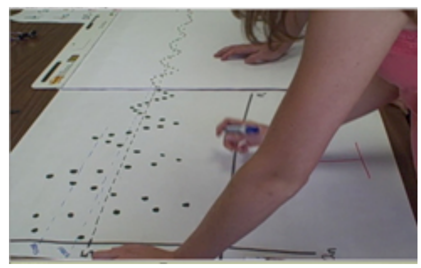

Figure 8: Decimal barriers on damped oscillating sequence

Although Joann's and David's interpretation of absolute value as a transformation was potentially viable for creating a rigorous definition, the facilitators determined that the approach would be inefficient and could create conceptual difficulties when producing limit-related proofs. Thus, on Day 3, using a ready-made graph (see Figure 3), Joann and David were asked to explain the meaning of $| a_n - 5 |$ for the sixth dot without transformations. Joann replied, "[a_1 and a_6] are the same *distance* from 5. $| a_1 - 5 |$ is the same as $| a_6 - 5 |$." David responded, "Exactly." Yet, on Day 4, when again asked to explain without transformations, Joann exclaimed: "All I can think of is shifting and reflecting!" After some time, David articulated absolute value as distance and marked the graph to show a term's distance from 5. Joann said, "So the absolute value is … your distance."

On Day 5, though David was repeatedly asked to explain $| a_n - 5 |$ without transformations, he continued to do so. He explained, "I'm not trying to reflect. This is how I would represent positive distance." Later, after Joann and David had drawn barriers around 5 on the damped oscillating graph, they no longer shifted and reflected as they coordinated absolute values with decimal barriers. From then on, they consistently referred to absolute value as distance.

Conclusion and Discussion

Graphs were essential for the constitution of relevant quantities, but addressing problems provided the context. For both pairs, absolute value as a quantity emerged as they contemplated problems involving notions of "closeness." Unfortunately, their quantitative reasoning did not emerge to resolve *explicit* problems, otherwise they would not have reverted to prior rejected interpretations. For example, Joann and David initially indicated that they *did* understand absolute value as distance but their transformation conception served them well as they were addressing early problems. Even by Day 5, David's need to shift and reflect before talking about a term's distance from 5 suggests that absolute value had not yet been reified as a quantity.

The little progress that Megan and Belinda made when using dynamic interpretations, their sudden shift to absolute value as quantity after introducing "error language," the swift incorporation of symbols, and their ability to coordinate errors and error bounds, suggest that the systematization provided by approximation instruction supported the emergence of relevant quantities not supported by dynamic interpretations. Joann and David's struggles and lack of experience with approximation activities supports this conclusion.

In contrast to Megan and Belinda, Joann's and David's eventual adoption of absolute value as a quantity resulted from *repeated* facilitator intervention, as well as the conceiving of and the repetitive coordination with other quantities such as decimal barriers. This suggests the need for other quantities to aid the emergence of related quantities. Had the facilitators been able to produce cognitive conflict causing Joann and David to identify a problem with their transformational reasoning, they may have conceived of absolute value as a quantity sooner.

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IMPLEMENTATION FOR JUSTIFICATION: SUPPORTING MIDDLE SCHOOL STUDENT ARGUMENTATION

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This study explores how four middle school math teachers support student justification on a "scaling" task implemented over two years. This research building on the framework developed by Stein and colleagues (1996) by providing evidence for three types of supports for justification: high level performance modeled, sustained pressure for explanation and meaning, and task specific scaffolding. In addition, this research reports three additional practices that support justification: reviewing as an opportunity to justify; cyclic iterations of investigation and discussion; and building on incomplete, incorrect or misunderstood justifications.

Keywords: Reasoning and Proof; Middle School Education; Instructional Activities and Practices

Literature Review and Theoretical Framework

Many studies have reported difficulties associated the implementation of justification tasks (Bieda, 2010; **Henningsen & Stein, 1997)**. Stein, Grover, and Henningsen (1996) studied a sample of 144 reform-oriented middle school task implementations, 61 of those tasks declined in cognitive demand between set-up and implementation stages, with a greater decline for high cognitive demand tasks. The researchers identified seven attributes that were common among the tasks that maintained faithful implementation to high cognitive demand: building on students' knowledge, appropriate amount of time, high-level performance modeled, sustained pressure for explanation and meaning, scaffolding, student self-monitoring, and teacher drawing conceptual connections. Bieda (2010) applied the mathematical task framework to examine student opportunities to prove within the middle school setting. Bieda found that student proving opportunities were limited, and student proofs were even more rare: of the 109 student conjectures 59 resulted in justifications, although 31 of those 59 were empirical or non-proof arguments. This research extends the work of both Stein and colleagues (1996) and Bieda (2010) by identifying components of task implementation that led to student justification.

Study Background

The data for this study is drawn from an NSF funded project called "Justification and Argumentation: Understanding Algebraic Reasoning (JAGUAR)." Twelve teachers in two states were selected to take part in the two year-long JAGUAR project which included two one week long summer session, four prescribed justification lessons, and three working sessions through each of the two school years. The primary research questions of the project focus on how teachers develop mathematical knowledge about justification and how teachers transform that knowledge into their classroom practice. The teachers selected to participate in the JAGUAR project were intentionally selected based upon previous professional development and are not representative of typical middle school teachers.

The analysis for this study draws upon classroom data from four of the twelve case teachers: Audrey Tompkins, Bruce Hummel, Paige Davilla, and Cynthia Littrel (all teacher and student names are pseudonyms). The analysis focuses on the implementation of a single prescribed task: "The Scaling Task." The task as presented to the teachers was: "How does scaling various 2-dimensional [and 3 dimensional] figures impact the perimeter, area, [surface area and volume]?" Data for this analysis include copies of student work, written teacher reflections on the lesson, classroom video, and classroom video transcripts.

Methodology

In the first stage of analysis justification episodes were identified. Justifications are mathematically valid and connect previously accepted facts or definitions to conclusions using valid forms of reasoning (see Stylianides & Stylianides, 2009). A simple statement of an accepted fact or algorithm did not constitute a justification. For example, when finding the perimeter of a rectangle with length 10 and width 4, "I added 10 plus 4 times 2 to get 28 centimeters" is not considered a justification because it only describes a procedure without supporting the validity of the procedure. On the other hand, the following explanation is considered a justification because it justifies the operations by connecting them to structural features of square: "Well, because *s* is like the side length, and then *k* is the scale factor, so we have *s* times *k* equals the similar side length. And then you times it by 4 because there is 4 sides."

Once the justification episodes were identified, social and environmental scaffolds leading to those justifications were identified. Identification of these factors was guided by, but not limited to, the Stein and colleagues' (1996) seven traits for maintenance of cognitive demand. In the final phase of analysis, three emergent themes were identified as independent from the original themes: reviewing as an opportunity to justify; cyclic iterations of investigation and discussion; and building on incomplete, incorrect or misunderstood justifications. This list of themes represents only a subset of the possible factors that encourage the expression of student justification within a classroom.

Data and Analysis

Support for Previously Identified Themes

High level performance modeled. Stein and colleagues (1996) found that when teachers or more capable students modeled high levels of performance high levels of cognitive demand were more likely to be maintained. In this study, three student justifications were modeled after a previous correct justification given by another student. For example, in Mr. Hummel's second implementation of the task, Rick, one of the more capable students, gave a pictorial argument for the relationship between the area of the original square and a scaled-up square. The next day of class, students explored rules for different shapes. Clay modeled his justification for the scaling of a parallelogram directly off of Rick's justification from the previous day of class (Figure 1).

Clay: Alright, I got original area times scale factor squared.
Mr. Hummel: Okay. And can you explain your picture? Why that works there?
Clay: Uh- this is the original one right there [pointing to the bottom left parallelogram]. That one is the original and then times 2, 3, 4... [outlining each scaled up parallelogram on his drawing as he counts] And then times 3 squared would be 9.

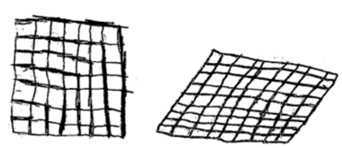

Figure 1: Clay's picture of a scaled up parallelogram

It is important to note that Clay was not simply mimicking the images from Rick's previous work. In this passage, Clay develops an independent justification for why the area of a parallelogram is 9 times bigger when the side lengths are scaled up by three.

Sustained pressure for explanation and meaning. Throughout the implementations of each of these tasks, teachers maintained strong pressure for explanation and meaning making, primarily through their use of questioning. Based upon analysis of teacher questioning patterns at the local level, four different types of teacher questions emerged. "Why" questions such as "Why is this happening?" ask students to explain a mathematical relationship or process. "Explain" questions ask students to more fully explain their work or their thinking. “Representation” questions ask students to alter, add, or improve their representation of a problem. Finally, “Skeptic” questions cast doubt on student claims and prompted students to provide further reasoning. For example, Ms. Littrel states, “I’m not totally sold yet.”

Task specific scaffolding. According to Stein and colleagues (1996), task specific scaffolding breaks a task down into smaller or more manageable tasks, while at maintaining the cognitive demand of the task in spite of the decomposition. Several teachers used representational scaffolding to help students access and reason about the task. For example, in Ms. Littrel’s first year of task implementation the class co-constructed a completely generalized diagram representing an original figure and a scaled-up figure. This diagram served as a transformational record, which is a “notation, diagram, or other graphical representation that [is] intentionally used to record student thinking and then are later used by students to solve new problems” (Rasmussen & Marrongelle, 2006).

Emergent Theme 1: Reviewing as an Opportunity to Justify

Teachers used review of pervious material as an opportunity for students to justify. For example, Ms. Tompkins noticed that many students were having trouble calculating the surface area of a rectangular prism even though surface area had been a content topic from the previous week. Tompkins asked a student to explain for the class how to find the surface area.

Ms. Tompkins: Would you come up? Would you explain to us? And again, if you've done this, this is a review, but maybe it's um, *we need to review it again* [emphasis added].

Sharon: You're just multiplying the length of the sides by the width or the height, whichever one it is. And so then you multiply it by two because there's one on the opposite side that's exactly the same. And so … each one is times two.

Students may have had better access to resources required for constructing justifications for review topics, because review contexts allow students to focus on the act of justification apart from conceptually challenging material. These findings are consistent with the results of Niemi (1996): stronger student content understanding is associated with stronger student explanation and justification.

Emergent Theme 2: Cyclic Iterations of Investigation and Discussion

In all of the task implementations across both years, all of the teachers utilized some variation of a “Think-Pair-Share” protocol in which students were given time to think individually, then work with a partner or small group, and then discuss as a whole class. Many valid student justifications emerged after multiple cycles of whole class and small group discussion time. For example, in the first year of Ms. Davilla’s scaling task implementation, students suggest that doubling the scale factor doubles the perimeter of the rectangle. Ms. Davilla asks students to consider this in small groups: “I'm not sure I'm ready to move on until people agree with this, understand this… I’m going to give you 30 seconds to a minute, discuss these questions in your group.” In the next whole class discussion, several students give empirical evidence for the rule, but fall short of a justification. Ms. Davilla again returns to small group discussion and asks the students to probe deeper into why the relationship holds. In the following small group time, several groups formulate valid justifications, one of which is shared with the class, “I think this is happening because maybe it’s the scale factor you have to multiply everything so like everything doubles, the perimeter will double as well.” The iterative cycles of student work time and whole class discussion allow the teacher to draw student attention toward increasingly sophisticated components of the task, and allow students to synthesize and build upon the ideas of their peers presented in whole class discussion.

Emergent Theme 3: Building on Incomplete, Incorrect or Misunderstood Justifications

In contrast to the previous theme of *high level of performance modeled*, eight justifications in this data were given in response to incomplete, incorrect, or misunderstood justifications given by other students. For example, in Ms. Littrel’s class there was a debate about whether $3s$ is equal to s^3. Janis offers her justification to the class, but her justification is not understood by the class. In response Zoe provides a different justification.

Janis: Because that one [points to 3*s] is multiplying 3 by s, and that one [points to s^3] is multiplying s three times.
Ms. Littrel: Ok, but I am not convinced.
Student: That's the same thing. You just said the same thing.
Janis: Ok, well Zoe can help me.
Ms. Littrel: Do you want to come support Zoe?
Zoe: Ok, so if $s = 2$, then 3 times s would be 3 times 2 which equals 6. And then 2 cubed which would be 2 cubed equals 2 times 2 times 2 which equals 8.

Although the first justification offered by Janis was correct, it was not understood by the class, and it allowed Zoe the opportunity to provide an empirical proof. Empirical reasoning is often favored by students (Stylianides & Stylianides, 2009), and although it is typically invalid, in this case it provides a valid counterexample to disprove the conjecture that $3s = s^3$. Building on student contributions is an important part of advancing the mathematical agenda as well as an important opportunity for student justification.

Conclusions and Implications for Further Research

This paper provides evidence in support of five of the themes identified by Stein and others (1996) and describes additional three elements of implementation that appear to support justification. This work has implications both for teacher practice and professional development. For example, allowing students to share incomplete, incorrect, or misunderstood justifications can provide strong opportunities for further student justifications. Many teachers are uncomfortable and resistant to using wrong answers within the classroom (Hoffman et al., 2009). A challenge in professional development is to help teachers create a critical classroom environment in which productive incomplete ideas are leveraged for deeper student learning.

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PUSHING SYMBOLS: TEACHING THE STRUCTURE OF ALGEBRAIC EXPRESSIONS

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We describe an intervention being developed by our research team, Pushing Symbols. PS is designed to encourage learners to treat symbols as physical objects, which move and change over time according to dynamic principles. By providing students with the opportunities to physically manipulate colored symbol tiles and interact with a new touchscreen software technology, we aim to help students learn the structure of algebraic notation in general, and in particular learn to simplify like terms. We present preliminary findings from a study with 70 middle-school students who participated in the intervention over a three-hour period.

Keywords: Algebra and Algebraic Thinking; Technology; Instructional Activities and Practices

Students have great difficulty in mastering basic algebra content and notation (NAEP, 2011). Often, instruction emphasizes content while students struggle to understand basic notation (Koedinger & Alibali, 2008), which is often presented as the memorization of abstract rules. However, algebraic literacy—the fluent construction, interpretation, and manipulation of algebraic notations—involves not just memorizing, but learning appropriate perceptual processes (Landy, 2010; Kirshner, 1989; Kellman, Massey, & Son, 2010).

Our theoretical framework comes from work in cognitive psychology and perception. Successful students often use perceptual and visual patterns available in notations to solve mathematical problems. Like many skills learned from long practice, learning algebra involves perceptual training—learning to *see* equations as structured objects and chunks (Kirshner, 1989; Kellman et al., 2008). Because rigid motion is a powerful perceptual grouping mechanism (Palmer, 1999), and real-world motion is naturally memorable and easy to acquire, it is anticipated that training students to see correct algebraic structures through dynamic transformations may be a promising approach to teaching algebraic ideas (Landy, 2010). The purpose of this study is to describe and present preliminary findings from an intervention exploring this approach to improve student learning of algebra notation.

Pushing Symbols: Teaching the Structure of Algebraic Expressions

The purpose of the PS intervention is to explore an alternative method of algebra instruction that focuses student efforts on the visual structure of notation. The core idea of this approach is that symbolic strings can be usefully thought of as physical objects located in space, and that proofs or derivations can often be thought of as dynamic transformations of those objects. The intervention sustains this idea by allowing students to physically and dynamically interact with elements of algebraic expression, through a set of in-class discussions, activities, and a dynamic computer-based visualization method by which students manipulate expressions with their hands. The symbol strings respond in mathematically appropriate ways. The strong role of the expression in maintaining its own integrity is analogous to the ways that real physical objects maintain their own integrity: fluids flow when poured, while solids maintain their shape, for instance. This internal integrity allows the physical world to become intuitive and predictable. The PS program is designed likewise to build efficacy and engagement in students by creating a formal symbolic structure that can be intuitive, predictable, and even fun.

The first component of the PS manipulative system uses colored magnets and tiles to model and decompose the structure of algebraic expressions. Each color represents a specific mathematical object (number, variable, coefficient, symbol). After modeling an expression, the tiles can be rearranged, transformed, and simplified into equivalent expressions—this component is intended to help students see pieces of symbol strings as physical objects, with real-world properties. The second component of the PS system uses a new touch-based computer application*, Algebra Touch: Research (ATR),* developed in collaboration with Regular Berry software and based on the commercially available *Algebra Touch* system. In ATR, students perform arithmetic functions by tapping on a sign and carry out algebraic rearrangements by touching appropriate symbols and moving them into the desired location. An example of *ATR* can be seen at http://davidlandy.net /PushingSymbols/RPS--12-1-11-Like-Terms-Automatic-1.mov**.** ATR does not allow students to make mistakes; if they attempt to do something that violates the laws of mathematics, a brief side-to-side motion (a “shake”) provides immediate feedback that their desired action was illegal. As a result, students immediately see how the rules result in legal transformations or manipulations in a way that is impossible with a traditional lesson. Problems in ATR can be presented in either an untimed *list* mode or a *game* mode. At the end of each problem, the program provides immediate feedback to students about the number of errors they made and the speed to which they simplified the expression.

In this present study, we report preliminary results from a study implementing the PS intervention in 3 middle school classrooms. We aim to improve students’ understanding of algebraic structure through engaging students in perceptual training. We anticipated that the intervention would decrease the number of structural errors that students made on procedural problems, but not a task that required primarily bridging formal notation with situations (equation modeling). Second, we hypothesized that pre-test scores, self-efficacy, engagement, and performance on the iPad would positively contribute to post-test scores, while math anxiety would negatively contribute to post-test performance.

Methods

Participants

Seventy eighth-grade students from an urban public middle school in the mid-east United States participated in this study during their regular mathematics instruction time. Student assent and parental consent were obtained prior to participation in this study.

Procedures

The study took approximately 3 hours and occurred over three class periods. On the first day (90 minutes), students completed the simplifying expressions pretest and the self-efficacy & anxiety questionnaire. Next, the teacher led a whole-group lesson and led a series of discussions and activities using colored magnets and tiles to demonstrate algebraic structure, followed by 20 minutes of familiarization with the iPad and ATR. On the second day (90 minutes), students were given 40 minutes to simplify both simple and complex expressions, followed by the engagement questionnaire and post-test. Two weeks later, students completed a retention test assessing simplification.

Measures

Simplifying Expressions Assessments. Each child completed an 18-item pre, post, and retention test involving procedural facility with simplification (10-items) and expression construction (word problems) (6 items). For each assessment (pre, post, and retention), we calculated 2 composite scores: (1) proportion of attempted procedural problems that were free of structural errors (i.e., combining unlike terms, over-combination, or partial structural errors); and (2) proportion of attempted word problems that were modeled correctly.

ATR Performance. ATR Performance was measured by calculating total points at 2 different levels (simple and complex), using the *Algebra Speed* game. Level 1 asked students to simplify a series of 36 *simple* expressions (e.g., 5+7+3; x+2+6), while Level 2 asked students to simplify a series of 40 *complex* expressions (e.g., 7+2x+5x+4y+1+-2y). Students could receive a maximum of 3 points for each problem solved. The points system accounted both the number of errors that they made and the speed to which they simplified the expression.

Mathematics Self-Efficacy and Anxiety Questionnaire. Students were administered a set of 10-items pertaining to their self-efficacy (Midgley et al., 2000; 5 items, α=.82) and anxiety in mathematics (5 items, α=.61)

Student Engagement in Mathematics Questionnaire. Student engagement during the lesson was measured using 18 subjective rating items that were adapted from the Student Engagement in Mathematics Questionnaire (Kong, Wong, & Lam, 2003).

Results

Analysis 1: Does the Pushing Symbols Intervention Improve Student Understanding of Algebraic Structure?

On average the intervention increased students' knowledge of algebraic structure (Figure 1). At pretest only 9.4% of problems were solved without structural errors. At post-test 54% of problems attempted were solved without structural errors (Improvement of 44.6%, $t=10.48$, $p<0.01$). At retention 41.4% of the problems were solved without structural errors (overall improvement of 32%, $t=6.81$, $p<0.01$). After 2 weeks students retained 72% of their structural learning. As expected, the intervention did not appear to affect equation modeling at post-test ($t=-0.87$, $p>0.05$) or retention ($t=-0.07$, $p>0.05$).

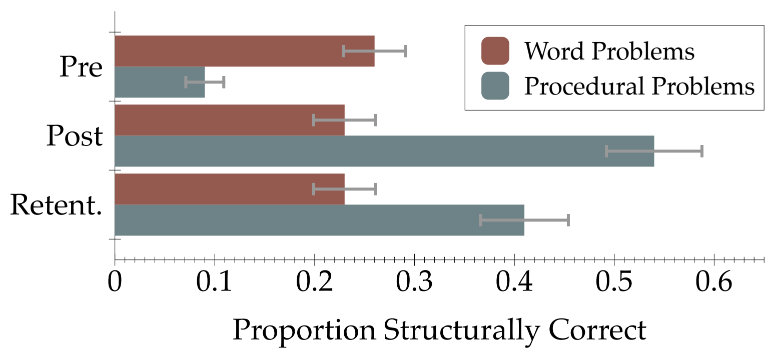

Figure 1: Performance on assessments

Analysis 2: Relations between Structural Performance, Efficacy, Anxiety, Engagement, and Performance on *ATR*.

A regression analysis was conducted to examine predictors of structural performance on the post-test. Results indicate that math efficacy was related to higher performance on the post-test (β=1.27). Second, successfully completing more problems (both simple and complex) on ATR was related to higher scores on the post-test. Further, students who reported being more engaged during the PS intervention performed higher on the post-test (β=1.80). Interestingly, neither students' performance on the pre-test or levels of math anxiety predicted post-test performance.

Discussion

We have described an approach to algebra instruction that emphasizes perceptual and manual interactions with dynamically realized models of algebraic notation. These preliminary results demonstrate that a short intervention based on this framework can help students become fluent with algebraic structure and substantially improve student performance at simplifying expressions. Furthermore, this work adds to a small literature suggesting that touchscreen-based learning tools can successfully lead to student learning (Segal, 2011).

The current findings demonstrate that using a hands-on approach to teaching the structure of algebra may benefit students. Given the clear demonstrations that students struggle to understand basic algebraic notation (Koedinger, Alibali, & Nathan, 2008) and existing evidence linking teaching algebraic structure to improved student understanding of algebraic expressions (Banerjee & Subramaniam, 2011), we believe that there is good reason to pursue manipulative systems that expressly communicate algebraic structure through engaging perceptual and motor interactions. The current system contrasts with many popular algebra manipulative systems, such as Algebra Tiles and Hands-on Equations, in its emphasis on the structure of mathematical expressions rather than models of the concepts referred to by them. It is also worth noting that the intervention seemed to increase student interest, participation, and interactions. Both observational and student reported engagement during this intervention was high. Virtually all students reported that the intervention was highly engaging, fun, and helped make the mathematics easier to understand.

These results are clearly preliminary, and the conclusions that can be drawn from the empirical results are limited. It is unclear how the learning that results this intervention differs from typical classroom learning, and how such differences may impact learning of future topics.

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BECOMING A MATHEMATICAL AUTHORITY: THE SOLUTION LIES IN THE SOLUTION

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This paper focuses on the development of three skills underlying mathematical authority: (1) explanation, (2) justification, and (3) assessment. An intervention was designed to help students develop these skills through explicit engagement with assessment in the classroom. Preliminary results from this ongoing study indicate that students had improved meta-level understandings of solutions, which supported greater levels of explanation in their solutions of problems.

Keywords: Design Experiments; Instructional Activities and Practices; Metacognition

Introduction

When a mathematician solves a problem or submits a proof to a journal, he or she doesn't wonder whether or not the work is correct; he or she *knows* it is. Most mathematicians self-assess using highly-internalized mathematical standards. In contrast, mathematics students routinely submit assignments with little sense of how well they did, relying on their instructor to be the arbiter of mathematical truth. For these students, mathematical authority is something that exists externally to them. This paper is focused on how students internalize mathematical standards.

Mathematical authority relates to positioning and identity (cf. Boaler & Greeno, 2000; Engle & Conant, 2002), as well as specific skills and domain knowledge. This paper focuses on three mutually supportive skills, hypothesized to be crucial to mathematical authority (as conceptualized in Figure 1): (1) explanation, (2) justification, and (3) assessment. Mathematicians use these skills to derive authority from the logic and structure of mathematics (internalized authority), rather than relying on some other authoritative source like a teacher or textbook (external authority). These skills are widely recognized as part of the multi-faceted nature of mathematical proficiency (e.g., NCTM, 2000) with explanation and argumentation specifically emphasized by the Common Core State Standards (Common Core State Standards Initiative [CCSSI], 2010).

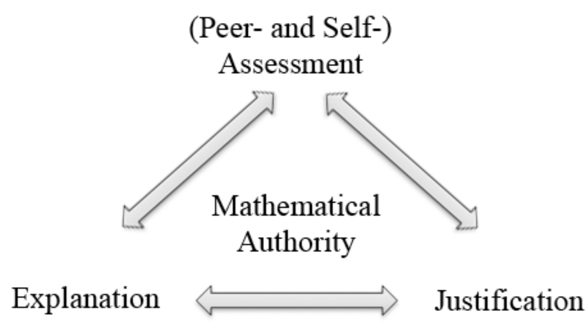

Figure 1: Three skills hypothesized to underlie mathematical authority

The design of this study's intervention draws on research showing that when students self-assess, they are unlikely to spontaneously generate information to test their understanding, which impedes accurate self-assessment (Dunlosky & Lipko, 2007). Just as perceiving constructive and deductive geometry as unrelated hinders mathematical performance (Schoenfeld, 1988), I hypothesized that perceiving explanation and justification as extraneous parts of a solution inhibits accurate self-assessment. When students are unable to use their own reasoning to justify their work, they are forced to rely on an external mathematical authority. Sadler (1989) suggests that standards of a high-quality solution should be communicated through a combination of descriptive statements, exposure to exemplars, and direct evaluative experience. Crucially, as students analyze others' work, they develop the required objectivity and skills to assess their own work (Black, Harrison, & Lee, 2003). Thus, this study used peer-assessment to promote the development meta-level understandings of solutions that are crucial to self-assessment.

Methods

This paper draws on preliminary data collected during from an ongoing design research study with elementary algebra students (N = 20) at a community college in the San Francisco Bay Area. Data were collected from classroom videos, student written work, and the instructors' daily reflections. As a pre-test, students assessed sample written work of two hypothetical students solving the problem: "*If a tortoise is traveling at an average of 1 2/3 miles per hour, how long would it take the tortoise to travel 6 miles?*" (see Figure 2). Students were presented with two solutions sequentially, and after seeing each solution were asked to explain the hypothetical student's reasoning, and why it was correct or incorrect. Finally, students were asked to reconcile the two conflicting solutions, and explain how they could determine which solution was correct.

$$1\tfrac{2}{3} = \frac{5}{3}$$
$$\frac{5}{3} \cdot 6 = \frac{30}{3} = 10$$

a.

$$1\tfrac{2}{3} = \frac{3\cdot 1 + 2}{3} = \frac{5}{3}$$
$$6 \div \frac{5}{3} = 6 \cdot \frac{3}{5} = \frac{18}{5} = 3\tfrac{3}{5}$$

b.

Figure 2: (a) Initial sample solution; (b) Sample solution presented after initial assessment

As an intervention, students were introduced to a framework for assessing mathematical solutions. The framework emphasized that a solution should answer three questions for the reader: (1) What did you do?; (2) Why did you do it?; and (3) Did you do it correctly? These relate to three parts of a solution: (1) the execution, (2) the explanation, and (3) the justification. Guided by the instructor, students discussed features of high-quality solutions to generate a rubric based on the above framework. Students also engaged in various peer- and self-assessment tasks using the student-generated rubrics.

The results presented here document students' changes in their perceptions of solutions. Because the data are preliminary, and peer-assessment can be seen as a precursor to self-assessment (Black et al., 2003), this paper focuses on the development of understandings that would support self-assessment, but not their actual application to self-assessment. This brief report considers the development of a few focal students, to highlight trends within the larger data corpus.

Results and Analysis

In the pre-test, students articulated *what* steps the hypothetical students took to solve the problem, but could not explain *why* they took them, even when pressed by the instructor (e.g., "why did the student multiply rather than divide?"). When asked to determine which solution was correct, only one student generated an answer. This student recognized that if the tortoise was traveling faster than 1 mph, then 10 hours for a travel time was much too long, so therefore the first solution must be incorrect. Other students either responded that it was impossible for them to determine which solution was correct, or that they didn't know how to figure it out. Students asked the instructor to resolve the mathematics for them.

The pre-test provided the basis for classroom discussions about important qualities of a complete mathematics solution. In these discussions, students articulated that the sample solutions lacked detail, thus providing limited access to the hypothetical students' reasoning. Students were presented with a framework for high-quality solutions, and were guided to generate a rubric using this framework. Students suggested 8 important features of a solution, such as: a written statement explaining why the solution path was chosen, checking units, and estimating what a reasonable answer would be. Students then discussed how the sample solutions would have been easier to assess if they had these features. The instructor introduced 5 additional features of high-quality solutions to the class to complete the rubric.

Two weeks later, students were once again presented with one of the sample solutions from the pre-test (see Figure 1, sample a). After analyzing the solution using the rubric they had developed, students explained how generating a more complete solution would have helped their classmate. Some students, like Tanya, focused on specific solution features:

Tanya: It would have helped him if he put the units down on his paper to check what to cancel out, since the problem gives you miles and miles per hour.

Tanya seems to understand that units are not just part of a complete solution, but actually a tool for problem solving, because they help determine which arithmetic operations are meaningful to perform. Other students, like Enrique and Jason, focused on solutions holistically:

Enrique: A more complete solution would have made him catch his mistakes.
Jason: The execution is well done, but there's no explanation of any sort. The only thing that *seems* good is the answer.

Enrique's response emphasizes that careful solutions are important because they make our thinking (and thus mistakes) more evident. Jason alludes to the fact that the lack of explanation makes it difficult to say much about the student reasoning (e.g. "the only thing that *seems* good"). In sum, students transcended the specifics of the solution given, and exhibited meta-level understandings of solutions in general. These are the types of understandings that would allow students to begin to act as authorities themselves, rather than referring to an external authority.

As students develop a sense of high-quality mathematics solutions, it should also become evident in their written work. A comparison of students' solutions to the first two homework assignments (one week apart) provided evidence of such growth. (Note: the first homework assignment had 10 problems, and the second assignment had 11 problems but was of comparable length.) In general, solutions for the second homework assignment were more verbose and began to include explanations of reasoning (the first assignments contained little to no explanations). These changes were particularly striking for two of the students highlighted above, Jason and Tanya, whose solutions doubled in length (from 2 to 4 pages) between these two assignments. The increase in length was due to an inclusion of much more significant explanations and justifications in the second assignment.

Evidence of a more sophisticated understanding a solution was also evident in students' daily reflections. At the end of each class session, students were asked to answer a number of reflection questions, both in general and specifically related to the given lesson. When asked, "What does a good explanation in a math solution look like and why is it important?" Jason cogently responded:

Jason: A good explanation can help someone understand the problem just by redoing the steps you took. After reading the steps they know why you took those steps and what you were doing.

This response seems to indicate a transition to seeing the solution to a math problem as an explanation of one's reasoning, not just "finding an answer." Jason's initial homework assignment included little to no justification or explanation, whereas his second homework assignment and responses to in-class questions were much more complete. Although we can only infer how Jason *sees* mathematics, there is evidence of changes in how he *does* mathematics. By explicitly turning students' focus to important features of solutions, it is possible to improve the quality of solutions that they submit.

Conclusion

By making the analysis of solutions an explicit focus of classroom activity, students were supported to develop meta-level understandings of solutions to mathematics problems. Students were able to articulate why including certain aspects of a solution can be essential, rather than an extraneous requirement imposed by the teacher. Evidence of growth was also apparent in students' homework solutions, which included greater explanations and justifications. Thus, preliminary results from this ongoing study show evidence of students' nascent development of internalized mathematical authority. These results provide the basis for the further refinement of classroom activities for promoting and studying students' development of skills of explanation, justification, and assessment. Moreover, the continuation of this work will allow for the study of students' application of these skills to self-assessment.

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INVESTIGATING A STRATEGY FOR SOLVING INDEFINITE INTEGRATION PROBLEMS: AN fMRI STUDY

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Recently the emerging field of educational neuroscience has begun to use neuroscience methods to investigate questions of interest to both education and neuroscience. This study used behavioral measures and fMRI to examine the first step of integration problem solving—selecting an appropriate technique. In Phase 1, college-aged subjects reviewed integration techniques; in Phase 2, subjects were trained in an expert-like strategy for selecting an integration technique. Results indicated that training improved the accuracy in selecting an integration technique. fMRI results indicated increased activity in brain regions associated with visuospatial processing, attention, and working memory. This study extends what is known about the neural basis of integration problem solving and raises questions about the role of working memory and visuospatial processing in the calculus classroom.

Keywords: Post-Secondary Education; Metacognition; Research Methods

For over 100 years, efforts to reform the teaching and learning of calculus have focused on curricular, pedagogical, and technological changes to improve calculus education (Ganter, 2001). Less emphasis has been placed on how students develop expertise in calculus. Recently, education-based neuroscience research using functional magnetic resonance imaging (fMRI) has been used to complement traditional educational research into student understanding (e.g., Delazer et al., 2005; Lee et al., 2007; Qin et al., 2003, 2004; Thomas et al., 2010). Thus, exploring the neural regions associated with selecting an integration technique may ultimately provide an avenue for future research aimed at improving the teaching and learning of calculus.

Although recent calculus curriculum reform efforts placed increased emphasis on students acquiring conceptual understanding of calculus, facility with techniques of integration continues to be a goal of first-year calculus instruction (Sofronas et al., 2011). Research (Schoenfeld, 1985; Kallam & Kallam, 1996) along with anecdotal evidence suggested that while first-year calculus students may be able to successfully implement individual integration techniques, determining an appropriate technique was difficult. In contrast, college calculus instructors did not appear to have similar difficulties (Kaczmarczyk, 2005).

Schoenfeld (1985) argued that students' difficulties with integration stemmed from issues of control associated with strategy selection and allocation of cognitive resources. To test this theory, Schoenfeld (1980, 1985) developed a strategy for selecting an integration technique consisting of three broad stages: simplify, classify, and modify the integrand. Research suggested that supplemental instruction using this strategy resulted improved student performance (Kallam & Kallam, 1996; Schoenfeld, 1985). Yet questions remain about how students become more expert-like in the selection of an integration technique.

Theoretical and Methodological Framework

The work of Dehaene et al. (2005) and Arsalidou and Taylor (2011) form the theoretical foundation for this research. The triple-code model of numerical processing (Dehaene et al., 2005) posits that a three-part parietal network consisting of the horizontal part of the interparietal sulcus, the left angular gyrus, and a posterior superior parietal system is responsible for processing numbers and numerical operations. In a meta-analysis of neuroimaging studies in mathematics, Arsalidou and Taylor (2011) argued that this model should be extended to include regions (e.g., the prefrontal cortex) which are associated with executive functions such as working memory and attention. Although these regions are not part of the triple-code model, behavioral evidence has linked working memory and attention with mathematical performance (LeFevre et al., as cited in Arsalidou & Taylor, 2011, p. 2390).

Atherton and Bart (2001) proposed a methodological framework to guide research studies involving fMRI. In the *Discovery* stage, imaging studies focus on the brain regions associated with a particular cognitive function, identifying *the* brain region most strongly activated by a given task. The *Functional Connectivity* stage addresses the more complex nature of cognition and investigates the coordination of activity between brain regions (Atherton & Bart, 2001; Varma & Schwartz, 2008). Since fMRI research into calculus is in its infancy with one published study of calculus using fMRI (Krueger et al., 2008), this study takes place at the *Discovery* stage. Specifically, this study examined the brain regions identified in the literature (e.g., Arsalidou & Taylor, 2011; Dehaene et al., 2005) to determine which were significantly active while selecting an integration technique and if there were differences in the activation in these regions after subjects were trained to use Schoenfeld's (1980, 1985) strategy.

Methods

Eight right-handed, native English speaking undergraduate students who had completed Calculus II participated in this two-phase study. During the two Phase 1 sessions subjects reviewed four integration techniques (i.e., algebraic manipulation of the integrand, *u*-substitution, integration by parts, and the method of partial fractions) in a manner aligned with typical classroom instruction. At the end of Phase 1, subjects participated in computer-based test post-test and a block-design fMRI experiment. For both participants were shown an indefinite integration problem and told to select the most efficient integration technique. Response times to select a technique and proportion correct data were collected during both the computer-based test and the fMRI experiment.

Phase 2 began one week after the Phase 1 fMRI experiment, and consisted of two sessions where participants were taught Schoenfeld's (1980) strategy. After Phase 2 training, participants participated in a computer-based post-test and fMRI experiment which were identical in format to the Phase 1 post-tests described earlier. The integration problems presented in the Phase 2 post-tests were isomorphic to those used in Phase 1.

The response time and proportion correct data were analyzed using paired *t*-tests ($\alpha = .05$) to compare means for dependent groups. Analysis of the fMRI data was done in two stages. The first stage was a within-subject analysis for the contrast of interest (i.e., *integration vs. control*) using SPM8 (Wellcome Trust Centre for Neuroimaging, 2009). A second-level random effects analysis was implemented in MarsBar (Brett, Anton, Valabregue, & Poline, 2002) using the regions previously identified in the mathematics-related neuroscience literature (Arsalidou & Taylor, 2011; Dehaene et al., 2005). The second-level fMRI data analysis identified the regions for which the contrast were significant and then compared Phase 1 to Phase 2 in each of these regions for this contrast.

Results

The analysis of the response time data indicated that there was no significant difference ($p = .161$) in the mean response time between Phase 1 ($n = 32$, $\mu = 4.714$, SD = .827) and Phase 2 ($n = 32$, $\mu = 4.288$, SD = .595) during the computer-based post-test. There was also no significant difference ($p = .709$) in mean response time between Phase 1 ($n = 64$, $\mu = 3.509$, SD = .620) and Phase 2 ($n = 64$, $\mu = 3.617$, SD = .362) during the fMRI experiment. These results suggested that the training in an expert-like strategy did not have an effect on response time to select an integration technique.

The analysis of the mean proportion correct data for Phase 1 and Phase 2 computer-based post-tests indicated that there was a significant difference ($p = .035$) between the mean proportion correct in the Phase 1 ($n = 32$, $\mu = .660$, SD = .084) and Phase 2 ($n = 32$, $\mu = .746$, SD = .078). In the fMRI experiment, there was also a significant difference ($p = .014$) in the mean proportion correct in Phase 1 ($n = 64$, $\mu = .709$, SD = .112) and Phase 2 ($n = 64$, $\mu = .805$, SD = .074). These results suggest that the Phase 2 training in a more expert-like strategy had an effect on the subjects' ability to select an appropriate integration technique.

Phase 1 fMRI data indicated that there for the contrast *integration vs. control*, there was a significant difference bilaterally in 3 regions of interest: the horizontal part of the interparietal sulcus (hIPS), posterior superior parietal lobule (PSPL), and middle frontal gyrus (MFG) These results suggested that the hIPS, PSPL and MFG were more active during the selection of an integration technique than during the control task. Phase 2 fMRI results were similar to that of Phase 1. Results indicated that there was no significant difference between Phase 1 and Phase 2 in any region of interest for the contrast *integration vs. control*.

Discussion

The results of this study adds to the growing number of studies (e.g., Krueger et al., 2008; Lee et al., 2007; Stavy et al., 2006; Stavy & Babai, 2009; Thomas et al., 2010) which have begun to use neuroscience methods such as fMRI to investigate questions linked to mathematics education. The results of this study found activations in the hIPS and PSPL which were consistent with an earlier study involving integration (Kreuger et al., 2008). This study also associated activation in the posterior superior parietal lobule (PSPL) and middle frontal gyrus (MFG) with the first step of integration problem solving—selecting an integration technique. Prior neuroscience research into mathematical cognition indicated that the MFG was involved with working memory and executive functioning (Fehr et al., 2007; Kong et al., 2005; Lee et al., 2007) and the PSPL generally supported visuospatial processing, attention and spatial working memory (Dehaene et al., 2005; Delazer et al., 2003). The PSPL is also a key component in the central executive network which showed increased activation during cognitively demanding tasks (Sridharan et al., 2008). Activation in the PSPL and MFG while selecting an integration technique may be attributed to the working memory demands of the integration problems, specifically the need to maintain and manipulate information in working memory in order to select an appropriate technique.

Finally, the results of this study raise several questions for the researchers: (1) What is the difference in functional connectivity before and after training in Schoenfeld's (1980) strategy? (2) How does the visuospatial processing and working memory capacities of students impact their learning of calculus and (3) How should understanding of visuospatial processing and working memory capacity shape college classroom instruction?

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ACTIVITIES WHICH FOSTER HIGH SCHOOL STUDENTS' CONCEPTUAL UNDERSTANDING OF THE NOTION OF LIMIT

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In this article we describe how we carried out a sequence of activities designed to foster the acquisition of the concept of limit with a group of high school students who had not taken a Calculus course. The students had the opportunity to work in small groups, to present their ideas to their peers and thus modify their conceptions as a result of the critiques and opinions made by their classmates; at the end of the sessions they again had to solve the activity individually. We present "Yaya's leaps" an adaptation of the paradox of Achilles and the turtle.

Keywords: Advanced Mathematical Thinking; High School Education

Calculus is the field of mathematics that studies change phenomena which occur in nature and in everyday life. It involves concepts and procedures such as that of the limit, which are difficult to learn by most students or in some cases, never are understood. The limit of a function is one of the most important concepts of general mathematics and specifically, of calculus; it is the cornerstone for the study of change phenomena and several important numbers are defined as limits. In addition, such concept allows one to know the performance of a function towards a point, even though its performance at that point is unknown.

Intuitively, a function has a limit *L* at a point x_o if at any point, each time closer to x_o, the function has a value which is each time closer to *L*. However, the formal definition with "epsilon-delta" becomes inaccessible for most: *L* is the limit of function $f(x)$, when *x* gets closer to *p*, if and only if for all $\varepsilon > 0$, there is a $\delta > 0$ such as for any real number *x* which complies with $0 < |x - p| < \delta$, we have $|f(x) - L| < \varepsilon$. The preceding has originated the creation of teaching strategies, such as David Tall's, who proposes that formal definitions and concepts be introduced taking into account the student's context, in subtle and "sensitive" ways which appeal to intuition for acquiring these notions, since he sustains that mathematical rigor during the early instruction phases produces unfavorable learning results. What kind of activities and experiences further the understanding of the concept of limit for high school students? What mathematical and heuristic processes do the students use when facing problems which involve the concept of limit?

Theoretical Framework

The mathematical concept of limit is an especially difficult notion. Cornu (1994) mentions that in a teaching environment it is very important to distinguish between the definition and the concept itself. Additionally, he recommends being attentive to the spontaneous ideas which students may have of the concept since they do not disappear when they move on to a new mathematics lesson. On the other hand, Tall (2010) affirms that the ideas which belong to this area of mathematics, such as variation, continuity, slope, etc., must follow a teaching route which is close to the natural way of human thought. Tall (2010) strives for a "sensible approximation to Calculus." He classifies the evolution of mathematical thought in three stages: The first is the natural growth of ideas (which corresponds to the "world of the senses" associated to the sensory cortex, of basic primitive ideas); the second stage are the actions which help us to transform such ideas into symbolic handling and computation (corresponding to the "operational world" which takes place in the secondary brain cortex and allows for control and proficiency of short term memory); and the third, in which we can formulate logical definitions and carry out formal demonstrations (related to the "formal world" which takes place in the superior cortex).

Tall (2010) proposes the use of technology so as to view a graph on the computer screen, in such a way that it is visibly flat when maintaining the same vertical scale and increasing the horizontal scale (Figure 1); that is, to get the image so close for it to be possible to view the pixels forming a single line, thus insuring that the graph is continuous.

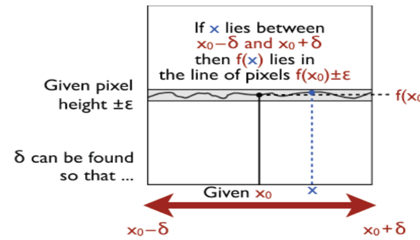


Figure 1: Tall's geometric interpretation

Participants, Research Method, Procedures

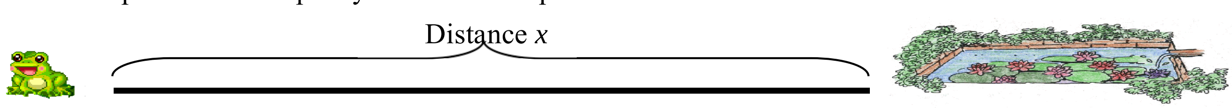
This report documents one of the four activities which were designed to offer a close approach to the concept of limit to students who had not studied Calculus. These activities were applied to students of high school (second semester) of Cecytem in the Municipality of Tzintzuntzan, state of Michoacan, Mexico who participated voluntarily in extra class sessions. The purpose of this study is to find out how these activities contribute to the initial building of such concept. The design of activities was carried out following the suggestions of the Balanced Assessment Package for the Mathematics Curriculum (2000), trying to make them attractive to the students.

Activity: **Yaya's Leaps**

Instructions: Analyze the situation and answer the questions, do not erase.

Yaya is a little frog that lives happily in a pond; she likes to count the leaps she does so as to reach the objects around her. One day, tired of always counting in the same manner, Yaya decides to modify the way in which she goes forward, and says: "From now on, every one of my leaps will be exactly one half the distance I need to cover to reach my goal."

Observe the picture and help Yaya count her leaps.



Yaya wants to reach the pond and w ith her first leap she travels half the distance required to get to it.

1. What distance did Yaya cover with the first leap?
2. What distance does she have to travel in order to reach the pond?
3. Remember that with each new leap Yaya travels one half of the distance which is missing? Yaya jumps her 2nd leap, what distance has she traveled between leaps 1 and 2? How much more does she need to travel?
4. Draw Yaya's Leaps and complete the following table according to the distance to be covered with each leap.

Leap number	1	2 3		4	5 6 7					
Distance to be covered	½	¼								
Expression	(½)	$(½)^2$								

5. Observe the table: after 5 leaps how m uch more does Yaya have to jum p to arrive at the pond? Express it algebraically.
6. If Yaya jumped "n" leaps, what distance must she still travel?
7. How many leaps does Yaya need to get to the pond? Justify your answer.
8. What happens to the distance that Yaya needs to travel as the number of leaps increases?

Results

The students worked by teams during 45 minutes. The group discussion practically started as the teams presented their results. It stands out that graphic representation seems to be the form which favors the approach to the concept of limit; when they referred to verbal representation, the students thought it was a problem of logic or guesswork.

Yaz: since this is the distance, then we have to divide it in… in half.
Dany: but, this seems almost logic, isn't it?...
Yaz: … **but** [looking at Dany], **she would never arrive.**
Dany: what I mean, she would never arrive because, it is pure logic…
Yaz: **no, because she has to get to here, mijo**, they are one, two, [counting the leaps] …, seven, she jumped seven times, **eight, no look** [counts again] **she jumped eight times**.

Apparently, the above dialogue shows the moment when Yazmin has a brilliant idea and, as mentioned by Schoenfeld (1992), the student's own experiences will lead him/her to understanding. The students that did not divide the segment have not noticed that very soon it will be physically impossible to make the partitions. Yazmín tries to explain it:

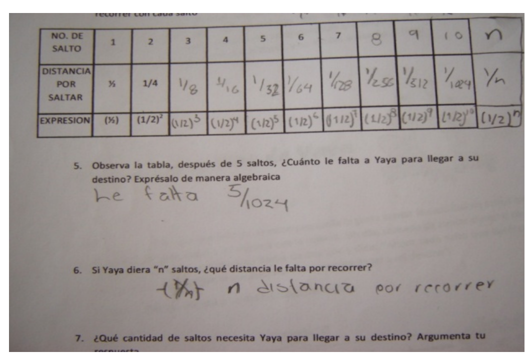
Yaz: **one half of one half is remaining, she will leave more and more space behind, have you understood me or not yet? she has to get there; she does get there, doesn't she?**
Erandi: **it reduces more each time**.
Yaz: **it's that… uhm it reduces more each time**.
Dany: **yes it's true that it reduces more and more**.
Yaz: **whether the space is large or small, she has to arrive**
Erandi: **there's no more space**.
Yaz: **there will be no more space which can be cut by half, she simply has to get there, do you or don't you understand?**

When Yazmín explains to her peers she is confronting her peer's ideas in relation to the notion that problems are solved by completing tables:

Juan C: **but then the table would only have eight boxes**…
Fidel: **Because… we only went for ten… and for completing the table**. [and shows more complex issues] **I'm telling you that here she would give two equal leaps and on this page it never says that it will jump the same distance twice.**

However, although Yazmín tries to explain by showing the partitions on the floor with the aid of markers and pencils which represent each of Yaya's leaps and the pond, she does not convince them to visualize:

Yaz: **it's that she is not jumping the same distance, there will simply be no more space to leap with her nails… it reduces and reduces until she has no more space to jump one half.**
Fidel: **but she also jumps into the water**. [laughs]
Yaz: **oh, but she has arrived at the pond.**
Ely: then how many jumps does Yaya need to get to the pond?
Dany: [very slowly] I say it is infinite.



NO. DE SALTO	1	2	3	4	5	6	7	8	9	10	n
DISTANCIA POR SALTAR	½	1/4	1/8	1/16	1/32	1/64	1/128	1/256	1/512	1/1024	1/n
EXPRESION	(½)	$(1/2)^2$	$(1/2)^3$	$(1/2)^4$	$(1/2)^5$	$(1/2)^6$	$(1/2)^7$	$(1/2)^8$	$(1/2)^9$	$(1/2)^{10}$	$(1/2)^n$

5. Observa la tabla, después de 5 saltos, ¿Cuánto le falta a Yaya para llegar a su destino? Exprésalo de manera algebraica
le falta 5/1024

6. Si Yaya diera "n" saltos, ¿qué distancia le falta por recorrer?
(1/n) n distancia por recorrer

7. ¿Qué cantidad de saltos necesita Yaya para llegar a su destino? Argumenta tu

Figure 2: Fidel's team's response

In general, there was no numerical difficulty in the teams when partitioning the segment which represented the distance Yaya had to travel; they were also able to complete the table without problems; some even added a columns for leap "n" and they answered correctly the expression on the table, however they could not answer question 6, as seen on Figure 2. This problem arose with all the teams when having to find the algebraic expression for leap "n." In some teams we even observed rejection or frustration towards algebra which does not permit them to at least make an attempt.

Juan C: which would be the algebraic answer?, would it be here? [pointing to the table]
Dany: … I don't know [takes the sheet and writes $(½)^n$].
Yaz: **but, why to the nth power? we're supposed to be obtaining…**
Dany: **one half to the 2nd, one half to the 3rd… one half to the nth.**
Yaz: **but what is the value of "n" so it can be reduced to this? and so it results in "n".**
Dany: but here it goes, and goes, and goes.
Erandi: **"n" es infinite.**
Yaz: **"n" es infinite? "n" is any number**.[laughs]

However Yazmín and many of her classmates do not identify that "n" represents the number of leaps and cannot be used to represent another variable; they understand that "n" is a variable which can have any value but they use it indiscriminately to indicate the number of leaps and the distance to be traveled.

Ely: what does "n" represent?
Mando: the distance and the number of leaps.
Ely: Both? [Armando nods affirmatively].
Fidel: well, it represents one thing in each question.
Ely: Different things?
Mando: ... because it is a distance and also the leaps.

Barbara's team did not have a problem to find the limit, although when they counted the partitions they made, they counted up to nine leaps:

Juan C: [starts to count with the aid of the drawing, followed by Penelope] one, two, three, four, five, sex, seven, **eight and from this half, it got there** [laughs], how many leaps did we do? nine [answer his peers]…, then it wouldn't need to jump anymore; [insists] **ten leaps, if you place another pencil there would be one, two, three… Ten leaps.**
Yaz: **but another pencil doesn't fit.**

Final Comments

It seems that Yazmin had the best approach to the concept of limit. At the start her thoughts were similar to those of her peers assuming that Yaya would never reach the pond; however, she discovered the contrary when partitioning the segment on the drawing. After that she tried to explain her classmates with several examples using the argument: how is she going to leap again if there is no more space to jump? She either leaps to the pond or remains in her place. This convinced most of the students since there was a consensus in the last question; all answered that the distance which Yaya had to travel was reduced inasmuch as the number of leaps increased. These are essential prerequisites to reach understanding of the concept of limit.

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PRODUCTIVE STRUGGLE AND ITS ROLE IN TEACHING AND LEARNING MIDDLE SCHOOL MATHEMATICS

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Mathematics educators and researchers suggest that struggling to make sense of mathematics is a necessary component of learning mathematics with understanding. This study examined students' productive struggle as students worked on tasks of higher cognitive demand in middle school mathematics classrooms. Observations of 186 episodes of student-teacher interactions revealed types of struggles students encountered, the ways teachers responded to these struggles, and the kinds of interaction outcomes that were productive or not. A productive struggle framework was developed to examine the phenomenon of student struggle from initiation, interaction, to its resolution.

Keywords: Instructional Activities and Practices; Middle School Education; Classroom Discourse; Problem Solving

Introduction

Students' struggle with learning mathematics is often cast in a negative light and viewed as a problem in mathematics classrooms (Hiebert & Wearne, 2003; Borasi, 1996). Teachers, parents, educators and policymakers routinely look for ways to overcome the "problem," seen as a form of learning difficulty, and attempt to remove the cause of the struggle through diagnosis and remediation (Adams & Hamm, 2008; Borasi, 1996). From this one would hardly expect that focusing on students' struggle in mathematics could be viewed in a positive light and as a learning opportunity.

Mathematics educators and researchers James Hiebert and Douglas Grouws suggest, however, that *struggling to make sense of mathematics* is a necessary component of learning mathematics with understanding (Hiebert & Grouws, 2007). The idea that struggle, that is to expend intellectual effort, is essential to intellectual growth has a long history (Dewey, 1933; Piaget, 1960; Polya, 1957; Hatano, 1988). More recently, Hiebert and Wearne (2003) stated, "all students need to struggle with challenging problems if they are to learn mathematics deeply" (p. 6).

While the phenomenon we call struggle may be internal, it is also observable in most classrooms. This study identified episodes during instruction where students made mistakes, expressed misconceptions, or claimed to be lost or confused, and to which teachers responded. Interactions between students and teachers generally advanced toward some resolution of the students' difficulties and attempts at sense-making. Using an embedded case study methodology with instructional episodes as the unit of analysis, the study identified and described the nature of the students' struggle as well as the instructional practices of teachers that either supported and guided or did not support or guide the students' sense- and meaning-making of the mathematics in the lesson episodes.

Conceptual Framework

The conceptual framework was built on three main components: (1) The role of struggle in learning mathematics with understanding; (2) The nature and types of mathematical tasks and their relationship to students' struggle; and (3) The ways teachers' respond to students' struggle in classroom interactions. This study used the perspective of mathematics as a social phenomenon, where people create objects and study the patterns and relationships of these objects within a social culture (Hersh, 1997; NCTM, 2000). In addition, mathematics is viewed as a dynamic discipline that involves exploring problems, seeking solutions, formulating ideas, making conjectures, and reasoning carefully.

Methods

This exploratory case study used embedded multiple cases (Yin, 2009) in order to study the role of productive struggle in learning and teaching mathematics. Specifically, the research questions addressed were:

1. *What are the kinds and patterns of students' struggle that occur while students are engaged in mathematical activities that are visible to the teacher and/or apparent to the student in middle school mathematics classrooms?*
2. *How do teachers respond to students' struggle while students are engaged in mathematical activities in the classroom? What kinds of responses appear to be productive in students' understanding and engagement?*

Participants

The participants were 6th and 7th grade middle school students and their teachers from three middle schools located in mid-size Texas cities.

Procedure

Data collection. Each teacher was videotaped teaching six to eight classes in a one-week period with each class ranging from 60 to 90 minutes. Thirty-nine class sessions were observed among the six teachers and 327 students for a total of 52.5 observation hours.

Data analysis. An excerpt file of video clips of instructional episodes was created guided by Erickson's (1992) methods for analyzing video data. An instructional episode consisted of a classroom interaction about a mathematical task that was initiated by a student struggle that was in some way visible to a teacher or another student whether voiced, gestured, or written. The transcripts of the class observations and interviews were coded using the open-coding process (Strauss & Corbin, 1990) to identify and analyze (1) the kinds of struggle that occurred, (2) the level of cognitive demand within which each struggle occurred, and (3) the nature and kind of responses each teacher made to the students' struggle.

Findings

Students' Struggle

In the analysis of the 186 episodes, four main types of struggle emerged as students engaged in mathematical tasks. The struggles centered about students' attempts to: (1) Get started, (2) Carry out a process, (3) Give a mathematical explanation, and (4) Express a misconception or errors.

Findings confirm that occurrences of struggle depended on students' engagement in the prescribed tasks that challenged them and had some element of difficulty.

Teacher Response

Findings showed four main ways that teachers responded to student struggles situated along a continuum that includes *telling, directed guidance, probing guidance,* and *affordance*. The analysis of the teacher responses in the student-teacher interactions focused on three dimensions based on the conceptual framework: (1) level of cognitive demand of the mathematical task, (2) attention to the student's struggle, and (3) building on student's thinking.

The study found that teachers seem to constantly strike a balance between trying to sustain student engagement and maintaining the cognitive demand of the task (Kennedy, 2005).

Interaction Resolutions

Resolutions were classified as *productive* if they (1) maintained the intended goals and cognitive demand of the task, (2) supported students' thinking, and (3) enabled students forward in the task execution; *productive at a lower* if productive in points (2) and (3) above but the cognitive demand of the intended task was lowered; or *unproductive* if students continued to struggle without showing signs of making progress towards the goals of the task.

The Productive Struggle Framework below in Figure 1 incorporates the aspects of student–teacher interactions about student's struggle from initiation, interaction, to resolution.

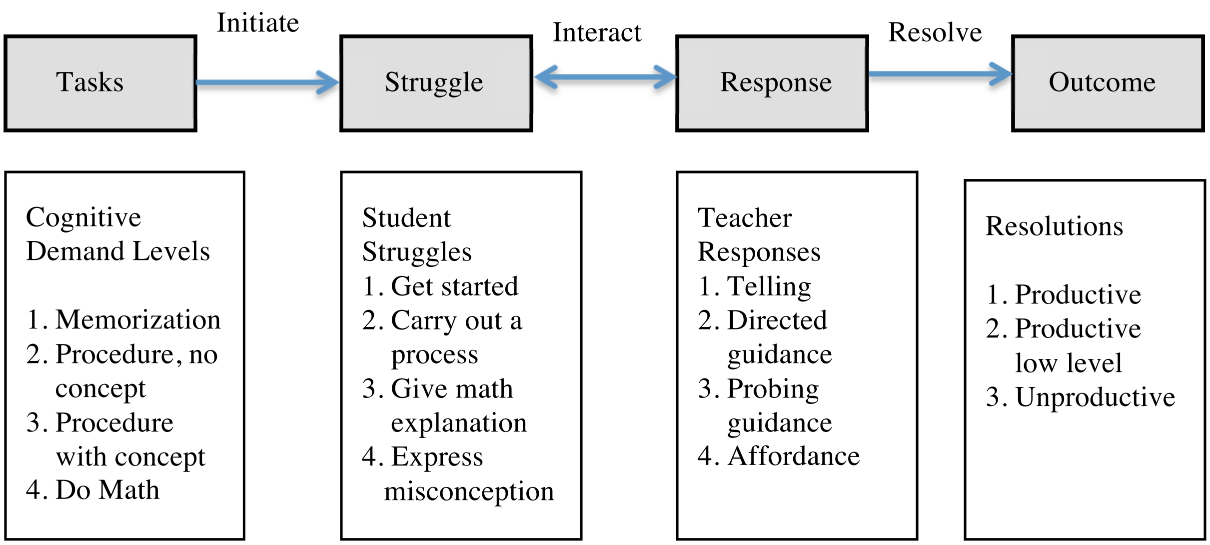

Figure 1: Productive struggle framework

Conclusion and Implications

The aspects of the interactions that helped direct and support student struggles productively and toward student understanding of mathematics was the joint student actions, teacher actions, and the physical and cultural contexts established by the norms in the class. The encouragement to communicate with teacher responses such as, "Tell me what you mean" and "Talk about it some more" or insistence on sense-making with "Why is that?" provided opportunities for students to elaborate on what they understood and clarify the source of their struggles. Responses that encouraged continued effort such as, "Try that" and "Well, what if you do…" gave positive reinforcement for engagement without student worrying about whether the result was right or wrong. Posing problems of high cognitive demand gave the students opportunities to think, reason, and problem-solve in ways that meant the students had to think deeply about the problems and not just find routine methods to apply. For future research, we can build on the descriptions of struggle, interaction, and resolution to assess how productive the students' struggles were and what the students learned through their struggle.

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PROSPECTIVE TEACHERS REINVENTING INTEGRATION USING MEANS

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The ubiquitous Riemann sum definition of integration is not the only way to define integral. Here we present an alternative mean-based definition of integration that we conjecture is more accessible to students. In support of this proposition, we first present a theoretical argument pertaining to how human beings think about infinity, followed by a discussion of results from a small-scale classroom-teaching experiment. This teaching experiment was conducted with a group of pre-service elementary school teachers all of whom had no prior calculus experience. These students, who are arguably much weaker mathematically than typical students entering college calculus, made considerable success in reinventing the mean-based definition.

Keywords: Post-Secondary Education; Curriculum; Instructional Activities and Practices; Advanced Mathematical Thinking

Toward an Alternative Pathway

The limit concept relies on notions of infinity. Núñez (2005) argued that the human mind understands the various instantiations of the concept of infinity as a conceptual blend of a finite process, which has a final state, and an ongoing process, which has no perceivable end (Fauconnier & Turner, 2002). Both of these processes are understood through real-world experiences. Students have more tangible experiences with ongoing processes of growth than processes of shrinking. One can imagine regularly depositing $100 into a bank account and watching the account balance grow without bound. However, it is much harder to imagine continually zooming in on an object. After the subatomic level is passed, there is nothing tangible left to imagine. Since we have more real-world experiences with processes of growth than processes of shrinking, if we are to accept Núñez's blending model, a natural consequence is that the concept of infinitely small is harder to grasp than the concept of infinitely big. Thus, if all other factors remain constant, a formulation of a concept that uses only notions of infinitely big is potentially more accessible to learners than a formulation of that concept which involves notions of infinitely small.

Students struggle with understanding processes that involve infinitely small quantities. Many students make sense of integration through something Oehrtman (2009) refers to as "the collapsing metaphor." This is when students reason that a dimension is lost as increments get smaller. For example, when finding the area underneath a function students may reason that to get an exact answer they need to shrink the rectangles used in the Riemann sum until they become lines. When the rectangles become lines a dimension is lost.

In light of this theoretical argument, we propose an alternative pathway to the definition of integration. Rather than relying on the concept of infinitely small, which is something students have little tangible experience with, we propose developing a definition that builds on students' understanding of mean. In this alternative approach, students reason about infinitely large sample sizes instead of dealing with infinitely small rectangles.

The Mean Based Definition of Integration

The following definition is a mean-based formulation of integration:

Uniform Sampling: $x_{i-1} - x_i = h = (b-a)/n \qquad x_i = a + ih$, $i=0,1,2,\ldots,n$

Sample Data: $y_i = f(x_i)$, $i=0,1,2,\ldots,n$

Statistical mean: $\bar{y}(n) = 1/(n+1)\sum_{i=0}^{n} y_i$

Definition of Integral: $I \equiv \lim_{n\to\infty} \bar{y}(n)(b-a)$ is the integral of *f(x)* over the interval [*a*,*b*].
This is written as $I = \int_a^b f(x)dx$.

Instead of rectangles that become infinitely narrow this definition takes a uniform sample of heights on an interval [*a*,*b*]. The mean of these heights provides an estimate for the mean height of the function. Multiplying this estimate by the width of the interval, *(b-a)*, provides and estimate for the area under the curve on the interval [*a*,*b*]. As the size of the sample is increased to infinity, the estimate of the area becomes the actual area, the integral of *f(x)* over the interval [*a*,*b*]. Unlike the Riemann sum definition, this formulation includes both end points of the interval [*a*,*b*], which is consistent with how it was reinvented by the participants in this study.

The Teaching Experiment

In the previous section we presented a theory-based argument grounded in how individuals think about infinity. Educational theory however, and the realities of classroom practice do not always line up with each other. A small scale teaching experiment was conducted to provide an existence proof of the accessibility of the definition (Steffe & Thompson, 2000). This teaching experiment was conducted with four prospective elementary school teachers who had no prior calculus experience. This lack of exposure minimized interference from previous instruction.

The first author was the instructor for all four sessions of the teaching experiment, which took place over four 50-minute sessions. Instruction focused on facilitating the guided reinvention (Gravemeijer, & Doorman, 1999) of the majority of the concepts involved.

The First Session

In the first session students were exposed to the mean height-line technique for finding areas composed of blocks. This technique involved dissecting and rearranging blocks to form an equivalent rectangular area (Figure 1). This technique was introduced in the context of students' own techniques for finding areas, all of which involved some kind of systematic counting. The technique was initially introduced as Joey's height-line technique, so that students could make their own connections between the technique and means. An exercise where students were asked to find the height-line of a shape that was described numerically, but not drawn, was the catalyst for students making this connection. This left students with some initial intuitions about the relationship between mean and area, which was exploited in the subsequent sessions.

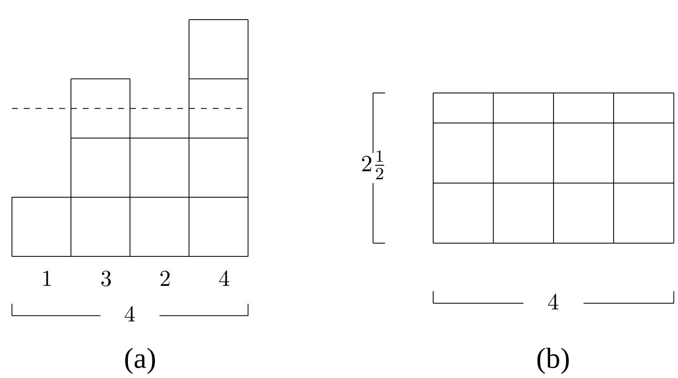


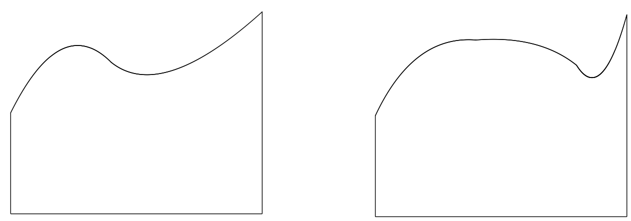
Figure 1: The mean height-line technique

The Second Session

The focus of the second session was to extend the mean height-line technique to estimating the area of "rounder" shapes. This specifically focused on shapes that would appear, to an expert, to be the areas underneath functions. The functions context was, however, suppressed until the third session.

After several warm up exercises which involved estimating the mean height-lines for rounder shapes, students were prompted to come up with a technique that would provide consistent estimates that others could replicate. This resulted in what we view as the pivotal episode in the teaching experiment. In it the group, with some guidance from the instructor, managed to co-operatively modify the techniques that they were exposed to up to this point into one that is applicable to function-like shapes.

After several failed attempts at creating a technique that could be used to find consistent estimates of the heights of mean height-lines, two techniques emerged. The first was jointly constructed by two of the four students in the group; it involved averaging the heights of the maxima, minima and end points of the function like shape. Another student, in an attempt to understand this way of generating the estimates, inadvertently proposed a second method when she asked, "Did you add up the height or total the heights from each centimeter?"



(a) (b)

Figure 2: Deciding between two methods

The instructor drew two pictures on the white board (Figure 2a and 2b). The two shapes had the same width, maxima and minima; however, the widths of the peaks and valleys differ. The instructor noted that the maxima/minima method would produce the same estimate for the area of both of these shapes, because the high and low points were the same in both cases. Then the group was asked what would happen with the regular interval method. The group answered that this would produce a larger estimate for Figure 2b than for Figure 2a. Since the students agreed that they thought the shape in Figure 2b had a larger area than the shape in Figure 2a, the group naturally concluded that the regular interval method was more desirable. The group used this technique to estimate the mean height-lines of two more figures.

The Last Two Sessions

The last two sessions served as a formalization process of the ideas developed in the first two sessions. In the third session the technique developed in the second session was adapted to a functions context. The group successfully applied the mean height-line technique to estimate the area under a number of different functions. The group then turned to formalizing these notions. Summation notation was used to translate the step-by-step technique that the students were comfortable using into a mathematical formula.

The last session addressed the idea of increasing sample size. Students had little trouble adapting to using the technique to estimate quantities by finding a height every half unit instead of every whole unit. They quickly pointed out how tedious such calculations were in anticipation of applying them on even smaller increments. This was used as motivation for formalizing these techniques in a manner that would allow us to tell a computer how to do work for us and students' ideas were translated into a formal limit definition of integral.

Conclusion

This teaching experiment served as an existence proof that the connection between mean and area, necessary for a deep understanding of the mean-based definition of integration, could be developed in a short period of time. As such, this approach to defining integral appears promising for students entering calculus, who on average have much stronger mathematics backgrounds. The theoretical arguments posed in the first section of this paper further support the accessibility of this definition. These arguments contend that the notions of an infinitely large sample used in the mean-based definition are more accessible to students than the notion of infinitely small intervals used in the Riemann sum definition. We hope in the future to be able to study how the mean-based definition can be utilized on a larger scale in a college calculus course.

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USING MULTIPLE REPRESENTATIONS AS AN ADVANCED ORGANIZER FOR SOLVING PROBLEMS RELATED TO PASCAL'S PYRAMID

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Keywords: Learning Progressions; Learning Theory; Problem Solving; Reasoning and Proof

This report is a 16-year case study of the mathematical reasoning of a student who participated in a longitudinal study of the development of mathematical ideas. We trace the counting reasoning from grade 5 through young adulthood of Robert, guided by the questions: (1) What representations were used to solve the strand of counting tasks leading to problems related to the properties of Pascal's Pyramid? (2) How were the representations useful in solving increasingly challenging tasks? and (3) What connections were made to the structure of solutions of other tasks? The representations and robust counting skills that Robert used and then later elaborated unveil the growth in understanding and the increased sophistication in representing earlier ideas. We suggest how Robert's earlier ideas grew over the years. In grade 5 (1993) Robert worked on a problem where students were asked to find as many different towers, four-cubes tall, when selecting from two colors of Unifix cubes. Robert observed his classmate Stephanie share the idea of a tree diagram to keep account of all towers. In grade 11 (1998), Robert worked on extensions of the same activity and sketched a tree diagram to convince the researcher that he had accounted for all towers. Later, in a post-graduate interview (2009), Robert built a 3-D model using *Zome tools* (connecting spheres and rods) for the Pascal's Pyramid. He then extended the representation of family tree using Unifix cubes and *Zome tools* to illustrate how subsequent layers of the Pyramid are derived from the preceding ones. In grade 7 (1994) Robert was asked to find all possible five-candle arrangements choosing from two colors of candles, red and gold. He created a binary list to find sixteen such arrangements, using 0 for red and 1 for gold. Five years later, in grade 12 (1999) Robert used this same binary notation to solve the *World Series Problem*, a baseball competition asking what is the probability that a series between two equally matched teams would end in four, five, six or seven games. This time around, Robert used 0 to represent one of the teams winning and 1 to represent the other team winning. In grade 11 (1998) Robert had to find all six-tall towers choosing from yellow and blue blocks that have exactly two blue blocks in them. Robert controlled for the spaces where the two blue blocks can be placed to methodically list all towers. He repeated the same technique in grade 12 for the *World Series Problem* to list all the possible ways a series could end in six games by controlling for where the two losses of a team could be placed in his binary list. Implications for further study will be offered.

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RECONSIDERING "OFF-TASK" MOMENTS

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Keywords: Early Childhood Education; Elementary School Education; Instructional Activities and Practices

This poster focuses on the mathematical thinking that emerged during off-task moments in a kindergarten classroom. Research has emphasized the importance of time-on-task for student learning (e.g., Anderson & Walberg, 1993); however, there has been little attention paid to students' time-off-task and the mathematical thinking that emerges while the children are off-task. Our study focuses on the main question: In what ways can some off-task activities promote mathematical thinking?

This study is grounded on Vygotsky's socio-cultural theory (Vygotsky, 1978). He focused on children's learning within cultural context and considered that children's learning is mediated not just by others (and knowledgeable others) but also by the available cultural and material tools. We pay close attention to social and cultural contexts of children's off-task moments and explore how the tools in those moments provided learning opportunities not being made available by the teachers.

The study was situated in the kindergarten of Taylor County Public School, a small rural school with a predominantly black population. We videotaped around 40 hours of classroom time, which we transcribed into fieldnotes. Following traditional methods of ethnographic analysis (Erickson, 1986), we coded the data for formal and informal learning, mathematical topics, materials used, and off-task moments, and used these codes to make assertions.

We found that off-task moments provided opportunities for students to explore, develop and discover mathematical concepts. These opportunities arose mostly when students had manipulatives or objects to play with during a lesson or activity. For example, while Ms. Perry was teaching a lesson about measurement, Maria had her fleece jacket with her at her desk and decided to folded and unfolded her jacket three times while the lesson was taking place. As she experimented with folding her jacket in different ways, she was developing conceptual understandings for symmetry, fractions, estimation and measurement. Another example of off-task learning was during a class activity where Carter needed to create an ABC pattern using small colorful dinosaurs. Immediately after he finished his pattern, he turned his attention to his own project: he found all the purple, orange and blue Brachiosaursus dinosaurs he could and grouped them according to their color. While Carter was waiting for further instructions, he was able to experiment with sorting and patterning in his own way. Though Maria and Carter were off-task, they were still able to learn and develop mathematical concepts.

The purpose of this paper is to bring to light the learning that can happen during off-task moments especially since the academic discourse focuses mainly on the learning that happens during on-task moments. We are not arguing that all off-task moments are educational, nor are we arguing that off-task moments should take place over on-task moments. Rather, we are bringing attention to an educational moment that is often overlooked in academic discourse, a moment which educators often times see as uneducational. Through this research, we can see valuable mathematical concepts being explored and developed while children are "off-task," playing with objects and manipulatives. Rather than leaving these spaces as "off task," teachers should create a space where children can freely play and explore mathematical concepts with available social, cultural, and material tools.

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TASK DESIGN TO CAPTURE MATHEMATICAL MODELING SKILLS

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Keywords: Modeling; Cognition; Advanced Mathematical Thinking; Post-Secondary Education

The goal of this research was to produce a tool sensitive to both mathematical and cognitive activities attendant to mathematical thinking by adopting mathematical modeling as a lens for observing mathematical thinking-in-use. This instrument development project is embedded within a larger research endeavor designed to study influential factors on engineering students' decision making processes when engaged in mathematical modeling tasks (MM) that draw on the content of differential equations (DEs). DEs were selected as targeted content for task development for three reasons: (1) it provides a natural site to study modeling since the subject arose from the study of change in physical systems over time; (2) for STEM students, the course marks a transition from studying the fundamentals of change-over-time to using the calculus as a tool for addressing life-like problems; and (3) the course is largely populated by engineering majors—individuals who must coordinate both mathematical and non-mathematical knowledge. A cyclical schematic (e.g., Blum, 2011) is often used to describe an individual's MM activity as he develops an idealized version of the problem, represents it mathematically, analyzes the mathematical model, interprets the results in terms of the real world, and then validates the model. Using the modeling cycle as a theoretical framework, researchers have identified some of the most challenging aspects of MM for students: framing the task (Schwarzkopf, 2007), making transitions between the real world and the mathematical model (Crouch & Haines, 2004), and articulating reasons for why models are or are not valid (Borromeo Ferri, 2006). The objectives for the instrument design were to study cognitive activities associated with these challenging mathematical activities. An item pool was created based on an extensive review of literature. Each item was mapped against the modeling cycle and critiqued by a panel of mathematics educators and mathematicians, critiqued by educators at a national conference, and field tested. The items ranged from those targeting specific stages of the modeling cycle to those intended to evoke multiple cycles. The tool allows the researcher to examine how individuals move among mathematical modeling stages while explicitly acknowledging how a student's cognitive resources influence his mathematical thinking as it is being used. The instrument is suitable for use with STEM majors in one-on-one cognitive interviews.

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AN INTERACTIONAL ANALYSIS OF STUDENTS' MATHEMATICAL ARGUMENT BUILDING IN AN AFTER-SCHOOL ENRICHMENT PROGRAM

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We adopted an interactional analysis perspective to examine a single class session on the development and use of argumentation among a group of 7th graders. Results indicated that students and teacher interacted in significantly different ways in the construction of mathematical meaning, which affected their learning.

In this work we focused on the development and use of argumentation in a 7th grade classroom to examine how the students and teacher develop and negotiate taken-as-shared mathematical meanings and ways in which the students and teacher viewed the use of argumentation in mathematics. For the episode, a basic transcript was produced for one whole class session. The verbal portion was divided up into message units and the timing and speaker labeled for each message unit. The nonverbal activities of the participants were recorded. The first round of analysis focused on the interactional moves at the message unit level—including who was speaking to whom as well as the discourse moves such as initiating, responding to, or closing a topic. At the second round, we used Halliday's function of language categories (1975) to catalogue the message units. We labeled message units in terms of instrumental, regulatory, interactional, personal, heuristic, imaginative, or informational in order to foreground important issues. Lastly, data was grouped into interactional units (Bloome et al., 2008) and then examined with respect to taken-as-shared mathematical meaning making, sociomathematical norms, and views on argumentation.

Findings

The students' interactional moves differed from those of the teacher's. In particular, the teacher was found initiating, validating, and closing a topic and the students responding to the topic. The students' utterances were found to function most commonly as interactional and informational, less as heuristic and imaginative. The teachers' utterances were often interactional and regulatory. The students accepted conjectures with respect to the concept of mathematical operation unless challenged either by the teacher or another student. The sociomathematical norm of re-explaining one's solution when challenged by a classmate or the teacher was prominent throughout the entire session. When discussing relational positioning in argumentation, students located themselves as dissimilar to the teacher. They interpreted argumentation in the mathematics class as something different than what they have been participating in with the other students. They also determined they are unable to challenge the teacher's power when arguing because they saw themselves lacking resources mathematically and with respect to time (i.e., the teacher controlled the time).

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HOW STRATEGIES AND CONCEPTS CO-DEVELOP THROUGH MATHEMATICAL PROBLEM SOLVING PROCESSES

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This poster focuses (at a micro-scale) on modeling transitions along the continuum of student learning by presenting a model for how strategic and conceptual knowledge co-develop in mathematical problem solving. The model was developed through a grounded analysis of six hours of videotaped interaction between a pre-algebra student and a researcher in which the student independently constructed a deterministic and essentially algebraic algorithm for solving algebra word problems of an underlying linear structure. The method he constructed, that can be recognized as linear interpolation/extrapolation, came about from incremental refinements to his earlier approaches to solving problems of similar structures, based on means-end analysis.

The analysis presented in this poster involves the development of an alternative and complementary perspective to the approach taken by the existing strategy construction literature (see Siegler, 2006, for a review). Like the existing strategy change and construction literature, this analysis tracks observable changes in strategy usage through an analysis of the problem solving actions of the individual. A novel aspect of the present analysis is the consideration of and elaboration of the nature of the knowledge needed to implement strategies. In particular, both strategies and the conceptual knowledge that undergirds their implementation are modeled as complex knowledge systems. The analysis is informed by and elaborates the Knowledge in Pieces (diSessa, 1993) theoretical perspective and the analytic methodology employed involves coordinating knowledge analysis (diSessa, 1993; Sherin, 2001) and microgenetic learning analysis (Parnafes & diSessa, submitted; Schoenfeld, Smith, & Arcavi, 1993).

In the particular case presented, the most significant class of conceptual knowledge that the solver consistently drew upon and developed over the course of the sessions was knowledge of how to control the variation of linear functions. The model of co-development of strategic and conceptual knowledge developed is one of mutual bootstrapping: (1) Strategy enactment promotes conceptual changes by seeding the development of new conceptual schemes, and (2) The creation of new conceptual schemes results in the creation of new conceptual categories and relations (e.g., and hence the creation of a new strategy).

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STUDENTS' USES OF SMALLER PROBLEMS WHEN COUNTING

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Research (e.g., Hadar & Hadass, 1981; Lockwood, 2011) suggests that there is a need for mathematics education researchers to examine students' work in combinatorics, attending to potential ways in which students may improve. In this poster, I report on one specific element of students' work with counting problems—the problem solving strategy of solving smaller, similar problems. Researchers in problem solving (e.g., Polya, 1945; Schoenfeld, 1979) and combinatorics education (e.g., Eizenberg & Zaslavsky, 2004) indicate that such a strategy may be effective for students as they engage in problem solving.

I interviewed 22 post-secondary mathematics students in individual, videotaped, 60–90 minute-long sessions. The students were first given five combinatorial problems to solve on their own, and they were subsequently given alternative answers of those same problems to evaluate. One such task was: "A password consists of 8 upper case-letters. How many such 8-letter passwords contain at least three Es?" A correct answer involves a case breakdown of counting passwords with exactly 3 through 8 Es. A tempting, incorrect solution first places 3 Es among the 8 positions of the letters, and then completes the password with any of the 26 letters in the alphabet (this answer overcounts). Using the methodology of grounded theory (Strauss & Corbin, 1998), data analysis included carefully reading transcripts and watching video, identifying and categorizing episodes that involved students using smaller, similar problems in their work.

I found three ways in which students used smaller, similar problems to their benefit. Students used smaller, similar problems (a) to facilitate systematic listing (which helped them to detect useful patterns in generating a solution and to identify an overcount), (b) to tackle one particular aspect of a problem, and (c) as a means of articulation and explanation. I also observed potential pitfalls of which students should be aware as they employ this strategy. Finally, I emphasize that the strategy has particular potential in the domain of combinatorics, and that the nature of counting problems makes them especially appropriate for the use of smaller, similar problems. Potential directions for further study include targeting how students think about and reflect upon their uses of smaller problems and investigating effective ways to develop this strategy among students.

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ANALYSIS AND INFERENCE TO STUDENTS' RESPONSES ABOUT PROGRESSION OF UNDERSTANDING IN PROBLEM SOLVING TASKS

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The development of problem solving ability has been among the primary goals of school instruction. In order to develop such ability, "students should acquire ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations that will serve them well outside the mathematics classroom" (NCTM, 2000, p. 52). While this goal has been a persistent part of mathematics education community for over a century, issues regarding how to develop problem solving skills among learners through instruction continue to be a major dilemma. Schoenfeld (1992) indicated that the instruction on application of Polya-style heuristic strategies had not been proven to be successful, and further explained that it may due to the lack of knowledge about problem solving activities. In our previous study (Zhang et al., 2010), we reported a categorical analysis of problem solving schema between two transitional grade levels (5th and 8th). Improvement in the area of "understanding of problems" and preference to more "mathematical" approaches was observed, while standardized approaches and representations were identified as constraints for higher grade level.

To further contribute to the understanding of students' problem solving development, we administrated a series of problem sets to 566 students from grades 5 to 7. The content of problems concerned patterns and functions, geometry and visualization, and data analysis. These areas were selected since they have been identified as core content areas in K–8. Each set of problems was designed specifically for each grade level while coherently for the three grade levels, aiming to capture the progression of understanding for the concept exhibited on the problem items. The research questions we aim to answer are: (1) What are some different types of modes of reasoning exhibited in students' responses? (2) What factors may contribute to each type of reasoning? (3) What is the overall progression of understanding for each concept throughout the grade levels?

Two researchers will categorize half of the students' responses in each grade level separately, then discuss ambiguous responses together and finalize the categories. The modes of reasoning proposed by Stacey and Vincent (2008) will serve as the basic framework for our analysis, while any types of reasoning distinct from the model will be recorded and discussed. The second and third research questions will be analyzed based on the initial analysis.

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Chapter 4

Student Learning and Related Factors

Research Reports

Brief Research Reports

Poster Presentations

SITUATING REVOICING WITHIN BROADER TASK AND SOCIAL STRUCTURES

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This article explores how situating the linguistic move of revoicing within broader structures helps to explain why researchers and practitioners attribute a variety of forms and functions to revoicing, and shows how revoicing may be described as consisting of and situated within a broader set of discourse structures that represent a continuum with respect to positioning students in epistemic roles. We present analysis from classrooms of three teachers enacting the same task in both small group and whole class activity structures. The results show how revoicing took on a variety of functions within longer exchange sequences, which themselves functioned to position students as active contributors and as participants in mathematical discourse practices. The implications are that broader exchange sequences can provide the functionality that O'Connor and Michaels attributed to revoicing.

Keywords: Classroom Discourse; Teacher Practices; Positioning

Revoicing has become part of the lexicon in mathematics education since O'Connor and Michaels (1993, 1996) introduced the idea. They contended that teachers' use of revoicing created *participant structures* in which students took on serious intellectual roles with respect to mathematical argumentation (Foreman, 2003). Subsequently there has been considerable research that has focused on the various forms and functions of revoicing (Enyedy et al., 2008; Foreman et al., 2008; Moschkovich, 1999) and on the ways practitioners perceive the form and function in their classroom practice (Herbel-Eisenmann, Drake, & Cirillo, 2009). Notably, revoicing is not simply seen as creating opportunities to engage students in argumentation (Foreman, 2003) but also to create opportunities for typically marginalized students to participate in complex and valued mathematical discourse practices (Enyedy et al., 2008; Moschkovich, 1999).

A theme that has emerged from the research on revoicing is the multiplicity of forms and functions evident in practice, with Enyedy et al. (2008) identifying at least seven functions and the teachers in the Herbel-Eisenmann et al. (2009) study identifying 25 potential intended and unintended functions. These findings point to potential pitfalls of looking at a single instructional move in isolation of broader task, activity, and discourse structures: characterizing any instructional practice may portray a sense of clarity that rarely exists in practice, with consequences for how such practices are presented to and taken up by practitioners. Consequently, it is necessary to situate any discussion of revoicing (or any other instructional move) within broader chronological, instructional, and discursive structures, particularly those that contribute to the development of participant structures in which students take on serious epistemic roles (i.e., student contributions drive the development of mathematical content).

In this paper, we present analyses from classrooms of three teachers in order to show how situating revoicing within broader structures helps to portray why researchers and practitioners attribute a variety of forms or functions to revoicing, and to show how revoicing may be described as situated within a broader set of discourse structures that represent a continuum with respect to positioning students in epistemic roles. For example, moves characterized as revoicing may: serve as an extension of teacher-controlled forms of discourse (e.g., IRE-dominated exchanges) in which the teacher 'hijacks' a students' explanation to launch into a related explanation; or resemble *animation* in which the teacher narrates students' actions and explanations, squarely attributing the explanation to the students; or constitute a brief clarifying move to get students to elaborate and refine their explanations. These moves focus the discussion on students' contributions, but differ according to how the primary responsibility for explaining the mathematics shifts among the participants. These examples differ with respect to both form and to their location in longer exchanges.

We explore what we see as the primary function of revoicing relative to more constrained forms of discourse (IRE), which is to help students see themselves as knowers and doers of mathematics by creating spaces for students' contributions to serve as the focus of classroom discourse and as the primary mechanisms by which mathematical content is developed. We explore how the various shades of revoicing affect this primary functionality. We also explore moves that, in conjunction with revoicing, place the responsibility for mathematical explanations (i.e., the *work* of mathematics classrooms) almost fully on students.

In this study, we address the following research questions:

1. How do exchange sequences—broader than revoicing—influence the social task structure (i.e., position students as active contributors and participants in mathematical discourse)?
2. How do exchange sequences—broader than revoicing—develop the mathematical goals of the lesson (i.e., the academic task structure)?

Defining Revoicing

Revoicing involves a dual function of creating a social task structure (positioning students as active contributors to the development of mathematical ideas) and an academic task structure (positioning students' contributions with respect to academic content) (O'Connor & Michaels, 1993). When describing a teachers' revoicing move, O'Connor and Michaels stated:

> What [the teacher] is doing here is creating a participant framework in which (a) she herself has taken the opportunity to draw a further inference from [Student A's] utterance; and (b) [Student A] has the right to validate [the teacher's] inference and, thus, take on a position himself with respect to an aspect of the current academic task …; and (c) [Student A] has been positioned in opposition to [Student B] in an activity that involves discussion of the relative merits of two proposals. (p. 322)

Typically revoicing involves (1) rephrasing or rebroadcasting a student explanation, (2) attributing intellectual contributions to the student, and (3) checking back with the student to see if the teacher described the explanation accurately. Performing these actions puts the teacher "on relatively equal footing" with the student (p. 324) and allows the student to "challenge or affirm" any claim attributed to him. In terms of the social task structure, this allows the teacher to "induct students into a discourse community, by getting them to adopt roles in the ongoing thinking practices that she wishes them to develop" (p. 325). In effect, revoicing coordinates the academic and social task structures.

Methods

We employed discourse analysis techniques to study three teachers enacting the same task in both small group and whole class activity structures. The three teachers were selected because they displayed distinct patterns in their discursive routines and because we had data of their enactments of the same instructional sequence, providing a common mathematical and curricular context. We characterized the teachers' discourse practices in terms of the extent to which they engaged in *accountable talk* (Michaels, O'Connor, & Resnick, 2008) as operationalized in the Instructional Quality Assessment Toolkit (Boston & Wolf, 2006). Accountable talk involves discourse practices that facilitate the development of participant structures that position students in substantive epistemic roles, and includes revoicing as one of a broader set of discourse moves. Other accountable talk moves include pressing for accuracy or pressing for reasoning. We also documented occurrences of the IRE discourse pattern and similarly monologic practices (Lemke, 1990; Nystrand, 1997)—such as extensive teacher explanation and direction—that were evident. We then situated the revoicing moves for how they functioned within the broader academic and task structures by considering their functionality in the immediate turns surrounding the move and within longer interactional patterns.

The teachers were observed enacting *Comparing and Scaling* from CMP, which focuses on helping students develop methods for comparing quantities using multiple strategies, including fractions, ratios, and percents. The task that is the focus of this analysis is the *Orange Juice* task, in which students were given four water/concentrate mixtures (e.g., 2 cups concentrate to 3 cups water, 5 cups concentrate to 9 cups water, 1 cup concentrate to 2 cups water, and 3 cups concentrate to 5 cups water) and asked to determine which were the most and least "orangey" mixes. This task offered opportunities for students to choose from a range of strategies to make their comparison and to make connections between fractions, ratios, and percents as forms of comparisons.

Results

Revoicing functioned within the longer exchanges most prominently to establish *common ground* (Staples, 2008) at a given point in time, with two primary patterns in terms of what followed. The two teachers who most frequently and productively used revoicing to establish common ground either subsequently: (a) pressed the student or group of students to refine, revise, or elaborate their explanation; or (b) elicited comments from other students about the explanation. A second use of revoicing, particularly within the group activity structures, was to conclude a set of exchanges (which are analytically akin to Mehan's [1979] *Topically Related Set*), to establish common ground for one strategy before students moved to recording that strategy or developing a second strategy.

The longer sets of exchanges in which the revoicing moves were located had distinct functions in terms of the social and academic task structures. In terms of the social structure (e.g., the participant frameworks), the longer exchanges helped to make public students' explanations in ways that clearly attributed the origins of the explanations to particular students or groups of students, and marked the mathematical qualities of the explanations. At least implicitly, this positioned students as competent problem solvers and active contributors to the collective development of the core mathematical concepts. In terms of the academic task structure, the longer sets of exchanges served as the primary vehicles by which the teachers explored the mathematical ideas they identified as the primary goals for the unit.

Although we provide more detail shortly, Table 1 summarizes the exchange patterns across the three classrooms and indicates the location of revoicing within those patterns. The table shows differences in group and whole class exchange patterns. A common occurrence in the group exchanges was the use of revoice and press routines, in which the teacher revoiced the student contribution as a means to continue pressing students to clarify or revise their explanations. Granville used revoicing more sparingly, engaging instead in extended press sequences that often resulted in a fairly complete student explanation. Pless, by contrast, used revoicing to inject some explanatory features before continuing to press the students. In her class, the students' contributions were less evident, though she often explicitly attributed the content of her revoiced explanation to a student or group of students. In the classes of both Granville and Pless, the exchange sequences typically resulted in an articulation of a coherent strategy, though, as noted, the responsibility for articulating that strategy was distributed differently in those two classes. Sadosky's group exchange sequences were not as productive in terms of producing a coherent strategy, though she too employed the revoice and press routines. Below, we present examples of the *revoice and press* (Example 1) and *revoice to conclude* (Example 2) patterns.

Table 1: Occurrences of Revoicing in Exchange Sequences

Teacher	Small Group		Whole Class	
	Exchange patterns	**Location of revoicing**	**Exchange patterns**	**Location of revoicing**
Granville	Press and extended presses followed by student explanations; Exchanges conclude by revoice or teacher explanation (in 4 of 10 exchanges)	The revoicing move concluded an exchange (3 of 10 exchanges)	Student explanation, teacher press for others to interpret, with press and explanation, and revoice and press sequences (with short revoicing turns); interspersed IRE sequences	Short instances inside of longer press, revoice, and student explanation sequences (4 of 6 lengthy exchanges)
Pless	Press and extended presses followed by student explanations; Press, revoice, and more press (in 4 of 8); Press, teacher explanation, and more press (in 2 of 8)	In the midst of longer revoice and press sequences (4 of 8); The revoicing move concluded an exchange (3 of 8)	Press and student explanations; revoice and press exchanges (with long revoicing turns); interspersed IRE sequences	In the midst of longer exchanges as catalyst for press and explanation sequences (4 of 5 lengthy exchanges)
Sadosky	Press, revoice, and press (in 2 of 5); Extended known-answer press (in 1 of 5); Press followed by student explanation (1 of 5); Teacher explanation (1 of 5)	In the midst of longer revoice and press sequences (2 of 5)	Primarily IRE sequences, with some student explanation and teacher explanation	Concludes exchange (2 of 8 mostly brief exchanges)

Example 1: Revoice and Press Routine

Pless: Guys can you listen to him while we do this? So look at what he did? He wrote two over three and said it is 66.6 repeating percent but as he looked at this now he realizes, he originally thought this meant 66.6 percent was concentrate right? (She looks at Adam for confirmation. Adam nods.) [**Revoice**]

Pless: But as he looked at this he is thinking that is not really the case anymore. Do you agree with him ... that is not the case? [**Pressing for agreement**]

...

Pless: Because what do I have to do if I want to find out what percent the concentrate is that of the juice? [**Pressing for accuracy**]

Adam: You have to add them together and then do the concentrate out of the total amount.

In Example 1, Pless revoiced the student's strategy as a precursor to a further press for him to explain the strategy. In Example 2, below, Granville revoiced the student's explanation to provide the strategic context for the procedural description provided by Tim. Her revoicing marked the end of the discussion on one strategy before proceeding to discuss a second strategy.

Example 2: Revoicing to Conclude

Tim: You multiply that to get your base-90 and top of it you multiply by the same amount to get the top of it ... to get ... to get the numerator. [**Student explanation**]

Granville: So what you are saying is that since these are all different, might as well try to get all out of the same amount. [**Revoice**]

The whole class exchange sequences were oriented toward explaining and collectively reflecting on strategies developed in the small group activity structure, as opposed to developing a strategy; consequently, the exchanges reflected that difference. In Granville's class, for example, the exchanges typically began with a student explanation, followed by a press for other groups to interpret the strategy or a press back to the initiating group (sometimes in the form of multiple IRE sequences) if clarification was required. When Granville employed revoicing, it was typically a brief move serving to clarify a key feature of the explanation before a further press to the group or to other groups to interpret the strategy. Pless, by contrast, used longer revoicing moves that involved narration of the group's processes and thinking before continuing with the press back to the group or class as a whole about the strategy. In both classes, revoicing helped to focus students on a particular strategy in which there were clear attributions back to the groups that developed the strategies. In Sadosky's class, the summary discussion was poorly organized and there was little opportunity to extensively discuss a particular strategy. Furthermore, she took on a greater role in explaining strategies, using revoicing to conclude exchanges rather than as a clarification before a subsequent press. Below, we present examples of the use of revoicing in the summary discussions.

Example 3: Granville's Use of Revoicing as Short Turn Followed by Press

*Granvill*e: So first they wrote the ratio of OJ to water, and then what was the second part they did? Why wouldn't I say they got a common denominator? … So what did you get a common—what? [**Pressing for Accuracy]**

Student: Uh, we got a common uh amount of cold water. [**Student explanation**]

Granville: Common amount of cold water. Ok. So you got them all out of—or compared to 90 cups of cold water. [**Revoicing**]

…

Granville: So here's the next question. How is this strategy related to our first strategy**?** Talk in your group. How is this strategy two related to strategy one [students start to talk]? [**Asking students to interpret peer's strategies]**

In Example 3, the revoicing move by Granville was brief and primarily functioned to establish a common description for a strategy before pressing the students to compare two strategies. In Example 4 below, Pless's revoicing turn was lengthy and explained both the strategic and technical aspects of the strategy before she pressed the students on further aspects of the strategy.

Example 4: Pless' Use of Revoicing as Longer Turn with Explanatory Qualities

Pless: So you started with a fraction and what type of comparison is your fraction? [**Pressing for Accuracy]**

Student: Part to whole.

Pless: Part to whole. And what part and what whole are you talking about? [**Pressing for Accuracy]**

Student: Concentrate to juice.

Pless: Concentrate to juice okay ... So I asked [the group] to keep going because I wanted you guys to see this. Now this might not be the easiest one to do common denominators with, however, they wanted, they were having trouble finding what they thought might be the smallest common denominator so they found a common denominator, they knew it would work by multiplying all of the denominators together and they came up with a denominator of 1,680. [**Revoicing**]

Pless: Do you think that's the lowest common denominator? [**Pressing for Accuracy]**

A key feature of revoicing (O'Connor, 2009) that was not as evident in these classrooms was the move where the teacher checks back with the student to ensure that any interpretation reflects the student's intentions. O'Connor and Michaels (1993) noted that this move allowed the student "to validate [the teacher's] inference and, thus, take on a position himself with respect to an aspect of the current academic task" (p. 322). One question that our data leads us to ask is the role of such a move with respect to the norms established in the respective classes. Given the way that Granville pressed her students to fully articulate their strategies, for example, it is possible that students felt freer to disagree with the teacher's interpretation of their strategy than in the classes of the other two teachers, who more strongly controlled the discourse with respect to the academic task structure. In those classes, however, it is important to ask whether attempts to rephrase or interpret students' explanations, even with attribution, constitute revoicing as envisioned by O'Connor and Michaels. This leads us to the bigger question of how the longer exchange sequences, in which revoicing served an important but limited role, potentially developed the participant structures that could ostensibly be created through revoicing.

Discussion

We reflect on how focusing on longer exchange sequences helps us to consider how revoicing and its proto-forms (i.e., those moves with some but not all of the features of revoicing described by O'Connor and Michaels) contribute to the development of social and academic task structures, especially structures in which students take on serious epistemic roles.

How Exchange Sequences Influenced Social Task Structure

Even though the range of accountable talk moves at times constituted a constrained form of positioning in that the teacher controlled how explanations were articulated and attributed, the moves still contributed to the portrayal of students as competent actors and thinkers. However, it should be noted, the lack of the third move limited student agency in terms of how their claims were taken up, with decreased roles especially in Pless's and Sadosky's classrooms.

Instances in which at least two of the three features were evident involved a continuum of control over responsibility for mathematical explanations. On the one end, teachers used the proto revoicing forms in ways that functioned as teacher explanation. In these cases, the teacher used an interpretation of a student's strategy as a beginning point to expand the mathematical claim but did not attempt to press the student or class to specifically focus on how the student to whom the claim was attributed may have interpreted the claim. That is, the teachers' interpretation became the focus of discussion as a tool to advance the academic task structure. This function of revoicing, as well as other functions that allow the teacher to control the content and flow of discussion, is seen by teachers as ‘muddying’ the ostensibly clear description of what revoicing is intended to accomplish (Herbel-Eisenmann, Drake, & Cirillo, 2009). In practice, revoicing, as seen by teachers, potentially describes a continuum of practices, which in part argues for looking at the characteristics and functionality of longer exchange sequences to see patterns in how the teacher positions students.

On the other end, the teachers, particularly Granville, used revoicing as a brief clarifying move to keep the discussion focused on the students’ explanation, including helping the student to elaborate her explanation more fully and helping other students to interpret that explanation for themselves. Granville seemed particularly skilled at using revoicing to coordinate the social and academic task structures. However, she did not explicitly scaffold the participant structure in the same way as the teacher in O’Connor and Michaels (1993) study; instead, her use of the extensive press for explanation made student thinking explicit features of classroom discourse and students took on the responsibility of aligning themselves in relation to the claims.

The regular presence of consecutive revoice and press moves in the exchange sequences was an interesting development. In these cases, the teacher used the revoice move not simply to attribute or to draw students into the discussion, but to establish common ground before continuing to press for explanation or for other students to interpret explanations.

How Exchange Sequences Influenced Academic Task Structure

Although it could be argued that the more teacher-focused forms exhibited by Pless and Sadosky allowed them to control the academic task structure and thus advance their didactical goals for the lesson, Granville’s skillful and persistent press for explanation and peer interpretation of strategies provided arguably the same or greater opportunities for students to make sense of the key mathematical ideas. That is, her use of accountable talk moves resulted in strongly coordinated social and academic task structures. She pressed students until the procedural and strategic features of the explanations were clear, which Granville supported by recording these features concisely on the same sheet as other strategies. This strategy allowed her not only to collectively press the class to interpret each strategy but also to compare strategies according to the primary concepts of the unit (e.g., the nature of comparison and nature of quantities being compared).

Revoicing as Situated Within Broader Structures

The revoicing moves used by the three teachers served a variety of purposes, not all of which were consistent with O’Connor and Michaels (1993) description of the move. In part, this was due to our interpretation of the move, which included proto forms that did not include all of the features listed by O’Connor and Michaels. However, our interpretations are consistent with those of other researchers and practitioners who have attempted to identify instances of revoicing in a wide range of classrooms. In part, the ambiguity stems from trying to isolate revoicing from broader discursive and activity structures and in part from trying to map messy data onto theoretically driven descriptions of discourse moves.

The broader exchange sequences, particularly those used by Granville, served many of the same functions as the revoicing moves described by O’Connor and Michaels (1993), suggesting that the broader set of accountable talk moves can create participant structures that position students as active contributors in ways that provide agency to the students. Much of the work in the longer exchange sequences was more implicit than in a more singular instance of revoicing, in terms of attributions and students’ opportunities to verify the interpretations of their claims. That is, there were extended opportunities for students to revise and clarify their explanations, notably during the revoice and press sequences.

It should be noted, however, that in many cases, especially in Sadosky’s class and occasionally in Pless’s class, that the proto forms of revoicing placed much of the work of interpreting and explaining on the teacher, with presses for verification and clarification often in the form of a known-answer questions. A question that arises from this research is how to help teachers develop awareness of the potential for transforming these proto forms into exchange patterns that provide greater opportunities for students to do the intellectual work in mathematics classrooms.

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VISUAL REPRESENTATIONS IN MATHEMATICS PROBLEM-SOLVING: EFFECTS OF DIAGRAMS AND ILLUSTRATIONS

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Many types of visual representations are used in math textbooks but not all of them contain mathematically relevant information. Little research directly addresses the effects of different types of representations on mathematics performance. Theories offer differing perspectives about how visual representations such as illustrations influence student learning. Here, we investigated the effects of diagrams and contextual illustrations on trigonometry problem solving. Diagrams helped all students, but the effect of contextual illustrations depended on students' backgrounds. Additionally, not all subgroups of students accurately assessed the effect of illustrations on their performance. We emphasize the need to consider how different types of visual representations interact with student characteristics and the problem-solving task.

Keywords: Problem Solving

Mathematics textbooks use a wide variety of visual representations, including diagrams, tables, graphs, decorative images, and photographs. Given students' frequent use of textbooks and the large number of visual representations in these textbooks, understanding how different types of visual representations affect problem solving and learning is critical.

Most research on the effects of visual representation has been conducted using scientific texts. It has focused primarily on diagrams and illustrations accompanying expository texts about causal phenomena. In such contexts, graphically integrating visual and verbal information is found to be beneficial, as is removing irrelevant information (e.g., Mayer, 2009). However, mathematical and scientific problem solving differ in many important ways, including different emphases on causality, spatial relationships, procedural and conceptual knowledge, and analytic methods. Thus, findings from research on science learning may not apply straightforwardly to math.

Existing research about the effects of visual representations in mathematics is based primarily on studies with elementary-age students, and it presents a complex and mixed picture. Some studies suggest that contextual illustrations hurt performance for particular subgroups of students (e.g., Berends & van Lieshout, 2009). Other studies suggest that decorative illustrations do not affect performance (e.g., Berends & van Lieshout), or that certain types of illustrations can benefit performance (e.g., Hegarty & Kozhevnikov, 1999; McNeil, Uttal, Jarvin, & Sternberg, 2009). Many studies also suggest that the usefulness of visual representations depends on students' ability levels (e.g., Booth & Koedinger, 2011; Berends & van Lieshout).

Theoretical Frameworks

In making sense of research on visual representations, two theoretical frameworks are particularly relevant: the Cognitive Theory of Multimedia Learning (e.g., Mayer, 2005, 2009) and Cognitive Load Theory (e.g., Sweller, 2004, 2005). Both theories address the processing and learning of information presented in different formats.

The Cognitive Theory of Multimedia Learning (e.g., Mayer, 2009) is based on three assumptions: (a) a limited capacity for processing information, (b) separate visual and verbal pathways through which information enters the cognitive system, and (c) meaningful learning arising from active processing. Cognitive Load Theory (e.g., Sweller, 2005) focuses on the cognitive load—the mental effort from the task itself, the processing required to integrate new and old material, and the processing required to work with a task's format. Overall, one idea is that the structure of the cognitive system imposes limits on how learners select, organize, and integrate information. These approaches have been used to guide instructional design.

Two principles derived from these theories are particularly relevant to the research reported here. The *multimedia effect* holds that words and pictures are better than just words (e.g., Butcher, 2006; Mayer & Anderson, 1992) based on the assumption of separate visual and verbal channels which can then be integrated for deeper learning. The *coherence effect* captures the performance benefits that occur when extraneous or seductive features of the material are eliminated (e.g., Harp & Mayer, 1997, 1998). Adding interesting but irrelevant material can overload the visual or verbal pathways or create too much extraneous load, thereby disrupting learning (Sweller, 2005). The coherence effect applies to both text and visual material.

Contextualization Perspective

Another theoretical perspective applicable to the current study focuses on how contextualizing or "grounding" math problems in real-world scenarios can help learners (Goldstone & Son, 2005; Koedinger & Nathan, 2004). Contextualization is thought to help students build a model of the situation underlying a problem. In addition, realistic content or greater familiarity with the content may promote generalization or facilitate reasoning because it fosters integrating the current problem with prior knowledge. Some studies have suggested that contextualization is more beneficial for simpler problems (Koedinger, Alibali, & Nathan, 2008), whereas other studies have suggested that it is more beneficial for difficult problems or lower ability students (Walkington, 2012). This body of research has typically involved contextualizing problems by adding verbal information to text, but contextualization can also be accomplished through accompanying visual representations.

Current Study

It remains an open question as to how the multimedia principle, the coherence principle, and the notion of contextualization apply to visual representations used in mathematics. The current research involves trigonometry problems accompanied by 4 types of visual representations: combining diagram presence (or not) with the presence of contextual illustrations (or not). The contextual illustration could add extraneous details through the graphics, but it also could ground the problem situation. We use the term *contextual illustration* since the illustrative features correspond to the spatial layout necessary to solve the problem. The perspectives discussed above vary in their predictions about which visual representations will be most helpful (see Table 1). We consider these effects in terms of student performance and evaluations of the problems.

Table 1: Predictions from Applicable Theoretical Frameworks

	Theoretical Prediction		
Multimedia Principle	Visual representations will help	→	Text by itself will be hardest
Coherence Principle	Extraneous information hurts performance	→	Problems with illustrations will be harder
Multimedia + Coherence Principles	Provide visuals but avoid extraneous information	→	Diagram by itself will be easiest.
Contextualization Perspective	Illustrations further ground problem solving	→	Problems with illustrations will be easier.

Method

Participants

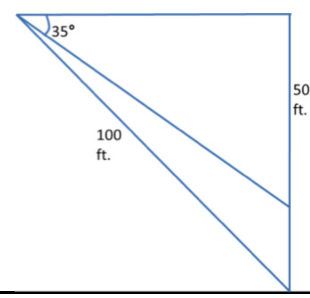
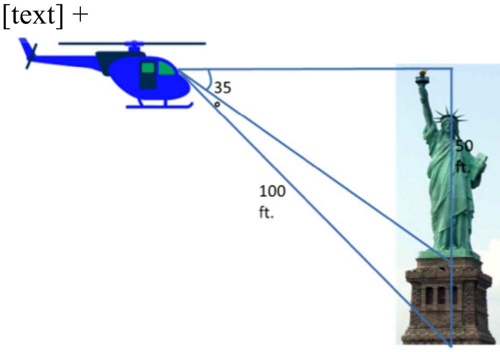
Participants were 93 undergraduates, who received credit in introductory psychology for their participation. The majority (63%) had completed middle school math in the United States. Of those who had not, most (82%) had their earlier math education in an Asian country. Over two-thirds (69%) intended to major in a math or science field.

Participants were divided into subgroups based on their intended major (math/science field or not) and where their previous math education occurred (U.S. or non-U.S.). Students who were math/science majors were fairly evenly split into those who were previously educated in the U.S. (n = 34) and those who were previously educated outside the U.S. (n = 30). The vast majority of participants who were not math/science majors were educated in the U.S. (n = 25). Only 4 participants previously educated outside the U.S. were not math/science majors; this small group was excluded from the analyses reported here.

Design and Materials

Each participant received 4 problems based on a 2 (Diagram Presence) × 2 (Illustration Presence) within-subjects design, yielding 4 conditions: text alone, diagram alone, illustration alone, and illustration with diagram overlay (see Table 2). Condition order was counterbalanced across participants.

Table 2: First Background Story, Shown for Each Visual Condition

	No Diagram	**Diagram**
No Illustration	The parks department is putting a statue on a base. The statue is some distance away, and you are in a helicopter, eye level with its top. The angle of depression to the bottom of the statue (i.e., the top of the base) is 35 degrees. The height of the statue is 50 feet. If someone were to stretch a string from the bottom of the base directly to you, it would be 100 feet long. How tall is the base?	[text] + 
Illustration	[text] + 	[text] + 

Each of the 4 problems each participant received involved a different cover story. All required applying trigonometric relations to overlapping right triangles to solve for an unknown dimension. The different stories had varied combinations of sides and angles, such that the solution processes were not identical for any two problems. These types of problems were selected as they lend themselves well to concrete situations and are at an appropriate difficulty level for undergraduate participants. The order of the cover stories (and thus of the mathematical solutions) was held constant across participants. Each problem was on its own page, with the text and visual representation (if present) at the top of the page.

The illustration corresponded to the problem situation. As shown in Table 2, although it did have decorative features, it was also mathematically relevant because it indicated the spatial layout of the components of the story problem.

Procedure

Participants received a reference handout (with text and equations but no diagrams) of information about triangles and trigonometric formulas. The information was available throughout the study, and participants were told that not all of it would be needed. Participants worked through each of the four problems at their own paces. After completing the problems, they rated how difficult each problem was, how clear it was, and how willing they would be to do more problems like it. They assessed these characteristics on a 5-point Likert scale. While making these ratings, participants were permitted to look back over the problems but not to change any of their answers. Finally, participants completed a questionnaire about their attitudes towards mathematics, their math abilities, and their math background.

Results

A sizeable proportion of participants answered all or none of the problems correctly, and these rates depended on participant subgroup. For instance, 40% of the students who were not math/science majors answered *no* problems correctly and 30% of the math/science majors who received their previous education outside of the U.S. answered *all* the problems correctly. The results below include all participants; however, the patterns also hold for the subset of participants who did not perform at floor or ceiling (i.e., correctly answered 1–3 of the 4 problems).

Did Visual Condition Affect Accuracy?

We analyzed the dichotomous measure of accuracy on each problem using mixed models logistic regression (Bates & Maechler, 2009) in R. The best fitting model included the fixed factors of diagram presence, illustration presence, educational background / major (henceforth *participant subgroup*), and the interaction between illustration and participant subgroup. We also included participant and cover story as random factors; cover story significantly improved the model's fit ($p < .0001$). Coefficients and odds for the model are reported in Table 3.

Table 3: Coefficients from Regression Model for Accuracy Ratings

Fixed effects:	Estimate (logit)	SE	Odds	z value	Sig
Intercept	-0.53	0.68	0.59	-0.78	0.44
Diagram – no	Reference				
Diagram – yes	1.49	0.28	4.42	5.23	<.0001
Illustration – no	Reference				
Illustration – yes	0.996	0.47	2.71	2.1	0.04
Subgroup – outside US & math/science major	Reference				
Subgroup – US & math/science major	-0.39	0.64	0.68	-0.61	0.54
Subgroup – US & not math/science major	-1.15	0.70	0.32	-1.65	0.10
Illustration – yes x Subgroup – US & math/science major	-0.62	0.64	0.54	-0.96	0.34
Illustration – yes x Subgroup – US & not math/science major	-1.56	0.73	0.21	-2.13	0.03
Model: Accuracy ~ DiagramPresence + IllustrationPresence * Subgroup + (1 | CoverStory) + (1 | ID)					
Random effects (Intercepts): Variance of participant = 3.26 Variance of cover story = 0.91					

Participants performed significantly better on problems with diagrams than without, $p < .0001$ (see Figure 1; *SE* are corrected via procedure in Morey, 2008, to reflect within-subjects design). This effect existed for all three subgroups and did not interact with illustration presence.

However, the effect of illustration varied across participant subgroups. Participants who received their math education outside the U.S. (and were also math/science majors) performed significantly better with illustrations ($p < .04$) than without. This improvement differed significantly ($p = .03$) from the slightly negative effect of illustration on the US subgroup not majoring in math/science. The accuracy level of the subgroup who were not math/science majors was significantly lower than the accuracies the other two subgroups when there was an illustration (*p*s < .02), but this pattern did not reach significance when there was no illustration.

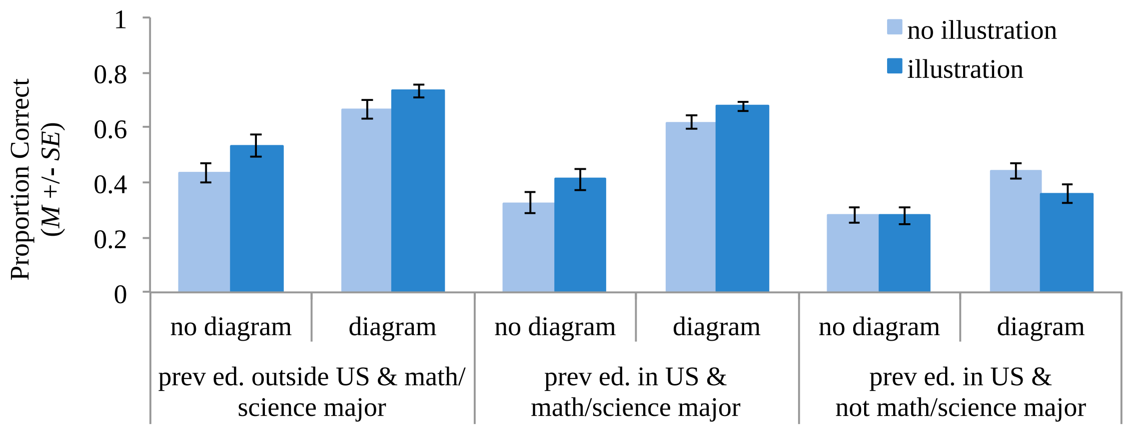


Figure 1: Average accuracy (+/– *SE*)

Did Visual Condition Affect Participants' Ratings of the Problems?

We combined participants' ratings of each problem's clarity and difficulty (reverse-coded) as well as their ratings of how willing they would be to do more problems similar to those completed. This composite measure offered an assessment of a participant's overall favorability towards a problem type. Correlations among the three measures ranged from .34 to .56, *p*s < .0001. The best fitting mixed effects model for participants' ratings included the fixed factors of diagram presence, illustration presence, and participant subgroup. We also included participant and cover story as random factors; cover story significantly improved the model's fit ($p < .001$).

As shown in Figure 2, participants viewed problems with diagrams significantly more favorably than those without, and they viewed problems with illustrations significantly more favorably than those without; respectively, each of these factors improved the fit of the model, $\chi^2(1) = 7.79$, $p = .005$ and $\chi^2(1) = 10.5$, $p = .001$. However, the magnitude of these effects was relatively small. Comparisons of the subgroups indicated that participants who were not math/science majors rated the problems significantly lower than participants with math/science majors (*t*s > 2.71), whose subgroups did not differ from one another ($t = 1.70$).

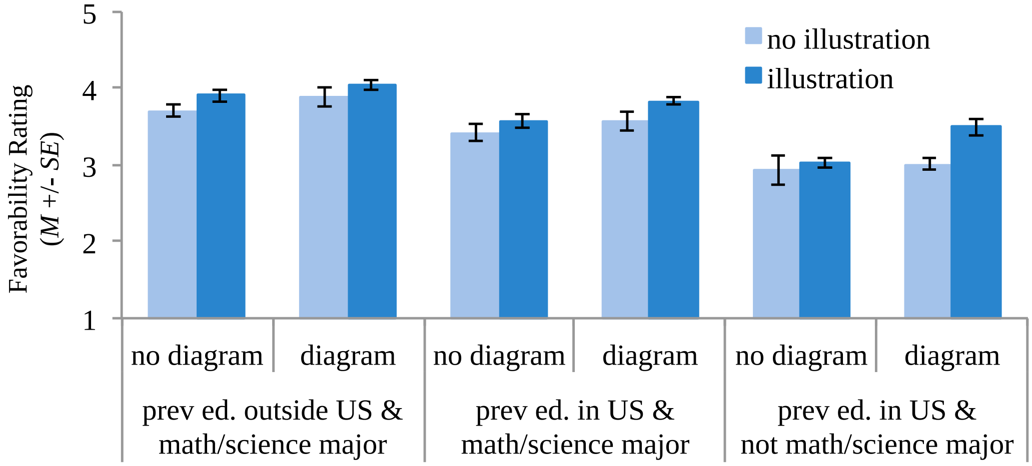


Figure 2: Average favorability (+/– *SE*)

Discussion

In this study, participants performed more accurately on trigonometry problems with diagrams. The effect of illustrations was mixed. Illustrations yielded a slight improvement in performance for students who intended a math/science major, but illustrations slightly hurt performance among students who were not intending to major in a math- or science-related field. These findings highlight that ability differences affect the use of visual representations.

The *multimedia principle* predicts that problems with visual representations would be solved more successfully than problems presented as text only. This was clearly the case for diagrams. The more mixed influence of contextual illustrations can be considered with respect to the *coherence principle* and the *contextualization perspective,* which make opposite predictions. Indeed, each prediction fit a subset of the participants. As predicted by the contextualization perspective, having an illustration benefited performance for participants who were math/science majors. In contrast, as was predicted by the coherence principle, illustrations hurt performance for those who were not math/science majors. Overall, though, the effects of illustration presence were relatively small.

These findings indicate that the coherence principle, which has been supported in multiple studies using science material (see Mayer, 2009), may not apply so straightforwardly in math. However, the coherence principle stresses the removal of extraneous details. Not all details are extraneous, and added visual details do not necessarily harm everyone's performance. This research used contextual illustrations that could have assisted students in mapping the problem content to the visual representation and thus does not necessarily contradict the coherence principle. It is also worth noting that the contextualized illustrations we used were more relevant to mathematics than the majority of illustrations that are found in American mathematics textbooks (Cooper et al., 2012; Mayer et al., 1995). Addressing the impact of purely decorative illustrations will be an important extension of this research.

Focusing on the cognitive load required by these problems offers a possible way to combine the two perspectives on the effect of illustrations and understand the dependence of the effect on subgroup. The cost of encoding and integrating the extraneous information (such as the design of the base of the statue) conveyed in illustrations may outweigh any possible benefits from contextualization if cognitive load surpasses the available cognitive resources. Illustrations might be more helpful for individuals with more math experience because such individuals can construct a contextualized mental representation of the problem scenario without exceeding their available cognitive resources. However, other research on contextualization has found grounding problems to offer greater benefits for students of lower math ability (see Walkington, 2012).

Diagram presence increased the favorability with which participants viewed the problems, as did illustration presence. Comparing this with performance data indicates that all participants' metacognitive beliefs about problems with diagrams matched their actual performance. However, only students intending a math/science major accurately perceived the effect of illustration presence. Participants who were not math/science majors performed the same or worse when an illustration was present, despite their more favorable view of these problems. This pattern of findings is particularly important to consider in light of the motivation-based argument that textbook visuals will help engage learners, particularly those with low math interest (see Durik & Harackiewicz, 2007, for related findings). However, it aligns with the research arising from the Cognitive Theory of Multimedia Learning and from Cognitive Load Theory, which hold that these extraneous but interesting details can be problematic for learning. As noted above, this may hold true especially when an individual's resources are taxed, which is more likely to occur for individuals with lower background knowledge.

The overall differences we observed in accuracy between students of different backgrounds are not surprising in light of the well-documented finding that students from many foreign countries outperform American students in math (Fleischman et al., 2010). What is more interesting is that students of different backgrounds were differentially affected by visual representations. The underlying constructs tapped by our measures of students' backgrounds need to be characterized with greater precision. We collected data on the intended majors and the country in which they received their middle school education. These measures may simply reduce to experience and interest in math; however, further research on students' backgrounds and how they affect performance is needed.

It is also worth noting that overall levels of performance in this study were not high, even in the highest performing subgroup. The problems we used were quite complex, and many components needed to be performed correctly in order to reach an accurate final answer. Students needed to know how to map information from the problem content to the visual representation and from the visual representation to their mental representation of the problem. Students also needed to identify what quantity to solve for, figure out the steps needed to reach the solution, and correctly apply the trigonometric formulas to reach a final answer. Understanding the differential effects of the type of visual representation on these different components of problem solving is an important arena for future research (see Butcher, 2006; McNeil et al., 2009).

In sum, this work highlights the need for a continued focus on the ways in which visual representations support learners' strategic problem solving and learning. Rather than asking simply which types of illustrations serve learners better, it is important to identify how learners with different backgrounds and skill levels utilize visual representations when solving problems.

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GESTURES AS FACILITATORS TO PROFICIENT MENTAL MODELERS

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Gestures are profoundly integrated into our communication. This study focuses on the impact that gestures have in a mathematical setting, specifically in an undergraduate calculus workshop. There was strong correlation between diagramming and the two types of gestures identified in this study (i.e., dynamic and static gestures). Dynamic and static gestures were part of the students' constructive thinking, whether it was related to the manner in which they viewed the problem or the construction of their diagrams. Nonetheless, gestures played a strong role in the students' problem solving and the manner in which the gestures were utilized provided insight into their constructive thinking.

Keywords: Classroom Discourse; Instructional Activities and Practices; Modeling; Problem Solving

Background

There have been numerous studies conducted on gestures and their presence in the educational environment especially in math and science related fields (Rasmussen, Stephan, & Allen, 2004; Chu & Kita, 2011; Goldin-Meadow, Cook, & Mitchell, 2009; Scherr, 2008). For example, Rasmussen, Stephan, and Allen (2004), studied gesturing in a differential equations class where they observed mathematical classroom practices become what they call taken as shared (TAS) ideas for the participants. Through their theoretical perspective they formed a gesture/argumentation dyad, which they used to analyze the gesturing that occurred among the classroom community. In short, their analysis was predicated upon students and teachers as opposed to just students.

Goldin-Meadow, Cook, and Mitchell (2009) explored the impact of teaching third and fourth graders how to gesture while solving a specific type of problem such as 3+4+6=__+6. In their study the participants were separated into three groups: the first was taught a correct gesture, the second group was taught a partially correct gesture, and the third group was not told to gesture at all. Although Goldin-Meadow, Cook, and Mitchell (2009) based their research on the manipulation of gesturing during a math lesson, their findings shed light on the idea that gestures act as an aid when it comes to problem solving for the children who gestured correctly or partially correctly as opposed to those who were not taught to gesture during the lesson.

Studies conducted by Engelke (2004, 2007), were based on understanding students' thought processes in related rates problems. She found that many students fail to understand these types of problems because there is a lack of transformational/covariational reasoning, which pinpoints students' deficiencies in geometry as well as being able to apply mathematical concepts to problems (i.e. similar triangles, substitution, and function composition). As revealed in Engelke (2007) function composition is a necessary tool when dealing with related rates problems. Engelke, Oehrtman, and Carlson's (2005) study highlighted the fact that not much research has been conducted in regard to student understanding of function composition. Other studies indicate that students tend to develop their notion of functions throughout their undergraduate years (Carlson, 1998).

Visualization and representation are dynamic in nature and are an important part of being able to solve word problems (Booth & Thomas, 2000; Carlson & Bloom, 2005; Cifarelli, 1998; Gravemeijer, 1997; Johnson-Laird, 1983; Lucangeli, Tressoldi, & Cendron, 1998; Simon, 1996). Johnson-Laird (1983) extensively discussed three types of mental representation of which two we considered "mental models which are structural analogues of the world, and images which are the perceptual correlates of models from a particular point of view" (p. 165). It has been theorized that in order to understand word problems, students benefit from drawing a diagram and trying to understand the relationships their diagrams represent of the given situation (Engelke, 2007). Visual and analytical skills are essential for students to understand mathematical concepts and construct mental models (Haciomeroglu, Aspinwall, & Presmeg, 2010). Hegarty and Kozhevnikov (1999) described visual-spatial representations as schematic or pictorial. The latter obstructs students from fully understanding the mathematics behind the problem, while the former encourages students to think about the problem in a more abstract manner. Gestures are a form of visual-spatial representation and we investigate how such representations facilitate the problem solving process. We seek to answer the question: How do students' gestures facilitate contextual problem solving in calculus? Through observing students' use of gestures while solving related rates and optimization problems, we will better understand the mental models and diagrams being created during the problem solving process.

Methods

In this study, we used open and axial open coding to observe three different supplemental instruction (SI) workshops, which consisted of undergraduate students taking first semester calculus. Pseudonyms were given to the participants for privacy purposes. The workshops were led by peer instructors. We focused on related rates problems as well as some optimization problems. We watched the videos specifically attending to students making hand gestures and the diagrams they drew during the problem solving process. The observed groups usually consisted of three students. The groups were given specific problems to complete as a team, although some students worked individually and then shared their ideas with their group members. Students often asked SI leaders for help. We took into account student-to-student and student-to-SI leader interactions.

Results

We identify diagramming as a visual tool, including drawing a picture, which is used during the problem solving process and is intrinsically linked to the construction of the mental model. The definition of gestures varies in the literature. For instance, Roth (2001) defined gestures as hand movements made with a specific form where "the hand(s) begin at rest, moves away from the position to create a movement, and then returns to rest"(as cited in Rasmussen, Stephan, & Allen, 2004). Although there are many definitions present today, we define gestures similarly to Roth: gestures are hand movement(s) where the hand(s) extends in an outward position, makes a movement or movements consisting of icons, symbols, and indices, and then returns to its normal position. Icons include gestures that demonstrate a thing, symbols are those that describe a thing, and indices are gestures indicating a thing (Clark, 1996 as cited in Rasmussen, Stephan, & Allen, 2004).

There are two types of gestures we identified: dynamic and static gestures. Dynamic gestures consist of moving the hands to describe the action that occurs in the problem or movements made to represent mathematical concepts. Within dynamic gestures there are two subcategories: dynamic gestures related to the problem (DRP) and gestures that are not related to the problem (DNRP). Static gestures are done to illustrate a fixed value (length, constant radius, etc.) or to illustrate a geometric object. Static gestures consist of static gestures related to a fixed value (SRF) and gestures related to the shape of an object (SRS).

Dynamic Gestures

We define DRP as hand gestures that consist of movements describing parts of the students' diagrams, whether it is the motion of an object or changing rates/values. DRP is further broken down into two subcategories. The first subcategory identifies hand gestures used to answer/clarify a concept/question to a classmate. A student may use a hand gesture to reason the problem out. For instance, a group of students, Jackie, Josh, and Cathy, were trying to solve the following related rates problem (the boat problem):

> A boat is pulled onto a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8m from the dock? (Stewart, 2009, p. 132)

Although the problem in the textbook has an image depicting the situation, students were not provided with the image. They had to construct the diagram and solve the problem.

Josh: You know what a pulley is right?
Jackie & Cathy: No
Josh: It's a little thingy [HG*: raises his right hand in the air, rotates his right index finger inward, then raises left hand to the same height as his right, puts both hands together, and pulls downward*] you pull…so it's gotta be on top.
Jackie & Cathy: Ohh….
Jackie: So it's going to be like that?
Josh: So… [*Drawing diagram*]
Cathy: I did it like that [*laughs a little*]
Jackie: See, how do they expect us to not know what a bow is
Cathy: Yeah…
Josh: You don't know what a boat is?
Jackie: I know what a boat is…
Cathy: But not a bow…or whatever

The gesture made in this clip is also characterized as a symbol because Josh describes a pulley with his hands. Although his peers, who did not know what a pulley was, prompted Josh's gesture, the gesture revealed his mental representation of the problem. After he makes the gesture, he begins drawing his diagram and concludes that the pulley must be on top. The gesture influenced Josh's perception of the problem (i.e. placement of the pulley), which also had a role in how he labeled his diagram. After this exchange, Jackie constructs an appropriate mental model of the situation as is evidenced by her subsequent exchange with Cathy.

The second subcategory deals with hand gestures done in order to understand and reason about the problem. For instance, Jackie is trying to solve the boat problem by first attempting to understand the dynamic element of the problem.

Jackie: [HG*: Jackie moves her hands in a circular inward motion (Figure 1)*] Is the…when its being pulled, it's being pulled from there?
Cathy: Yeah
Jackie: [HG*: Jackie points her left index finger towards Cathy's diagram (Figure 2)*] so then it's approaching on *x*, so okay we are looking for *dx/dt*, but then we're looking for this rate…

Figure 1: Jackie's gesture for boat

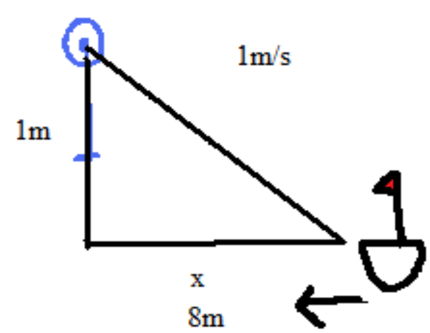


Figure 2: Cathy's drawn diagram for problem

Jackie's hand gesture is prompted because she is trying to understand the problem both conceptually and physically. In the first hand gesture, she is portraying the pulley with the use of her hands, which Clark (1996) classified as a symbol, because she is describing the pulley with her hands (as cited in Rasmussen, Stephan, & Allen, 2004). With the second hand gesture, she references Cathy's diagram (Figure 2) to understand what is happening to the boat. Here, Jackie is utilizing an index gesture, because she is indicating Cathy's diagram to observe the behavior of *x* as the pulley is pulled. Through the first and second hand gestures, she sees that as the rope is pulled, the boat moves closer to the dock. She recognizes that the change is occurring on *x*, and hence, she relates the change in *x* with *dx/dt*. The gesture made here, appropriated in part from Josh's earlier gesture, allows her to conceptualize the diagram described in the problem. The gesture acts alongside Jackie's constructive thinking as she tries to comprehend the situation at hand.

For DNRP, students and/or SI leaders use hand gestures to refer to mathematical concepts. The SI leader assisted Susan, Brian, and Cesar with the trough problem which states:

> A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of 12 ft^3/min, how fast is the water level rising when the water is 6 inches deep? (Stewart, 2009, p.132)

SI Leader: Now, what do you do to find what… cause you're trying to find *dh/dt* [HG*: He first moves his right hand right to left, with his fingers curved in the shape of a c, and then he changes the position of his hand and moves it up and down*] right?
Susan: Yeah. So we have to take the derivative of each side.
SI Leader: and that is when you plug in what your paused… Do you see it Cesar?

When the SI leader makes the hand gesture, his motion depicts the fractional aspect of the derivative (i.e., *dh* over *dt*). Additionally, Susan seems to associate finding *dh/dt* with implicit differentiation because she immediately thinks about taking the derivative of both sides when the SI leader mentions *dh/dt*. In another clip, Jackie explains to Cathy the difference between taking the derivative in a related rates problem, and taking the derivative of a (usual) function of *x*.

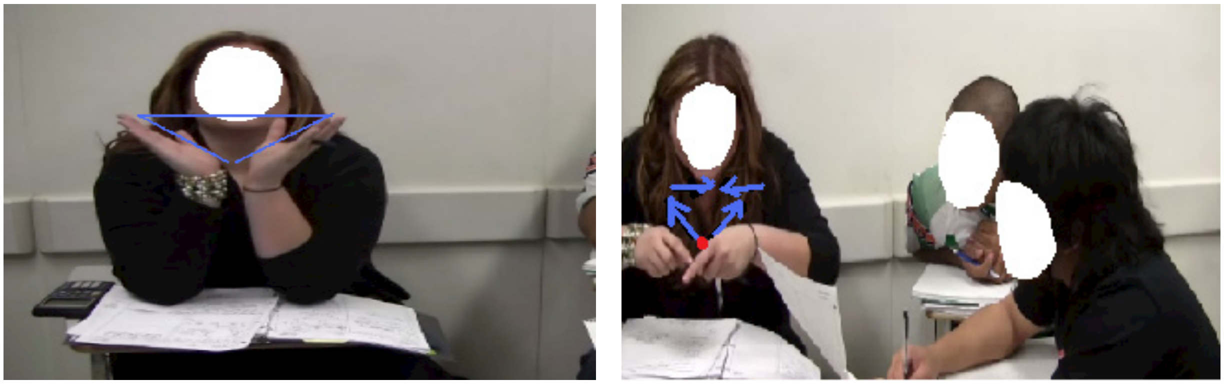
Cathy: Jackie I have a question [*laughs*]. Remember how last time you said we always keep *dy/dt*
Jackie: Mhmm.
Cathy: Do we also keep *dx/dt* cause it's not the same?
Jackie: Yeah…no, because we're not solving for uhmm… [HG*: lifts right hand up then moves it downward in a diagonal manner*] it's no longer like a just taking the straight out derivative of it, cause we have different properties that we need to relate together.

The hand gesture was a symbolic representation of a derivative. Along with her hand gesture and her explanation to Cathy, it can be seen that Jackie is able to discriminate between the derivatives in the context of a related rates problem and generally taking the derivative of some function of *x*. Although she does not explicitly state that the difference is based on taking the derivatives with respect to time, that underlying concept somehow triggered Jackie to separate the two. The gesture, in this case, did not act as part of Jackie's constructive thinking about the problem; rather the gesture was used as an explanatory aid.

Static Gestures

Apart from dynamic gestures, we identified static gestures, which consist primarily of gestures that illustrate a geometric object or refer to a fixed value (length of one the sides, constant radius, etc).We will distinguish them as static related to the shape of an object (SRS) and static related to a fixed value (SRF). SRS is focused on students or SI leaders utilizing their hands to depict the geometric shape of an object; we also consider referencing the general diagram as SRS. Here we see Susan trying to process the trough problem. Figure 3 shows Susan's initial image of the trough problem. She traces with her fingers the outline of tip up standard equilateral triangle. As seen in Figure 4, however, Susan changes the orientation of the triangle to a downward position as she reasons out the scenario in a realistic setting. Although not much is said, Susan's facial expression and gestures illustrate her thought process of trying to understand the general shape of the trough.

Susan: [HG*: elbows are bent and on top of desk, wrists are touching, hands are open diagonally, and pointing in opposite directions, flickers her left hand as she moves her pencil between her fingers, puts hands together, then pulls them apart in a diagonal direction Figure 4)*] so it's not the sides…it's the width [*SI Leader interjects and provides insight on the problem*]



Figures 3 & 4: Susan gestures to describe the trough

Her first gesture illustrates a cross-section of the ends of the trough; as she continues to think, she forms the length of the trough by pulling apart her two hands. Susan, along with many students, struggled to understand the geometric aspect of related rates problems. Engelke (2005, 2007) indicated that students tend to adopt a procedural way of thinking when they approach related rate problems. This may be why students have a hard time understanding problems such as the trough problem, which deal with a three-dimensional object, as well as applying the concept of similar triangles. For instance in the trough problem, Susan automatically drew an upright equilateral triangle (Figure 3) as opposed to visualizing the triangle upside down or oriented in a different way. Students with geometric misconceptions tend to construct incorrect diagrams, leading them to the wrong solution. If the SI leader had not intervened, the students would have attempted to solve the problem with an incorrect diagram, hence leading them to the wrong answer. That said, we consider SI leader intervention is necessary at times to provide students with guidance on challenging problems. The SI leader's help can start the students on the path to solving the problem correctly, without just giving them the answer. However, sometimes intervention by SI leaders, as also revealed in Scherr (2008), may actually interrupt a student's thinking.

The discussion below provides a glimpse as to how students think about a challenging problem.

Susan: Okay so like the square one is ten feet [HG: hands are both raised, fingers spread out flat, she then moves them down, putting hands in a diagonal position] like if you look at this from the top

Brian: It's like this right? [*HG*: *mimics Susan's gesture (Figure 5)*]

Susan: Yeah, the sides are cause you have the square part [*HG: hands are moved front to back then she brings them together, to denote the shape of a square (Figure 6)*] and that's the base [*HG: moves her left hand, which is extended and flat back and forth in a vigorous manner*], which is ten, but [*HG: she lifts her hands, elbows bent, hands in a diagonal manner, with hands in this position, she moves both index fingers back and forth*] then you have the three feet wide triangles that are coming down into it [*HG: moves hands that are elevated and in a diagonal position, down , closer to each other*], so [*HG: hands laid flat, palms facing the ground, she moves her hands left to right*] that's why it's not just a flat, it's not like a square box [*HG: moves both hands vertically*], it has that extra [*HG: elbows bent and tilted, hands are lifted up in a diagonal manner, then moves them a bit towards each other*] side coming in… so from if you look at it kind of from the top [*HG: hands are slightly up, moved up and down swiftly*] its coming down like that [*diagramming, then shows it to group members*] like that's what the inside looks like

Brian: Yeah

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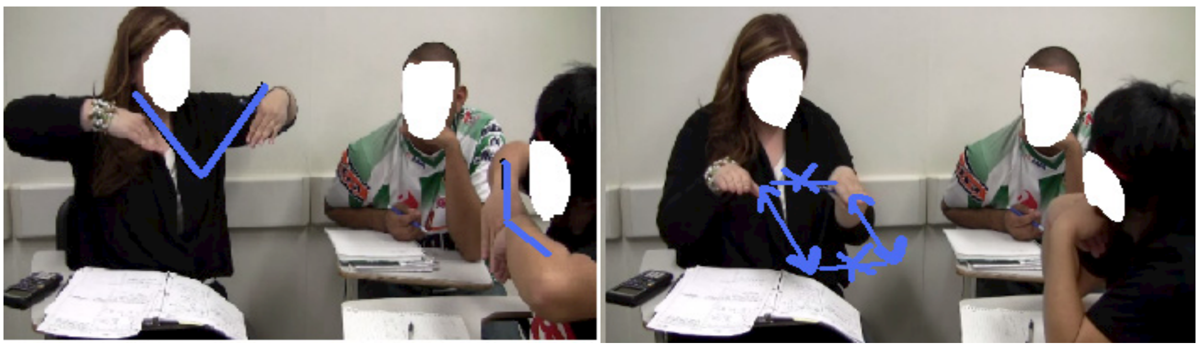
Susan: …if you are looking at it and this is your bottom length, that right there, which is ten, [*HG: hands flat, lifted from desk, she moves them swiftly together and apart, then lifts them up*] so you can't just say its ten on the top and bottom, which is like what we were doing, on, in class…

Although Susan's mental picture of the diagram itself is not correct, her gestures show that she is engaged in constructively thinking about the solution to the problem. She gestured to describe the picture she was visualizing in her mind. She was able to transfer that mental image into a more realistic perspective with the use of gestures. Since she visualizes the diagram with a square bottom, she reasons that the top and bottom could not be the same because they have different lengths. She realized that the trough cannot be a square box, because the sides are diagonally positioned. She also seemed to recognize similarity between this problem and those discussed in class. The gesture that Susan made also prompted one of her group members to engage in the problem solving process. When she made the first gesture, Brian mimicked the same gesture (Figure 5), which showed that he was also thinking about the problem in an abstract manner.



Figures 5 & 6: Susan engages in describing the shape of the trough with her hands

For SRF we take into consideration the students' gestures done with respect to a fixed value, such as length or constant radius. These gestures are usually associated with students' diagrams. As mentioned above, SRF deals mostly with students referencing a fixed value. In this session, Brad and Mark work on a related rates problem that deals with the distance between two cars moving in different directions.

Brad: [*diagramming*] Specific, its saying, one is traveling south at 60 miles per hour
Mark: [*mumbles something*]
Brad: I don't know. So you times it, so 2 times 60 [*labels*], cause the two hours, [HG: *moves his left hand top to bottom*] that's the length, and 25 is the top.

Brad's gesture was done in order to describe a fixed value, in this case, length. Although his gesture was quick, it was done in an effort to explain to Mark the values corresponding to their specific diagram. This type of gesture shows how students associate given values to their diagrams.

Conclusion

A strong relationship between diagramming and the two types of gestures identified in the study is evident. Static and dynamic gestures were often used in regards to the students' diagrams, but static gestures seem to have a stronger relationship to diagramming as they deal with the diagram itself. In order for the student to even attempt solving the problem he or she began with drawing a diagram. It appears that when the students were stuck with their diagram or parts of their diagram, they gestured while trying to reason out the part about which they were confused. Several students gestured because they were trying to obtain a better understanding of how the diagram corresponded to geometric terms in the given problem. The more challenging the problems were, the more the students gestured. Some gestures were influenced by prior gestures and students quickly adopted and adapted gestures made by their peers. For instance, in the boat problem with Jackie, Josh, and Cathy, before Jackie made the gesture to describe how the boat was being pulled by the pulley, Josh's gesture, which described the pulley, had to occur first. Other times gestures arose because there was a lack of knowledge that was needed to even begin the problem. An example would be the trough problem that Susan's group was assigned. Her gestures were caused partly because neither she nor any of her peers knew what a trough was. In her mind she pictured an object with a square bottom and triangular sides. Since she did not know what a trough was, most of her gestures were done to figure out what the trough looked like. Most gestures have one thing in common; they were made to solve the problem by first understanding the problem abstractly. Although not much research has been done on the impact that gesturing has on undergraduate mathematics students, one must wonder whether or not the gesturing that occurs is beneficial to students. There is no denying that gesturing does influence the way students approach a problem, but to what extent?

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CONCEPTUAL CONNECTIONS BETWEEN STUDENT NOTICING AND PRODUCTIVE CHANGES IN PRIOR KNOWLEDGE

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In this report, I examine what students noticed as they participated in an instructional unit on quadratic functions and how a shift in what they noticed was conceptually connected to productive changes in their prior knowledge about linear functions. My results show that, over the course of the instructional unit, students' attention shifted toward noticing changes in the quantities involved in quadratic functions. Furthermore, I identify two conceptual connections between the shift toward noticing changes in quantities in quadratic contexts and the productive changes in reasoning that students exhibited on post-interview linear function tasks.

Keywords: Algebra and Algebraic Thinking; Design Experiments; Mathematical Knowledge for Teaching

Purpose of the Study

Multiple studies in mathematics (and science) education have examined aspects of how prior knowledge changes as new knowledge is acquired (e.g., Hohensee, 2012; Smith, diSessa, & Roschelle, 1993; Vosniadou & Brewer, 1987). My study extends the research in this area by identifying explanations for why prior knowledge changes in productive ways as a consequence of new knowledge being acquired about a related but different topic. The purpose of the current research is to account for the role that features of the instructional environment played in bringing about the previously reported finding that five out of seven middle school students that participated in an instructional unit on quadratic functions exhibited productive changes in their prior understanding of linearity as a result of participating in the unit (Hohensee, 2012).

In the earlier research in which productive changes in prior knowledge were observed, students went from reasoning in non-proportional ways with the quantities from linear functions prior to participating in a quadratic functions instructional unit, to reasoning proportionally with the changes in linear function quantities after the unit. For example, in the pre-interview, when students were shown the graph of the linear relationship between the hours a cell phone was used and the cost of using the cell phone, Jenn,[1] one of the participants, reasoned *univariately* (i.e., she reasoned exclusively with the cost variable) and concluded incorrectly that the rate of the function was non-constant. However, on a post-interview task about the graph of a linear relationship between the number of employees in a business and the cost of running the business, Jenn reasoned proportionally with the changes in quantities (i.e., after finding a 4 employee change in the number of employees from 8 to 12 employees and a corresponding $2500 change in cost from $6000 to $8500, she multiplied the rate of $625 per employee by 4 employees to see if it produced the $2500 change in cost).

I considered various explanations for why students' prior knowledge about linear functions had changed as a result of participating in instruction on quadratic functions. Of the explanations I considered, *student noticing* offered the greatest promise as an underlying mechanism behind the productive changes. According to Lobato, Rhodehamel and Hohensee (2012), student noticing is defined as "selecting, interpreting, and working with particular mathematical features or regularities when multiple sources of information compete for students' attention" (p. 9). One important reason why student noticing recommended itself to my purposes was because research has already shown that student noticing possesses explanatory power for changes in how students think about novel tasks as a result of instruction (Lobato et al., 2012). Although what students notice and what students understand in a particular context is likely closely related, looking at what students notice offers a unique perspective on their thinking. In this paper, I examine what students noticed during an instructional unit on quadratics to see if it is conceptually connected to the productive changes in their prior knowledge about linear functions.

Theoretical Foundation: Noticing and the Focusing Framework

Much cognitive and psychological research has examined attention (e.g., McCandliss, Beck, Sandak, & Perfetti, 2003; Posner & Fan, 2008; Treisman & Gelade, 1980). However, only a small body of mathematics education research has examined attention from the perspective of what students notice in more realistic educational settings (e.g., Lobato et al., 2012; Radford, Bardini, & Sabena, 2007). Building on the limited prior work on student mathematical noticing, I used Lobato et al.'s definition of *noticing* as stated above. Furthermore, I used , et al.'s four-part *focusing framework*, which was specifically designed to characterize what students notice in mathematics instructional settings. The four parts are: (a) the centers of focus, which are the objects that students attend to within a given perceptual or conceptual domain; (b) the focusing interactions, which are the discursive practice that influence what students notice; (c) the features of the mathematical tasks, which are the attributes of the activities that students participate in that "afford and constrain" (Lobato et al., 2012, p. 12) what they notice; and (d) the nature of the mathematical activity, which is the participatory structure of the classroom environment (i.e., the norms that get established in the classroom). The center of focus represents the psychological aspect of noticing while the other three parts refer to the social structures of mathematics classrooms that influence what gets noticing. Thus, the focusing framework coordinates the psychological and the social, to develop a comprehensive picture of student noticing in realistic educational settings.

Methods

I employed a design-based research (DBR) methodology for this study (Design-Based Research Collective, 2003). Specifically, my quadratics instructional unit became the third iteration of the unit, the previous two iterations being part of a larger study conducted by the research team with which I was associated.[2] Thus, I continued the refinement of the activities that had been developed in the previous two iterations. My instructional unit was also similar in duration to the previous iterations (16 hours of instruction, spread over two weeks).

Seven students were recruited from an ethnically diverse urban middle school set in a middle class neighborhood. I, the author of this paper, served as the teacher. This is consistent with the principles of DBR in education, where the line between the teacher and researcher is often blurred to make in-the-moment refinements of the instructional design possible (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003).

Each class was recorded by two video cameras operated by graduate students and another researcher. The camera operators also served as external observers, presenting their observations to the teacher during debriefing sessions at the end of class. These observations further contributed to the refinement of the instructional unit. Incorporating observer feedback while the instructional unit is still being conducted is consistent with the principles of DBR (Steffe & Thompson, 2000).

I began data analysis by creating a descriptive account of the things that were said by students and the teacher with minimal interpretation (Miles & Huberman, 1994). I also identified episodes that seemed potentially rich with respect to student noticing during this pass through the data. To prevent data overload, the data was then reduced (Miles & Huberman, 1994) to these rich episodes. Next, multiple analytic passes were made through the reduced data, one pass for each part of the focusing framework. In the first pass, I identified the emergent *centers of focus* (i.e., what students appeared to be attending to mathematically). In the second pass, I used a priori codes to categorize the *focusing interactions* that occurred in and around the time in which each center of focus emerged. In the third analytic pass, I identified a connection between the features of the *mathematical tasks* that students engaged in and the centers of focus that emerged. In the fourth pass, I analyzed the *nature of the mathematical activity* that appeared to be related to the centers of focus that emerged in the instructional intervention.

Finally, I looked for conceptual connections between what students were noticing (their centers of focus) and productive changes that I had discovered in the students' prior knowledge about linearity during an earlier analysis of the post-interview (Hohensee, 2012). Looking for conceptual connections is consistent with the *realist view of causation* (Maxwell, 2004). Researchers who subscribe to a *realist view* assume that the actual causal mechanisms (processes) underlying regularities between events can be observed. In contrast, researchers who subscribe to the *regularity view of causation* assume that the causal mechanisms that underlie events are unobservable. The realist view aligns with taking a *process-oriented approach* to research (i.e., conducting qualitative research and creating causal explanations). Therefore, in looking for conceptual connections between noticing and productive changes in prior knowledge, I oriented my analysis toward noticing as a potential process underlying the productive changes in prior knowledge I had previously discovered.

Results

In this section, I present classroom evidence that shows there was a shift over the course of the quadratics instructional unit in what students noticed (see Table 1). Three types of evidence from the classroom data will be presented: (a) evidence of students' initial center of focus prior to the shift; (b) evidence of the new center of focus after the shift; and (c) evidence of the how the shift in center of focus was socially organized by particular kinds of focusing interactions, features of the mathematical tasks and nature of the mathematical activity (see Table 2).

Initial Centers of Focus

Students initially appeared to be focused on accumulated distances and accumulated times and in some cases on the changes in distance as well. Kendra,[3] Jenn, Armando, and Nicolas were initially focused on all three quantities. For example, on a Lesson 2 task, which involved a SimCalc Mathworlds™ computer simulation of a fish swimming according to a quadratic distance-time function, Kendra wrote, "From 0-2, it's 1 second. And then in the 2nd second it goes 6 feet, in the 3rd second it goes 18 feet, which is 10 feet. Then in the 4th second it goes 32 feet, which is 14 feet. In the 5th second it goes from 32 feet to 50, which is 18 feet." In this response, Kendra focused on the accumulated distances (i.e., 2, 18, 32 and 50 ft), the changes in distance (2, 6, 10, 14 and 18 ft) and the accumulated times (i.e., 1st, 2nd, 3rd, 4th and 5th second). To an adult, Kendra's talk about the 1st or 2nd second may seem like she was also attending to the 1-second changes in time. However, previous work in this area has shown that, unless middle school students explicitly refer to the 1-second changes in time (e.g., the fish went 6 ft in 1 second), they are likely not focused on those quantities (Lobato et al., 2012).

Other students like Peter, George, and Brady appeared to be initially focused on accumulated distances and accumulated times only. For example, on the same task described above, George and Brady recorded accumulated time/accumulated distance fractions (i.e., 1 s/2 ft, 2 s/8 ft, 3 s/18 ft, 4 s/32 ft, 5 s/50 ft, 6 s/72 ft, and 7 s/98 ft) and then reduced all to equivalent fractions with a 1 in the numerator (i.e., 1/2, 1/4, 1/6, 1/8, 1/10, 1/12 and 1/14). Despite George and Brady reducing their time/distance fractions to a numerator of 1, I did not count this as an instance of focusing on 1-second changes in time and the corresponding changes in distance because (a) they dropped the units for the reduced fractions; (b) they did not refer to the numerators as representing 1-second changes in time or to the denominators as changes in distance; and (c) they produced fractions that represented the set of *average distances* the fish travelled in the first 1, 2, 3, 4, 5, 6 and 7 seconds, rather than the changes in distance over each second.

Table 1: Summary of Shifts in Centers of Focus

Lesson when shift occurred	Student	Initial Center of Focus	New Center of Focus
Lesson 2	Jenn	Focus on accumulated distances and times and, in some cases, changes in distance	Focus on changes in distance and changes in time
Lesson 8	Armando		
Lesson 9	Nicholas		
Lesson 11	Kendra		
Lesson 8	Peter	Focus on accumulated distances and accumulated times	
Lesson 8	George		
Lesson 8	Brady		

Table 2: Summary of Social Organization of Individual Noticing

Individual component of shift in noticing	Social component of shift in noticing		
Center of focus after shift	Focusing interactions	Features of the mathematical tasks	Nature of the mathematical activity
• Focusing on changes in distance *and* changes in time	• Naming • Highlighting	• Specifically targeting an exploration of changes in quantities • Varying the relationships between changing quantities	• Presenting diagrams • Noticing features in others' diagrams • Asking *Can you explain why?* questions

New Center of Focus

As Table 1 shows, all students exhibited a shift in their center of focus with respect to the quantities they noticed in quadratic function distance-time data. Specifically, by Lesson 8, six students had converged on noticing the changes in distance and the changes in time. The other two students began consistently recording changes in distance and time in their diagrams for quadratic distance-time functions by lessons 9 and 10, respectively. One way that students exhibited this new center of focus was by recording changes in distance and time in their diagrams of quadratic distance-time data. For example, in Lesson 8, when students were presented with tabular quadratic distance-time data representing the motion of a remote-control (RC) car (i.e., [0 s, 0 yds], [4 s, 16 yds], [8 s, 64 yds]) and asked to produce a diagram showing speeds, George, Armando, Jenn, Brady, and Peter recorded changes in distance and the corresponding changes in time in their diagrams (see Figure 1 for Peter's diagram).

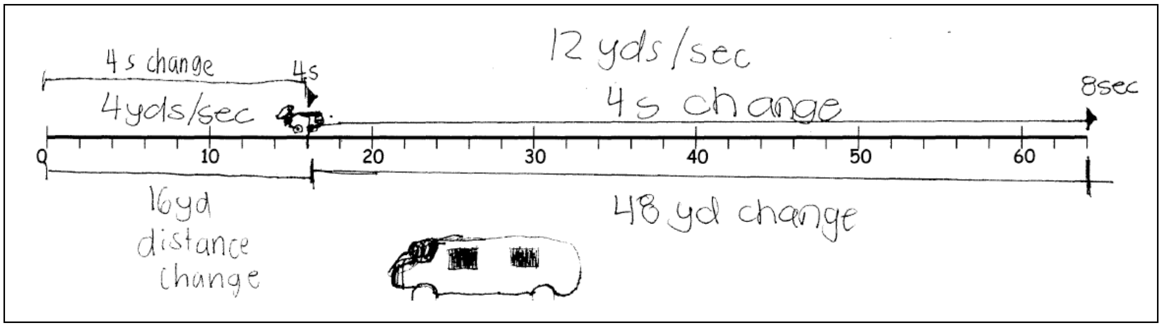

Figure 1: Peter's RC car diagram highlighting changes in distance and time

Students also verbally referred to changes in distance and time when explaining or reasoning about their own and other's diagrams. For example, when Kendra explained what she noticed in Brady's diagram, which had been projected for the class to see, she said, "Umm, he put the *change in time* like I think it's like the box thing and then the *change in distance* on the bottom where 0 to 16 yards. And then over here [points to another part of his diagram] he did the same." Jenn used similar language when describing her diagram:

> I have the *change in time* up here [points at change in time labels] 1 second for all of them, and *change in distance* [points at change in distance labels] and you can see like it's getting faster because the blocks are bigger [points from left to right].

Except for Armando, each student provided at least one example of similar dialogue.

After Lesson 8, there were only two instances in which students appeared to *not* focus on changes in distance and time. One instance occurred during Lesson 10, when Armando recorded changes in distance on his diagram, but not changes in time. The other instance occurred during Lesson 16, when Peter did not record changes in distance or time on his diagram. However, his diagram was not completed. It is possible that he would have added them had he had more time.

The new center of focus represents the psychological aspect of student noticing that emerged during the quadratics instruction. Next, I provide evidence of the interactions and educational structures that represent the social aspect of what students noticed. Later, I discuss how the new center of focus was conceptually connected to the productive changes in students' prior knowledge for which noticing is posited as an underlying process.

Focusing Interactions

Several kinds of focusing interactions appeared to influence what students noticed. In particular, analysis of the data revealed that *naming* and *highlighting* contributed to the shift in the center of focus.

Naming. The teacher used naming, which is defined as "the act of using a category of meaning from mathematical practice to classify and label some mathematical characteristic or property" (Lobato et al., 2012, p. 35), in introducing the new quantities *changes in distance* and *changes in time* (Lesson 3). In particular, the teacher said:

> So now I wanna ask you something . . . So if this is how clown or rabbit runs, could anybody see a place where the change in distance is 6 when the change in time is 1? . . . So maybe think about that just a second. I'll say it again. Can anybody see a place where the change in distance is 6 when the change in time is 1?

After introducing these new names and talking about them, the teacher encouraged students to use the new names:

> OK, so see if you can use the words that I'm using. I said what's the change in distance, or sorry, can you see a change in distance of 6 when there's a change in time of 1? So see if you can use the word "change" in how or what you're saying.

This kind of encouragement appeared to be effective because all students began using both names regularly in small- and whole-group discussions.

A hypothesis for why the names *changes in distance* and *changes in time* may have influenced what students noticed is that the names appeared to evoke for students a strong image of making comparisons. For example, the first time the teacher asked, "Can anybody see a place where the *change in distance* is 6 when the *change in time* is 1?" all but one student verbalized or gestured in a way that indicated they were making comparisons between two distances and/or two times (e.g., Peter and Jenn pointed from one distance to another on a number line; Armando said "The distance changes from 6 to 18 by 12 meters and 2 seconds"). This hypothesis is consistent with other findings that have shown that mathematical terms can create powerful images (e.g., Siebert & Gaskin, 2006).

Highlighting. The teacher and students used *highlighting*, which is defined as "methods used to divide a domain of scrutiny into a figure and a ground, so that events relevant to the activity of the moment stand out" (Goodwin, 1994, p. 610), to foreground changes in distance and changes in time on student-generated diagrams. An example of gestural highlighting occurred when Jenn, who was explaining to the other students her number-line diagram representing the distance and time for a swimming fish, swept her finger back and forth between positions of the fish to highlight particular changes in distance and time as she said, "I have the *change in distance* and the *change in time*, from each point [points to change in distance and change in time labels] that takes *one second*." An example of written highlighting occurred when the teacher annotated student work with arrows that highlighted the changes in distance and time and then projected the student work for the class to see and discuss.

In each of these examples, highlighting foregrounded the changes in distance and time and likely backgrounded accumulated quantities. In summary, highlighting and naming appeared to be focusing interactions that figured prominently in the emergence of the new center of focus on the changes in distance and changes in time.

Mathematical Tasks

There were at least two features of the mathematical tasks that likely contributed significantly to the emergence of the center of focus on changes in distance and changes in time. A common feature of tasks in Lesson 3 and Lesson 8 was that students were explicitly asked to find changes in distance and time. A common feature of tasks in Lessons 8, 9 and 10, was that they all involved students drawing diagrams of the same quadratic distance-time data, but with different equal partitions of the time variable (i.e., over 4 s, over 2 s and over 1 s intervals). Whereas the first feature explicitly directed attention to changes in distance and time, this second feature likely directed attention to changes in time. Interestingly, it was during Lessons 8, 9 and 10 that all students came to focus on changes in quantities. Thus, changes in distance and time appeared to become particularly salient for students during these tasks.

Nature of the Mathematical Activity

I characterized the nature of the mathematical activity that likely contributed to the shift in what students noticed as defining three related classroom norms (Cobb & Yackel, 1996): (a) presenting student work to the class, (b) noticing features of other students' diagrams, and (c) students asking each other *can you explain why?* The norm of presenting work provided students with examples of what others were noticing when creating their diagrams. In particular, students provided verbal evidence that they saw how others were recording changes in distance and time. Furthermore, the norm of noticing features of each other's diagrams meant that students publically identified the recorded changes in distance and time that they noticed on each other's diagrams. Finally, the norm of asking each other *explaining why* questions meant that students engaged in a close scrutinization of the quantities that were represented in each other's diagrams.

Discussion

In the results section, I provided evidence that all seven students in my study shifted their center of focus toward noticing changes in distance and time over the course of the instructional unit. Furthermore, I showed that particular focusing interactions, features of the mathematical tasks and the nature of the mathematical activity appeared to influence their shifts. However, an additional goal of my study was to determine if what students noticed in the instructional unit was linked to the productive changes I observed in five of the seven students' prior knowledge about linearity when I compared their understanding before and after the quadratics instructional unit. To achieve this goal, I compared what students noticed during instruction with the productive changes that five students exhibited when they reasoned proportionally with changes in quantities on linear function tasks during their post-interviews. This comparison led me to discover two important conceptual connections.

The first conceptual connection was that noticing changes in quantities in a quadratic context and reasoning proportionally with changes in quantities in linear function contexts both involve the same focus on changes in quantities. In other words, the new center of focus that emerged during instruction on quadratics persisted into the post-interviews, where students reasoned proportionally with changes in quantities in linear contexts. For example, on a post-interview linear function task about a water-pump, all but one student recorded both the changes in water volume and the corresponding changes in time. Thus, changes in quantities, which were established as a new center of focus in the instructional intervention, also appeared to be a center of focus during the post-interview.

Second, the students who appeared to most quickly establish a focus on both changes in distance and changes in time during quadratics instruction provided the greatest increase in proportional reasoning with changes in quantities on linear tasks in the post-interview. In the instructional intervention, Jenn and Nicholas were the first students to attend explicitly to the changes in distance and changes in time; George, Armando, Brady, and Peter required an extra lesson before they also began to focus on changes in distance and time; Kendra required three extra lessons. In the post-interview, Jenn, Nicholas, and Brady changed from reasoning non-proportionally to reasoning proportionally with changes in quantities, while Peter, George, and Armando's reasoning changed less, and Kendra provided no evidence of reasoning with changes in quantities. Therefore, the quicker the establishment of a focus on changes in distance and time, the greater the productive change in prior knowledge seemed to be. Brady was the exception because he exhibited similarly substantial changes in reasoning during the post-interview as Jenn and Nicholas did, despite not attending as quickly to changes in distance and time in the instructional intervention.

These conceptual connections suggest that noticing is an underlying mechanism for productive changes to prior knowledge that occur as a result of learning something new. Nevertheless, further investigation into the connection between student noticing and productive changes in prior knowledge, which has thus far been under-researched, is warranted.

Endnotes

[1] All participant names are gender and ethnicity preserving pseudonyms.

[2] NSF-funded 3-year collaboration between researchers at San Diego State University and University of Wisconsin-Madison (Joanne Lobato, PI; Grant REC-0529502).

[3] Participant names are gender and ethnicity preserving pseudonyms.

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MATHEMATICS, THE COMMON CORE STANDARDS, AND LANGUAGE: MATHEMATICS INSTRUCTION FOR ELS ALIGNED WITH THE COMMON CORE

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This paper outlines research-based recommendations for mathematics instruction for English Learners (ELs) aligned with the Common Core Standards. The recommendations focus on improving mathematics learning and teaching through language for all students, and especially for ELs. These recommendations are intended to guide teachers and teacher educators in developing approaches to support mathematical reasoning and sense making for ELs.

Keywords: Classroom Discourse; Equity and Diversity; Standards

Introduction

This paper outlines recommendations for meeting the challenges in developing mathematics instruction for English Learners (ELs) that is aligned with the Common Core Standards. These recommendations for teaching practices are based on research that often runs counter to commonsense notions of language. The first issue is the term language. There are multiple uses of the term language: to refer to the language used in classrooms, in the home and community, by mathematicians, in textbooks, or in test items (Moschkovich, 2010). It is crucial to clarify how we use the term, what phenomena we are referring to, and which aspects of these phenomena we are focusing on. Many recommendations for teaching academic language in mathematics classrooms reduce the meaning of "language" to single words and the proper use of grammar (for an example, see Cavanagh, 2005). In contrast, work on the language of specific disciplines provides a more complex view of mathematical language (e.g., Pimm, 1987) as not only specialized vocabulary (new words or new meanings for familiar words) but also as extended discourse that includes syntax and organization (Crowhurst, 1994), the mathematics register (Halliday, 1978), and discourse practices (Moschkovich, 2007b). I use a socio-cultural and situated framework to frame these recommendations (Moschkovich, 2002). From this perspective, language is a socio-cultural-historical activity. I use the phrase "the language of mathematics" not to mean a list of vocabulary words with precise meanings but the communicative competence necessary and sufficient for competent participation in mathematical discourse practices.

It is difficult to make generalizations about the instructional needs of all students who are learning English. Information about students' previous instructional experiences in mathematics is crucial for understanding how ELs communicate in mathematics classrooms. Classroom instruction should be informed by knowledge of students' experiences with mathematics instruction, language history, and educational background (Moschkovich, 2010). In addition to knowing the details of students' experiences, research suggests that high-quality instruction for ELs that supports student achievement has two general characteristics: a view of language as a resource, rather than a deficiency, and an emphasis on academic achievement, not only on learning English (Gándara & Contreras 2009). Research provides general guidelines for instruction for this student population. Overall, students who are labeled as ELs are from non-dominant communities and they need access to curricula, instruction, and teachers proven to be effective in supporting the academic success of these students. The general characteristics of such environments are that curricula provide "abundant and diverse opportunities for speaking, listening, reading, and writing" and that instruction "encourage students to take risks, construct meaning, and seek reinterpretations of knowledge within compatible social contexts" (Garcia & Gonzalez, 1995, p. 424).

Research on language and mathematics education provides several guidelines for instructional practices for teaching ELs mathematics (Moschkovich, 2010). Mathematics instruction for ELs should (1) address much more than vocabulary; (2) support EL's participation in mathematical discussions as they learn English; and (3) draw on multiple resources available in classrooms (objects, drawings, graphs, and gestures) as well as home languages and experiences outside of school. Research shows that ELs, even as they are learning English, can participate in discussions where they grapple with important mathematical content. Instruction for this population should not emphasize low-level language skills over opportunities to actively communicate about mathematical ideas. One of the goals of mathematics instruction for ELs should be to support all students, regardless of their proficiency in English, in participating in discussions that focus on important mathematical concepts and reasoning, rather than on pronunciation, vocabulary, or low-level linguistic skills. By learning to recognize how ELs express their mathematical ideas as they are learning English, teachers can maintain a focus on mathematical reasoning as well as on language development.

Alignment with Common Core State Standards

The recommendations provided here describe teaching practices that simultaneously align with the Common Core State Standards (CCSS) for mathematics, support students in learning English, and support students in learning important mathematical content. Mathematics instruction for ELs should align with the CCSS, particularly in these four ways: (1) *Balance conceptual understanding and procedural fluency.* Instruction should balance student activities that address important conceptual and procedural knowledge and connect the two types of knowledge; (2) *Maintain high cognitive demand.* Instruction should use high cognitive demand math tasks and maintain the rigor of tasks throughout lessons and units; (3) *Develop beliefs.* Instruction should support students in developing beliefs that mathematics is sensible, worthwhile, and doable; (4) *Engage students in mathematical practices*. Instruction should provide opportunities for students to engage in mathematical practices such as solving problems, making connections, understanding multiple representations of mathematical concepts, communicating their thinking, justifying their reasoning, and critiquing arguments.

According to a review of the research (Hiebert & Grouws, 2007), mathematics teaching that makes a difference in student achievement and promotes conceptual development in mathematics has two central features: one is that teachers and students attend explicitly to concepts and the other is that teachers give students the time to wrestle with important mathematics. Mathematics instruction for ELs should follow these general recommendations for high quality mathematics instruction to focus on mathematical concepts and the connections among those concepts and to use and maintain high cognitive demand mathematical tasks, for example, by encouraging students to explain their problem-solving and reasoning (AERA, 2006; Stein, Grover, & Henningsen, 1996).

The CCSS and the NCTM Standards provide examples of how instruction can focus on important mathematical concepts (i.e. the meaning of equivalent fractions or the meaning of fraction multiplication, etc.) and how students can show their understanding of concepts (conceptual understanding) not by giving a definition or describing a procedure, but by using multiple representations, reasoning, and justification. For example, students can show conceptual understanding by using a picture of a rectangle as an area model to *show* that two fractions are equivalent or how multiplication by a fraction smaller than one makes the result smaller, and pictures can be accompanied by oral or written explanations.

The preceding examples point to several challenges in connecting *language* to content that students face in mathematics classrooms focused on conceptual understanding. Since conceptual understanding and mathematical practices are often made visible by showing a solution, describing reasoning, or explaining "why," instead of simply providing an answer, the CCSS implies an expectation that students will communicate their reasoning. Students are expected to (a) communicate their reasoning through multiple representations (including objects, pictures, words, symbols, tables, graphs, etc.); (b) engage in productive pictorial, symbolic, oral, and written group work with peers; (c) engage in effective pictorial, symbolic, oral, and written interactions with teachers; (d) explain and demonstrate their knowledge using emerging language; and (e) extract meaning from written mathematical texts. The main challenges teachers of ELs face are, first, to teach for understanding and then to support students in using multiple representations and emerging language to communicate about mathematical concepts. Since the CCSS documents already provide descriptions of how to teach mathematics for understanding, below I will focus on how to connect mathematical content to language, in particular through "Engaging students in mathematical practices" (Focus #4 above).

A Classroom Transcript

This transcript is intended to illustrate the recommendations and show how they play out in classroom interactions. The excerpt (Moschkovich, 1999) comes from a third-grade bilingual classroom in an urban California school with 33 students identified as Limited English Proficiency. In general, this teacher introduced students to topics in Spanish and then later conducted lessons in English. For several weeks the students had been working on a unit on two-dimensional geometric figures. Instruction had included using vocabulary such as "radius," "diameter," "congruent," "hypotenuse" and the names of quadrilaterals in both Spanish and English. Students had been talking about shapes and the teacher had asked them to point, touch, and identify different shapes. The teacher identified this lesson as an English as a Second Language mathematics lesson, where students would be using English in the context of describing and talking about geometric shapes.

1. *Teacher*: Today we are going to have a very special lesson in which you really gonna have to listen. You're going to put on your best, best listening ears because I'm only going to speak in English. Nothing else. Only English. Let's see how much we remembered from Monday. Hold up your rectangles . . . high as you can. (Students hold up rectangles) Good, now. Who can describe a rectangle? Eric, can you describe it [a rectangle]? Can you tell me about it?
2. *Eric*: A rectangle has . . . two . . . short sides, and two . . . long sides.
3. *Teacher*: Two short sides and two long sides. Can somebody tell me something else about this rectangle, if somebody didn't know what it looked like, what, what . . . how would you say it.
4. *Julian*: Paralela [holding up a rectangle, voice trails off].
5. *Teacher*: It's parallel. Very interesting word. Parallel. Wow! Pretty interesting word, isn't it? Parallel. Can you describe what that is?
6. *Julian*: Never get together. They never get together [runs his finger over the top side of the rectangle].
7. *Teacher*: What never gets together?
8. *Julian*: The parallela . . . they . . . when they go, they go higher [runs two fingers parallel to each other first along the top and base of the rectangle and then continues along those lines], they never get together.
9. *Antonio*: Yeah!
10. *Teacher*: Very interesting. The rectangle then has sides that will never meet. Those sides will be parallel. Good work. Excellent work.

The transcript shows that English language learners can participate in discussions where they grapple with important mathematical content. Students were grappling not only with definitions for quadrilaterals but also with the concept of parallelism. Student were also engaged in mathematical practices as they were making claims, generalizing, imagining, hypothesizing, and predicting what will happen to two lines segments if they are extended indefinitely. To communicate about these mathematical concepts students used words, objects, gestures, and other student's utterances as resources. This transcript illustrates several instructional strategies that can be useful in supporting student participation in mathematical discussions: asking for clarification, re-phrasing student statements, accepting and building on what students say, and probing what students mean. It is important to notice that this teacher did *not* focus directly on vocabulary development but instead on mathematical ideas and arguments as he interpreted, clarified, and rephrased what students were saying. This teacher provided opportunities for discussion by moving past student grammatical or vocabulary errors, listening to students, and trying to understand the mathematics in what students said. He kept the discussion mathematical by focusing on the mathematical content of what students said and did.

Recommendations for Connecting Mathematical Content to Language

Recommendation #1: Focus on students' mathematical reasoning, not accuracy in using language

Instruction should focus on uncovering and supporting students' mathematical reasoning, not on accuracy in using language (Moschkovich, 2010). Understanding the mathematical ideas in student's talk can be difficult. However, it is possible to take time after a discussion to reflect on the mathematical content of student contributions and design subsequent lessons to address these mathematical concepts. But, it is only possible to uncover the mathematical ideas in what students say if students have the opportunity to participate in a discussion and if this discussion is focused on mathematics. For teachers, understanding (and re-phrasing) student contributions can also be a challenge, perhaps especially when working with students who are learning English. It may not be easy (or even possible) to sort out which aspects of a student utterance are due to the student's conceptual understanding or the student's English language proficiency. However, if the goal is to support student participation in a mathematical discussion, determining the origin of an error is less important than listening to students to uncover the mathematics in what they are saying.

As we can see in the transcript, uncovering the mathematical content in Julian's contributions was certainly a complex endeavor. Julian's utterances in turns 4, 6, and 8 are difficult both to hear and interpret. He uttered the word "parallela" in a halting manner, sounding unsure of the choice of word or of its pronunciation. His voice trailed off, so it is difficult to tell whether he said "parallelo" or "parallela." His pronunciation could be interpreted as a mixture of English and Spanish; the "ll" sound being pronounced in English and the addition of the "o" or "a" being pronounced in Spanish. The grammatical structure of the utterance in line 8 is intriguing. The apparently singular "parallela" is preceded by the word "the" which can be either plural or singular and then followed with a plural "when they go higher." In any case, it is clear that Julian made several attempts to communicate a mathematical idea in his emerging second language. If we only focus on accuracy, we would miss his mathematical reasoning. Julian is, in fact, participating in mathematical practices and attempting to describe a property of parallel lines. This teacher moved past Julian's unclear utterance, he focused on uncovering the mathematical content in what Julian had said. He did not correct Julian's English, but instead asked questions to probe what the student meant.

Recommendation #2: Shift to a focus on mathematical discourse practices, move away from simplified views of language

In keeping with the CC focus on *mathematical practices* (Focus #4) and research in mathematics education, the focus of classroom activity should be on student participation in mathematical discourse practices (explaining, conjecturing, justifying, etc.). Instruction should move away from simplified views of language as lists of words, phrases, vocabulary, or definitions (Moschkovich, 2010). In particular, teaching practices need to move away from oversimplified views of *language as vocabulary*. An overemphasis on correct vocabulary and formal language limits the linguistic resources teachers and students can use in the classroom to learn mathematics with understanding. Work on the language of disciplines provides a complex view of mathematical language as not only specialized vocabulary (new words and new meanings for familiar words) but also as extended discourse that includes syntax, organization, the mathematics register, and discourse practices. Instruction needs to move beyond interpretations of the mathematics register as merely a set of words or phrases that are particular to mathematics. The mathematics register includes styles of meaning, modes of argument, and mathematical practices. Looking at the transcript, we can ask: What mathematical practices did Julian display? Julian was participating in three central mathematical practices, abstracting, generalizing, and imagining. He was describing an abstract property of parallel lines and making a generalization saying that parallel lines will never meet. He was also imagining what happens when the parallel sides of a rectangle are extended. If we only focused on vocabulary, we would miss Julian's participation in these important mathematical practices.

While vocabulary is necessary, it is not sufficient. Learning to communicate mathematically is not merely or primarily a matter of learning vocabulary. The question is not whether students who are ELs should learn vocabulary but, instead, how instruction can best support students as they learn both vocabulary and mathematics. Vocabulary drill and practice is not the most effective instructional practice for learning vocabulary. Instead, vocabulary and second-language acquisition experts describe vocabulary acquisition as occurring most successfully in instructional contexts that are language-rich, actively involve students in using language, require both receptive and expressive understanding, and require students to use words in multiple ways over extended periods of time (Blachowicz & Fisher, 2000). To develop written and oral communication skills students need to participate in negotiating meaning and in tasks that require output from students (Swain, 2001). In sum, instruction should provide opportunities for students to participate in mathematical practices, actively using mathematical language to communicate about and negotiate meaning for mathematical situations.

Recommendation #3: Recognize and support students to engage with the complexity of language in math classrooms

Language in mathematics classrooms is complex and involves multiple modes (oral, written, receptive, expressive, etc.), multiple representations (objects, pictures, words, symbols, tables, graphs, etc.), different types of written texts (textbooks, word problems, student explanations, teacher explanations, etc., different types of talk (exploratory and expository), and different audiences (presentations to the teacher, to peers, by the teacher, by peers, etc.). "Language" needs to expand beyond talk to consider the interaction of the three semiotic systems involved in mathematical discourse—natural language, mathematics symbol systems, and visual displays. Instruction should recognize and strategically support EL students' opportunity to engage with this linguistic complexity. Looking at the transcript, we can ask: What modes of expression did Julian and the teacher use? Julian used gestures and objects in his description, running his fingers along the parallel sides of a paper rectangle. The teacher also used gestures and visual displays of geometric figures on the blackboard. This example shows some of the complexity of language in the mathematics classroom.

Instruction needs to distinguish among multiple modalities (written and oral) as well as between receptive and productive skills. Other important distinctions are between listening and oral comprehension, comprehending and producing oral contributions, and comprehending and producing written text. Different mathematical domains, genres of mathematical texts, for example word problems and textbooks. Materials need to support and consider how artifacts serve as mediators. Instruction should support movement between and among different types of texts, spoken and written, among texts such as homework, blackboard diagrams, textbooks, interactions between teacher and students, and interactions among students.[1] Instruction should recognize the multimodal and multi-semiotic nature of mathematical communication, move from viewing language as autonomous and instead recognize language as a complex meaning-making system, and embrace the nature of mathematical activity as multimodal and multi-semiotic (Gutierrez et al., 2010; O'Halloran, 2005; Schleppegrell, 2010).

Recommendation #4: Treat everyday language and experiences as resources, not as obstacles

Everyday language and experiences are not necessarily obstacles to developing academic ways of communicating in mathematics (Moschkovich, 2002, 2007c). It is not useful to dichotomize everyday and academic language (Gutierrez et al., 2010; Moschkovich, 2010). Instead, instruction needs to consider how to support students in connecting the two ways of communicating, building on everyday communication, and contrasting the two when necessary. In looking for mathematical practices, we need to consider the spectrum of mathematical activity as a continuum rather than reifying the separation between practices in out-of-school settings and the practices in school (Gutierrez et al., 2010). Rather than debating whether an utterance, lesson, or discussion is or is not mathematical discourse, teachers should instead explore what practices, inscriptions, and talk mean to students and how they use these to accomplish their goals. Instruction needs to shift from monolithic views of mathematical discourse and dichotomized views of discourse practices and consider everyday and scientific discourses as interdependent, dialectical, and related rather than assume they are mutually exclusive. Looking at the transcript, we can ask: What language resources did Julian use to communicate his mathematical ideas? He used colloquial expressions such as “go higher” and “get together” rather than the formal terms “extended” or “meet.” These everyday expressions were not obstacles but resources.[2]

Recommendation #5: Uncover the mathematics in what students say and do

Looking at the transcript, we can ask several questions that illustrate this recommendation: How did the teacher respond to Julian’s contributions? The teacher moved past Julian’s confusing uses of the word “parallela” to focus on the mathematical content of Julian’s contribution. He did not correct Julian’s English, but instead asked questions to probe what the student meant. This is significant in that it represents a stance towards student contributions during mathematical discussion: listen to students and try to figure out what they are saying. When teaching English learners, this means moving beyond vocabulary, pronunciation, or grammatical errors to listen for the mathematical content in student contributions. (For a discussion of the tensions between these two, see Adler, 2001.) What instructional strategies did the teacher use? The teacher used gestures and objects, such as the cardboard geometric shapes, to clarify what he meant. For example, he pointed to vertices and sides when speaking about these parts of a figure. Although using objects to clarify meanings is an important ESL instructional strategy, it is crucial to understand that these objects do not have meaning that is *separate* from language. Objects acquire meaning as students talk about them and these meanings are negotiated through talk. Although the teacher and the students had the geometric figures in front of them, and it seemed helpful to use the objects and gestures for clarification, students still needed to sort out what “parallelogram” and “parallel” meant by using language and negotiating common meanings for these words.

Overall, the teacher did not focus on vocabulary instruction but instead supported students’ participation in mathematical arguments by using three instructional strategies that focus more on mathematical discourse: (1) Building on student responses: The teacher accepted and built on student responses. For example in turns 4–5, the teacher accepted Julian’s response and probed what he meant by “parallel.” (2) Asking for clarification: The teacher prompted the students for clarification. For example, in turn 7 the teacher asked Julian to clarify what he meant by “they.” (3) Re-phrasing: The teacher re-phrased (or re-voiced) student statements, by interpreting and rephrasing what students said. For example, in turn 10 the teacher rephrased what Julian had said in turn 8. Julian’s “the parallela, they” became the teacher’s “sides” and Julian’s “they never get together” became “will never meet.” The teacher thus built on Julian’s everyday language as he re-voiced Julian’s contributions using more academic language.

Researchers and practitioners alike need to recognize the emerging mathematical reasoning that English learners construct in, through, and with emerging language. To focus on the mathematical meanings English learners construct—rather than the mistakes they make or the obstacles they face—curriculum materials, professional development, and training for researchers needs to focus on recognizing emerging mathematical reasoning that expressed through emerging language. Professional development should support teachers in uncovering the mathematics in student contributions, when to move from everyday to more mathematical ways of communicating, and when and how to develop mathematical precision (Schleppegrell, 2010).

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Endnotes

[1] Topics for further research include defining linguistic complexity for mathematical texts and providing examples of linguistic complexity that go beyond readability (such as the syntactic structure of sentences, underlying semantic structures, or frequency of technical vocabulary, verb phrases, conditional clauses, relative clauses, and so on).

[2] The question of whether mathematical ideas are as clear when expressed in colloquial terms as when expressed in more formal language is highly contested and not yet, by any means, settled.

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FACTORS INFLUENCING MIDDLE SCHOOL STUDENTS' SPATIAL MATHEMATICS DEVELOPMENT WHILE PARTICIPATING IN AN INTEGRATED STEM UNIT

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This study examined differences between two groups of students' spatial-scientific reasoning from pre to post implementation of an Earth/Space unit. Using a quasi-experimental design, researchers explored how instructional method and gender affected learning. Treatment teachers employed an integrated STEM curriculum while the control teacher implemented her regular Earth/Space unit. The Geometric Spatial Assessment (GSA), the Purdue-Spatial Visualization Rotation Test, and the Lunar Phases Concept Inventory (LPCI) were used to assess learning. Experimental groups made gains on periodicity LPCI domains while the control made gains on geometric spatial visualization LPCI domains. Only females made gains on GSA items. This is the first quasi-experimental study to examine students' spatial reasoning as they participate in Earth/Space units and to discover gender's role in this spatial development.

Keywords: Spatial Visualization; Sex Differences; Middle School; STEM Integrated Curriculum

Objective and Theory

Research studies have shown links between students' spatial reasoning ability and their understanding of scientific phenomena (Rudmann, 2002; Black, 2005). This is particularly true in the areas of Earth/Space phenomena. For example, Rudmann (2002) found that students' propensity to learn scientific explanations for phenomena such as the cause of the seasons was limited by their spatial aptitude. Similarly, Wellner (1995) reported that students were more likely to describe a correct cause of lunar phases when they had a strong spatial sense. Black (2005) claimed that "mental rotation is the most important in understanding Earth science concepts that are associated with common misconceptions … humans are handicapped by their single vantage point from Earth of the moving bodies in outer space" (p. 403).

We claim that one cannot understand many astronomical concepts without a developed understanding of four spatial mathematical domains defined as follows: (1) *Geometric Spatial Visualization*—Visualizing the geometric spatial features of a system as it appears above, below, and within the system's plane; (2) *Spatial Projection*—Mentally projecting to a different location on an object and visualizing from that global perspective; (3) *Cardinal Directions*—Distinguishing directions (N,S,E,W) in order to document an object's vector position in space as a function of time; and (4) *Periodic Patterns*—Recognizing occurrences at regular intervals of time and/or space.

The *Geometric Spatial Visualization* domain also involves *mental rotation* since as one visualizes a system, such as the Moon/Earth/Sun, one must consider and manipulate the motion of the system itself. *Spatial Projection* has a mental rotation derivative as well since one must mentally maneuver the sky throughout a day's viewing due to Earth's rotation.

Research on students' understanding of spatial concepts shows gender differences. Kerns and Berenbaum (1991) reported that males performed better than females on spatial tests and outcomes were significantly different in the area of 3D mental rotations (p. 391). Silverman, Choi, and Peters (2007) conducted a study that assessed the universality of sex related spatial competencies. They found that men scored significantly higher than women on a 3D mental rotations test in all ethnic groups with 40 countries participating in their research study.

Not only has literature shown gender differences on spatial assessments (in favor of males), but one study conducted by Rahman and Wilson (2003) also found significant main effects of gender and sexual orientation. Large differences were found on mental rotation spatial assessments between male groups in favor of heterosexual men while modest differences were found between female groups favoring homosexual women. Rahman and Wilson claimed “variations in the parietal cortex between homosexual and heterosexual persons” explained the results (p. 25).

Previous research on gender differences on spatial assessments were conducted by the first author. Wilhelm (2009) found that pre-teen female students scored significantly lower than pre-teen male students on spatial pre-tests. However, following an intervention that utilized integrated STEM curricula with many opportunities to experience 2D and 3D stimuli, females achieved significantly higher gain scores than their male counterparts. The study speculated that the initial sex differences (on pretests) could be explained by the faster maturation (during preteen years) of the male brain’s anatomical regions that handle spatial visual reasoning (Giedd et al., 1999). The implication of the study was that the 2D and 3D instructional intervention allowed females to develop their spatial skills resulting in significant achievement.

This study builds on earlier research conducted by Wilhelm (2009) and examines differences between two groups of sixth-grade students’ mathematical spatial reasoning and scientific knowledge from pre to post implementation of Earth/Space units. Using a quasi-experimental design, researchers evaluated how the curricular choice and instructional method affected learning outcomes. Treatment teachers employed an integrated STEM curriculum while the control teacher implemented her regular Earth/Space unit. Differences in understanding by gender groups were also investigated within and between control and experimental groups.

Participants

Research subjects were sixth-grade students from a south-central US school. The school’s demographic make-up was 84% White, 7% Black, 3% Hispanic, 3% Asian, and 3% Other; and 25% eligible for reduced-price lunches. One sixth-grade group ($N = 70$), taught by Ms. Glover (29 years experience), served as the control group. The experimental group ($N = 124$) was taught by two teachers (Ms. Stevens and Ms. Castle) with 3 and 8 years teaching experience, respectively. Both groups studied Earth/Space concepts related to the Solar System within their units. Treatment teachers employed an integrated NASA-based curriculum over a six-week period while the control teacher implemented her regular Earth/Space lessons for the same time duration. This was the first time that the NASA-based curriculum was being implemented by teachers in this state. Table 1 outlines the time spent on Earth/Space content by each (control/experimental) group, the content implemented, and the instructional format.

Table 1: Unit Timeline by Group with Lesson Content and Method of Implementation

Week	Control Teacher		Experimental Teachers (with NASA-based curriculum)	
	Lesson Topics	**Method**	**Lesson Topics**	**Method**
Week 1	How Planets Compare in Size with Sun?	Video (NASA Cosmic Voyage) Fill in blank Worksheet Mnemonics	*Overview of Universe** Why does the Moon appear to change its shape?	Poster Project “Many Moons” by Thurber, Moon Journaling (five weeks) Stellarium (planetarium software)
Week 2	Sun and Stars	Video Reading Note Taking PPT	How do I measure the distance between objects in the sky? Altitude and Azimuth Angles	Measurement and graphing
Week 3	Rotation/Revolution and Predictable Motions	PPT Worksheet	How can I say where I am on the Earth? Introduction to Longitude/Latitude	Longitude and Latitude Worksheet
			*Rotation/Revolution and Seasons**	PPT Modeling Activity

Week	Control Teacher		Experimental Teachers (with NASA-based curriculum)	
	Lesson Topics	**Method**	**Lesson Topics**	**Method**
Week 4	Moon Phases	PPT & Worksheet Phase Animations 3D Activity of Earth/Moon/Sun system for various phases	What can we learn by examining the Moon's surface?	Exploration of Lunar Images
Week 5	Eclipses and Seasons	Videos & Worksheet Mnemonics	Scaling Earth/Moon/Mars	PPT Scaling Activity using Balloons
Week 6	Tides and Planets Review	Video –(Tides; Sun/Earth/Moon) Planets Scavenger Hunt PPT	Modeling Earth/Moon/Sun System for various phases *Tides**	PPT 3D Modeling Activity

* *Not part of the NASA-based curriculum*

Research Methods

This research focused on the development of students' mathematical spatial reasoning and scientific content knowledge from pre to post unit implementation. Students were assessed pre and post intervention via survey responses given to experimental and control science classes. Table 2 outlines each of the research questions pursued and data collection method.

Table 2: Research Questions and Methods of Data Collection and Instrumentation

Research Questions	Data Collection and Instrumentation
What science and spatial content knowledge and skills will students develop through Earth/Space unit experiences? *How will Earth/Space curricular choice and instructional method affect students' learning outcomes?* *What gender differences will be observed in learned science and spatial content knowledge and skills within and between the control and experimental groups?*	Pre and Post Content Surveys: – Lunar Phases Concept Inventory (LPCI) – Geometric Spatial Assessment (GSA) – Purdue Spatial Visualization-Rotation Test (PSVT-Rot)

This quasi-experimental study utilized quantitative measures to document students' understanding before and after project implementation. The quantitative data sources used to assess students' pre and post understandings were the Lunar Phases Concept Inventory (Lindell & Olsen, 2002), a multiple-choice survey which assessed eight science domains as well as four spatial domains (Table 3); the Geometric Spatial Assessment (Wilhelm, 2009), a multiple-choice survey which assessed the same four spatial domains (Table 3); and the Purdue Spatial Visualization-Rotation Test, which assisted with diagnosing the level of students' mental rotation reasoning (Bodner & Guay, 1997).

Table 3: Concept Domains: LPCI Science Domains and Corresponding GSA Math Domains

LPCI Scientific Domains			GSA Mathematics Domains
A - Period of Moon's orbit around Earth	B - Period of Moon's cycle of phases		Periodic Patterns (*occurring at regular intervals of time and/or space*)
C - Direction of the Moon's orbit around Earth	E - Phase due to Sun/Earth/Moon positions	G - Cause of lunar phases	Geometric Spatial Visualization (*visualizing the geometric spatial features of a given system as it appears in space above/below/within the system's plane)*
D - Moon Motion from Earthly Perspective	F - Phase-location in sky-time of observation		Cardinal Directions (*documenting an object's vector direction in space as a function of time from a given position*)
H - Effect of lunar phase with change in Earthly location			Spatial Projection (*projecting one's self to a different location and visualizing from that global perspective)*

A one-way analysis of variance (ANOVA) was conducted on pre-test scores to determine if there were significant differences between control and experimental groups and between gender groups. A repeated measures ANOVA (RMANOVA) was also conducted with the factor being gender and the dependent variables being pre/post scores, and again with the factor being control/experimental group with pre/post scores as dependent variables. This was conducted for each domain within each assessment as well as for the overall scores of each assessment.

Data and Analysis

Assessments

All quantitative assessments were given to both the experimental and control groups immediately prior to and at the conclusion of their Earth/Space unit implementation. Reliability was calculated using the Cronbach's alpha; this measures the instrument's internal consistency. The coefficient alpha was calculated for 0.72, 0.79, and 0.53 for the LPCI, the PSVT-Rot, and the GSA assessments, respectively. LPCI and PSVT-Rot values were high and acceptable; the GSA value was considered moderately acceptable. The control group scored significantly higher on all content pretests than the experimental group (Table 4). No significant differences between male and female groups were observed within the control group or the experimental group on the pre-tests for the LPCI, PSVT, or GSA.

Table 4: Percentage Correct on Pre-Assessments for Control and Experimental Groups Showing Control Group Scoring Significantly Higher than Experimental on All Assessments

Assessment	*n*	Con All Pre (SD)	*n*	Exp All Pre (SD)	*p* value	*n*	Con Male Pre (SD)	*n*	Exp Male Pre (SD)	*p* value	*n*	Con Female Pre (SD)	*n*	Exp Female Pre (SD)	*p* value
LPCI	66	26.6 (14.1)	124	21.2 (9.20)	0.002*	37	27.6 (14.8)	68	21.5 (8.68)	0.009*	29	25.2 (13.6)	56	20.9 (9.87)	0.101
GSA	58	46.3 (16.1)	124	41.0 (13.6)	0.022*	27	47.2 (15.1)	64	42.5 (15.0)	0.173	31	45.6 (17.0)	60	39.5 (11.9)	0.05*
PSVT-ROT	70	43.7 (20.2)	111	35.6 (17.4)	0.005*	35	45.9 (22.8)	61	38.4 (17.1)	0.075	35	41.6 (17.1)	50	32.2 (17.2)	0.015*

* $p < 0.05$

LPCI Results

Control. The LPCI pre/post tests were given to 66 control students. A RMANOVA revealed a significant increase in the mean values from pre (26.6%) to post (38.5%) on overall test scores, $F(1, 65) = 48.1$, $p < 0.001$, partial $\eta^2 = 0.422$. The significant gain scores for control males and control females were 11.3% and 12.7%, respectively.

Experimental. The LPCI pre/post tests were given to 124 experimental students. A RMANOVA revealed a significant increase in the mean values from pre (21.2%) to post (33.7%) on overall test scores, $F(1, 123) = 72.7$, $p < 0.001$, partial $\eta^2 = 0.371$. The significant percentage gain scores for experimental males and control females were 12.1% and 13.0%, respectively. Table 5 illustrates gain scores by domain for each group.

To test for significant differences from pre to post on individual science domains, a RMANOVA was conducted for the control and experimental groups. Table 5 displays the percentage correct on each science domain. Results included experimental males achieving nearly triple the significant gains of the control males on Domain A (*orbital period*). Experimental females also made a significant gain on Domain A from pre to post whereas the control females did not. Domain B (*phase cycle period*) showed only experimental males with gain scores and Domain C (*orbital direction*) showed both control and experimental females and experimental males with significant gain scores. Only the control group made significant gains on Domain E (*phase and Sun/Earth/Moon positions*).

Table 5: Percentage Correct on Pre and Post LPCI by Science Domain for Control and Experimental Gender Groups

Science Domain	Con Male Pre (SD)	Con Male Post (SD)	Con Male Gain *n* =37	Con Female Pre (SD)	Con Female Post (SD)	Con Female Gain *n* =29	Exp Male Pre (SD)	Exp Male Post (SD)	Exp Male Gain *n* =68	Exp Female Pre (SD)	Exp Female Post (SD)	Exp Female Gain *n* =56
A-Period of Moon's orbit around Earth	**29.7 (34.3)**	**43.2 (41.1)**	**13.5**	**24.1 (39.2)**	**43.1 (39.5)**	**19.0**	**16.9 (29.4)**	**46.3 (38.0)**	**29.4****	**11.6 (23.3)**	**37.5 (38.4)**	**25.9****
B-Period of Moon's cycle of phases	**30.6 (28.7)**	**43.2 (27.1)**	**12.6**	**34.5 (30.2)**	**46.0 (27.3)**	**11.5**	**30.9 (29.0)**	**45.6 (27.6)**	**14.7****	**28.6 (28.0)**	**39.3 (29.2)**	**10.7**
C-Direction of the Moon's orbit around Earth	**41.9 (38.2)**	**55.4 (36.9)**	**13.5**	**24.1 (31.7)**	**53.5 (44.2)**	**29.4****	**41.1 (37.6)**	**72.1 (36.0)**	**31.0****	**41.1 (33.2)**	**79.5 (31.3)**	**38.4****
D-Motion of the Moon	**37.8 (39.8)**	**37.8 (39.8)**	**0.00**	**32.8 (33.5)**	**34.5 (38.0)**	**1.70**	**19.1 (30.0)**	**26.5 (31.7)**	**7.40**	**27.7 (35.6)**	**37.5 (36.0)**	**9.80**
E-Phase and Sun/Earth/Moon positions	**31.5 (27.2)**	**55.9 (36.1)**	**24.4****	**23.0 (28.3)**	**49.4 (37.4)**	**26.4****	**19.1 (20.2)**	**27.0 (28.9)**	**7.90**	**19.6 (24.4)**	**24.4 (27.3)**	**4.80**
F-Phase-Location in sky/time of observation	**10.8 (15.8)**	**11.7 (21.1)**	**0.90**	**14.9 (19.1)**	**11.5 (20.5)**	**-3.4**	**9.31 (19.0)**	**7.35 (16.1)**	**-1.96**	**11.9 (19.5)**	**14.3 (21.9)**	**2.4**
G-Cause of lunar phases	**20.3 (27.5)**	**27.0 (30.3)**	**6.70**	**19.0 (28.1)**	**25.9 (36.9)**	**6.90**	**12.5 (21.8)**	**24.3 (37.1)**	**11.8**	**9.82 (22.2)**	**19.6 (31.2)**	**9.78**
H-Effect of lunar phase with change in Earth location	**12.2 (27.4)**	**28.4 (38.3)**	**16.2**	**20.7 (28.4)**	**36.2 (42.0)**	**15.5**	**13.9 (24.2)**	**19.1 (30.0)**	**5.20**	**12.5 (23.8)**	**20.5 (34.1)**	**8.00**

**$p < 0.001$

GSA Results

Control. The GSA pre/post tests were given to 58 control students. A RMANOVA revealed a significant increase in the mean values from pre (46.3%) to post (52.0%) on overall test scores, $F(1, 57) = 9.005$, $p = 0.004$, partial $\eta^2 = 0.136$. A RMANOVA also revealed a significant increase (7.5%) in the control female mean values from pre to post on overall test scores, $F(1, 30) = 10.7$, $p = 0.005$, partial $\eta^2 = 0.234$. Control males did not achieve a significant increase in scores.

Experimental. The GSA pre/post tests were given to 124 experimental students. A RMANOVA revealed a small significant increase in the mean values from pre (41.0%) to post (43.5%) on overall test scores, $F(1, 123) = 4.107$, $p = 0.045$, partial $\eta^2 = 0.032$. Like the control group, a RMANOVA revealed a significant increase (4.6%) in the experimental female mean values from pre to post on overall test scores, $F(1, 59) = 8.434$, $p = 0.005$, partial $\eta^2 = 0.125$. Experimental males showed no significant gains.

To test for significant differences from pre to post on individual spatial domains, a RMANOVA was conducted for the control and experimental groups (Table 6). Results show control females achieved significant gains on *Periodic Patterns* and *Geometric Spatial Visualization* whereas experimental females made a significant gain on *Cardinal Directions*. No male groups made significant gains on any GSA domain. Similar to Wilhelm's previous study, females in both control and experimental groups scored lower (not significantly) than their male counterparts on three of the four spatial domains on the pre-tests; and by the time of the post-tests, females ended with higher post-scores on three of the four spatial domains (see Table 6).

Table 6: Percentage Correct on Pre and Post Geometric Spatial Assessment by Domain for Control and Experimental Gender Groups

Spatial Domain	Con Male Pre (SD)	Con Male Post (SD)	Con Male Gain	Con Female Pre (SD)	Con Female Post (SD)	Con Female Gain	Exp Male Pre (SD)	Exp Male Post (SD)	Exp Male Gain	Exp Female Pre (SD)	Exp Female Post (SD)	Exp Female Gain
Periodic Patterns	53.7 (22.7)	59.3 (28.7)	5.6	48.4 (26.6)	62.1 (24.0)	13.7*	47.7 (25.1)	49.2 (24.4)	1.6	47.1 (26.1)	43.8 (18.8)	-3.3
Geometric Spatial Visual.	43.5 (30.7)	54.6 (31.8)	11.1	49.2 (33.2)	60.5 (34.6)	11.3*	45.3 (24.8)	43.4 (28.3)	-2.0	39.2 (29.6)	46.3 (27.6)	7.1
Cardinal Directions	48.2 (21.8)	47.2 (27.2)	-0.9	45.2 (19.8)	45.2 (26.9)	0.0	41.4 (22.4)	40.2 (23.0)	-1.2	35.0 (20.2)	45.8 (25.7)	10.8*
Spatial Projection	43.5 (22.6)	42.6 (29.3)	-0.9	40.0 (24.2)	44.2 (27.6)	4.2	35.6 (21.7)	39.5 (21.3)	3.9	36.7 (25.8)	40.4 (22.1)	3.8

*$p < 0.01$

While the interaction effect between gender and time was not significant for either the control or experimental groups, one cannot help but notice the similarity in the plots shown in Figure 1 for both control and experimental groups where girls began with lower GSA scores and ended with higher scores than the boys.

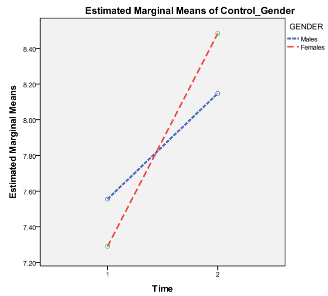

Figure1A: GSA control pre and post mean scores by gender

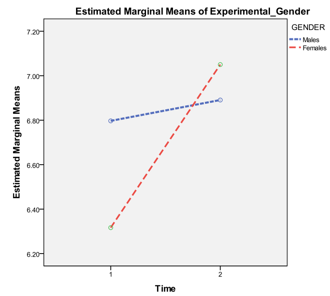

Figure1B: GSA experimental pre and post mean scores by gender

PSVT-Rot Results

The PSVT-Rot pre/post tests were given to 70 control and 111 experimental students. A RMANOVA revealed a significant increase in the mean values from pre to post for both control and experimental groups on overall test scores. Significant increases were also achieved by all gender groups except for experimental males (Table 7). These results indicate that *mental rotation* abilities are increased as a result of learning about Earth/Space science dealing with lunar phases no matter the curriculum or the instructional approach.

Table 7: Percent Scores on PSVT-Rot for Control and Experimental Groups

	n	Mean Pre % Correct (SD)	Mean Post % Correct (SD)	% Gain Score	F	p-value	Partial η^2
Control All	70	43.7 (20.2)	49.5 (21.6)	5.8	10.8	0.002*	0.135
Exp. All	111	35.6 (17.4)	40.1 (20.3)	4.5	7.035	0.009*	0.060
Control Males	35	45.9 (22.8)	52.9 (23.4)	7.0	6.26	0.017*	0.156
Exp. Males	61	38.4 (17.1)	42.9 (22.4)	4.5	3.04	0.086	0.048
Control Females	35	41.6 (17.1)	46.1 (19.3)	4.5	4.47	0.042*	0.116
Exp. Females	50	32.2 (17.2)	36.7(17.0)	4.5	4.53	0.038*	0.085

*$p < 0.05$

Conclusion

The authors claimed that one must have well-developed spatial skills in order to understand astronomical phenomena having to do with the Moon and its phases. Students could come to the classroom already equipped with strong spatial reasoning, ready to understand complicated Earth/Space phenomena; *or* students will begin to develop the necessary spatial ways of thinking as they make sense of the patterns, geometries, and motions.

As we compared control and experimental groups' LPCI learning outcomes, we found the experimental group made significant gains on the periodicity of the Moon's orbit and phases. The authors attribute these gains to their five-weeks of lunar observations since students had the opportunity to notice patterns and lunar orbital direction. Control females also made significant gains with direction of the Moon's orbit, and both control males and females made significant gains on domain E (*phase and Sun/Earth/Moon positions*). This was not surprising since domain E was emphasized during instruction through worksheets, simulations, and modeling.

In analyzing the GSA results, other interesting features emerged. Only experimental females made significant gains from pre to post in the area of *cardinal directions*. The integrated STEM curriculum emphasized documentation of the Moon's position in terms of cardinal directions. Like the experimental group, only control females made significant GSA gains; however, theirs were on *periodic patterns* and *geometric spatial visualization*. The emphasis on Sun/Earth/Moon configurations for various phases could explain the *geometric spatial visualization* development.

The PSVT-Rot showed all groups (except experimental males) achieving small but significant gains from pre to post. This assessment tested students' mental rotation ability, which we claimed was linked to *geometric spatial visualization* and *spatial projection*. A correlation test was run on the post assessments to see how well the PSVT-Rot correlated to the GSA and the LPCI, and how well the LPCI correlated to the GSA. Table 8 displays significant correlations between these assessments with every group except for the control males with PSVT-Rot versus LPCI. This supports our original claim regarding the connection between students' spatial reasoning and lunar-related understanding.

Table 8: Correlations Between Post-LPCI, GSA, and PSVT-Rot Results by Group

	LPCI vs. GSA		PSVT-Rot vs. GSA		PSVT-Rot vs. LPCI	
	r	***p*-val**	***r***	***p*-val**	***r***	***p*-val**
Control All	0.543	0.000*	0.511	0.000*	0.431	0.000*
Control Males	0.437	0.024*	0.409	0.042*	0.305	0.075
Control Females	0.63	0.000*	0.6	0.000*	0.593	0.000*
Exp. All	0.315	0.000*	0.462	0.000*	0.403	0.000*
Exp. Males	0.285	0.024*	0.495	0.000*	0.421	0.001*
Exp. Females	0.367	0.005*	0.413	0.004*	0.36	0.014*

Significance

This study is unique because it is the first quasi-experimental study that examines students' spatial reasoning as they participate in Earth/Space units. This study also extended previous research that examined the role gender plays in the development of spatial reasoning. Similar to Wilhelm's (2009) previous study, females scored lower and ended higher on three of four spatial domains (for both control and experimental groups). As noted earlier, brain developmental differences between gender groups during these preteen years could explain these results.

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CONNECTIONS ACROSS REPRESENTATIONS IN STUDENTS' GROUP DISCUSSIONS OF A NON-ROUTINE PROBLEM

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This research report examines how two groups of bilingual algebra students made connections among representations while solving a non-routine generalization problem. Using a socio-cultural orientation to mathematics learning, together with a semiotic lens on students' joint mathematical activity, this report details the type of connections among representations each group of students made as they solved the problem. Follow-up analysis shows that some connections afforded making more productive conclusions while other connections may have constrained the groups' solution processes. Finally, analysis of change across time reveals that the initial connections made by each group persisted across six weeks, despite intervening instruction that suggested other possible connections to solve the problem. The conclusion contains implications for researchers and practitioners.

Keywords: Generalizing; Connecting Representations; Collaborative Learning; Algebra and Algebraic Thinking

This paper reports on how two groups of bilingual algebra students made connections among multiple representations of an algebra problem (a story, diagrams, a table, equations, and a graph) while solving a non-routine problem about generalizing a linear relationship. This analysis addresses the continuum of student learning in school mathematics. Making connections—whether across presentations, among mathematical concepts, and/or between situations and mathematical representations—is a critical component of mathematical understanding (Hiebert & Carpenter, 1992; National Council of Teachers of Mathematics, 2000; National Research Council, 2001; Presmeg, 2006). Given the centrality of connections in students' conceptual understanding (Hiebert & Carpenter, 1992), prior research on how students learn about and reason with linear functions has examined the ways that students connect or coordinate representations (e.g., Brenner et al., 1987; Lobato, Ellis, & Muñoz, 2003; Moschkovich, Schoenfeld, & Arcavi, 1993; Presmeg, 2006; Radford, Bardini, & Sabena, 2007).

This paper extends prior research on mathematical connections by examining the affordances of different connections made by two groups of students as they solved a non-routine generalization task. The data for this empirical report are from a study that investigated how bilingual students learned to reason about the rate of change of linear functions through engaging in peer discussions. This research report focuses on findings related to the question, How did each group of students use and connect multi-semiotic tools to solve generalization questions during peer discussions of a non-routine algebra problem? Initial answers to this question led to a follow up analysis of the affordances of the different mathematical connections each group made and the relative persistence of the connections made by each group.

The primary finding, which is outlined in more detail below, is that each group connected two or more representations as they reasoned through the given task, but there were important differences in the combination of connections that each group relied upon. The connections each group made afforded (and at times constrained) making accurate generalizations. Moreover, the combination of connections each group discussed remained mostly stable over a time period of six weeks.

Framework and Prior Research

This study is grounded in a sociocultural approach to thinking and learning (Vygotsky, 1978; Wertch, 1998). Under this approach, learning mathematics may be examined as a process of appropriating culturally shared tools for engaging in mathematical activities (Forman, 1996; Moschkovich, 2004; Rogoff, 1990). For example, in the domain of school algebra, learning is evidenced by the increasingly skillful use of tools for algebraic problem solving—where tools include things such as mathematical inscriptions, standard algorithms, and mathematical discourse practices. This framework acknowledges that the meaning of these tools for thinking is not static. For this reason, many researchers in this tradition often refer to *inscriptions* rather than *representations* (Sfard & McClain, 2002). This report uses the term representations to align with prior research and with standards documents in mathematics education, while retaining the notion that the meaning of representations is constantly under negotiation.

When examining whether and how students make connections, a critical question that arises is, *what* do students connect? This analysis focuses on the connections that these students made between the multiple semiotic resources available in the problem (a story, diagrams, a table, a graph, and questions) as well as representations constructed by the students (equations, numerical answers, alterations to the diagrams, et cetera). Evidence of connections is visible when students use or coordinate multiple representations (Hiebert & Carpenter, 1992) and this can be observed in their talk, gestures, and writing (Moschkovich, 2008; Radford, Bardini, & Sabena, 2007). For example, at one point in this study, one student referred to the problem story while pointing at a diagram as his group debated which numbers to fill in to a table of values. In this case, I argue that the students connected the story, the diagrams, and the table.

The content focus of this study is how students generalize linear relationships and reason about the slope of linear functions as a rate of change. While understanding the relationship between slope and rate of change is a critical topic in school mathematics, prior research suggests that many students struggle with this concept, and that the origin of these difficulties may lie in how students and their teachers use representations and computational procedures to reason about the slope of linear functions (Leinhardt, Zaslavsky, & Stein, 1990; Lobato, Ellis, & Muñoz, 2003).

In a review of the literature on student learning of mathematical functions, "making connections" was identified as a key component of fluent reasoning with mathematical functions (Wlimot, Schoenfeld, Wilson, Champney, & Zahner, 2011). The centrality of connections is also highlighted in the descriptions of mathematical discourses as multi-semiotic (O'Hallaran, 2003; Radford, Bardini, & Sabena, 2007). From a semiotic perspective, generalizing from particular cases requires seeing the particular (e.g., the first three terms of a geometric pattern) as representative of more than the particular (e.g., the *n*th term of the pattern). Therefore generalizing is intimately related to making connections (Radford, Bardini, & Sabena, 2007). This work extends prior work reported in Wilmot et al. (2011) by examining (a) how connections developed across time during group discussions, and (b) how the quality of connections mediated each group's success in generalizing a linear relationship during a group discussion.

Methods

Setting and Participants

This study examined the mathematical reasoning of two groups of ninth grade students enrolled in a bilingual algebra class at a comprehensive high school in an agricultural region of California. Over 90% of the students at the school were Latino/a and 35% of the students were classified as English Learners. Seventy-seven percent of the school population was eligible for a free or reduced price lunch. The two groups were enrolled in a bilingual algebra class taught by an experienced teacher who has been recognized for her excellent teaching and for her skillful use of group work. Thirty percent of the students in the class spoke primarily Spanish, and the remaining students spoke both Spanish and English. The bilingual setting was chosen intentionally because it is a site where attention to language and meaning in mathematics was likely (Sierpinska, 2005).

The algebra curriculum focused on reasoning and problem solving in real-life contexts, and this data collection coincided with a unit focused on interpreting data and reasoning with linear functions (Fendel, Resek, & Alper, 1996). In consultation with the researcher, the teacher selected two focal groups of four students each. The groups were chosen to be representative of the class and each group had students with a broad range of prior mathematics achievement.

Group 1 consisted of two boys and two girls who all reported speaking Spanish at home, but who primarily spoke English in class. Two of the students in Group 1 moved to the U.S. from Latin America as children, but they were classified as “Fully English Proficient” by their school at the time of the study. Group 2 included four students who were all recent immigrants from Latin America. All members of Group 2 were classified as English Learners. The members of Group 2 spoke mostly Spanish in class, and the teacher provided them with copies of the curricular materials in Spanish. Two members of Group 2 left the school halfway through the data collection, and two other Spanish-dominant students in the class replaced them in the group.

Data Collection

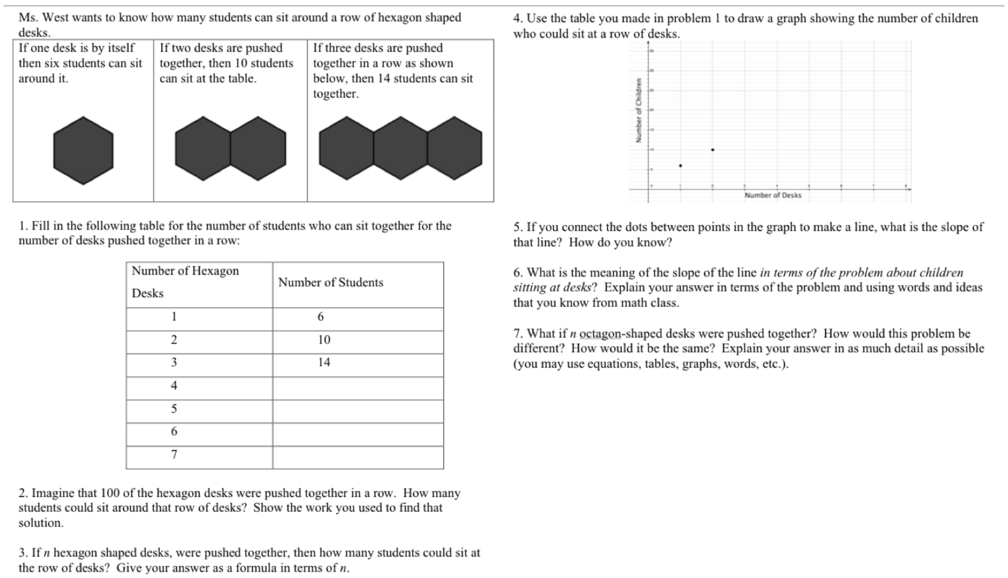
The principles for data collection were derived from Moschkovich and Brenner’s (2000) naturalistic paradigm for research on mathematical thinking, as well as the microgenetic method for examining learning across time (Chinn, 2006). Following these principles, the data collection included six weeks of in-class observations as well as a series of three out-of-class group problem solving discussions. The out-of-class group discussions were designed to systematically document change across time in the students’ reasoning on non-routine problems that required using important concepts related to rate, slope and reasoning with linear functions. The in-class observations are regarded as naturalistic observations that reveal how the students’ mathematical reasoning developed in relation to ongoing activity (Moschkovich & Brenner, 2000). Due to length restrictions, this report focuses on the out-of-class group discussions.

Each group participated in three out-of-class discussion sessions: one near the start of the unit, one near the middle, and one a week after the unit was finished. The problems that the students solved during these discussions were adapted from previous research and piloted before the data collection. Each group worked on the same problem multiple times across the six weeks, allowing for direct comparisons of changes and similarities in the groups’ reasoning across time (Chinn, 2006). In the protocol for these discussions, the students were instructed to discuss each problem as a group, come to agreement, and write one agreed-upon answer on the group’s paper. These discussions were video recorded, and copies of the students’ final answers and scratch work were collected. The videos were transcribed with a focus on capturing the propositional content of the students’ talk, and gestures were included in the transcript when students made deictic statements. This analysis focuses on the groups’ discussions of one task, Hexagon Desks. This task was chosen because (a) each group discussed it thoroughly during each discussion session, (b) both groups appeared to come to consensus on their answers to this question, and (c) this question invited the most “real life” connections for the students.

Hexagon Desks asked the groups to construct a generalized linear relationship describing how many people could sit around a row of 1, 2, 3, and more Hexagon Desks arranged in a row. Figure 1 contains a copy of the problem in English (Group 2 received the problem in Spanish). Variations of this task have appeared on the National Assessment of Educational Progress, among many other venues. The type of questions and the order of the questions in Hexagon Desks were written to parallel questions from mathematics problems that the students completed in class (e.g., observe a numerical pattern, make a table, and then generalize).

The first question on Hexagon Desks asked students to complete a table showing the number of people who could sit at a row of 4, 5, 6, and 7 desks. Question 2 required the students to find how many people could sit at a row of 100 desks. Question 3 required generalizing the pattern for *n* desks. In Question 4, the students graphed points from the table on a given graph. Question 5 asked the students to imagine connecting the points with a linear function and to compute the slope of the liner function (the question was worded to sidestep the issue of discrete and continuous functions since that was not a topic of discussion in the students’ class). Question 6 asked the students to explain how the slope of the linear model related to the story about desks. Finally, Question 7 was an open ended question asking the students to explain how their answers would change if the desks were octagons rather than hexagons.

Some questions on Hexagon Desks demanded making at least one connection, while other questions invited, but did not require, making multiple connections. For example, Question 6 asked the students to explicitly connect the slope of the linear model back to the story about pushing desks together (e.g., “the slope is 4 because each new desk adds four new places at the row of desks”). This question required making a connection between two representations. In contrast, Question 1 invited, but did not require, making connections because it could be solved without connecting representations by noticing the numerical pattern within the table.



Ms. West wants to know how many students can sit around a row of hexagon shaped desks.

If one desk is by itself then six students can sit around it.	If two desks are pushed together, then 10 students can sit at the table.	If three desks are pushed together in a row as shown below, then 14 students can sit together.

1. Fill in the following table for the number of students who can sit together for the number of desks pushed together in a row:

Number of Hexagon Desks	Number of Students
1	6
2	10
3	14
4	
5	
6	
7	

2. Imagine that 100 of the hexagon desks were pushed together in a row. How many students could sit around that row of desks? Show the work you used to find that solution.

3. If *n* hexagon shaped desks, were pushed together, then how many students could sit at the row of desks? Give your answer as a formula in terms of *n*.

4. Use the table you made in problem 1 to draw a graph showing the number of children who could sit at a row of desks.

5. If you connect the dots between points in the graph to make a line, what is the slope of that line? How do you know?

6. What is the meaning of the slope of the line *in terms of the problem about children sitting at desks*? Explain your answer in terms of the problem and using words and ideas that you know from math class.

7. What if *n* octagon-shaped desks were pushed together? How would this problem be different? How would it be the same? Explain your answer in as much detail as possible (you may use equations, tables, graphs, words, etc.).

Figure 1: The task the students discussed

Analysis

This analysis relied primarily on the transcripts of the group discussions and the copies of the students' written work. However, the video recordings were used throughout the analysis process to clarify ambiguities in the students' talk and to document the students' gestures. The transcripts were divided into segments corresponding with each group's work on a particular problem in Hexagon Desks. For example, each group's talk about Question 2 was one segment in each transcript. Some segments were divided into sub-segments when the group discussed subparts of a question separately. For example, as Group 1 discussed Question 1of Hexagon Desks, they engaged in several sub-discussions to decide which values to add in each cell in the table.

The first stage of analysis required documenting the connections among representations made by each group. Connections were coded by noting when each group made verbal, gestural, or written references to more than one representation during a particular segment in the transcript. For example, when completing the table of values on Question 1, Mateo in Group 1 explained why the net result of adding a new hexagon is adding four new spaces: "you add another one [hexagon] and nobody's gonna be sitting on that one." As he said "sitting on that one," he pointed to a vertical line at the intersection of two hexagons on Krystal's paper. With this utterance the group was coded as making a connection between the numerical table, the diagram, and the story about seating students at desks.

In addition to documenting connections, each group's final written answers were analyzed to examine whether the final result of the group's discussion was a correct response to each question. This analysis led to claims about the relative affordances of different connections made by each group. Finally, the connections made by each group and each group's agreed-upon final answers were compared across the three discussion sessions to analyze whether and how the groups' connections developed across time.

Results

Each group made multiple connections as the students worked through the questions on Hexagon Desks. In general, Group 1 had longer discussions and they made connections among multiple semiotic resources as they solved each question, while Group 2 tended to be focus on pair-wise connections. For example, in the quotation from Mateo in Group 1 above, Mateo connected the table, the given diagrams, and the story about seating students at desks. In contrast, when Group 2 discussed the same question on Hexagon Desks, they focused exclusively on the numerical pattern within the table. This is illustrated in the following excerpt from their first discussion of Hexagon Desks. (Note: in the transcript, comments are in double parentheses, while translations are in double parentheses and quotation marks).

1.	Hector	Son seis catorce, son seria ((“it’s six, fourteen, they are a series”))
2.	Iris	Yo puse xxx de cuatro ((“I put xxx by four”))
3.	Hector	Dieciocho ((looks at Iris and points at Graciela's paper)) Dieciocho ((“eighteen eighteen”))
4.	Graciela	Por qué? ((“why?”))
5.	Hector	Dieciocho, son cuatro-- cada uno tiene cuat- ((“Eighteen it is four, each one has four”)) Este tiene seis, este tiene diez, y este tiene catorce ((pointing at the table on Graciela’s paper)). Cuánto es la diferencia? ((“This one has six this one has ten, and this one has fourteen. How much is the difference?”))
6.	Graciela	Cuatro. ((“four”))
7.	Hector	Son cuatro ((lifts up four fingers)) ((“They are four”))
8.	Graciela	Um
9.	Hector	Entonces son dieciocho ((“Then they are eighteen”))
10.	Iris	Son dieciocho ((”they are eighteen”))
11.	Hector	Son veintidos(.) Son vientiseis(.) Son treinta(.) ((“They are twenty two, they are twenty six, they are thirty”))
12.	Graciela	Ah hum ((nods her head up and down))

This trend in the way each group made connections held across all three discussions. Group 1 repeatedly made multiple connections among three representations for the problem: they referred to the given diagram, the story, and the table as they solved Questions 2, 3, and 7, and they discussed how these representations were related. Group 2 adopted a narrower focus, they consistently connected each question back to the numerical patterns from the table. For example, when Group 1 generalized the pattern for 100 hexagons (Question 2), they focused on the contributions of the top, bottom, and sides of the diagram of a chain of hexagons to calculate the answer of $200 + 200 + 2 = 402$. In contrast, each time Group 2 attempted Question 2, they attempted to generalize the numerical pattern using only the table. Their answers across all three discussions were $100 \times 4 = 400$, 150, and $42 \times 10 = 420$ respectively, showing that they were not able to use this numerical pattern to successfully generalize to the hundredth case.

A first follow up analysis compared the connections across representations that the groups made with each group’s agreed upon final answers. While both groups made some connections, not all connections proved equally useful for solving the problem or reaching a generalization. Group 2’s responses to Question 2 show that their ways of focusing on the numerical pattern in the table was not useful for developing a generalization about how many students could sit at a row of 100 desks. By the third discussion both groups were able to use the graph to successfully compute the slope in question 5. Group 2 was able to correctly answer Question 6 (interpret the meaning of the slope) by noting that the slope of the linear model was 4, and they also used a connection between the linear function and the story to describe the meaning for the slope, saying that this slope was the same as the “add four” that results from adding an additional desk to the row of desks. Meanwhile, although Group 1 was able to calculate the slope using the graph, they did not describe the meaning of the slope in relation to the problem, thus there was no evidence that they were connecting representations to justify their response to the slope interpretation question. One possible explanation for Group 1’s difficulties interpreting the slope is that the net change of “add four” was not as readily apparent when focusing on the “adding and subtracting” action of adding a new hexagon on the end of the diagram. Thus, the different connections made by each group provided different affordances for justification and for generalization.

Finally, the second round of follow-up analysis examined how the connections made by each group shifted as they solved Hexagon Desks on three distinct occasions across the six-week data collection timeline. In general, each group consistently drew upon a similar set of connections each time they discussed the problem. For example, during Discussion 1, Group 1 repeatedly made connections between the table, the diagrams of hexagons, and the story as they completed the table in Question 1. During Discussions 2 and 3 Group 1 again referred to the table, the diagrams, and the story as they reasoned through which values to put in each line of the table. Likewise, Group 2 consistently used the connection to the numerical "add four" pattern in the table all three times they solved Question 1. While there were some changes in the groups' responses across time, the relative consistency in the connections made by each group indicates that once a group makes connections among representations, these may remain consistent for a particular problem.

Discussion

This study has illustrated the connections across representations made by two groups of students as they solved a non-routine generalization problem, explored the affordances of making different connections, and illustrated that each group's initial connections remained fairly stable across six weeks.

Implications for Research

These findings indicate that *making connections* may be a necessary, but not sufficient, characteristic for describing how students develop conceptual understanding in mathematics. Just as Wertsch (1998) noted that some number systems afford calculation by hand using standard algorithms, some connections may afford more mathematical insight than others, especially for developing generalizations. One connection, such as the recursive rule of "add four" in the table is common, but it is not necessarily the most useful connection for making generalizations about linear models (see also Kaput 1992 on the particular issue of recursive rules). Moreover, this study indicates that the connections that students initially made were relatively robust across time. For mathematics educators, this study invites a more systematic examination of which connections among which representations, and for which purposes or goals, have the most affordances for students' mathematical reasoning.

Two possible follow up studies might investigate (a) what sequences of instructional activities promote productive shifts in the connections that students make while generalizing about linear functions, and (b) whether the affordances of different connections can be incorporated in assessments to better understand students' emergent mathematical understandings.

Connections to Practice

The data collection for this study coincided with a classroom unit on interpreting data, graphing, and reasoning with linear functions. While this analysis focused on the groups' out of class discussions, the stability of the connections made by each group across time was surprising to both the researcher and teacher because, to us, the students' in-class work appeared related to the goal of generalizing linear functions from data. For teachers and instructional designers, this study illustrates the well-known fact that student thinking can be oriented toward different goals than those intended by curriculum designers (Newman, Griffin, & Cole, 1989). Student thinking may also remain consistent from the learners' own perspectives and thus appear to teachers as resistant to change through instruction. In this study, although the students did reason with linear functions, make sense of slope, and solve real life problems in the classroom, they did not seem to draw upon those classroom experiences while solving Hexagon Desks. While this study does not necessarily show how to help students draw on their classroom experiences, it does indicate that there is a continuing need to address this issue in both research and practice.

The difference in how each group made connections (and the affordances of those connections) is not suitable evidence to make generalizations about all bilingual students learning math in Spanish or English. First, these groups were chosen to illuminate two particular cases of students using social and linguistic tools, but not to represent of all bilingual students. Second, the data show that both groups were successful in different ways. For example, Group 1 was able to solve Question 2 by connecting the diagrams and story, but they were not able to solve Question 6. Conversely, Group 2 used a connection to the "add four" pattern in the table solve Question 6 but that did not work for solving Question 3.

Conclusion

This analysis of the connections among representations made by two groups of students illustrates a critical issue faced by students as they navigate transitions along the learning continuum of school algebra. The connections that students make between representations are primary mediators of students' learning and understanding (National Council of Teachers of Mathematics, 2000; National Research Council, 2001). The findings of this study suggest that simply making connections across representations is not enough. Researchers and teachers need a better understanding of which connections students make, for which purposes, and how connections develop across time. An improved understanding of these issues can affect students' success navigating the continuum of school mathematics.

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ASSESSMENT OF ELLS IN MATHEMATICS USING EPORTFOLIOS: NAVIGATING USER-INTERFACE AND PARENT ENGAGEMENT

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Portfolio assessment amongst general populations in mathematics and other disciplines has been suggested to be a useful tool for tracking emergent student understanding and, thus, has been proposed to also be a useful tool for English language learners (ELLs). In this research, 10 teachers implemented an "ePortfolio" with 30 primary (grades one to three) ELLs. Results suggest that while conceptually an ePortfolio is a promising tool, user-interface challenges limited effective pedagogy and these challenges were not seen as unique to the tool used in the research. Additionally, parent-level factors that were unanticipated also emerged that were influential in the overall research.

Keywords: Assessment and Evaluation; English Language Learner; Elementary School Education; ePortfolio

As global mobility continues to increases over the next decade, the percentages of English language learners (ELLs) can be expected to grow (Cummins, 2007). ELLs are defined as "students in . . . English language schools whose first language is a language other than English, or is a variety of English that is significantly different from the variety used for instruction . . . , and who may require focused educational supports to assist them in attaining proficiency in English" (Ministry of Education, 2007, p. 8). Robinson (2010) points out that many current strategies used to support and assess ELLs have historically been appropriated from other student populations (i.e., special education), and do not address the barriers that linguistic challenges present. Also, traditional assessment practices in schools have been found to be more advantageous for some students over others (Gutiérrez, Bay-Williams, & Kanold, 2008).

This research explored the implementation of an "ePortfolio" in mathematics for primary ELLs in grades one to three. The objectives of this research were to: (a) work with teams of teachers to implement ePortfolios for identified ELLs in grades one to three, and (b) evaluate the extent to which ePortfolios potentially increased the "reliability and validity of the evaluation of student learning" (Ontario Ministry of Education, 2010, p. 39) both empirically and holistically.

Literature Review

As numerous researchers suggest, ELLs should be able to demonstrate what they know with limited or no English (Ministry of Education, 2007). Rivera and colleagues (2006) call for "ELL-responsive" approaches and sensitive assessment strategies that allow ELLs to effectively demonstrate their knowledge using a variety of methods. ELLs require more time and frequent assessment over time. Opportunities for assessment should occur in ELLs first language, and use of translation where possible is recommended (Rabinowitz, Ananda, & Bell, 2005; Robinson, 2010). Portfolios are often identified as a useful tool for assessment of ELLs that facilitate both collection and creation of artifacts where progress over time can be tracked and documented (Stiggins & Chappuis, 2011). Although portfolios are strongly recommended throughout the literature for ELL their efficacy has not been well evaluated. However, promising results about the usefulness of portfolio assessment amongst general school populations have been found in mathematics education and other disciplines (Fukawa-Connelly & Buck, 2010).

Theoretical Frameworks

Culturally responsive pedagogy has been suggested to be effective at improving outcomes of marginalized groups or populations in schools such as Hispanic, African-American, linguistically diverse, and so forth (Gay, 2000; Torres-Velasquez & Lobo, 2004). According to Howard and Terry (2011), culturally responsive pedagogy

> embodies a set of professional, political, cultural, ethical, and ideological disposition that supersedes mundane teaching acts, but is centered in fundamental beliefs about teaching, learning, students, their families, their communities, and an unyielding commitment to see student success become less rhetorical and more of a reality. (p. 347)

In the case of the present research, our particular interest is directed towards shifting classroom practices in assessment, which have been historically shown to be particularly under privileging to ELLs in two ways. Cummins (1984), for example, says that IQ assessment tools have shown to be a highly unreliable indication of intelligence for ELLs in that the tests consistently show lower than normal intelligence. Cummins shows that these results are reflective of language proficiency as opposed to intelligence. Alternatively, there is also the risk that some forms of assessment may suggest students know more than they do because there may be an illusion of competency based upon perceived levels of language proficiency.

Methods

Participants

Participants for this research were from an urban school known for its high levels of diversity and immigrant populations. The research was implemented for one school term during the second half of the year. The school had up to 50 languages spoken at the school. There were 47 ELLs invited to participate of which 30 agreed to participate. There were 15 males and 15 females. In total there were 10 teachers (all teachers teaching grades one to three), three English as a Second Language Teachers (ESL), one principal, two university researchers, and one teacher/research assistant that participated.

Data Sources/Instruments

All parents completed a demographic questionnaire. School-based data of the ELLs' language proficiency were also obtained. To obtain a measure of the students' mathematical ability, a non-verbal arithmetic problem solving task using manipulatives was administered by a trained teacher/research assistant to assess students' mathematics knowledge without relying solely on their language competence. It was initially proposed that standardized tests (i.e., Woodcock-Johnson III Tests of Achievement battery (WJ III ACH)) would also be collected at the beginning and the end of the research; however, this was not completed due to some of the early findings related to parents and to the user-interface challenges which we explain in our results shortly.

Student generated artifacts (e.g., drawings and mathematical work done on the computer or scanned into the computer through a document camera and scanning device), audio, and video recordings were collected through the ePortfolio. Teacher collaborations/professional development took place in person, informally day-to-day, and electronically. Field notes were collected by research assistants during face-to-face meetings.

For this research, we partnered with *Desire2Learn* (D2L) who provided the web-based Learning Environment (LE) and ePortfolios proposed by them to be suitable for K-12 student populations. The LE provided common space for students and teachers to post to discussion rooms, access articles or other artifacts (e.g., video links), engage in assessment, and gather assessment data. The ePortfolios were predominantly student-driven. The ePortfolio also was available in Canadian English, Canadian French, U.S. English, UK English, Spanish, Arabic, Simplified Chinese, Indonesian. This provided an important opportunity for some of our ELLs to benefit use of their first language (Torres-Velasquez & Lobo, 2004).

Each classroom involved in the research was provided with a laptop computer, a document camera, and set of headphones which included microphones. Students could access these tools during class time when appropriate. Support for the students, when needed was provided by the classroom teacher, an ESL teacher, or the teacher/research assistant. The ePortfolio was used for a ten-week period.

Procedures

A parent information session was hosted at the school where the research was introduced to parents. Translators were present to answer any questions that arose. Parents were also informed that anonymity could not be assured as a result of their participation given that video data and audio data would be used as part of the analysis, for dissemination, and for future knowledge mobilization efforts emerging from the research. Those parents who did not attend had information letter and consents translated to the home language where possible sent home with their child. Follow-up phone calls were conducted by classroom and ESL teachers to secure consent and answer questions.

Teachers gathered face-to-face to plan on eight-half days, and three-full days over the course of the entire two years with two half days and one full day occurring prior to the ePortfolio being implemented with students. These opportunities for interaction involved planning curriculum and assessment tasks, videotaping virtual asynchronous “conversation” starters for students, engaging in assessment, some professional development related to mathematics learning (e.g., problem solving, generating multiple solutions, error analysis, conversation starters, etc.), and ELL (e.g., assessment challenges, role of first language communication, culturally sensitive pedagogy, etc.). These sessions were jointly led by teachers and researchers in the project.

Students were introduced to the ePortfolios in small groups by their ESL teacher and additional training was provided, as needed. Students were trained on how to customize their screens, upload documents using document cameras purchased for the research, record voice and audio files, and record small video clips to be used for the asynchronous virtual conversations with their teachers , and as we explain shortly, sometimes with each other.

Results and Discussion

Two main results emerged. First, our potential participant group was reduced from 47 to 30 because parents were reluctant to allow their child to be involved because their child was “Canadian” and not an ELL. These concerns were voiced during the information session and to teachers following the session. Despite these children being identified as an ELL and receiving school-based ESL resources and support, many parents at the information session and even after to their teachers were very adamant that their child was not “ESL.” Parents’ identity, both their own and that of their child’s, prevented our intentions for culturally sensitive pedagogies to advance to the extent that we had planned. Second, and most critically, while the LE and the ePortfolio were proposed by D2L to have been used effectively in a primary setting, the overly complex work-flow challenges proved to be unmanageable for teachers and often were contrary to effective pedagogy and assessment practices. Moreover, students required support each time they tried to engage with their ePortfolios because of the extensive navigation required within the tool. The product did not match good pedagogy and was primarily reduced to a presentation tool. It is very important to note that as a result of the challenges we were experiencing, we reviewed other available “ePortfolio” products. While some were perhaps easier to use, they all had similar pedagogical deficiencies that reduced their overall potential to support teaching and learning.

Conclusions

Culturally sensitive pedagogy discourse must be extended to include parents—currently a perspective that is absent from the scholarship in this area. Parents are important pedagogical resources and thus must be considered for all students. Many of the ePortfolios available appeared to be further behind the “Web 2.0-ready” students in classrooms today, and lacked pedagogical grounding. Product developers are encouraged to work closely with teachers and to partner with universities to develop ePortfolios. While the notion of ePortfolios is promising, still there is little empirical research in their efficacy in classrooms and should be introduced with caution about the proposed added value. Even our own attempts in this research were thwarted while attempting to navigate product developers' views of pedagogy and those from within the discipline.

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STRONGER ARGUMENTS WITHIN INDUCTIVE GENERALIZATION IN MIDDLE SCHOOL MATHEMATICS

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Using examples as a means of justification is a common strategy in mathematical and non-mathematical domains. However, in domains such as biology, significant research has explored the justification of generalizations through strategic reasoning about examples. We suggest that while example-based inductive reasoning cannot generate conclusive proof in mathematics, middle school students do engage in strategic inductive reasoning in math. We find that children have preferences for the use of particular example-based arguments in mathematics, much as they do in other domains. These strategies are also influenced by their knowledge about examples. Having a better understanding about how students employ example-based inductive reasoning in mathematics could ultimately help students' deductive proof development.

Keywords: Reasoning and Proof; Middle School Education

Curricular standards emphasize constructing and evaluating conjectures and proofs as central to mathematical understanding across grade levels (Common Core State Standards Initiative, 2010). However, it is well documented that students frequently use examples to justify the truth of conjectures, treating empirical evidence as sufficient proof (e.g., Bieda, 2001; Healy & Hoyles, 2000). We suggest that students' inductive strategies, by which we mean the use of examples to "prove" conjectures, actually reflect fairly sophisticated mathematical reasoning.

Inductive reasoning, or generalizing from examples in order to justify a conjecture, is also widespread in domains such as biology (e.g., Hayes et al., 2010; Osherson et al., 1990). Induction is a powerful form of reasoning, and considerable research has focused on strategies used in intuitive inductive reasoning. From this body of research, we know that having more examples, using a diverse set of examples, and using highly typical examples all increase children's and adults' judgments of the strength of an inductive argument. Inductive reasoning in these domains is strategic; some types of arguments are viewed as stronger evidence.

In mathematics, however, students' example-based justifications are viewed as erroneous and tangential to the desire to help students construct deductive proofs (e.g., Harel & Sowder, 1998). However, inductive approaches could also be viewed as a starting point to better understand conjectures and to provide possible insights for proofs (e.g., Simon & Blume, 1996). While inductive strategies do play an important role in mathematics, particularly in helping students develop generalizations (e.g., Ellis, 2007), much remains to be understood about how to effectively help students leverage their inductive strategies into insights that can support deductive arguments (Bieda, 2011; Stylianides & Stylianides, 2009).

Thus, our approach considers strategic inductive reasoning in mathematics as a topic worthy of investigation in its own right, in addition to its role in the transition to deductive reasoning. In this research, we examine middle school students' differentiation of example-based arguments and relevant strategic reasoning. This will inform researchers and educators about how students' knowledge and beliefs guide their current approaches to generalizations. As a starting point, we ask if principles of inductive inference in mathematics are similar to those from other domains.

Methods

Participants

Middle-school students in 6th to 8th grade (n = 433, 49% Female) from a school in a midsize Midwestern city participated in this study. All classrooms, aside from one class of Algebra I students, used a reform-oriented curriculum, CMP 2 (Lappan et al., 2006).

Materials and Procedure

Participants received a survey consisting of 24 problems. Each problem presented participants with a conjecture and two narratives, each with a fictional student who tested example(s) related to a particular characteristic to figure out if the conjecture was always true. The participant was asked to evaluate the relative strength of each fictional student's argument.

Participants received problems from two of three **domains** (numbers, shapes, and birds). Half of the students received specific conjectures (e.g., "If you add any number to four times itself, you always get a multiple of 5"), and half received conjectures that only referred generically to a mathematical property.

After each conjecture, participants read two narratives, each with an example-based reasoning strategy tried by a fictional student in order to justify the truth of the conjecture. Each narrative was based on the level of the relevant **characteristic** manipulated through the fictional students' example choices. Table 1 gives the assessed characteristics (here we focus on 4 of the 6 used in the study), along with the levels that were used by each of the fictional students. These characteristics were chosen based on both their applicability to inductive reasoning outside mathematics and students' reported explanations for example choice.

Table 1: Levels Used for Each Characteristic (Within–Subjects), Number Domain

Characteristic	1st Level	2nd Level
Quantity of Examples	two numbers	five numbers
Diversity of Examples	really similar numbers	really different numbers
Commonness of Examples	a really common number, one you see every day	a really unusual number, one you do not see every day
Sampling Method	picked a number at random	picked a number that was really, really big [or small]

The narratives by the fictional students were presented in three different **labeling formats** (manipulated between-subjects). The examples each fictional student used as part of their argument were described by a label, as shown in Table 1, (e.g., "I tried this with a really common number"), the item (e.g., "I tried this with **9**"), or both the label and item. The items selected for the levels of each characteristic were based on previous studies (Williams et al., 2011). Reported analyses compare the two formats with a label to the format without a label.

Each fictional student's narrative ended with the statement, "It worked for [this number] so I think this means the property is always true." Participants selected which fictional student made them more likely to think the property is true for all cases in that domain. Then, participants rated how convincing each person's demonstration was on a 1–7 Likert scale. Here, due to space, we only address their numerical ratings of the demonstrations.

Design and analyses. Putting the different variables together, the 24 problems each student received (2 domains $\times$ 6 characteristics $\times$ 2 instances of each characteristic per domain) were presented in different randomized orders. Demographic information and math attitudes were gathered but not reported here. Linear mixed-effects regression (Bates & Maechler, 2009) was used in R to model student preference for each characteristic; MCMC sampling was used to determine significance levels (see Baayen, 2008).

Problem domain, conjecture type, labeling format, and their possible interactions were treated as fixed factors.

Results

Quantity of Examples

Arguments with five examples were preferred over arguments with two examples, $p = .0001$. This was true for problems across domain, labeling format, and conjecture generality. Also of note, narratives testing five examples were rated as the most convincing among all characteristics ($M = 5.32$, $SE = 0.045$), whereas narratives with two examples were rated as the least convincing ($M = 3.87$, $SE = 0.046$).

Diversity of Examples

Unlike the characteristic of quantity, students' differentiation of arguments based on example diversity depended on labeling format. In all domains, students preferred diverse examples over similar examples only when a label was present ($p = .01$).

Commonness of Examples

Like diversity, students preferred an uncommon example over a common example only when a label was present ($p = .02$). This overall preference for uncommon examples was stronger when reasoning about numbers ($p < .05$), even without labels.

Sampling Method

As with diversity and commonness, student preference for one level of a characteristic over the other depended on labeling format. When a label was present, students preferred a random example over an example chosen because it was an extreme size, $p < .001$.

Reasoning When No Label Was Presented

We saw above that students' preferences for particular levels of characteristics were clearly established when a label was present. In a few instances, they applied similar types of reasoning to cases without labels. In order for the item only condition to affect performance, students needed to infer characteristics of the items and then make judgments about the relative argument strength. This was easily accomplished with *quantity*, as that is a visible feature of the example sets. With *diversity* and *commonness*, preferences for particular levels extended to the item condition only for the number domain.

Discussion

The middle school students we studied recognized different qualities of example-based arguments in mathematical and scientific domains. That is, they strategically applied principles to distinguish stronger and weaker inductive arguments. Testing more examples was broadly recognized as increasing the strength of an argument. In addition, arguments based on dissimilar sets of examples, uncommon examples, and randomly selected examples were stronger than their complements when labels were present.

However, without the labels of the characteristics, students' preferences for stronger inductive arguments were often limited to the number domain. It is possible that these were cases in which students could infer, to some degree, the features of relevant characteristics. Thus, students recognized the significance of the characteristics but did not spontaneously consider examples outside of the number domain. Our selection of examples could have influenced this or students may not have been familiar enough with the features of the examples in order to infer the relevant dimensions. Learning how examples relate or differ is a key factor in inductive reasoning (Christou & Papageorgiou, 2007).

While example-based justifications cannot conclusively prove a conjecture's truth, previous research has shown that students often use these approaches. Thus, this study informs educators and researchers about how children think about these example-based approaches—we see that they think strategically about their justifications. In general, students applied the same principles of evidence, distinguishing between more and less convincing empirical arguments, across domains. These principles match those identified as reliable and effective in the psychological literature on inductive reasoning (e.g., Osherson et al., 1990). Knowing that the same principled strategies are being employed during inductive reasoning in mathematics and science is a good starting point. These strategic judgments about different types of example-based arguments may form the basis for introducing formal deductive arguments. Students might be able to harness their strategic inductive reasoning to appreciate the limits of even the strongest empirical argument. Building on that these existing strategies and knowledge may result in deeper connections as students become fluent with both inductive and deductive reasoning strategies.

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DEVELOPMENT OF AN EXPLANATORY FRAMEWORK FOR MATHEMATICS IDENTITY

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This paper explores the construct of mathematics identity by testing a hypothesized framework based upon students' beliefs and experiences related to mathematics. This study examined data drawn from the Factors Influencing College Success in Mathematics (FICSMath) project, a national survey study of college students enrolled in single-variable calculus at 2- and 4- year institutions across the United States. Structural equation modeling was used to confirm the salience of the mathematics identity framework, indicating that students' mathematics interest, their being recognized by others in mathematics, and beliefs about their ability to perform and understand mathematics were directly related to their mathematics identity.

Keywords: Beliefs; Modeling

Purpose

The purpose of this study is to test an explanatory model for mathematics identity by examining responses to a survey of college students enrolled in single-variable calculus across the United States. Mathematics identity is a construct that has the potential to improve our understanding of complex mathematics classroom environments, the broader context of mathematics education, and what it means to be a mathematics learner (Lester, 2007). This is because it simultaneously accounts for the individual's perspective, the influence of the communities with which the individual identifies, and the social context in which the learning occurs. Although the concept of mathematics identity promises to aid in examining these complex connections and in better understanding students' experiences and persistence in mathematics, Cobb (2004) stated that mathematics identity is underdeveloped as an explanatory construct in research. Research on the construct of identity in relation to mathematics has begun to develop an explanatory framework (Holland & Lave, 2001; Sfard & Prusak, 2005; Solomon, 2007), but these research efforts have been mostly confined to a micro-identity approach (moment-to-moment), as opposed to a macro-identity approach (global view). This study will develop an explanatory model for mathematic identity in a macro-level approach using structural equation modeling (SEM) on national survey data.

Theoretical Framework

This study defines mathematics identity as how students see themselves in relation to mathematics based upon their perceptions and navigation of everyday experiences with mathematics (Cass, Hazari, Cribbs, Sadler, & Sonnert, 2011). This definition focuses on students' views about themselves in relation to mathematics and how their experiences with mathematics have influenced their perceptions. The framework in this study takes into account the sociocultural link that Sfard and Prusak (2005) state to be an important component to identity research as well as the perspective that a person has multiple identities that overlap and influence one another (Gee, 2001). Further, mathematics identity is seen as being comprised of multiple sub-constructs. Using results from previous research on science identity (Carlone & Johnson, 2007; Hazari, Sonnert, Sadler, & Shanahan, 2010), this study hypothesizes a framework for mathematics identity that is comprised of the sub-constructs interest, recognition, competence, and performance. The sub-constructs of competence and performance have been combined, based on results from a prior study (Cass et al., 2011), as seen in Figure 1.

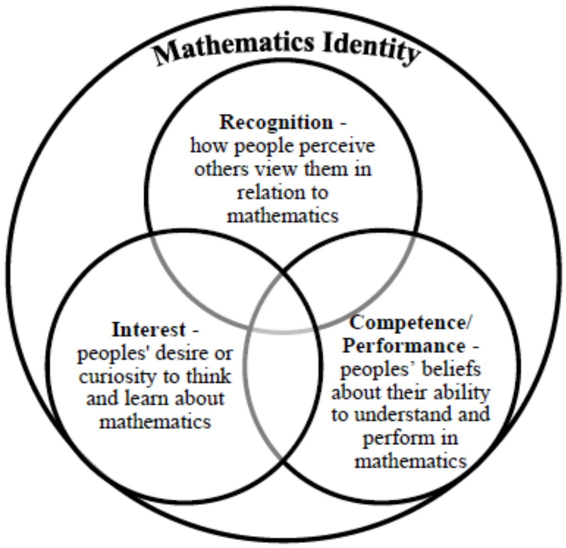

Figure 1: Mathematics identity framework

The research question guiding this study is: to what extent do the data map onto the sub-constructs of interest, recognition, competence/performance, and how these sub-constructs relate to the construct of mathematics identity?

Methods

This study is part of the Factors Influencing College Success in Mathematics (FICSMath) project, a national survey study of 10,437 students enrolled in 336 single-variable calculus classes at 134 colleges and universities across the United States. The survey included items on students' experiences in high school mathematics, students' background information, students' attitudes and career goals, as well as performance in their college calculus classes.

Validity of the survey was established through pilot testing of the survey with 45 students and a focus group with experts in science and mathematics education. Further, a test re-test study with 148 students was conducted to examine the stability (a form of reliability) of the survey. Results indicated an overall reliability coefficient of 0.71 for linear variables and 94 percent agreement for binary and categorical variables.

SEM was used to investigate the construct of mathematics identity. Nine items from the FICSMath survey were used in the measurement model as detailed in Table 1. All the indicator variables for interest and competence/performance were dichotomous variables (0 = disagree, 1 = agree), while the indicator variables for recognition were on a Likert-scale (1 = No, not at all, 6 = Yes, very much).

Table 1: Items from FICSMath Survey

Latent Variable	Indicator Variable	Survey Item
Interest		*Do you agree or disagree with the following statements?*
	Q44enjoy	I enjoy learning math.
	Q44interest	Math is interesting.
	Q44lookforward	I look forward to taking math.
Recognition		*Do the following people see you as a mathematics person?*
	Q45mathpersonp	Parents/Relatives/Friends
	Q45mathpersont	Mathematics teacher
Competence/ Performance		*Do you agree or disagree with the following statements?*
	Q44understand	I understand the math I have studied.
	Q44nervous	Math makes me nervous.
	Q44persist	Setbacks do not discourage me.
	Q44exam	I can do well on math exams.

The variable Q45mathpersons was used as a scaling variable for the latent variable mathematics identity. This variable asked participants to rate themselves to what degree they see themselves as a mathematics person (1 = No, not at all, 6 = Yes, very much).

Results

The goal in SEM is to achieve the best model fit without compromising the theory being represented. Figure 2 details the final structural model along with the corresponding fit indices.

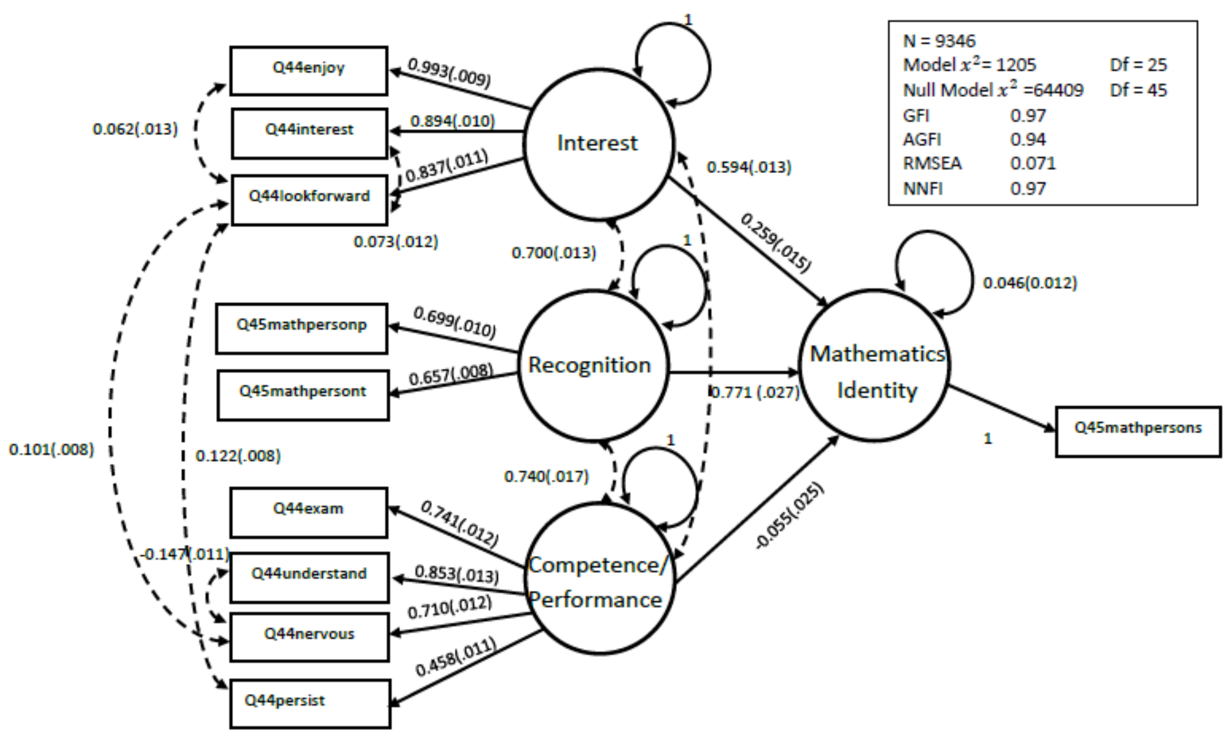

Figure 2: Final model and fit indices

By looking at the fit indices, it can be seen that the final model is a good model with fit indices falling within recommended levels. Furthermore, all pathways in Figure 2 were highly significant ($p < 0.001$).

Discussion and Conclusions

The mathematics identity framework was strongly supported by the data given that the sub-constructs of interest, recognition, and competence/performance had a significant direct effect on mathematics identity, though the negative effect of competence/performance was unexpected. The effect is strongest for the sub-construct of recognition, which indicates that students who believe that their parents, peers, relatives or teachers see them as a mathematics person are more likely to develop a mathematics identity. Interest also has a significantly positive effect on mathematics identity indicating that students who have an interest in mathematics have a higher mathematics identity. Competence/performance has a negative effect on mathematics identity, which is so tiny (–0.055), however, as to be practically negligible. This result may be a consequence of the lack of variability in this specific sample with respect to competence/performance beliefs, i.e. college calculus students may have very similar levels of such beliefs when they enroll in these courses. It is also important to note that competence/performance also had strong indirect effects through both the sub-constructs of interest and recognition. Another study is currently being conducted with a different population of students. This study will test the theoretical framework again and unravel whether the current choice of sample explains the competence/performance finding in this study.

Because mathematics identity is associated with student persistence in STEM (Cass et al., 2011; Cribbs, 2012), it is important to consider the beliefs of students related to mathematics identity, and how exactly these beliefs could positively or negatively influence student choices. These results have broader implications for mathematics researchers and educators. In terms of the classroom, educators can incorporate practices that help to draw student interest, as has been advocated by the National Council of Teachers of Mathematics (2000). Being recognized as a mathematics person plays an even more vital role for students' mathematics identity. However, how students come to feel recognized needs to be further explored. More broadly using the mathematics identity framework established in this paper, future research can investigate the impact of teacher practices on mathematics identity.

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AN EXAMINATION OF NON-MATHEMATICAL ACTIVITIES IN THE MATHEMATICS CLASSROOM

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Stein and Smith (1998) classified mathematics tasks by their cognitive demand. In this paper, I examine tasks in which the cognitive demand falls outside of the classifications proposed by Stein and Smith into what they refer to as non-mathematical activity. I explored instances of non-mathematical activity in three secondary mathematics courses. This examination led to three classifications of non-mathematical activity—word activities, transcription activities, and motivation activities. I discuss each of these activity types in detail and suggest strategies to bring a mathematical focus back to these activities. I close with a discussion of implications for mathematics educators involved in the training of preservice and inservice teachers.

Keywords: Curriculum; High School Education; Instructional Activities and Practices

A large portion of teachers' jobs entails selecting and enacting curriculum materials for their students. Ben-Peretz (1990) is one of several researchers who have asserted, "The ways in which teachers handle the curriculum determine, to a large extent, the learning processes in their classrooms" (p. 23). For this study I have chosen to examine one aspect of mathematics curricula, mathematics tasks. I have adopted Stein and Smith's (1998) definition of a task as "a segment of classroom activity that is devoted to the development of a particular mathematical idea" (p. 269). Tasks are the activities teachers select for students to enact. Doyle and Carter (1984) wrote, "The study of tasks, then, provides a way to examine how students' thinking about subject matter is ordered by classroom events" (p. 130).

A number of researchers have created classifications to describe mathematics tasks. Stein, Smith, Henningsen, and Silver (2009) classified mathematical tasks into four different categories. Each of the four categories related to the type and depth of mathematical thinking required of students as they solve mathematics tasks, what Stein et al. referred to as cognitive demand. Stein et al. also briefly discussed instances where tasks seemed to fall outside of these categorizations due to their non-mathematical nature.

I define non-mathematical activities as tasks where the mathematical goal of the lesson is lost or misaligned. I was able to identify such tasks by attending to the tasks' cognitive demand. The non-mathematical tasks were those tasks that fell outside of the cognitive demand classifications proposed by Stein et al. (2009). Rather than examining only those tasks that support meaningful mathematics instruction, I suggest that there is value in examining those activities that fail to create meaning. Although several studies have examined teachers' uses of mathematics tasks (e.g., Stein et al.), a review of the literature found no studies specifically examining tasks Stein et al. refer to as falling into the non-mathematical classification. For this paper I examined tasks of this type. In particular, I examined characteristics of these activities and the ways in which teachers might bring mathematics back to the forefront.

Methods

This study is part of a larger study in which I examined teachers' selection and enactment of curriculum materials for English language learners. I employed a qualitative, multiple case study methodology. The participants—Ms. Thomas, Ms. Hunter, and Mr. Dubois—were secondary mathematics teachers who taught a ninth grade, mathematics class. Each of the teachers was in their sixth year of teaching. The primary data sources for this study are surveys, interviews, observations, and classroom artifacts. I administered a survey to each teacher prior to conducting interviews or observations. I observed each teacher's ninth grade mathematics course daily for two weeks. Each observation was video recorded and partially transcribed. I conducted daily interviews with each teacher prior to observing their teaching and conducted two extended interviews after the two weeks of observations. Each interview was audio recorded and transcribed verbatim. The classroom artifacts consisted of student work and the tasks selected by the teachers.

I analyzed the data using the constant comparison method decoupled from grounded theory. This involved many rounds of inductive coding. I first analyzed each teacher individually and identified emerging themes using analytic memos. I then collapsed these themes into codes as I analyzed each of the different data sources for each teacher. I then performed a cross case analysis looking across the three teachers to identify those codes which were relevant to all the teachers. I consulted with my major professor in developing and verifying the codes. In the following section I discuss findings related to the teachers' task modifications.

Findings

The cognitive demand of the tasks I observed in this study frequently devolved into non-mathematical activity. The transfer of cognitive demand from mathematics to something other than mathematics is a phenomenon Stein et al. (2009) briefly addressed in their book, although they did not describe how this transfer occurs or what it looks like as it is happening. In this section I describe the different types of non-mathematical activities I observed.

Word Activities

During my time in the teachers' classrooms, I witnessed three categories of non-mathematical activities. The first of these categories is word activities. During the interviews, each of the teachers stated the need to build their students' mathematical vocabularies. Although the stated teachers' goals for many tasks were to build academic vocabulary, the tasks they selected seldom achieved this goal. The activities chosen by the teachers were frequently solitary exercises requiring no mathematical understanding. The teachers' failure to use the terms in context and allowance for resources that reduced many of the tasks to transcription activities led to my classification of these tasks as non-mathematical activities. I have termed tasks of this type *word activities*. This terminology highlights the difference between a focus on words without meaning versus activities that help build academic language (Coggins, Kravin, Coates, & Carroll, 2007).

Each of the teachers in this study presented his or her students with word activities. Guy Dubois' lengthy list of terms required students only to copy the definitions out of the textbook's glossary. Similarly, both Natalie Hunter and Meg Thomas asked their students to create flashcards for important mathematical terms. In order to create the flashcards, the teachers expected the students to find definitions for the terms in their notes or textbook and copy them onto note cards. This transfer of words from one source to another is something I term a transcription activity, an idea discussed in the following section. The teachers frequently coupled word activities with transcription activities.

Transcription Activities

The second category of non-mathematical activity I witnessed is what I have termed transcription activities. Perhaps the best description of this category is through the example of a court stenographer. Stenographers create written records of court proceedings. Stenographers sit in on a multitude of different cases and listen to testimony that witnesses provide in technical language. Though a stenographer witnesses and records the courtroom proceedings, it is unlikely he or she understands the details of all of the recordings. This is understandable as the goal of the stenographer is to create a written record in real time, not to understand what he or she writes.

In the mathematics classroom, a similar situation exists in transcription activities. The teachers often attached a mathematical learning goal to these lessons, but the mathematics tended to get lost in implementation. The purpose of many of these transcription activities was for students to explain or understand a mathematical concept. There were different ways in which these activities appeared in the mathematics classrooms. One such way, which I frequently encountered in the teachers' classrooms, was the copying down of teacher work. In such scenarios, the teacher was frequently at the board working out a problem. As the teacher worked out the problem, he or she expected the students to copy down the solution. In addition to copying down the teacher's solution, transcription activities include those tasks in which teachers ask students to restate an idea in their own words. In this case, the student is simply recanting an experience but does not have to have an understanding of the experience. Each of the teachers had instances of tasks that devolved into transcription type activities as they were implemented. The classroom norm seemed to be that when students struggled, the teachers would take over the mathematical thinking, thus transforming tasks into transcription activities.

Motivation Activities

Motivation activities are the third category of non-mathematical activity I encountered and involved activities where students were engaged in crafting or other hands on activities that did not have a mathematical goal. I chose the term motivation activities because the teachers selected these activities in order to engage and motivate students to learn mathematics. In many cases these motivation activities, on the surface, appeared to be mathematical in nature due to their enactment in a mathematics classroom and the inclusion of mathematical words. Upon closer examination, however, it was clear that these activities lost the mathematical focus leading to their categorization as non-mathematical activities.

One such activity was the creation of the quadrilateral mobiles in Ms. Thomas' classroom. Though the students were required to write down the properties from the board to their mobile pieces, this was simply a transcription activity. The students spent the vast majority of this time cutting the pieces into the appropriate shape and decorating the mobile with markers.

Classroom games can also fall into the gimmicky activities category. Ms. Thomas played "trashketball" with her students. This game, meant to serve as a review, required students to answer questions in small groups. If a group's answer was correct, they received the opportunity to throw the trashketball from one of three lines into a waste bin. Clearly, throwing a trashketball into a waste bin does not engage students in mathematical activity, though it did motivate the students to participate. This was not the only non-mathematical portion of the game. The questions Ms. Thomas had students answer were from the prior night's homework assignment. Therefore, students had only to write down a response from a paper.

Discussion

Teachers can learn to avoid non-mathematical activities if provided the proper resources and support. Non-mathematical activity stems from the absence or misalignment of mathematical goals. In order to bring the mathematics back to non-mathematical tasks, teachers should first clearly identify mathematical learning goals on which they would like their lesson to focus. The task must then be selected to support the development of the mathematical goal. Throughout the implementation of the task, the teacher must remain carefully attuned to the mathematics to avoid the transfer to non-mathematical activity, a process that frequently occurred in the teachers' classrooms in this study.

In the case of the word activities, teachers must commit to helping students attach meaning to the mathematical terms. The challenge is to create vocabulary activities that attach meaning to terms rather than simply creating word activities. Building students' academic language is an especially important and challenging aspect of teaching mathematics. Academic language means more than learning words; it suggests the meaningful use of words in mathematical contexts. Barnett-Clarke and Ramirez (2004) explicated, "not only do students need explicit instruction to read and write mathematical symbols and words, they also need to learn how to express mathematical ideas orally and with written symbols" (p. 57). Understanding the importance and meaning of academic language may help teachers develop activities that go beyond the learning of words and help students to learn the language of mathematics.

Teachers should avoid activities that require students to simply transcribe material rather than engage in mathematical thinking. Teachers should select tasks that allow students to extend and evaluate thinking rather than to simply restate what the teacher has already stated. Teachers should avoid transferring tasks to transcription activities by eliminating the productive struggle and taking on the mathematical thinking for students because productive struggle is a necessary part of engaging with higher cognitive demand tasks (National Council of Teachers of Mathematics, 2010).

Many of the motivation activities could be rethought to retain the "fun" while bringing in rigorous mathematics. Teachers may find that students can find motivation in solving tasks focused on real world phenomena related to their interests. In terms of games such as Ms. Thomas' "trashketball," teachers could include new tasks for the teams of students. These tasks could include high cognitive demand problems requiring students to work together toward a solution.

Defining types of non-mathematical tasks provides researchers and teachers with a common vocabulary when discussing these tasks. Understanding the aspects of the tasks that fail to support students' mathematical engagement provides a starting point for discussions related to strategies that add the mathematics back into such tasks. Teachers may be able to use the categorizations I have presented in order to critically examine their own activities to ensure the tasks they implement support students' interactions in mathematical thinking.

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STUDENTS' OF COLOR MATHEMATICAL IDENTITY IN AN INTENSIVE SUMMER PROGRAM

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Additional opportunities are available for students of color who have historically been marginalized by the system of schooling in summer programs that can support them to be successful in mathematics. Yet, we know little about students' experiences in these opportunities. In this empirical study, the author employed a case study approach to better understand African American students' developing mathematical identity in an intensive summer program. Data was collected through audio journals, interviews, and student written work to gauge students' beliefs about their mathematical identity. Preliminary results suggest students' conceptual understanding, confidence, and peer interaction in the summer program positively affected their mathematical identity.

Keywords: Equity and Diversity; Informal Education

It is commonly known that many students of color underperform in mathematics (Secada, 1992; Ladson-Billings, 1997), and they rarely enter mathematics-related fields (Bailey, 1990), in comparison to their white counterparts. Those who demonstrate competence in mathematics are better positioned to be economically independent (Secada, 1992). Some research has shown that in some of the most successful schools for these students (Kitchen, DePree, Celedon-Pattichis, & Brinkerhoff, 2007), teachers are working extraordinarily long hours to provide experiences that level the playing field. As Apple (1992) pointed out, however, the intensification of teaching makes it difficult to put additional demands on teachers. Although much research in mathematics education focuses on these students' in-school experiences, we cannot rely solely on K–12 schools. There is a dearth of empirical studies concerning out-of-school and summer programs, especially programs designed for students who have historically been marginalized in school mathematics.

In this report, I focus on students' developing mathematical identity in the context of a summer program designed to support students who have been historically under-represented in higher education and in STEM fields. Following the work of Martin (2006), I define mathematical identity as the "participants beliefs about: (a) their ability to perform in mathematical contexts, (b) the importance of mathematical knowledge, (c) the constraints and opportunities to enter mathematics fields, (d) the motivation and strategies used to obtain mathematics knowledge" (p. 19). The research questions are: In what ways does an intensive summer program shape four African American students' mathematical identities? What impact do the students think this intensive summer program will have on their schooling experiences in relation to their mathematical identity?

Relevant Literature

Identity

Mathematics education researchers have a growing interest in student identity because the identities that students construct have an impact on their learning (Nasir & Cobb, 2007). "[Identity] is also important with respect to equity in students' access to significant mathematical ideas. Mathematics is typically framed in terms of students' cognitive abilities (or the assumed lack thereof), mathematics as it is realized in the classroom also appears to function as a powerful filter in terms of identity" (Cobb, 2004, p. 333).

Mathematical Identity

To understand how African American students' mathematical experiences were internalized at a psychological level, Martin (2000) analytic framework focuses on the construction of mathematics identities. He examines sociocultural factors and how they relate to in-school mathematics experiences. He found that sociohistorical factors (historical discrimination regarding policies and practices among African Americans), community factors (parent and guardian beliefs about the importance of mathematics), school factors (influence of teachers and institutional beliefs), and individual agency (students' personal beliefs about their abilities to do mathematics, motivation to learn, and the importance of learning mathematics) are important in developing mathematical identity (Martin, 2000). For the scope of my study, I focus specifically on the students' individual agency because this study is of smaller scale than Martin's work. The research on mathematical identity and identity in mathematics education has been conducted in the context of *school* mathematics. Out-of-school experiences designed for students who are typically under-represented in mathematics, however, can also have an impact on students' mathematical identities.

The Promise of Summer Programs

Shapiro, Gatson, Hebert, and Guillot (1986), for example, found that after only two hours of mathematics and two hours of literacy per day, low socio-economic status (SES) students of color improved in computation and conceptual understanding. This study shows that summer programs have the potential to improve a student's ability to do mathematics, which provides evidence that at least one aspect of mathematical identity (i.e., ability to perform mathematics) can be influenced by out-of-school experiences. Yet, it remains unclear whether additional aspects of math identity might also be impacted.

Some empirical studies of summer literacy programs allow me to speculate further about out-of-school learning experiences. For example, in a program focused on reading skills, Chaplin and Capizzano (2006) found improvements in test scores and increases in the amount of time students spent reading books. These findings lead me to speculate that a summer program might increase interest in doing mathematics or the desire to take more mathematics courses. Shapiro et al. (1986) showed their summer program helped reduce the rate of reading comprehension that is typically lost in the summer. These types of impacts could help develop students' motivation and strategies related to literacy. The same kind of benefits might occur in mathematics-based summer programs, but no research seems to have focused on this area.

Methods

This study comprises qualitative case studies (Yin, 1984) of four rising 10th grade African American students. Although case studies are not generalizable to make decisive claims, they do provide an in-depth examination of the phenomena being studied (Yin, 2003).

Context

This summer program offers Latino/a and African American students an intensive four-week summer program. One goal of the program is to improve students' math readiness by teaching Algebra, Geometry and Pre-Calculus to rising ninth, tenth, and eleventh graders. In the past seven years, the program has had 98% of its students attend college, with 80% attending a four-year university and 63% of university-attending students majoring in a STEM field (Hargrave, 2011). This program provides a successful out-of-school context for mathematics education researchers to examine summer programs and their specific benefits for students of color.

Data Collection

In consultation with the program director, I selected the four focus students and contacted each student to collect some background information. During the program, I conducted interviews and collected audio journals and student written work. During each of the four weeks, the students were asked to record an audio journal Monday through Wednesday to talk about their experiences. On each Thursday, I listened to each journal and generated a set of interview prompts, clarified questions, and probed with respect to the four components of mathematical identity. On each Friday, I conducted 45-minute interview with each student using a semi-structured format. The final interview took place four weeks into fall semester to help me understand the impact on their schooling experience. Additionally, I collected students' written work to gauge their mathematical progress in course requirements.

Data Analysis

Using Martin's framework, I focused on understanding and unpacking students' individual agency. Since Martin did not elaborate his codes, I articulated codes for the interview and audio journals to accompany each of the four dimensions of mathematics identity. For example, I drew on Jansen's (2006) analysis framework that used modal verbs, expressions of affect, and repetition to understand which aspects of an experience are seen as important to students. The data collected from the audio journals will provide personal narratives of students' strategies and motivation, and their beliefs about the significance of the mathematics as it pertains to the development of their mathematical identity. As part of the analytic process, I wrote weekly memos, based on information gained from the audio journals and interviews, which were the basis of a case report for each student at the end of the four weeks. The student work was analyzed by comparing the weekly tests to the pre- and post-tests. When possible, I conducted member checks (Glesne, 2006) to make sure the students' perspectives were accurately portrayed in this study, and I looked for discrepant events within the data, allowing me to have trustworthy results with the codes that were derived from the interview data.

Results and Discussion

The analysis for this paper is currently ongoing, but I share some tentative results here, which will be expanded upon in the presentation. I will focus what ways does an intensive summer program shape two African American students' mathematical identity. In the program, students were able to focus on conceptual understanding in mathematics. Students' confidence to do mathematics also improved when they returned to school by their successes experienced in the program. Such programs are intense, so in order for students to be successful, they must persist through the difficulties of learning a large amount of mathematics in a short time frame. An emphasis is placed on learning about careers in the STEM fields was stressed, so students were able to learn about careers they had no idea existed. Students worked with so many other students of color who also wanted to be successful in school, which helps them be more motivated, which is something they say they did not witness when they were back in their home schools. Although I have observed these changes in students, little empirical research in this area has been done to date.

Mathematical identities afford and constrain different opportunities for learning and participation in wider contexts (Anderson & Gold, 2006), which could exist in out-of-school contexts. Given that almost every university offers such programs for students of color, this work will help to understand better the role an intensive summer experience in developing underrepresented students' mathematical identity. Such work can inform the design of related programs and policies, as well as help the public school system better understand the kinds of experiences they might develop or encourage students of color to be involved in.

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STUDENTS' EXPERIENCES OF SEEKING INJUSTICE-INDUCED HELP IN A MATHEMATICS CLASSROOM

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In this paper, we describe the phenomenon of a need for help that comes about as a result of problematic classroom practices, as re-lived and described by Tanzanian high school mathematics students. Through the students' experiences, we demonstrate how unequal power relations in the classroom can suppress students' voices, rendering their attempts at seeking justice futile. The students' experiences were characterized by pain and resignation.

Keywords: Equity and Diversity; High School Education; Affect, Emotion, Beliefs, and Attitudes

Introduction

In maths classrooms the world over, students seek help with their maths learning when they experience a lack in their knowledge or understanding. Seeking help in maths may be considered unique in its own way for at least two reasons. First, most maths concepts are abstract by nature, and thus many students experience difficulties not only in relating the concepts to their own prior knowledge and experiences, but also in establishing the connections between the concepts. Even in maths "word" problems, which may be argued to closely resemble problems in other academic disciplines, the language used is different from the one used in ordinary speech, and so many students inevitably encounter ambiguities and difficulties when solving such problems. Secondly, maths often serves as a gatekeeper to many careers and fields of study. This privileged status of maths over other academic disciplines serves well to perpetuate the popular myth of maths as a difficult subject. Seeking help is thus ascribed to persistence in the face of a challenging task, as well as the ability to recognize when help is needed, to decide to seek help, and to secure the needed help from others (Ryan & Pintrich, 1997). By seeking help, students increase their chances of understanding the content, thereby achieving the learning outcomes. Unfortunately, some students are reluctant to seek help even when help is needed and available, because they do not want to appear "dumb" in the eyes of others (Butler, 2006; Ryan & Pintrich, 1997), or out of a desire for independence, or even out of a concern about the competence or utility of available help (Butler, 2006). Although these and other research findings reveal a great deal about students' behaviour when seeking help, they do not offer us a deep appreciation of the student's experience of seeking help. The purpose of the study from where this paper is drawn was to explore and understand the essence and meaning of help in the context of maths learning from high school students' perspective.

Background

The Tanzanian education system has a 2-7-4-2-3+ structure; that is, 2 years of pre-primary education, 7 years of primary education, 4 years of ordinary level secondary education (also known as "O-level"), 2 years of advanced level secondary education (also known as "A-level"), and at least 3 years of university education (United Republic of Tanzania, 2011). The participants reported in this paper were A-level maths students. Only a small proportion of O-level students go on to A-level. In 2009, for example, only 15.4% of the candidates who sat for the final O-level examination proceeded to A-level (p. 55). Thus, these participants may rightfully be described as very resilient, hardworking and very much interested in succeeding. However, as shall be demonstrated in this paper, the phenomenon of seeking help is normally manifested in the complexity of the contexts within which the students learn.

Methodology

This study was informed by the research methodology of hermeneutic phenomenology, whose aim is to deepen our understanding of what it is like "from the inside" to live through an experience, by describing the "lived-through quality" of the experience (van Manen, 1997, p. 25). Data consist of concrete descriptions of lived events provided by those who have lived through the experience of the phenomenon. These raw data are commonly obtained through open-ended interviews with individual participants, as well as through participants' own written accounts of their lived experiences. The transcriptions of the interview data and participants' own written accounts are then transformed into short concrete anecdotes that are aimed at reawakening the reader's basic experience of the phenomenon being described (p. 122).

Seeking Help or Seeking Justice?

What is it like for students to ask for help in maths? What meanings do students ascribe to their lived experiences of seeking help in maths? And what is the pedagogical significance of understanding these meanings? These were some of the questions we set out to examine in the study from where this paper is drawn. We expected to learn about the diversity of frustrations, concerns, successes and strategies attendant to seeking help in maths. But instead of, or in addition to, describing a moment they sought help with a mathematical task, some students described experiences of injustice and inequity in their maths classrooms. And so in this paper, we explore the domains of social justice and lived human experience.

Orienting Oneself Toward Injustice: Re-orienting One's Way of Being with the Other

In the middle of his first year in A-level, Phineas transferred to a new school, and was shocked to find that the maths teacher was charging additional tuition fee for some of the mandated lessons. The teacher was teaching two topics simultaneously, one of which was to be paid for. Phineas describes his experience of being required to leave the class for lack of money:

> *It pained me to see him kick me out of class for lack of money when he was being paid by the government to teach me. But even more painful was the fact that he would not be repeating this topic during the regular class time.*

The phenomenon of seeking help begins with a sense of dissatisfaction with an aspect of our life-world's condition, emanating from a realization that we lack what we consider vital or important for our life-worldly existence. Only in the midst of a felt need or want do we seek help. A need may draw our attention to the fact that we do or do not have a right, something we may have all along taken for granted. Phineas' inability to pay for the extra tuition fee being levied by the teacher appears to have drawn Phineas' attention to his entitlement. Because he does not have full access to the prescribed maths curriculum, Phineas feels that he has been unjustly treated, and so he stands in need of justice. He is aware that his teacher is legally and socially restrained from acting unjustly towards the students. In other words, Phineas knows that his teacher is legally and morally obligated to follow the terms of the contract between the teacher and the government, in which the teacher, in exchange for his salary, is to teach maths to students in accordance with the approved curriculum documents, and to address students' learning difficulties in an equitable, fair and impartial manner.

The awareness of our right to something that is brought about by a need may alter our way of being with the other. We may, for instance, begin to question the other's indifference to his/her moral obligations as they affect our rights. Our response to perceptions of injustice may include feelings of pain. Phineas says that it pained him to see the teacher kick him out of the class. Pain—from the Latin *poena*, meaning punishment or penalty (Harper, 2001)—is unpleasant and hurtful. Phineas is being punished in spite of his innocence. He is thus being subjected to the injustice of an undeserved and unjustifiable punishment.

Injustice Demands to Be Heard

Students may react to perceptions of injustice in the classroom by seeking help from the school administration. But what the conversations with some of the students in this study revealed is that unless one appeals to one's moral conscience, any external imposition of ethical responsibility cannot guarantee the authenticity of a pedagogical enterprise. When Kalunda observed that her maths teacher was frequently missing his teaching duties, she reported the matter to the headmaster. She describes the outcome of the headmaster's intervention:

> *Now the class monitor has an attendance register. When a lesson is taught, the monitor remarks: "Taught" and appends his signature against the remark. If the teacher assigns someone to copy some notes for the class, the monitor remarks: "Notes Written" and appends his signature. The teacher is then expected to countersign against each of the monitor's signature. But I have several experiences where the teacher comes to class, gives you questions, hangs around for a few minutes, and then leaves. The questions will not be graded nor discussed. Of course the teacher will not agree to sign if the monitor remarks: "Untaught" And so when the headmaster or his deputy delves through the attendance register, he will find that the teacher's signature is there, almost everywhere. And the game is over. But it is we the students who really know what goes on inside the class.*

Although the attendance register was meant to enforce justice in the teacher, he is somehow managing to get around it. A scenario is eventually established where a masquerade of adherence to one's ethical obligation goes unchallenged by the other. Who, then, is to blame for the injustice in Kalunda's classroom? The headmaster, to whom Kalunda's teacher is subordinate, may have done his job by warning or reprimanding Kalunda's teacher. In fact the headmaster may be thinking that Kalunda's classroom is running smoothly. The ball, as it were, is in the students' court. But then there is a problem: The playing ground may not be levelled for a fair play. If the power relations between oneself and the subjects over whom one is ethically responsible are skewed in one's favour, then any pedagogy based on other-rather-than-self-monitoring is bound to break down even under the other's keen and watchful eye.

In another interview we learned Jacinta's teacher would regularly miss his lessons. At times he would come to class and appoint one of those students who had attended private tuition to teach some topics to the class. Jacinta recounts her experience when she responded to these unjust practices:

> *It reached a point where everyone was dissatisfied with how we were learning maths. One day, the Deputy Headmaster came to our class and, in a very friendly manner, asked: "What problems are you facing in this class?" Now I just said to myself: "This is our administrator. If we don't tell him what we are going through, whom shall we tell?" So I decided to be honest. And it was as if everyone else was waiting for someone to initiate. So we said we had this and that problem in maths. The Deputy promised to look into the matter. But then I don't know how he presented the issue to the maths teacher, because the next time the teacher came to class, he was very angry with us. Thereafter, we became the marked group. Anytime you went to the staff room, a teacher would always find something to punish you for—your blouse, tie, shoes, socks, finger nails. And there was enmity between us, the "bad" class, and the teachers. From that incident, I learnt to persevere, whatever the case. Now if someone comes and asks how we are doing, I'll just look at them. And so people are just dying like that, each one on her own, quietly but surely.*

To speak is to be a person, to be unique, to be recognized. But the act of speaking is relational; it is something between oneself and an other. When we speak, we are voicing a desire to be listened to, to be heard, to be understood. But what happens when our speaking turns out to be merely an act of losing our ideas? What is it like to have our right to be heard violated? Jacinta knows that she is capable of learning much more maths than she is learning at present. She believes that she is not learning as much maths as are required to meet the intellectual demands of modern life and work. She knows that at the end of A-level, her demonstrated competencies in maths will determine her prospects for job opportunities and/or her admissibility to various programs in institutions of higher education. The main cause of Jacinta's dissatisfaction with her progress in maths is the indifference of her maths teacher to his ethical obligations.

The teacher is not adhering to the recommended instructional time guidelines for the maths curriculum. He is not monitoring the progress of the students and the course as required by the ministry. And contrary to his professional ethics, he is sanctioning an unqualified person to perform the duties of a maths teacher, and without compensating this person accordingly. Jacinta's act of seeking help from the Deputy headmaster produces in Jacinta's teacher a disinterest in Jacinta's welfare. Jacinta's teacher manages to mobilize his colleagues in the maths departmental office and together, they wittingly adopt a hostile "we-against-them" attitude towards the students. To whom can Jacinta now turn for justice? Or has she been officially consigned to silence? Has her need for help been rendered a taboo, something not to be spoken of?

Pedagogical Implications

When we think of students seeking help in maths, we most often imagine the object of their need for help being specific to a particular mathematical task. Yet students in this study raised a much more nuanced notion of help than has been written about in the literature. We knew that issues of self-confidence and esteem might be woven into the students' experiences of seeking help, but we didn't expect to hear about students' experiences of seeking justice in their search for help. What these students' experiences do is, on the one hand, to remind us that the "text" of the phenomenon of help cannot be isolated from its larger context and, on the other, to highlight some of the challenges that hinder students' realization of their full learning capacities. The onus, then, is on educational leaders to be more pedagogically sensitive to the challenges that students encounter in their classrooms. One sure way of guaranteeing authentic and meaningful learning experiences for the students is ensuring that the classroom is an arena for justice and voice.

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MATHEMATICAL LISTENING: SELF-REPORTS OF HOW STUDENTS LISTEN AND ITS RELATION TO HOW THEY ENGAGE IN MATHEMATICAL DISCUSSION

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Mathematical listening is an important aspect of mathematical discussion. Yet, relatively few examinations of this phenomenon. Further, existing studies of students' mathematical listening come from observational data, lacking the student perspective. The present study examined student replies to an open-response question regarding what happens to their thinking about mathematics when they listen to their peers' mathematical talk. Results suggest varying ways of listening that range from more passive to more active forms. Additionally, relationships were observed between ways of listening and perceived forms of engaging in mathematical discussion.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Classroom Discourse

Overview and Purpose

The literature focusing on mathematical discussion has suggested student engagement in discussion allows them to reflect more deeply about mathematical concepts and improve their understanding of the content (e.g., Hiebert & Wearne, 1993; Kazemi & Stipek, 2001; Mercer & Sams, 2006). Yet, simple participation in discussions does not necessarily transfer to a deeper understanding of mathematics (Kosko & Miyazaki, 2012; Manouchehri & St. John, 2006). Students' active listening has been posited as a defining characteristic of productive engagement in mathematical discussions (Hufferd-Ackles, Fuson, & Sherin, 2004; Otten et al., 2011). Researchers examining student listening have suggested that as students transition from a more passive to a more active form of listening, they are observed to engage more interactively with their peers during mathematical discussions (Hufferd-Ackles et al., 2004; McCrone, 2005). This transition can be scaffolded by the teacher, such that they have students revoice (restating or rephrasing another student's statements) their peers (Otten et al., 2011).

The various studies on students' mathematical listening have a common theme in that they all examine students' interaction to determine degrees or forms of mathematical listening. While these studies describe active listening as being inherently connected to student interaction in discussion, with specific focus on the ability to revoice, they pose the risk of observer-related bias. Specifically, listening is not a directly observable act, and an additional perspective is necessary to confirm or disconfirm findings from prior studies. I suggest that this additional perspective should be that of the student. In order to assess the student perspective, I used high school students' written responses to an open-response survey question asking how they listened to their peers and what happened to their thinking. I also gathered closed-response data regarding the manner and frequency that these students perceived they engaged in mathematical discussions. Comparison of students' responses to the open-response item regarding mathematical listening with closed-responses to the manner and frequency students participate in mathematical discussions was meant to address several gaps in the literature, as well as confirm or disconfirm findings of observational studies thus described. First, I sought to examine how students listened by assessing their own perspectives of what listening did to their mathematical thinking. This examination provides the student perspective, which I hypothesized would deepen our understanding of mathematical listening. Next, I examined how these perceived ways of listening related to the manner in which students reportedly engaged in class discussions. This examination was meant to uncover the degree to which passive participants in discussion were passive or active listeners (i.e., does a student have to talk frequently in a discussion in order to listen reflectively). Finally, the examinations of this data were meant to extend those of previous studies, by providing an additional perspective of the phenomenon.

Methods

Data were collected from 62 high school geometry students in five classes taught by the same teacher. Students were enrolled in a rural high school in the Southeastern United States. At the end of the year, students completed a set of open-ended questions, of which I examined responses to the question: *When you listen to other students talk about math in class, what happens to your own thinking about math? Please explain your answer*. Students also completed a closed-response which included items assessing students' perceived engagement in mathematical discussion (e.g., "When talking about math in class, I explain what I mean in detail"). Responses for items were on a 6-point Likert scale (*1= Strongly Disagree* to *6 = Strongly Agree*), and demonstrated sufficient internal reliability (M = 3.93, SD = 1.08, α = .73).

A main aim in studying students' perceptions about mathematical listening is to better understand how mathematical listening relates to mathematical thinking. To investigate this relationship, I examined the grammatical processes students used in their responses to the prompt, and the participant references associated with those processes. Processes are part of the grammatical system of transitivity (Halliday & Matthiessen, 2004), whose basic function is to convey meaning, or to describe "a particular domain of experience" (Halliday & Matthiessen, 2004, p. 170). Since the item prompt specifically asked about students' thinking, I also examined responses for references to such thinking. Five categories emerged based on similar meanings conveyed. The analysis and emergence of these categories is described below, by category.

Doesn't Listen emerged from two responses where students explicitly stated that they did not listen to their peers' mathematical discussions. *Nothing Happens* was the classification used to organize 15 responses where students conveyed that their thinking remained the same. For example, one student stated that: *I can't really say hearing others talk about math <u>warps</u> **my opinion of it***. The process <u>*warps*</u> conveys change, or lack thereof, to the participant ***my opinion***, which sits in as a reference for thinking. Several other students made the explicit statement, ***it** <u>stays</u> the same*, or a variation of such statement. Both classifications *Doesn't Listen* and *Nothing Happens* suggest, at most, limited engagement on the part of the student. While 29.1% of students conveyed that their thinking does not change when engaged in listening to others' during mathematical discussions, the remainder of responses suggested that students were engaged, in some degree.

The classification *Learning Happens* was used to classify 30.9% of student responses where students indicated that they did engage in some form of cognition, but provided little detail beyond such description. One such statement in this classification was: ***My own thinking** <u>is changed</u> and it <u>teaches</u> me different **things that I didn't understand** at first.* In describing their thinking, the student conveyed that it changes, but also by listening, the student suggests they are taught ***things that*** [they] ***didn't understand***. Another student stated: *It <u>can help show</u> me **what I don't know***. The student conveys some sense of learning from the process <u>*show*</u>. In both statements, some form of learning is conveyed, but the details of the learning is loosely structured. While it is tempting to suggest the lack of specificity is due to poor communication on the part of the student, I suggest that students are signaling a particular type of engagement with mathematics. Specifically, mathematics is treated as an entity that can be right or wrong, possessed or not. In short, it is treated in a simplistic manner that may value answers and products over processes and concepts.

Math Happens represented 10.9% of student responses and included responses where students described their thinking as changing in such a way that they discovered or learned new mathematical information. In the student response, *I try to <u>**understand**</u> what they are saying and try their way of solving the problem*, represents a subtle departure from the conveyed meaning in statements classified as *Learning Happens*. The student uses the process and reference to student thinking <u>***understand***</u> to describe their learning of mathematics. However, mathematics is treated as a set of processes or procedures, rather than a product or answer, as is the case in *Learning Happens*. Another student response, *It <u>opens</u> **my mind** to how to do other kinds of math*, conveys that the student's ***mind*** is opened to do math. Doing math conveys a process rather than a product. While seemingly subtle, this difference between the *Learning Happens* and *Math Happens* categories is crucial. Statements categorized as *Math Happens* describe changed thinking in regards to methods and strategies while statements categorized as *Learning Happens* describe changed thinking in regards to products or solutions.

Another feature of statements in *Math Happens* is that the references to thinking were more often conveyed as processes than in *Learning Happens*. Specifically, variations of the process "understand" were more consistently used in *Math Happens* statements. However, to "understand" is quite different from other mental processes one might engage. In distinguishing reflection from thinking, Dewey (1910) noted that "reflection thus implies that something is believed in (or disbelieved in), not on its own direct account, but through something else which stands as witness, evidence, proof, voucher, warrant; that is, *as grounds of belief*" (p. 8). Such a distinction characterized the classification of *Reflection Happens*, which represented 29.1% of student statements. *Reflection Happens* classified statements that used processes such as *compare*, *reflect*, *contrast*, etc. Such mental processes reflected the distinction made by Dewey, in that they conveyed meaning associated with cognition that represented some form of comparison in search of "grounds of belief." Following are two examples of statements classified in this manner: (1) *I **compare** [my thinking] to their* **thinking, (**2) *I listen to what they're saying, **consider** it and **compare** it to how I would solve it.*

Noticeably, in the first statement, "thinking" is present both as a process, in the form of ***compare***, and participant, in the form of ***thinking***. This particular example illustrates how the processes identified based on Dewey's conceptions relate to the notion of reflection as a form of metacognition. The student conveys the sense that they are manipulating the abstracted object of "my thinking" through the mental process of comparison. The second statement uses the same type of mental process but incorporates a procedural statement about the mathematics (i.e., *how I would solve it*). When examining how mathematics itself is described for the *Reflection Happens* category, there appears to be little observable difference with how it is described in the *Math Happens* category. Specifically, mathematics is referred to generally with procedures. Therefore, the defining distinction between these categories is not how mathematics is treated or referred to, but in how students conveyed they thought about the mathematics (e.g., "comparing" rather than "understanding").

The classifications presented in the previous section represented not only categorical, but ordinal data (1 = *Doesn't Listen*; 2 = *Nothing Happens*; 3 = *Learning Happens*; 4 = *Math Happens*; 5 = *Reflection Happens*). Specifically, statements in higher ordered classifications conveyed more active forms of cognition on the part of the student. Using the classification schemes as ordinal data, I correlated this student data with students' composite scores for *discussion*. The calculated Spearman Rho coefficient was found to be statistically significant with a moderate effect size (ρ = .38, p < .01). This result suggests that the more active form of listening students reported they engaged, the more they reported explaining and justifying their mathematics when they participated in discussions.

Discussion

Throughout the present study, I sought to examine what students reported happened to their own thinking while listening to other students talk about mathematics. Results suggested several different ways students reported their thinking was affected. Additionally, these different ways of thinking represented a transition from less to more active forms of thinking. The highest point on this continuum involved students in reflection, which included comparing and contrasting their own mathematical thinking to their peers'. This action is similar to the observable act of revoicing identifed by others (e.g., Hufferd-Ackles et al., 2004; Otten et al., 2011). Given the corresponding results presented in such previous work with the findings described in the current study, it seems that an important, practical implication of the present study is for teachers to facilitate active listening by having students rephrase and summarize their peers' mathematical contributions in class.

The findings presented here suggest that students' mathematical listening comes in many forms. Yet some forms appear more desirable than others. Therefore, the findings of the present study suggest a complicated picture of how students engage in mathematical listening, and what researchers and teachers may observe of this phenomenon when watching students talk about mathematics.

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IDENTITY AND EQUITY AMONG LATINA/O[1] UNDERGRADUATES IN AN EMERGING SCHOLARS MATH WORKSHOP

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This study uses an identity lens to explore how equity processes evolve for two Latina/o undergraduate students in a culturally diverse Emerging Scholars Program (ESP) Calculus I workshop at a predominantly White urban Midwestern university. Drawing on Gutiérrez's (2007, 2008) definition of equity, findings indicate that equity processes became strengthened for the participants through four dimensions (access, achievement, identity, and power) over time. Drawing on critical race theory and Latina/o critical theory, findings reveal that participants' participation through their racial identities—co-constructed with their gender, class, and math identities—aided in strengthening all four dimensions. Findings support that understanding how race, gender, and class function in Latina/o students' math experiences and identities, including in math classroom contexts, expands knowledge about how to develop equitable math practices.

Keywords: Equity and Diversity; Post-Secondary Education; Instructional Activities and Practices

Purpose and Background

In math education research and policy documents, equity issues have become more complex and received heightened attention in recent years (Allexsaht-Snider & Hart, 2001; Hodge, 2006; NCTM, 2000). Studies that investigate equity issues in math communities of practice often draw on sociocultural theories of learning (Lave & Wenger, 1991; Wenger, 1998). However, emerging equity math education research advocates for using a broader sociopolitical theoretical lens that addresses, for instance, contextual factors, identity, power, race, gender, and class (Gutiérrez, 2007; Martin, 2007; Nasir & Cobb, 2002). At a time when math achievement disparities among different racial, ethnic, and socioeconomic groups subsist, broader conceptualizations of equity would allow for a deeper understanding of the complex and often concealed mechanisms that support and impede students from experiencing equity in their math trajectories.

While equity definitions often discuss the importance of providing students with access, learning opportunities, and outcomes in relation to mathematics (Allexsaht-Snider & Hart, 2001), rarely do they include a dimension that addresses the mathematical empowerment of students and communities (Martin, 2003). However, Gutièrrez (2007, 2008) provides a definition of equity consisting of four dimensions that address access, learning opportunities, outcomes, and empowerment. Drawing on this definition, this study conceives equity in relation to the ESP Calculus I workshop community as:

(a) *access* – participants express satisfaction with how they are able to access the resources and opportunities that enable them to interact with quality mathematics;
(b) *achievement* – participants express satisfaction with the Calculus I grades they earn, they exhibit high participation rates, and they persist in mathematics;
(c) *identity* – participants develop stronger self-perceptions as math learners and positively co-construct this perception with other identities they possess; and
(d) *power* – participants exhibit agency related behaviors that contribute to transforming school or society (2008, p. 360).

Each dimension develops over time and is dependent upon each student's unique experiences, interactions, goals, and identifications as they engage in the ESP workshop's practices.

Too often math education equity research essentializes Latina/o students' experiences (Boaler, 2002) and fails to differentiate equity from enlightened self interest purposes (Secada, 1989). In contrast, this research adds to math education scholarship expanding meanings for equity among Latina/o students by acknowledging how their identities, agency, and sociopolitical constructs, such as race, gender, and class, function in their math trajectories. This is one of a few studies to adopt broader equity definitions to investigate how equity processes and equitable math practices develop for Latina/o students (Boaler, 2008; Gutièrrez, 2002).

Method

This project is part of a larger study that employed multi-case study and cross-case analysis (Yin, 2009) to explore undergraduate Latina/o students' identity constructions and participation in ESP (Oppland, 2010). In the larger study, multiple data sources, including interviews, reflections, and direct classroom observations, were analyzed to construct an in-depth case study for each participant that described how they co-constructed their math and racial identities, how their workshop participatory trajectories developed over time, and how their racial identities fortified their participation. This study expands upon prior findings by investigating how two Latina/o students' participation through their racial identities (and its co-construction with other identities) contributes to strengthening four dimensions of equity. Participants include Vanessa, an 18-year old, second-generation Mexican American immigrant, and Immanuel, a 19-year-old, first-generation Mexican American immigrant.

This study views both *narrative identity* (Sfard & Prusak, 2005) and *participative identity* (Wenger, 1998) as critical for interpreting how participants' racial identities inform equity processes. While narrative identities refer to stories that are generated as individuals contemplate and share their experiences (Sfard & Prusak, 2005), participative identities refer to the identities of participation and non-participation individuals construct as they engage in communities of practice (Wenger, 1998). Drawing on Wenger's (1998) social ecology of identity framework, participative identity in this study refers to how participants construct identities as ESP workshop members and how they negotiate and claim ownership of mathematical meanings through three modes of belonging (engagement, imagination, and alignment).

The ESP research site is located in Hall University, a large four-year research university in Chicago, Illinois. The ESP workshops were modeled after Uri Treisman's doctoral dissertation work, which aimed to discover explanations for why African American students were struggling to learn calculus at the University of California, Berkeley in the mid-1970s. Drawing on this model, optional ESP math workshops at Hall run parallel to students' standard math courses and encourage culturally diverse peer groups to collaboratively solve challenging problems. The ESP workshop in which this study occurred consisted of 27 students. Class demographics were roughly 41% Latina/o (4 females and 7 males), 30% Asian (5 females and 3 males), 22% White (2 female and 4 males), 7% African American (1 female and 1 male), and 44% female.

Data analysis consisted of four phases. Phases 1 and 2 involved coding interview data, constructing thematic memos, and writing narrative summaries that described how participants negotiated math and racial identity co-constructions (Phase 1) and constructed participative identities (Phase 2). Phase 2 also involved analyzing the remaining data sources and comparing that evidence to the narrative findings. Phase 3 involved identifying emergent themes related to how participants' racial identities strengthened their workshop participatory trajectories. Phase 4 involved identifying how participants' participation through their racial identities contributed to strengthening the four equity dimensions. When attempting to understand these connections, in addition to drawing on CRT and LatCrit theory, I used a grounded theoretical approach in terms of allowing relevant themes to emerge in the data.

Results

Results indicate that participants' participation through their racial identities—co-constructed with their gender, class, and math identities—powerfully contributed to strengthening the equity processes they experienced in relation to the ESP workshop context through all four dimensions.

Access refers to participants expressing satisfaction with how they are able to access the resources and opportunities that enable them to interact with quality mathematics. This study perceived the ways in which participants' racial identities contributed to positively shifting their participative identities as strengthening their access to mathematics. In a previous study (Oppland, 2010), results show that participants' racial identities (and their intersection with their gender, class, and math identities in some instances) contributed to optimistically shifting their participation in the ESP context by: (1) strengthening comfort levels with peers, (2) positively altering perceptions of their and peers' math abilities, and (3) allowing them to challenge racialized math experiences. For instance, in the ESP context, both participants challenged prior racialized secondary math experiences they managed while interacting with math teachers.

Achievement refers to participants expressing satisfaction with the Calculus I grades they earn, exhibiting high participation rates, and persisting in mathematics. Participants revealed that the ESP context aided them in earning higher Calculus I grades, that their workshop participation increased over time, and that peer interactions powerfully assisted in strengthening their participation and math appropriation. Participating through their racial identities contributed to optimistically shaping how participants interacted with peers in terms of positively altering their perceptions of their and peers' math abilities and strengthening their comfort levels with peers. For instance, sharing cultural backgrounds with Latina/o students contributed to both participants convergence towards practicing math with Latina/o students in the ESP context.

Identity refers to participants developing stronger self-perceptions as math learners and positively co-constructing this perception with their other identities. Both participants indicated that they positively co-constructed math and racial identities in the ESP classroom setting:

> It's good to see how other Hispanic people are so good at doing math. It's not what people usually think of. I think it makes me proud that there's a chunk of us…I'll put myself in that group that are willing to do whatever to be good at math… (Vanessa)

> I never met Hispanic or Latino people that are this high up like me. I was expecting to come to [Hall] and not be similar to people in that cultural background…I don't feel alone…when you get to this class you and I are breaking stereotypes. (Immanuel)

Power refers to participants exhibiting agency related behaviors that contribute to transforming school or society. Participants indicated that their ESP experience contributed to fortifying aspects of their identities that they perceived as important for creating positive change in their Latina/o families and communities. This included strengthening Immanuel's ability to tutor a younger sibling in math and Vanessa's ability to function as an academic role model for Latinas within her community.

The Latina/o participants' stories reveal the complex psychological and sociopolitical mechanisms underlying their math learning processes within a collaborative ESP Calculus I workshop that contributed to their mathematical success. Such knowledge can be applied to create equitable math learning environments that aid undergraduate students in effectively transitioning through gateway courses, that mathematically empower students in K–16 contexts, and that support underrepresented students in persisting through the K–16 math pipeline.

Endnote

[1] Latina/o is a political term produced by American culture that is used to describe people who have origins in the Hispanic countries of Latin America or Spain and people who self-identify as Latina/o (Ferdman & Gallegos, 2001).

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EXPLORING COGNITIVE DEMAND, STUDENT PARTICIPATION AND LEARNING IN MIDDLE SCHOOL ALGEBRA[1]

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This brief report analyzes mathematical task enactments in 12 middle school algebra classrooms using the complementary lenses of cognitive demand and participatory demand. Whereas cognitive demand is well known in the field as a conceptualization of the thought processes entailed in solving a mathematical task, participatory demand is a new construct that captures student involvement in mathematical discourse during enactment. Correlational analysis revealed a significant relationship between participatory demand and average student gain scores on a pre- and post-test, but not between cognitive demand and gain scores.

Mathematical tasks are the essential building blocks of much of mathematics instruction. Yet, enacting tasks in mathematics classrooms in ways that promote deep mathematical thinking and spur rich mathematical communication is a challenging endeavor (Henningsen & Stein, 1997; NCES, 2003). To illuminate aspects of this challenge, researchers from the QUASAR project analyzed *cognitive demand*—the kinds of thinking processes entailed in solving a task (Stein, Grover, & Henningsen, 1996). In particular, they traced shifts in cognitive demand throughout the phases of task enactments, such as a teacher focusing on correct answers, which can lower cognitive demand, or a press for justification, which can maintain a high level of cognitive demand. Such maintenance or declination of cognitive demand is important because high levels of cognitive demand have been linked to positive student outcomes (e.g., Stein & Lane, 1996).

This body of research focused on student thinking has had a widespread impact, informing pre-service and in-service teacher education as well as subsequent research in mathematics education. More recently, however, there has been a new and growing body of research focusing on classroom discourse (Ryve, 2011) and students' participation in a mathematics community as inseparable from their learning of mathematics. In this article, the construct of *participatory demand*—the extent and nature of student interactions during the enactment of a mathematical task—is used as a complement to cognitive demand, with both together providing a fuller picture of mathematical task enactments than either would provide separately. Using these complementary constructs, enactments of various mathematical tasks designed with the same learning goal were analyzed and investigated in relation to students' attainment of that learning goal as measured by gain scores between a pre- and post-test.

The research question guiding this analysis was as follows: *How does the cognitive demand and participatory demand of mathematical task enactments relate to students' learning of the content of those tasks?* This work is situated within an ongoing investigation of a larger set of questions that will not be reported here.

Theoretical Perspective

From a sociocultural perspective, learning is viewed not as the accumulation of knowledge in an individual's mind but as a collaborative process through which a learner comes to participate in particular discourse communities (Lave & Wenger, 1991). Cobb, Yackel, and Wood (1992) recognized the importance of collective processes but argued for striking a balance between the collective and the individual when considering learning in mathematics. Echoing Cobb, Yackel, and Wood, this study is an attempt to view students as both individual thinkers and collective participants in the mathematics classroom. In particular, the construct of cognitive demand is used to attend to potential thought processes during mathematical task enactments and the construct of participatory demand is used to attend to student interactions.

To structure the observations of task enactments in mathematics classrooms, a modified version of the Mathematical Tasks Framework (Stein, Grover, & Henningsen, 1996) is used. As Figure 1 depicts, a written mathematical task typically proceeds through several phases of enactment: the teacher sets up the task, students work on the task (individually, in small groups or as a whole class), and then the teacher and students look back at their work (perhaps to share solution strategies, summarize key ideas, or draw connections). The final portion of the framework highlights the fact that the written task and its enactment shape what students learn from the experience.[2] This study, in particular, is investigating the relationship that cognitive demand and participatory demand of mathematical task enactments have on one conceptualization of student learning, namely, their gain scores on a written pre- and post-test.

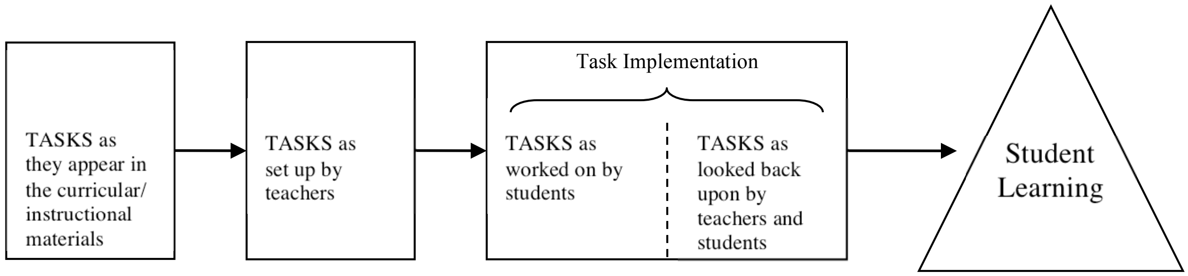

Figure 1. A modified version of the Mathematical Tasks Framework (MTF)

Method

Twelve sets of video recordings of lessons from nine middle school teachers in two states comprised the classroom data for this study (see Table 1). The analyzed lessons (n = 52) were those dealing with the learning goal of using variables to represent co-varying quantities. Although the learning goal of the analyzed lessons was constant, other characteristics varied, such as the geographical region, teacher's years of experience and educational background, and the curriculum materials being used. This variation provided a range of task enactments.

Table 1: Participating Teachers and the Data Collected for This Study

Teacher	State	College Major	Yrs. Exp.	Grade	Textbook	Students
Ms. Albert	DE	Md. Sch.[a]	7	8	*Math in Context*	23
Ms. Cavillon[b]	DE	Md. Sch.	9	7	*CMP*	22
Ms. Cavillon[b]	DE	Md. Sch.	10	7	*CMP*	19
Ms. Cesky	TX	Math[a]	6	8	*MaThematics*	16
Ms. DePalma	DE	Math	8	8	*Math in Context*	28
Ms. James	TX	Math	8	8	self-generated	20
Mr. Johnson	DE	Elem.	6	7	*CMP*	16
Ms. Mendoza[b]	TX	Math	13	8	self-generated	20
Ms. Mendoza[b]	TX	Math	14	8	self-generated	15
Ms. Mendoza[b]	TX	Math	15	8	self-generated	24
Mr. Milson	DE	Elem.	5	7	*CMP*	22
Ms. Wyncott	DE	Elem.	4	8	*Math in Context*	21

Note: [a] indicates a Master's degree at time of observation. [b] indicates observations during multiple academic years.

Main instructional tasks (i.e., excluding warm-up activities and homework review) were identified and parsed according to the modified MTF (see Figure 1). Each of the four phases were then coded for level of cognitive demand, using methods from past research (Stein, Grover, & Henningsen, 1996), and nature of participatory demand. Participatory demand codes focused only on verbal participation and consisted of two sub-codes: *level of student participation* and *focus of student participation*. The level of participation was high, medium, or low depending on the extent to which students were speaking full sentences (as opposed to single-phrase turns). The focus of participation was mathematical, semi-mathematical, or non-mathematical and used a technique called thematic analysis (Lemke, 1990; Herbel-Eisenmann & Otten, 2011) to determine the extent to which students were verbalizing semantic relations between mathematical terms (as opposed to stating mathematical terms in isolation or non-mathematical terms and relations). These aspects of participation are important because, as Lemke (1990) argued, coming to participate in a discourse community means being able to combine terms in ways that make sense in that context. Double coding was conducted on a sample of the videos and inter-rater agreement was greater than 80% on each dimension.

Student learning was operationalized as gain scores between a pre- and post-test measuring students' aptitude with respect to using variables to represent co-varying quantities. The tests consisted of 7 multiple-choice items, 8 constructed-response items, and 1 multi-part item and were piloted prior to this study. Inter-rater agreement was 94–99% on pre- and post-test scoring. The twelve sets of data were ranked separately by standardized average gain score, prevalence of high cognitive demand, and prevalence of high-mathematical participatory demand. Spearman's rank correlation coefficient (ϱ) was used to test the pairwise relationship between these variables.

Results

First, based on past research (e.g., Stein & Lane, 1996), one would expect cognitive demand to be positively correlated with average gain score. However, these two variables were weakly ($\varrho = 0.273$) and insignificantly ($p = 0.196$) correlated in this study, though this lack of evidence for a relationship is not necessarily evidence of a lack of relationship. For instance, the instruments used here may have been insufficiently sensitive to detect the association. Second, based on arguments for the importance of student discourse in mathematics classrooms (e.g., Ryve, 2011), one might also expect participatory demand to be positively correlated with average gain score. Indeed, a positive relationship was found ($\varrho = 0.406$), significant at the 10% level ($p = 0.095$). It is somewhat surprising that this relationship was detected with only 12 classes involved in the analysis and with verbal participation being coded in the enactments but a paper-and-pencil assessment of student learning.

Interestingly, nearly 30% of the squared deviations between participatory demand ranking and gain score ranking come from a single teacher, Ms. Cesky, who was ranked second to last with respect to participatory demand but had the fourth highest standardized gain score. The fact that the correlation between participatory demand and gain scores approached significance even with the inclusion of Ms. Cesky suggests that the other 11 sets of lessons exhibited an exceedingly strong link between these two variables. Indeed, calculating ϱ based on the rankings of the other 11 classes yields $\varrho = 0.664$ ($p = 0.0102$), which is highly significant.

Looking Ahead

What is the relationship between cognitive demand and participatory demand during task enactments? Conventional wisdom states that it is easier to get students talking about mathematics if they have something interesting to talk about, namely, a high-level task. In the present study, no significant relationship was found between cognitive demand and participatory demand, that is, there were numerous instances of high student participation around low cognitive demand processes and low student participation around high cognitive demand processes. This does not imply, however, that students working on high-level tasks did not have mathematical ideas to share verbally—it may just be that these ideas were not elicited during enactment. Perhaps coming to better understand these relationships between cognitive demand and participatory demand will allow us to better understand the broader relationship between enactment and student learning.

A correlation was found between participatory demand, operationalized as verbal utterances by students of mathematical semantic relations, and gain scores on a pre- and post-test of a middle school algebra learning goal. Although not necessarily a causal link, this finding does support claims about the importance of student participation in mathematics classroom discourse.

Endnotes

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[2] Although the arrow points unidirectionally toward learning, learning also occurs during the phases of enactment and students' learning also shapes the enactment itself.

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SECONDARY PRE-SERVICE MATHEMATICS TEACHERS' CONCEPTIONS OF EQUITY

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This study examines secondary pre-service teachers' (PSTs) conceptions of equity in the teaching of mathematics as part of a preliminary research investigation. Teachers' conceptions around equity in the mathematics classroom affect their classroom practices. Therefore, it is important to know what PSTs' conceptions around these issues are in order for teacher educators to help address any gaps or misunderstandings. We used three tasks during methods courses across three universities to elicit PSTs' conceptions. We found that unless specifically prompted, most students do not consider issues of race, power, ethnicity, SES, or language when they envision their future mathematics teaching.

Keywords: Equity and Diversity

The majority of pre-service teachers (PSTs) are educated to teach students from a White, Eurocentric middle-class culture (Kyles & Olafson, 2008). As a result, PSTs often hold a set of values that differ from the shared values of the student populations they will teach; these differences are referred to as a cultural discontinuity. The cultural discontinuity often leads to lower expectations for students from minority or low SES backgrounds (Gay, 2000; Ladson-Billings, 1994). Given that all mathematics students need high expectations and strong support (NCTM, 2000), we see it as the responsibility of mathematics teacher educators to prepare secondary mathematics teachers to provide these expectations and supports for their students.

Moreover, Barlow and Cates (2006) suggest, "beliefs affect how teachers see their students…thereby impacting [their] instructional practices" (p. 64). In other words, teachers' beliefs, or conceptions, act as a filter and shape how a teacher structures his/her classroom environment. Researchers suggest there is a strong relationship among teachers' beliefs, personal experiences, and how they teach (Levitt, 2001; Stuart & Thurlow, 2000). Therefore, understanding PSTs' conceptions of equity in the teaching of mathematics enables mathematics teacher educators to provide PSTs appropriate learning experiences, pedagogical tools, and knowledge of others' experiences and backgrounds in order to help minimize the cultural discontinuity and raise expectations. Consequently, the purpose of this study is to understand secondary mathematics PSTs' conceptions of equity.

Theoretical Perspectives

Cultural discontinuity can be described as the cultural mismatch between a teacher's and her students' cultural norms and values (Gay, 2000; Oakes, 1985). Cultural discontinuity is a problem for teachers and students, particularly for teachers with students from minority and low-income backgrounds (Gay, 2000; Lasdon-Billings, 1994). Discontinuity is often evident in teachers' differential treatment and expectations of students based on student demographics. Because many teachers view mathematics as a universal, culture-free subject (Rousseau & Tate, 2003), they do not connect their mathematics instruction with students' culture and background, therefore contributing to this discontinuity. According to Irvine (2003), "Students fail in school not because their teachers do not know their content, but because their teachers cannot make connections between subject-area content and their students' existing mental schemes, prior knowledge, and cultural perspectives" (p. 47).

Methods

This preliminary research took place during a single semester in three different secondary mathematics methods courses at three universities. Each class had 10–12 students enrolled, for a total of 33 participants, with 18 females, 15 males, and over 90% of the students being Caucasian. The participants were a mixture of undergraduate, post-baccalaureate, and master's students.

The authors developed and implemented three tasks that were designed to elicit secondary mathematics PSTs' attitudes, thoughts, and beliefs about issues of equity in teaching mathematics. The first task asked students to respond to the following prompt in 2–3 pages: "'All students can learn mathematics.' Do you agree or disagree with this statement? How do/don't you foresee this playing out in your future classroom? Be specific." The second task required students to respond to quotes related to equity and mathematics. Some examples include: (a) *The way teachers teach mathematics does not send any messages; mathematics is free of context*; and (b) *Teachers have different expectations of their students based off of the students' ethnic and socio-economic background*. The instructor at one university placed the quotes around the classroom, and students recorded their reactions on post-it notes. After responses were recorded, the PSTs summarized the class' reactions, discussed the meaning of each quote, and answered additional questions from the class. The other instructors assigned two quotes per student and asked them: (1) to write their interpretation of each quote in 1–2 paragraphs and (2) to write their reaction to the quote in 1–2 paragraphs. The final task required our PSTs to (1) design an equitable mathematics classroom environment, (2) explain the role of the teacher and the role of the student, and (3) describe why this mathematics classroom is equitable. This was a written assignment at two of the universities and a class discussion activity at the third university.

We individually analyzed the data during and after data collection, which allowed us "to focus and shape the study as it proceed[ed]" (Glesne, 1999, p. 130). Our initial analysis has focused on identifying if and how our students were thinking about race, socioeconomic status, home culture, English learners, and issues of power. Additionally, we looked for how students discussed how they might use race, socioeconomic status, home culture, English learners, and issues of power as resources to support student learning in the mathematics classroom, as well as the role of mathematics in changing or affecting cultural, civic, and social change. We individually wrote analytic memos (Maxwell, 2005) to identify themes within and among each task. We then discussed the themes that seemed most prevalent across the three contexts and tasks.

Results

As stated above, it is important that PSTs are thoughtful about issues related to equity in mathematics. As these three tasks were given to our students, we were hopeful each would elicit our students' thoughts, attitudes, and knowledge about equity and its role in teaching and learning mathematics. The following describes student responses to each task:

All Students Can Learn Task

Students' disabilities, motivation, learning styles, and ability are factors to students learning mathematics

Although the majority of PSTs agreed that all students could learn mathematics, the extent of students' learning is dependent on their disabilities (e.g., dyslexia and dyscalculia), motivation, or innate ability. PSTs responded that these students have the ability and capability of learning mathematics, but instruction must be differentiated to help them learn. PSTs contend, "Some people are math people and other people aren't," therefore "all students can learn math, just not the same math", and "students can learn math if they want to." The PSTs also stated it was the teacher's responsibility to differentiate his/her teaching, consider students' learning styles and strengths, and "make math fun" so students are motivated.

Equity Quote Museum Task

Quote 1: Minority and linguistically diverse students have not been construed as visible players within mathematical discourses either in or out of schools.

PSTs generally interpreted this quote to mean that minority and linguistically diverse students are not active participants in the mathematics classes. Some reasons PSTs mentioned included: (1) a teaching force is mostly Caucasian, which disadvantages minority; (2) these students do not participate because they "struggle asking question, discussing mathematical ideas, and working together;" and (3) "linguistically diverse students often fall behind due to their inability to communicate using the language of the teacher and other students."

Quote 2: The way teachers teach mathematics does not send any messages; mathematics is free of context.

Many PSTs disagreed with this quote because they contend a teacher's enthusiasm sets the tone to learn mathematics, which can either be "math is fun" or "math is boring." Other PSTs disagreed because mathematics should relate to students' lives and involve real world contexts. Real world contexts ranged from general (e.g., different jobs) to specific (e.g., calculating a tip). Lastly, some PSTs agreed with this quote because many textbooks present mathematics as a series of formulas and procedures without and context, and that's how teachers teach it.

Quote 3: "I thought math was just a subject they implanted on us because they felt like it, but now I realize that you could use math to defend your rights and realize the injustices around you… now I think math is truly necessary and, I have to admit, kinda cool. It's sort of like a pass you could use to try and make the world a better place" (a 9th grade student).

When reacting to this quote, PSTs focused on the usefulness of mathematics and how disconnected this student felt in the past. Some PSTs were attuned to the social justice elements mentioned in the quote (e.g., defend your rights and make the world a better place) and discussed the national budget, the impact of war, and the meaning of the national debt in their responses. Yet, other PSTs emphasized the teachers' role is to get students excited about mathematics.

Quote 4: Teachers have different expectations of their students based off of the students' ethnic and socio-economic background.

Many PSTs stated this quote represented an unfortunate truth. They contended that teachers are negatively biased towards students' capabilities based on their ethnic and socio-economic background, and teachers need to look past these stereotypes. Some PSTs also mentioned that these expectations are not necessarily negative and reference that the stereotype of Asian students might lead to unfounded high expectations. Other PSTs responded teachers should have different expectations for their students, but not just base it on ethnicity or SES.

Create an Equitable Classroom Task

PSTs' equitable classrooms included three themes. First, PSTs portrayed the role of the teacher as having high expectations for all students, being a facilitator, and creating a safe environment where students are respected and fully participate. Second, students would likely be sitting in groups, where students could help and learn from each other. Third, although minimal, some PSTs made some comments about gender, race, culture, and English language learners. For example, PSTs mentioned students need to be respectful of each others' race and culture and teachers need to plan lessons that allows all students access regardless of a student's cultural background and/or language strengths.

Discussion

Our study supported that PSTs generally have issues of cultural discontinuity when they consider their own mathematics teaching. In general, our PSTs discussed learning disabilities, inherent ability, learning styles, and motivation. They also discussed that a teacher's role is to be enthusiastic, make mathematics fun, care, foster a safe learning environment, and make mathematics applicable and engaging through context. We learned that, unless explicitly prompted, students do not readily connect phrases such as "all students" or "equitable" to issues of race, gender, culture, language, or socio-economic status. However, our PSTs responses to the quotes indicate some students are thinking deeply and thoughtfully about these topics. However, these ideas of equity in their mathematics classroom do not seem to be deeply embedded into their daily thoughts about teaching and learning.

This study was designed to give us initial insight of our PSTs conceptions of equity in mathematics. This preliminary analysis has confirmed that PSTs need guidance and support in developing these ideas and having them ingrained into their thoughts about the teaching and learning of mathematics.

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UNIFYING CURRICULUM INITIATIVES: TWO CAN BECOME ONE

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In Saskatchewan (a western Canadian province), teachers of mathematics face two major challenges with the implementation of a renewed curricula: the use of teaching and learning approaches based upon constructivist learning theory (CLT), and the infusion of First Nations and Métis (FNM) content, perspectives, and ways of knowing (CPW) into their enactment of the curricula. The teachers are not only being asked to think about the teaching and learning of mathematics differently, but to also find ways to infuse culturally relevant experiences that for most teachers in the province are foreign to their own culture. In this paper, we explore these two challenges facing the province's teachers by analyzing them through the theoretical lenses of two different worldviews, the traditional Western worldview and an Indigenous worldview, leading us to a proposal for how these two challenges might be addressed through one solution.

Keywords: Curriculum; Equity and Diversity; Teacher Beliefs; Learning Theory

In 2006, Saskatchewan began renewing its mathematics curricula. During the renewal process, two initiatives emerged as primary foci: the use of teaching and learning strategies based upon constructivist learning theory (CLT), and the infusion of First Nations and Métis (FNM) content, perspectives and ways of knowing (CPW) into the mathematics classroom. As the renewed curricula started to move from development to implementation, concerns were expressed about the magnitude of the challenge presented by these two initiatives for the province's mathematics teachers. At the same time, many people working on this project expressed the unsubstantiated "gut feeling" that if teachers embrace one of the two initiatives, the second could naturally occur. In this paper, we present a theoretical analysis that has led us to a justification for this "gut feeling" and to the identification of such a connection.

Literature Review

Infusing FNM Content, Perspectives and Ways of Knowing

For more than two decades, policies and action plans have been implemented with the aim to make the educational system more accessible and more responsive to FNM students and communities. One of the factors driving to these actions was the under representation of FNM students in high school science and mathematics classes which "leads to economic, resource management and sovereignty problems for First Nations, Inuit, Métis communities…; and this under representation defines an ethical problem of equity and social justice for the rest of Canada" (Aikenhead, 2006, p. 387). The under representation of FNM students in high school mathematics courses continues to be a concern within Saskatchewan as the percent of FNM students attaining Math 20 (giving them the grade 11 mathematics credit required for graduation) is well below that of their non-Aboriginal counterparts (Saskatchewan Ministry of Education, 2009b). Moreover, the final grades attained by the FNM students who do take Math 20 are consistently and significantly lower than their non-FNM counterparts (Saskatchewan Ministry of Education, 2010). These statistics significantly contributed to the move to infuse all new curricula with FNM CPW, ultimately bringing the renewal of the mathematics curricula into the arenas of ethnomathematics and culturally responsive mathematics education research.

Ethnomathematics research is concerned with the mathematical thinking and doing (including the teaching and learning of it) within cultures, the history and politics of the mathematics used and taught in schools around the world, and the interplay between the two. Researchers in the field of ethnomathematics, such as M. Ascher, U. D'Ambrosio, M. Frankenstein, and A. B. Powell, have demonstrated repeatedly that mathematics is a culturally defined artifact of human life, and that the mathematics taught in schools today is culturally biased. Non-ethnomathematicians who encounter different ways of thinking and doing mathematics, argue that the people and cultures performing these "acts" are not doing "real" mathematics (D'Ambrosio, 1997; Powell & Frankenstein, 1997). Emerging from ethnomathematic's rejection of this deficiency model of viewing other mathematics has led to research into culturally responsive mathematics education in which the findings of the ethnomathematics community are brought to bear within mathematics classrooms.

Constructivist Learning Theory

Within Saskatchewan's renewed curricula, the following statements form the basis of the expectations for the strategies used in the teaching and learning of mathematics: "Any learning in mathematics that is a result of the logical structure of mathematics can and should be constructed by students" (Saskatchewan Ministry of Education, 2009a, p. 15), and the inquiry approach, which is to be foundational in the teaching and learning of all subjects, "is a philosophical approach to teaching and learning, grounded in constructivist research and methods" (p. 23). The notion of students constructing their own mathematical understandings, and the research about teaching and learning strategies and approaches to achieving such construction, fall within the field of constructivist learning theory.

Constructivist learning theory emerged from the work of two researchers: Jean Piaget and Lev Vygotsky. Although the two researchers varied in how they believed children moved from one stage of learning to another, both based their work on the idea that, as humans, we generate our own knowledge through the interaction of our experiences and ideas. Out of Vygotsky's social and Piaget's cognitive constructivist learning theories, CLT strategies for supporting learners in the gaining of new knowledge have been emerging within research. These strategies vary from broad approaches to learning, such as inquiry and teaching through problem solving, to more specific methods, such as the use of children's literature, multiple representations, group discussions and the creating of open learning environments and experiences that allow students to enter into a dialogue and learn in ways that honour their past experiences and knowledges.

Theoretical Lenses

One's worldview, the overall schema through which one views and lives in the world, informs how one approaches all aspects of life, including mathematics and the teaching and learning of mathematics. We now consider two significantly different worldviews, or lenses: the traditional Western worldview and an Indigenous worldview.

The Traditional Western Worldview

The traditional Western worldview (TWW) is the foundation of Western society's knowledge, systems, and ways of being. Within this worldview, what knowledge and ways of knowing are of value corresponds to the following characteristics: knowledge is linear, singular, static and objective in nature, resulting in one correct answer and one right way of achieving it; specialization is an indication of greater knowledge; knowledge is gained through the Scientific method; measurability is essential to ascertain the truth; knowledge relates to physical objects and processes that are external to the individual; compartmentalization, isolation and categorization of knowledge is necessary for true understanding; and knowledge must be captured in written form in order for the truth of it to be maintained (Kovach, 2009; Little Bear, 2000).

An Indigenous Worldview

Although there is no one Indigenous worldview (IW), as each Indigenous group has fundamental differences from each other (Kovach, 2009; Little Bear, 2000), there are common characteristics across Indigenous groups that allow for the presentation of an overarching and broad Indigenous worldview. An Indigenous worldview holds that: knowledge is meaningful in terms of the place in which and for which it is attained; knowledge emerges through relationships (physical, social, emotional, and intellectual)—to people and to the physical and spiritual world (and beyond); knowledge is subjective (as well as objective) in nature; diversity in ways of knowing and of knowledge are valued and the individual is valued for what they contribute to the group; observation, personal experience and intuition are valid sources of knowledge; knowledge should be sought and gained in order to give back to the greater whole; and, truth can be captured and kept within oral language (Kovach, 2009; Little Bear, 2000).

Analysis

Infusion of FNM CPW Through the "Eyes" of the TWW and an IW

The infusion of FNM CPW into the teaching and learning of mathematics would necessarily imply that there are alternative approaches, even alternative answers, to mathematical questions and inclusion of contexts. Such an attempt would necessarily come in conflict with the notions of the singularity, objectivity, abstract and static nature of knowledge, which are valued characteristics of knowledge within the TWW. For a teacher grounded in TWW, the infusion of FNM CPW would at best seem a waste of time, and at worst a corruption of mathematics.

Inherent within an IW is the acceptance of diversity in the ways of knowing and in knowledge, the valuing of the place of the knowledge, and the direct connection between knowledge and relationships. In these ways, among others, a teacher grounded within an IW would not only see the importance of the infusion of FNM CPW, but would naturally seek the inclusion of any CPW that would honour the diversity of their students' cultures, experiences, and individuality.

CLT Strategies Through the "Eyes" of the TWW and an IW

For a teacher grounded in the TWW, alternate approaches to the teaching and learning of mathematics, such as CLT strategies, would seem frivolous, even contradictory, to how they know knowledge is and how it can be attained. Within the TWW, having children attempt to construct knowledge would be a waste of time, since knowledge is imparted from those who are already specialists. Thus, the very foundation of CLT strategies (students constructing knowledge rather than receiving it) cannot be fit into the TWW. As a result, teachers grounded in this worldview would not readily engage in the use of CLT strategies.

Being based in the belief in an individual's ability to construct knowledge, the CLT strategies endorse the IW valuing of the diversity of knowledge and ways of knowing. In addition, the CLT strategies focus on starting where the student is, the place of their knowledge, another parallel to an IW. Although an IW does not explicitly seek abstract knowledge, it also does not deny the possibility of such knowledge, moreover, it supports the connection of such knowledge to the place and relationships used in creating it.

Discussion

Based upon our (albeit brief) analysis, we now draw two conclusions: that there is a connection between the two curricular initiatives challenging teachers of mathematics in Saskatchewan and that the two initiatives become one through a change in worldview. Through the lens of an IW, the infusion of FNM CPW and the use of CLT strategies are intrinsically connected. From this perspective, one can see that the foundational beliefs for both initiatives are, in fact, the same. In both cases, the teaching and learning of mathematics is based upon the knowledge and ways of knowing that the students bring to the classroom and not upon pre-described steps and procedures. As well, both initiatives value the differences brought to the table by all individuals in the learning process. Finally, both approaches to the teaching and learning of mathematics are based upon context, place, and meaning for the learner.

On the other hand, both the use of CLT strategies and the infusion of FNM CPW are in contradiction to, or, at best, irrelevant within the TWW. The foundational beliefs of the TWW, in particular that knowledge of value is linear, static, objective, abstract, compartmentalized, and singular, all conflict with both of the curricular initiatives.

It would not be a far leap for one to conclude that when grounded in an IW, the teaching and learning of mathematics would be open to the use of CLT strategies as well as the infusion of FNM CPW. Alternatively, one can easily conclude that when grounded in the TWW, neither initiative is likely to be successfully engaged in. As a result, we contend that by grounding the teaching and learning of mathematics within an IW, the two curricular initiatives become one—that of embracing a different worldview. Moreover, we refer to this approach to the teaching and learning of mathematics as the *transreform approach*—an approach that not only reforms how mathematics is taught, but what is valued and acknowledged within a mathematics classroom. Finally, the transreform approach to the teaching and learning of mathematics is not just possible and desirable for FNM students and their teachers, rather such an approach would be supportive of all students, regardless of their ethnic, cultural, or experiential backgrounds.

Acknowledgments

Although we present these ideas for the first time in written (academic) form, there are unrecognized giants whose shoulders have given us the vantage point we have used—the ancestors of the Indigenous peoples, who over the millennia have created, refined, and shared openly their Indigenous worldviews. We hope that in doing this research and writing, we might begin to give back and contribute to the ongoing efforts of decolonization.

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TEXT RELEVANCE AND LEARNING FROM UNDERGRADUATE MATHEMATICS TEXTBOOKS

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We consider the role of text relevance in formulating an explanation for why undergraduate students do not read large parts of their beginning mathematics textbooks. In a previous paper (Shepherd, Selden, & Selden, in press), we asked why it is that good readers, who were also good at mathematics, did not read large parts of their beginning mathematics textbooks ***effectively****, that is, why they could not work straightforward tasks based directly on that reading. Here, we reanalyze that data in terms of text relevance to consider the role that students' personal implicit or explicit goals may play.*

Keywords: Post-Secondary Education; Instructional Activities and Practices; Curriculum

Introduction

The concept of text relevance has been proposed by reading researchers because readers "need literacy skills beyond those required for the comprehension of simple texts; they need to be able to identify, understand, and integrate ideas within and across documents." *Text relevance* refers to "the instrumental value of text information for enabling a reader to meet a reading goal" (McCrudden, Magliano, & Schraw, 2011, p. 2).

Only a little research seems to have been done on how students read their mathematics textbooks, but having to read one's mathematics textbook is an important and difficult transition to learning mathematics independently in college. Osterholm (2008) surveyed 199 articles on the reading of word problems, but found little about reading comprehension of more general mathematical text and did not consider text relevance. Weinberg and Weisner (2010) have introduced a framework for examining students' reading of their mathematics textbooks. While we see this framework as proving a useful perspective, in our previous research (Shepherd, Selden, & Selden, in press) we took a different perspective and considered whether, and how, students construct meanings very close to those of the author and mathematical community.

Research Question

A consideration of text relevance and individual goals rather naturally brings up the question: What are the reading goals of typical undergraduate students when reading their beginning mathematics textbooks? We considered this question somewhat indirectly in a previous study (Shepherd, Selden, & Selden, in press). In that paper, we asked why it was that good readers, who were also good mathematics students, did not read large parts of their mathematics textbooks *effectively*, that is, why they could not work straightforward tasks based on that reading. We now reconsider our data in terms of text relevance. In particular, we ask: What does the concept of text relevance have to offer in terms of explanatory power when analyzing why university students do not read their beginning mathematics textbooks effectively?

Our Previous Research on Students' Reading

Eleven volunteer precalculus and calculus students, who attended a U.S. Midwestern comprehensive state university, were interviewed. According to their ACT reading comprehension and mathematics scores, as well as according to their mathematics instructor, they were good at both reading and mathematics.

The interviewees each read aloud a new section of their respective textbooks, one selected by their instructor. These passages were selected because the students would be familiar with the notations and prior definitions used and because the students were judged to have the necessary prerequisites for reading them. The precalculus students read about "The Wrapping Function" from Barnett, Ziegler, and Byleen

(2000, pp. 336–343). The calculus students read about “Extrema on an Interval” in Larson, Hostetler, and Edwards (2002, pp. 160–164). Along with definitions, theorems, examples, figures, and discussions, the precalculus book had “Explore/Discuss” tasks and the calculus book had “Exploration” tasks to encourage students to become active as they read.

The interviewees were stopped at intervals during their reading and asked to try a task based on what they had just read, or asked to try to work a textbook example (task) without first looking at the provided solution. These were the places that the textbook authors would probably have assumed readers would independently pause for such activities. The tasks were straightforward ones based directly on the reading and required very little in the way of problem-solving skills. They were what might be called “routine exercises.” After the entire section had been read and a few final tasks were attempted, the students were questioned about how reading during the interview differed from their normal reading of their mathematics textbooks. For further details, see Shepherd, Selden, and Selden (in press).

All of the students in our study had considerable difficulty correctly completing some of the straightforward tasks based on their reading. The percent of tasks done correctly by individual students ranged from 13% to 76%. The students had trouble reading and understanding definitions and using theorems. All read the expository parts of the textbook since that was part of the interview, but upon questioning at the end, some students viewed exposition as of minor importance—something often to be skipped or skimmed. The students stated that they normally wanted to concentrate on exercises (tasks) and find similar worked examples in their textbook.

In our previous paper, we drew on the psychology literature on “zoning out” during reading (e.g., Smallwood, Fishman, & Schooler, 2007) to suggest that *cognitive gaps*, that is, periods of lapsed or diminished focus, during reading may explain some of our students’ ineffective reading. In this paper, we reexamine our data in terms of text relevance to consider whether our students’ reading behavior, may have been greatly influenced by their personal implicit or explicit goals.

Concepts of Text Relevance

There are a number of concepts that text relevance researchers have considered: goals, working memory capacity, standards of coherence, and academic purpose. We will consider these, in turn, to see how they might apply to our data on undergraduates’ reading of their precalculus and calculus textbooks in an interview setting.

Goals

Reading in instructional settings is often task-induced, and readers’ goals may affect their inferential processes while reading. Tasks can impact “how people judge information’s relevance to their goals and the strategies that they enact to meet their goals.” Readers’ goals can also affect their online processing (e.g., their strategy use and their attention allocation) as well as their offline products (e.g., their learning from, and memory of, the text) (McCrudden, Magliano, & Schraw, 2011, pp. 3 -4).

In addition, reading goals are the outcome of a complex interaction between external intentions and personal intentions. “Specific relevance instructions [can] prompt readers to focus on discrete text segments … whereas general relevance instructions [can] prompt readers to read for a general purpose (e.g., to read for study).” Also, because people have limited working memory capacity, they will devote more resources to relevant stimuli and fewer resources to less relevant stimuli. Skilled readers “can achieve optimal cognitive efficiency by formulating reading goals and developing criteria for determining information’s relevance to those goals.”

The above ideas naturally bring up the following potential research question: In reading their mathematics textbooks, do students attempt to achieve optimal cognitive efficiency by looking for sample worked problems to mimic, rather than by first reading the entire section? We do not have an answer to this question, but one of our students’ proffered comments is suggestive. Zoe said, “Usually I will read in between if it looks like it’s important, but if it just looks like it’s fluff, or explaining it, and I already understood it—like, I understood the definition pretty well [referring to her reading during the interview] so maybe I wouldn’t have read. ...”

This comment, and similar ones, seem to indicate that our students did not consider it worthwhile to read the entire section before attempting their homework exercises, especially the exposition at the beginning, if having them complete the exercises correctly was the instructor's goal (as perceived by the students), and they wanted to complete their homework assignments as quickly and completely as possible in order to get good grades.

Working Memory Capacity

It has been shown that "a reader's working memory capacity (WMC) affects online processing when they read for different purposes. …" Furthermore, working memory resources are important because they also allow the reader to integrate ideas across sentences, a process that involves the maintenance of previously translated [i.e., comprehended] text as attention is focused on new information that must be processed" (Linderholm, Kwon, & Wang, 2011, p. 201). Although we had information on our students' ACT reading comprehension and mathematics scores, we did not have information on their working memory capacities as measured by the reading span task (RST) (Daneman & Hannon, 2007), so it is possible that knowledge of their working memory capacities might have given us some additional insights.

Standards of Coherence

Standards of coherence refers to the types and strength of coherence that an individual reader aims to maintain during reading. These can be implicit or explicit and reflect that individual's desired level of understanding for a particular reading situation, and influence "the dynamic pattern of automatic and strategic cognitive processes that take place during reading. They [standards of coherence] are influenced by various aspects of the reader, the text, and the task" (van den Bock, Bohn-Goettler, Kenedou, Carlson, & White, 2011, p. 125).

Characteristics of the text include the "specific content of the text, the order in which the content is presented, gaps in the semantic flow, layout, [and] the presence of text signals such as titles and italics" (van den Bock et al., 2011, p. 125). We did not question our students about why they normally read their textbooks as they did, but one calculus student volunteered that the textbook was not "clear" and another referred to parts of the calculus textbook as "jibberish."

Characteristics of the reader include working memory capacity (discussed briefly above), and inadequate or insufficient prior knowledge (discussed in our previous paper). Readers can adopt standards of coherence that are incomplete, but adequate (in the eyes of the reader). They "at times may pursue less-than-maximal coherence. … Such 'good-enough' processing is not necessarily a matter of laziness but may be a matter of efficiency" (van den Bock et al., 2011, pp. 128–129).

Because they were asked to do so, our students read parts of the textbook, such as the exposition, that they said they would not normally have read. Winnie, one of the calculus students, commented in her final debrief, "I don't think it's [the textbook/author] clear a lot of times like, I don't know. It helps me if I see an actual [worked] example [like the homework exercise] that we're actually going to be working on."

We conjecture that our students did not have adequate standards of coherence, as their goal in the course was, very likely, to complete the homework exercises as quickly as possible.

Academic Purpose

Some reading researchers consider both academic purpose and goals, and view purpose as overall goals for reading, such as the one that our students probably had, namely to fulfill a requirement and to get a good grade, whereas specific reading goals are seen as being more relevant to the task at hand. For our students, that seems to have been to complete the assigned homework exercises as quickly and efficiently as possible. As Tara said in her final debrief interview, "I don't usually get the reading done before class, because I'm trying to do homework from the day before." In general, "Given the clear impact of reading purpose on reading process, reading purpose instruction should be included in the curriculum of those reading for academic purposes" (Linderholm, Kwon, & Wang, 2011, p. 219). However, given that the instructor of our precalculus and calculus students had provided our students some carefully considered instruction in the reading of their textbooks (Shepherd, 2005), this is no straightforward task.

Concluding Remarks

The ineffective reading we observed in our students had to do with *not* being able to consistently correctly work straightforward tasks, immediately after reading how to work them. It could be that our students were simply not accustomed to reading their textbooks in order to find out how to work tasks, but rather depended greatly on their instructor to illustrate such methods during class. Or perhaps a lack of self-efficacy led them to believe reworking tasks would not produce new results.

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WHAT'S THE DIFFERENCE BETWEEN DOUBLING AND SQUARING?: INVESTIGATING STUDENTS' DEVELOPMENT OF POWER MEANINGS OF MULTIPLICATION

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This paper reports on a clinical interview study conducted with 14 sixth grade students. During the study students were presented with Cartesian product, arrangement, and combination problems. The purpose of using these combinatorics problems was to investigate in what ways such problems could support students' development of a meaning for raising a quantity to a whole number power—a power meaning of multiplication. Two results of the study were: (1) a framework for investigating students' multiplicative reasoning in contexts that can potentially support their development of a power meaning of multiplication; and (2) a comparison of explanatory constructs necessary to explain students' multiplicative reasoning in contexts that can potentially support a linear operator meaning of multiplication and contexts that can potentially support a power meaning of multiplication.

Keywords: Cognition; Middle School Education; Number Concepts and Operation

Students commonly confuse a linear operator meaning of multiplication (i.e., doubling, tripling, etc.) and a meaning for multiplication that involves raising a quantity to a whole number power (i.e., squaring, cubing, etc.)—a power meaning of multiplication (Van Dooren, De Bock, Janssens, & Verschaffel, 2008). This confusion has been expressed in researchers' findings in a range of ways: as students "errors" when working with algebraic symbols like overusing squaring in notation to express doubling or tripling (MacGregor & Stacey, 1997) and concluding that "$(x + y)^2 = 2x + 2y$" or that "$(x + y)^2 = x^2 + y^2$" (Matz, 1982); and as students difficulty in differentiating linear and square units like confusing perimeter and area (Simon & Blume, 1994).

Despite the well documented difficulties that students experience with a power meaning of multiplication, K–8 curricula overwhelmingly contain problems that are aimed at supporting students' development of a linear operator meaning of multiplication (Confrey, 1994; Van Dooren, De Bock, Janssens, & Verschaffel, 2008); often one of the first times students experience problems that have the potential to involve a power meaning of multiplication is when they take their first algebra course where curricula include solving quadratic equations and reasoning about quadratic functions. This trend in curricula along with the findings outlined above raise the following questions:

(1) What is the relationship between students' development of a linear operator meaning of multiplication and a power meaning of multiplication?
(2) Does helping students develop a strong linear operator meaning of multiplication support their subsequent development of a power meaning of multiplication? If so, how?
(3) Do students need problems or experiences that are not emphasized in current curricula to develop a power meaning of multiplication?

Responding to these questions has the potential to profoundly impact the way that K–8 curricula are structured (Confrey, 1994), and to inform current discussions among researchers about what kinds of experiences students need prior to taking algebra courses in order to be successful in them (e.g., Kaput, Blanton, & Moreno, 2008). The purpose of this paper is to respond to these questions by outlining a framework that establishes how students' multiplicative reasoning differs in contexts that have the potential to support their development of a linear operator meaning of multiplication and contexts that have the potential to support their development of a power meaning of multiplication.

Two Meanings of Multiplication

One common problem that K–8 curricula use to help students' develop a linear operator meaning of multiplication—a meaning where one quantity acts or operates on the other—are equal groups problems like the Donut Problem.

> *The Donut Problem:* There are 4 packages of donuts. Each package has 6 donuts. How many donuts are there total?

The reason these problems have the potential to support students' development of a linear operator meaning of multiplication is because they can involve students in creating composite units and repeating these composite units to determine the total number of donuts (Steffe, 1992, 1994). In doing so, a student treats the quantities in the problem asymmetrically—one quantity acts or operates on the other.

In contrast to equal groups problems, researchers have identified that combinatorics problems like the Outfits Problem have the potential to support students' development of a power meaning of multiplication (Behr, Harel, Post, & Lesh, 1994; Confrey, 1994; Vergnaud, 1983).

> *The Outfits Problem:* There are 4 shirts and 3 pants. An outfit is 1 shirt and 1 pants. How many possible outfits can you make?

One reason that problems like the Outfits Problem can support students' development of a power meaning of multiplication is that in the statement of the problem the two quantities (shirts and pants) are symmetric—one quantity is not explicitly identified as the quantity that acts or operates on the other. A second reason that problems like the Outfits Problem can support students' development of a power meaning of multiplication is the outfits can be created from pairing one shirt with one pants. Pairing one shirt with one pants to create an outfit means that the outfits are a unit that contains two units, but they are counted as a single unit (Vergnaud, 1983). The formation of this type of unit contrasts with the units, donuts, which are counted as one in a problem like the Donut Problem. In fact, the formation of outfits can be symbolized as $1 \times 1 = 1$ (i.e., one squared equals one).

In order to investigate the difference between students' linear operator meanings of multiplication and power meanings of multiplication, I conducted an interview study with 14 sixth grade students. The interview study consisted of three interviews. The first interview was a selection interview whose purpose was to identify which of three multiplicative concepts students used to solve problems that I deemed were likely to support their understanding of a linear operator meaning of multiplication. The goal of the second and third interviews was to make models of the students' multiplicative reasoning in combinatorial contexts in order to identify whether and how the students created different units in these contexts. In the next section I outline the framework that I used to identify the multiplicative concepts that the students were using in the selection interview.

Conceptual Framework

Schemes

A *scheme* has three parts—an assimilatory mechanism, an activity, and a result (von Glasersfeld, 1995). The assimilatory mechanism involves a person in making an interpretation of a problem. The assimilatory mechanism triggers the activity of a scheme, in which a person carries out mental operations on physical or mental material or both. The activity, then, produces a result. When a person can use the result of a scheme in assimilation of a future problem without carrying out the activity that produces it, the person has *interiorized* the scheme and constructed a *concept* (von Glasersfeld, 1995).

Framework for multiplicative reasoning. Steffe (1992, 1994) has identified two mental operations—iteration and units coordination—that are central to the schemes that students' construct in their solutions of equal groups multiplication problems. *Iteration* involves the repetition of a unit and *units coordination* involves the insertion of units within units. He has used differences in how students use these two mental operations to identify three qualitatively distinct multiplicative concepts that students construct.

Students who use the first multiplicative concept (MC1) to solve equal groups problems like the Donut Problem by iterating a unit of one six times and inserting these six units into a containing unit to create a unit of six units in activity (i.e., one package of six donuts). They would continue with this activity, iterating six more units of one and inserting these six units into a containing unit to create another unit of six units in activity, as they continue to solve the problem. One behavioral indicator that a student is engaged in this activity is that they coordinate two counts as part of their solution of an equal groups problem. So a student could count: one, two, three, four, five, six that is one package; seven, eight, nine, ten, eleven, twelve that is two packages, etc.

Students who use the second multiplicative concept (MC2) to solve equal groups problems have interiorized the activity just described. Therefore, they can assimilate equal groups problems using a unit of units structure and operate on this structure as part of their solution of a problem. This means that to solve a problem like the Donut Problem they are likely to iterate a unit of six units four times, and as they iterate a unit of units structure they are likely to use strategic reasoning to combine the sixes together. For example, these students can reason that six and six is twelve because six and four is ten and two more is twelve. In doing so, they reasoned that the second unit of six units was composed of a unit of four units and a unit of two units, and used this to strategically combine it with the first unit of six units. To finish solving this problem, they are likely to continue this type of strategic reasoning to determine that the result of the problem is twenty-four. At this point, these students can insert the four units of six units into a containing unit to create a unit of four units of six units in activity.

Students who use the third multiplicative concept (MC3) to solve equal groups problem have interiorized the activity just described. Therefore, they can assimilate equal groups problems using a unit of units of units structure and they can operate on this structure as part of their solution of a problem. Because these students have interiorized a unit of units of units structure, they no longer have to engage in iteration or units coordination to solve equal groups problems. Instead they can solve an extension of the Donut Problem that involves twelve packages of six donuts by reasoning that ten packages of six donuts is sixty donuts and two packages of six donuts is twelve donuts so there are a total of seventy two donuts. In solving the problem in this way, they reasoned that a unit of 12 units of 6 units is composed of a unit of 10 units of 6 units and a unit of 2 units of 6 units, evaluated each of these structures using multiplication, and combined the result of each multiplication problem to get 72 donuts.

Methods and Methodology

The study was conducted at an urban middle school in partnership with a sixth grade teacher. The research team, which consisted of three researchers, spent the first three months of the school year observing and co-teaching with the sixth grade teacher in her regular classroom. The purpose of these activities was twofold: (1) to make an initial determination about which of the three multiplicative concepts students were using; and (2) to build trust between the research team and the students who were potential subjects of the interview study. At the end of the three-month period 14 students agreed to participate in the interview study. 11 of these students were African American, 2 were Hispanic, and 1 was Caucasian, 5 were boys, and 9 were girls. This demographic breakdown was representative of the demographic breakdown of the entire sixth grade population as well as the population of the school.

The interview study consisted of a sequence of three interviews. Students were interviewed individually in order "to collect and analyze data on mental processes at the level of a subject's authentic ideas and meanings" (Clement, 2000, p. 547). The first interview was a 45-minute non-video recorded selection interview, which was used to identify which of the three multiplicative concepts the students were using; 3 students were identified as using the first multiplicative concept, 6 were identified as using the second multiplicative concept, and 5 were identified as using the third multiplicative concept.

The second and third interviews were 45-minute video recorded interviews. The second and third interview protocols consisted of a sequence of increasingly difficult combinatorics problems, which included Cartesian product, arrangement, and combination problems. The initial findings from the study will be discussed as part of the presentation.

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ALGEBRAIC REASONING THROUGH COMBINATIONS AND PATTERNS: BILINGUAL LATINAS "EXPERIENCING" MATHEMATICS

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Research on gender and mathematics has often compared female and male students' attitudes, performance, achievement, and confidence levels. Although more recent research has moved beyond this dichotomy to explore how female students experience mathematics, it often places race, class, and nationality as secondary to gender subordination. This paper sets out to explore how group interactions, within the context of an afterschool mathematics club, mediate the way bilingual Latinas engage in and experience mathematics. Qualitative analysis of transcripts reveal, in same gendered groups (a) equal participation status when multiple languages and strategies are privileged; (b) exclusion of particular students when one language resource and certain strategies are privileged; and (c) the facilitators key role in shaping these interactions.

Keywords: Equity and Diversity; Gender; Algebra and Algebraic Thinking

Gender and Mathematics

Much of feminist analysis places race, class, and [nationality] as secondary to gender subordination (Wolf, 1996). Within the U.S., few studies explore the power relations of gender with race, class, and language around mathematical meaning making. Much of what we do know about gender and mathematics results from research comparing female and male students' overt behaviors such as answers on a mathematics test, items on a confidence scale, student-teacher interactions, or students' career decisions (Fennema, 1996). Although providing rich information, this approach is limited. Highlighting differences can be taken to be concluding that females are less able in mathematics (Fennema and Hart, 1994), mark males as "the standard" against which females are measured, can lead to programs seeking to fix perceived "deficits" (Kaiser & Rogers, 1995), and fail to recognize existing disparities within these gendered groups. More recent research investigates how females construct mathematical knowledge and experience mathematics (Fennema, 1996). This is consistent with a feminist perspective in mathematics education, which entails the question "how do women experience mathematics?" be addressed (Damarin, 1995); where the mathematics content itself takes a peripheral role. Viewing mathematics learning as a socially constructed process and recognizing that students tend to voluntarily work in same gendered groups, this paper sets out to explore how group interactions, in an afterschool context, around activities developing algebraic thinking through combinations and patterns, mediate the way bilingual Latina's experience mathematics.

Theoretical Framework

Through a sociocultural framework and perspectives from feminist standpoint theory and the politics and epistemology of location, this study examines combining sociopolitical factors influencing students' 'experiencing' of mathematics. Sociocultural theory suggests that development is mediated through social interaction, where language and dialogue are central to ones overall development (Vygotsky, 1986). This was an important aspect of the afterschool program, as participants collectively engaged in meaningful activities while learning higher-level mathematics (Khisty, 2004). Feminist standpoint theory suggests the existence of subjectivity through agency and authorship (the 'who' and 'what' of mathematical learning) (Burton, 1999), where females' voices, mathematical and nonmathematical experiences, and unique perspectives are valued (Fennema, 1996). My, as researcher, "locationality (historical, national, generational)" and "positionality (race, gender, class, nationality)" and the acknowledgment of how "the dynamics of where we are always affects our viewpoint and the production of knowledge" (Wolf, 1996,

p. 14), is informed through the "politics and epistemology of location" (Haraway, 1991). These perspectives inform my viewpoint on teaching and learning, informed by various positionalities and the process of reflexivity.

Los Rayos de CEMELA

An afterschool mathematics club, Los Rayos, was developed as part of the Center for the Mathematics Education of Latinos/as (CEMELA) at the University of Illinois at Chicago and in partnership with a Chicago public elementary school located in a working class and predominantly Mexican/Mexican-American neighborhood (Khisty, 2004). Los Rayos, an adaptation of The Fifth Dimension (Cole, 2006) and La Clase Mágica (Vásquez, 2003), took a dual-language (Spanish-English) approach to its non-remedial curriculum (e.g., pre-algebra, probability, and proportional reasoning). Over a three-year period, mainly one volunteer cohort of Latina/o students participated from 3rd to 6th grade. Approximately 14–20 Latina/o students participated, with a fairly equal number of females and males. Sessions ran twice a week for ninety minutes each session, where students had the autonomy to choose a group to work with, a mathematical task to work on, and the language(s) to work in.

Methods and Data Analysis

I focus on groups of female students (9 females total) and facilitators, ranging from combinations of 2 students and 1 facilitator to 4 students and 1 facilitator. All identify as bilingual (Spanish-English). Primary data consist of videotapes of individual interviews and student participation. Secondary data consist of student work and facilitator's field notes. The research methodology of critical ethnography, an in-depth study of a cultural group, where education is seen as political in nature (Trueba, 1999), supports the investigation of multiple aspects of student identity, including language use, in generating mathematical understanding. Focus is placed on dialogue and language use since in feminist research, language, as competing ways of giving meaning, is the place where subjectivity is constructed (Richardson, 2000). This resulted in 5 episodes of group interactions around tasks developing algebraic thinking using combinations and patterns ranging in time of 33 minutes to 94 minutes. These were informed by student interviews and student work (valuing these students' voices), and facilitator's field notes. Transcripts were coded for shifts in student participation, language use (Spanish, English, or both), and under what conditions (when, how, and how often) allowed for optimal student participation and use of multiple language resources.

Findings

Qualitative analysis of transcripts reveal (a) equal participation status in same gendered groups when multiple languages and strategies are privileged; (b) exclusion of particular students in same gendered groups when only one language resource and certain strategies are privileged; and (c) the facilitator's key role in shaping these interactions.

Equal Participation Status

The dynamic of same gendered groups often led to rich interactions and the equal participation of all group members. Here the facilitator acknowledged, valued, and built off of students' language resources and mathematical strategies. An example is when Cara, Miriam, Karmen, and Neyreda were working on a task: I know that you want to give your mother a floral gift (for Mother's Day) and you have $20 to spend. What flowers would you buy? [Priced: roses $2.50; margaritas $0.29; tulips $0.45; and basket $1.29] How many different combinations can you make with $20? Students' positionalities (gendered practices, class, and nationality) were key and allowed for familiarity with Mother's Day, flower arrangements, and $20 as a suitable price for an arrangement at the local floral store.

Cara: I already have five dollars.
Facilitator: Okay put more.

Karmen: Okay I want three of these…no, five of these, seven of these.
Facilitator: Good. [As she watches what Karmen is jotting down]…
Neyreda: Can I just spend this much?
Facilitator: No you have to do more…. *How much more do you need Neyreda?*
Neyreda: I don't know.
Facilitator: Let me see. [Picks up her paper to see strategy used]
Miriam: I have fifty-seven dollars and three cents!
Facilitator: How much?
Miriam: Fifty-seven dollars and three cents! *I bought three of each one…*
Facilitator: Look Karmen this is what you have right now: nineteen with nine.
Miriam: You can buy, you can buy one of each: nineteen zero nine plus each one.

Cara, Karmen, Neyreda, and Miriam devised individual and shared strategies (starting with a combination of flowers (using addition and multiplication); adding flowers to the combination; and applying subtraction strategies as needed [see English, 2003])—while using both Spanish (as shown italicized) and English. The facilitator drew on each student's strategy by asking for an explanation, clarification, or to see each student's work. The facilitator then valued and capitalized on individual strategies through appraisal words (Good) and on each student's language resource by using a similar language as students addressed her. This resulted in socially constructed algebraic reasoning where students' different combination outcomes were within cents of the $20 limit (Cara: $19.80; Karmen: $19.99; Neyreda: $19.94; and Miriam: $19.60).

Exclusion of Students

When the facilitator did not capitalize on all students' language resources and problem solving strategies, it led to the partial or full exclusion of certain students. An example is when Yolanda, Cara, Karmen, and Neyreda were working on a task: Juanita has pennies, nickels, and dimes in her purse. She has eight coins altogether, including more dimes than nickels, more nickels than pennies, and fewer pennies than nickels. What are the two different amounts of money she could have? Students' positionalities (gender, race, and national) were key and allowed for connecting with Juanita (in task) and being able to identify U.S. coins.

Yolanda: So it looks like she has more dimes than anything.
Facilitator: Yeah she has more dimes and she has eight all together.
Cara: Eight, five plus…
Facilitator: No eight coins, not eight cents. *Eight coins, she has eight coins*. Yes she has eight, so we have to figure out how many dimes…*How many coins of 5 are there, how many coins of 10 are there, so that there are 8 in total and that there are more of 10 than 5 and more of 5 than 1*. More dimes than nickels, more nickels than pennies.
Cara: [Places a number of coins down and then counts] ten, twenty, thirty, forty, forty-five, fifty, fifty-five, fifty-six.
Nadia: What are we doing?
Facilitator: Do you want to read it in Spanish? [looks at Neyreda and Karmen] *Let's read it in Spanish*. Do you need help Yolanda?
Yolanda: No I'm okay I got it.

In this example, Yolanda and Cara quickly got involved with the task. The facilitator solely used English at the beginning of the task, which excluded Karmen (who identifies as a more dominant Spanish speaker). And although, Yolanda was at first engaged, the facilitator was more focused on Cara (acknowledging her strategy and shifting to using mainly Spanish (as shown italicized)). Yolanda (identifying as a more dominant English speaker) became excluded. What resulted was that Yolanda and Karmen, although listening attentively, did not fully participate until another facilitator intervened. By failing to recognize the language resources each student felt most comfortable with and failing to acknowledge these students' strategies, led to the exclusion of students, no collective meaning-making, and loss of valuable student input.

Discussion

The afterschool program promoted bilingualism, higher-level mathematical thinking, and student autonomy through meaningful activities. As students tended to work in same gendered groups, this seemed to have a dual-effect. First, students engaged in rich mathematical meaning making where they collectively constructed mathematical knowledge around the decided tasks. This rich dialogue occurred particularly when the facilitator capitalized on students' language resources and problem solving strategies. On the other hand, even within same gendered groups where females had much to talk about and much in common, particularly with shared gender, race, and class, the differences became evident in terms of language and abilities. When the facilitator failed to use, acknowledge, and capitalize on students' language resources and problem solving strategies, this led to the exclusion from full participation of certain students. This study points to the importance of moving beyond the female-male dichotomy; beyond race, class, and nationality as secondary to gender; and beyond individual students as learners devoid of context. Further research is needed addressing approaches to investigating the complexities involved in gendered groups experiencing mathematics.

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A FRAMEWORK FOR ANALYZING MATHEMATICS LECTURES

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Despite efforts to make college classrooms more inquiry-based, typical college students still spend approximately 80% of their time in class listening to lectures (Armbruster, 2000). Consequently, it is important for us to develop an understanding of the opportunities students have for making sense of lectures in upper-level mathematics courses, where the focus is on definitions, proofs and examples. To do this, we propose to view the lecture as a "text" and describe both the aspects of the text and the students' beliefs that affect this sense-making process. This brief research report describes previous work in this area and outlines a framework for analyzing lectures.

Keywords: Instructional Activities and Practices; Post-Secondary Education

Introduction and Research Goals

Despite efforts to make college classrooms more inquiry-based, typical college students still spend approximately 80% of their time in class listening to lectures (Armbruster, 2000). Lecture listening is a difficult cognitive task for college students (Ryan, 2001), but it is important, since what students take away from lectures is closely linked to what they learn (e.g., Titsworth & Kiewra, 1998). In upper-level mathematics classes, where the focus is on definitions, proofs and examples, it is particularly important for us to develop an understanding of the aspects of lectures that shape the ways students are able to make sense of—and learn from—lectures.

The goal of our research is to develop and use a collection of frameworks to describe the various factors that influence and constrain students' opportunities for making sense of mathematics lectures. Specifically, our research questions are to identify:

1. The components of a lecture and how these relate to the instructional goals of the lecture.
2. Students' *observing models*: their beliefs about observing, taking notes during, and learning from lectures
3. The components of the *implied observer* of a lecture, how these components affect students' opportunity to learn, and how they relate to students' own lecture notes

Lecture Components

Lectures contain numerous components that students must attend to and interpret. These components can be broadly distinguished by their mode of presentation: written, spoken, and gestural; following Shein (2012), we further classify gestures as *pointing* (drawing attention to an aspect of the lecture) and *representing* (representing an object, process, or relationship). Beyond these modes, we divide lecture components into *communicational* and *mathematical* aspects.

Communicational aspects of a lecture include organizational cues, immediacy, and temporal-spatial components. Titsworth and Kiewra (1998) defined *organizational cues* as "detailed transition statements and signposts indicating macro- and microelements of lecture organization." We can separate these cues by their *temporal focus*: whether they are organizing transitions from moment-to-moment, from day-to-day, and from unit-to-unit or class-to-class. *Immediacy* refers to instructors' attempts to reduce the social distance between themselves and their students through the use of gesture and tone, use of the first-person plural (e.g., "*we* have noticed…"), and conveying a sense of care and well-being. In addition, instructors may write notes in a non-linear fashion, structuring their board-work to make connections to previous ideas, adding a *temporal-spatial* component to the lecture.

Mathematical aspects of a lecture include facts, procedures/algorithms, and processes. *Facts* include definitions, examples, and results of mathematical theorems. *Procedures and algorithms* include step-by-step methods that students are expected to know or follow. Mathematical *processes* include *problem-solving* (e.g. creating definitions, using definitions, using particular types of proof structures), mathematical *communication*, *representation* (e.g., using diagrams or formal symbols), and *justification* (e.g., conjecture or formal proof structures, such as direct, contradiction, induction, contrapositive, or counting).

Observing Models

We base our framework for observing models on an analogous idea from the literature on reading comprehension. Schraw and Bruning (1999) describe *reading models* as "set[s] of systematic beliefs readers bring to the act of reading that guide their goals and strategies" (p. 283); Weinberg and Wiesner (2011) described how students' reading models can impact the ways they read mathematics textbooks, and Schraw and Bruning (1999) found that particular reading models help students read more productively.

We define *observing models* analogously to the reading models identified by Weinberg and Wiesner (2011): The set of systematic beliefs that students bring to the act of observing a lecture that guides the ways they engage with the lecture. Students who have a *lecturer-centered* observing model believe that the lecture consists of a collection of idea units that they should record and internalize. In contrast, students with an *observer-centered* model believe that they generate a personal meaning through a transaction with the various lecture components (even if this transaction does not involve physical involvement in the lecture).

Many students believe that the only way they can learn from lectures is to take verbatim notes (e.g., Kiewra & Fletcher, 1984; Peper & Mayer, 1986), suggesting the prevalence of lecturer-centered observing models. However, research has suggested taking notes selectively, paraphrasing, and adding personal information—indicators of observer-centered models—leads to improved performance on exams (e.g., Van Meter, Yokoi & Pressley, 1994).

Implied Observers

In contrast to the static nature of a textbook, the temporal nature of mathematics lectures and the myriad components that they incorporate places significant demands on the observer. Weinberg and Wiesner (2011) described the *implied reader* of a mathematics text as the set of behaviors, codes, and competencies needed to engage with the text in a meaningful and accurate way. We build on their definition to define the *implied observer* of a mathematics lecture analogously and construct a framework for describing their behaviors, codes, and competencies.

Behaviors are actions that the implied observer takes. Some of these actions are *physical*, such as paying attention to the instructor and the board-work, or recording enough in notes to learn ideas or reflect on them later. Other actions are *mental*. These include: (a) monitoring your own understanding; (b) identifying "ideas" and "concepts" so they can be reified; (c) recalling examples, proofs, definitions, theorems, or proof structures; (d) noticing abstract structures in exemplars and connecting them to definitions and theorems; (e) recognizing and keeping track of the macro- and micro-structure of the lecture; (f) engaging with ideas in the order they are presented; (g) committing definitions (and other facts) to memory; and (h) being critical and skeptical of claims, as well as making conjectures along with the instructor.

Codes are systems of signification, or ways of ascribing meaning to components of the lecture. Implied observers of lectures use particular codes to interpret various aspects of the lecture. These codes can be classified as *formatting* (e.g. the layout of the board, temporal sequencing, fonts and colors), *symbol use* (e.g., commonly-used mathematical symbols, such as G representing a group), *diagrams* (i.e., understanding the ideas conveyed by various collections of symbols), *vocabulary* (both technical words, such as "abelian," and delimiters that signal the beginning or end of a section or idea), *verbal* codes (e.g., pauses and forms of emphasis), *mathematical grammar* (e.g., use of "we" or "recall"), and interpreting various *gestures*.

Competencies are the knowledge, skills, and understandings that enable the observer to understand and work within the established context. These may be mathematical, such as knowing particular definitions or theorems, or being able to perform specific procedures. These competencies may also relate to physically participating in the lecture, such as being able to write quickly and legibly, summarizing verbal presentations into written form, and instantly recalling the relevant facts, definitions, procedures (etc.) that the lecturer is addressing.

In addition to the more general mathematical and procedural competencies, the implied observer also has specific competencies related to mathematical proofs and their presentations, which play a central role in upper-level mathematics classes. There is relatively little research describing how proofs are actually presented in undergraduate mathematics classes or how students understood these presentations (Mejia-Ramos & Inglis, 2009). To address this, we adapt Mejia-Ramos et al.'s (2010) framework for describing proof comprehension. Their framework includes five main dimensions: (a) the meaning of terms and statements (including the meaning of theorems, of the individual statements, and the meaning of terms); (b) the justification of individual claims; (c) the logical structure; (d) higher-level ideas that provide form and direction; and (e) the general method used by the proofs.

Data Collection and Analytical Methods

We have begun collecting and analyzing data in a pilot study. The data corpus consists of videotaped classroom observations that expert observers have transcribed, notes from student volunteers (which Castello and Monero (2005) describe as "a symbolic mediator between the content taught by the teacher and the knowledge constructed by students" [p. 268]), and interviews with the students. Our analytical focus is on identifying opportunities for learning, which we hypothesize is affected by the degree to which the actual and implied observers coincide as well as the relationship between the structure and components of the lesson and students' observing models.

The classroom observations are designed to capture as much of the "text" of the lecture as possible. This enables us to identify the components and the implied observer of the lecture. We also collect students' notes. The notes, along with the interviews, help us identify observing models. In addition, incongruities between the notes and lecture components help us identify aspects of the implied observer.

For example, the course instructor may rely on various proof heuristics—such as an "onto proof"—and we will describe whether these heuristics are part of the implied reader, whether the students' notes are guided by this heuristic, and whether the students' notes and their interview responses indicate that they possess the underlying codes. In the interviews, students are asked to describe their beliefs about their role in a lecture-based classroom and how they use their notes as learning tools. Before each interview, we identify key excerpts from the lecture (where the instructor attempts to convey a difficult idea, where the board work isn't developed linearly, or where an aspect of the lecture didn't appear in the student's notes), present the corresponding video clips to the student, and ask them how they thought about these components of the lecture.

Summary

In order to understand undergraduate mathematics students' opportunities to learn class material, it is important for us to describe and better understand the constraints and affordances inherent in mathematics lectures and the ways students' beliefs might affect the way they construct meaning from mathematics lectures. In particular, we view students' opportunity to learn mathematics from a lecture as the interface between the implied observer and the actual observer—a student's own behaviors, codes, and competencies, along with the ways they comprehend proofs. Although the framework described here has not yet been subjected to revision based on data analysis, we hope that it provides a valuable theoretical lens for beginning to describe the difficulties students may have making sense of lectures and, by doing so, enable educators find ways to help students engage with and learn from mathematics lectures.

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THE RECRUITMENT AND RETENTION OF MATHEMATICS AND COMPUTER SCIENCE MAJORS

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Statistical data and current surveys show that the number of undergraduates interested and completing degrees in the Science, Technology, Engineering and Mathematics (STEM) fields is rising (National Research Council, 2006). Mathematics courses have been and continue to be gatekeepers to higher education and therefore to achieving STEM degrees (Blickenstaff, 2005). In addition to possibly being underprepared for mathematics and computer science majors, students often portray negative attitudes related to these fields (NRC, 2006). The focus of this study is to explore undergraduate Mathematics and Computer Science departments and the factors that influence the students to choose and complete degrees in these fields. By examining the recruiting and educational practices within these departments and linking those practices to students' perceptions and experiences, this ongoing study reveals successful practices for producing mathematics and computer science majors.

Ten comparable universities were selected to participate, within each university STEM department chairs were contacted through written and electronic communications. Only one university opted to participate, and within that university only 44 mathematics and computer science majors responded. Through the utilization of an online survey, demographic and qualitative data on students' experiences and perceptions were collected. Responses to open ended questions were coded using characteristics based on the Community of Practice framework (Wenger, McDermott, & Snyder, 2002). These characteristics were developed during a study on graduate mathematics departments (Lambertus, 2010). Additional characteristics were developed through open coding and development of content themes (Creswell, 2007).

Students interested in Mathematics and Computer Science highlighted five characteristics of their departments: (1) the University Departments (UDs) provide welcoming and diverse environments for undergraduate student success; (2) UDs provide a variety of structures to support students throughout their academic careers; (3) UDs facilitate opportunities for interaction among members; (4) alumni create opportunities for interaction among students, faculty, and alumni; (5) faculty members create opportunities for interaction among themselves and students and value resulting relationships. As a result UDs should: foster relationships among its members, encourage advisors to be mentors, provide support for success, and involve students in research early in their academic careers.

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UNDERSTANDING TEACHING AS ENACTMENTS OF CARE: CASE STUDIES OF AFRICAN-AMERICAN MATHEMATICS TEACHERS OF STUDENTS OF COLOR

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This study examines how teachers of color in urban schools socialize their students into being doers and learners of mathematics and how they support students' mathematics identity formation (Martin, 2000). It is a commonly held notion that "knowing" and "connecting with" one's students is critical for supporting their transition into being learners of mathematics. However, the ways mathematics teachers establish strong relationships with their students (particularly for diverse learners) remains less clear. Recent theoretical work by Bartell (2011), integrating theories of care with culturally relevant pedagogy, provides insight in this area. She argues that teachers who *care with awareness* "know their students well mathematically, racially, culturally, and politically" (p. 65) and leverage this knowledge to create substantive learning opportunities. Taking up the theoretical lens of care (Bartell, 2011; Hackenberg, 2010; Noddings, 1984), we build off this work and provide a fine-grained analysis of the practices of two African American high school Algebra teachers, April Lincoln and Floyd Lee, to demonstrate how they supported and engaged their students as mathematical learners.

Data from each teacher include 25 audio/video-recorded teaching observations and 9 interviews. Analysis of interview data focused on: views of students; descriptions of teaching practice; rationales for practices; and views of mathematics. Analysis of classroom observations attends to the presentation of mathematics; patterns of classroom discourse; and explicit references to students' personal experiences, beliefs, and identities. The case of Floyd Lee illustrates how one teacher builds relationships with his students that attend to issues of access and equity. Drawing on his ties to the local area (an urban, primarily African-American community), he connects his own racialized and class-based experiences with the lives of his students. Specifically, he discusses how he "knows" the kinds of students in his classes and what it takes for them to "make it." Floyd leverages this knowledge and explicitly communicates the importance of mathematics and how it bears directly on students' potential to achieve future success. The case of April Lincoln illustrates how one teacher, who identifies herself and is seen by others as having particular aptitude for working with students characterized as low-performing or with IEPs, establishes nurturing relationships by attending to students' affective and individual needs. She works to make the content accessible, positions students as capable and competent learners, and attempts to create a safe and respectful environment to learn.

This work seeks to develop an understanding of the heterogeneous ways two African American teachers conceptualize and enact caring in practice. Knowing more about how teachers develop and maintain relationships with their students can provide insight into how to better prepare teachers to support students' math identity formation and their mathematics learning.

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A STUDENT'S REACH SHOULD EXCEED HER GRASP, OR WHAT'S TECHNOLOGY FOR?

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The negative effect of poverty on student achievement in mathematics is well documented. So, too, is the positive effect of instruction focused on higher-order thinking, meaning, reasoning and understanding on the mathematics achievement of children of poverty. We delivered a lesson activity that uses TI-Nspire to support middle school students' interpretation of boxplots to 115 students in a high-poverty middle school. Preliminary data analysis indicates that the use of the technology provided helpful scaffolding for these students as they applied higher-order thinking and reasoning to understand the meaning behind these statistical representations of data.

Keywords: Data Analysis and Statistics; Equity and Diversity; Instructional Activities and Practices; Technology

One way in which the mathematics achievement of students in high poverty areas has been addressed is by providing instruction focused on higher-order thinking skills (Balfanz & Byrnes, 2006). However, McKinney and Frazier (2008), in a study of 64 mathematics teachers working in high-poverty middle schools, found that the vast majority of their sample reported frequently using lecture, drill and practice, and teacher-led lessons. Despite the influx of technology into U.S. mathematics classrooms over the last twenty years, Usiskin's (1990) vision of technology as an equalizer has not fully come to fruition, particularly not in high-poverty schools. Synthesizing these ideas we envisioned mathematics pedagogy for use in high poverty schools by focusing on higher-order thinking, reasoning, and understanding with technology to support teaching.

The 115 eighth-grade students in a high-poverty school whose work is reported in this paper were chosen by convenience sampling. Participating students took a pretest on day 1, engaged in the lesson activity on day 2, and took a posttest on day 3. The students working in pairs used TI-Nspire calculators and completed a worksheet that was designed to help them make sense of and interpret box-plots and make connections between two statistical plots. All of the responses to the two questions on both the pretest and posttest were analyzed quantitatively and qualitatively.

According to the quantitative results, there was a statistically significant effect on the test scores on three short-answer items, with students scoring higher on the posttest. Qualitative analysis revealed that even following a brief intervention, students were able to learn technical features of box-plots. Some students wrote longer answers that were much more mathematically mature on the posttest than on the pretest. We believe that technology by itself is not responsible creating these results, but valuing and focusing on the meaning of a box-plot via powerful, multiple-linked representations might be. In this way, perhaps the promise of Usiskin's (1990) vision of technology as a great equalizer can one day be fully realized.

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MATHEMATICAL LITERACY ASSESSMENT DESIGN: A MULTIVARIATE ANALYSIS OF PISA 2003 MATHEMATICS ITEMS IN THE U.S.

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Keywords: Assessment and Evaluation

This study draws on National Research Council's (NRC) assessment design framework that proposes the assessment triangle, where each corner represents: cognition, or model of student learning in the domain; observation, or evidence of this knowledge; and interpretation, or making sense of this evidence (Pellegrino, Chudowsky, & Glaser, 2001). The triangle representation signifies the idea of interconnectedness of the three elements as opposed to having them as isolated from each other, which is problematic and often times found in most of assessment designs. Of the five important recommendations for research outlined in NRC's assessment report, one urges for in-depth analyses of these three elements and their coordination. Thus, this study sheds light on these elements and their interconnectedness by analyzing data from a widely used international assessment design: Programme for International Student Assessment (PISA).

PISA assesses mathematical literacy in a multidimensional structure: content, processes, and situations (or context). The first, "content," is divided into 4 dimensions (overarching ideas): quantity, space and shape, change and relationships, and uncertainty. "Processes" consist of 3 competency clusters: reproduction, connection, and reflection. Lastly, "situations" are defined in terms of 4 dimensions: personal, educational/occupational, public, and scientific (OECD, 2009).

The purpose of this study is to investigate the validity of an international mathematics assessment (PISA) and its relationship with mathematical literacy based on following research questions: (1) *What content dimensions* do PISA *2003 mathematics items reflect?* (2) *To what extent do U.S. students' responses to mathematics items match the original assessment framework?*

A factor analysis (FA) was conducted for PISA 2003-US. The results could be summarized in 3 parts. First, results suggest that only one construct explains the most of the variance in students' responses to mathematics items. This result does not confirm the original multidimensional design of assessment items and implies weak connection between *cognition-observation* elements of assessment triangle. Secondly, the *space of individuals* shows no significant differences between males and females but some math items seem more correlated to low achievers. Lastly, the *space of variables* shows that few items were less correlated to the rest of items. Overall, the results show a rather low consistency among the three components of PISA's assessment design. More interpretations and deeper discussions will be presented in the poster. However, some causes for this weak interconnectedness are yet to be explored through further multivariate analyses (Carmona, Krause, Monroy, Lima, Ávila, & Ekmekci, 2011).

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FOREGROUND: TRANSITIONING STUDENT MOTIVATION FOR LEARNING MATHEMATICS

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Keywords: Affect, Emotion, Beliefs, and Attitudes; Equity and Diversity; Informal Education

This poster presentation results from a doctoral study in progress that aims to investigate possible changes in students' perspective of their future. This is important since it can provide them with reasons for learning. This subject has been treated from different perspectives for over 30 years, but in this study I emphasize the sociological concept of foreground (Skovsmose 2007). Foreground refers to how individuals see their social, economic, and political opportunities, that is, their perception of their future possibilities in life given its sociopolitical context. Foreground also includes how one interprets and reacts to these opportunities and to their expectations. The mathematics education research literature has several studies involving the concept of foreground (Skovsmose, Scandiuzzi, Alrø, & Valero, 2008; Alrø, Skovsmose, & Valero, 2009). An important outcome of this research literature is that hopeful foregrounds provide reasons for learning, or the opposite, in the case of broken foregrounds, that is, the case of ruined perspectives, which are common among students in unfavorable social situations. Since students' foregrounds motivate or discourage the learning of mathematics, it is vital to consider this question: How can mathematics education create mechanisms for the "reconstruction" of broken foregrounds? To better understand this problem, I will develop mathematical activities that involve social, political and cultural realities. I will conduct my investigation in two different environments: a Brazilian social institution that shelters economically poor children and adolescents in an after-school center, and a virtual learning environment in a school in the United States that serves immigrant students. The work at this virtual environment will occur in the context of a project directed by Arthur B. Powell which develops his investigation using a multi-user version of the GeoGebra, through the online environment, called Virtual Math Teams (Powell & Dicker, 2011). I believe that this comparative study—face-to-face, out-of-school environment in Brazil and virtual, in-school environment in the U.S.—contributes to debates about learning environments that provide opportunities for students to "reconstruct" their broken foregrounds.

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STUDENTS' CONCEPTIONS OF MATHEMATICS AS SENSIBLE AND RELATED INSTRUCTIONAL PRACTICES

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Students' conceptions about the nature of mathematics play a critical part in the development of mathematical proficiency. The productive disposition strand of mathematical proficiency is described as a "habitual inclination to see mathematics as sensible, useful, and worthwhile, couple with a belief in diligence and one's own efficacy" (National Research Council, 2001, p. 5). Despite the importance of students seeing mathematics as sensible, research on students' conceptions about mathematics indicates that the majority of students do not have a conception of mathematics as connected and coherent (Crawford, Gordon, Nicholas, & Prosser, 1994; Petocz et al., 2007). There is very little research about student actions that might signal that students see mathematics as sensible or about the instructional practices that might be effective for the development of such a conception in high school students.

Description of this Study

This study examines action-oriented indicators of students' conception of mathematics as sensible and the instructional practices that influence the development of this conception of mathematics as students transition into high school. In particular, this study examines action-oriented indicators that students in one purposefully-chosen 9th grade mathematics classroom conceive of mathematics as sensible and systematically examines the instructional practices within that classroom that may be associated with the students developing such a conception of mathematics. The first phase of this study was to adapt questionnaire and interview questions from research on conceptions of mathematics to create a list of action-oriented indicators that students conceive of mathematics as sensible. The list was then tested for usefulness and completeness by using it to code student actions in several classroom observations. The next phase of the study will be to examine one mathematics classroom for indicators that students conceive of mathematics as sensible, study the instructional practices in that classroom, and seek links between instructional practices and students' conceptions of mathematics as sensible.

Findings

I found that the list of action-oriented indicators is a useful tool for extracting from a classroom setting incidents in which students' actions indicate a conception of mathematics as sensible. I have also identified several instructional practices that appear to be associated with the development of a conception of mathematics as sensible as students make the transition from middle school to high school mathematics.

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ELEMENTARY TEACHERS' VIEWS OF AND EXPERIENCES WITH MATHEMATICS, MATHEMATICIANS, AND MEDIA

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Keywords: Teacher Beliefs; Affect, Emotion, Beliefs, and Attitudes; Elementary School Education

In North America, mathematics tends to have an image problem: People often dislike or fear the subject area (Boaler, 2008), and mathematicians are portrayed in stereotyped ways in the media (Applebaum, 1999). Negative attitudes toward mathematics are linked to decreased participation and achievement (Ma, 1999), but it is unclear how these attitudes form.

With these concerns in mind, I sought to understand how children may be impacted by growing up in a culture that is rife with negative views surrounding mathematics and mathematicians. My overarching goal in conducting this research project is to provide a better understanding of the complex interplay between outside sources (i.e., parents, teachers, and the popular media) and children's views of mathematics and mathematicians.

My study is framed by a social constructivist and feminist epistemological stance, wherein I understand the discipline of mathematics and views of mathematics and mathematicians to be socially constructed and gendered in nature. Following, my conceptual framework positions the actors in the study (i.e., students, parents, teachers, and media) as being both producers and (active) consumers of ideas about mathematics and mathematicians.

My study investigates elementary students' views of mathematics and mathematicians and the ways that parents' views, teachers' views, and popular media representations may impact students' views. Data collection consisted of online questionnaires, drawings of mathematicians, and focus group interviews (with media prompts) with Grade 4 and 8 students; interviews with parents; interviews with teachers; and document analysis of children's media.

For this presentation, I focus on the teacher interviews, in order to fully examine the ways in which teachers, a key socializing agent in children's lives, act as producers of ideas about mathematics and mathematicians. Additionally, the teacher interviews provide an understanding of the ideas about mathematics and mathematicians that teachers consume.

Teacher interviews took place in the 2010–2011 school year with 10 Grade 4 and 8 teachers from Ontario. The interviews (average duration: 1.5 hours) investigated the teachers' experiences with mathematics throughout their lives, their teaching philosophies, and their views of mathematics, mathematicians, and the media more broadly.

The interviews were transcribed verbatim, which resulted in more than 400 pages of data, and analyzed using Atlas software. Emergent coding was completed for each participant separately, and then comparisons were drawn across participants. Results will be discussed with regard to the conceptual framework of producers and consumers.

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WHAT COMMUNITY COLLEGE STUDENTS OF DEVELOPMENTAL MATHEMATICS THINK ABOUT LEARNING MATHEMATICS

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America has a growing need to address high school graduates' preparedness for the study of college-level mathematics. One-third of first-year college students in America need to take at least one remedial course, because they lack skills needed to succeed in college-level courses (Taylor, 2008). According to Stigler, Givvin, and Thompson (2010), community colleges have implemented various systemic reforms such as learning assistance centers or redesigned curricula, but community colleges have not instituted significant pedagogical changes. "Substantive improvements in mathematics learning will not occur unless we can succeed in transforming the way mathematics is taught" (Stigler et al., 2010, p. 5). To make such pedagogical shifts in the teaching of students of developmental mathematics, we need to examine the perspectives and needs of these students and the mathematical pitfalls experienced by them in their current courses (Stigler et al., 2010). In this study I investigated the perspectives of community college students of developmental mathematics on their current experiences in mathematics class and on what they think mathematics class *should* look like.

Eight community college students enrolled in intermediate algebra courses participated in semi-structured interviews. Participants recently graduated from high school and were at least 18 years of age. My sample was randomly selected from forty volunteers who indicated a willingness to participate in an interview. I asked the students to describe their current experiences in a developmental mathematics class.

Three main themes arose from the eight interviews: the classroom environment, views of learning math, and the purpose of math. Participants viewed the classroom environment as unengaging and unmotivating. Classes were characterized by inattentiveness and poor attendance. Students were expected to take notes on lectures and practice mathematical procedures. No group work or peer collaboration occurred in any classes, although several students indicated they would enjoy this. Students' responses conveyed various views of learning mathematics. Several students considered repetition crucial for learning; however, repetition in a lesson is "boring" or "tedious." Students wanted to be shown procedures, but several students expressed a desire for a deeper level of understanding of concepts. Consistently students saw no purpose for taking intermediate algebra, beyond satisfying a graduation requirement. All students wanted a patient teacher who explains things well and jokes around with them.

In analyzing students' perspectives, I gained insights about what students want and need from a mathematics class as they transition from high school to college. In addition to sharing these findings, I will also illustrate students' experiences with problem solving tasks that I conducted and will discuss how these influenced the students' perspectives.

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PROBLEM SOLVING THINKING CO-CONSTRUCTION OF LATINA/O ENGLISH LANGUAGE LEARNERS IN AN AFTER-SCHOOL MATHEMATICS ENRICHMENT COURSE

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This study aimed to trace and explain how problem solving, as one form of mathematical practice, is co-constructed in a classroom, using interactional analysis and drawing from in-depth case studies of two students, one Latino and one Latina English Language Learner (ELL) enrolled in an after-school program for urban students. Latina/o students in the United States perform consistently lower on average than their White peers on national and international mathematics assessments (e.g., NAEP, NCES, 2011). This underperformance of Latina/o ELL students in mathematics is discouraging, given that not only will the particular population continue to grow, but also the clear social and economic inequities created by such underachievement (cf. Gutstein & Peterson, 2005; Moses & Cobb, 2001). Attempts at improving the mathematics achievement of this population remain futile in the absence of a deep understanding of how classroom environment, instruction, and curriculum might provide space for their growth and development. Research in this area is currently lacking in mathematics education; this study was designed to address this gap. More specifically, the data collection and analysis was guided to address: How do Latina/o ELL students co-construct problem solving thinking through interaction with their teacher and peers in a mathematics enrichment course?

Our study supports that exploring the co-construction of mathematical problem solving thinking reveals multiple dimensions of students' thinking. For example, capturing what is happening (e.g., prosodically, physically, socially, materially, structurally, verbally, etc.), provides a picture of: the influences on a student's thinking, a student's use of verbal and non-verbal language, and the existence of power and influence of language and cultural hegemony. These aspects of classroom interactions contribute to what knowledge is honored, considered, taken-up, and further developed—greatly impacting students' growth in mathematical understanding. The use of discourse analysis revealed how the students' interactions with their teacher and peers, through language and related semiotic systems, influenced their mathematical thinking and knowledge building. Multiple layers of analysis (cf. Fairclough, 1995) uncovered that influences such as power, interaction structure, prosodics, and materials, play a significant role in both participants' mathematical work and in institutional sociomathematical norms (Yackel & Cobb, 1996) as depicted in the student/teacher interactions. The complete case studies of each participant reveal how their participation and that of their teacher and peers, in small and whole class discussions, shapes their mathematical thinking during problem solving.

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BUILDING MATHEMATICAL MODELS: IDENTIFYING COMMON ERRORS IN ENGINEERING STUDENTS' WORK WITH LINEAR PROGRAMMING PROBLEMS

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In this study, we aim to answer the following question: What difficulties to engineering students encounter as they build symbolic models of typical linear programming modeling tasks?

Linear programming problems are more complex than typical algebra word problems. The increased number of variables and level of complexity can contribute to learners' difficulties (Jonassen, 2000; dos Santos & Brodlie, 2004). Students often struggle with deciding what to include in a mathematical model (Wang & Brooks, 2007) as well as with deciding how to represent the chosen items. To better understand how students develop deeper conceptual understandings of LP modeling, it is important to understand the existing characteristics of different student errors across LP problems.

We collected data from students enrolled in an undergraduate engineering course in optimization who were learning LP problems for the first time. We collected responses for five quizzes focused on linear programming and analyzed them to understand the types of errors that students made while building mathematical models. Three coders worked together to identify and describe each individual error in a sample of ten randomly selected quizzes in order to build an initial coding scheme, which was further revised and applied to all quizzes.

Our analysis resulted in a taxonomy of errors that contains four primary categories. *Decision Variable Errors* include instances where students introduced their own new variables that were either disruptive or not clearly connected to the given variables. *Variable Relationship Errors* involve ways that students developed incorrect relationships among the variables and coefficients given. This category of errors was identified more often than any others. *Notation Errors* refer to mistakes specifically linked to a mathematical symbol such as summations, subscripts, inequality and equal signs. *Form Errors* involve the omission of the non-negativity constraints. Many of these errors do not seem context specific, but instead represent a conceptual difficulty where students may understand the problem, but cannot translate to a correct mathematical solution. As our work in this area continues, our goal is to develop a visualization tool that could help students better understand the complexity of LP problems, organize the given information, and check the validity of the models they create.

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AFFECTIVE COMPETENCE IN MATHEMATICAL LEARNING

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Categories present in affective domain in mathematical learning are analyzed (Gómez-Chacón, 2000; Maab & Schlöglmann, 2009; Reeve, 2003; Rivera, 2006, 2011). The relationships found among them are showed to provide an overview of the affective domain and to point out the aspects that allow the approach to emotional competency in the mathematical learning.

The leading elements: teacher, student, knowledge and people around them and the links that they have with the psychological and some socio-cultural aspect are displayed.

It is emphasized that to learn and to use mathematics, the affective aspect that goes implicitly in students' learning has an equal importance that the cognitive and skills aspects.

Theory of Social Representations (TSR) is applied (Moscovici, 1979), employing the technique of focus groups as dynamics form of collective in-depth interviews applied in several groups of Mexican teachers from primary education level until higher education level.

Categories of beliefs, emotions, attitudes, attributions and ethics and morals are considered for the implementation of the dynamics of focus groups, taking focus of attention in the epistemological, cognitive, didactic and social aspects (Rivera, 2006, 2011).

The representations obtained in this way are then presented schematically in relationship to the categories of the affective domain analyzed. This allows showing photography of the reality and it validates the structured conceptual frame that has been glimpsed by the investigators who have investigated in this aspect and that have been validated in this work.

The Social Representation founded indicate four weak points, left of side by the teachers: the emotional category, the ethics and morals aspect; to give a permanent emphasis to the indissoluble link that is present between epistemological and the cognitive beliefs, and finally cultural partner considers the environment where the learning is effected, with the complex and unpredictable implications that this have inside, in the affective and into the cognitive aspects and skills that the mathematical learning have inside too.

It can be foreseen how far teachers are from having in them and propitiating in their pupils an affective competency and a competency in mathematical learning, which involves also to the cognitive and skills aspects.

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TOWARD SUCCESSFUL TRANSITIONS FOR ALL: UNPACKING IDENTITY THREATS AMID AFRICAN AMERICAN STUDENTS' MATHEMATICS LEARNING EXPERIENCES

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As part of a broader "social turn" in mathematics education research (e.g., Lerman, 2000)—and, perhaps more recently, a sociopolitical turn (Gutiérrez, 2010)—there has been growing attention to the role of identity construction as a social element of mathematical thinking and learning (Martin, 2000; Sfard & Prusak, 2005). Although many conceptualizations of identity have emerged or been incorporated from other areas of inquiry (e.g., psychology, philosophy, sociology, anthropology), mathematics education researchers have recast and re-operationalized identity in mathematics-specific terms and contexts (cf. Bishop, 2012; Martin, 2000). Despite the increased attention within the field, however, there is still a need to deepen analytical perspectives on identity and to empirically explore the ways in which mathematics-specific identities may shift and respond to other social factors in certain situations and/or over time.

This poster presentation has three central aims: (a) to advance a framework for analyzing mathematics identities as narrative performances; (b) to present a study in which this framework was used to explore African American students' mathematics learning experiences in a university-level remedial course; and (c) to unpack the narrative version of a social-psychological phenomenon, stereotype threat (Steele, 2010) and discuss the various ways in which (and factors by which) students respond to identity threats (cf. Lindemann Nelson, 2001).

The poster draws on two study participants' narratives about their transitions from high school- to college-level mathematics courses, particularly as the transition includes a remedial mathematics course. Across series of semi-structured interviews, Cedric and Vanessa—two high-achieving, African-American, high-school graduates who upon matriculation to college were placed in a mandatory, non-credit-bearing, basic-skills algebra course—discussed the influence of socialization factors on their mathematics learning experiences over the course of several months (cf. Martin, 2000). These interviews were supplemented with recurrent observations during the same period. The findings of the study center on four responses to threat that emerged from the students' narratives: identity satisficing, contingency detection, masternarrative threats, and counternarrative "repair."

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ATTENDING TO IMPLICIT KNOWING IN THE MATHEMATICS CLASSROOM

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We sat with a small group of mathematics teachers working on a problem: "In a warehouse you obtain a 20% discount but you must pay a 15% sales tax. Which would you prefer to have calculated first, discount or tax?" (Mason, Burton, & Stacey, 1982, p. 1). All concluded that order does not matter (unless you're the seller or tax-collector). Although in agreement, one person commented: "It feels wrong that the solutions should come out the same." I (Martina), too, was unsatisfied and located my discomfort in an association with adding then subtracting a particular percentage to the cost of an item and not coming back to the starting point. I realized that a general notion of "coming back to where I started" was interfering–not with my (already strong) confidence in my solution to the warehouse problem, but by creating an uncomfortable feeling of non-resolution. This experience helped motivate this study: How might learners deepen their awareness of partially conscious feelings associated with doubt and certainty and use them as gateways to deeper mathematical understanding?

We conceptualized the study within an enactivist view of cognition, emphasizing autonomous, co-emergent, and embodied knowing (Thompson, 2007), and we used these principles to design the research classroom. Attending to aspects of understanding that dwelled beneath full awareness–even after stated problems were solved–was of particular interest. Varela's (Varela & Scharmer, 2000) notion of researcher as an empathic coach and Gendlin's notions of "felt sense" (1978) and "implicit intricacy" (1991) were helpful in bringing more of such understanding to awareness.

By attending to external indicators of felt meaning, learners interacted with each others' implicit understanding in ways that helped bring it closer to consciousness and into conversation. Here, the importance of directly referring to emerging understanding in ways that were broad enough to allow it to evolve became clear. Conversely, prematurely insisting on clarity and logic precluded awareness of the implicit.

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"THAT'S WHY PEOPLE DON'T ASK QUESTIONS, 'CAUSE THEY LOOK AT YOU WITH THAT BLAZING STARE"

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Researchers have begun to understand the ways in which students' developing identities as knowers and doers of mathematics shape their understanding of the discipline of mathematics and their place within it (Cobb & Hodge, 2007; Martin, 2000). As such, mathematics education researchers have recently employed the construct of *identity* to gain insight into how students' beliefs, attitudes, and motivations are negotiated within the mathematics classroom context. Current empirical studies have primarily focused on younger students in the K–12 setting and have taken into consideration the relationship between students' identities and classroom norms.

The purpose of this study was to contribute to the ongoing conversation about the ways in which students' mathematical identities are studied and interpreted. Specifically, I was interested in exploring the mathematical identities of college students. Having recently transitioned to the university setting, I hypothesized that early college students would be in a unique position to articulate their observations, interpretations, and evaluations of classroom norms in ways that younger students in other settings may not. Through a multi-layered qualitative case study of Malik—an African American male student enrolled in a college Algebra course at a large, Midwestern university—I sought to answer the following questions: (a) How does Malik enact his own mathematical identity through language? and (b) How does Malik use language to frame his relationship to other students in his class?

In the initial phase of analysis, I interpreted the interview data collected from four classroom participants through the lens of *normative identity*, or the "identity that students would have to develop in order to affiliate with mathematical activity as it is realized in the classroom" (Cobb & Hodge, 2007, p. 166). Out of all of the participants, Malik stood out as having an interesting and unique perspective on classroom norms. Malik's descriptions were distinctly different from the others' in that he consistently defined himself in opposition to his peers, describing himself as fulfilling a unique role in the classroom. For example, he described the ways in which speaking up in class can be scary—"'cause everyone looks at you with that blazing stare"—but that he overcame this fear to ask questions on behalf of his peers. He felt that he was a role model for other students. After reading and re-reading the data that I collected, I began to notice a connection between this identity and his language use, prompting me to do further investigations into his language. A subsequent discourse analysis, which focused on his use of "I-statements" (Gee, 2011) and pronouns, revealed the ways in which his identity was enacted through language. Taken together, these results provide unique insights into Malik's identity as a college Algebra student, which have implications for future work on identity.

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USING VIDEOS OF HOW CHILDREN LEARN MATHEMATICS WITH PARENTS

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Parents are often overlooked and could be one of the missing links in the continuum to help all children better understand mathematics. Videos of children learning math (along with associated commentary) can also be a useful resource for parents. This study reports the results of an online survey completed by parents after they viewed videos of children engaged in mathematical activity.

Videos have been shown to be a powerful tool in helping preservice teachers expand their content knowledge and in developing beliefs that promote developing mathematical understanding (Philips et al., 2007). One of the key features of using videos is in helping viewers of the videos *notice* (Jacobs et al., 2010) what is important in children's learning of mathematics. The videos and especially the expert commentary included at the beginning and end of these videos is designed to help viewers make sense of the video in a new way.

Our research looks at how we can help parents help their children with their mathematical homework. One of our goals is to encourage parents to focus on the way children think about the mathematics. Rather than providing parents with a multitude of resources to help their children learn mathematics, the primary hypothesis of our research is that children's understanding of mathematics can be improved by empowering parents with knowledge of how children learn mathematics.

We sent a letter home to all second and fifth grade parents in two small-town Midwestern elementary schools. In the letter we asked parents to view multiple students solving the same math problem in the grade respective to their own child's. At the conclusion of the videos, they were asked to complete an online survey.

The results of the survey indicate that: (1) many parents are actively involved in helping children with mathematics homework, (2) parents' confidence in helping their children with mathematics varies, (3) these videos were helpful to parents, (4) the videos reinforce the idea that children think about mathematics differently than adults, (5) for some of the parents, the videos helped them better understand how their own children think about mathematics, and for some parents, the videos helped them think about ways of helping their children learn mathematics.

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READING AND WRITING IN THE MIDDLE LEVEL MATHEMATICS CLASSROOM: A MULTI-LEVEL MODEL ANALYSIS

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NCTM's *Curriculum and Evaluation Standards* (1989) tied mathematical power to mathematical literacy and emphasized the central roles of reading, writing, and discussion. Although the Common Core State Standards for Mathematics (CCSSM) (CCSSI, 2010) do not mention reading and writing in the mathematics classroom directly, the standards for mathematical practices require, implicitly, fluency with the language of mathematics. We suggest that reading and writing in mathematics will be critical to the successful attainment of the mathematical practices proposed by the CCSSM. Although the mathematics education community has discussed the importance of and provided recommendations for incorporating reading and writing into mathematics classrooms (e.g., Barton & Heidema, 2002; Thompson et al., 2008), there remains a need for empirical research on the effects of reading and writing about mathematics, particularly in middle school (Bangert-Downs, Hurley, & Wilkinson, 2004).

Our research questions were as follows: (1) To what extent does the self-reported value that teachers place on reading and writing about mathematics influence the value that their students place on those same activities? (2) To what extent do the classroom pedagogical decisions regarding reading and writing about mathematics affect student attitudes toward those activities?

Data were collected over one year as part of the third edition evaluation studies of two textbooks developed by the University of Chicago School Mathematics Project (UCSMP), *Transition Mathematics* and *Algebra*. Analyses are based on data from mathematics teachers (n = 41) and their students (n = 931) in grades 6–9 from 12 schools (n = 10 public and n = 2 private) in nine states. Including the curricula used as comparisons to UCSMP, nine different curricula are represented in the study. Teachers provided intended plans for reading and writing on initial questionnaires and documented enacted decisions on final questionnaires; students completed an end-of-the-year questionnaire about their attitudes toward reading and writing mathematics.

Data analysis used multi-level logistic modeling. Results show that classroom time spent reading and writing were the predictors that most positively influenced the likelihood that students valued reading and writing about mathematics. These results have practical implications that can guide teachers to improved pedagogical practices that can maximize student transitions toward increased participation in the mathematical practices of the Common Core.

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THE MATHEMATICS PHILOSOPHY OF A PRESCHOOLER: EXPLORING MATHEMATICAL ACTIVITY AND BELIEFS

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There is a growing view in the mathematics education community that mathematical knowing is a cultural activity (Bishop, 1988; Burton, 1994; Millroy, 1992; Nasir, Hand, & Taylor, 2008). Bishop (1988) wrote that because mathematics is a cultural activity, children bring their own mathematical meanings to the classroom and thus play an active role in the cultural practice of mathematics education. So it is important that teachers—as well as families and children themselves—recognize how children might experience mathematics outside of school to help navigate the transition into school mathematics.

The purpose of this study was to investigate the mathematical activity children engage in and beliefs they have about mathematics before entering school. Through a qualitative case study of three-year-old Olivia, who had no formal school experience including daycare and preschool, I sought to answer the following questions:

- What is the nature of Olivia's mathematical activity according to Bishop's (1988) framework?
- What seems to be Olivia's philosophy of mathematics?

More specifically, I used Bishop's six types of mathematical activity—counting, measuring, playing, explaining, designing, and locating—as a framework for activity, and the what, who, where, when, and why of mathematics to gain insight into her mathematics philosophy.

Using two full weeks of home-based observation and two semi-structured interviews, I found that Olivia's mathematical activity was plentiful and widely varied, with multiple events in each of the six categories of activity. To her, however, mathematics is a far more limited activity in terms of what it is and where and when we engage in it. Namely, she believed mathematics is something "smart" people do "at schools" involving "writing numbers and letters," though science can be used to investigate and solve problems. The incongruence between her activity and beliefs could be problematic, and suggests some implications for instruction. For example, professional development for existing teachers and courses for future early elementary teachers could emphasize the nature of students' mathematical activity at home and how to recognize and mathematize those experiences in the classroom.

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IMPROVING MULTIPLICATION STRATEGIC DEVELOPMENT IN CHILDREN WITH MATH DIFFICULTIES

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Keywords: Assessment; Elementary School Education

The purpose of the present study was to test the effectiveness of a constructive Strategic Training (ST) program for improving students' performance in solving multiplication problems. Multiplication is a foundational skill that children are expected to master through third to fifth grade (National Council of Teachers of Mathematics [NCTM], 2000). However, students with mathematics learning disabilities often fail to achieve these academic standards (No Child Left Behind [NCLB], 2000). The participant was a third grader with math difficulties (Carl) who was recruited from a Midwestern elementary school in the United States. Carl had average or above average IQs but scored lower than the 35th percentile in the Math Fluency Subtest of the Woodcock Johnson Test of Achievement (Woodcock, McGrew, & Mather, 2001). First, we gave a baseline assessment to the participant, including a 1 × 1-digit multiplication calculation test and a 1 × 2-digit multiplication calculation. We found that Carl's accuracy in solving 1 × 1-digit problems was low (medium = 50% correct) and his accuracy in solving 1 × 2-digit problems was even lower (0%). Carl responded to more than half of problems with incorrect operations or replied with "I don't know." Therefore, the researchers designed an ST program for Carl to gradually transition from Level 1 (incorrect operations and don't know) to Level 2 (unitary counting) and then to Level 3 (repeated addition) and Level 4 (direct retrieval and algorithm). Wave A was designed to help Carl master unitary counting strategies by introducing small-number problems that allowed him to count by ones; When Carl had established consistent use of the unitary counting strategy, he was introduced to Wave B which was aimed to transition him to increased use of double counting and/or repeated addition. Results showed that Carl rapidly increased his use of double counting and repeated addition and took only one session to begin using repeated addition for 50% of all trials of the session. Correspondingly, Carl increased his accuracy in Wave B probes to 100% by the third session. Then Wave C intervention aimed to help Carl transition to greater use of higher order strategies such as algorithm and decomposition. With Carl's quick acceptance of using algorithm strategy, he improved his accuracy of solving Wave C problems up to 70% correct. In short, results showed that Carl began the intervention on the lowest strategic developmental levels and were consequently given individualized tasks to promote this strategic development during differential waves.

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Chapter 5

Teacher Education and Knowledge–Inservice

Research Reports

Brief Research Reports

Poster Presentations

PHOTO-ELICITATION/PHOTOVOICE INTERVIEWS TO STUDY MATHEMATICS TEACHER IDENTITY

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How do mathematics teachers think of themselves? The construct of identity—how teachers see themselves—is an important and understudied construct in understanding mathematics teaching. This study investigates the use of Photo-Elicitation/Photovoice Interviews with six high school Algebra I teachers. Each teacher captured or chose photographs of their "world," then presented them during a formal interview. Through stories elicited in this process, mathematics teacher revealed personal/private identities they often covered up with their professional identity. These mathematics teacher identities fell along a professional learning continuum with a conservative identity on one end and radical multiculturalist identity on the other end.

Keywords: Teacher Education–Inservice/Professional Development; Teacher Knowledge; Teacher Beliefs Equity and Diversity

Objective of the Research Study

Over the last several decades, mathematics educational researchers built a substantial body of work exploring how to support mathematics teachers. Spurred by Shulman (1987), researchers have focused on teacher's professional knowledge of mathematics (Hill et al., 2007), their beliefs about how mathematics is learned (Philipp, 2007), and their dispositions towards mathematics (National Research Council, 2001), among other things. However, Shulman also warned that taking this professionalization too far risks losing sight of the human element inherent to what makes teachers teach. Shulman wrote, "We must achieve standards without standardization. We must be careful that the knowledge-base approach does not produce an overly technical image of teaching, a scientific enterprise that has lost its soul" (Shulman, 1987, p. 20).

Many mathematics teachers might feel successful in their professional responsibilities of mathematics teaching, yet still feel disconnected, unfulfilled, and even powerless in connecting their personal identity with who they are in the classroom (de Freitas, 2008; Drake, Empson, & Dominguez, 2003; Van Zoest & Bohl, 2005). Most educational research focuses only on the person that exists in this professional role–the teacher in the classroom. By uncovering the personal story that lies beneath the professional story and allowing mathematics teachers to confront and reflect upon these aspects of their identity, we might begin to know the "soul" or identity of a mathematics teacher.

This study investigated teachers' conceptions, notions, and stories of themselves through the construct of *mathematics teacher identity*. In particular, the following research questions guided the investigation: What kinds of things does a photo-elicitation/photovoice interview capture about mathematics teachers in regards to identity? What does the photo-elicitation/photovoice interview reveal about mathematics teachers that may not be captured through other research methods?

Theoretical Framework

Many scholars have looked at the construct of the identity as a lens for research in education (Bishop, 2012; Boaler & Greeno, 2000; Cobb, Gresalfi, & Hodge, 2009; de Freitas, 2008; Drake, Spillane, & Hufferd-Ackles, 2001; Gee, 2000; Holland, Lachicotte, Skinner, & Cain, 1998; Van Zoest & Bohl, 2005). This extensive work on identity in education centers on the constructs of *dilemma, relationships, narrative, agency,* and the *photo-elicitation/photovoice interview* to understand the identity of teachers or students.

Dilemma

The work on teacher identity historically centers on the construct of "dilemma" (Connelly & Clandinin, 1995; Lampert, 1985; Lyons, 1990; Shulman, 1988). Lampert (1985), one of the first researchers to write about her teaching as scholarly work, framed her journey into connecting her teaching practice to her personal beliefs through the dilemmas she encountered in her classroom. Lyons (1990) also examined teachers' dilemmas, particularly analyzing how teachers resolved or dealt with moral dilemmas that came up in their relationship to students, finding that when teachers responded to dilemmas, they were forced to examine themselves, which led to growth—to change.

Clandinin and Connelly (1996, 1995) use the idea of dilemma as central to their construct of Teachers' Professional Knowledge Landscapes, finding that the *big* dilemma teachers face is between theory and practice. To navigate this dilemma, teachers tell both *Cover stories* and *Secret stories*. *Cover stories* are the stories, "In which they portray themselves as characters who are certain, expert people. These cover stories are a way of managing their dilemmas" (Clandinin & Connelly, 1996, p. 15). *Secret stories* takes place behind closed doors, often revealed only in private spaces (i.e., happy hours or parking lot chats), among like-minded colleagues.

Palmer (2007) uses the term *Divided Life* to mean the same thing—the separation of external from their internal, causing a psychic rift between teacher-selves and the way teacher actually perceive themselves. de Freitas's (2008) work exploring mathematics teacher identity through classroom discourse found the same warring elements, referred to as procedural registers versus personal narrative registers.

Relationships

Relationships and formed with other teachers is another construct researchers use in exploring teacher identity (Grier & Johnston, 2009; Holland et al., 1998; Van Zoest & Bohl, 2005; Wenger, 1998). Wenger (1998) uses the construct of *Communities of Practice* to look at identity formation, particularly within socially-constructed spaces. In this definition, a teacher creates his or her identity through learning how to be a member of a particular community of practice, in this case, the community of teaching. Van Zoest and Bohl (2005) build their definition of mathematics teacher identity through this idea of the various communities of practice that a person inhabits as they become a mathematics teacher. Holland, Lachicotte, Skinner, and Cain (1998) study identity within the aspect of creation; identity as enacted in the building, forging, and authorship of *Figured Worlds*. Grier and Johnston (2009), in looking at the identities of STEM career changers, found that socialization into the teaching culture was crucial to creating a teacher identity.

Narrative

Hiebert, Gallimore, and Stigler (2002) pushed forward the idea of using narratives as a way to understand teaching. This idea echoes Bruner's (1986, 1996) ideas about the need for a narrative construal as necessary for studying teachers and teaching. Doyle (1997) also showed that narrative-based research on teaching was just as valid and measurable as any other forms of research, especially quantitative endeavors. Drake and Drake, Empson, and Dominguez (2001, 2006) found that mathematics teacher identity was best elicited through the *Math Stories*, which were teacher narratives about prior and current experiences with mathematics. Sfard and Prusak even go so far to claim that identity and narrative are one and the same, "No, no mistake here: We did not say that identities were *finding their expression* in stories—we said they *were* stories" (Sfard & Prusak, 2005, p. 14).

Agency

Agency and power are other constructs used to understand mathematics teacher identity (Crockett, 2008; de Freitas, 2008; Gee, 2000; Gutstein, 2006). Gee (2000) defines a teacher *Discourse identity* as the individual traits of a teacher that are recognized through discourse/dialogue of/with the teacher. When a teacher's *Discourse identity* is legitimized/seen by others, the teacher moves away from a socially defined professional identity, and is thereby open to understand the hegemony of the system they live within.

Palmer (2007) similarly says that the choice to live "divided no more" by rectifying the dilemma of warring identities, is the first step towards true teacher social revolution.

Crockett (2008) outlines a culturally-based continuum to categorize mathematics teacher identity. Teachers often start with a *Conservative* identity, characterized by practice and beliefs that mimic the "traditional" mathematics education they encountered as a student. Then, teachers move to a *Liberal* identity, characterized by inquiry-based, "reform" teaching practices mirroring the ideas promoted by NCTM (National Council of Teachers of Mathematics, 2000). Finally, an identity that teachers rarely ascend to is the *Radical Multicultural* identity, which involves teachers empowering students to act independently, make free choices, and use mathematics to not just see, but also do something about the injustice in their own communities and worlds (Freire, 1970; Gutstein, 2003).

The Photo-Elicitation/Photovoice Interview

The Photo-Elicitation/Photovoice Interview is a research method that anchors an exploration into identity by adding a visual structure to teachers' narratives. Researchers introduce photographs, either selected by the participant or the researcher, into the interview context (Clark-Ibanez, 2004; Gauntlett & Holzwarth, 2006; Harper, 2002). The method successfully elicits identity for a number of reasons. First, visual and creative methods like this are especially useful for studying identities (Brown, Wiggins, & Secord, 2009; Clark-Ibanez, 2004; Gauntlett & Holzwarth, 2006). Second, visual and creative research methods open up imaginative spaces in a non-invasive way that honors teachers' busy schedules and responsibilities (Clark-Ibanez, 2004; Gauntlett & Holzwarth, 2006; Harper, 2002). Third, this method has a strong history of studying identity within other social sciences, such as nursing and anthropology research (Hansen-Ketchum & Myrick, 2008). It is only recently being used within education research (Clark-Ibanez, 2004). Fourth, the PEI excels in generating a narrative structure authored by the research participants themselves (Brown et al., 2009; Clark-Ibanez, 2004; Gauntlett & Holzwarth, 2006). Finally, photovoice techniques elicit teachers prior knowledge by focusing on what they know rather than what they do not know; this empowering reflection leads to social action and critical consciousness (Freire, 1970; Wang & Burris, 1997).

Methods and Procedures

The six teachers who participated in this study taught Algebra I at six different high schools located in large, urban cities in a large Southwestern state. I had previously interacted with all six teachers through research, professional development workshops, and classroom observations for a previous research project. I chose teachers who expressed interests in exploring and talking about their identities as it connected to their mathematics teaching and whom I felt would tell "good" stories.

The Process

I first visited each teacher to give them digital cameras and a loose prompt to "capture your world as a mathematics teacher" in at least twenty photographs. We then set up a time and date to sit individually for a formal Photo-Elicitation/Photovoice Interview two weeks in the future. During these two weeks, I observed at least one Algebra I class period for each teacher to get a feel for their teaching practice and style, to get to know their classroom and school culture, and to be available to answer any questions each teacher might have about the study, the research method, or what types of photographs they should capture. A day or two before the scheduled interview, I sent an email to each teacher reminding them about the interview and also prompting them to choose the ten most important photographs out of original twenty in order them by importance. This forced editing of the photograph pool right before the interview provides a space for each teacher to reflect upon each photograph.

I then sat with each teacher for the actual Photo-Elicitation/Photovoice Interview, each one taking place after school in the teacher's classroom and lasting at least 90 minutes. In the interview, each teacher shared one photograph at a time in order of self-selected importance. I used a minimal interview structure, using non-judgmental and non-evaluative language such as, "Tell me more about that," or, "How does that connect to you as a mathematics teacher?" (Johnston, 2004). I also used clinical interview strategies to get teachers to elaborate more on how each image connected to their identity (Ginsburg, 1997).

Data Sources

Audio data from each interview was captured with digital recorder and then transcribed for analysis using a grounded theory coding structure. I coded for emerging themes of mathematics teacher identity centering on the overlap of professional and personal identities (Corbin & Strauss, 2007; Merriam, 2009). The main data source was the interview of each teacher, which I transcribed in InqScribe to build an emerging coding scheme. Each interview consisted of at least 90-minutes of transcribed audio. While the actual photographs that teachers shared and the notes I took during classroom observations added tremendous depth to the interview, I did not consider them as primary data and did not thoroughly analyze them.

Analysis Approach

Analysis began during transcription. InqScribe allowed the inclusion of time codes for every line of interview data, as well the tagging of thematic codes in order to start build a general grounded theory (Corbin & Strauss, 2008). These first-pass codes formed the initial stages of a categorical scheme to organize the data, (Corbin & Strauss, 2007). Then, using a textual conversion script I wrote in Perl, I imported the transcripts from InqScribe into Transana.

I then built a theoretical sample by analyzing the two longest interviews, listening to the audio and reading the transcripts in order to create individual audio clips that seemed important (Corbin & Strauss, 2008). Each clip captured something that eluded to mathematics teacher identity, which allowed me to create a deeper, more robust code set grounded in a more general understanding of the data now that I had transcribed all the interviews. For the first interview, I created 103 clips. For the second interview, I created 69 clips. Both of these clips generated a total of 446 codes within 35 categories.

At this point, I started to see the need to combine similar categories or drop categories to make the analysis more manageable. I went through the code set with a specific lens of looking at the how teachers were telling *Cover* or *Secret stories* (Clandinin & Connelly, 1996; Connelly & Clandinin, 1995) in order to collapse the categories. I then used the second-pass code set to analyze the remaining four interviews, adding codes and categories as necessary. This allowed me to focus specifically on the construct of mathematics teacher identity in a way that was not specific to any one teacher. It also allowed me to see initial conclusions from the data as we noticed how certain codes or categories were used again and again. This third-pass (and final) code set contained 206 codes within 14 categories.

I then went back and wrote up a four-page summary of the main categories and keywords I found for each teacher. I emailed each teacher their summary, asking them to add to, edit, or clarify anything in the summary they wanted in order to create a form of member-check validity (Corbin & Strauss, 2007). I specifically asked each teacher to ensure that what I was finding was true to his or her "voice" and to check that I was being "real." I also asked each teacher to choose a pseudonym that was both personal and unique. Feedback from each teacher was incorporated back into the analysis.

Results

This paper looked specifically at the usage of the Photo-Eliciation/Photovoice interview in eliciting teacher identity. Specifically, I examined what was revealed through the interviews and what this method captured that other research methods might miss. Deeper analysis into the actual construction of mathematics teacher identities and formation of agency will be forthcoming in further papers.

Cover Stories Initially

In each interview, the first few photographs usually involved the teacher talking about something in their past that was important to them in terms of their mathematics identity. Usually it was a picture of their family, which elicited a narrative of a parent who was "good" at mathematics and therefore taught them that they could be too. This led to a narrative where teachers revealed they chose to be a mathematics teacher because of this prior experience, because they liked math.

In fact, this "I like math" was the most dominant *Cover story*. The following transcript shows how one teacher, Mr. Ginobili, initially creates a story that his sister and him became mathematics teachers because they inherited a "math gene" from their accountant father. All of this comes from a photograph Mr. Ginobili shares of his sister.

Mr. Ginobili: Just the fact that, for both of us [Self and Sister], math is our passion.
Interviewer: For her as well?
Mr. Ginobili: Yeah. I mean, she became the math lead, and they told her, like, what would you want to do and stuff like that. And she was like, math was definitely her thing. And I mean, I see that coming from my father because of the accounting. And the fact that number sense really came, is an easy trait for us to attain.
Interviewer: So tell me about that, then. So you feel like you were instilled with a high sense of number sense?
Mr. Ginobili: Yeah, I think so. I mean, and it's probably, you know, a gene that we developed, that we got from our dad or anything like that.

Photographs Reveal Secret Stories

A traditional interview or belief survey might end here, taking up this teachers' cover story as valid and concluding that this particular teacher holds an entity-based view of mathematics as genetically predestined—the belief in a "math gene." This might lead to conclusions of his teaching practice as "traditional." Yet, through the anchoring structure of the Photo-Elicitation/Photovoice interview, teachers were forced to stare at the photograph and continue to reflect on what the image meant to them. This opened up space to go beyond the *Cover story* and reveal the *Secret story* that connected to their personal/private identity.

For instance, minutes after talking about this "math gene," Mr. Ginobili reveals a *Secret story* about his competitive, yet supportive, teaching relationship with his sister. Mr. Ginobili reveals his passion for working with struggling students.

Interviewer: What do you see in yourself in your sister?
Mr. Ginobili: I think we really have the same passion for the kids. You know, I'll go over to her house every once in a while and she'll be thinking about, "Oh, well this kid is really struggling with this and I need to have an activity for them." And she's really passionate for each one of her kids. And I, I don't want to say that I have that same passion, because she really goes far beyond what's necessary of her. But you know, and like, I still, I do have a little bit of passion for my kids, you know. If I see one of them struggling, you know, you always want to talk to them and figure out what's wrong and stuff like that.

A few minutes later, another *Secret story* further reveals that, through Mr. Ginobili's relationship with his sister, he understands the importance of listening to his students. He reflects upon a teaching belief of valuing listening to students' prior knowledge.

Mr. Ginobili: I think the skills that are really helped me as a teacher and more as a mentor to the kids, is something that she sees in me and she likes and that's why she comes to me more than she does my brother, right? Because she's told me before, she's like, well, if I need somebody that I can talk to and will listen to me, I'm going to call you because, you know, you have that in you. And it's one of the things that makes me want to be a teacher, is the fact that I can listen to kids and I like listening to people, the stories and really try to help them out whenever possible. And so she sees that in me.

Through these Photo-eliciting/Photovoice interviews, teachers revealed the *Secret stories* that described their mathematics teaching practice, weaving narratives that reflected how they saw themselves as a mathematics teacher rather than the usual *Cover stories* they are used to living with.

Discussion and Conclusion

This study fills a current gap in the research on understanding mathematics teacher through the lens of identity. This research extends previous research on teacher identity by incorporating the structure of the Photo-Elicitation/Photovoice Interview to ground the teacher's space for authorship. I successfully explored a research tool that captured teachers' voices without hovering, disrupting their practice, or looking for only what I wanted to see. I attempted to answer Shulman's (1988) call that the best forms of teacher support must involve meeting teachers where they are. This study introduces a professional development tool that gets teachers to reflect on themselves in a minimally invasive and easily implementable way. Mathematics teacher identities are unique and varied; so getting teachers to reflect on their own identity as mathematics teachers helps them become more aware of their own teaching practice.

With more time and resources to do follow up interviews or observations, I might have been better able connect the results with actual teacher practice or student reports. I could also have incorporated multiple narratives into building a mathematics teacher identity, a limitation of singular interview structure.

In the end, this experience allowed teachers to get past their *Cover* stories, remove the "mask" (as one teachers put it), and reveal how they really see themselves. Perhaps if there were more developed constructs to understand these identities, we as a research community can better show the value in knowing what makes each mathematics teacher human, unique and special.

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HONORING TEACHER'S IDENTITY: A JOURNEY TOWARDS NON-EVALUATIVE LISTENING

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An evaluation of the impact of a professional development experience on participants' ability to explore student voices as input for improving the teaching of mathematics evolved into a self-study of our growth as non-evaluative listeners. This paper specifically describes our emergent awareness of the evaluative stance implicit within our attempt to examine teachers' writing samples with the goal of developing a framework, denying teachers agency and identity. This presented us with a living contradiction since this stance conflicted with our belief that learners deserve both.

Keywords: Equity and Diversity; Teacher Education–Inservice/Professional Development; Instructional Activities and Practices

Introduction

This paper exemplifies transition; it is the story of our journey along a continuum of professional growth. It is told in three parts, parts that defy the typical organization of a research report. We begin at our genesis: an evaluation of the impact of a professional development experience on participants' ability to explore student voices as input for improving the teaching of mathematics. We then describe the transition of our work from an evaluation project to a self-study of our growth as non-evaluative listeners. Our self-study resulted in an awareness of the evaluative stance implicit within our attempt to develop a framework by which to classify teachers' writing samples, thus denying teachers agency and identity. We end the paper with a discussion of the theoretical stance that grounds our work as we consider future teaching and research activity and the "living contradictions" (Whitehead, 1989) that have emerged creating new dissonances in our practices.

Genesis

We, the authors of this paper, were involved in the planning and implementation of a large scale Mathematics and Science Partnership for professional development. Our goals, identified in concert with district faculty and administrators, were to support teachers in becoming better listeners and in understanding the importance of listening to students as a major component of their practice. In conducting the workshops for teachers we were operating under norms for best practices for professional development as defined by the larger mathematics education community. Lesson study (Yoshida, 1999), using student interviews (Schifter & Fosnot, 1993), and Thinker-Doers (Hart, Najee-ullah, & Schultz, 2004; Hart, Schultz, & Najee-ullah, 2004) were all integral components of our program that are defensible with tomes of literature.

We began this project in an effort to evaluate one cycle of this professional development. Our research question was: "How effective had we been in supporting teachers to become better listeners and to understand the importance of listening to students as a major component of their practice?" Participants in our professional development had conducted clinical interviews with their students and had written a reflection paper summarizing their interpretations of students' mathematical understanding and the implications for their teaching. Therefore, we decided to use these data to explore our research question.

At this stage we were framing our work according to the norms of action research (Lewin, 1946) with an emphasis on a qualitative analysis of the teacher reflections. As action researchers we were looking for indicators of how our cycle of PD practices had impacted our teachers' listening strategies. We thought that our analysis would provide insights as to how teachers used the voices of students to make sense of students' mathematical understanding.

Early in our work we found ourselves positioning the teachers on two dimensions, as to whether they seemed teacher-centered or student-centered and whether they were analytic or descriptive in their reflections. In this positioning, we attempted to keep individual reflections intact and carried through the individual contexts in which those teachers were working. As our work progressed we found that some of the data allowed us to make clear decisions as to these two dimensions. However, some cases were much more difficult to categorize. Keeping the analysis at the level of the teacher became unwieldy and in our second attempt we agreed to work with excerpts or "chunks" from the papers. By reducing the grain size to passages rather than entire papers, we tried to keep the focus on abstract ideas rather than individuals. In Figure 1 we present two iterations of our framework. The early stage of analysis resulted in sorting the data according to a framework with two dimensions and multiple levels of nuance. As our analysis of the data evolved, we recognized the need for more encompassing and detailed categories leading from the framework on the left to that on the right.

Preliminary Framework		
Analytical Reflection		
Descriptive Observation		
	Teacher-Centered	Student-Centered

Extended framework			
Inference			
Critique			
Description			
	Teacher	Student–Affective	Student – Mathematical Understanding

Figure 1: Evolving framework

As our work progressed we sensed personal disappointment in the work of the teachers. We had inadvertently understood the diagonal (from lower left to upper right) of our new extended framework to indicate growth along a listening continuum. We had hoped that more of our teachers' chunks would have been placed in the student-centered inference cell. To us this would indicate a teacher who listened and reflected on the child's understanding of mathematics. How could a professional development program grounded in best practices have had so little impact on teachers' listening strategies? We began to conclude that we had failed in the mission and goals of our program.

During a professional conference we received positive feedback from members of the mathematics education community about the framework and the way we were analyzing our data. The exercise of discussing and negotiating with colleagues regarding where to place teachers' work on the framework proved to be stimulating and educational for us. It wasn't until colleagues suggested that this framework could be used to create vectors that characterized teacher growth over time that we started to sense a discomfort in the goals of our actions. This interpretation of our work, both in the moment and in its future retelling within our group, reflected to us like a mirror the true nature of our work. Juxtaposed with teaching teachers to listen non-evaluatively was our own story as teachers, listening in judgment of our students.

In hindsight, having someone challenge our framework could have caused us to realize the nature of our evaluative posture and pushed us further along the continuum of our professional growth. It wasn't until we started writing our findings and results that we became increasingly aware of our living contradictions. In theory we believed in (and taught project participants about) listening non-evaluatively to students in order to gain insights into their mathematical understanding. Yet, we were unable to enact the same non-evaluative listening practice with our own students (participants in our PD). We were listening non-evaluatively to our teachers' sharing of conceptual understanding during their mathematical activities, but were unable to suspend doubt (Harkness, 2009) and judgment when they shared genuine reflections about their practice.

In looking back now it is interesting to note that nearly a year went by during which we were naïve about the contradictions in our beliefs about teaching and our practices. Throughout that year we had engaged in activities that we believed would push us toward deeper understanding of our practice. We collaboratively reflected on our program, participated in a reading group on postmodern and critical theories, and attended professional conferences as a venue for vetting new ideas and receiving challenging feedback. However, to become more aware of our evaluative nature and the contradiction we were living, it would take three key catalysts: A personal reflection from a colleague, a revisiting of postmodern thinking, and efforts to situate our research within a shifting paradigm.

Major Transitions

Three Key Catalysts

A personal reflection, told in first-person narrative. A group of mathematics educators attending an international conference were invited to observe and experience local mathematics classrooms. The purpose was to understand the local context and culture of mathematics teaching, a context and culture unfamiliar to attendees. I traveled with a small group of mathematics educators to observe a day at a government-funded elementary school. We were given a warm welcome by children dressed in their best uniforms, wearing fresh flowers in their braided hair, performing traditional songs and dances. The lesson I observed was in a classroom studying 3-D geometry. There were interesting artifacts on display including local containers used to measure milk along with tins and boxes presumably used to talk about the volume of prisms. The teacher appeared proud of the lesson and artifacts used and eager to give students a stage on which to demonstrate what they knew. We witnessed many recitations and demonstrations by eager students who waived their hands wildly to signal to the teacher that they were ready to shine. Both the teacher and the children had worked hard to impress the visitors. At the end of the lesson, we were given the opportunity to ask the teacher questions about the lesson and about the school. Few questions were asked, and those few were along the lines of "How long has this lesson been going on?"

The next morning, the group of mathematics educators reconvened outside of the context of the school. Immediately, the conversation turned to a discussion of what we had seen. We had not been there in the capacity of evaluation, yet we automatically assumed this role. The criticisms flew around the table indicting not only the actions and decisions made by the teacher, but also the skill of the students. *"The lesson was taught by rote. The students were memorizing and not reasoning. There was too much focus on multiplication facts and too little on measurement concepts or problem solving. If the lesson had been rehearsed (it must have been), then who knows if the students even understood what they had been asked to recite and demonstrate?"* I, like others in the group, was comfortable dissecting the lesson and took license in judging what we had seen without any further context or background.

Once this story was shared within our current community, our group began to reflect on the act of observing and studying teaching and learning. This particular story evoked concerns about the evaluative stance that is so natural to this work. As we discussed the story together as a group, we discovered empathy for the teacher and the students and regretted the missed opportunity to understand the complexity of a specific act of teaching. The opportunity had been given to uncover that complexity and truly understand the dynamics of the lesson; the teacher had invited questions and discussion, yet no one had thought to ask about the specific needs of the teacher, students and community and why this particular lesson could help fulfill those needs. The group had denied the teacher and students reason. Who gives mathematics educators the right to judge teachers and their enactments? Are we such experts that we can, on first sight and without economic, political or cultural context, determine the value of an instructional episode? How quickly we strip teachers of agency (Valero, 2004) and identity (Brown, Jones, & Bibby, 2004).

Revisiting postmodern thinking. Concurrently with our evaluation project, we were all involved in a book study of *Mathematics Education Within the Postmodern* (Walshaw, 2004). Each week we met to discuss a different chapter, each of us taking turns facilitating that discussion. Some chapters we discussed for multiple weeks, arriving at insights we valued and took personally. These discussions were humbling for many of us as we began to see similarities in our thinking about our "students" and the structures in our educational system that oppress students and teachers. The constructs of power, agency, privilege, identity and oppression were particularly central to our discussions and seemed relevant to our work with students and teachers.

This was all in a general abstract sense. It was not until we began considering these issues in our research practice and the reflection above was shared that our thinking on these matters became concrete and available for application. It was as if the pieces of a jigsaw were flying about in the ether, but had finally begun to arrange themselves in a way to create a picture of our research practice. It was very much like the experience shared by Valero (2004), "my postmodern attitude did not result from a conscious paradigm selection; rather, it was constructed as I met school leaders, teachers and students in different schools in the world whose lives shook me in significant ways" (p. 36).

Our colleague's personal reflection was an obvious example in which we could apply these new principles and identify the power structures that existed. Much more challenging was the application of these principles to our practice. As we continued to revise and reconsider our work in framing the work of teacher listening, we faced this challenge head on. Revisiting our previous discussions and readings from the study group caused us to question the act of characterizing individuals within any framework, and particularly the one we had developed. We expected our teachers to gain respect for the whole student and not parcel their perceptions into evaluative boxes like "mathematically correct." Yet, we were doing this for them. We were being evaluative listeners and positioning them according to our critical lens, denying them voice and reason in their own practice. At this point, our conversation and the purpose of our project shifted in substantive ways. As one member stated, "*As I analyzed reflections, I felt more aware of the difficulty of what we were asking them to do and the vulnerability it required giving me more empathy for the teachers.*"

Acknowledging living contradictions in our work. According to Whitehead (2009), the practitioner addressing the question "How do I improve what I am doing?" will engage in a reflection that will illuminate their living contradictions. As he explains: "I am thinking here of 'I' existing as a contradiction in the sense of holding together a commitment to live certain values with the recognition of the denial of these values in practice" (p. 87). We frame this discussion of our living contradictions as it relates to our practice as researchers and teachers.

We chose a qualitative research design to best address our research purpose. The qualitative research design that we adhered to denied us our values—to respect and honor teachers' voices—the very values that we wanted our teachers to accept as a critical component of good teaching. In our quest to be scientific and methodical in our research process, we identified a data set, i.e., teachers' written reflections, that we analyzed and interpreted using the tools of qualitative inquiry. As warranted by the norms of academic research involving human subjects, we were concerned about preserving anonymity and remaining unbiased in our interpretations of data. This led us to devise coding mechanisms that masked teachers' identities. Also, in an effort to make more of their statements fit our framework, we cut up entire reflection papers into smaller chunks. All of this manipulation of data fragmented the teachers' work and thus created an abyss between the teachers' reflections and the context in which they had been operating. In concealing the teachers' identities we were no longer able to honor their voices and engage in non-evaluative listening. We realized that our chosen research paradigm denied us the opportunity to listen. We had interpreted teachers' writing without considering the social, political and cultural realities of teaching.

Just as we denied our values of respecting and honoring teacher voices in our research, we realized that the same could be said in relation to our teaching. What began as a study of our teacher's writing samples became this story about the development of a faulty framework – one that revealed to us the limitations of our thinking and the contradiction between assuming an evaluative stance (that gave teachers neither agency nor identity) and preaching that learners deserve both. As constructivist teachers, when teaching mathematics, we have, for the most part, learned how to give reason (Duckworth, 1996) to our students as we listen to their mathematical voices. We have learned how to embrace the mathematics of students in shaping our knowledge of mathematics. We are effective in suspending doubt (Harkness, 2009) as our students describe their mathematical thinking. For the most part, we honor and respect the mathematical voices of our students. For this reason we create a learning environment where we are co-constructors of mathematics with our students. However, our analysis of our work with teachers revealed to us another glaring contradiction—Why were we able to give reason to learners when dealing with mathematics, but so unable to give reason to the learner when dealing with teaching? We seemed to have a pre-conceived vision of what constitutes good teaching and were unable to hear the voices of teachers with alternative perspectives—perspectives that grew out of living within a social, political and cultural reality to which we were strangers.

Moving Forward on the Continuum of Professional Growth

How do we live with our living contradictions? We face the personal challenge of positioning ourselves as mathematics education researchers within a new research paradigm that is more aligned with our values. The pressures of our discipline require adherence to a strict code of long standing expectations regarding what counts as valued research. In fact, these constraints sometimes feel oppressive as we work to align our values to our practice. Still the awareness of the living contradiction in our research will guide our future projects.

The living contradiction in our teaching has caused us to question many of our typical practices as mathematics educators, especially in the role of professional development providers or math consultants to districts and schools. We have often engaged in practices such as:

- Accepting the challenge of helping a teacher "improve" her practice based on just a few observations;
- Watching short video-clips of teachers at professional conferences and drawing inferences about their practice as a whole;
- Making judgments about teacher practices from knowing the textbooks adopted by their districts;
- Consulting with schools or districts and accept the administrator's assessment of their staff; and
- Designing professional development experiences based upon our expert analysis of student performance data.

In hindsight, we realize that in each of these instances we have positioned ourselves as experts and denied our teachers agency and identity. The challenge that remains for us is to find a way to enact our new perspective on our role in professional development. What does it mean to engage in professional development with teachers without assuming an evaluative stance; to go into our work together with teachers without a preconceived notion of what is to be learned or taught? We want to do work that respects and maintains the dignity of our teachers and gives them autonomy in crafting a picture of ideal practice. We have begun to acknowledge the value of co-constructing meaning alongside teachers, but need to explore models for how this can be accomplished. We want to move from being imparters of teaching knowledge to being co-conspirators in the act of defining good practice. Perhaps the best next step we can take is to talk about our own learning and to continue to document a living theory (Whitehead, 2009).

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"WRONG, BUT STILL RIGHT" – TEACHERS REFLECTING ON MKT ITEMS

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The mathematical knowledge for teaching (MKT) measures have become widely used among researchers both within and outside the U.S. Despite the apparent success, the MKT measures and underlying framework have been subject to criticism. The multiple-choice format of the items has been criticized, and some critics have suggested that opening up the items might be an option. One way of opening up the items is to include commentary boxes that allow teachers to explain their thinking. This paper reports on a Norwegian study where commentary boxes were added to MKT items in order to investigate the connection between teachers' responses to the items and their written reflections. The results indicate that there is a mismatch between the answers given by the teachers on the MKT items and their written reflections. Teachers' written reflections do not always support their responses to the MKT items.

Keywords: Mathematical Knowledge for Teaching; Teacher Knowledge; Teacher Reflections

Introduction

Knowledge about mathematical topics and teaching tasks with which teachers struggle is useful when preparing professional development (PD) programs (Hill, 2010). Various methods have been used to study and assess different aspects of teachers' knowledge (e.g., Hill, Sleep, Lewis, & Ball, 2007). The work by Ball and colleagues at the University of Michigan (e.g., Ball, Thames, & Phelps, 2008) is well known, and they have developed the concept of "mathematical knowledge for teaching" (MKT). MKT is defined as "the mathematical knowledge used to carry out the work of teaching mathematics" (Hill, Rowan, & Ball, 2005, p. 373). They have also developed sets of multiple-choice (MC) items to measure MKT. These MKT measures were designed from studies of the work of teaching mathematics in the U.S. (e.g., Hill, 2010). The results from these researchers' efforts are encouraging. MKT appears to make a difference to the mathematical quality of instruction (Hill et al., 2008), as well as to students' achievements in mathematics (Hill, et al., 2005). Morris et al. (2009, p. 492) have described MKT as: "the most promising current answer to the longstanding question of what kind of content knowledge is needed to teach mathematics well."

Many researchers have built upon the efforts of Ball and colleagues, and the MKT measures have been widely used both within and outside the U.S. (e.g., Ng, 2012). Despite the apparent success of this research, there have also been critics (e.g., Schoenfeld, 2007). It is suggested, for example, that the knowledge required for teaching may be more culturally based than simply pertaining to mathematical knowledge (Stiegler & Hiebert, 1999; Stylianides & Delaney, 2011), and that cultural aspects have not been taken into consideration in the development and application of the MKT measures. There have been efforts to study the challenges of translating and adapting the items into a different cultural context (e.g., Fauskanger, Jakobsen, Mosvold, & Bjuland, in press; Ng, 2012) and to compare some of these challenges (Ng, Mosvold, & Fauskanger, 2012). Substantial additional investigation is needed to learn more about the cultural issues related to the translation, adaptation and use of MKT items in different cultural contexts. Another criticism relates to the MC format of the items. Schoenfeld (2007) claimed that the MC format has the potential to complicate the items for test-takers. This claim was supported by findings from a Norwegian study (Fauskanger, Mosvold, Bjuland, & Jakobsen, 2011). In the Norwegian study, teachers suggested that the items include commentary boxes to enable the teachers to explain their thinking, which has been proposed as one way to open up the items. Thus, an extended discussion of the validity of the items appears to be necessary.

The present paper contains a discussion of the criticism as addressed by adding commentary boxes to enable the inclusion of teachers' written reflections to the MKT items. One expectation would be that teachers responding correctly to an item would also provide reflections that support their responses, but to our knowledge no previous attempts have been made to investigate opening up of MKT items in this way. We address herein the following research question:

> What is the connection between teachers' responses to MKT items and their written reflections concerning the content of the items?

Given the importance of developing students' fluency in multi-digit arithmetic as a foundation on which to build a proper understanding of the decimal number system (e.g., Verschaffel, Greer, & De Corte, 2007), we have chosen to analyze data from a MKT testlet, including four items where different methods of decomposing a three-digit number are presented.

Methods

The research reported in this paper is part of a larger project focusing on teachers' MKT and their corresponding beliefs about MKT (e.g., Fauskanger, 2012). For the purpose of this paper, data from 30 teachers' responses to MKT items and their written reflections will be analyzed, taken from one testlet including four MC items. This testlet has not been released for publication,[1] therefore we are only able to provide a description of it. The stem presents a context dealing with groups of students who have decomposed a three-digit number (e.g., 456) into hundreds, tens, ones and tenths in different ways. The question posed is which answer the teacher should accept as correct. In the first item (1a), the students have answered incorrectly (e.g., 456 decomposed into 4 hundreds, 50 tens and 6 ones). The remaining three items represent correct decompositions including hundreds, tens and ones (1b), hundreds, tens and tenths (1c) and tens and ones (1d). The decomposition that strictly follows the positions (e.g., 456 divided into 4 hundreds, 5 tens and 6 ones) is not present in any of the items.

Although MKT items are normally used in a testing situation, other alternative uses have been applied. In our study, the teachers responded to the items at home. It is important to consider the advantages as well as disadvantages of allowing teachers to work with the MKT items at home (see e.g., Hill, 2011). One obvious consequence is that the teachers in our study had the opportunity to discuss the items with others. Their written reflections, however, were individual. They were asked to reflect on the following questions in the commentary boxes: (1) What do the students responding as in items (a) to (d) know? (2) What, if anything, do they need to learn more about? (3) Do the items in this testlet reflect a content that is relevant for the grade(s) you teach? (Why?/Why not? Please provide an illustrating example from your classroom). The reflections were provided for the entire testlet, not for each individual item.

The 30 teachers (8 men and 22 women) were participating in a one and one-half year PD course, and the written work reported on was given as an assignment after their first day in this PD. Sixteen of these teachers worked in grades 1–4, nine in grades 5–7 and five in grades 8–10. Their working experience as teachers varied from less than 5 years to more than 20 years, and their formal education in mathematics (education) varied from 0 to 60 ECTS.[2]

The analysis was conducted with the aid of the computer software NVivo9 (QSR International). The teachers' written reflections were first divided into two groups. One group contained reflections from teachers who had identified the correct key for all four items in the testlet (main group 1, table 1), whereas the second group contained reflections from teachers who had responded incorrectly to one or more of the items (main group 2, Table 1). In the next phase of the analysis, a grounded theory approach (Strauss & Corbin, 1998) was applied in order to analyze the reflections from these two groups of teachers according to how they had commented on the items. Based on several cycles of reading and re-reading the data, the teachers' reflections were refined into codes that were revised several times to establish consistency. The codes were based on well-established findings from the literature concerning place value (e.g., Jones, Thornton, Putt, et al., 1996; Verschaffel, Greer, & DeCorte, 2007). Two sub-categories (a and b in table 1) were discovered for each of the two main groups as a result of this analysis.

Results

In this section, we present the results from our analysis of data regarding the connection between teachers' responses to the four MKT items and their written reflections on the content of the items. Item 1a presents a context where a group of students had given an incorrect response to the MKT item. All 30 teachers in our study identified the key in this item, but 10 teachers had incorrect responses to at least one of the other items in the testlet. If the teachers selected the alternative "I am not sure," their answer would then be coded as incorrect (see Table 1).

Table 1: Teachers' Reflections Regarding Multiple Decompositions

	Think multiple decompositions are correct (1)	Think multiple decompositions are incorrect (2)
All correct (a)	7	13
At least one incorrect (b)	1	9

It is important to notice that it was not always evident in which groups the teachers should be placed; when placing the teachers in groups we allowed them the benefit of doubt if their reflections were ambiguous (e.g. Gerd's reflection below). Two among the seven teachers in the first group could have been placed in the second group, but were placed in the first group although some thoughts in their written reflections were incomplete.

Correct Responses with Supporting Reflections (Group 1a, Table 1)

Although 20 teachers identified all four correct keys in this testlet, only seven displayed supporting thinking in their written reflections. Oda[3] was one of the seven teachers who gave the correct answer to all four items, and she wrote this in her reflections on item 1c:

> I think we have before us an advanced solution in relation to the place value system (in this item). This student has a well-developed number concept and is able to use his fantasy when replacing the one with tenths. In this way, his knowledge about tenths is displayed.

Two of the seven teachers indicate that the students display a very good understanding when they make a decomposition of numbers that differs from the standard decomposition. Tor writes:

> They could have given a more simple solution by using 4 hundreds, 5 tens and 6 ones, but we can say that some (students) are clever in the way they don't necessarily use the correct decomposition but still get the right answer.

The reflections of some teachers were less clear, and as a result it was difficult to evaluate whether or not they displayed a correct manner of thinking. In relation to item 1b, Gerd writes:

> This is, consciously or unconsciously, written in a more advanced way. It might be that he wants to show that he has complete mastery of this, or it might just be a coincident.

These reflections indicate that Gerd believes that the students might have a more advanced understanding, but she is not certain about whether or not their responses are conscious. Later in her reflections, however, she writes that: "none of these solutions are perfect according to the place value system." This last assertion is not explained further.

Correct Responses with Non-Supporting Reflections (Group 2a, Table 1)

Of the 20 teachers who had correct responses to all four MC items in this testlet, 13 provided reflections that did not entirely support their responses. Dina writes:

> The students need to learn about the standard place value system and the proper exchange between digits.

Brit writes:

> The students need to learn more about exchange, learn to fill up the ones, tens, hundreds, etc. Know that each position has its (distinct) value. When the value exceeds 9, they should shift position.

In relation to item 1d, Frida writes that the students:

> ... lack an understanding of the place value system, and the student only understands the tens place and ones place in the place value system. The total sum is still 456, so the student obviously understands decomposition and the value of the number (...) All students are on their way towards an understanding of how a number can be written in extended form. They have to learn more about the place value system. They need to reach an understanding of which number belongs where, one number in each position.

The suggested solutions in items (b) through (d) are all mathematically correct, and these teachers have identified the correct solutions in the MC items. In their reflections, however, they seem to insist on the mathematical convention: “When the value exceeds 9, they should shift position.” Although they recognize that the students’ solutions were mathematically correct, they do not regard them as “the answer the teacher is seeking,” and, therefore, their reflections do not support their responses to the MC items. The written reflections of these teachers are in line with those given by the nine teachers in the last group (2b in Table 1), which consists of teachers who have identified one or more incorrect keys and who have written reflections supporting their responses to the MC items. For example, Erna identified three correct keys and writes this in her reflections:

> Student b) is wrong, but still right. Incorrect decomposition, but the correct total (amount). The student has understood how to decompose the number so that it doesn’t increase or decrease in value, but still hasn’t placed it correctly according to the place value system.

Eli identified the correct key for item 1a only, and she writes this in connection with items 1b-d:

> b) This student manages to decompose the number 456, but apparently hasn’t completely understood the value of the digits in the place value system. It is indeed correct that you can decompose 456 into 3 hundreds, 15 tens and 6 ones, but this is not the answer the teacher is seeking.
>
> c) This student has understood the value of the numbers 4 and 5, but is mixing up the ones. It is a little bit funny to see that the student makes it so hard on himself. This student knows that there are 10 tenths in 1 one.
>
> d) This student knows how many tens there are in 456, but hasn’t understood the value of the digits.
>
> Which of these answers you should approve as correct depends on how long the students have been working on this topic. If the students have been working with this for an extended period of time, I wouldn’t have approved any of the answers. If, however, this is the introduction to decomposition of numbers, I would have approved b) through d).

In their reflections, the teachers in the fourth group appear to insist on the same mathematical convention as the teachers in the second group do.

Incorrect Responses but “Correct” Reflections (Group 1b, Table 1)

Out of the ten teachers who gave incorrect responses to one, two or three items, or who gave the response “I am not sure” to some of these items, one teacher showed an understanding of the MKT being measured in her reflections. Laura marked all three items with “I am not sure” (which is coded as an incorrect response), but she argued in her reflections that the testlet stem could be interpreted in different ways and that the key for each item would be dependent on how the stem was understood. The following is an excerpt from Laura’s reflections:

> Item a) is wrong by all means. Items b), c) and d) are wrong if it (the problem presented in the stem) is a closed problem, but they are correct if it is an open problem.

By “closed problem” this teacher means using the positions given (e.g., 456 = 4 hundreds, 5 tens and 6 ones) and by “open problem” the teacher means open to other ways of decomposing three-digit numbers. This teacher’s written reflections are in line with some of those from the seven teachers in the first group.

Discussion and Conclusions

Four groups (as presented in Table 1) emerged in our analysis, and the results from our study indicate that there is not always a clear connection between the teachers’ responses to the MKT items and their written reflections. Researchers who use the MKT items would probably expect, or at least hope, that teachers who answer the MC items in the measures correctly also have an appropriate understanding of the content, and the other way around. In our study, there are some teachers who follow this pattern. We have, however, identified an apparent mismatch between the responses to the MC items and the written reflections of several teachers. We have seen an example of one teacher (group 1b) who provided incorrect responses to the MC items but who displayed a high level of understanding in her written reflections. She appears to know the mathematics but she is still unable to determine for what answer the test-makers are looking. Another group of teachers (group 2a), were interesting as they were able to select the correct answer, but appeared to still believe that there is a particular name for the number that is better. These teachers are able to see that “4 hundreds, 5 tens and 6 ones” is the same value as “4 hundreds, 15 tens and 6 ones,” but still believe that the name that matches the place values is better. If we consider the example of using the standard Norwegian algorithm for calculating 456 minus 37 (Figure 1), we have an example where the “place value name” is clearly not the best name for the number. As a result, teachers who hold such beliefs could be seen as in transition along a continuum. First, we have the teachers who are not able to understand non-standard decompositions of numbers. Second, there are teachers who can understand multiple decompositions of numbers, but who still believe that the standard decomposition is best. Third, there are teachers who understand and value multiple decompositions of numbers. Fourth and finally, there is a possibility that some teachers understand, value and can explain the use of alternate decompositions.

$$\begin{array}{r} {}^{10} \\ 4\cancel{5}6 \\ -\ \ 37 \\ \hline 419 \end{array}$$

Figure 1: Standard Norwegian algorithm for subtracting 37 from 456

The findings may be explained in a variety of ways. We present four possible explanations as follows.

1. The findings are incidental. This might be connected with our study being limited by the number of participants as well as the limited focus of the items. More research is necessary in order to investigate whether or not the same tendencies can be found in a larger number of participating teachers.
2. This apparent mismatch is specific to this particular topic. It would be pertinent to also investigate whether or not, or if the same pattern can be found for all sets of MKT items.

3. There are cultural differences involved in how teachers reflect upon these items. One such difference might be related to how the decimal number system is taught. If such cultural differences are involved, it would be of great interest to conduct additional research to investigate this further. Researchers have already adapted and used MKT items in different countries (e.g., Ng, 2012), and if there are cultural differences related to the connection between teachers' responses to items and their reflections, great care should be taken when it comes to how the results from such studies are interpreted. We suggest that efforts should be made to investigate these issues both inside and across cultures to learn more about the connection between teachers' MKT, their corresponding beliefs and the educational culture(s).
4. A final possible explanation of the differences between these Norwegian teachers' responses to the MKT items and their written reflections is that there are indeed, as Schoenfeld (2007) argued, constraints resulting from the MC format of the items. Such possible difficulties with this format might be specific to culture, as Fauskanger and colleagues (2011) suggest. From the results of the present study, however, it does not appear that the MC format itself makes it more difficult for the teachers. The complicating connection between the teachers' responses to the MKT items and their written reflections only indicates that the MC items are harder to interpret than they might appear. The inclusion of commentary boxes along with the items, or other ways of opening up the items, should be investigated further. It would also be relevant to include interviews with teachers to further investigate teachers' reflections as well as the connection between these reflections and their MKT as measured by their responses to the items.

Our study indicates that researchers have to be careful concerning the conclusions they draw when measuring teachers' MKT. Particular care should be taken when using these measures in other cultural settings and more research is needed in this area. We argue that it is important to include the teachers' reflections in order to learn more about their MKT, and more research is needed to investigating teachers' epistemic beliefs (Fives & Buehl, 2008) related to the different aspects of MKT. Analyses of teachers' reflections concerning MKT items can be particularly useful in this regard. Follow-up interviews with teachers in groups 1b and 2a would also be relevant for future studies.

Endnotes

[1] The numbers have been changed in our descriptions of the item in order not to reveal the entire item, and these details in the teachers' reflections have been changed accordingly.

[2] ECTS stands for European Credit Transfer and Accumulation System. One year of full-time studies in Norway gives 60 ECTS.

[3] The teachers' names have been changed to ensure anonymity.

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A PROCESSES APPROACH TO MATHEMATICAL KNOWLEDGE FOR TEACHING: THE CASE OF A BEGINNING TEACHER

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This study examines the connection between mathematical knowledge (described as a teacher's engagement in mathematical processes and actions on the products of those processes) used by a beginning secondary mathematics teacher (Fiona) in her personal mathematical problem solving and the mathematics in which she engaged her students in her classroom instruction. This Process and Action approach involved analysis of Fiona's use of four mathematical process/product pairs (justifying/ justification, generalizing/generalization, defining/definition, and representing/representation). Two themes arose in the analysis of interview and classroom observation data: (a) Although able to do so, Fiona did not regularly engage in processes in her personal mathematics or classroom mathematics, and (b) Fiona focused on selected features of a product or mathematical object rather than attending to all relevant features.

Keywords: Teacher Knowledge; Mathematical Knowledge for Teaching

Purposes or Objectives of the Study

Research interest has recently burgeoned regarding the relationship between teachers' mathematical knowledge and the ways that that knowledge impacts what happens in the classroom. Of particular interest is how teachers' knowledge affects both what teachers do and what students learn. Over the years, researchers (e.g., Eisenberg, 1977; Hill, Rowan, & Ball, 2005; Monk, 1994) have investigated the relationship between teacher knowledge and student achievement, typically using proxies for teachers' mathematical knowledge. These studies have often found that teacher knowledge is related to student achievement, but they have not shed light on how teacher knowledge affects what is happening in classrooms. This question has received much less attention from researchers, and many of the studies of that relationship do not separate content knowledge and pedagogical content knowledge (e.g., Hill, Ball, Blunk, Goffney, & Rowan, 2007; Lehrer & Franke, 1992; Swafford, Jones, & Thornton, 1997), leaving one to wonder about the effects of content knowledge itself on instructional practice. Studies that examined the relationship between content knowledge and classroom practice (e.g., Baumert et al., 2010; Rowland, Martyn, Barber, & Heal, 2000; Tchoshanov, 2011; Wilkins, 2008) have found relationships, but these studies focused on narrowly defined aspects of classroom practice (e.g., cognitive demand of tests and homework, students' opinions about instruction, teachers' self-reports of reform practices) and used written tests of predetermined categories of teacher knowledge (e.g., high and low content knowledge, cognitive type of content knowledge). Although the studies identified relationships, they did little to explain why these relationships might have occurred. The study reported here used extensive sets of interviews and observations focused on teachers' mathematical knowledge and its use in the classroom to characterize and explain the relationship between a teacher's mathematical knowledge and classroom practice. This study addressed the question of what characterizes a beginning secondary mathematics teacher's engagement in personal mathematics and classroom mathematics and the relationship between them.

Perspective

We describe mathematical knowledge in terms of four mathematical processes and their respective products: defining, justifying, generalizing, and representing, and actions on (or uses of) definitions, justifications, generalizations, and representations (Zbiek, Peters, & Conner, 2008). Using the processes and products to characterize a teacher's mathematical knowledge allows us to examine the mathematics demonstrated in his/her problem solving (*personal mathematics*), the mathematics in which the teacher engaged his or her students (*classroom mathematics*), and the relationship between the teacher's personal and classroom mathematics. A second affordance of using mathematical processes is that it transcends both mathematical content areas and grade levels, since the use of these processes and products is not dependent on either of these factors. Examining mathematical knowledge as knowledge evidenced by engagement in mathematical processes allowed us to examine mathematical knowledge over a period of three and a half years across three different content areas.

Methods

Our data consist of five task-based interviews and 16 teaching observation cycles of a secondary mathematics teacher, Fiona (a pseudonym). At the beginning of data collection, Fiona was enrolled in a secondary mathematics teacher certification program at a large Mid-Atlantic university. She was one of several in the program who volunteered to participate in the study. The task-based interviews (*Area*, *Count*, *Cube*, *Wrap*, and *Defining*) were conducted during Fiona's teacher preparation program for the purpose of understanding her use of mathematical processes and products in her personal mathematics. To understand Fiona's use of processes and products in her classroom teaching, teaching observation cycles were conducted during her student teaching (pre-calculus), first-year teaching (algebra), and second-year teaching (geometry). An observation cycle consists of a pre-observation interview, an observation, and a post-observation interview.

All the task-based interviews were videorecorded and audiorecorded, transcribed, and annotated. Teaching observation cycles were audio-recorded, transcribed, and annotated. Still photos from the teaching observation cycles were also collected. The task-based interviews were coded line-by-line for Fiona's use of processes and/or products by the research team. The coded instances were elaborated, categorized into the four process/product categories, and analyzed for emerging themes. Any disagreements were resolved by review of the data by the entire team. This procedure was repeated for the teaching transcripts, but included the coding and analysis of mathematical activity and pedagogical choices. After the initial coding and analyses, the team then compared Fiona's use of process and/or products in her personal mathematics with their use in her classroom mathematics.

Figure 1 illustrates parts of the Cube and Area tasks. In Cube, Fiona was asked to describe the pattern and determine the surface area and volume of the model shown in the left panel of Figure 1. In Area, Fiona was asked to describe the mathematical relationship between the sum of the area of the circles and the area of the equilateral triangle shown in the right panel of Figure 1 as the number of circles on the base increased. [One side of each equilateral triangle passes through the centers of the circles on that side and the endpoints of the same side lie on circle(s).]

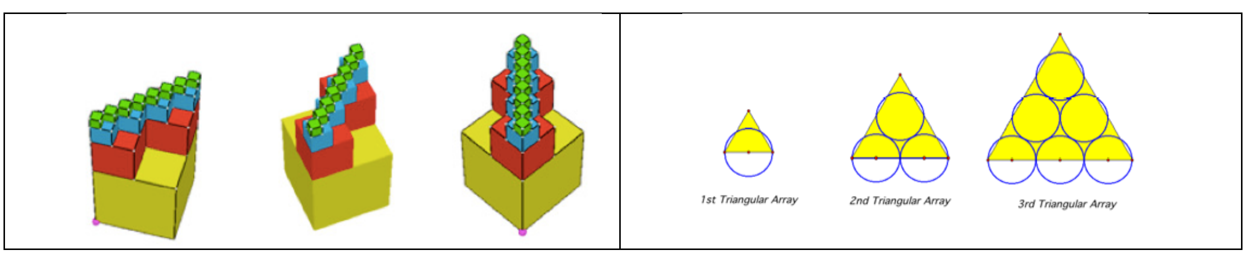

Figure 1: Some of the illustrations accompanying the cube and area tasks

Results

Representing

Fiona tends to notice and pay attention to only select features of representations rather than accounting for the representation's complete set of relevant characteristics. In Area, when Fiona was given the graph of the function defined as the difference between the area of the triangle and the sum of the areas of the circles, she focused on the x-value of the minimum point of the function as invariant and interpreted it as the point at which the area of the triangle exceeds the sum of the area of the circles. She seemed to recognize that the graph of the differences required a change from negative to positive without recognizing that the negative-to-positive change captured by the minimum was a change in slope rather than a change in the output value of the function. She did not attend to the x-intercepts, the feature of the graph most relevant to the question she was asked, until she was specifically asked about them.

In her classroom mathematics, Fiona missed opportunities that might have engaged her students in linking different representations. For example, in student teaching, she purposefully did not interpret for students a graphical representation of derivative that appeared on an activity sheet. During the post-observation interview, Fiona ably linked this graphical representation to a symbolic representation of derivative but she stated that "they don't know that, and if I would explain it to them I would have confused them, I think endless amounts" as the reason for not discussing the graphical representation. She seemed to have made a conscious choice not to include this in her lesson. This might have been because her goal was only for students to be able to apply the limit definition of derivative in order to complete exercises and she thought that trying to develop further understanding of the limit definition was not worth confusing students.

In both her personal and classroom mathematics, Fiona often focuses on local features of representations and seems not to grasp the entire representation. This tendency of localization and inattention to connections is frequently observed in her personal and classroom mathematics and suggests that mathematics as an integrated system is not central to her conception of mathematics.

Justifying

In her personal mathematics, Fiona regularly makes initial mathematical claims for which she provides no or limited mathematical rationale. Fiona offers mathematical justification unprompted only when she recognizes an error and engages in correcting the error. Otherwise, Fiona justifies her mathematical claims only after she is prompted by the interviewer with questions such as, "How might you convince someone of your claim?" Moreover, when Fiona justifies by referencing properties of mathematical objects, she often fails to complete a valid mathematical argument. In these instances, Fiona often attends to one property of the mathematical object, but fails to attend to other relevant and necessary properties. In the Area interview, for example, Fiona engaged in justifying that the sum of the areas of the circles in an array is larger than the area of the triangle in the same array. She identified two differences in the symbolic representations of the two areas, but based this argument on one difference in the formulas (one area formula involved multiplying by π and the other involved dividing by the square root of three) without accounting for the other difference (($x + (x - 1) + (x - 2) + \ldots + 0$) versus x^2) (see Figure 2).

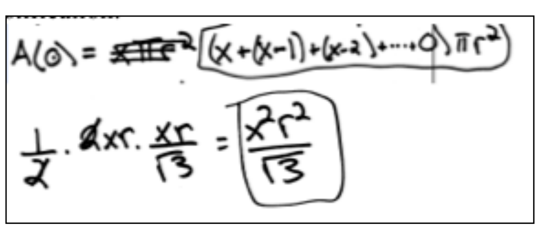

Figure 2: Fiona's representations of the sum of areas of the circles and the triangle area

In the context of Fiona's classroom mathematics, Fiona seldom engages in mathematical justification or asks students to justify. Across all three years of teaching, Fiona regularly misses opportunities to engage students in justifying. One of these instances occurred as Fiona taught rules for derivatives in student teaching. Fiona followed her presentation of the product rule with a presentation of the quotient rule. Even when a student pointed out the similarity between the product rule and the quotient rule, Fiona did not use the comment as a segue to recognizing and justifying that the quotient rule can be viewed as an instance of the product rule.

When Fiona engages in justifying or asks students to justify she often accepts superficial rationales. These superficial rationales were often rules or step-by-step procedures Fiona taught the students to use for a set of homework exercises. In first year teaching, for example, Fiona asked the students to justify the claim that 156 is the y-intercept of the equation $y = 78x + 156$. Fiona accepted a student response of "Because I know the equation is y equals $mx + b$ and then b is the y-intercept," echoing a fact that Fiona had taught during the previous class.

Instances of justifying in Fiona's personal mathematics and teaching seem to indicate that Fiona sees the role of justifying as verifying an assertion rather than as a critical process in her mathematics. Although Fiona ably demonstrates her ability to justify in her personal mathematics, she rarely does so without prompting. Often her justifications are invalid, because she attends to only some of the relevant features or properties of a mathematical object. In Fiona's classroom mathematics, she justifies or has students justify in the context of reviewing homework exercises or in-class examples. These justifications are usually superficial rationales or rules that do not provide mathematical connections.

Generalizing

In Fiona's personal mathematics, she generalizes but rarely uses generalizations. Although she generalizes, she does not tend to generalize without being prompted even when it seems reasonable to do so. For example, in Cube, Fiona was asked to find the volume and surface area of a stack of cubes. Fiona recognized that the volume of a cube in one particular layer was one-eighth the volume of a cube in the previous layer, but was hesitant to conclude that this was true for all layers and did not do so until she was prompted. Fiona's generalizations are not always correct, and these incorrect generalizations are often due to her focusing on a limited domain or set of examples and not accounting for all possibilities. For example, in Count when asked about a three-dimensional analogue of the circles situation shown in the right panel of Figure 1, Fiona generalized that no matter the size of a pyramid constructed of spheres, there are no interior spheres. She based this incorrect generalization on having examined only a single case.

In Fiona's classroom mathematics, there were many more instances of generalizations than generalizing. Similar to her personal mathematics, Fiona does not engage in generalizing when it seems that it would be appropriate to do so. For example, rather than giving students a single equation that can be used in various exercises, Fiona directed students to use three different equations for three different, but clearly related, cases: the equation $y/x = k$ to find the value of k, the equation $y = kx$ to find the value of x, and the equation $f = kd$ to find the value of k in an exercise involving Hooke's Law. Fiona discusses and implements activities that potentially provide students the opportunity to engage in generalizing. However, when she implements activities aimed at generalizing, she usually leads students to reach a generalization that she has predetermined rather than allowing for generalizations she has not anticipated. The theme in her teaching seems to be that generalizations are finished products, suggesting that she may consider the universe of possible generalizations as fixed and known. Also, Fiona states generalizations that are false, often based on a limited domain or set of examples. For example, when introducing a lesson on graphing lines, Fiona states the incorrect generalization, "There is a y-intercept and an x-intercept for every single line," not accounting for horizontal or vertical lines.

Although Fiona's personal mathematics did not make use of generalizations and her classroom mathematics contained almost no generalizing, Fiona's personal mathematics and her classroom mathematics have two main commonalities. First, Fiona often does not generalize when it seems as if it would be appropriate to do so. Second, Fiona often incorrectly generalizes or states incorrect generalizations because she is focusing on a limited domain or examples and is not accounting for all possibilities. The absence of further commonalities in Fiona's generalizing and use of generalizations may be attributed to the lack of the use of generalizations in her personal mathematics and the lack of generalizing in her classroom mathematics.

Defining

As with other processes, the process of defining does not seem to play a central role in either Fiona's personal mathematics or her classroom mathematics. We hardly ever observed her engaging in defining and most often we saw her engaged with the product of defining, namely definitions. In her personal mathematics and teaching, she tends to focus on elements of definitions rather than thinking about the definition as a whole. She seems to compartmentalize definitions and not to coordinate them in her teaching or in her personal mathematics. In the task-based interviews, she was presented with six ways in which people may talk about a parallelepiped (see Figure 3). She was asked which of the six descriptions are most similar to each other. If Fiona were to be thinking of these six statements as defining six separate objects, we would expect her to consider the mathematical entity each statement defines and to compare those entities. However, she chose to examine parts of each statement and to compare them to parts of other statements. For example, she stated that descriptions B and F are similar to each other because they both describe a six-sided polyhedron. However, she never endeavored to examine each description as a whole.

a. **A parallelepiped is a box with a lid.**
b. **A parallelepiped is a polyhedron with six faces that are parallelograms, which could be rectangles, squares or rhombuses but not trapezoids.**
c. **A parallelepiped is a cube whose sides are rectangles.**
d. **A parallelepiped is a prism the base of which is a parallelogram.**
e. **A parallelepiped is a hexahedron each face of which is a parallelogram.**
f. **A parallelepiped is a hexahedron with three pairs or parallel faces.**

Figure 3: Six ways people may talk about a parallelepiped

This tendency to focus on parts of a definition, rather than on the whole definition, is also reflected in her teaching. For instance, Fiona presented a definition of a vertex as "a point at which three or more faces meet." However, later on during the same lesson, Fiona introduced the phrase, "a vertex of a cone," offering a description: for a cone, "a curved surface connects the base to the vertex." One of her students pointed out that what she has labeled as a vertex is not actually a vertex, given the original definition of a vertex. Fiona agreed with the student without offering an explanation for her agreement. In the post-observation interview, Fiona explained that the student who asked the question is "very smart" and answering his question would just confuse the other students in the class.

The previous example involving the vertex also highlights how Fiona seems to view mathematics as a static and fixed body of knowledge, rather than something that can be discovered using the processes. If Fiona were to privilege a perspective of mathematics that encourages involvement in mathematical processes such as defining, we would have expected her to engage the student in a discussion of the mathematical properties of the two definitions of a vertex. This view is further illustrated in her first year of teaching when she presented her students with a definition of a y-intercept of a line. Fiona asked one of her students to read the textbook definition of a y-intercept, "The y-value of the point where the line crosses the y-axis." However, when Fiona repeated this definition and subsequently used it, she re-worded it as, "The y-intercept is the point where our line crosses the y-axis." Fiona seemed unaware of the change she made to the definition of a y-intercept, or she may not have seen a difference in the two definitions presented. Either possibility points to her seeing mathematics as comprised of static and disconnected pieces of information, and thus, if these pieces of information conflict with each other, she does not seem perturbed by it.

Conclusions

Two themes characterize Fiona's use of processes and actions on the products of those processes. These themes cut across several processes and are evidenced both in her personal mathematics and in her classroom mathematics.

Themes

The first theme is that, although Fiona has demonstrated competence in engaging in all four of the mathematical processes studied, the processes of generalizing, justifying, and defining are not central to how she engages in mathematics or how she engages students in mathematics. In her task-based interviews, Fiona generally engages in these processes only when prompted. Similarly, while teaching, she seldom engages in processes or requires that students do so even though we observed several occasions (e.g., students asking Fiona for justification or Fiona providing students with activities designed to lead to generalizing) in which it would have seemed reasonable to engage in processes. Fiona is far more likely to engage in actions on the products of processes than to engage in processes or to engage her students in those processes. The possible exception to this is the process of representing. In her personal mathematics, Fiona seems to use representing to help her in problem solving, often using one type of representation to create another and connecting representations to provide justifications. In her classroom, although Fiona occasionally directs students to use different representations in problem solving (e.g., directing students to draw a graph if they are struggling with writing an equation of a line or having them use geometric figures to generate tables of values in order to look for a generalization) she often misses opportunities to have students examine multiple representations even when it would seem to make sense to do so (e.g., not showing students a graph to explain a limit definition of derivative).

A second theme is that Fiona has a tendency, when working with processes and actions on the products of those processes, to focus on some features of a product or mathematical object and not attend to other relevant features. In her personal mathematics, many of Fiona's justifications are incorrect because she has not attended to all of the relevant characteristics of the object in question. In the Defining interview, Fiona incorrectly defines a particular set of polyhedra as having exactly one pair of parallel faces without recognizing that some of the polyhedra in the set have more than one pair of parallel faces. In her classroom mathematics, she uses the term vertex as having universal applicability and fails to distinguish between definitions of a vertex of a polyhedron and a vertex of a cone.

Conceptions of Mathematics

Much of the work to date on teachers' and students' conceptions of mathematics has been focused on describing and categorizing these conceptions. For example, Ernest (1988) categorized conceptions of mathematics into three broad categories: an instrumentalist view of mathematics as a set of unrelated but utilitarian rules and facts, a Platonist view of mathematics as a static body of knowledge, and a problem solving view of mathematics as dynamic and continually expanding. Lerman (as cited in Thompson, 1992) identified two different prevailing conceptions of mathematics: the absolutist perspective, that is, the perspective that "mathematics is based on universal, absolutist foundations" and the fallibilist perspective that "mathematics develops through conjectures, proofs, and refutations and is accepted as inherent in the discipline" (Thompson, 1992, p. 132).

There is growing evidence that teachers' conceptions of subject matter have an influence on their classroom instruction and there have been a few studies that provide evidence of the link between teachers' conceptions about mathematics and their instruction (e.g., Cross, 2009; Raymond, 1997; Thompson, 1984). It is evident that this influence is not direct or simple and results are inconsistent about how strong that influence may be.

Although Fiona did not speak directly about it, it is conceivable that Fiona's conception of mathematics could explain her approach to processes and actions on the products of those processes in both her personal mathematics and her classroom mathematics. The pervasiveness and consistency of the two themes we identified across the four processes and in both Fiona's personal and classroom mathematics seem to indicate that Fiona has a conception of the nature of mathematics that is not centered on the use of mathematical processes and does not require attention to connections and consistency. It seems likely that Fiona views mathematics as a fixed body of facts, ideas, and rules that are not necessarily or easily connected. Within such a concept of mathematics many of Fiona's actions make sense. For example, if mathematics is a fixed set of ideas then it is reasonable that Fiona does not encourage creative generalizing but chooses to focus attention on the generalization students are supposed to be learning. Fiona's apparent lack of attention to the fact that mathematics needs to be connected and coherent helps to explain why she seems not to notice or be concerned about contradictory definitions of vertex. Meanwhile, a view that mathematics is fixed and static may explain why, when a student points out the discrepancy, she might find it adequate simply to tell students that "it's part of the definition."

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MATHEMATICAL KNOWLEDGE FOR TEACHING HIGH SCHOOL GEOMETRY

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This paper documents efforts to develop an instrument to measure mathematical knowledge for teaching high school geometry (MKT-G). We report on the process of developing and piloting questions that purported to measure various domains of MKT-G. Scores on the final set of items had no statistical relationship with total years of experience teaching, but all domain scores were found to have statistically significant correlations with years of experience teaching high school geometry. We use this result to propose ways of conceptualizing how instruction-specific considerations might matter in the design of MKT items.

Keywords: Mathematical Knowledge for Teaching; Geometry; High School Education; Assessment

Overview

In his description of paradigms for research on teaching, Shulman (1986a) had called for a focus on teacher knowledge. With particular reference to mathematics, Ball, Lubienski, and Mewborn (2001) responded to Shulman's call by, on the one hand, reviewing research that showed that traditional measures of teachers' content knowledge (e.g., degrees obtained or mathematics courses taken) had not shown to make a difference on students' learning and, on the other hand, arguing that the kind of teacher knowledge needed to focus on was a particular kind of mathematical knowledge, mathematical knowledge for teaching (MKT). This MKT is knowledge of mathematics used in doing the work of teaching and it includes but also goes beyond the pedagogical content knowledge that Shulman (1986b) himself had proposed. The theoretical and empirical work on Ball's brand of MKT that followed such proposal has been vast, showing among other things that the possession of MKT can be measured, that MKT is held differently by teachers and non-teachers, that MKT is held differently by teachers of higher grade level experience than those of lower grade level experience, that it makes a difference in students' learning, and that scores on MKT correlate with scores on an observation measure of good teaching (Hill, Schilling, & Ball, 2004; Hill, Rowan, & Ball, 2005; Hill et al., 2008). The work on constructing measures of MKT has been concentrated mostly on the mathematical knowledge of elementary and middle school teachers (Hill & Ball, 2004; Hill, 2007); a more recent effort has developed MKT items in algebra (Mark Thames, personal communication, 6/15/11). The purpose of this paper is to report on a parallel effort to develop an instrument that measures mathematical knowledge for teaching high school geometry. Our effort has attempted to follow the theoretical conceptualization of MKT and item development procedures of Ball and Hill's group. The paper provides pilot data that compares high school teachers with and without experience teaching geometry in terms of their possession of mathematical knowledge for teaching geometry, and it uses these results to raise some questions about the content specificity of the notion of mathematical knowledge for teaching.

A crucial element in our development of items to measure the mathematical knowledge for teaching high school geometry has been Ball, Thames, and Phelps (2008) conceptualization of the different domains of mathematical knowledge for teaching. According to Ball et al. (2008), the mathematical knowledge used in teaching can be conceptualized as the aggregation of knowledge from six domains. These domains include Common Content Knowledge (CCK), which is the mathematical knowledge also used in settings other than teaching, including for example knowledge of canonical methods for solving the problems teachers assign to students. The domains also include knowledge that is specific to the work of teaching. Thus Specialized Content Knowledge (SCK) is knowledge of mathematics used particularly in doing the tasks of teaching, such as, for example, the knowledge a teacher needs to use in writing the problems they will assign to students or figuring out "whether a nonstandard approach would work in general" (Ball, et al, 2008, p. 400). A third domain, KCT, or Knowledge of Content and Teaching is defined as a combination of knowledge of teaching and knowledge of mathematics and includes the knowledge needed to decide on the best examples and representations to use for given instructional objectives. And KCS, or Knowledge of Content and Students, includes a blend of knowledge of mathematics and of students' thinking, such as the capacity to predict what students might find confusing or what kind of errors students might make when attacking a given problem. In our effort to construct measures of mathematical knowledge for teaching high school geometry, we developed items that purport to measure each of those four domains CCK, SCK, KCT, and KCS. Ball et al. (2008) also include Horizon Content Knowledge (HCK) and Knowledge of Content and Curriculum (KCC), but our work has not included those domains.

Ball and Hill's Learning Mathematics for Teaching project has developed items that measure the different domains of MKT and that has included, over time, attention to different content strands, including number and operation, patterns, functions and algebra, and geometry. These instruments have also included items that purport to measure mathematical knowledge for teaching middle school mathematics as well as for teaching elementary school mathematics. The extensive item development has yielded numbers of validated items that can be put together into forms that assess MKT for particular content strands. But there has not been, as of yet, a systematic development of items to measure MKT in different content strands or deliberate theoretical consideration about how content-strand differentiation might interface with the domains of MKT (Heather Hill, personal communication, 2/8/12). In particular, how would the specific practice of teaching particular mathematics courses be considered and featured in the process of designing measures of the mathematical knowledge for teaching those courses? In this paper we present our beginning attempts to conceptualize such instruction-specificity within the framework of MKT, by reporting on our development of an instrument to measure the mathematical knowledge for teaching high school geometry.

Our interest in MKT originated from our attempts to contribute to a theory of mathematics teaching that accounts for what teachers do in teaching in terms of a combination of, on the one hand, individual characteristics of practitioners and, on the other hand, practitioners' recognition of the norms of the instructional situations in which they participate and of the professional obligations they must respond to (Herbst & Chazan, 2011). While our earlier work focused completely on the conceptualization and empirical grounding of the latter, the present effort was part of a larger project in which we'd develop measures of the constructs that we had contributed (particularly *norms* and *obligations*) as well as measures of other constructs that would give us measures of individual resources. The conceptualization and disciplined approach to measuring MKT spearheaded by Ball and Hill (Ball et al, 2008; Hill and Ball, 2004) provided us with important guidance for the development of MKT measures. Hence, we developed multiple choice items following the definitions of the domains provided by Ball et al. (2008).

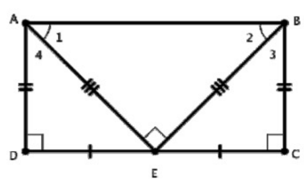
Development of MKT-Geometry

Our item development process covered a relatively wide range of topics from the high school geometry course. We consulted curriculum guidelines in various states and on that basis sought to develop items dealing with definitions, properties, and constructions of plane figures including triangles, quadrilaterals and circles, parallelism and perpendicularity, transformations, area and perimeter, three-dimensional figures, surface area and volume, and coordinate geometry. Those topics by themselves were good enough a guide to create items of Common Content Knowledge. But the definitions of the MKT domains, particularly the definition of Specialized Content Knowledge, calls for items that measure knowledge of mathematics used in the tasks of teaching. To draft these items we found it useful to create a list of tasks of teaching in which a teacher of geometry might be called to do mathematical work. The list included elements like designing a problem or task to pose to students, evaluating students' constructed responses, particularly student-created definitions, statements, explanations, and arguments, creating an answer key or a rubric for a test, and translating students' mathematical statements into conventional vocabulary. As we sought to draft these items, we noted that those tasks of teaching could call for different kinds of mathematical work depending on specifics of the work of teaching geometry. For example, the task of designing a problem would involve a teacher in different mathematical work if the designed problem was a proof problem versus a geometric calculation. While the former might involve the teacher in figuring out what the givens should be to make sure the desired proof could be done, the latter might involve the teacher in posing and solving equations and checking that the solutions of those equations represented well the figures at hand. Thus while a list of generic tasks of teaching was useful to start the drafting of items, this list appeared to grow more sophisticated with attention to tasks that are specific of different instructional situations in geometry teaching (Herbst, 2010).

The tasks of teaching were also useful in drafting items that measured knowledge of content and teaching. To draft these items we used as a heuristic the notion that the item should identify a well-defined instructional goal and the possible answers should name mathematical items that, while correct in general, would be better or worse choices to meet the specified goal. For example, teachers often need to choose examples (and justifications) for the concepts (viz. statements) they teach. While different examples (viz. different justifications) might be mathematically correct, they might not all be pedagogically appropriate to meet particular instructional goals: One example may be better than others as a first or canonical example while another example may be better as an illustration of an extreme case; one argument may require less prior knowledge and thus be more appropriate when students don't know many of the properties of the figure at hand, while another argument may illustrate how all the properties of a figure interrelate.

Finally, to create items that measured knowledge of content and students, we were attentive to the definition provided by Ball et al. (2008) and sought to draft especially items that tested for knowledge of students' errors. As in the case of other domains, there were specifics of the high school geometry class that shaped the items we developed. Thus, while we did create items that probed for teachers' knowledge of students' misconceptions about geometric concepts (e.g., angle bisector), we also created items that probed for their knowledge of students' misconceptions about processes or practices that are specific to geometry—such as the notion that empirical evidence is sufficient proof or that definitions are exhaustive descriptions.

While proving a claim on the board about the figure below, Joe wrote "1 + 2 = 90." Ms. Staples ponders how to correct that statement. Of the following, what is the best alternative?



A	Do nothing. The statement is correct as is.
B	Add the degree symbol so that the statement reads 1 + 2 = 90°.
C	Replace what Joe wrote; write instead that "m∠A + m∠B = 90°".
D	Replace what Joe wrote; write instead that "m∠EAB + m∠EBA = 90°".

After teaching her students the base angles theorem (which states that base angles of an isosceles triangle are congruent), Ms. Wellington is pondering what example she could give to illustrate how this theorem can help find new information. Which of the following would be the best example?

A	If you have a triangle that you know is isosceles and you know two of the sides are 8 and 10, then the third side will be 8 or 10.
B	If you have a triangle that you know the sides have length 8, 10, and 12 you know it does not have two angles that are congruent.
C	If you have an isosceles triangle and two of its angles are 30 you can calculate the third angle by subtracting 60 from 180.
D	If you have a kite and draw its minor diagonal, the kite is divided into two isosceles triangles and those have two congruent angles each.

Figure 1: Example of an SCK-Geometry Item (left) and KCT-Geometry Item (right)

Our research group drafted and revised an initial set of questions including 13 CCK, 20 SCK, 26 KCT, and 16 KCS questions; this drafting and revision process relied among other things on general guidance and comments on specific items by Deborah Ball, Hyman Bass, Laurie Sleep, and Mark Thames.[2] The questions drafted took the form of multiple-choice items, as well as multiple-response items within a single question (e.g., a single stem with 3–4 yes/no questions following). These items were submitted to a process of cognitive pretesting (Karabenick et al., 2007), by way of interviewing teachers and asking them to comment on what they thought each item was asking. Data from the cognitive interviews was also used to examine the content validity of the items, as well as improving such validity. Items were revised to improve interpretability and validity. A revised set of items was pilot tested with inservice secondary mathematics teachers from the same Midewestern state between July and October of 2011. Ten questions from each domain were uploaded into the LessonSketch online platform and completed by participants who took them either by coming in person to a computer lab (37 participants) or by responding to the items online from their homes or workplace (10 participants). For the purposes of this chapter, all data reported is pooled from both samples (n = 47). Participants were predominantly Caucasian (96.4%) and female (56.4%). Participants varied in the amount of mathematics teaching experience (M = 13.02, SD = 7.30), mathematics content courses (M = 10.78, SD = 4.46), and mathematics pedagogy courses (M = 3.04, SD = 2.54). Additionally, 67% of participants had taught Geometry for 3 years or more. Participants completed other questionnaires including one in which they reported on their years of experience teaching secondary school mathematics and teaching high school geometry. Our goal was to use the pilot to select five questions from each domain, as well as additional public-release items.

Item analysis for the MKT-Geometry test was conducted separately for each domain (CCK, SCK, KCS, KCT). We also used the pilot data to select the public-release questions (see Figure 1). In examining the fit of items for each domain, we used biserial correlations (Crocker & Algina, 2006) to measure item discrimination or how well the items discriminated between higher scoring test-takers and lower scoring test-takers. Crocker and Algina (2006) note that in performing classical item analysis such as the one we present here should "…have 5 to 10 times as many subjects as items" (p. 322). Since we conducted item analysis per MKT domain, this suggests a sample of approximately 50 participants (5 × 10 items per domain).

The item analysis of all 10 CCK questions yielded an initial Cronbach's alpha coefficient of .54. We used low biserial correlations (below .30) as one indicator for possible item removal. This resulted in the removal of 3 questions and an acceptable level of internal reliability
(α = .64). The final set of seven questions had biserial correlations ranging from .30 to .48, suggesting sufficient item discrimination. Additionally, item difficulty, in the form of percentage of the sample selecting the 'correct' answer, ranged from 30% to 83%.

We applied the same process to the study of the 10 questions that purported to measure the SCK domain. Item analysis resulted in the removal of three questions. The internal reliability of the remaining questions was found to be sufficient (α = .68), with item difficulties ranging from 19% to 96%. These results suggest both sufficient item discrimination and range of difficulty levels. Item analysis for the KCT domain led to removal of 3 items with a Cronbach's alpha of .57 with item difficulties ranging from 17% to 60%. Item analysis of the ten KCS items resulted in the removal of 3 items. Item difficulties ranged from 17% to 74% (α = .62).

Table 1: Composite Scores and Descriptive Statistics

Domain	M	SD	N	α
CCK – Geometry	0.68	0.22	48	0.64
SCK – Geometry	0.64	0.19	48	0.68
Subject Matter Knowledge of Geometry (CCK & SCK)	0.66	0.18	48	0.74
KCT – Geometry	0.39	0.24	47	0.57
KCS – Geometry	0.44	0.25	47	0.62
Pedagogical Content Knowledge of Geometry (KCT & KCS)	0.41	0.21	47	0.66
MKT – Geometry	0.54	0.18	47	0.84

Table 2: Correlations Between MKT-G Domain Scores

	CCK	SCK	KCT	KCS
CCK	-			
SCK	.44**	-		
KCT	.41*	.59**	-	
KCS	.68**	.55**	.48**	-

*$p < .05$, **$p < .01$

Items chosen through item analysis were used to compute scores for each domain (CCK, SCK, KCT, and KCS) shown in Table 1 above. Correlations between the domain scores are presented in Table 2, and suggest moderate to strong relationships between the different domains. These results show similar trends to those found by Hill et al. (2004) for CCK and KCS, which suggest that the different domains are, to a degree, interrelated. Thus, we interpret the findings from Table 2 similarly in that such relationships make sense, as it would be unusual for a teacher with higher KCS or KCT scores to have significantly lower CCK and SCK scores.

Following the notion proposed by Ball et al. (2008) that some of the MKT domains (notably KCS, KCT, and Knowledge of Content and Curriculum) operationalize the notion of Pedagogical Content Knowledge while the other MKT domains (CCK, SCK, and Horizon Content Knowledge) operationalize Subject Matter Knowledge, we created two additional scores: PCK-G which aggregates scores in KCT and KCS and SMK-G which aggregates scores in SCK and CCK. These are also summarized in Table 1.

Relationships Between MKT-G Scores and Teaching Experience

Our interest in MKT contributes to a larger project that investigates the influence that individual factors (such as mathematical knowledge for teaching) and socialization to the work demands of teaching a particular high school course (in this case, high school geometry, as indicated by teachers' recognition of instructional norms and professional obligations) have in the decisions that a teacher would make. A question we posed to the pilot data is what is the relationship between mathematical knowledge for teaching geometry and experience teaching the high school geometry course. Therefore, we correlated scores for each domain with teachers' years of experience teaching high school, but also with teachers' years teaching mathematics in general. These results are presented in Table 3.

Results indicated a statistically significant and positive relationship for each domain examined. These results show that the more years of experience a participant had teaching high school geometry, the higher their scores were for each domain. While that relationship was statistically significant for years of experience teaching geometry, such a relationship was not found to be statistically significant, or particularly meaningful in size for most measures, when looking at years of experience teaching mathematics in general. Therefore, these results suggest that while teaching experience may affect MKT-Geometry scores, it is the particular experience of teaching the geometry course. To the extent that mathematical knowledge for teaching is the knowledge of mathematics used in the work of teaching, the results lead to ask how the specifics of the instructional work a teacher does in a course matter in the mathematical knowledge for teaching the teacher has.

Table 3. Correlations Between Experience-Type and Score

	Years Teaching Geometry	Years Teaching Mathematics
CCK-G	.32*	.03
SCK-G	.31*	.11
SMK-G	.37**	.08
KCT-G	.36*	.27
KCS-G	.37*	.13
PCK-G	.42**	.23
MKT-G	.43**	.17

$p < .05$, $**p < .01$

Discussion and Conclusion

In our earlier and parallel work we have argued that the particular nature of the didactical contract (Brousseau, 1997; Herbst & Chazan, in press) for a course creates conditions of work that make the teaching of geometry different than the teaching of other mathematics courses, including algebra. The data shown above seems to suggest that teachers of geometry have more mathematical knowledge for teaching geometry, while the difference does not seem to be accountable to general experience teaching secondary mathematics. While at one level one might not find that result surprising, the fact that three of the four domains of mathematical knowledge for teaching we tested for (SCK, KCT, and KCS) are defined as mathematical knowledge used in the work of teaching helps raise questions for future inquiry.

As we noted above, the current conceptualization of MKT has not addressed content differentiation within domains. A natural way of thinking about differentiation could be the topical content of the item—items drawing on knowledge from different branches of mathematics might aggregate into different scores. But that approach seems to apply well only to differentiation within the domain of Common Content Knowledge. To the extent that the other domains are defined in relation to the work of teaching, it is plausible that differentiation within each domain will require considerations of the specifics of the teaching involved. The results from this study suggest that the teaching of high school geometry may entail specific mathematical knowledge demands.

In particular, SCK is defined as the knowledge of mathematics used in doing the tasks of teaching. One could expect that some of those tasks will not be course specific: The task of creating a grading system, for example, involves a teacher in making a mathematical model that feeds from grades in individual assignments; but there is no reason for this mathematical work to be different for teachers of different high school courses. Other tasks of teaching, however, while amenable to generic statement (e.g., choosing the givens of a problem for students), may involve practitioners in different mathematical work depending on the specifics of the task (e.g., choosing the numbers for a word problem in algebra involves different mathematical work than constructing a geometric diagram to include in a geometry worksheet). Are those differences merely differences in mathematical strand (algebra vs. geometry) or do they also reflect differences in the instructional situations (Herbst, 2006) to which those tasks contribute?

We suggest that the management of instructional situations involves teachers in singular mathematical work. An instructional situation has been defined (Herbst, 2006) as a frame for exchanges between types of mathematical work that students will be doing and the knowledge claims that a teacher can make on their behalf based on their accomplishing that work. The teacher's management of instructional situations includes in particular the choosing of the various tasks that constitute that work, the observation of the proceeds (what students actually do), and the effecting of exchanges between such observed actions and the knowledge at stake (identifying at least for herself but possibly also publicly to the class how what students have done indicates their knowledge of what is at stake). While the definition of these tasks of teaching is general, the mathematical knowledge called forth in doing them would be different across different courses, as long as the specific exchanges were different.

A case in point that helps argue that instructional situations matter comes from one SCK question in our instrument. This was a multiple-response question with two items; the stem spoke of a teacher needing to choose algebraic expressions for the sides of an isosceles triangle where the students would be expected to find the lengths of the sides of the triangle after solving an equation. Each item provided expressions for the three sides and asked whether or not they were appropriate expressions. A quick examination of the responses to the item indicated that teachers with more or less years experience teaching geometry (≥ 3 years and < 3 years, respectively) did not respond much differently for the item where the equation could not be solved. However, the two types of teachers' responses did show differences for the item where the equation could be solved: the less experienced teachers tended to answer that the expressions were appropriate while experienced teachers that they were not. In fact, the numbers obtained after solving the equation would not work to represent the sides of a triangle in that the triangle equality would not hold for those numbers. We conjecture that the experienced teachers' familiarity with the instructional situation of "calculating a measure" (Herbst, 2010) mattered in their decision to check that the expressions would yield sides with positive lengths and that they would satisfy the triangle inequality. Our conjecture is not that the non experienced teachers did not know the triangle inequality, but that they did not know it mattered in this task of teaching, possibly because they only saw the problem as an exercise in algebra rather than also as an exercise in triangle properties. More generally, we conjecture that tasks of teaching that are subservient to instructional situations specific to a given course of studies might involve teachers in mathematical work that teachers who are experienced in managing those situations would know better how to do. We suggest that considerations of the nature of the instructional situations in a course could lead to analogous differentiation within the domains of KCT and KCS as well.

Endnotes

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[2] Daniel Chazan, a co-PI of this project was also involved in design discussions. Individuals involved in the drafting of items included Michael Weiss, Wendy Aaron, Justin Dimmel, Ander Erickson, and Annick Rougee.

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MATHEMATICS TEACHERS INVESTIGATING REASONING AND SENSE MAKING IN THEIR TEACHING

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The purpose of this study was to generate an understanding of the experiences of mathematics teachers examining recommendations for Reasoning and Sense Making (NCTM, 2009) and investigating them in their practice. Narrative inquiry incorporates the voices of teachers and illustrates the phenomenon studied through narratives of participants' experiences. This paper presents the findings through four analogies that convey abbreviated narratives of teachers' experiences enacting recommendations for Reasoning and Sense Making.

Keywords: Standards; Teacher Education/Professional Development; High School Education

Recent recommendations for improving the nature of teaching and learning mathematics across the United States can be traced back to 1980 with the National Council of Teachers of Mathematics (NCTM) publication of *An Agenda for Action* (NCTM, 1980). In subsequent years, NCTM published a series of standards documents (1989, 1991, 1995, 2000) to clarify new goals and curricular recommendations for mathematics education. When evaluating the state of mathematics classrooms, discourse within the field often focuses on the deficits, making broad generalizations pointing to a gap between the state of mathematics classrooms throughout the nation and the classroom environments promoted by these recommendations. Hiebert (1999), for example, declared that "the same method of teaching persists, even in the face of pressures to change," (p. 11). Similarly, the Conference Board of the Mathematical Sciences (1975) asserted that "teachers are essentially teaching the same way they were taught in school," (p. 77) referring to the lack of impact of the earlier "new math" reform movement of the 1960s.

A contributor to the gap between curricular recommendations and classroom practice is the complexity of learning to teach mathematics differently. The changes proposed by reform efforts such as the NCTM standards have the underlying assumption "that teachers will change their world view of mathematics, mathematics teaching, and mathematics learning" (Shaw & Jakubowski, 1991, p. 13). Even when such changes are desired or instigated by the teacher, many have described difficulties they encountered as they attempted them in their own teaching (e.g., Ball, 2000; Cady, 2006; Chazan, 2000; Heaton, 2000). In short, making changes to one's teaching is a complex process.

To add to the conversation surrounding teachers' responses to NCTM recommendations, this study sought to develop an understanding of the experiences of mathematics teachers attempting to enact recommendations for mathematics teaching from *Focus in High School Mathematics: Reasoning and Sense Making* (NCTM, 2009). This document proposed that "reasoning and sense making are the foundations of the NCTM Process Standards" (NCTM, 2009, p. 5), and should be incorporated into "every mathematics classroom every day."

A collaboration of seven mathematics teachers was formed by recruiting teachers interested in investigating their practice and incorporating recommendations into their teaching. The purpose of this study was to learn about the experiences of mathematics teachers as they investigated NCTM recommendations for *Reasoning and Sense Making* (NCTM, 2009) and attempted to make changes in their practice through informal teacher action research. Particularly I focused on five aspects of the experience: conceptions of reasoning and sense making, actions that the teacher took in their teaching, challenges, opportunities, and the teacher's interpretations of the results of their actions. Teacher action research was conceptualized as a self-critical inquiry into one's practice with the goal of improving practice as well as developing a better understanding of that practice (Carr & Kemmis, 1986). Stenhouse (1975) promoted applying curricular recommendations to the formation of one's action research inquiry, suggesting that "the crucial point is that the proposal is not to be regarded as an unqualified recommendation but rather as a provisional specification claiming no more than to be worth putting to the test of practice" (p. 142).

Teachers represented six high schools and ranged from 0 to 11 years of teaching experience (mean of 3.5). They agreed upon the theme of *Reasoning and Sense Making* as the focus of their work together. We met regularly throughout the school year, a total of nine times. Teachers initially read and discussed *Reasoning and Sense Making* and began to focus their action research inquiries in individual ways by selecting specific actions to take in their practice to incorporate their interpretation of the recommendations. Teachers learned informally about the methods of action research through PowerPoint presentations, reading excerpts of methods handbooks, and narrative examples of teacher action research. I created a library of practitioner readings that were related to their goals, from which they selected additional readings. Meetings served as a time for them to discuss readings and share their goals, challenges, and successes.

Data Analysis

During this study a variety of data sources, or *field texts* (Clandinin & Connelly, 2000), were collected to generate an understanding of teacher's experiences. See Figure 1 for an illustration of the data sources that inform this analysis.

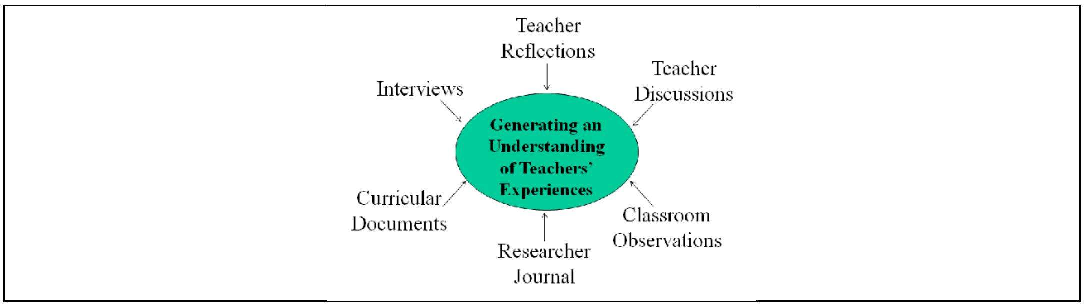

Figure 1: Data sources that inform the research question

This study used narrative inquiry to investigate the ways teachers incorporated recommendations into their practice. Narrative inquiry was selected to allow the voices of teachers to be heard and expand our understandings of "what the experience is like." Clandinin and Connelly (2000) describe narrative inquiry as "a way of understanding experience. It is a collaboration between researcher and participants, over time, in a place or series of places, and in social interaction with milieus" (p. 20). Narrative researchers illustrate a phenomenon studied through creating unified, coherent narratives that convey the meaning of their experiences working together. The method of analysis is emplotment and narrative configuration (Polkinghorne, 1995), in which data snapshots are pieced together to develop a plot. This requires a synthesis of the data rather than separating it into its constituent parts.

As data was collected, I continuously reviewed it. After all data was collected, I organized the data pieces pertaining to each teacher into chronological order in spreadsheets. I coded data pieces according to the aspects of the experience previously identified: conceptions of reasoning and sense making, actions that the teacher took in their teaching, challenges, opportunities, and the teacher's interpretations of the results of their actions, plus the additional category of contextual information. Coded data for each teacher was then reorganized into condensed spreadsheets according to category. I continuously reviewed these consolidated spreadsheets until recurring ideas and connections developed to synthesize the information into the "plot" of the narrative. Then the process of writing of *interim texts* (Clandinin & Connelly, 2000) or smaller drafts of the research text, was an important element of the emplotment and narrative configuration. Through repeatedly experimenting with the writing process by writing interim texts, and then sharing those texts with the teachers, I eventually produced the final research texts. More details about the analysis will be shared in the presentation.

Findings

The analysis revealed the complexity of each teacher's experience. Teachers varied in their past experiences as a mathematics teacher, and in the awareness they held of the ways they influenced student's opportunities to reason and make sense of mathematics. Teachers also varied in the actions they took to adapt their teaching in response to the recommendations. Their action research foci varied from improving their questioning strategies, curriculum, role in discussion, prompting students' justification, prompting writing about mathematics, and incorporating student creativity into the doing of mathematics. As I developed narratives of the teachers' experiences, it became apparent that within the different journeys, subtle similarities existed. I compared the plotlines of narratives that held similarities to clarify my understanding, and I examined the differences across groups. As I read and reread my data, I tested different categorization schemes in my own sense making process to understand the ways I grouped teachers' narratives. Two aspects of their experiences emerged that provided a way to categorize their narratives. The first aspect was their level of *awareness*—both at the time they entered our collaboration and the development of their awareness over time—of the ways that they influenced students' opportunities to engage in reasoning and sense making. The second aspect was their evolution, or development in any direction, of the ways they acted on this awareness by developing *strategies* to promote students reasoning and sense making.

These two aspects, teachers' *awareness* and their *strategies*, were intertwined within each teacher's experience. Particularly, teachers' *awareness* of the ways they impacted students' engagement in reasoning, and their *strategies* for fostering students' reasoning, evolved in response to each other over time. Evolution in teachers' awareness was only recognizable when teachers self-reported new things they had come to realize about their teaching. Evolution in teachers' strategies was more easily identifiable, through teachers sharing new strategies they were developing and through my own observations during classroom visits. As I made sense of differences in teachers' journeys, I generated four analogies to represent their journeys. There isn't sufficient space to present the narratives of teachers' experiences here, but I will offer a glimpse through the four analogies: a linear function, a piecewise function, a step function, and a scatterplot. The independent variable in this mathematical relationship is the time spent studying one's teaching practice. The dependent variable is the evolution of strategies to support students' engagement in reasoning and sense making. While these analogies attempt to illustrate teachers' experiences over the seven months we collaborated, this was a brief stretch amidst their longer journey as a mathematics teacher. In the proceeding sections, I introduce each analogy and provide illustrations from the data of the teachers that they represent.

A Linear Journey

Teachers Peter and Alexis both entered the collaboration having already problematized many aspects of mathematics teaching that *Reasoning and Sense Making* sought to change. Both talked openly about the problematic consequences of teaching mathematics through providing a list of procedures, consequences they had seen firsthand. Peter used humor to tell stories illustrating the negative effects of students' reliance on procedures or the teacher's authority, instead of reasoning and sense making. "I really want my students to start critically thinking. I swear that I could say, 'Your lesson today is to learn that 5 + 8 = 22.' And they will just write 5 + 8 = 22, and not even think a thing about what they're actually writing, whether it even makes sense at all" (12/9/10, meeting 3). Peter talked often about how "we're fighting a decade's worth of ingrained math," after seeing indications that his students were well practiced at learning mathematics *without* reasoning. As Peter and Alexis read *Reasoning and Sense Making*, they agreed wholeheartedly with the proposition of the document that teaching mathematics through steps and procedures did not produce positive student learning outcomes.

Along with identifying certain teaching practices as discouraging to students' reasoning and sense making, Peter and Alexis began their action research with a similar awareness of the ways that their role as teacher influenced student's engagement in reasoning and sense making. Both agreed with the philosophy of the recommendations and shared ways they had already made improvements to their teaching that aligned with recommendations. For instance, Alexis shared:

> I don't teach the $\frac{y_2-y_1}{x_2-x_1}$ formula to find slope. I use t-charts and put six graphs up on the board when I want to start teaching them about slope. … So [the students] figured out when you put it in y=mx+b form, where that [slope] number was coming from. And they realized, you know, if it was negative it went left, positive went right. … And so, we look at all the graphs, and we talk about the change in y over the change in x, and how it goes up and over, and where those numbers came from, and then we just call it change in y over change in x. (11/16/10, meeting 2)

Despite examples of shifts away from a focus on procedures, upon reading *Reasoning and Sense Making* both Peter and Alexis saw themselves as guilty of reliance on practices that did not promote students reasoning and sense making. They both identified room for growth to align their teaching with these recommendations. While their described approaches varied, they each developed their own ways to "transfer the deliverance of my lesson to my students" (Peter, 2/22/11, reflection). Peter focused on removing opportunities for students to rely on his authority instead of their own reasoning, through habits he developed such as "keeping silent," "firing students' questions back at the class," and "going along with wrong ideas." Alexis focused on developing her questioning; restructuring lessons so that students uncovered the mathematical ideas through class discussions facilitated by her questions.

To illustrate the similarities among their journeys and the differences between their journeys and those of others, I draw on the analogy of a linear function (see Figure 2). While Peter and Alexis faced challenges, I conceptualized their evolution of strategies as being fairly linear when compared with that of others. They conceptualized a vision for their teaching and developed their strategies to move continuously towards their goal. Their awareness of their own impact on students' opportunities to engage in reasoning and sense making facilitated a steady progression in the direction of their vision.

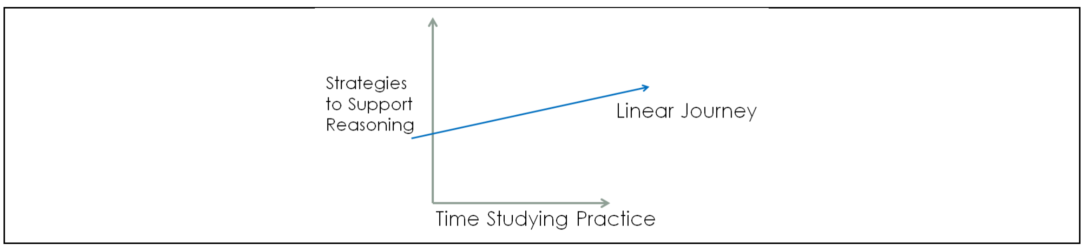

Figure 2: The analogy of the linear function

A Piecewise Journey

Teachers Logan and Melinda both expressed interest in the theme of *Reasoning and Sense Making* as they joined the study, but they did not cite examples of ways that their teaching methods influenced students' opportunities to engage in reasoning and sense making. They hoped to learn more strategies to foster reasoning and sense making as a result of their collaboration in the group.

After reading the recommendations, both Logan and Melinda formulated goals that were related to improving their classroom discussions. Both were interested in changing the structure of lessons to move away from direct instruction by incorporating questions and using student-generated ideas to move a lesson forward. Both identified their initial changes in their teaching as successful based on their students' responses. However, at different points during the school year, each teacher experienced frustration as they encountered students responding to their questions with increasing silence. When their best efforts were met with resistance from students, they became discouraged and wondered if some of their students were not capable of reasoning.

One day after observing Melinda teaching an Algebra lesson, she asked if I would teach the same lesson to the next class walking in. I agreed, and this proved to be a valuable opportunity to foster her thinking about her teaching. After watching me teach her lesson, and noticing the ways her students responded to my questions, she said, "I thought the problem was that my students couldn't reason. But now I see that I was just asking the wrong questions." After that episode, I observed noticeable differences in the questioning that Melinda used. Rather than questioning patterns that resembled those described as "funneling" (Wood, 1998), her questioning changed to resemble more closely the pattern described as "focusing" (Wood, 1998). For example, previous questions had directed students towards a particular procedure such as "Which fraction should we use? What if we use this one? Can we cross anything out?" Her new questioning tended to be more open to allow students to determine their own solution methods, such as "How can you find the side length of a square with an area of five?" and "Steve subtracted and then divided. Do we have to do it in that order?" The following year, Melinda continued to e-mail to share ongoing successes she saw as a result of long-term use of her new questioning strategies.

A similar experience happened in Logan's action research. He became discouraged for several months during the spring semester, and began to wonder if the juniors and seniors in his "intro" level Algebra II courses were capable of reasoning. After persuading him to allow me to teach one of his lessons, I attempted to make an "existence proof" that his students *could* reason mathematically. The following is an excerpt of his reflection:

> When watching Lindsay teach my class, I noticed how she was able to get everyone involved. She was calling on students who had not volunteered to share an idea in months. I have made a point to call on each and every student in my class since then. I also do not let students get away with just saying, "I don't know." They were actually saying, "I don't want to think right now," so I have to make them tell me something that they do know. (5/6/11, final reflection)

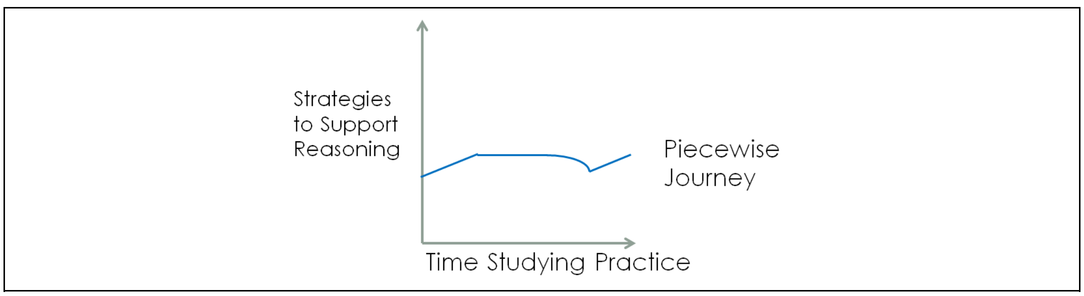

Figure 3: The analogy of the piecewise journey

To illustrate the similarities among Logan and Melinda's journeys, I draw on the analogy of a piecewise function. While Logan and Melinda initially saw short-term improvement in their students' engagement in reasoning and sense making, both also experienced a plateau. They overcome the obstacle when they developed a heightened awareness of ways they impacted students' opportunities to reason. A new awareness of their teaching prompted the development of new actions to support student's reasoning and sense making.

A Step Function Journey

Sarah, a fourth year teacher of high school geometry and algebra, shared that she had not previously considered the importance of fostering reasoning and sense making opportunities until reading these recommendations. The authority of the document convinced her of the importance of developing such practices in students to prepare them for their future. Beginning with suggestions pulled from the document, through trial and reading other practitioner articles, she narrowed the focus to asking more questions and requiring students to justify all ideas. These changes increased the amount of student talk in Sarah's classroom, opening up opportunities for students to "surprise" her with their mathematical ideas. Through studying her teaching, these unexpected incidents became learning opportunities that increased her awareness of how to support students' reasoning and sense making.

> You remember the Algebra class where they wanted to use synthetic division? *(laughing)* I was so caught off guard because I've never thought of using that method [in that context] before in my *life*. I

was like, "Okay let's go with it." But I was really surprised. And I should've been more calm about it… because then they wanted to know what "my way" was. But it totally caught me off guard. (5/18/11, final interview)

Each new unexpected finding fueled further development of her actions. One thing she learned from her students was the value of allowing them to determine their own solution path:

Before, I wouldn't let them [solve problems] the way that they wanted to. … I think a lot of times I would just be like, "Well didn't you see this method," instead of just letting them do it their way. I think its okay now just to let them do it a different way, even if it's the hard route. Just let them be, because that's the way they understand. Giving them that freedom. (5/18/11, final interview)

The analogy of a step function illustrates Sarah's experience (see Figure 4). Each step in the function represents actions she tested in her teaching and subsequently learned from, resulting in new knowledge and a heightened awareness of strategies to support students reasoning. The heightened awareness facilitated her in developing her actions further, represented by the next step in the function. Sarah's experience was unique from the others by the pattern of repeated instances of surprise that resulted in new awareness that fueled developments to her action strategies.

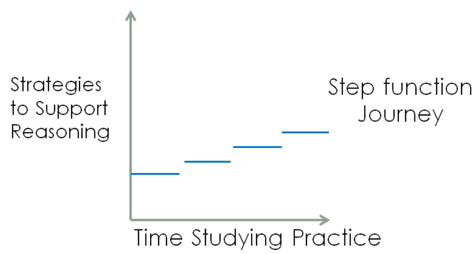

Figure 4: The analogy of the step function

A Scatterplot Journey

Claudia and James were in their first year of teaching, and both juggled many new responsibilities. It took them more time to develop the focus of their actions, and their initial actions changed frequently as they experimented with a variety of different strategies. Claudia reflected on these early months and discussed the challenge of trying to focus her actions:

With it being my first year and everything, I didn't know what my teaching style was and how I wanted to change or improve it... I kept kind of trying the different things I heard people talking about, thinking, "Is this what I need to work on? Is this something that interests me?" (4/28/11, meeting 9)

Both teachers eventually narrowed their efforts to posing open-response prompts on assessments. This approach to incorporating reasoning was more like an add-on to their teaching than a part of their everyday routine. James explained in a written reflection why he picked a subtle approach:

I would love to hold classroom discussions and ask questions where students learn from their mistakes, discuss problems with one another, and problem solve when they do not get the correct answer (Eggleton et al., 2001). That type of classroom environment is one that I envision for the future, but I do not believe my classes are ready for such radical changes all at once. To me, writing seems like a natural and subtle way for students to convey their reasoning and sense making. (1/12/11, reflection)

Both teachers also dealt with school-wide pressures to raise students' scores on the state-wide algebra exam. With the many other things vying for their attention, Claudia and James at times would "forget" their focus. Over time Claudia and James recognized the need to incorporate reasoning beyond assessments and into their mathematics lessons. They each tried fostering reasoning through occasional student-centered activities. However, limitations in time and resources hindered them from incorporating activities on a daily basis. Each saw room for improvement and made plans to continue their actions in subsequent small steps in the future.

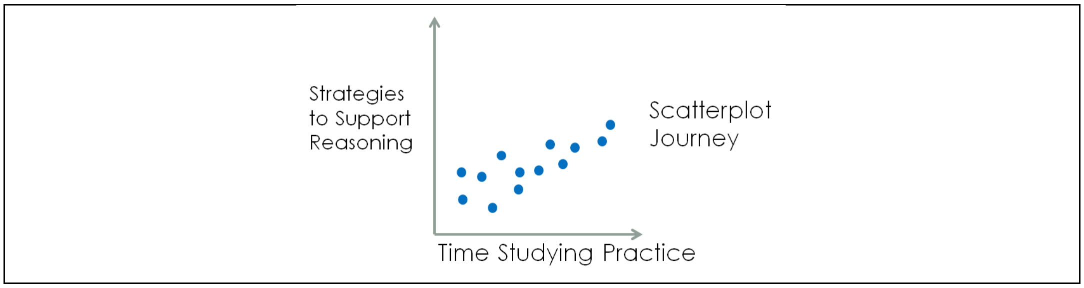

Figure 5: The analogy of the scatterplot

Their journey is illustrated by the analogy of a scatterplot with a positive correlation which became stronger over time. This analogy is distinct from the others as it illustrates the variety of seemingly disconnected actions that Claudia and James tested in their practice but also indicates a progression towards developing more focused and refined strategies.

Conclusion

While each teacher focused their efforts to foster reasoning and sense making in unique ways, the elements they chose to take up and test in their practice were a reflection of those that held meaning for them in the context of their teaching. Common gains among all teachers were a heightened awareness of the ways they impacted students' opportunities to engage in reasoning. Given the trend to focus on the deficit between NCTM recommendations and mathematics classroom practices, this research expands the discourse by illuminating the experiences of teachers attempting changes in their practice. Past research on mathematics teacher change has measured changes in practice along continuums or stages that gauge the degree to which teachers' instructional practices adhere to preconceived change objectives (e.g., Fennema et al., 1996). Alternatively, this study approached teacher change by seeking to understand the complexity of teachers' attempts at change from their perspective. Narrative inquiry offers a valuable perspective to the discourse surrounding mathematics teacher change, validating the knowledge and experiences of teachers and seeking to learn from them.

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PROMPTING MATHEMATICAL KNOWLEDGE FOR TEACHING THROUGH PARENT-TEACHER LEARNING COMMUNITIES

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Parents, K–8 teachers, and 4th–8th grade children participated as equals in math-focused learning communities through the Math and Parent Partners (MAPPS) program. Pre/post testing and qualitative interviews revealed that the learning communities served as a platform for improvement in mathematical knowledge for teaching of participating teachers. Moreover, teachers learned about parents' knowledge and strategies, a construct analogous to Knowledge of Content and Students that we describe as "Knowledge of Content and Parents."

Keywords: Mathematical Knowledge for Teaching; Teacher Knowledge; Informal Education

Background and Research Questions

Student achievement lags in many economically disadvantaged schools. Two factors associated with this achievement gap include inadequate teacher knowledge and low parental involvement (Hill, Rowan, & Ball, 2005; Jackson & Remillard, 2005). A school district in the Southeast partnered with the local university to boost student achievement in Title I schools through Math and Parent Partners (MAPPS) parent-teacher learning communities in mathematics (MAPPS, 2009). We asked,

> Does parental involvement in a standards-based mathematics program such as MAPPS carried on at Title I K–8 schools improve student understanding and achievement in mathematics? Secondarily we asked, how might this improvement occur? In particular, do parents and teachers in MAPPS develop mathematical knowledge for teaching?

Students were found to improve standardized test scores significantly over a three-year period (Knapp, Jefferson, & Landers, in press). However, this paper focuses on factors that may have prompted the student improvement. In particular, we describe *teachers'* development in mathematical knowledge for teaching as they participated in MAPPS learning communities.

Theoretical Framework and Literature Review

Hill, Rowan, and Ball (2005) reported a study in which teachers' *mathematical knowledge for teaching* (MKT) was linked to student achievement in first and third grade. Moreover, they found that teachers in economically disadvantaged schools tended to possess lower MKT. The framework of mathematical knowledge for teaching (MKT) relates to the knowledge and habits of mind needed to teach mathematics well (Ball, Thames, & Phelps, 2008). In the framework, MKT includes six constructs of which we focused on the following four in investigating the Math and Parent Partners learning communities. Common content knowledge (CCK) is basic, lay-person knowledge of the mathematical content. Specialized content knowledge (SCK) is the way the mathematics arises in classrooms, such as for building representations. Knowledge of content and students (KCS) indicates a teacher's knowledge about how students think in mathematical contexts. Knowledge of content and teaching (KCT) indicates a teacher's knowledge of advantageous representations or teaching sequences. MKT encompasses both content knowledge (CCK & SCK) and pedagogical content knowledge (KCS & KCT).

Studies have additionally shown that parent involvement in their children's education is linked with children's academic outcomes (D'Agostino, Hedges, Wong, & Borman, 2000; Epstein, 1994; Kellaghan, Sloane, Alvarez, & Bloom, 1993). As Henderson and Mapp (2002) stated, "The evidence is consistent, positive and convincing: families have a major influence on their children's achievement. When schools, families, and community groups work together to support learning, children tend to do better in school, stay in school longer, and like school more" (p. 7). Low-income parents may be untapped resources for the mathematical achievement of their children. Henderson, Mapp, Johnson, and Davies (2007) asserted that districts serious about closing the achievement gap would have to address the school culture gap that expects parents to remain relatively uninvolved in their children's mathematics learning. Although parental involvement may be linked to student achievement, parents are often not accessed as resources for helping children learn mathematics in standards-based school environments (Jackson & Remillard, 2005; Perissini, 1998). In this paper, we describe a study of a parental involvement program that engaged parents and teachers in mathematics learning communities.

Participants and Context

The Math and Parent Partners (MAPPS) program equips families to act as mathematical resources for their children and for schools. MAPPS curriculum was developed with National Science Foundation funding to engage K–8 parents in exploring with peers the concepts and skills behind the mathematics that their children are learning in schools (see http://mapps.math.arizona.edu/). Currently, the MAPPS program serves sites in six states and the Virgin Islands. One MAPPS site, located in the Southeast and the focus of this article, worked toward improving the mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008) of both parents and teachers in Title I schools within its school district. All parents, teachers, paraprofessionals, and children from selected schools were invited to participate. The local university partnered with MAPPS and the school district to offer Mini-courses for parents and teachers, while young children participated in related mathematical activities and games. Children in 4th-8th grade accompanied their parents in the Mini-course classes. Mini-course sessions convened two hours per week for eight weeks. Over the course of three years, eight separate 8-week Mini-courses, centered on the National Council of Teachers of Mathematics' (NCTM) (2000) content and process standards, were offered. These Mini-courses were hosted by the University's Office of Continuing Education, and instructors were graduate students in mathematics education who were also practicing teachers.

8-week Mini-course Title	NCTM Content Standard Addressed
Thinking About Numbers (offered two times)	Number & Operations
Thinking About Fractions, Decimals, and Percents (offered 3x)	Number & Operations
Thinking in Patterns (offered once)	Algebra
Geometry for Parents (offered once)	Geometry and Measurement
Data for Parents (offered once)	Data Analysis & Probability

Figure 1: Math for parents Mini-course curriculum

In all, 115 children, 59 parents, and 33 teachers from primarily four Title I elementary schools attended at least one Mini-course on a regular basis. Nearly twice that many participants attended sporadically. Approximately 75% of attendees were single parents, and those that attended the Mini-courses did so with one to three children. Most of the parents had graduated from high school with some technical training, and they typically held low-income jobs. Attendees were approximately 40% Caucasian, 40% African-American, and 20% Hispanic. Teachers who attended faithfully received stipends and professional learning units.

MAPPS Mini-courses engage parents in doing mathematics using hands-on materials, working in small groups to solve problems, and presenting their solutions to the whole group as outlined by the NCTM process standards (NCTM, 2000). Both content knowledge and pedagogical content knowledge are intertwined into the instruction for parents (Ball, Thames, & Phelps, 2008), with pedagogical considerations made relevant by Mini-course instructors depending on grade levels of participating children.

To illustrate the intervention and details of the MAPPS program, we describe a learning activity from the two-hour Week Eight session from the Fractions, Decimals, & Percents Mini-course (see Figure 1). For the task, participants were to have shaded a given percentage of various grids for homework from the previous session. The first grid, a bar divided into fifths, required 60% to be shaded (see Figure 2). Participants had to figure out what percent each fifth represented for the entire grid to equal 100%, and they discussed their findings at the beginning of the session.

Shade 60% of this grid.

Figure 2: MAPPS homework task

A father and his 6^{th} grade daughter found that each rectangular fifth must be 20%. The father held up his hand to demonstrate his fingers as the rectangle saying, “Each finger is 20, so we shaded three of them to make 60, see (pointing to his fingers) 20, 40, 60.” Later in the session, parents, teachers, and children made percent strips that they then compared to the fraction and decimal strips made during previous sessions. At the end of each task, group members reported their various solutions and strategies to the entire class. Sometimes the *children* presented unique strategies allowing parents and teachers to learn from the children, and visa versa.

Data Analysis

To assess the impact of the MAPPS Mini-courses, parents and teachers took pre/post tests on mathematical knowledge for teaching (Hill, Schilling, & Ball, 2004) and pre/post attitude surveys (Tapia, 1996). Pre/post tests and surveys were administered before and after each 8-week Mini-course. A focus group of parents, teachers, and children also participated in 95 pre/post interviews. Interviews lasted approximately 15 minutes, and questions were such as these: (1) Have you learned anything about mathematics that you did not know before? Explain. (2) Have you learned anything in MAPPS that helped you help your child or students with math? Explain. Interviews were coded for evidence of improved student understanding, achievement, and factors that might affect that improvement, such as the elements of mathematical knowledge for teaching: CCK, SCK, KCT, SCK (see Table 1). After coding the interviews and pre/post surveys, we tallied the 59 codes to identify the salient areas of participant growth as well as factors prompting that growth. We looked for clusters in the data each year, producing primary and secondary results for each year. At the end of the study, we compressed codes and identified themes based on the primary and secondary codes. Themes arising from the coding process included strengthened teacher content knowledge, improved teacher Knowledge of Content and Teaching, and benefits of the learning community.

Table 1: Teachers’ Results from 34 Teacher Interviews

Code	Freq	Description of Result
Primary:		
Knowledge of Content and Teaching (KCT)	56	
Content Knowledge SCK(16) CCK(6) *GLM(6)	28	Primarily SCK for teachers *GLM-General learning of mathematics reported that could not be identified as CCK or SCK
Enjoyment of/Valuing MAPPS	40	High value placed on program.
Secondary:		
Learning Community	23	The learning community was valued.
Broader impact of program	10	Program impacted non-MAPPS students.
Student learning/achievement	8	

Results and Discussion

We present interview data from several teachers to amplify our coding process and themes that emerged. Examples of codes are in bold. Teacher A, a primary teacher, shared the following:

AK: Well can you just talk to me about how the program went for you?
Teacher A: Well, coming in as a teacher, it really helped me see, uh…just a little more in depth look at the math. Because math isn't one of my strengths, I will tell you, it's not one of my strengths. So I came out to actually deal with like ideas, like fractions, and really at my own pace, kind of look at what is it, what is a fraction.
AK: …Can you give me a specific one [example] that maybe you understood superficially and you could teach it, but now you understand it in a different way?...
Teacher A: One of my favorite ones was when we talked about fractions. She gave us strips and she said, "Okay, fold this strip into one third; fold it into one eighth, …twelve. Fold your strip into twelfths. Eighths and thirds." And first, when you wanted to make the equal sections you kind of thought well this is going to be my stopping point. For that line, which is what we put on our strips as the stopping point. It's the space between that makes that one third. Just that space. Even with measurement, when they looked at the ruler, you know they have all the little increments between 1 and 2, but she kind of let me see that that space. It's not as important as what the numbers are, it's that space, the little spaces between the numbers that, I hadn't really [considered].

In this Mini-course, Teacher A developed **Specialized Content Knowledge (SCK)** when she learned what was mathematically significant about a fractional increment using fraction strips. She came to understand the content more deeply as it arises in a classroom. Moreover, she and other teachers stated that after attending the MAPPS Mini-course on fractions, decimals and percents, that the MAPPS materials became their handbook for teaching the unit on fractions in their classrooms. After attending MAPPS, this teacher and numerous others made the decision to pursue graduate degrees in Elementary Mathematics Education (**Higher Education**).

Another teacher who attended MAPPS described the benefit of the learning community for herself, her parents, and her students.

AK: Is there anything in MAPPS that has helped you to better explain math to kids?
Teacher B:…when we did fractions, decimals, and percents I had so many kids that would not understand that, and so I would literally I would have one child in my room on this side of me and their parent would be right here. And then I would have another one from my room on this side and we would literally work through exactly what we were doing in class with adding and subtracting decimals. We would take the unit cube, and it would be the one whole or the units, rods, and
AK: flats.
Teacher B: flats. And that helps too with the kids when they go home and say, "Well our teacher told us that 5 and 50 are the same." And the parents are going, "No," you know.

This episode exemplifies the parent-teacher interaction enshrined in the **learning community**. The teacher could see first-hand the disconnect between school learning and home learning. It became evident to her that children appeared to understand the concept in class, but they were unable to verbalize their misconceptions adequately to their parents who did not know about the base-10 block representation for decimals. She realized that students were making confusing comments such as, "Our teacher told us that 5 and 50 are the same," to the parents. The experience allowed the teacher to develop **Knowledge of Content and Students (KCS)** about student misconceptions, in this case not understanding that 5 rods were the same as 50 unit cubes. The MAPPS session amplified the student and parent misunderstandings for the teacher, and at the same time, helped the parent to develop **Specialized Content Knowledge (SCK)** about how base-10 blocks can represent decimal operations.

Teacher B additionally expressed that the MAPPS instructors modeled good explanations for her and helped her be reflective. Other teachers stated that MAPPS helped them learn connections between mathematical topics such as fractions, decimals, and percents. Paraprofessionals and substitute teachers who participated also reported that MAPPS equipped them to assist with instruction in mathematics. Finally, the MAPPS environment helped teachers add rigor to their teaching practice. Teacher A stated, "I think I wasn't really going in depth as much as I could." She learned to facilitate conceptual understanding at a deeper level than had been afforded through teaching fractions by rote.

The qualitative result that teachers improved their content knowledge was substantiated by MKT test results. Significant changes were noted when the first Mini-course to the last Mini-course scores were compared (n=20; p=.052). The content knowledge tests were designed such that a well-prepared elementary teacher would get 50% of the questions correct (Hill, Schilling, & Ball, 2004). Although the test scores improved significantly, the average scores did not rise above this 50% benchmark. This data suggests that teachers involved in the program were in need of further instruction in mathematics for teaching and highlights the importance of the result that the MAPPS parent-teacher learning communities built teachers' **confidence** as mathematics learners and emboldened some to attend graduate school in mathematics education.

The next aspect of mathematical knowledge for teaching that developed for both parents and teachers during MAPPS was Knowledge of Content and Teaching (KCT). Teachers reported learning to model problems with tasks and manipulatives, instead of relying as heavily on direct instruction and drill. We considered this shift to be KCT because teachers were expanding their repertoire of effective examples and teaching sequences that they over and over reported taking back to their classrooms, sometimes as an entire grade level (**broader impact**). An interview displayed a teacher learning about the instructional advantage of a dynamic representation to help her teach subtraction with regrouping.

Teacher B: The other thing that I used that they showed us talked about the virtual manipulatives, was the website where you can, there are the little unit cubes, the rods, and the flats…

AK: The base ten blocks?

Teacher B: Yeah and you can drag them over and show and the kids can go up to the active board and manipulate those around and they just loved that because I needed a tool when I was teaching the kids even just when we were learning subtraction with regrouping.

Teacher B went on to explain that her students were confused when using the concrete base-10 blocks and that her static drawings were inadequate. However, after attending MAPPS, she engaged students in a MAPPS task using the virtual base-10 blocks. She said, "But I pulled that website up, and I could just move it right around. And it was just so convenient, and it was easy for them to see because it was color coded too where my little drawings were crude…" Thus, this teacher developed **KCT** related to choosing effective examples and representations.

We found that although some teachers had access to manipulatives, they were unaware of how concrete manipulatives could undergird young children's understanding of mathematics content. A special education teacher said, "Since MAPPS, I've done a lot more work with manipulatives. I make a point to go to the manipulatives quickly and then the abstract." He additionally said, "To approach it [content] rather than drill it and kill it that [MAPPS activity] was a problem solving model that I wouldn't help them with except you know I would get them over humps and stuff. But it was a way to get them to think and to realize, 'Oh I get it, I get it,' and the light bulb [would] click on. You could see it happening, and it was really good." This teacher began using MAPPS activities exclusively for Saturday school instruction. This school's standardized mathematics test scores rose from 64.3% passing in 2008 to 81.3% passing in 2011.

Aside from improved, purposeful manipulative and task use the classroom, Knowledge of Content and Teaching in general improved, as evidenced by teacher statements such as this: "It [MAPPS] gave me ways to bridge that gap between what I know and really making it something that they know." Teacher learning continued to develop after MAPPS, as teachers reported adapting materials to other grade levels and mathematics content areas from year to year.

Teacher C: I think I learned as much from it as the parents did.

RL: And that is what it does for you.

Teacher C: I don't think I learned as much as I did *until I brought it back into my classroom* [emphasis added].

Thus, although teacher learning did occur during MAPPS, there seemed to be a delayed amount of learning that took place. Teachers adapted MAPPS tasks to their own grade levels and state standards, and in the process of enacting the tasks in their own classrooms, or collaborating about the tasks with their colleagues, they furthered their Knowledge of Content and Teaching.

In addition to strengthening teachers' mathematical knowledge for teaching, the learning community afforded by MAPPS strengthened parent-teacher relationships as well. Bonding formed because teachers got to know parents in a different way than in the negatively-connoted position of power, telling parents what to do or not to do in regards to their children. Teachers and parents enjoyed a level playing field in which all were learning for the desired end of helping children (**Enjoyment of/Valuing MAPPS**). One teacher said, "When they saw me get excited about something, they were like, 'Wow, she didn't know this. We're learning this together.'"

Moreover, parents appreciated teachers' extra effort to help children learn, and teachers came to view parents as dedicated individuals, invested in the academic success of their children. The light-hearted nature of the Mini-courses drew families and teachers back for not only more mathematics learning, but relationships fueled by a desire to learn mathematics. Even 4th-8th grade children participated as equal learners of mathematics content, often presenting solutions or strategies that parents and teachers learned from. The casual, non-threatening learning environment served as a relationship-building environment. Finally, teachers helped each other make the tasks and rigor relevant to their respective classrooms. The MAPPS learning community forged a Parent-Teacher-Child triangle of knowledge and respect (see Figure 3).

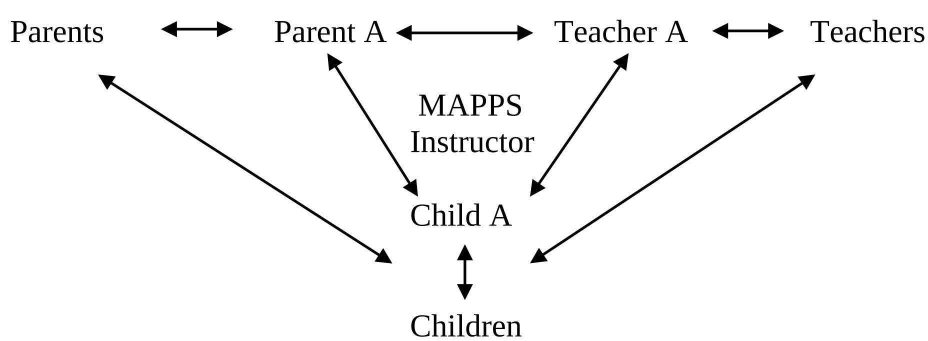

Figure 3: MAPPS learning community

A notable benefit of the learning community was *teachers learning about parents'* knowledge and strategies, a construct we call "Knowledge of Content and Parents." An interview evidenced this.

RL: Tell me, how did you feel about working in groups with other parents and other teachers?
Teacher D: It enlightened me a lot. I didn't know that they didn't know so much of the vocabulary. I had no clue. They really had no clue of how to talk about manipulatives and use them. They didn't know what that meant when their children came home and discussed it.

Just as Teacher B learned that parents didn't understand how to use manipulatives to teach decimal operations when the child said, "5 and 50 are the same," this teacher learned that parents did not possess the *vocabulary* related to the manipulatives. Another teacher explained that she learned about parents by listening to them talk with their children about mathematics. The MAPPS environment allowed teachers to see *why* parents struggled to assist their children with mathematics. Moreover, teachers learned about parent content knowledge and strategies as parents and teachers collaborated to solve problems. Teachers also listened to parents present their strategies, such as when the parent presented his strategy of finding 60% of the fraction grid (see Figure 2). We believe that this *content-focused teacher learning about* *parents' knowledge and strategies,* or *Knowledge of Content and Parents,* is analogous to Knowledge of Content and Students and is also an aspect of Schulman's (1987) category of teacher knowledge, "knowledge of educational contexts" (p. 8).

Through the clinical experience of the MAPPS learning community, teachers gained Mathematical Knowledge for Teaching and Knowledge of Content and Parents. As such, teachers learned what misconceptions parents had, that parents did not know about many manipulatives commonly used in the classroom, and that parents lacked vocabulary needed to connect manipulatives to conceptual understanding of content. Furthermore, teachers learned about parents' content knowledge, problem-solving strategies, explanations to their children, and desire to help their children. Thus, the parent-teacher mathematics learning community provides a unique professional development environment for teachers. Teachers learn both mathematics for teaching and how to access parental involvement in a way that enhances student learning. This study implies that Knowledge of Content and Parents can and should be taught through parent-teacher mathematics learning communities. Further research is needed on the nature of Knowledge of Content and Parents and its relationship to student achievement in mathematics.

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TEACHERS' IDENTIFICATION OF CHILDREN'S UPPER AND LOWER BOUNDS REASONING

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This research investigates the question of what growth, if any, is shown by teachers in identifying the components of children's reasoning using an upper and lower bounds argument for a fraction task. Specifically, it reports on assessment outcomes from design-based research in teacher education that measures teachers' identification of children's reasoning from studying videos. We describe the nature of the instructional intervention as well as the video-based assessment used as a pre and post measures for identifying children's mathematical reasoning, and report on the nature of teacher growth in recognizing components of children's arguments.

Keywords: Reasoning and Proof; Teacher Education–Inservice/Professional Development; Rational Numbers; Design Experiments

Introduction

The research presented here comes from an ongoing, interdisciplinary research and development project[1] at a large public university. Work includes the development of a digital repository that provides open access to a seminal video collection of children's mathematical reasoning that accumulated through a quarter century of research on the development of mathematical thinking and reasoning in students.[2] Videos from the repository have been used to conduct design research in teacher education, specifically for the purpose of examining how the opportunity to study videos may help teachers augment their abilities to recognize mathematical reasoning as it emerges from children's explanations and justifications of their problem solving. Instructional interventions for teachers were created for implementation in courses or workshops, typically based on one of two models (Palius & Maher, 2011). We report here on a different kind of intervention model that was created specifically for implementation in the context of online learning with digital resources.

Theoretical Perspective

Learning occurs in complex contexts and it is important that it be studied in the way it naturally occurs (Brown, 1992; Greeno & MAP, 1998; Spiro, Feltovich, Jacobson, & Coulston, 1992). However, teachers and those preparing to be teachers do not ordinarily have the opportunity to study in detail the learning of individual students in classrooms. Collections of video offer a rich source of data for careful analysis and reflection on children's learning. Choosing subsets of videos from large collections can provide a rich resource for addressing particular research questions. Our work and the work of others have demonstrated that there is much to gain from studying episodes of children's learning from videos (Cobb, Wood, & Yackel, 1990; Maher & Davis, 1995; Fenemma, Carpenter, Franke, Levi, Jacobs, & Empsom, 1996; Tirosh, 2000). Further, video offers an excellent medium for teachers' development of what Bransford et al. (2006) refer to as "adaptive expertise," that is, an ability to spontaneously and flexibly identify, critically evaluate, and respond in appropriate ways to instances of children's learning. It is from this perspective that our study was designed.

Yackel and Hanna (2003) discuss the importance of reasoning and proof in mathematics learning and their functions of verification, explanation, and communication. They point to the need for mathematics educators to be able to support students' development along the continuum from reasoning, explaining, and justifying towards articulation of formal proof, as well as to the need for teachers to create a classroom atmosphere that support such development (Yackel & Hanna, 2003). Mathematics teacher education, therefore, is faced with the challenge of helping teachers to attend to emerging forms of reasoning as children express justifications using their own language. Making use of episodes and transcripts of video data of children's reasoning from a major collection, we sought to investigate whether teachers could build the mathematical knowledge for recognizing components of children's reasoning. Specifically, the question that guided our research was whether and to what extent teachers successfully identified components of children's reasoning using an upper and lower bounds argument for a fraction task.

Methodology

As part of the design research in teacher education, three of the authors developed a new, online course in mathematics education, entitled *Critical Thinking and Reasoning*, to be taken as an elective by graduate students. Its purpose was to focus teachers' attention to how children reason about fraction ideas through study of videos children's reasoning, while engaged in problem solving with fraction tasks (Yankelewitz, Mueller, & Maher, 2010). Research literature connected to the video content was assigned as readings to comprise course units around which online discussions were focused. As a component of the design research, we examined teachers' attention to children's reasoning before and after the intervention. For this report, we investigate the nature of teacher growth in identifying upper and lower bounds reasoning in children from videos.

The first implementation of the course was during a semester with 12 students participating in the research. The second iteration was done as a four-week summer session course with 10 students participating in the research. Both courses contained a unit that focused specifically on children's mathematical reasoning about the fractions task in the video assessment. Specifically, students were assigned to study two videos, *Fractions, Grade 4, Clip 1 of 4: David's upper and lower bound argument* (http://hdl.rutgers.edu/1782.1/rucore00000001201.Video.000054465) and *Fractions, Grade 4, Clip 4 of 4: Designing a new rod set* (http://hdl.rutgers.edu/1782.1/rucore00000001201.Video.000054751). The reading assignment from the unit was a book chapter that discussed children's mathematical exploration that leads toward proof-like reasoning, which included the example of David's upper and lower bounds argument (Maher & Davis, 1995). The prompt for group online discussions was open-ended and suggested that attention be paid to forms of children's arguments and the evidence they provide, as well as consideration of what may be evidence of understanding or evidence of obstacles to the children's understanding of the mathematics. Students were assigned to small groups for engaging in online discussions about the videos they were viewing and the related literature.

Consistent with methodology of the larger research project, participants were administered pre and post-tests to measure change from before to after the intervention. We focus here on a video-based assessment for identifying children's mathematical reasoning on a particular task in the fractions strand. The assessment video includes footage from research conducted in an after-school enrichment program for 6th graders in an urban community, where children engaged in many of the same tasks that were explored by children in the 4th grade classroom study (Maher, Mueller, & Yankelewitz, 2009). It contained short clips of children working in groups on a task to find a Cuisenaire rod in the set that could be given the number name one-half when the blue rod has been given the number name one. It also contained short clips of children explaining their solution ideas with rod models as justification to the whole class (Maher, Mueller, & Palius, 2010).

The children in the assessment video offered various explanations for why they found that there is no rod in the set that can be called one half when the blue rod is called one. Some of the explanations took the form of reasoning by cases; however, one of the arguments took the form of reasoning by upper and lower bounds (Yankelewitz, Mueller, & Maher, 2010). More than one child's discourse contributed to the articulation of this argument form, which, along with the mathematical sophistication of the argument, made it particularly interesting as focal point of analysis after coding the assessment data. That is, we were curious about the extent to which teachers would recognize that children were expressing in their own language that the solution for half of Blue is bounded by the Yellow and Purple rods, with Yellow being the least upper bound and Purple being the greatest lower bound (i.e., that there is no rod in between them).

A highly detailed rubric was developed by our research team in order to code the data by the components of the arguments that were articulated by the children in the assessment video. The assessment prompted study participants to describe as completely as they can the reasoning that the children put forth, whether each argument offered by children is convincing, and why or why not are they convinced. Participants were provided with a transcript for the video and were not restricted in the amount of time spent working on the assessment. The assessment prompt also informed participants that their responses would be evaluated by the following criteria: recognition of children's arguments, their assessment of the validity or not of children's reasoning, evidence to support their claims, and whether the warrants they give are partial or complete.

Two researchers scored assessment data with 90.4% inter-rater reliability. For the upper and lower bounds argument, there were four components of the children's reasoning that could combine in three different ways to be a complete argument (a, b, and c; a, b, and d; or a, b, c, and d):

a. The Yellow rod is (1/2 of one White rod) longer than half of Blue; (AND)
b. Purple is (1/2 of one White rod) shorter than half of Blue; (AND)
c. There is no rod with a length that is between Yellow and Purple; (OR)
d. The White rod is the shortest rod and the difference between the Yellow rod and the Purple rod is one White rod.

Participant responses that did not mention any of the above components or that mentioned only one or two of them were deemed to be incomplete. The coded data were analyzed quantitatively.

Results

Analysis of the video assessment data yielded the following results with regard to the upper and lower bounds argument. Tables 1a, 1b, and 1c describe the distributions of pre-assessment argument components, showing results for the two classes combined and then disaggregated by the two implementations of the course. In Table 1a, we note that 13 of the 22 students in the combined courses provided an incomplete argument description in the pre-assessment, while 8 of these 13 students provided none of the 3 essential argument components (a, b, and c or d) of a complete upper and lower bounds argument. A total of 11 out of 13 excluded argument component a; 12 out of 13 excluded argument component b; and 10 out of 13 excluded either argument component c or d. Table 1b shows that 8 of 12 students in the intervention provided an incomplete argument description in the pre-assessment; 5 of these 8 students provided none of the 3 essential components (a, b, c or d) of a complete argument description. A total of 3 out of 8 excluded argument component a; 7 out of 8 excluded argument component b; and 6 out of 8 excluded either argument component c or d. Table 1c shows that 5 of 10 students in the summer course intervention provided an incomplete argument description in the pre-assessment; 3 of these 5 students provided none of the three essential components (a, b, c or d) of a complete argument description. A total of 4 out of 8 excluded argument component a; 5 out of 5 excluded component b; and 4 out of 5 excluded either component c or d.

In summary, the pre-assessment results indicate that 59% of the students in the two courses did not provide a complete upper and lower bounds argument description on the pre-assessment. Of the students with an incomplete argument description, over 75% from the two combined courses failed to describe each of the three essential upper/lower bound argument components.

Table 1a: Distribution of Pre-Assessment Argument Components: Two Courses Combined

Students with Incomplete Argument			Students with Complete Argument		
Components	Count	Frequency	Components	Count	Frequency
None	8	0.6154	a, b, c	4	0.4444
a	1	0.0769	a, b, d	2	0.2222
c	1	0.0769	a, b, c, d	3	0.3333
d	2	0.1538			
a, b	1	0.0769			
Total	13	1.0000	Total	9	1.0000

Table 1b: Distribution of Pre-Assessment Argument Components: Semester Course

Students with Incomplete Argument			Students with Complete Argument		
Components	Count	Frequency	Components	Count	Frequency
None	5	0.625	a, b, c	2	0.25
c	1	0.125	a, b, d	1	0.50
d	1	0.125	a, b, c, d	1	0.25
a, b	1	0.125			
Total	8	1.000	Total	4	1.00

Table 1c: Distribution of Pre-Assessment Argument Components: Summer Course

Students with Incomplete Argument			Students with Complete Argument		
Components	Count	Frequency	Components	Count	Frequency
None	3	0.6	a, b, c	2	0.4
a	1	0.2	a, b, d	1	0.2
d	1	0.2	a, b, c, d	2	0.4
Total	5	1.0	Total	5	1.0

Tables 2a, 2b, and 2c describe the distributions of post-assessment argument components, showing results for the two classes combined and then disaggregated by the two implementations of the course. In Table 2a, we note that of the 10 of the 22 students in the combined courses provided an incomplete argument description in the post-assessment, while only 1 of these 10 students provided none of the three essential argument components (a, b, and c or d) of a complete upper and lower bounds argument. A total of 4 out of 10 excluded argument component a; 4 out of 10 excluded component b; and 7 out of 10 excluded either component c or d. Table 2b shows that 6 of 12 students in the intervention provided an incomplete argument description in the post-assessment. Of these 6 students, at least one the three essential components (a, b, c or d) were provided. Three of the 6 students excluded argument component a; none excluded component b; and 5 out of 6 excluded either argument component c or d. Table 2c indicates that 4 of 10 students in the summer course provided an incomplete argument description in the post-assessment, 1 of these 4 students provided none of the three essential components a, b, c or d of a complete argument description. A total of 2 out of 4 excluded argument component a, 3 out of 4 excluded argument component b, and 2 out of 4 excluded either argument component c or d.

Table 2a: Distribution of Post-Assessment Argument Components: Two Courses Combined

Students with Incomplete Argument			Students with Complete Argument		
Components	Count	Frequency	Components	Count	Frequency
None	1	0.1	a, b, c	4	0.3333
b	2	0.2	a, b, d	3	0.2500
d	1	0.1	a, b, c, d	5	0.4166
a, b	4	0.4			
a, d	1	0.1			
a, d	1	0.1			
Total	10	1.0	Total	12	1.0000

Table 2b: Distribution of Post-Assessment Argument Components: Semester Course

Students with Incomplete Argument			Students with Complete Argument		
Components	Count	Frequency	Components	Count	Frequency
b	2	0.3333	a, b, c	2	0.3333
a, b	3	0.5000	a, b, d	2	0.3333
b, d	1	0.1667	a, b, c, d	2	0.3333
Total	6	1.0000	Total	6	1.0000

Table 2c: Distribution of Post-Assessment Argument Components: Summer Course

Students with Incomplete Argument			Students with Complete Argument		
Components	Count	Frequency	Components	Count	Frequency
None	1	0.25	a, b, c	2	0.3333
d	1	0.25	a, b, d	1	0.1667
a, b	1	0.25	a, b, c, d	3	0.5000
a, d	1	0.25			
Total	4	1.00	Total	6	1.0000

In summary, the post-assessment results indicate that 45.5% of the students in the two courses combined were not able to provide a complete upper and lower bounds argument description, compared to 59% on the pre-assessment. Of the students with an incomplete argument description on the post-assessment, 40% failed to describe each of the components a and b, and 70% failed to describe component c or d. This is in contrast to over 75% who failed to describe each of the three argument components on the pre-assessment.

Table 3 classifies the pre-assessment argument descriptions into three categories: (1) a Complete Argument description containing components a, b, and c or d; (2) a No Components description which lacks all three essential argument components; and (3) a Partial Argument description which contains at least one essential argument component but lacks all three. The respective frequencies for the two combined courses are: 40.9% Complete Argument, 36.4% No Argument Components, and 22.7% Partial Argument.

Table 3: Upper-Lower Bound Pre-Assessment Argument Frequencies

Pre-Assessment Argument Components	Combined Courses		Semester Course		Summer Course	
	No.	Freq.	No.	Freq.	No.	Freq.
Complete Argument	9/22	40.9%	4/12	33.3%	5/10	50.0%
No Components	8/22	36.4%	5/12	41.7%	3/10	30.0%
Partial Argument	5/22	22.7%	3/12	25.0%	2/10	20.0%
Total Number Students	22	100%	12	100%	10	100%

Table 4: Post-Assessment Transition Frequencies

Pre-Assessment	Post-Assessment	Combined Courses No.	Combined Courses Transition Frequency	Semester Course No.	Semester Course Transition Frequency	Summer Course No.	Summer Course Transition Frequency
NONE		N=8		N=5		N=3	
	No Growth	1/8	12.5%	0/5	0%	1/3	33.3%
	Partial Growth	6/8	75%	5/5	100%	1/3	33.3%
	None to b	2		2		0	
	None to ab	4		3		1	
	Complete	1/8	12.5%	0/5	0%	1/3	33.3%
	None to abcd	1		0		1	
PARTIAL		N=5		N=3		N=2	
	No Growth	1/5	20%	0/3	0%	1/2	50%
	d to d	1		0		1	
	Partial Growth	2/5	40%	1/3	33.3%	1/2	50%
	a to ad	1		0		1	
	d to ad	1		1		0	
	Complete	2/5	40%	2/3	66.7%	0/2	0%
	ab to abd	1		1		0	
	c to abc	1		1		0	

Table 4 provides the post-assessment transition descriptions and frequencies. For example, the 4th data row of Table 4 indicates 2 students in the combined courses exhibited a pre-to-post argument description transition of “No Components” on the pre-assessment to a post-assessment description with only the argument component “b” (transition labeled as “none to b”). In examining the transition frequencies for the combined courses in Table 4 we note the following: (1) 75% of students with no upper and lower bounds argument components on the pre-assessment provided a partial upper and lower bounds argument description on the post-assessment and 12.5% provided a complete argument description, and (2) 40.0% of students with a partial argument on the pre-assessment provided a complete upper and lower bounds argument description on the post-assessment. In the semester course, it is important to note that 2/3 of the students with a partial argument description on the pre-assessment transitioned to a complete argument description on the post-assessment. This is in contrast to the summer course, where one half of the students with a partial pre-assessment description exhibited no growth on the post-assessment and the other half exhibited only partial growth.

Conclusions and Discussion

The effectiveness of using video examples in online courses to stimulate the growth of teachers’ ability to recognize and describe upper and lower bounds arguments of students is evidenced by the fact that 2/3 of the semester course students transitioned from a partial to a full upper and lower bound argument description on the post assessment, and 2/3 of the summer course students transitioned from a recognizing no components of the upper and lower bounds argument description to a partial or complete argument description. Some teachers recognized the yellow rod as an upper bound and the purple rod as a lower bound, but did not attend to the detail of the child’s argument that there was no rod in between, so that the yellow rod was the smallest upper bound and the purple rod was the largest lower bound. Although there was some growth in teachers’ recognition of components of children’s arguments after studying the videos, there is still a need for improvement. The research suggests that a video-based approach for teacher education has the potential to be effective, but that a single-unit intervention may not be adequate for developing satisfactory adaptive expertise with regard to this particular form of reasoning. Future studies might include interventions that give greater attention to the variety of arguments, partial and complete, that children naturally develop in the process of problem solving so that there may be increased opportunities for teacher evaluations of the validity of the arguments posed. With regard to online courses, research also is needed to investigate the role of threaded discussion as a tool to develop adaptive expertise in recognition of children's emergent mathematical reasoning and what kinds of scaffolds may serve to stimulate group discussions that address important aspects of the process as can be observed through studying video data.

Endnote

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[2] The repository for the project, Video Mosaic Collaborative, is accessible at the website: http://videomosaic.org/

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AT THE CROSSROADS OF MATHEMATICAL VOCABULARY, WRITING, AND THE SECONDARY MATHEMATICS TEACHER

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Although much research has been done citing the benefits of using writing in mathematics lessons, little has been done that examines teachers' responses to writing about mathematics and how those responses may shape teacher attitudes about using writing in the classroom. In this study, I examined the experiences of six teachers as they explored mathematics in a graduate class in secondary mathematics education and prepared written reports for Internet publication. After analyzing their work and their responses on questionnaires and in interviews, I found that several of the participants struggled with using the technical vocabulary of mathematics in their writings and that some of them had differing beliefs about the use of the vocabulary. Based on these findings, I recommend that mathematics educators use activities that challenge teachers to view vocabulary as mathematical content and that hone their skills in using it in their writings.

Keywords: Teacher Education–Preservice; Teacher Education–Inservice/Professional Development; Teacher Knowledge; Teacher Beliefs

Objective

During their careers, teachers will frequently transition between being a student and being a teacher. It is a necessary move that offers teachers the chance to experience for themselves the lessons they want their students to learn. As mathematics educators, we bear the burden of insuring that those moments the teachers spend as students are filled with rich, thought-provoking experiences that provide them with the knowledge they will need to help their students learn. This need for teachers to have rich experiences as students is particularly important in the area of mathematical language and writing in which teachers are asked to navigate yet another transition—when they are asked to transition from using informal language in writing about mathematics to using the formal language of mathematics.

For many years now, mathematics educators and researchers have promoted the use of writing in the mathematics classroom. In 1977, Geeslin reported on the benefits of using writing in mathematics "as a learning device for the student" (p. 113). In 1989, the National Council of the Teachers of Mathematics (NCTM) suggested in its *Curriculum & Evaluation Standards* that "all students need extensive experience…writing about…mathematical ideas" (p. 140). Since those early years, much research has been done noting the benefits of using writing in the mathematics classroom (Porter & Masingila, 2001), but there is evidence that suggests these recommendations have not been embraced by a majority of secondary mathematics teachers. In a national survey of secondary mathematics teachers conducted in the United States in 2000, 55% of those teachers surveyed indicated that they never use reflective writing in their classrooms (Weiss, Banilower, McMahon, & Smith, 2001).

As intuitively expected, Flores and Britain (2003) suggested that mathematics teachers are likely not to use writing in their lessons "unless they have had the experience themselves of writing in relation to mathematics" (p. 112). However, this suggestion seems to overlook the nature of the experience and creates a question about how teachers respond to writing about mathematics. Before we as mathematics educators can help preservice and inservice mathematics teachers transition to an effective use of writing in their classrooms, we must first understand how teachers themselves respond to the writing in terms of what they can do and what they believe. Principally, we need to know how they respond to writing about mathematics when they are acting from the perspective of a student. In this study, I endeavored to explore and examine those responses. In this paper, I report on one area of the study in which I focused on how the participants responded to writing in terms of the language they used and what they believed about the type of language they should use.

Theoretical Framework

I conducted this exploratory study from the view that to effectively write about mathematics, teachers and students must give attention not only to accurate descriptions of concepts and procedures but also to the proper use of mathematical language. Arguably, students do not have a competent understanding of mathematics unless they are fluent in its language which Ball and Sleep (2007) characterized as "both mathematical content to be learned and [a] medium for learning mathematical content" (p. 19). Essentially, to write about mathematics in a manner which showcases understanding, students must first have a working knowledge of mathematical language.

The language of mathematics is often defined as the mathematics register. Foley (2008) characterized the mathematics register as "the formal academic approach to mathematical speaking and writing" (p. 1). Schleppegrell (2007) separated the mathematics register into two categories: multiple semiotic representations and grammatical patterns. Multiple semiotic representations address symbolic notation, oral and written language, graphs, and other visual displays. Grammatical patterns cover technical vocabulary, dense noun phrases, and "implicit logical relationships" (p. 141). In this paper, I focus on elements drawn from both categories. I specifically focus on the use of technical vocabulary within the written language of mathematics.

In terms of using the technical language of mathematics in the classroom, teachers can often feel two opposing forces at work within themselves: the urge to use students' informal language in order to be relevant and the need to foster the development of the technical vocabulary of mathematics. Jill Adler (1997) characterized this delicate balancing act as one of the "dilemmas of mediation" (p. 235) in which mathematics teachers have the burden of "shaping informal, expressive and sometimes incomplete and confusing language, while aiming towards the abstract and formal language of mathematics" (p. 236). How teachers balance this tension, however, is often influenced by what they believe about the use of mathematical language in the classroom.

In this study, the word *belief* is being used in a broad sense to encompass the idea of *attitude* which Philip (2007) defined as "manners of acting, feeling, or thinking that show one's disposition or opinion" (p. 259). Although there are distinctions between the two concepts, it can be argued that belief and attitude are deeply connected and that what people believe does influence how they act and what they say. In teacher education, beliefs play an important role in how preservice and inservice teachers approach their training and what they glean from it. Cooney (1998) stated that mathematics educators must consider such beliefs in order to "create activities that encourage teachers to wonder, to doubt, to consider what might be, to reflect, and most important, to be adaptive" (p. 332). In this paper, I focus on those beliefs about the use of the technical language of mathematics that seemed to influence how the participants in the study wrote about the mathematics.

Methodology

In this qualitative study, I examined the responses of five preservice teachers and one inservice teacher in a graduate course in secondary mathematics education as they completed 11 explorations of various mathematical topics using technology. After completing the explorations, they posted their findings on the Internet in written reports called "write-ups." In addition to preparing these formal reports, I asked the participants to take notes while they explored the mathematics and to complete a written reflection after they finished each activity. My objective was to have three forms of writing to which the participants could respond: formal, informal, and reflective. In this paper, I focus on their responses to the formal writing or to the write-ups they prepared for Internet publication.

Maxwell (2005) noted in his book *Qualitative Research Design: An Interactive Approach* that "the typical way of selecting setting and individuals" (p. 88) is "purposeful selection" (p. 88). He described this method as "a strategy in which particular settings, person, or activities are selected deliberately in order to provide information that can't be gotten as well from other choices" (p. 88). In an effort to collect unbiased data, I solicited participation from a class in secondary mathematics education in which writing about mathematics was frequently used but was not a focus of instruction. In so doing, I diminished the risk that the biases of the instructor about writing in mathematics were frequently passed on to the participants. Throughout the study, I also endeavored to refrain from offering my opinion about writing in mathematics, about what the participants had to say, or about the quality of their work. My objective was to study the responses of the participants in an atmosphere as free as possible from instructor or researcher bias.

The semester-long class met weekly for three hours, and after a brief introduction of the relevant topic by the instructor, most of the class time was devoted to individual explorations of mathematical topics at computers. Students prepared their Internet reports based on 11 activities covering topics in algebra, geometry, data analysis, precalculus, and calculus. These activities presented a wide range of tasks that students could explore using software such as Geometer's Sketchpad (Version 4.07) and Graphing Calculator (Version 3.5). In each activity, students were given several tasks from which they could choose one to explore and about which they could write a report. For example, in one activity they could choose to describe what happens to the graph of a quadratic equation in standard form when the value of *a, b,* or *c* is varied as the other two values are held constant in the equation. Students were free to work through the activities at their own pace and post their reports to the Internet at any time throughout the semester.

At the first class meeting, I requested that all master's level students in a class of 31 complete the initial questionnaire. Twenty-three students signed a consent form and 18 students returned their responses via email or at the next class meeting. From these 18 students, I asked 10 if they would agree to participate based upon their responses to the questionnaire. My goal was to ask participants to volunteer who offered differing opinions on the use of writing in the mathematics classroom. Throughout the semester-long class, I tracked the participants' progress with the write-ups by checking their Internet postings, informally speaking with each participant during class, and by conducting formal interviews of each participant at the beginning, midpoint and end of the semester. At the end of the semester, I also asked the participants to complete a post-questionnaire about their experiences with the various writings in the class.

Near the end of the study, I determined that four of the participants had finished less than half of the write-ups. This lack of progress meant that they would complete the bulk of the course in two weeks which ran counter to an initial request I had made at the first meeting that they work at a steady pace throughout the semester to insure they had an adequate amount of time for reflection. In good faith, I could not compare their work with those who had steadily worked their way through the course and were primarily done with the course at the end of the study; therefore, I eliminated these four participants from the study. After data collection, I began the analysis of the data collected from the six remaining participants: Gwen, Amy, Claire, Grace, Lisa, and Kim.

During the analysis phase of my study, I performed two different types of examinations. During the first examination, I studied all notes, interview transcriptions, questionnaire responses, and written reflections to categorize participant responses according to various topics such as background, experiences with the course and the writings, and their beliefs about writing in mathematics. This categorization allowed me to situate the participants according to their various experiences. I prepared a report for each participant in outline form which addressed these topics. After I completed a report for each participant, I carefully examined each report noting emerging themes across the documents about participant responses to writing in mathematics. Once I identified these themes, I reexamined all the data, making note of any new evidence to support or contradict these major ideas. During the second type of examination, I studied each write-up posted on the Internet to determine the soundness of the mathematics used and the quality of the writing in terms of style, grammar, and language usage. In this paper, I specifically focus on two themes which emerged from these two examinations: the quality of the participants' use of technical vocabulary in the write-ups and the participants' differing beliefs about the use of technical vocabulary when writing about mathematics.

Results

Several participants struggled with the use of technical vocabulary in their write-ups. In one write-up, Grace described ellipses as "tall up and down" or "long left to right" rather than as *vertical* or *horizontal*. In another write-up, she characterized the areas of triangles as congruent. Kim characterized the graph of an inverted parabola as a "negative" graph. Gwen described the number of "humps" in the graph of a parametric equation. Amy described graphs as merging “after the domain of –3 [and] 3” which she seemed to want to mean that the graphs merged after the points with the *x-coordinates* of –3 and 3 (see Figure 1). However, her phrasing technically means that the domain consisted only of –3 and 3 which is not a true statement. Although a knowledgeable reader could reasonably infer what these participants intended when they used these words and phrases, the use of mathematical vocabulary in the write-ups ranged from informal at best, imprecise on average, and incorrect at worst. For example, the word *congruent* is customarily used in reference to two geometric figures that have the same size and shape. The concept of area, as Grace used it, is typically not included in that description.

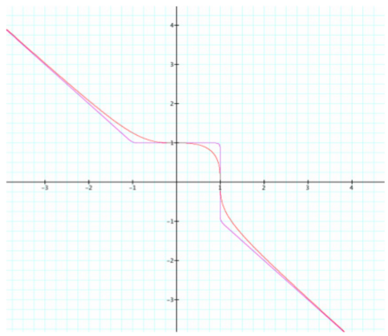

Figure 1: Amy’s merging graphs

The use of technical vocabulary also brought out differing beliefs and attitudes in three of the participants. Claire stated in the second interview that she had recently learned about the mathematics register in one of her graduate classes and implied that the lesson had helped her to become more aware of the language she was using in her write-ups. She implied it was important that the readers understood the technical language she was using in the write-ups so they could make sense of her explanations. Kim, however, expressly stated that she wanted to avoid the use of technical language. During the second interview, she offered the opinion that she thought her write-ups were “mathematically written” but were not like a textbook which seemed to be what she wanted. She stated during the interview that she believed textbooks contain “just too much mathematical language” and implied that she wanted to “use just normal conversational language.” In the final interview at the end of the study, she noted again that she “wanted to make sure that [her] words were universal.”

In contrast to Kim, Amy showed a desire to use technical vocabulary in her writing, but she noted on several occasions that she struggled with the language. During the second interview, she commented that “a lot of why I can’t communicate mathematically [is] sometimes I don’t know the language.” She clarified the comment by stating that she had a problem with “using the right math terminology” but conceded that doing the write-ups up to that point had helped her to build her mathematical vocabulary. She commented that completing the write-ups was helping her “think about the math language and how should I write this or how should I explain what’s going on with this in words….” It was not entirely clear, however, that Amy believed that the mastery of the vocabulary was part of the mathematical content she needed to know. She stated in the final interview that “my…my problem isn’t math, it’s writing about math or writing, I think, about anything period….” When asked if she would have preferred to have done oral rather than written presentations, she stated that she would have preferred the oral presentations because it would have been easier for her to “show you why versus trying to explain in words why.”

Discussion and Conclusions

Although researchers have claimed for many years that writing is a beneficial tool to help students learn mathematics, a gap exists in the research which informs us about how teachers respond as students to writing about mathematics. In this study, I sought to examine the responses of preservice and inservice teachers as they engaged in an intensive exploration of various mathematical topics and published their written findings on the Internet. In this paper, I focus on participant response in terms of technical vocabulary. The results of this study tend to suggest that preservice and inservice teachers struggle with the use of mathematical language in terms of vocabulary and that they also have varying beliefs about the role technical vocabulary should play in writing about mathematics.

Ball and Sleep (2007) described mathematical language as “mathematical content to be learned” (p. 19). Viewed from this perspective, this study shows that several of the participants were deficient in this area of their mathematical content knowledge in varying degrees. For example, Grace knew how to describe the orientation of the ellipse as “tall and long” but she did not reach for the content word *vertical*. In effect, she did not take her informal language and translate it into mathematical content. At the other end of the spectrum, Amy’s struggle with the use of mathematical language showed a clear deficiency in mathematical understanding. In other words, she could not use the content word *domain* properly in her writing because she did not appear to fully understand the concept.

The study also tends to show that not all of the participants agreed with the notion that mathematical language is mathematical content as demonstrated by their attitudes or their “manners of acting, feeling, or thinking…” (p. 259). After learning about the mathematics register in one of her graduate classes, Claire embraced technical vocabulary as part of mathematical content and endeavored to make her writing precise and technically correct. Kim, on the other hand, expressly fought against it. Kim desired that her writings contain what she called “universal” language. For her, there seemed to be no “dilemmas of mediation” (Adler, 1997) between formal and informal language but instead a belief that technical language is an unnecessary obstacle to understanding mathematics. Amy seemed to share a similar opinion. Although Amy frequently acknowledged her struggles with knowing the vocabulary of mathematics, she nevertheless declared at the end of the study that she knew the mathematics but had problems expressing it in writing. This stance seems to imply that she believed someone could know the mathematics without being able to use its technical language in writing.

When we as mathematics educators ask secondary teachers to use writing in their classrooms, we sometimes assume that they believe in the use of technical vocabulary in writing about mathematics and that they have mastered the skill themselves. However, this study indicates we must first consider the importance of teachers’ beliefs about the use of technical language in writing about mathematics. We must do as Cooney advised and “create activities that encourage teachers…to consider what might be…and most important, to be adaptive” (p. 332). Specifically, we need to develop lessons that directly challenge both preservice and inservice teachers to confront their beliefs about the role of technical language in mathematics and that encourage a consideration of mathematical language as an integral part of mathematical content knowledge. The simple inclusion of lessons about the mathematics register may raise awareness for some teachers such as Claire experienced in one of her classes. Next, we need to provide guidance and practice through activities and assessments designed to help teachers transition from the use of informal language in their writing to the effective use of the formal language of mathematics. We also need to raise their awareness that a misuse of technical vocabulary in writing may indicate that teachers do not fully understand the mathematical concept behind the terms. In providing these types of activities, we better equip teachers to navigate the “dilemmas of mediation” (Adler, 1997, p. 235) which they will face in their own classrooms.

The results of this study also suggest an area in need of further research. We need studies devoted to how teacher beliefs about the use of technical vocabulary in writing may influence their teaching. Specifically, we need to probe the depths of how those beliefs may influence how teachers structure their lessons, what they expect from their students in terms of language use, and how those decisions and expectations influence student learning. In pursuing such research, we provide a connection to practice that informs both the researcher and the practitioner in the ways language use in writing may shape what students learn and do not learn in their mathematics classes.

The most important transition educators make is between the roles of student and teacher. It is a transition that occurs frequently during one’s career from the preservice phase to the inservice stage, to advanced schooling, and then on to years of professional development. It is during those times when teachers are in the role of students that we as mathematics educators must make the most of their experiences. Essentially, we cannot ask teachers to teach what they have not experienced themselves as students. This is particularly true in the area of mathematical language and writing. Teachers at all stages in their development need a rich, thorough experience of using the technical vocabulary of mathematics in their writings. By participating in activities and lessons focused on the use of technical vocabulary in writing about mathematics, they have a chance to confront their beliefs and work on their skills. In essence, we as mathematics educators help them navigate the transition from the use of informal language to the use of the formal language of mathematics. By providing these rich experiences, we also increase the odds that these students will become master teachers who feel more comfortable with writing about mathematics and, in turn, are more likely to use writing in their classrooms. Ultimately, we create teachers who can effectively guide their students in becoming fluent in the use of mathematical language in writing about mathematics.

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FOSTERING STRATEGIC COMPETENCE FOR TEACHERS THROUGH MODELING RATIONAL NUMBERS PROBLEM TASKS

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The purpose of the study was to examine how teachers enhance their knowledge of rational numbers focused on modeling problem tasks using multiple representations. The professional development summer institute and the follow-up Lesson Study (Lewis, 2002) throughout the academic year focused on engaging teachers in rational numbers and proportional reasoning problem solving tasks, exploring pedagogical strategies, utilizing mathematics tools and technology, and promoting connections in the elementary and middle school curricula. This research report has two aims: (1) identify ways in which focusing on modeling rational numbers with multiple representations impacted teachers' understanding of rational numbers and proportional reasoning concepts; and (2) examine what strategic competence (NRC, 2001) looks like in teachers as they learn to model rational numbers concepts using multiple models.

Keywords: Modeling; Rational Numbers; Teacher Development; Professional Development

Theoretic Framework

Strategic competence has been defined as the "ability to formulate, represent, and solve mathematical problems" (NRC 2001, p. 116). The National Research Council define "mathematics proficiency" as having five strands that include strategic competence along with conceptual understanding, procedural fluency, adaptive reasoning and productive disposition. This study uses the term strategic competence as a competence we want to develop in teachers and expand the definition to include specific criteria in AMTE's standards for Pedagogical Knowledge for Teaching Mathematics, which include the ability to "construct and evaluate multiple representations of mathematical ideas or processes, establish correspondences between representations, understand the purpose and value of doing so; and use various instructional tools, models, technology, in ways that are mathematically and pedagogically grounded" (AMTE, 2010, p. 4). Modeling mathematics and developing representational fluency are key mathematics practices emphasized in the common core standards for math (CCSSI, 2010).

Research on rational numbers has also shown that representational fluency is critical in developing a conceptual understanding of the topic (Lamon, 2007; NRC, 2001). Representational fluency, the ability to use multiple representations and to translate among these models, has been shown to be critical in building students' mathematical understanding (Goldin & Shteingold, 2001; Lamon, 2001). The Lesh Translation Model highlights the importance of students' abilities to represent rational numbers in multiple ways, including manipulatives, real life situations, pictures, verbal symbols and written symbols (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). Translations among the different representations assess whether a student conceptually understands a problem. Such abilities to be able to translate within and among multiple representations indicates an aspect of strategic competence. Some of the ways to demonstrate translation among representations in mathematics is to ask students to restate a problem in their own words, to draw a diagram to illustrate the problem, or to act it out. In teaching and learning, representations can play a dual role, as instructional tools and learning tools. As Lamon (2001) states, representations can be "both presentational models (used by adults in instruction) and representational models (produced by students in learning), which can play significant roles in instruction and its outcomes" (p.146). Another way to think about representations is that they allow for construction of knowledge from "models *of* thinking to models *for* thinking" (Gravemeijer, 1999). The *Principles and Standards for School Mathematics* (NCTM, 2000) emphasizes that representations serve as tools for communicating, justifying, sense making and connecting ideas by stating, "Representations allow students to communicate mathematical approaches, arguments, and understanding to themselves and to others. They allow students to recognize connections among related concepts and to apply mathematics to realistic problems" (p. 67).

Research Questions

This study explored the following research questions:

1. How does focusing on modeling rational numbers with multiple representations impact teachers' understanding of rational numbers and proportional reasoning concepts?
2. How do teachers exhibit strategic competence, in terms of the ability to construct, use and evaluate multiple representations and models of mathematical ideas and establish correspondences between representations?

Methods

Sixteen elementary and middle grades teachers from grades 3–8 met for a one-week summer institute and continued as school-based Lesson Study teams during the academic year. A majority of the teachers (78%) taught in Title One schools that served underrepresented and underserved populations. The daily topics included reasoning up and down, direct and inverse thinking, unitizing, and, ratios and proportional thinking. For this research report, we focused our analysis in the summer content institute and the lesson study data sources to demonstrate the progression of development in teachers' strategic competence as they emerged from the critical incidents as reported in teachers' reflections and instructors' memos and field notes and artifacts.

Data Sources

The data sources included teacher reflections, posters of solution strategies, videotapes of the class sessions, instructors' memos and field notes.

Teacher daily reflections from content institute. Teachers reflected daily on the problem solving tasks and wrote about new strategies and representations that were shared in class by other teachers. The teacher reflections focused on the understanding, reactions, and feelings of the individual teachers. The purposes of the daily reflections were to elicit responses in teachers focused on rational number problems which asked teachers to explain their thoughts and solution strategies; identify any differences in their own understanding, approaches, and thinking which resulted from the day's activities; and, illuminate any modifications to their teaching content and approach which they intend to employ.

Artifacts from class sessions–Poster proofs and concept map posters. The data collected, teacher reflections, posters of solution strategies, videotapes of the class sessions, and field notes, was focused on the development of conceptual knowledge, not procedural skill. Each group discussed the problem, and recorded their thought processes on large poster paper. The poster proofs were used to explain their reasoning, their discussions, their mistakes, and their conclusions with the class. Others in the class could comment or ask questions. These poster proofs used to document teachers' progression of ideas.

Video class sessions and instructors' memos. The researchers collected instructors' memos each day to serve as a record of the professional development. Researchers also took photographs and video recorded daily sessions focused on teachers' sharing their representations, class discussions, and studying teachers' work and collaborative poster proofs.

Through the use of multiple data sources our goal was to capture, teachers' strategic competence, namely: (a) the connection between teachers' content knowledge and the use of representations;

(b) teachers' use of mathematics models, tools and technology and pedagogical strategies; and
(c) teachers' rationale for choosing tools and representations to represent their thinking. The researchers included the faculty and knowledgeable others who recorded their observations in a consistent format that helped us analyze and identify evolving themes and misconceptions.

Data Analysis Procedure

Critical Incident Analysis (Tripp, 1992, 1994) was used to analyze key events that evoked teachers to reflect on their math knowledge for teaching rational numbers and their teaching practices. Tripp (1992, 1994) defines critical incidents, which emerge through the critical reflection process, in the following way: *"Incidents happen, but critical incidents are produced by the way we look at a situation: a critical incident is an interpretation of the significance of an event. To take something as a critical incident is a value judgment we make, and the basis of that judgment is the significance we attach to the meaning of the incident."* Tripp also describes critical educational events are catalysts for transformative development of both students and teachers. As researchers, we took inventory of critical incidents that occurred throughout the content institute and the Lesson Study and collected the data sources from those episodes to analyze them for their meaning, relate the incidents to a broader analysis to understand how those critical incidents developed teachers' strategic competence.

Findings

Critical Incident 1: Letting Go of Formulas and Modeling Division of Fractions: What's All These Partitive, Quotitive Models?

A challenge that teachers encounter in their curriculum is having to model division of fraction. This requires understanding of the partitive and quotitive model of division. It was evident in many of our teachers that they had learned math rules without conceptual understanding and were challenged to reason about the mathematics they were teaching. To understand modeling division of fractions it is necessary to appreciate the different meanings such as measurement division, sharing, finding a whole given a part, and missing factors etc. Two different conceptual models that often evolve in modeling fractions include a fair-share (partitive) or measurement (quotitive) model. In the fair-share partitive model, the goal is to share out the same number of object to a fixed number of groups. On the other hand in a measurement quotitive model, a measurement unit is chosen and is repeated as many times to yield the quantity being measured. While the former leads to an invert and multiply algorithm, the latter leads to a common denominator algorithm. One of the instructors focused her module on this notion of division of fractions and in helping teachers model story structures that represented partitive and quotitive models. She writes in her instructor's memo:

> *I did see discussions between models of division that showed that participants did not have two equally robust models of division that they could use in their models. There was a debate between two participants that suggested that one participant had a model of division that was partitive, but the table-mate was showing a quotitive model. I hope that through modeling division problems tomorrow they will have the opportunity to figure out each division model from a set of problems. Having participants use manipulatives to model quotitive and partitive expressions challenges their views of division. (Excerpt-instructor's memo Day 3)*
>
> *The next thing I learned today is that having participants use manipulatives to model quotitive and partitive expressions challenges their views of division. For example, students who were comfortable with modeling $34 \div 14$ were stumped by $45 \div 2$. However, the reverse was also true: participants comfortable with $45 \div 2$, could not figure out a way to model $34 \div 14$. It is fascinating to me that this occurred at most tables, and I think it is something that could be followed up on. From a teaching point of view, setting up the confusion over division models and then resolving them by naming the modeling process made it much easier to teach the idea of partitive and quotitive. The participants knew that there was something "fishy" going on, but couldn't name it, and therefore couldn't work*

with their models. I know that many want to skip teaching these ideas explicitly, but I think it is an essential understanding in learning to model rational number operations, and more importantly, learning to teach students modeling! (Excerpt-instructor's memo Day 4)

These excerpts from the instructor's memo were revealing of how this class activity elicited a relearning experience for teachers. One teacher commented in her reflection, "I am grappling with the process of modeling the process of dividing by a fraction. Since I learned to multiply by the reciprocal over forty years ago, and it has always worked that way, I have never questioned that process really works. I still don't fully get it. But I will continue to examine the model until I "get it!" Other teachers also echoed their awareness of being too reliant on learned procedures and how they needed to let go of the formulas to relearn the conceptual models of operation with rational numbers. "I need to constantly use the manipulatives or I revert back to my happy place with algorithms."

Critical Incident 2: Developing Conceptual Maps and Poster Proofs with Multiple Models

The conceptual posters were used to document teachers' progression of ideas. In addition, posters showed how people in a group approached problem solutions in a variety of ways. The reflections gave insight into how the individual teachers were feeling about the sessions but they also documented how they were adding new models on to their conceptual maps for rational numbers. In class sessions, groups were required to strategize solutions by at least three of the five possible representations. The teams would affix their poster proofs to the wall, but, before verbal explanations from the teams, the class would do a "gallery walk," a walk around the room stopping to look at and to analyze each poster then they would take time considering different representations. Several themes were present in the majority of the reflections about the poster proofs. These were: the importance of clarity in the models, seeing the connection between the various models, the advantage of building multiple models, the benefit of collaboration, and recognizing that there are multiple valid approaches to problem solving, which leads to viewing student work with new eyes. Several teachers reported "Aha" moments concerning ideas about rational numbers, which they had formerly accepted but now actually understood, giving them a feeling of liberation. A teacher wrote, "I wish more classroom teachers fostered an environment where students can struggle with problems and work together to solve problems. Struggling through and listening to strategies of others has really opened up my thinking." As the teachers' conceptual knowledge deepened, the teachers began to question their own knowledge and assumptions. Classroom discussions of problems and sharing solution strategies was seen as a valuable approach both to clarify problems as well as to develop their conceptual thinking.

The teachers rediscovered the use of a ratio table to solve a problem called the Robot and Cars problem. Teachers reported that the reasoning up and down strategy helped them to break problems into chunks and build on those chunks. One teacher wrote that she would use reasoning up and down to help her students focus on what they already know and then guide them in building on that knowledge. Several teachers remarked on the importance of labeling processes so that students have a clear picture of how the concepts tie together; this leads to the development of conceptual understanding and the internalization of concepts and processes for the students. The teachers recognized the crucial importance of thinking about the question before crunching numbers. Additionally, as can be seen in the posters, the teachers gained an appreciation for the validity of multiple approaches to problem solution (see Figure 1).

One of the teacher's reflection commented on how the poster proofs allowed for colleagues to share different models of proportional reasoning. *"Even though people have different approaches on problem solving. Not one person thinks alike. The robot/hrs/cars problem had multiple ways to get the answer. Some were very basic and others more complex."*

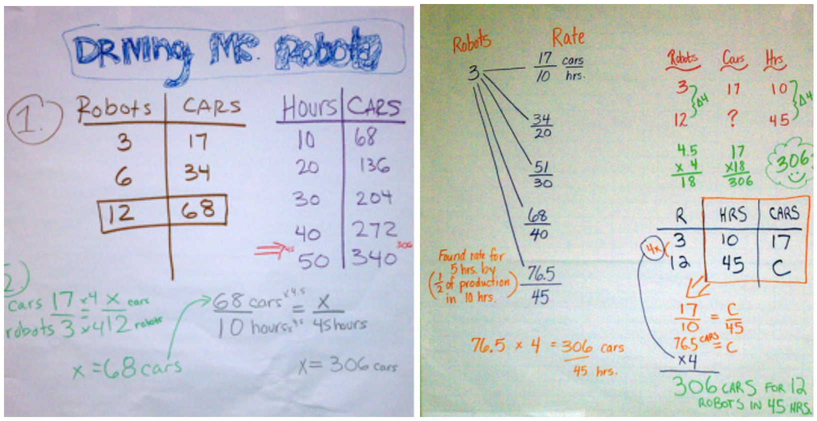

Figure 1: Using a ratio table and counting vs. finding multipliers

Mistakes and confusion allowed the teachers to use mathematical reasoning and arguments to do side-by-side comparisons of solutions, or just talk through comparisons of solutions to find where they did not match up. Then, the teachers would strategize to determine not only how to proceed but also to determine why one method did not work. For example, "1 robot can make 1 car in 1 hour" does not mean "2 robots can make 2 cars in 2 hours." Teachers discussed why a simple "multiply through" technique did not work. Teachers benefited from these discussions in several distinct ways. First, they began to see that real problems involving rational numbers are not simply plug-and-play exercises; they are multi-layered challenges, which require analysis, sound reasoning, and understanding of the relationships among quantities. Second, they recognized the profound importance of conceptual understanding as a baseline for strategizing approaches to problem solving. And, third, they gained an acute appreciation for the frustration of their students who apply incorrect procedures and cannot understand why their answers are incorrect. Several teachers mirrored that idea in their writings. Lastly, another teacher reflected, *"I am also starting to think differently about analyzing student work. When problems have the opportunity of yielding a variety of correct answers, it is important to consider what the student is doing and what math they can do and understand."*

Critical Incident 3: Using Lesson Study to Observe How Students Modeled a Problem

One critical class episode during the summer institute surrounded a problem called the *Mango Problem*. The problem is as follows: *One night, the King went down into the Royal kitchen, where he found a bowl full of mangoes. Being hungry, he took 1/6 of the mangoes. Later that same night, the queen was hungry, found the mangoes and took 1/5 of what the King had left. Still later, the first Prince awoke, went to the kitchen, and ate ¼ of the remaining mangoes. Even later, his sister, the Princess, ate 1/3 of what was then left. Finally, the youngest Prince woke up hungry and ate ½ of what was left, leaving only 4 mangoes for the kitchen staff. How many mangoes were originally in the bowl?* Teachers initially had difficulty approaching this problem because they were fixated on the whole being one mango or figuring out a formula. The researcher noted how the instructor reminded the teachers to "letting go" of rules and figure out ways to approach problems through modeling without getting fixated on the numbers. Video analysis revealed a group of teachers, Sunny, Jane and Al act out their solution. As they acted out the scenario, they asked questions like, "Is a mango the whole or are 4 mangoes the whole?" "What role are the fractions playing?" They started to wrestle with the idea of their previous math task called the Candy Bar and Circle Problem, which focused on the varying definitions of the "whole" and they had to negotiate and determine different meaning of fractions of that whole. Some teachers were observed having obstacles because they started with one mango as the whole; but, halfway through started to think as 4 mangos as the whole. This indicated a misconception that the teachers seemed to have about part-whole vs part-part interpretation. In addition, we observed teachers solving problems by working backwards using the manipulatives. The idea of unitizing that involves mentally constructing quantities in different chunks appeared to be somewhat problematic even for teachers. Although this group seems to be quick to catch on, they seem to be having problems truly grasping the concepts and applications of unitizing.

Because we noted this episode to be a critical incident, we were interested to see how this group of teachers who planned a Lesson Study (Lewis, 2002) with the *Mango Problem* would elicit models from their students. For the *Lesson Study Reflections,* we asked teachers to reflect on the process of developing and refining a research lesson, creating assessments items, and analyzing students' learning. The formal reflection assignment included teachers' evaluation of instructional strategies that promoted rational numbers and proportional reasoning through modeling, teachers' analysis of student thinking and what was learned from the process of collaboratively planning, teaching, observing and debriefing with colleagues. One of the Lesson Study teachers taught the mango problem to her 5th grade students and commented on the multiple models and representations that were used in her class.

> *Students approached the task in numerous ways. Some students tried to employ algorithmic approaches base on their current knowledge. This strategy often highlighted misconceptions they were having in regard to the relationships of fractions. Students would add all the numerators and then add on the number of mangos that remained. Others drew pictures or a model of 6/6 and took 1/6 away, but got stuck with where to go next. Others used the unifix cubes and represent this model the same way and were not sure how to proceed either. Still others quickly drew a model of 6/6 and identified the last box as having three mangos in it. They saw at that point that because fractional parts are of equal size all the boxes would have three mangoes in them. From there, they eliminated 1/6, 1/5,1/4, etc. recognizing that each time they took away one-sixth their whole changed. Drawing seemed to be the strategy that worked the best.*

Her reflection continued with an analysis of her students' work and how she asked her students to use their models of understanding the problem to justify their answers. This teacher reflected upon this Lesson Study and reported that the planning of the *Mango Lesson* helped bring deeper understanding of the importance of unitizing or the changing of the unit as one proceeds through a task. In addition, it solidified the meaning of fractional parts being of equal size. The planning session, also, brought to the forefront for her the multiple approaches that could be utilized by students to solve the task. Developing pictorial representations and then discussing the processing behind each solution with a collegial group allowed her to see thinking that was different from hers and yet valid. They looked at the process of working backwards and the relationship of parts to the whole. Collaboratively discussing misconceptions with her lesson study group also aided her in developing open-ended guiding questions to assist students in navigating through the task if and when they get stuck while modeling the task.

Discussion and Conclusion

Our study operationalized the notion of teachers' strategic competence using the NRC's (2001) description "as the ability to formulate, represent, and solve mathematical problems" and AMTE's standard (2010), "as the ability to construct and evaluate multiple representations of mathematical ideas or processes, establish correspondences between representations, and understand the purpose and value of doing so; and use various instructional tools, models, technology, judiciously, in ways that are mathematically and pedagogically grounded".

In our analysis we observed that teachers needed multiple opportunities to construct and evaluate multiple representations of mathematics ideas. In the critical incidence described above, teachers recognized that certain models afforded different opportunities for mathematizing. For example, the ratio table allowed teachers to bring out the ideas of reasoning up and down and highlight the multiplicative structures in proportional reasoning. In addition, the notion of "establishing correspondences between representations" came up a lot as an important theme when making connections between tabular, numeric and graphical approaches to representing a problem. *"Because I am so comfortable with mental math and using numbers, I find it arduous to think in terms of manipulatives and pictures. However, I can see the* *value of hands-on manipulatives for my math students. Today I used a ratio table and Kathy showed me how to "pull apart" a ratio so that I could manipulate it more easily."*

During the lesson study, the planning and debriefing phases revealed teachers pedagogical dilemmas with the "use various instructional tools, models, technology, judiciously, in ways that are mathematically and pedagogically grounded." For example, teachers who presented the mango problem wrestled with the pedagogical dilemmas of determining which manipulatives should be available for students and what model of fractions would be important in the lesson.

Most importantly, we gathered from their multiple reflective entries, teachers' sense of "understanding the purpose and value of doing so (representing and connecting representations). Teachers reflected on how the opportunity to struggle with problems in order to develop deep understanding of rational numbers. While many teachers expressed frustration with the homework problems as well as the in-class problems, they also recognized that their frustration led them to think about rational numbers in ways which they had not employed previously. This led to deeper understanding. Several teachers reported that they now "get" rational numbers and are gaining appreciation for the connections between concepts; they attribute this to the experiences of struggling through the investigative problems without the crutch of plug-and-play procedures. Teachers questioned each other's thinking and would not allow unsubstantiated assumptions. The focus was on mathematical reasoning, not the answer. We repeatedly heard teachers asking each other, "please explain that again, I don't understand where you are going with this" or "why would that be reasonable way to solve this?" Knowing that numerous approaches to problem solution were both possible and valid freed the teachers to concentrate on the soundness of their approaches, resulting in the teachers being able to develop more profound understanding. Participants valued the learning process and the opportunity to collaborate with other mathematics educators in translating their learning into practice. This study contributes to the growing body of knowledge on documenting how professional development serves as a catalyst for change in teachers as they reflect on developing their strategic competence for teaching and modeling rational numbers concepts in elementary and middle grades.

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TEACHERS' ATTRIBUTIONS FOR STUDENTS' MATHEMATICAL WORK

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This paper reports on a design experiment within a professional development context purposefully planned to teach teachers about students' mathematics thinking and learning. We examine the factors to which participating elementary teachers attributed student mathematics success or failure when engaging with the projects' professional learning tasks.

Statement of the Problem

In his attempt to explain how people think, Schoenfeld (2011) put forth the following claim: "People's decision making in well practiced, knowledge-intensive domains can be fully characterized as a function of their orientations, resources, and goals" (p. 182). Defining orientations as including a myriad of concepts such as dispositions, beliefs, values, tastes, and preferences, Schoenfeld explained that orientations shape what we perceive, the meaning we make of what we see as relevant, the goals we establish in a particular situation, and the resources we bring to bear to achieve those goals. Further, he claimed that in mathematics classrooms, teachers' actions were shaped by their orientations toward mathematics, students, learning and teaching.

Using the broad definition of orientation that Schoenfeld put forth, our study examines elementary teachers' orientations toward students' mathematics. More specifically, we attend to teachers' attribution as one aspect of orientation and examine the following question: *to what factors do elementary teachers' attribute students' mathematical work when working on professional learning tasks designed to teach them about students' mathematics thinking and learning?* The project consisted of a design experiment within a professional development setting purposefully planned to teach teachers' about students' mathematical thinking and learning. This design experiment allowed for the study of changes over time in teachers' attributions for students' mathematical successes and failures. The initial conjecture under investigation stated that learning about students' mathematical thinking would add a new attribution to teachers' repertoire, thus changing the array of attributions available for teachers to use as they examined student work within the professional learning tasks used in the professional development. As a first step in the investigation of this conjecture, the various factors teachers used in the professional development to attribute students' successes or failures were documented.

We begin this paper by briefly reviewing the literature that defines the theoretical framework of our study. Then, we present our research methodology, describing the professional development setting in which we work. Next, we define the attribution factors we observed in our professional development and share examples of how these attributions were present in our work with elementary teachers. We conclude with a set of next steps for our research.

Framework

Thompson, Phillip, Thompson, and Boyd (1994) first used the concept of orientation to describe what they called a calculational and a conceptual approach to teaching mathematics. The authors incorporated teachers' knowledge, beliefs and values within the concept of orientation and, much like Schoenfeld (2011), proposed that these orientations shape teachers' images, views, intentions, and goals for mathematics instruction. Magnusson, Krajcik, and Borko (1999) included orientation as a component of teachers' pedagogical content knowledge. They considered that teachers' orientation influenced instructional practice by shaping teachers' knowledge and beliefs about curriculum, students, teaching, and assessment. Phillip (2007) noted teachers' orientations were operationalized through attention to teachers' language and actions.

When analysing teachers' language in a professional development organized around student work, Kazemi and Franke (2004) reported a shift in the ways teachers attended to the details of student mathematical thinking. They explained that in the initial meetings of the professional development group, teachers focused their analysis of student work on students' mistakes, could not provide detailed explanations on how students completed the problem posed to them, and were surprised that the problem posed was difficult for the students. However overtime, teachers conversations became more detailed regarding the work their students were doing and teachers able to note various levels of sophistication in students' mathematics reasoning.

Sowder (2007) cited various examples of professional development projects in mathematics that used student thinking to promote teacher learning and noted that these projects often provided teachers with opportunities to examine student work. As indicated in Little (1999), the sustained and systematic study of student work provides one of the most powerful and least expensive opportunities for teacher learning.

When working with teachers in professional development that offered opportunities to examine student work, we observed that an important aspect of teachers' orientation toward students was the ways in which they talked about students' successes or failures in completing the mathematics tasks under examination. The attribution aspect of teachers' discourse turned our attention to attribution theories as one facet of teachers' language when examining student work.

Bar-Tal (1978) defined attributions as the inferences made about the causes of one's own or someone else's behaviours. Weiner (1985) noted that attributions were classified in relation to its locus of causality (internal or external) as well as stability (fixed or not) and controllability (who can change it). Classification of attribution along these dimensions usually leads to the examination of ability, effort, luck or the difficulty of the task as the causes for one's successes or failure.

Middleton (1999) noted that teachers' attributions of their students' successes and failures were reflected in the ways teachers interacted with their students during mathematics instruction. Examining pre-school settings, Dobbs and Arnold (2009) claimed that teacher's attributions of the students' behavior shaped the teacher's behavior toward the child, which in turn often elicited the expected behavior from the child, having a self-fulfilling prophecy effect.

Because our work is in professional development settings, we extend the discussion of the role of teachers' orientations and attributions in instruction to professional development settings. We consider that teachers' orientations toward students' mathematics play a fundamental role in teachers' engagement with professional learning tasks, with teachers' attributions of students' successes and failures shaping professional conversations around student work used in these learning tasks.

Methods

Professional Development

Our work with teachers is based on the concept of learning trajectories (LTs). When Simon (1995) coined the expression "hypothetical learning trajectory," he indicated that teachers create representations of the "paths by which learning might proceed" (p. 135) when students progress from their own starting points toward an intended learning goal. He named these trajectories hypothetical because each student individual learning path was not knowable in advance. However, he suggested that these learning paths represented expected tendencies and that commonalities across students allowed teachers to develop expectations about how learning might proceed.

Over time, the concept of LTs has developed to go beyond the notion that teachers have expectations about how learning might proceed to include an empirical search for the highly probable sets of levels through which students progress as their learning of specific mathematics topics evolve. Thus, current work on LTs uses research on student learning from clinical interviews and large-scale assessment trials to seek clarification of the intermediate steps students take as learning proceeds from informal conjectures into sophisticated mathematics.

Recently, research on LTs has progressed from an agenda for studying student learning to an agenda for research on teaching. Daro, Mosher, and Corcoran (2011) called for the translation of LTs into "usable tools for teachers" (p. 57). They indicated the need to make these trajectories available to teachers so that they can guide classroom instruction.

Research Design

The overarching purpose of our research is to understand the ways in which teachers come to learn about one particular LT as a representation of students' mathematics in the context of a professional development setting. Inasmuch, we examine both teacher learning and the set of professional learning tasks that support their learning experiences. As we teach teachers' about students' mathematics through the concept of LTs, teachers' orientations toward students shape the ways in which teachers engage with the professional learning tasks proposed to them, with teachers' attributions playing an important role in their discourse.

We use a design experiment methodology within a school-based professional development setting to accomplish our research goals. Design experiments are "iterative, situated, and theory-based attempts simultaneously to understand and improve education processes" (diSessa & Cobb, 2004, p. 80). They are used to develop "a class of theories about both the process of learning and the means that are designed to support that learning" and they "entail both 'engineering' particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them" (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003, p. 9).

In order to analyze the data, we engaged in a grounded theory approach to data analysis (Strauss & Corbin, 1989). In doing so, we coded our data (field notes and group discussion transcripts) using open coding, which enabled us to create concepts from raw data. In addition to creating data-driven codes, we also used theory and research goals to help create several of the codes. Once all of the codes were created, we then engaged in axial coding in order to make connections between the initial codes. This allowed us to create larger categories or themes (see results section for categories). In line with the grounded theory approach to data analysis, we used the constant comparison method in that we were comparing various project data sources including field notes and group discussion transcripts as well as the research literature (Glaser & Strauss, 1967). The constant comparison method allows for the creation of emerging categories in the data analysis and the refinement of these categories as they are contrasted with new project data. Various sources of data are used for the ongoing analysis and for triangulating information (Miles & Huberman, 1994) in search of both confirming and disconfirming evidences.

Context and Participants

The professional development comprised of both a summer institute and academic-year monthly meetings. These two components of the intervention were designed with different goals in mind. The summer institute offered teachers opportunities to learn about the LT and develop an appreciation for the role of the trajectory in understanding student mathematics. In contrast, the academic-year monthly meetings focused on establishing connections between the trajectory and instructional practices. The two components of the professional development totalled 60 hours of face-to-face, whole group interactions over one school year.

The professional development was offered in partnership with one elementary school in a mid-size urban area in the southeast of the United States. The school had approximately 600 students, 35% Caucasian, 29% Hispanic, 25% African American, 7% Asian, and 4% other; 54% of the children qualified for free or reduced lunch. Teachers at the school volunteered to participate and all professional development meetings were conducted at the school, in times selected based on convenience to the teachers. Of the 24 teachers who started the professional development in July 2010, 21 completed the program one year later in June 2011. The initial group of teachers included six Kindergarten teachers, three

Grade 1, five Grade 2, three Grade 3, two Grade 4, and one Grade 5 teacher. Four teachers taught multiple grade levels.

Results

Teachers' attributions of students' mathematics emerged very early on in the professional development, calling our attention to its importance for our work. Here, we offer one example from a professional learning task posed to teachers in the beginning of the professional development to demonstrate how teachers' attributions shaped discourse in the professional development setting.

The task we are using as an example engaged teachers in watching videos of clinical interviews with students from different grade levels solving similar mathematical problems. Teachers were asked to describe the ways in which each child solved the problem, conjecture about each student reasoning for that particular solution, consider the sophistication of the various strategies, and examine what surprised them about each student work. In the discussion that followed, despite the facilitator's effort to focus the discussion on what each child did and why, teachers' discourse focused mostly on whether what each child did aligned (or not) with what teachers' thought a student at that grade level was expected to do. That is, they attributed what the children did to grade level. The information about each child's grade level, offered to teachers as part of the context for the clinical interviews, became the center of the discussion as if grade level defined for the teachers what a child could or could not do mathematically. Thus, in the case of this particular task, teachers' attributions for students' work shaped the discussion around the professional learning task.

Through the examination of teachers' discourse when asked to engage with a collection of professional learning tasks, we documented the various attributions teachers' brought forth. In what follows, we present each attribution, a short working definition for it, and two or three quotes that exemplify how the attribution was represented in our data.

1. *Ability*: Considers personal traits of students and characteristics that define the student as fixed qualities related to students' aptitude in mathematics. Often times, teachers use achievement to consider students' abilities, attributing students' performance to an innate capacity.

   "We had evaluated this student and we were convinced there was a learning disability. The work was really low. But we were working on tangrams and this student put the 7 shapes into a square; he did immediately, first one to have done it and did it quickly."

   "I had a lot of math genius and they can figure things out when they are so young."

2. *Effort*: Refers to the level of student attention and engagement with a particular task at a particular moment. It indicates that performance does not always represent a fixed characteristic of the student, but depends on how carefully or how speedy that particular student progressed through the work at a particular moment.

   "Well, he just zipped through all this, so, no wonder…"

   "He worked on this so carefully."

   "In my mind, this kid just wasn't paying attention to me while I was teaching and he played connect the dots."

3. *Luck*: Includes the idea that what students do has no intentionality behind it. Also implies that students do things that have no real explanation for what or why they did something, or knew what they were doing.

   "I thought she was just guessing and she was just lucky."

   "When questioned how did you know, that is when I realized she really randomly chose to give each one two pieces. It was not that she had the number fact or she understood."

4. *Difficulty of task*: Expresses the notion that what students do is determined by the clarity or lack of clarity of the question posed to them. Has an embedded idea that there is a perfect way to ask a question so that students would not make a mistake.

   "The proctor asked her to put things together and then divide them, so, she shared differently because the proctor asked a different question."

   "When we teach a group of students and over half of them make the same mistake, then we have to go back and look at the way we presented it and ask ourselves…is it some fault in the way the question was presented?"

5. *Grade level*: Includes the notion of development and the expectations teachers have for students' performance given normalized definitions for what the generic student should be able to do at certain point in his or her development. Indicates that grade level groups students at similar developmental levels.

   "I taught Kindergarten and I would have guessed she would share using one for you, one for you, one for you; what she did was more advanced because she counted two plus two plus two."

   "I had expected the third grader to not share dealing it one by one."

6. *Cultural context*: Indicates that teachers take into account the experiences students bring with them from their own lives. Includes outside school understandings and explanations that students generalize to the academic context.

   "She just shared and she thought "it is fair because we each got some", and that is because of how we use the word share in the real world. She thought we both have some so we have shared."

   "I think that was a problem for a lot of these kids, dishing out the whole birthday cake (to fair share it). I just wonder if you called it something else besides a birthday cake if they would have seen the whole differently."

7. *Teaching*: Implies that what students do depends on what teachers have presented to them. Depending on whether or not a teacher has already taught a particular topic to the students, teachers expect students to know a topic taught. On the other hand, it indicates that teachers consider that students have no way of knowing a topic not yet taught.

   "Sometimes students can say something even when we had not taught it, like, this is ½ of 10 so that part has to be 5 as well. It seems simplistic, but I don't know how they would have known that already."

   "It used to be that students would do what teachers taught and we would follow it. But now students generate their own ideas and can do it in a way that is different from my own. They know how to come with the right answer by themselves."

Next Steps

In this paper, we documented seven different factors brought forth in the context of our professional development as teachers attributed students' mathematics successes or failures when examining student work. These attributions go beyond ability, effort, luck and difficulty of tasks to also include grade level, cultural context and teaching. They represent teachers' orientation toward students, and indicate the knowledge, dispositions, beliefs, and values teachers activated to examine student work in the context of our professional learning task.

In continuing our research, our conjecture is that the array of attributions available for teachers examining student work will change as teachers learn about student mathematics represented by LTs. Thus, we will examine whether our professional development on LTs added a new attribution to teachers' repertoire, one that includes recognition of students' mathematics successes and failures in relation to the level represented in LTs. This attribution recognizes that students' mathematics requires interactions between internal and external factors such as previous knowledge and opportunities to learn. Further, this attribution is not fixed and both students and teachers are responsible for changing it. We also conjecture that as the professional development unfolds and teachers come to better understand LTs, they will use the learning trajectory attribution more often. Examining these conjectures are the next step in the development of our work.

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TEACHERS' NOTICING OF CHILDREN'S UNDERSTANDING OF LINEAR MEASUREMENT

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This article is a report on the findings of a case study that focuses on a first grade teacher's noticing of children's understanding of linear measurement along a learning trajectory, extending Jacobs and her colleagues' framework (Jacobs, Lamb, & Philipp, 2010). It documents what the teacher noticed in terms of attending to and interpreting student strategies in four different contexts during her participation in a lesson study. The findings indicate that the teacher was overall more successful in attending to student strategies than interpreting mathematical understanding reflected in the strategies when she used a learning trajectory as a tool to notice student understanding. More interestingly, we found that her level of noticing differed depending of the role that she took in the process of lesson study.

Keywords: Learning Trajectories (or Progressions); Teacher Education–Inservice/Professional Development; Teacher Knowledge)

Understanding children's mathematical thinking is one of the key factors for teachers to provide effective instruction. More specifically, teachers' knowledge about how children's thinking progresses over time and what conceptual mile stones indicate is critical to support children's mathematical learning. The National Research Council (NRC, 2001) asserted, "Familiarity with the trajectories along which fundamental mathematical ideas develop is crucial if a teacher is to promote students' movement along those trajectories" (p. 370). Many research studies (e.g., Cobb et al., 1991; Confrey, Mojica, & Wilson, 2009; Gearhart & Saxe, 2004; Schifter, 1998, 2001) investigated teachers' instruction that builds on children's mathematical thinking and its progression in the domain of numbers and operations, and some studies reported improvement in student learning by building on children's thinking (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fennema et al., 1996; Jacobs, Franke, Carpenter, Levi, & Battey, 2007; Villaseñor & Kepner, 1993).

Several groups of researchers (Barrett & Clements, 2003; Barrett et al., 2012; Sarama & Clements, 2009) have documented children's thinking and learning trajectories in the domain of measurement, more specifically linear measurement. A study by Barrett and his colleagues (Barrett, Jones, Thornton, & Dickson, 2003) discussed benefits of instruction when teachers design tasks and questions that recognize where students are in the learning trajectory and help move children to a more sophisticated level. However, more research studies are needed to document how teachers make sense of a learning trajectory and how it may impact their teaching. This initial study may help to fill this gap through a case study of a first grade teacher who participated in a professional development program that focused on knowledge of children's thinking about linear measurement through the use of a learning trajectory. The teacher was supported in her effort to put the knowledge into practice through a lesson study.

To capture how teachers use their knowledge of a learning trajectory of linear measurement in practice, we used *noticing* (Mason, 2002) as a main framework. Noticing allows us to highlight the nature of knowledge that teachers need to actively respond to complex and challenging environment in practice. Teachers may have knowledge on children's thinking, but if it is not active they may not notice it in practice, which in turn will result in difficulty in taking appropriate instructional actions to improve it. In mathematics education, Jacobs and her colleagues (Jacobs, Lamb, & Philipp, 2010) recently studied what teachers notice in terms of children's mathematical thinking in the domain of whole numbers and operations with a goal of unpacking teachers' in-the-moment decision making. They defined the professional noticing as a set of skills including how they attend to children's strategies, how they interpret mathematical understanding reflected in the strategies, and what decisions teachers make to respond to the understanding in the strategies. Jacobs et al. analyzed what teachers noticed in children's strategies presented in a video clip and also a collection of students' written work during their professional development activities. They then compared teacher noticing with different levels of teaching and professional development experiences. They concluded that teachers' noticing expertise grew with teaching and professional development experiences, which indicate that this expertise can be learned and supported through professional development. Although Jacobs' study provides a framework on how to analyze teachers' noticing of children's thinking, the question of how teachers' noticing in a professional development context is related to teaching context still remains unanswered. In this study, we aimed to extend Jacobs' study to an actual classroom, and to put a step closer to teachers' in-the-moment decision making.

This study examines teachers' noticing in the context of lesson study as a part of a professional development program. The context of lesson study allows teachers to develop *knowledge-in-practice*, which Cochran-Smith and Lytle (1999) described as the practical knowledge of teaching "embedded in practice and in teachers' reflections on practice" (p. 250). Sowder (2007) discussed lesson study as an example of learning-in-practice because "in lesson study teachers deliberate on the practices they observe with others." Fernandez and Yoshida (2004) described lesson study as a well-defined common practice in Japanese schools. This process involves three processes and three additional processes that some groups follow. The first three processes are for teachers and professionals in the community to collaboratively plan a study lesson, to observe the lesson study in action, and to discuss the lesson. Sowder (2007) pointed out that the first two steps are not new to U.S. teachers, although they rarely involve other teachers, but the last step is uncommon. Ball (2002) and Lewis (2000) discussed that these processes allow teachers to attend to and learn what each child understands, organize instructional tasks based on mathematics, and make adjustments as needed. Three additional processes that are optional are to revise the lesson, to teach the new version of the lesson, and to share reflections about the new version of the lesson. Our study situates teacher's noticing of children's understanding of linear measurement concepts along a learning trajectory in the context of all six processes of lesson study.

The purpose of this study is to contribute in making sense of teachers' learning of children's thinking in the domain of linear measurement. More specifically, we aim to examine one teacher's noticing through a case study in the context of a lesson study, which supported teachers' learning and use of a learning trajectory as tool to make sense of children's understanding. Our research question is:

- How do teachers use a learning trajectory as a tool to notice students' measurement understanding in the context of lesson study? In their noticing, how do teachers attend to and interpret students' strategies?

Methods

Participant

Here we report a case study that focuses on one teacher, Ms. Smith, from a larger study involving 24 teachers. At the time of the study, Ms. Smith was teaching first grade with 16 years of teaching experience. Ms. Smith taught at a K–4 elementary school. The school was classified as a Title 1 school, where 34% of the students were qualified for free or reduced lunch, and 59% were minority.

Professional Development

The aim of the larger study was to introduce teachers to a learning trajectory on length measurement and support their use of it in assessing students and designing instructional tasks. All of the participants were from an urban school district in the Midwest. The teachers participated in two summer professional development conferences for a total of ten days. During the first professional development, which lasted six days in June, the teachers were introduced to the Length Learning Trajectory developed by Sarama and Clements (2009). The teachers learned about each level of the trajectory and student understanding at each level. They designed assessment tasks using the trajectory and also tested their tasks with children from a local summer program. The second professional development, which lasted four days in August, introduced the concept of lesson study.

The teachers worked together in six groups of four to develop lesson plans. Ms. Smith's lesson study group designed a lesson to develop students' understanding of linear measure focusing on non-standard units. The teachers in Ms. Smith's lesson study group designed a lesson about a postman delivering mail. The students were asked to measure and compare routes on their classroom floor using cutouts of the postman's foot.

Prior to the first instruction of the lesson, each teacher was asked to interview six students of varying abilities from their classroom using length tasks he or she designed or tasks given during the summer professional development. Following the interviews, each group of teachers participated in the processes of teaching or observing the lesson, and discussing the lesson. Based on the discussion, the teachers revised the lesson and iterated the processes four times. In this process, each teacher was asked to re-interview his or her six students following the classroom lesson. The teachers were asked to write reflections for each iteration of the lesson study as well as reflections on pre- and post-student interviews. These reflections prompted the teachers to describe the tasks or lesson posed, discuss student responses and thinking in relation to the learning trajectory, and prescribe future instructional tasks for the students.

Data

In this study, we analyzed video and journal accounts of Ms. Smith's reflections for the second and third iterations of the lesson. Ms. Smith taught the second iteration of the lesson and observed the third iteration. We transcribed the videotapes of the second lesson that Ms. Smith taught and the post lesson discussions of the second and third lessons that she participated in. The journal accounts included her reflections of the lessons and discussions as well as the pre- and post-student interviews. This provided us with four main data sources: Ms. Smith's report of pre-lesson interviews with the six students, her reflection of her own teaching, post-lesson interview with the six students, and Ms. Smith's reflection of third iteration of the lesson taught by another teacher in her group.

Data Analyses

Each of the four main data sources was analyzed with attention to two of the professional-noticing skills from Jacobs, Lamb and Philipp (2010), including *attending* to student strategies and *interpreting* children's understanding. With regard to attending to student strategies, we used two codes of showing or not showing evidence when we analyzed her reflections on pre- and post-student interviews. If she was able to provide mathematically significant details on how a student measured or used tools to measure then it was coded as showing evidence of attending to student strategies. We used three codes, attending to individual student strategies, attending to group strategies, or not showing evidence when we analyzed her reflections on discussions or lessons during the lesson study. This two-tier coding scheme was used because in the interview context, Ms. Smith worked with the students one-on-one, and in the lesson study context she worked with a classroom of students. We decided to use the additional code for the data from the lesson study context to account for the difference in the nature of the contexts.

With regard to interpreting student strategies, we used three codes including robust evidence of interpretation, limited evidence of interpretation, or lack of evidence of interpretation. We used the same set of codes for both contexts. When Ms. Smith made specific comments about her interpretation of mathematics in students' strategies, it was coded as robust evidence. When Ms. Smith made general comments of mathematics in student strategies, it was coded as limited evidence. When Ms. Smith provided little to no comments of mathematics in student strategies, it was coded as lack of evidence. For instance, her comments focusing on other issues within her classroom such as her teaching style, improving teaching, or student behavior were coded as lack of evidence.

Results

In this section, we share Ms. Smith's noticing of children's understanding of linear measurement from the four different contexts. We describe our observation of her noticing with sample statements from her reflections.

Ms. Smith's Noticing in the Context of Pre-Lesson Student Interviews

In the pre-student interviews, Ms. Smith interviewed six students one-on-one and described their responses to each of the three tasks. After describing the student response to the task, Ms. Smith provided her interpretation of their responses.

Attending to student strategies. Ms. Smith's reflections showed evidence of attention to student strategies for each of the six students. When describing student strategies, she noted how individual students responded to the task with very detailed descriptions of the strategy. She typed up about one-page descriptions of each student. Consider Ms. Smith's following statements that showed evidence of attention to student strategies:

> With the *Length Comparer* activities, she lined up the first two objects and identified them correctly as a big one and a small one. Then she took the five objects and lined them up in correct order but there were not all starting at the same zero point.

In the statements, Ms. Smith captured mathematically important details. Specifically, she included descriptions about how the student compared the length of multiple objects and made a mathematically significant note that the student did not line them up with the same starting point.

Interpreting students' understanding. Ms. Smith exhibited limited evidence of interpretation of students' understanding. Her statements showed her intention of interpretation but they were rather broad and general. The following is an example of showing limited evidence of interpretation:

> In the Indirect Length Tasks, he was able to identify the shorter and taller of two fixed objects…he took the thread and measured the first cabinet and saved his place on the string. When he held it up to the longer cabinet, he said it was longer because it was longer than his arms… I would place student 6 in the Indirect Length activities.

Ms. Smith provided detail descriptions of what the student did to compare the height of two cabinets, but she concluded that the student's strategy would be at level 6 without providing evidence or justifying why she came to the conclusion.

Ms. Smith's Noticing in the Context of Teaching

Attending to children's strategies. When her group met after she taught the postman "Bob" lesson as the second of the four iterations of the lesson, Ms. Smith shared what children's strategies she noticed during the lesson. Unlike her detailed descriptions of individual student's strategies in the context of pre-student interview, Ms. Smith provided description of strategies that she noticed a group of students used:

> Most of them just slid the foot [paper cutout] along counting as they went. Some of them slid it longer than other ones. … Students seem to be at the beginning of the end-to-end trajectory. They were moving their foot [paper cutout] along the street [marked on the floor] and counting as they went. Some were actually putting a finger down to mark their place but most were just moving it in jerky, supposedly iterated movements.

Although she thought that children's strategy of sliding the foot cutout to measure lengths of delivery routes was invalid, Ms. Smith provided a detailed description of the strategy including, the motion that children took, length of the motion, and jerkiness of it. However, she did not discuss which students used the strategy, but rather said "most of them," referring to a large of group of students. We found that in her journal account Ms. Smith also reported her observation of the whole class, instead of individual students. We coded her noticing of students' strategies as attending to group strategies.

Interpreting children's understanding. Ms. Smith demonstrated lack of evidence in interpreting student understanding during the discussion following the teaching and in her written reflection. In both contexts, she focused on student behaviors unrelated to mathematical understanding or aspects of the lesson related to her teaching. The following are examples of Ms. Smith's responses to children's understanding:

> I was pleased with how the lesson flowed. The students were enthusiastic…. In retrospect, I guess I needed to model that a little more thoroughly…. I expected some of the students to use this as a time to play more than focus on the learning part of what they were doing and this is precisely what happened.

Ms. Smith's Noticing in the Context of Post-Lesson Student Interview

Attending to children's strategies. After teaching the lesson, Ms. Smith was also asked to re-interview the six students she initially interviewed and reflect on what they said about how they attempted the task and learned from the lesson. During the second interview, Ms. Smith showed evidence of attending to student strategies for only one student. For the other five Ms. Smith did not comment on how the student attempted the task. Ms. Smith did not reference her findings from the initial interview. Again she wrote about each student individually but this time she only wrote a few sentences and rarely referenced students' mathematical strategies. In this reflection she shifted from making specific comments about students' understanding to commenting about general behavior and teaching and learning. These are several of her comments from her post-interview with students.

> JH said it was fun. She said she had worked as a team with her friend who helped her measure the lines…TD did not iterate. He said he had compared it to driving and counted up that way as he moved his foot.

Interpreting children's understanding. In the post-lesson interviews, Ms. Smith demonstrated lack of evidence of interpreting student understanding. Miss Smith mainly focused on non-mathematical student behavior and she did not try to link student's individual behaviors to the levels in the trajectory following the lesson.

Ms. Smith's Noticing in the Context of Classroom Observer

Attending to children's strategies. As an observer Ms. Smith demonstrated evidence of attention to individual student strategies. Ms. Smith commented during the reflection that she was able to watch several students closely as she followed them around the classroom as they attempted to measure the length of several paths. In this instance, Miss Smith considered individual students within the group and the mathematical strategies that they used to measure a line.

> The team that I followed used their fingers to mark where they needed to move the foot forward from and count. One girl was more accurate with this than others… One of the boys didn't iterate, instead he just moved his foot along and counted… At one point they realized that it did not matter if they started at one end or the other when counting"

Interpreting children's understanding. Following the lesson, Ms. Smith demonstrated robust evidence of interpreting student strategies. Ms. Smith discussed with the group that that she had considered why the students were measuring in different ways and had formed a hypothesis based on student reasoning. She was able to link interpretations to specific student behaviors. In the post-discussion, she reflects on one student's struggles with measuring the path and she attributes this to his understanding of the number line.

> *Ms Smith*: I got the feeling that they were confusing how they were measuring with the foot. The student (that demonstrated) that came up with the incorrect answer was thinking of that first foot placement not as something he would count but he was using that as a starting off point and that is why their answer is less than the other. Instead of saying its one, two, three (moving her hand along a line). He started here, and you know how we teach the first step is one, two, three so he ended up

saying its three…it is kind of how you teach the number line counting to the kids at the start of the year. So, that's one observation I made on that initial thing. Trying to look at what the kids were thinking in their minds.

Discussion

In this case study of Ms. Smith, several themes emerge. First, there is evidence that when Mrs. Smith was introduced to a learning trajectory, it provided her with a language to describe student thinking. The findings indicates that Ms. Smith was able to use the learning trajectory to focus on student strategies, share knowledge about students with other teachers, and reflect on student strategies and responses. Ms. Smith was able to use appropriate mathematical language from the trajectory to communicate her understanding of students. Although Ms. Smith was not always proficient in using the trajectory, in several instances, she was able to correctly link student strategies with the appropriate level in the learning trajectory.

A second finding that emerged was that the lesson study provided a context that allowed the researchers to see differences in Ms. Smith's ability to notice student strategies. During the lesson study process Ms. Smith was able to take on several roles apart from her normal role as classroom teacher. Throughout the lesson study, Ms. Smith's noticing varied depending on the role that she took in the processes. Ms. Smith was more successful in attending to student noticing when she assumed the role of interviewer or observer. It may have been easier for her to observe and record student behavior because her focus was solely on one student at each interview. In the observer role during the third iteration of the lesson study, Ms. Smith focused on a small group of several students, instead of a whole class. Her attention to students' thinking may have been better because she was not responsible for student learning or classroom management. It seemed that this role of the observer allowed her to direct her focus to a few students for the entire class period and pay closer attention to their strategies and responses. When Ms. Smith taught the lesson, she did not attend to student strategies as well as in other contexts. This could be because the complexity of a classroom environment made it difficult for Ms. Smith to notice details of students' strategies or recall them in reflection.

Third, there seems to be connection between attending to student strategies and interpreting student understanding. When Ms. Smith provided more clear evidence of individual student strategies, she was more successful at interpreting student strategies. When Ms. Smith was able to attend to individual student strategies in the assumed role of interviewer or observer, she was able to interpret mathematics reflected in the strategies. We wonder if her close attention to individual strategies allowed an access to more concrete examples, which in turn helped her interpretations of student thinking. When Ms. Smith taught, she had difficulty attending to individual student strategies. In that context, she provided limited interpretation of student thinking and instead the focus was on her teaching or children's non-mathematical responses.

Lastly, we note the challenge of prompting teachers to use a learning trajectory as a longitudinal tool to assess children's progression over time. In the initial interview, Ms. Smith was able to use the trajectory to evaluate what level of the trajectory she thought students exemplified. However, we observed no evidence of her making connections of the information she gained from the pre-lesson student interviews to reflecting on the same students' thinking in a classroom lesson, and then to the post-lesson student interviews, although we had called on teachers to do so. It makes us wonder if she thought of the trajectory as an assessment tool prior to the lesson and not a tool to help promote student growth before, during, and following the lesson.

This case study of Ms. Smith provided us with a preliminary but very complex picture of what and how teachers notice children's thinking and how they use a trajectory to assess and make sense of student thinking. The results signify that the act of teacher noticing using a learning trajectory may become increasingly more complex when teachers move from observing and analyzing one or two students to working with an entire classroom. Further studies need to be conducted to analyze how classroom teachers develop in their ability to notice using a learning trajectory and how teachers connect knowledge of individual student strategies to classroom instruction. The findings and themes that emerged in this initial study gave us a glimpse of the multiple factors involved in improving teachers' noticing using a learning trajectory and provide a direction for future research.

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PROSPECTIVE TEACHERS' CONCEPTIONS OF PROOF

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What types of mathematical justifications do pre-service elementary teachers find convincing? To investigate this question, a task-based interview which was designed to elicit arguments of what students find convincing was administered to two female students who were enrolled in a geometry course at a large Midwestern university. These arguments were categorized according to the proof schemes crafted by analyzing different studies dealing with proof. A qualitative analysis of the data revealed that the two pre-service elementary teachers (PSTs) who were interviewed have difficulties in following or constructing formally presented deductive arguments and in understanding how deductive arguments differ from inductive arguments. They also held explicit misconceptions about proving (or disproving) statements such as: "a couple of examples constitute proof" or "one counterexample is not sufficient to disprove a statement."

Keywords: Inductive Argument; Deductive Argument; Proofs; Preservice Elementary Teachers

Introduction

It is difficult to overstate the importance of proofs in mathematics. If you have a conjecture, the only way that you can be completely sure that it is true is by presenting a valid mathematical proof. However, the mathematics education community worldwide is facing the challenge of improving students' abilities to prove and to reason mathematically at all grade levels. Despite the fundamental role that proof and refutation play in mathematical inquiry (Lakatos, 1976) and the growing appreciation of the importance of these concepts in students' mathematical education (Hanna, 2000; Reid, 2002), students hold various misconceptions not only about proof, but also about refutation (Chazan, 1993; Simon & Blume, 1996).

Several studies have reported that formal deduction among students who have studied secondary school geometry is nearly absent (Burger & Shaughnessy, 1986; Dreyfus, 1999; Chazan, 1993). Many students accept inductive arguments as valid mathematical proof (Martin & Harel, 1989; Chazan, 1993) or they fail to recognize that using a larger set of examples still does not constitute proof (Knuth, Choppin, & Bieda, 2009). In addition, students have difficulty in understanding that a valid proof confers the universal truth of a general statement; thus mathematical proof requires no further empirical verification (Fischbein, 1982; Martin & Harel, 1989; Chazan, 1993). Some students believe that counterexamples do not really refute; instead they tend to treat valid counterexamples to general statements as exceptions that do not really affect the truth of the statements (Balacheff, 1988). Similarly, Simon and Blume (1996) show that many students think that giving one example is not enough to refute an argument.

Despite students' current lack of knowledge, as well as interest, in proof and proving, the topic is central to mathematics, so it should be a key component of mathematics education (Bell, 1976; Hanna, 2000; Martin & Harel, 1989). Not only is proof at the heart of mathematical practice, it is an essential tool for promoting mathematical understanding (Martin & Harel, 1989; Hanna, 2000; Knuth, 2002). Stylianides (2007) has shown that young children can make legitimate mathematical arguments and even formal arguments that count as proof. He claims that proof should be part of students' mathematical experiences even in early elementary grades (Stylianides, 2007). Similarly, Harel and Sowder (1998) argue that instructional activities that educate students to reason mathematically about situations are crucial to students' mathematical development, and that these activities must begin at an early age. Thus, calls for improvement in mathematics education in the U.S. have increasingly emphasized the importance of proof and reasoning by recommending that reasoning and proof should be a part of the mathematics curriculum at all levels from pre-kindergarten through grade 12 (NCTM, 2000).

The purpose of this paper is to describe pre-service elementary teachers' attempts to construct proofs and also to examine different arguments regarding proofs in order to better understand their conceptions of proof. The following research questions guided the study:

- How do pre-service elementary teachers support claims, warrants, and backings as elements of their argumentations?
- What are pre-service elementary teachers' conceptions of proof?

The Framework

Various studies that include a description of proof schema ideas at both pre-college and college levels are evaluated with an effort to craft the framework used in this study (see Table 1). Hanna (1989) argues that proofs can have different degrees of validity and still gain the same degree of acceptance. To document different types of mathematical justifications attempted by each participant, mathematical arguments constructed to examine pre-service teachers' conception of proof are assessed according to a hierarchy of levels of mathematical justifications explained in the framework.

While many studies have focused primarily on distinctions between the inductive and deductive justifications (Chazan, 1993; Martin & Harel, 1989; Morris, 2002), some researchers have posed questions such as: What might make one example or empirical justification stronger than another? Or can all mathematical arguments be categorized as inductive or deductive? As a result, they have divided inductive and deductive justifications into further subcategories (Balacheff, 1988; Harel & Sowder, 2007; Simon & Blume, 1996; Quinn, 2009) and proposed another type of justification along with inductive and deductive justifications (Harel & Sowder, 1998; Simon, 1996). The framework crafted for this study focuses not only on the distinction between the inductive and deductive justifications, but also further subdivides those categories as well as includes the justifications that are neither inherently inductive nor deductive.

Many researchers define several stages or levels in which students' reasoning skills vary in terms of the justifications they are able to produce (Bell, 1976; Simon & Blume, 1996; Quinn, 2009). Harel and Sowder (1998, 2007) categorize those levels into three classes in their taxonomy—the external conviction proof scheme class, the empirical proof scheme class, and the deductive proof scheme class—with some subschema for each class. Similarly, Balacheff (1988) describes two main categories—pragmatic justifications and conceptual justifications—which play complementary roles. The first justification type (pragmatic justification) is divided into three subcategories: naïve empiricism (justification by a few random examples), crucial experiment (justification by carefully selected examples), and generic examples (justification by an example representing salient characteristics of a whole class of cases) and the second justification type (conceptual justification) into two subcategories as "thought experiment" and "symbolic calculation." Weber and Alcock (2004), on the other hand, focus only on deductive reasoning and divide deductive justifications as syntactic proof scheme (manipulating correctly stated definitions and facts in a logically valid way) and semantic proof scheme (use instantiations of the mathematical objects to which the statement applies to suggest and guide the formal inferences) which aligns with Hanna's (2000) distinction of proofs that prove and proofs that also explain.

The framework used in this study summarizes the proof schemes explained above, by merging those categories—from external to analytic—along with different levels in which provers demonstrate different level of mathematical justifications. The framework outlines various strong background work, thus, provides a powerful as well as useful tool for an analytical assessment of PSTs' conceptions.

Table 1: Types of Mathematical Justifications

Level 0		Responses that do not address justification.
EXTERNAL	**Level 1**	**Appeals to external authority** • ***Authoritarian proof*** : depends on an authority such as a teacher or a book • ***Ritual proof*** : depends on the appearance of the argument • ***Non-referential symbolic proof***: depends on some symbolic manipulation, often without reference to the symbols' meaning
Level 2		***Naïve reasoning, usually with incorrect conclusions.*** Although provers use some deduction, the arguments start with an analogy or with something that provers remember hearing, often incorrectly. Provers generally reach an incorrect conclusion or, if they reach a correct conclusion, they have used the wrong assumptions.
EMPIRICAL	***Inductive Frame***	**Level 3A*: Naïve Empiricism:*** an assertion is valid from a small number.
		Level 3B: ***Crucial Empiricism*** deals more explicitly with the question of generalization by examining a case that is not very particular. If the assertion holds in that case, it is validated.
	Rudimentary Transformational Frame	**Level 3C:** ***Perceptual Proof:*** Provers make inferences that are based on rudimentary mental images that are not fully supported by deduction.
Deductive frame expressed in terms of particular instances		
TRANSFORMATIONAL	**Level 4**	***Generic Example:*** Deductive justification that is expressed in terms of a particular instance (examples might be used to generalize the rules, but unlike an empirical proof scheme, the general rules are predicted based on the inference rules.) Simon (1996) defines transformational proof schemes an enactment of an operation (or set of operations) on an object (or set of objects) that allows one to envision the transformations that these objects undergo and the set of results of these operations.
Deductive frame that is independent of particular instances		
ANALYTIC	**Level 5**	• ***Syntactic:*** a verification of a statement is evaluated according to ritualistic features. • ***Semantic/Conceptual:*** a judgment is made according to causality and purpose of argument.

Method

Participants

Two pre-service elementary teachers, Sara and Dacey (pseudonyms), volunteered to participate in this study. Both of the participants had enrolled in a Geometry-content course designed for elementary majors at a large Midwestern university. Both participants satisfied the course requirements and passed the course with a grade of B or above.

Data Sources

The study uses a qualitative approach, mainly participant classroom observation and task-based interviews to investigate pre-service elementary teachers' conceptions of proof. Every class in which the participants were enrolled was audiotaped and notes were taken by the author of this study.

Each participant was interviewed individually for about an hour in a semi-structured manner using an interview script consisting of three phases. The interviews took place near the end of the semester, in each case. Thus, the interviewees were expected to have learned most of the course topics and have had some practice justifying different statements by the time the interviews took place.

Phase 1. During this phase, each student was presented the written tasks A, B, and C, described below. The PSTs were asked to explain in their own words what the statements said, and to decide whether the statements were always, sometimes, or never true and how they would know. And then, they were asked to produce a justification in cases where they believed the statements to be true.

Phase 2. After letting the participants try to justify the statements by themselves first, the participants were presented with four brief arguments for both Task A and B, varying in terms of level of justification, one after the other, and asked to think out loud as they read each one, to judge the correctness, and to say to what extent each argument was convincing.

Phase 3. Having seen and thought about all four arguments, one after the other, the students were provided "Always," "Sometimes," "Never" cards and asked to assign the appropriate card to each argument presented. For instance, if the participants thought that the conclusion derived from one of the arguments would always hold true then they needed to put an "Always" card on the argument.

The data collected consisted of the audiotaped interviews, the interviewer's notes and the students' work on the "proof" sheets provided during the interview.

Interview Tasks

The interview tasks were designed to provide, first, an indication of pre-service elementary teachers' competence in constructing proofs, and then, an overview of their views as to what constituted a proof. The interview tasks included three types of items (from familiar to unfamiliar) to probe pre-service teachers' views of proof from a variety of standpoints.

Task A. This task was adopted from the course textbook. Thus, it was expected that by this time of the year, the interviewees were familiar with it and could reproduce the proof on their own. The task appears on the sheet presented to the participants as follows:

> A kite is a quadrilateral with two distinct pairs of adjacent sides that are equal in length. Given the definition, justify whether or not the following statement is true. "*In a kite, one pair of opposite angles is congruent*."

Task B. The same structure as in Task A was used to construct Task B. This task was adopted from Chazan (1993), but it was modified such that four arguments, varying in terms of level of justification, were added to present to the participants. The task appears as follows:

> Justify whether or not the statement is true: "*In any triangle, a segment joining the midpoints of any two sides will be parallel to the third side*."

Task C. Task C was a non-familiar case to the participants. Thus, it was expected that this task might be challenging for them. This task was adopted from Simon and Blume (1996). The task appears as follows: Find the area of the shape below.

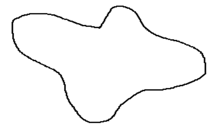

Figure 1

Arguments for interview tasks. The theoretical framework explained before (see table 1) governed the choice of arguments included in both of task A and B. Arguments for task A and B were characterized as empirical, subdivided as Naïve Empiricist with a small number of cases (Arguments 1 labeled as Level 3A) and Crucial Empiricism with an extended number of cases including non-particular cases (Arguments 2 labeled as Level 3B), argument requiring concrete demonstration or explanation written in everyday style (Arguments 3 labeled as Level 4), and a deductive proof, written in a formal style (two-column) (Arguments 4 labeled as Level 5). For task C, the following method was presented to the participants:

> "*If you take a piece of string and measure the whole outside of the area and then pull that into a shape like a rectangle, you can easily calculate the area of the figure*." Justify whether or not the above method will work to find the area of the figure.

Results

Sara's Proof Scheme

When Sara was presented task A, she attempted to dissect the kite into two triangles in order to use triangle congruency to justify the statement. However, because she drew a diagonal that produced two isosceles triangles instead of a diagonal that produced two congruent triangles, she failed to proceed from there to justify the statement. Even though, she started to use some deduction such as congruent triangles that she remembered hearing from her class, she failed to reproduce it correctly. Similarly, she attempted to use what she learned about parallel lines in her class to justify the statement in task B. However, she failed to proceed from using her previous knowledge to construct a justification. Thus, her proof scheme was coded as Level 2 (Naïve Reasoning) for both task A and B.

Even though Sara failed to construct a proof, she correctly distinguished the deductive arguments from the inductive arguments when she was presented the arguments for both task A and B. Sara understands that a couple of examples do not qualify as proof. She was aware that the conclusion that was arrived at from direct measurements of specific cases was approximate and that the generalization which was arrived at without examining every possible case might be highly probable but not certain. Additionally, Sara understands the role of justification in mathematics: that is, to provide an argument that holds for every case. She knows that providing examples will hold only for those specific examples and she chooses "Sometimes" for argument 1 and 2 and "Always" for argument 3 and 4. Thus, her proof scheme was coded as Level 5 in phase 2 and 3 for both tasks.

Dacey's Proof Scheme

Dacey, on the other hand, did not attempt to reproduce the proof she learned in her class for task A. Instead, she tried to justify the statement by saying that if two sides are equal, then the angles between them are going to be equal since where those sides will meet will be the same. Dacey did not attempt to provide examples nor attempted to use logical deduction to justify the statement. Rather, her attempt to prove the statement in Task A was driven by her perceptual observation of the figure provided to her. Thus, her proof scheme for this task was coded as Level 3C based on the framework. When Dacey was presented task B and asked to decide whether or not the statement was correct, she quickly concluded that the statement was correct, because, as she explained, the instructor recently showed the same statement and justified why it was correct in the class. However, because Dacey did not understand why the statement was correct or how to justify that it was correct in class, she failed to reproduce the proof. Dacey remembered that the proof included corresponding angles, but she could not proceed from using corresponding angles to conclude that the statement is true. Her response for this task was coded as Level 2.

Dacey found arguments 3, 2 and 1 more convincing than argument 4. She claimed that seeing actual measurements or illustrations was more convincing than providing a logical argument. She insistently claimed that arguments 4 were not convincing for her at all.

Task C was an unfamiliar case; none of the participants had experienced this type of task in their classrooms. Thus, both participants struggled with the task and neither of them could come up with a method to find the area of the figure presented. After they presented the method, their answers also differed. Even though Sara confirmed the method would work, after more thought she realized that there might be two rectangles with the same perimeter and different areas or vice versa. However, she also concluded that being able to refute argument (the method in this case) requires more than one counter example. Dacey, on the other hand, was certain that the method would work and she justified her conclusion by stating that if the outside of two shapes are equal so must the inside.

Conclusion and Discussion

The findings outline a mixed picture of what constitutes proof in the eyes of those two pre-service elementary teachers. When asked to define proof, it was clear that pre-service teachers had some experience of proof and were using this to inform their judgments about what constituted a good proof. They had experience of seeing a proof being performed and were quoting these as examples of what was required. However, despite their experience of seeing proofs in their classrooms, both participants failed to produce a proof for task A and B. This result aligns with Senk's (1989) argument that students need to be at higher levels in order to perform a proof than to be able to follow a proof. In addition, Healy and Hoyles (2000) provide evidence that students are better at choosing correct mathematical proofs than at constructing them.

In this study, even though Sara failed to apply her understanding of logical necessity to construct a proof, she was aware of the fact that inductive conclusions as in arguments 1 and 2 provide probable conclusions while, in deductive inference, the prover reaches a conclusion that is certain. Additionally, Sara exhibited different levels when she was asked to prove the statement by herself than when she was asked to evaluate different arguments constructed by others. Even though she failed to prove the statements, she recognized and selected the deductive arguments correctly.

Dacey, on the other hand, relied on examples as her primary means of justification for task A and B. She consistently justified the generalizations by stating that it worked for all the cases tested. She did not realize the limitations of such reasoning. Stylianides (2007) argues that considering empirical arguments as proof is a threat to students' opportunities to learn how to prove a proposition. Thus, one can argue that Dacey might lead students to believe that two examples would qualify as proof in her future classroom. Balacheff (1988) distinguishes between two large categories of proofs that students produced—pragmatic proofs and conceptual proofs. Pragmatic proofs are those having recourse to actual action or showings, and by contrast, conceptual proofs are those which do not involve action and rest on formulations of the properties in question and relations between them (p. 217). As in Balacheff' s definition, Dacey stated that actual action or showing was more convincing than the one which did not involve action.

This study reveals that students' levels of mathematical justification are not static. Rather, students might demonstrate different levels on different tasks depending upon their familiarity with the tasks. Fischbein (1982) argues that students choose to believe in something that seems more natural to them, subjectively, intuitively as an intrinsic property of the object. Nothing in the direct experience of the student needs such an explanation and leads to intuition. It was clear that Dacey was intuitively convinced that the outside of the shape also determines the inside in Task C. Thus she believed that if the perimeters of two shapes are equal, then the areas should be equal as well, so the method should work.

Implications and Suggestions

This study reveals that the participant PSTs do not always see the need of justifying a statement if the statement is intuitively appealing to them. In addition, they referred to the authority (the math teacher in this case) a couple of times to support their answers. Thus, it would be necessary to develop the shift of authority from teacher and textbook to the whole class. A classroom environment where mathematical ideas are not only constructed individually but also socially as students participate in meaningful activities (Cobb, Wood, Yackel, & McNeal, 1992; Yackel & Cobb, 1996) has the potential of generating mathematical justifications among prospective teachers.

I believe that a balance between visual reasoning and deductive reasoning seems to be a direction to pursue—discussing with students the role that examples play in proving statements in mathematics (Knuth, Choppin, & Bieda, 2009) while also creating learning opportunities for students to encounter both inductive and deductive proofs, so that students may develop not only a deeper understanding of proof but also a deeper understanding of the underlying reasons for using deductive proofs (Knuth, 2002).

If the goal is to help students develop a strong understanding of proof—especially in a deductive manner—teachers should assess students' current knowledge (common difficulties or misconceptions) in order to help them gradually refine their knowledge (Harel & Sowder, 2007). The framework used in this study may be a useful tool for teachers not only for assessing students' development in order to seek ways in which to help students gradually refine their perceptions but also for examining their perceptions of the nature of proof.

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"AFTER ALL, MATH WAS ONLY NUMBERS, RIGHT?" TRANSITIONS IN TEACHERS' BELIEFS ABOUT EQUITY

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In this paper I examine the beliefs about equity of a cohort of 14 upper elementary teachers participating in their first year of a 5-year project aimed at developing elementary mathematics teacher leaders. Teachers' responses to a beliefs survey and their written reflections indicate a greater recognition of the role of equity in the teaching and learning of mathematics.

Keywords: Equity; Teacher Beliefs

There is a growing body of work focused on explicitly incorporating issues of equity, social justice, and diversity into mathematics teaching. One body of work focuses on using mathematics as a tool for analyzing inequity in our society (M. V. Gutiérrez, 2009). Related work focuses on issues of power surrounding the role of mathematics in our society (R. Gutiérrez, 2007). A second body of work emphasizes the role of culture and out-of-school experiences in the learning of mathematics, such as building on students' home and community Funds of Knowledge (González, Andrade, Civil, & Moll, 2001) or teaching mathematics in culturally responsive ways (Leonard & Guha, 2002). A third body of work focuses on mathematics as a fundamentally cultural activity emphasizing the accomplishments of groups around the world and throughout history (Zaslavsky, 2001) or on how socially significant constructs—such as race—are partially constructed in and through mathematics (Martin, 2009; Tate, 1994).

There is also a growing field of research focused on preparing prospective teachers to grapple with these issues in their teaching of mathematics (e.g., Rodriguez & Kitchen, 2005). While there are some exceptions (e.g., Wager, 2012), there has been minimal work in mathematics education examining how practicing teachers learn to incorporate the social and political nature of mathematics described above into their teaching practice. In this paper I examine transitions in a cohort of 14 elementary teachers' beliefs about the social and political dimensions of mathematics teaching and learning during one semester near the beginning of their participation in a 5-year grant focused on developing greater leadership in elementary mathematics education.

Theoretical Framework

I use the term "equity" to cover the broad range of perspectives focusing on the social and political nature of mathematics detailed above. I draw on the *What, How,* and *Who* (WHW) framework of mathematics (for more detail see Felton, 2010a, 2010b). The WHW considers:

- *What* messages are sent through the teaching of mathematics. Specifically,
  - mathematics is *co-constructed* with other important social constructs such as race, gender, culture, etc., and
  - mathematics can and should be a tool for *social analysis* by examining complex real world issues, and more specifically, social and political issues.
- *How* mathematics *concepts* and real world *contexts* are related in mathematics teaching. Specifically, we can:
  - use real world contexts as a tool for learning about mathematical concepts, or
  - use mathematics as a tool for learning about real world contexts.
- How people (the *Who*) relate to the mathematics we teach. Specifically,
  - mathematics can serve as a *mirror* reflecting back, or
  - as a *window* into broader perspectives (R. Gutiérrez, 2007).

Methods

Description of Program and Course

The Arizona Master Teachers of Mathematics (AZ-MTM) is an NSF Noyce grant to work with a single cohort of 14 upper elementary teachers over a 5-year period to develop leadership in elementary mathematics education. AZ-MTM focuses on developing teachers' knowledge of content, pedagogy, equity, and leadership. The grant has two primary strands: (1) each semester the teachers take a course at The University of Arizona taught by a mathematics educator focused on a particular content theme, such as Numbers and Operations, and (2) the teachers participate in professional development opportunities focused on teacher leadership.

This paper documents the teachers' participation in the mathematics education course taught by the author during the Fall 2011 semester. This was the first semester that the entire 14 teachers participated in the grant (8 of the teachers began the grant the previous semester). The course focused on (1) children's learning of whole numbers and early algebra, and (2) an introduction to issues of equity through readings and discussions (e.g., Felton, 2010b; M. V. Gutiérrez, 2009; Leonard & Guha, 2002; Martin, 2009; Tate, 1994; Zaslavsky, 2001).

Data Sources and Analysis

There are two data sources used in this paper. First, the teachers completed a beliefs survey designed around the themes of the WHW framework at the beginning and end of the semester. Second, I analyzed the teachers' responses to their final written reflection assignment in the course in which they were asked to (a) consider how their definition of mathematics has changed, (b) describe their views of equity and how they have evolved over the semester, (c) reflect on what they had learned during the semester, and (d) reflect on an interview project they completed. The teachers' reflections were coded for instances of *transitions*—a stated change from a past belief to a present belief. I coded the beliefs were based on the WHW framework and I engaged in open coding to capture new themes raised by the teachers (Strauss & Corbin, 1998).

Results and Discussion

Beliefs Survey

I focus on two instances of change across the pre and post surveys. One survey prompt stated "no matter how mathematics is taught it sends messages about…" followed by several sub-prompts. The number of teachers who selected "agree" or "strongly agree" for each sub-prompt is shown in Table 1. As can be seen, there were fairly consistent increases in agreement with the idea that mathematics sends messages about our social world. Thus, the course appears to have been successful in shifting many of the teachers' reported beliefs about the relationship between mathematics and important social constructs.

Table 1: Teachers Who Agree That Math Teaching Sends Messages About . . .

Sub-prompt	Pre	Post
Society in general	9	13
Race/ethnicity	5	10
Gender	5	10
Socioeconomic background	5	11
Culture	9	11
Social/political issues	6	11

A second pair of prompts stated "it is impossible to separate mathematics and politics in school" and "mathematic and politics should remain separate in school." Table 2 shows the number of teachers who selected "agree" or "strongly agree" for these two prompts (the results from the first prompt are reversed due to the phrasing). While many of the teachers indicated that mathematics and politics *can* be separated in school, a smaller number felt that it *should* be separated. Moreover, both prompts showed a substantial decrease from the pre to post survey.

Table 2: Teachers Who Agree That Math and Politics *Can* and *Should* Be Separated in School

	Pre	Post
Can be Separated	10	6
Should be Separated	6	3

Final Reflections

I identified three types of transitions in the teachers' reflections. *Adding* a new belief occurred when the teachers indicated that they had not previously thought about issues of equity as can be seen in the following quote.

> This semester I've thought more about math and equity with regard to how math is taught in schools. For a long time, I've looked at equity and social justice issues with regard to instruction in literacy, but I've never really thought about it in math, besides how math is taught. After all, math was only numbers, right? At the beginning of the semester, I started thinking more about how math was taught and not necessarily the messages behind what content was taught. Now, I think there is so much behind both how and what we teach in math.

Replacing a past belief occurred when a teacher indicated a change from a prior belief to a new belief about equity. The following quote provides an example of this.

> My views of equity…. [have] changed from the beginning of the class. I believed equity was just being fair to students in class and making sure all students had equal opportunities in class…. Although, I do know those things are important I know being fair means so much more. All the articles we read and discussed allowed me to see equity in an entirely different light. Even though I do understand it, I am not too sure how to fix it.

Finally, there was one instance of a teacher *expanding* on a previous belief. In the quote below the teacher describes how her prior valuing of "real world applications" was expanded to include more explicit connections to the political and social issues of students' lives.

> Prior to the readings and discussions I've never thought there was a place for politics in math, or even in the classroom at all. I'm always looking for real world applications to help students understand mathematical concepts, so why not include the real goings-on in their lives.

Conclusion

The results above indicate that the course served as a powerful introduction to issues of equity in the teaching and learning of mathematics. The teachers showed shifts in their beliefs about whether mathematics sends messages about our social world and about the relationship between mathematics and politics in the classroom. In their written reflections the teachers exhibited transitions in their thinking about equity over the course of the semester. One common theme across the quotes above, and across many of the reflections, was a lack of specificity in the teachers' beliefs about equity and uncertainty about how to address equity in their practice. On their final reflection 8 of the 14 teachers indicated that they had not considered issues of equity in the past and/or that one or more ideas in the course were new to them. Considering the diverse range of perspectives on equity in mathematics education and the newness of these ideas to many of the teachers, it is not surprising that they often provided vague or uncertain responses regarding their own beliefs about equity. This indicates the need for additional opportunities to grapple with these ideas. Future research in this area should focus on how teachers' beliefs evolve over extended periods of time and across differing contexts, including their classroom teaching.

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CONSTRUCTING A MODEL ELICITING ACTIVITY FOR MATHEMATICS TEACHERS: SUPPORTING TEACHERS IN THE COMMON CORE ERA

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Model Eliciting Activities (MEAs) are powerful tools for teachers that support their knowledge development in ways that productively and proactively impact their practice. However, the upcoming Common Core State Standards (CCSS) will redefine the knowledge base required for teachers in the classroom, and the large role that the teacher education programs must serve in order to support the CCSS. This study details the construction of an MEA designed to develop mathematics teachers' models of quantitative reasoning, one of the standards for mathematical practice given by the CCSS. An online summer course was the focus for the MEA, where secondary teachers completed a mathematics education course in master's program. By documenting the development process as well as conducting observations of the implementation of the MEA, the author will describe the MEA and suggest improvements based on this data.

Keywords: Design Experiments; Instructional Activities and Practices; Teacher Education–Inservice/ Professional Development

Introduction and Literature Review

With the impending incorporation of the Common Core State Standards (CCSS), K-12 mathematics education in the United States will change. In addition to laying out what content each grade is expected to cover, the CCSS include eight standards for mathematical practice, which are expected proficiencies with longstanding importance in mathematics education. These reform efforts place added pressure on mathematics teacher education, as teacher expectations shift. At the same time, research on mathematics teacher educators is lacking, especially concerning how professional development aligns with the CCSS educational reform goals in ways that productively impact teacher practice (Confrey & Krupa, 2010; Krupa, 2011; Sztajn, Marrongelle, & Smith, 2011.

A models and modeling approach to professional development has received increased attention by mathematical educators due to this perspective's ability to challenge teachers to develop ways of thinking that productively impact teacher practice while simultaneously documenting the development for research purposes. This approach uses Model Eliciting Activities (MEAs), which are tasks that engage teachers in thinking about realistic and complex problems embedded in their practice in order to foster ways of thinking that can be used to communicate and make sense of these situations (Doerr & Lesh, 2003; Lesh & Zawojewski, 2007). Model eliciting activities contribute to teacher development by encouraging teachers to think more deeply about student thinking, engage in mathematics, and reflect on prior held beliefs about problem solving (Chamberlin, Farmer, & Novak, 2008; Schorr & Koellner-Clark, 2003; Schorr & Lesh, 2003). These studies have implemented successful MEAs for teachers, but there is a need for additional activities given the recent demands the CCSS have placed on professional development (Confrey & Krupa, 2010; Garfunkel et al., 2011).

In response to that specific research need, this study documents the construction of a teacher MEA focusing on the quantitative reasoning standard for mathematical practice. Specifically the research questions were: (1) what is the process of developing a teacher MEA about quantitative reasoning; and (2) what are challenges specific to the design of an MEA that must be implemented within a professional development course conducted online during the summer? This study could be classified as an instrumental case study (Stake, 1995), with the creation of the MEA being the event under analysis that incorporates contextual factors in ways that convey understanding in other contexts. Aspects of a holistic case study are incorporated, since literature-based artifacts and feedback from mathematics educators, mathematicians, and teacher MEA experts were incorporated in the construction of the MEA.

Methods

The setting for this study was a mathematics education course for 23 secondary teachers in a master's program. The course was offered online during the summer, lasting four weeks, and met synchronously online four times a week. During each of these three hour meetings, the instructor, a mathematics educator, led class discussions with PowerPoint and application sharing. The teachers were also split into smaller groups for additional discussion. Audio and video transfer allowed communication to occur in real-time and facilitated group work despite the geographical distance between teachers in the course. Asynchronous technology such as email and Blackboard was used for file transfer and assignment submissions.

This course was titled Quantitative Reasoning in Secondary Mathematics and used aspects of a models and modeling perspective to promote teacher development of quantitative reasoning. The main course objectives included teachers being able to understand the meaning of quantities, quantitative relationships, and quantitative reasoning. They also needed to be able to identify each in secondary mathematics curriculum and deepen their understanding of secondary mathematics content involving quantities and quantitative reasoning; and be able to develop model-eliciting activities to support and document the development of student understanding and reasoning. The course reading list was comprised of articles focused on MEA development and quantitative and mathematical reasoning for students. The MEA which I describe in this proposal constituted 50% of the course grade, with the other 50% coming from task analyses.

The theoretical perspective I used for this study is a models and modeling perspective, as described by Lesh and colleagues. In addition to having a powerful lens for examining professional development, a models and modeling perspective also provides guidelines for the methods that support significant findings given the current research question. Given these methods, a models and modeling perspective offers a framework for understanding teachers' models, their development, and provides a mechanism for analyzing and piecing together findings, which may help future studies that investigate the implementation of the MEA (Koellner-Clark & Lesh, 2003; Hiebert & Grouws, 2007; Silver & Herbst, 2007; Sriraman & English, 2010).

Data collection sources include documents from the development of the MEA and feedback from other faculty members familiar with the teachers in the course, the setting, and teacher MEAs. Multiple versions of the MEA were developed to conform to the MEA guidelines established by literature, the setting restrictions, and input from these individuals. The faculty members used is summarized in Table 1, along with rationale for why their input was considered valuable in the development process. The course used for this study begins in June. At this time observations will be conducted document how the MEA was used and how it can be further improved. Interactions with the teachers enrolled in the course and participants in the table during and after the implementation of the MEA will provide data for a measurable response of the research question.

Table 1

Pseudonym	Position	Position/Expertise
James	Mathematics Educator	Instructor of the Quantitative Reasoning Course
Michael	Mathematician	Instructor of previous content courses within the master's program
Nikkea	Mathematician	Principal Investigator of the grant, familiar with the teachers and previous content instructor within the master's program
Jennifer	Mathematics Educator	Familiar with the master's program, and published multiple studies on MEAs for teachers
Germaine	Mathematics Educator	Expert researcher and theorist on MEAs and models and modeling perspectives

Preliminary Results

The purpose of the MEA is to reveal the way teachers think about quantitative reasoning tasks. Teachers' models of quantitative reasoning tasks can be evidenced by their development of a task that, from the teachers' perspective, captures quantitative reasoning skills of their students. The studied MEA for teachers is purposeful, sharable, and reusable, and also upholds the five principles stated in literature to guide the design of activities within teachers' practice (Doerr & Lesh, 2003; English, 2003); these include the reality, multilevel, multiple contexts, sharing, and self-evaluation principle. These principles as well as the purposeful, sharable, and reusable aspects of teacher MEAs were used to first structure the development of the MEA.

The version of the MEA implemented in the course included 5 iterations of the task, each prompted by feedback, where teachers work in groups of three. This process, outlined in Table 2, was determined by the guidelines in literature and input from the individuals detailed earlier. The rationale behind the development of the MEA was included to provide evidence that this data collection task meets the five principles of a quality teacher MEA, and that it upholds the final product is purposeful, shareable, and reusable. These components also allow the researcher to examine how teachers' interpretations, reasoning, and expressions develop during the iterations of the MEA. Details on how the MEA developed will be included in the presentation of this research, as well as results from the implementation of the MEA that occurs this summer.

Table 2

Assignment Name	Short Description of Components
Pre-Assignment	Document including initial models of QR, QR tasks, QR course
Version 1	Four documents including (a) Quantitative Reasoning Task; (b) Facilitator Instructions; (c) Assessment Guidelines; (d) Decision Log
Instructor's Feedback	Instructor's comments and suggestions to Version 1.
Version 2	Updated Version 1 in response to the instructor's feedback.
Teachers' Feedback	Groups swap Version 2 and offer comments and suggestions
Version 3	Updated Version 2 in response to the teachers' feedback.
Undergraduate Work	Student work after completing QR task (part (a) of Version 3)
Version 4	Updated Version 3 in response to student work, plus evaluation of student work.
Post-Assignment	Models of QR, QR tasks, relation to Continuous Math Course.
K12 Implementation	Each teacher implements the QR task (part (a) of Version 4) in classroom
Version 5	Updated Version 4 in response to student work, plus evaluation student work.

Implications

The development of this activity contributes to both research and teaching methods. Teacher MEAs are designed to promote development as well as documentation, the construction of such an activity would have benefits to other researchers in the field of mathematics teacher education, as the design of the activity could be shared and reused in similar contexts (Lesh & Doerr, 2003; Lesh & Lehrer, 2003). The selection of quantitative reasoning, a vital component of mathematics education that is receiving increased attention by the CCSS, increases the demand to make this specific MEA. Documenting the MEA development process can be helpful to other mathematics educators designing professional development (Pope & Mewborn, 2009).

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EXAMINING WHAT IN-SERVICE ELEMENTARY TEACHERS NOTICE WHEN PRESENTED WITH INSTANCES OF STUDENTS' MATHEMATICAL THINKING

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This report presents the findings of a study that was designed to investigate what in-service elementary teachers notice when watching video clips of task-based interviews of elementary mathematics students. Four second-grade teachers met with the researcher to watch these video clips and discuss their thoughts in reaction to them. Sessions were conducted without any prior professional development on noticing. They were video recorded and videos were analyzed for emerging themes. Over the course of the study, teachers displayed an ability to notice students' mathematical thinking to varying degrees of sophistication.

Keywords: Mathematical Knowledge for Teaching; Teacher Education–Inservice/Professional Development

Introduction

How does such a common word, like *noticing,* become powerful? What is it about the word that carries weight among teacher educators today? *Noticing*—also called *professional vision* (Goodwin, 1994)—happens when a teacher takes in his or her surroundings and is able to determine what is worth paying attention to, and what can be disregarded depending on the current purpose. Then they must integrate this information about the students with their own knowledge of the content.

This study examines what in-service elementary teachers notice about students' mathematical thinking without any direct professional development to guide them. By determining a baseline level of noticing, it is possible to examine avenues for developing teachers' abilities to identify, interpret and analyze students' thinking and subsequently use that thinking to guide their instruction. Recent research indicates that teachers who develop the ability to examine students' thinking with the aid of video are better able to respond to and make use of that thinking when it arises in a lesson (Cohen, 2004), in line with reform mathematics thinking. This report shares the findings of a video-based study that was conducted during the fall semester of 2011 at an urban, Title one, public elementary school in the United States.

Literature Review

When teachers can see exactly what students know, they can develop appropriate plans of action to ensure that everyone in the classroom reaches their full potential as a learner. Teachers with well-developed noticing skills are able to respond to students, and use the students' thinking to guide learning. Reform mathematics initiatives and curricula encourage teachers to work closely with students to understand and use student thinking to guide learning experiences in the classroom (National Council of Teachers of Mathematics [NCTM], 2000).

Teachers' knowledge of content and pedagogy is closely tied to their noticing skills. Because many elementary teachers are at a disadvantage as evidenced in their lack of confidence in their own mathematical knowledge (Ball, 1990), one can infer that it must be difficult for them to notice students' mathematical thinking. Teachers may be able to identify—to notice—an area of misunderstanding that a student exhibits without knowing how to remedy the confusion.

Teacher noticing as a field of study is gaining ground in three main areas of mathematics education (Sherin, Jacobs, & Phillipp, 2011): teaching by responding and adapting to what happens in the classroom; ongoing learning by teachers; and deconstructing teaching practice into essential elements for practice and improvement. The second focus responds to the idea that teachers can learn to be better "noticers" of their own practice in order to grow as professionals. They can learn to identify the particulars of students' mathematical thinking in a classroom setting, or to see the ways that they should change a lesson to better impact student learning. The seminal author on this topic is John Mason (2002), whose book *Learning to Notice: Researching Your Own Practice* has been a guide for many education researchers interested in this field.

Methods

As part of her work, the researcher serves as a math specialist for K–6 teachers at an urban public school supporting teachers' mathematics instruction, planning, and professional development. One of the ongoing professional development projects with the second grade teachers involved watching video clips of students solving math problems in a one-on-one interview setting. The teachers watched the clips and responded to them in an adaptation of a video club (Sherin & Han, 2004), a kind of professional learning community where video serves as the medium for launching discussions about teaching and learning. The goal of the project was to determine what the teachers were noticing about the video clips without guidance from the researcher, in order to better formulate future professional development around noticing students' mathematical thinking. As such, the sessions were dominated by teacher discussion, with little guidance or commentary from the researcher. Sessions were video recorded to capture the teachers' conversation and interaction, as well as to match specific video clips to teachers' comments.

Sherin and Han (2004) define video clubs as "meetings in which groups of teachers watch and discuss excerpts of videotapes from their classrooms" (p. 163). In order to determine what teachers notice about students' mathematical thinking when faced with it directly, the design of this study purposefully eliminated many of the elements that other noticing studies have included. Rather than having teachers bring in clips from their own classrooms, which would likely contain many elements unrelated to student thinking, video clips where student thinking is paramount were provided. The video clips used in this study come from a project conducted at San Diego State University, led by Randolph Philipp, Ph.D. The study, called IMAP (Integrating Mathematics and Pedagogy), was funded by the National Science Foundation and the Department of Education from 1999–2002.

Aligned with research done by Elizabeth van Es on video clubs (van Es, 2011), teachers' responses were examined and categorized into levels of noticing and an overall session level was determined. For instance, if teachers commented on students' confusion but then also made comments about specific mathematical thinking that took place, the session would be categorized as a Level 2, or Mixed, level of noticing (see Table 1 below—modified from van Es, 2011, p. 139).

Table 1: Levels of What Teachers Noticed

	Level 1 *Baseline*	*Level 2* *Mixed*	*Level 3* *Focused*	*Level 4* *Extended*
What teachers notice	Notice students' confusion or clarity about the math task	Begin to attend to students' mathematical thinking	Attend to students' mathematical thinking	Attend to the relationship between students' mathematical thinking and possible pedagogical strategies or learning experiences
Sample teacher comment	"She doesn't understand place value." "He's really got it."	"He added on." "She groups by tens and ones."	"You can tell how flexible his understanding of tens and ones is by…"	"I wonder if she's had many opportunities to build the quantities with Base-10 blocks so she can see how the place value relationships work."

Findings

At the outset of the study, the researcher expected to hear many comments that could be categorized as Level 1 comments, in which teachers would remark about a student's confusion or understanding without much else added to it. Indeed, many of these comments were made. They stated exactly what a student did, saying, "He counted up and ignored the manipulatives," and they questioned another student's understanding, remarking, "Maybe that was just a guess?" However, even in the first session, teachers made comments that showed their attention to students' mathematical thinking. After one student showed an invented subtraction method, one teacher said, "He found the easiest way to make the biggest number possible, and then added on…it really shows he knows place value—he can trade the tens out." This comment shows the beginnings of an attention to students' thinking, bringing them to Level 2 on the continuum, Mixed Noticing. Their comments didn't stay focused on students' thinking, but instead often led back to their own classrooms—"That's really inefficient. I wouldn't want my kids to do it that way."

By the second video club session, the teachers were more consistently noticing students' thinking, operating at a Level 3, or Focused level of noticing. They often wondered what experiences students were bringing to the interviews, whether the way they solved problems were related to their teachers' expectations of how they should work. "Could she solve it in her head, but her teacher makes her use the algorithm, so that's what she does here?" one teacher asked. Another teacher reacted to a student's use of the traditional subtraction algorithm to solve 1000 – 4 by saying, "I want my students to think about number combinations so that they immediately think about putting that four with a six to make a ten." By getting a glimpse into the minds of these children, the teachers were thinking more critically about their own instructional practice. What they failed to do was think about how the specific student thinking displayed could be addressed with particular teaching strategies. How, for instance, would Elizabeth get her students to think about those number combinations that make ten?

In the third and fourth sessions, teachers continued to notice students' thinking. When a student struggled to connect the subtraction algorithm to the Base-10 blocks with which he had also solved a problem, the teachers noted that, "the five is not a fifty for him, and the zero isn't part of a hundred, so he really thinks he's taking five from nothing." They wondered how they might help him make that connection, and discussed lessons and activities they might use to develop a better understanding of place value. This was a clear example of Level 4, Extended Noticing. However, after another clip where a student clearly misunderstood what the numbers in the traditional algorithm represented, Kate commented that she felt students should be taught to add this way before they learned about larger numbers. She made this comment without recognizing the lack of understanding of quantity that the student had displayed. This missed opportunity to notice was telling—her understanding of student thinking is fragile.

Discussion

In general, the teachers participating in this study clearly showed their ability to notice students' mathematical thinking. Their comments showed a range of noticings, from basic retellings of what students in the video clips had done, to beginning identifications of what students are thinking mathematically, to eventually talking about instructional strategies to develop students' understanding of basic number concepts. Teachers were able to notice students' thinking at the most sophisticated level of the framework, Extended Noticing, some of the time. This occurred within a group discussion and without guidance from the researcher.

What is less clear is how much high-level noticing was taking place on an individual level. Due to the presence of a more experienced teacher who had a strong knowledge of the school's mathematics curriculum, it is possible that the other teachers' lack of noticing abilities were sometimes masked. Since no written data collection took place, this cannot be confirmed. Future studies should include a written component to be completed before discussion in order to analyze individual noticing abilities.

It should also be noted that the small size of the video club group makes generalization about teacher noticing impossible. Other groups might show more or less sophisticated noticing of students' thinking. With a larger group of participants, it would be possible to examine teachers' noticing abilities relative to their level of classroom experience, understanding of mathematics concepts, depth of pedagogical content knowledge, and many other factors.

Continued research on teachers' noticing of students' mathematical thinking might investigate the following topics: how does collaborative noticing compare or contrast with noticing in the absence of peers; how does noticing progress over significant periods of time; what strategies can professional developers utilize to increase teachers' noticing abilities; how can the simple framework in this study be further augmented? Research on teacher noticing is a field that is rich with possibility. This study provides a single focus.

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ELEMENTARY TEACHERS' UNDERSTANDING OF COGNITION BASED ASSESSMENT LEARNING PROGRESSION MATERIALS FOR MULTIPLICATION AND DIVISION

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Research-based learning progressions (LP) of children's mathematical development represent a valuable resource to help teachers make sense of children's thinking, and research on teachers' use and understanding of such materials is needed. This study investigated teachers' understanding of the Cognition Based Assessment (CBA) LP materials for multiplication and division concepts. Data sources included clinical interviews based around teachers' use of the CBA LP materials while analyzing student work episodes. Analysis of teachers' understanding of the CBA materials consisted of grounded theorizing and retrospective analysis. This paper reports on one key finding; as the complexity of either the CBA LP level, or a student's mathematical reasoning, so did the inconsistency and variation in teachers' interpretations of the student thinking.

Keywords: Learning Trajectories (or Progressions); Teacher Knowledge

Introduction and Literature Review

Teachers' knowledge of mathematical content and children's mathematical thinking have been identified as critical elements related to teachers' ability to effectively teach mathematics (Peterson, Carpenter, & Fennema, 1989). According to the National Research Council (2007), "Learning progressions are descriptions of the successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic" (p. 214). The LP conceptualization has provided a way to encapsulate research-based findings about not just how children think about specific mathematical topics, but how learning about those topics develops (Battista, 2007; Daro, Mosher, & Corcoran 2011; Maloney & Confrey, 2010). Research-based learning trajectories/progressions of children's mathematical development are a valuable resource for teachers to help make sense of children's thinking, and research on teachers' use and understanding of such materials is needed.

Although there is a significant amount of research in the LP realm, limited research has investigated teachers' learning about research-based learning progressions. Research indicates that teachers need a solid understanding of the mathematics they teach, the common conceptions children hold about mathematics, and how to help children make connections to increasingly sophisticated mathematical ideas (Fennema & Franke, 1992). Research also shows that teachers struggle with exactly these things, but professional development focused on children's thinking can help them improve in these areas (Jacobs, Lamb, & Philipp, 2010).

Methodology

Because research on teachers' understanding of learning progressions is in an exploratory stage, qualitative and descriptive research can explicate some of the core issues in this area. Although quantitative methodology is not uncommon in the research design of projects focused on designing learning progressions, the available research on teachers' learning of learning progressions is largely qualitative in nature (Bardsley, 2006; Wilson, 2009). Quantitative counts of qualitative trends and patterns are included to help encapsulate ideas that emerged in the data.

The larger project that this research is a subset of is CBA2—Cognition Based Assessment, Phase 2 (An Investigation of Elementary Teachers' Learning, Understanding, and Use of Research-Based Knowledge about Students' Mathematical Thinking). CBA2 began in 2006, with the goal to investigate how elementary teachers make sense of and use research-based knowledge about the development of students' reasoning about particular mathematical topics.

Participants for this study included 17 elementary teachers responding to as many as four samples of student work (a total of 65 teacher episodes of student work analysis are included in the study), in the state of Ohio or Michigan, who were involved in the CBA2 research project. By utilizing transcribed clinical interviews, detailed descriptions, grounded theories, and brief vignettes of teachers' understanding of CBA learning progressions were developed.

CBA2 researchers collected a significant amount of data from participating teachers from 2006–2010. The data described for the current inquiry represents a portion of the collected data from the CBA2 project. Interview protocols consisted of episode(s) of student work and subsequent questions. Of interest in this paper were teachers' responses to the question: What CBA level of sophistication do you think the student's strategy is?

Data analysis utilized a retrospective approach as described by Steffe and Thompson (2000) which calls for the use of videotaped interviews, and individuals' written work in order to build an historical account of the individuals' actions and interactions. Data was analyzed and re-analyzed in ways that can explicitly support or disconfirm the evolving theories and hypotheses. This retrospective analysis informed the continuation of model building, hypothesis generation, and analytic category construction.

Results

Recognizing and understanding children's mathematical thinking in terms of CBA levels is an important component of making sense of the CBA LP materials. We investigated teachers' ability to do this when students' reasoning exactly matched a CBA level, or not, and when students' reasoning was simple and complex. Here is the relevant portion of the CBA MD LP:

Level 3. Operates on Numbers by COMBINING/SEPARATING (without Counting or Skip-Counting)

3.1 Recalls Facts	*3.3 Uses Distributive Property to Decompose Numbers by Place-Value into 2 Partial Products*
3.2 Uses Number Properties	*3.4 Uses Distributive Property to Decompose Numbers by Place-Value into 4 Partial Products*

In identifying CBA levels, sometimes teachers focused more on perceptually salient characteristics than on the technical, and conceptually more precise, definitions given in CBA documents. As an example, in evaluating RR's reasoning on the following problem, T2 relies heavily on keywords describing the number of partial products computed, as opposed to recognizing the precise place-value decomposition of these partial products.

Task: 46×5. RR: 20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.

T2: [T2 had used CBA MD materials]

(Reads problem with RR's strategy) See that's interesting. Because, there is no 3 parts [in the CBA levels]…but they broke it up into 3 parts almost, so that's kinda interesting. The difference between them is that they broke down the 200 into smaller parts, and maybe that is because it is a single digit on the other number, so they couldn't break it down any further. I would definitely put them at least at an MD 3.3, but I would say that they…well I would pursue whether they were at the 4 part level.

T2's interpretation of *partial products* focused on the number of concrete computations in the decomposition, but does not explicitly include mention of the distributive property or the specific place value nature of the partial products as described in CBA levels 3.3 and 3.4. Although T2 indicates that the students' strategy might not exactly match the description of level 3.3 when she states, "The difference between them is that they broke down the 200 into smaller parts," she does not explicitly recognize or articulate the fact that it is not representative of the technical definition of place-value partial products in CBA (in this case, 2 place value partial products would mean 46×5 would be computed as 40×5 plus 6×5).

Even so, T2 understands that RR's reasoning is at Level 3. She also seems to understand that RR's reasoning is some type of legal decomposition into parts (although she does not mention the distributive property). She might even consider RR's decomposition based on place value because RR decomposes 46 into 20 + 20 + 6 – which has tens parts and a ones part. This understanding brings her to consider Levels

3.3 and 3.4. At this point, she examines the number of parts in the decomposition, because that is what distinguishes Levels 3.3 and 3.4. And finally, taking almost a rubric scoring approach, she says that RR is "at least at an MD 3.3 [level]." We hypothesize that because use of non-place-value distributive property decomposition is one of many strategies discussed in Level 3.2, T2 does not recognize RR's reasoning as such.

Looking at overall teacher responses to four samples of student work can help to frame the discussion for this type of teacher use of the CBA MD materials. CBA teachers were asked to evaluate students' mathematical thinking about multiplication on the following three tasks.

Task 1. There are 6 soccer teams in a tournament. Each team has 12 players. How many players are there altogether? SX: 12, 24, 36, 48, 60, 72 [raising one finger after each number]. So, there's 72 players.

Task 2. 46×5 RR: 20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230. QR: 40 times 5 is 200; 6 times 5 is 30. So it's 200 plus 30 equals 230.

Task 3. 45×23 Sally: 45 times 10 is 450. 45 times another 10 is 450; that's 900. 45 times 3 is 120, plus 15. So it's 1020 plus 15, or 1035.

Task 1 describes a less complicated strategy involving iteration (skip-counting) of a two-digit number. This strategy is a relatively simple strategy to recognize based on perceptually salient actions, as the CBA MD level of "skip-counting all" always involves a student writing or saying all multiples in the skip count sequence without decomposing numbers in any way.

Task 2 has two samples of student thinking. QR's strategy represents using the distributive property to decompose 46 by place value parts into 2 partial products. Much like the "skip-counts all," CBA MD level 3.3 always involves decomposing one factor by place value and the distributive property to create 2 place-value partial products. RR's strategy, however, represents using a non-place-value distributive-property decomposition, one of many strategies included in Level 3.2. Because students at this level do not use strict place value decomposition of numbers, their decompositions can take unique forms based on particular students' comfort with certain numbers. For this reason, level 3.2 might be challenging for teachers to learn at a conceptual level because there are many variations; indeed, in the language of research on concept learning, level 3.2 is a superordinate category and might be more difficult to learn than level 3.3, which is at the basic level.

Task 3 represents a fairly complicated episode of student thinking about multiplying a 2-digit by 2-digit multiplication problem. In consideration of the CBA MD levels, Sally's thinking is quite complex, as it involves instances where Sally decomposes by place value (40 times 3 is 120, plus 15…where Sally decomposes 45×3 into 40×3 and 5×3 implicitly), and instances where Sally does not decompose strictly by place value (45 times 10 is 450 and another 10 is 450; that's 900…where Sally decomposes 45×23 into 45×10 and 45×10 instead of simply 45×20). Sally's thinking is best described as applying 2-partial products twice (Level 3.3), with further decomposition of one of the partial products using a non-place value decomposition (Level 3.2). Because Sally's reasoning is a combination of CBA MD levels 3.2 and 3.3, this episode represents a situation in which it would be expected that teachers would struggle to make sense of her thinking by matching it to CBA descriptions and examples of student work.

Quantitative Summary

A quantitative summary of teacher responses indicated that episodes of student thinking the directly matched a single-strategy encompassing CBA level were correctly identified more often. Those situations requiring deeper analysis of student thinking, or matched a level describing several possible strategies showed higher variation in teachers' determination of students' CBA level. Below are the counts of teacher responses to each episode. Although the sample sizes are small, this exploration of teachers' interpretations of various complexity of children's thinking supports the idea that mathematical and cognitive complexity of the CBA levels as well as complexity of student thinking both play a role in teachers' ability to appropriately level students within the CBA MD framework.

Table 1: Student Work and CBA Complexity Comparison

SX and QR (less complex work and CBA levels)	Percent Completely Consistent	Percent Partially Consistent	Percent Inconsistent
Total of Teachers with CBA MD use	89.3% (25/28)	7.1% (2/28)	3.6% (1/28)
Total of Teachers with **No** CBA MD use	77.8% (7/9)	11.1% (1/9)	11.1% (1/9)
Total of ALL Teachers	86.5% (32/37)	8.1% (3/37)	5.4% (2/37)

Sally and RR (more complex work and CBA levels)	Percent Completely Consistent	Percent Partially Consistent	Percent Inconsistent
Total of Teachers with CBA MD use	47.6% (10/21)	42.9% (9/21)	9.5% (2/21)
Total of Teachers with **No** CBA MD use	0% (0/7)	28.6% (2/7)	71.4% (5/7)
Total of ALL Teachers	35.7% (10/28)	39.3% (11/28)	25% (7/28)

Discussion

There is no doubt that with any LP materials, or any materials intended to help teachers with the process of teaching, certain aspects of the materials will be related to more complex components of mathematics or children's thinking about mathematics. As was anticipated, CBA teachers struggled far more to effectively characterize student work that was more complex and that represented more complex CBA levels, or did not fit into the CBA level descriptions in a straightforward way. One interesting finding in the data was that teachers with experience using the CBA MD materials seemed to more effectively choose CBA aligned levels than teachers without use of the CBA MD materials on the student work sample that was most complex.

The CBA materials can be understood at varying levels of conceptual understanding. Obviously, it would be ideal for teachers to develop a formal conceptual understanding of CBA levels. Understanding the conceptual and computational delineations between each level could allow teachers to have fine-grained knowledge about children's mathematical thinking that could serve to inform instructional decisions. This could be especially important for a teacher working with a student one-on-one who is struggling to make progress, for whom conventional instruction does not seem to benefit. However, teachers do not always have the time or flexibility to work with their students individually to utilize CBA to its fullest. Therefore, using CBA more efficiently, in manners that lead to identifying big-picture conceptualizations of children's thinking (potentially using less formal and more fuzzy conceptions of CBA) could serve a valuable purpose as well, especially for whole-class LP use.

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SUPPORTING TEACHER LEARNING THROUGH DESIGN RESEARCH

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This study reports on a design research approach used to support elementary teachers to use reformed approaches in teaching mathematics. Findings from a three-year Math and Science partnership grant are presented. Specifically, the impact on content/pedagogical content knowledge and student learning are presented in addition to the mechanisms for supporting teacher learning. A theoretical framework for designing professional development aimed at using reformed approaches is presented.

Keywords: Teacher Education–Inservice/Professional Development

Theoretical Framework

Studies have found that content focused professional development is the most effective form of professional development to support student achievement (Hill, Rowan, & Ball, 2005). Content focused professional development should provide teachers with pedagogical content knowledge for effective teaching (Ball, 1991; Schulman, 1986, 1987). Pedagogical Content knowledge integrates content knowledge and knowledge of how students learn math. In other words, it requires understanding how students think and learn. Once the teacher has an understanding of how the student is thinking, then the teacher has to challenge the student's current thinking in order to facilitate new learning. This requires carefully selecting tasks, posing problems and figuring out what representations can help this student make sense of the problem.

Unfortunately, most teachers do not teach for conceptual understanding. The TIMSS (1999) Study points out that a typical American teacher relies on teaching students' strategies and formulas as opposed to conceptual understanding. The Common Core State Standards for Mathematics (CCSSI, 2010) point out that students not only should learn procedures but also develop conceptual understanding of mathematical ideas. The NCTM standards recommend that students should develop these understanding through a problem solving approach that builds on student reasoning. This means that professional development must support teachers to teach for mathematical understanding through a problem solving approach advocated by NCTM and the Common Core State Standards for Mathematics.

Designing professional development that provides teachers with content and pedagogical content knowledge is a challenge. This is particularly the case when teachers are using more traditional or procedural approaches in teaching. Therefore, simply having students take more math content courses are not effective (Hill, Rowan, & Ball, 2005). We need to consider ways we can shift their practice from teaching mathematics as procedures to focus on students understanding mathematical concepts.

A design research (Lamberg & Middleton, 2009; Kelly, 2003; Cobb, diSessa, Lehrer, & Schauble, 2003; Collins, Joseph, & Bielaczyc, 2004) approach is useful for designing professional development. This is because it allows us to generate conjectures on how teachers might learn, design tasks, try these tasks and modify subsequent tasks with teachers. When using a design research approach, tasks are designed based on theoretical principals from prior research (Lamberg & Middleton, 2009, Kelly, 2003; Cobb, diSessa, Lehrer, & Schauble, 2003; Collins, Joseph, & Bielaczyc, 2004). The theory provides an overarching framework for the design team to generate tasks, select and validate alternative designs (DiSessa & Cobb, 2004). However, when tasks are used with teachers, they may make sense of the tasks differently than the intention of the design team (Collins et al., 2004). The goal of design research is to understand why teachers interpreted the tasks differently. In addition, the process in which learning happens is also documented. The underlying focus of a design experiment is to understand the *process* or *mechanisms* that promote development of one state of learning to another (Lesh & Kelly, 2000). Therefore, we investigated how to support teacher learning through a design research approach in addition to its impact on teacher knowledge and student achievement.

Methods

We received funding for three years from a Math Science Partnership grant to provide teacher professional development. A design research approach was used to design tasks and implement professional development to support teacher learning during the first two years. In the third year, a lesson study approach was used. Thirty-seven teachers from 5 school districts in a large rural area participated. The project team consisted of a mathematician, a math educator, regional professional development trainers and a math consultant from the state department. The teachers participated in three week long institutes with 6 follow-up sessions during the academic year. We used the following cyclical design approach to work with teachers. Conjectures were generated; tasks were designed, and implemented in professional development sessions. Ongoing adjustments were made based on what was happening during the session, and a retrospective analysis of data took place to modify, confirm and generate new conjectures.

The data to measure the impact of this professional development included a teacher pedagogical content knowledge pre and posttest. The mean of the gain scores of the pre and post tests were compared. Student achievement was measured using a pre and post math content test. The percentage of growth of students gain scores was measured over three years. Data that were collected to examine professional design decisions and its impact included videos of professional development sessions, field notes, video analysis of teacher professional development data, field notes from and professional development team meetings. The qualitative data was analyzed using Strauss and Corbin constant comparative method (2001). The following framework was used to make sense of teacher learning and sense making. Specially, we examined how teachers interpreted the tasks, the professional development activities that supported the sense making and the context in which the professional development took place. An interpretive framework listed below was used as an analytic lens for making sense of the learning of teachers.

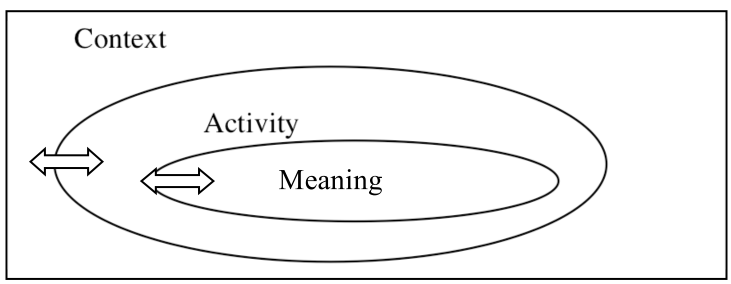
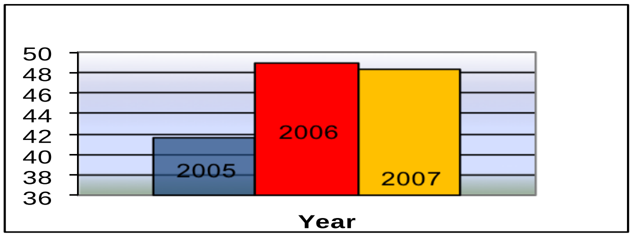


Figure 1. Interpretive framework to make sense of teacher learning

Results

The data revealed that not only teacher pedagogical content knowledge increased, but also teachers changed their teaching practices towards more NCTM reformed practices. In addition, student learning was impacted as indicated in the gain scores of the pre and posttest.

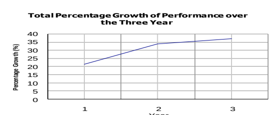
Mean of Teacher Pedagogical Content test



Teacher's pedagogical content knowledge significantly increased from year 1 to 2. Year three pretty much stayed the same. It is interesting to note that a design research approach was used the first two years, while the third year we used a different approach.

Comparison of Percentage growth of Student Gain Scores

	Percentage Growth (%)		
Grade	*Year One*	*Year Two*	*Year Three*
3rd	87.1	110.1	102.1
4th	33.7	42	41.9
5th	15	21.7	33.5
6th	20.6	45.4	40.9
7th	7.3	8.5	-1.1
8th	2.2	0.3	N/A
Total	21.5	33.8	37



Student achievement data also reveals that learning increased. The data presented in the chart above represent a comparison of the percentage of gain score that occurred each year across three years. The following framework captures the three phases of professional development that took place to support shifts in teachers' practices. Professional development tasks were designed around these three phases in teacher development.

Phase I: Teachers experience thinking about mathematics conceptually.
Raise awareness of student reasoning and how students learn.
Examining norms of how a lesson is structured – Supporting a problem-based approach to teaching.

Phase II: Interconnections between lessons-sequencing – *Building on student thinking and supporting mathematical connections*

Phase III: Reflection on student learning and teaching – *Refine teaching*

Discussion

The study is significant because it elaborates how a design research approach can be a useful approach for making design decisions to develop effective professional development. Furthermore, this study demonstrates that design research approach had the most impact on teacher knowledge and teaching. This was because we adapted the PD based on the needs of the teachers. Whereas the third year, we had laid out the PD plan and followed the plan of lessons study. Therefore professional development must not only provide activities for teachers to do but should be adapted based on teacher learning. A design research approach is useful for making these decisions. In addition, effective professional development not only pays attention to teacher learning but also examines shifts in teacher's classroom practice to make changes.

Hill, Cohen, and Ball (2005) point out that the most effective professional development pays attention not only to providing teachers' content knowledge but also pedagogical content knowledge. However they point out there is very little research that says what this should look like. The phases outlined in the framework above provide a starting point for future professional development work. Phase I points out that teachers should experience mathematics conceptually through a problem solving approach. If teachers don't understand what problem solving is, they are unable to transfer this way of thinking to their students. Teachers also examined student reasoning and how it related to understanding. During Phase II, teachers began to use a more problem solving approach to their teaching; however we discovered that even though they had made changes in their practice, they were not sequencing lessons. Therefore, during the second phase we focused designing experiences for teacher to reflect deeply on how lessons are interconnected mathematically. The third phase involved refining the first two phases and implementing in the classroom. These phases can be a starting point for designing future professional development for teachers.

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TEACHER SUCCESS: DEEP LEARNING AND PRACTICE

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This study explores the effectiveness of a hybrid professional development model in supporting a teacher's success in their professional learning and practice. The findings present the impact of the program on teachers' practice, beliefs about math, perspective about themselves, as well as the nature of the virtual interactions, including frequency and content of virtual communication. Furthermore, the study investigates the depth of learning and variables influencing the depth of learning. The conceptual discussion includes potentials and limitations to be considered in further dialogue, development, and research in designing and evaluating a hybrid program for teacher success in deep learning and practice.

Purpose of the Study

Ongoing professional development is critical in teacher success. By "teacher success," we mean the improvement of a teacher's understanding of mathematical content and practice within their classroom. Teacher quality is closely related to student achievement. Gains in student achievement cannot be made without ongoing professional development. The research question that guided this study was "How should a professional development program support teacher success in their professional learning and practice?"

Theoretical Framework

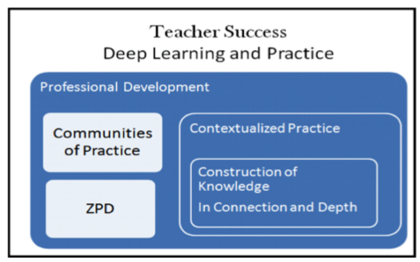

Figure 1: Theoretical framework

Figure 1 illustrates the theoretical framework for teacher success in deep learning and practice through a blended professional development involving contextualized practice and active construction of knowledge. Since Dewey, the social nature of all human learning and the role of communication skills and abilities in the human development process are the most often discussed concepts in higher education (Feldman, 2000; Heinze & Procter, 2005). Learning in Communities of Practice, networks of practitioners, relies on communication between individuals and occurs as interaction among practitioners takes place (Heinze & Procter, 2005; Kahan, 2004). Communities of Practice involve "creating a Zone of Proximal Development with capable peers. There is no one on the stage who is the knowledge source, but all individuals have an equal right to share their experience, and their stories are valuable contributions to the community" (Heinze & Procter, 2005, ¶ 3).

Mere interaction among participants, however, does not guarantee meaningful cognitive engagement or facilitate meaningful learning and understanding. Garrison, Anderson, and Archer (2000) provide a model of a community of inquiry, the integration of cognitive, social, and teaching presence. A community of inquiry involves more than a social community and interaction among participants: "Interaction directed to cognitive outcomes is characterized more by the qualitative nature of the interaction and less by quantitative measures. There must be a qualitative dimension characterized by interaction that takes the form of purposeful and systematic discourse." (Garrison & Cleveland-Innes, 2005, p. 135)

Rovai (2002) found a "positive significant relationship between a sense of community and cognitive learning" (p. 328). Even though researchers recognize various views in defining learning communities, there are common themes that link the definitions and uses. Professional development programs should provide opportunities for participating teachers to "learn mathematics around specific content and teaching situations that may arise in practice" (Brown & Benkenp, 2009, p. 56), as well as opportunities to implement/practice their learning in their own context. The key concerns, therefore, are to ensure peer exchange of ideas and information (Hammond, 1998); the creation and assimilation of knowledge (Thomas, 1992); and providing a climate of accelerated change in which participants need to access up-to-date knowledge and apply new skill flexibly in changing circumstances (Hargreaves, 1994). The learning community should be more than a forum for the exchange of information. It must evolve in response to the diverse needs of learners and the communities in which they work.

Methods: Data Collection and Analysis

This study analyzed data from the *Teaching Algebra in Context, Community, and Connections (TACCC)* project funded by the Improving Teacher Quality Program in 2010–2011 and 2011–2012. The professional development program of this study was designed to improve student mathematics knowledge and attitude by (1) increasing teacher content knowledge, and (2) improving instructional skills to teach mathematics concepts in an environment rich in context, community, and connections. The major activities of the project consisted of three components: face-to-face workshop courses, online conversations between face-to-face meetings, implementation, and sustaining a professional community. Teachers explored how algebra is taught at different grade levels, is related to other math content, and applies to our daily life. The workshops were conducted through discussions; collaborative group work; hands-on activities; problem-solving opportunities; and presentations by participants. Table 1 summarizes the two projects and the data collected for the study.

Table 1: Participants and Data Collection

Project Year	Online support system	Participants (MS teachers)	Data				
			No. of online Postings	Pre-survey	Post-Survey	Interviews	Observation notes/reflection
Group 1 (2010-2011)	Desire2Learn	29	1149	Yes	Yes	Yes	Yes
Group 2 (2011-2012)	Online Social Network (MSP2)	22	1002	Yes	Yes	Yes	Yes

Surveys, interviews, and filed observation notes were analyzed, and mean scores were compared. The written document data were analyzed in two ways: (1) purpose and content, and (2) depth of learning. The final codes were developed after two analyses. The depth of learning was analyzed using the Gerbic and Stacey (2005) model. Gerbic and Stacey proposed a generic framework for the content analysis of deep and surface learning that can be used both for face-to-face and various online learning environments. The discussion threads in each online assignment were used as the unit of data collection and analysis. All of the postings among the 51 participants were analyzed and coded. Three coders were trained and checked for inter-rater reliability at the end of training by using a sample set of coding. SPSS was used for the quantitative elaboration of online posting data, and chi-square tests were used to examine the relationship between two categorical variables.

Results

Changes in Opinions about Mathematics and Classroom Practices

Overall, most of the group revealed a high level of comfort with innovative instructional practices on the pre-survey. Teachers' responses (2010–2011) pointed to increases in their use of inquiry-based instructional practices and alternative assessment. They had also expanded their professional development activities at their schools and online. Teachers from the 2011–2012 program expressed the desire to continue their growth as math instructors who were equipped with a wide range of strategies that would address the needs of their students. Their responses on the follow-up survey demonstrated their progress. Comparisons of the teachers' pre- and post-responses to questions about the frequency of their use of inquiry-based instructional practices and involvement in various professional development activities revealed significantly higher use in several areas.

A descriptive analysis on teachers' beliefs about mathematics and their perspectives about themselves will be shared during the presentation. In summary, with regard to teachers' opinion about themselves, teachers from the 2010–2011 program show significant changes in their confidence about their preparation to answer students' questions. Responses indicated that teachers were more comfortable about their skills in mathematics at the year's end. Teachers in 2011–2012 show numerous significant improvements about their understanding of effective strategies for teaching math and confidence about ways to help their students learn math. They were more knowledgeable about how to turn students on to math and gave more value to the role of effective math teaching.

Teachers altered their beliefs about math during the course. The 2010–2011 group of teachers gave responses on their confidence level in preparing to answer students' questions that indicated that teachers were more comfortable about their skills in mathematics at year's end. They came to value working on problems that do not have a precise answer. They increasingly viewed strategies as a way to help students learn math. Finally, they were less likely to view math as a solitary activity. This is consistent with the results of another survey in which participants were significantly more likely to have students working in groups. The 2011–2012 group of teachers was significantly less likely to agree that only certain people could do well in math and to view math as a solitary activity. They gave greater value to students working hard to learn math, to taking a longer time if needed to learn math concepts, and working on problems that do not have a precise answer.

Depth of Learning

In order to measure participants' depth of learning, messages posted for four online assignments were analyzed: (1) Book Reading Discussion and Assignment, (2) Video Critique, and (3) and (4) Discovery Lesson Creation, which required creation of lessons and implementation, and commenting on the work of peer teachers.

Descriptive Analysis: The average number of postings per assignment and individual teachers' total and average postings will be compared within the group and between the two groups.

Purpose of Messages and their Content: In addition to communication frequency, our study analyzed the data to find the purpose of messages and their content. Our codes were founded in Gareis and Nussbaum-Beach (2007), Interstate New Teacher Assessment and Support Consortium (1992), and Joint Committee on Standards for Educational Evaluation (1988) and finalized after two initial analyses. The final codes were: Reflection, Sharing experience, Issues/Problems, Questions/Suggestions, Instructional ideas, Concerns regarding students, Guided advice, Simple agreement, and Acknowledgement. Sample messages and descriptive comparisons between groups and within group will be presented during the presentation.

Depth of Learning: The Gerbic and Stacey (2005) model was used to analyze teachers' depth of learning. According to Gerbic and Stacey's analytical framework, deep learning is defined in this study as the ability of learners to demonstrate critical thinking skills by (a) looking for meaning in course content; (b) relating course topics to prior knowledge and real world examples; (c) interpreting content through synthesis, analysis, and evaluation; and (d) utilizing internal motivation to learn (2005, p. 55). Surface learning is defined as a learning approach where learners simply dwell inside (a) a reproducing approach; (b) course boundaries; (c) an unthinking approach; (d) fear of failure; and (e) are extrinsically motivated (2005, pp. 55–56). The depth of learning is divided into two categories, deep learning and surface learning, with four subcategories in deep learning and five subcategories in surface learning. It was noted that online communication mechanisms help teachers not only to develop critical thinking skills and deepen their learning of teaching, but also to demonstrate their abilities to teach mathematics in different ways. The study further investigated variables that are related to the depth of learning. Chi-square test results for the relationship between the level of discussion threads and the depth of learning, the relationship between discussion topics versus the depth of learning, and the relationship between the online support system and the depth of learning will be presented during the presentation.

Discussion

This study investigated the effectiveness of a hybrid professional development program and online communications. By year-end, it was evident that participants had made an effort to make their classrooms more student-centered and inquiry-based. Students were doing more group work and the teachers had begun requiring students to explain their answers and to respond to “why” inquiries if their explanations were not sufficiently clear. Overall, teacher participants increased their math content knowledge and improved their instructional skills. Furthermore, teachers enthusiastically described changes that they have made in their classrooms to enrich their students’ math learning.

With regard to the effectiveness of the online part of the program, there were some trivial findings and unexpected outcomes that can provide insight to mathematics teacher educators. Most face-to-face discussions centered on the activity or mathematics problems that were presented in class; whereas online discussions reflected on and shared experiences in their own classrooms. The content of online discussions made no distinction between different assignments. The main focus of discussions was sharing their experience about teaching mathematics instead of focusing on what they were required to discuss. It would be important to investigate how different online discussion topics and assignments contribute to teachers’ active learning and participation. Another interesting finding was the differences in the total number of messages for each assignment. First-year teachers had more to share when they were asked to reflect on book readings or video-taped best practices than to exchange their own ideas and experiences about planning and implementing a discovery-based lesson. What teacher educators can learn from this finding is the sequential order of the online professional development program. Practitioners also need plenty of time to become acquainted with the website and technology that the online PD uses.

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TEACHERS' KEY DEVELOPMENTAL UNDERSTANDINGS FOR MEASUREMENT FRACTION DIVISIONS

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Contemporary research on mathematical knowledge for teaching (MKT) has predominantly been conducted under the framework suggested by Ball and her colleagues (Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008). Recently, alternative perspective of MKT is proposed by Silverman and Thompson (2008), which centers on key developmental understandings or KDUs (Simon, 2002, 2006). The preliminary analysis of 14 middle grade teachers' reasoning with fraction division problems indicates that 'common partitioning operations' and 'three-levels-of-units structure', would be essential KDUs for measurement fraction divisions. The study extends previous research on KDUs for fraction operations (Simon, 2006; Lee & Shin, 2011; Lee & Shin, 2012) by investigating teacher knowledge through a semester-long professional development.

Keywords: Mathematical Knowledge for Teaching; Rational Numbers

Purpose of the Study

Since Shulman's (1986) proposal of pedagogical content knowledge, in mathematics education, Ball and her colleagues (Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008) have used the phrase *mathematical knowledge for teaching* to emphasize knowledge teachers use when solving problems that arise in practice. Research on teacher knowledge (e.g., Hill, Schilling, & Ball, 2004) on this line shows that there is a positive correlation between teacher knowledge and student achievement. They have also stressed the conceptual demands of teaching mathematics by defining MKT as a special kind of knowledge, which the 'man on the street' does not need in their work. Nonetheless, further research is needed to elaborate mathematical knowledge for teaching particular topics (Izsák, 2008) and to reveal, "what it is, and how one might recognize it, and how it might develop in the minds of teachers." (Silverman and Thompson, 2008, p. 499). The Silverman's and Thompson's perspective of MKT is grounded in the idea that "teachers teach what they know" (Thompson, 1994, p.3), where "to know" means to have a scheme of meanings (i.e., mathematical ways of knowing and mathematical ways of understanding) that express themselves in action. In their view, powerful mathematical knowledge for teaching involves pedagogical understandings that are grounded in significant, coherent personal understandings, and they cited Simon's (2002, 2006) key developmental understandings (KDUs) as a type of powerful personal understandings. The purpose of this paper is to extend previous research on MKT and to investigate a particular topic (measurement fraction division) in detail by delineating KDUs of fraction divisions.

Theoretical Framework

In a measurement fraction division problem in the form of $a/b \div c/d$, one may ask oneself "How many groups of *c/d* are in *a/b*?" The fundamental operation one might use to find an answer to the problem is a *unit-segmenting operation* (Steffe & Olive, 2010). This entails the operation of segmenting the dividend by the divisor. To elaborate, consider the problem of grouping a pile of fifteen books by three. One could use three as a segmenting unit to divide the composite unit, fifteen and measure the fifteen by threes. While it is enough for one to reason with only two levels of units[1] in such a simple problem situation, our claim is that one who has flexibilities with forming and transforming three-levels-of-units structures could better adapt his or her reasoning to more complex problem situations, especially, those where one needs to reason with fractions. We use the term *common partitioning operations* (Steffe & Olive, 2010) to refer to the partitioning operations that one uses when one's goal is to coordinate the common number to be used in partitioning. This operation entails various partitioning operations that support one to achieve the goal of coordinating the common number (on an actual bar or mentally). When one uses common partitioning operations, one could find commensurate fractions[2] for the dividend and the divisor using a co-measurement unit.[3] For instance, to solve the problem $\frac{2}{3} \div \frac{1}{7}$ using a number line model, one can use common partitioning and measure two-thirds of a unit stick using one-seventh of the stick by finding 1/21 as a co-measurement unit for both one-third and one-seventh. Using the co-measurement unit as a base, one could find commensurate fractions, fourteen-twenty first for two-thirds and three-twenty first for one-seventh as in Figures 1a and 1b.

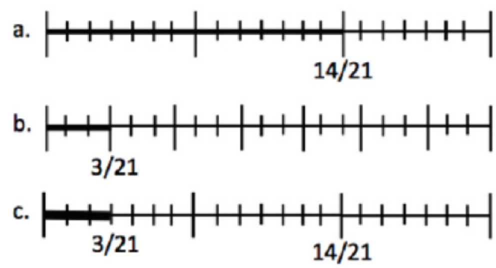

Figure 1: Determining 2/3 ÷ 1/7

(a) A three-level structure of 2/3. (b) A three-level structure of 1/7.
(c) Two three-level structures of 2/3 and 1/7.

Therefore, finding commensurate fractions entails reasoning with three-level structures. One coordinates two three-level structures to determine the quotient for 2/3 ÷ 1/7 as in Figure 1c. One may also calculate a common denominator between the divisor and the dividend quantities and then represent one's reasoning using drawn quantities. As long as one can clearly explain the co-measurement unit and commensurate fractions for the divisor and the dividend quantities, we include calculating the common denominator beforehand as part of common partitioning operations. In these cases we stated that the person made a conceptual association between procedural knowledge and common partitioning operations.

Methods

Professional development on rational numbers was offered to 14 middle grades teachers in an urban district. In this paper, we consider six of those teachers including three seventh grade and three sixth grade teachers. The course met once per week for three hours at a time across 14 weeks. All meetings were videotaped using two cameras and combined into a single restored view (Hall, 2000). The primary data for this analysis included two class meetings focused on measurement fraction division. In these two class periods, the teachers engaged with several tasks that relied on number line and area model representations as well as discussions about using those representations for fraction division. This analysis also relied on data from three interviews with the teachers about their answer selections on pre and post-course assessments. From the assessment, we only focused on two fraction division items: One using an area model to illustrate quotative division of fractions, and the other using a number line item. Each interview asked participants to explain how they selected their response on the multiple-choice items. These interviews were also videotaped. From the restored view, we created lesson graphs, a document that parsed the lessons and the interviews into episodes. From the lesson graphs, we memoed teachers' mathematical reasoning to identify the two primary categories, then we coded the lesson graphs using an emergent set of categories to generate hypotheses that were then united into comprehensive accounts.

Results

From the initial analysis of teachers' knowledge of quotitive division, we observed positive correlation between a sequence of problems and teachers' struggles to make sense of the problems using the drawn representations. The three sequences were (1) When the divisor partitions the dividend evenly; (2) When the divisor does not partition the dividend evenly; (3) When the divisor is bigger than the dividend. No matter what sequence the problem falls in, common partitioning operation with three-levels-of-units structures appeared as KDUs for measurement fraction division. The teachers' coordination of two three-levels-of-units structures activated more sophisticated partitioning operations and supported teachers making sense of more complex fraction division situations. While the teachers could use two levels of units across each measurement fraction division sequence to determine the quotient using drawings of quantities they chose, those teachers could not use more sophisticated partitioning operations to activate unit-segmenting operations. In addition, when the teachers established part-whole reasoning between the two quantities, their ability to use the measurement unit as the referent unit was critical, and it was impossible without the teachers' coordination of two three-levels-of units structures.

Conclusion

Conceptual analysis of teachers' knowledge at this grain size allows us to develop a stronger understanding of teachers' capacities to reason about fraction division in detail. By building on understanding of teachers' mathematical concepts and operations in the ways illustrated here, we propose that a richer understanding of teachers' understandings was identified, which could be used for developing stronger professional learning opportunities for teachers.

Above all, the present study indicates that the knowledge components found in the previous research literature about children's fractional knowledge appeared in the participating teachers' mathematical activities with fraction problems, and further turned out to be essential for their mathematical thinking in the context of division problems. Further, we began to realize that the development of teachers' knowledge (might or necessarily) differ from that observed in children even though applying the results from research with children could be a viable way to start. Teachers are already well equipped with procedural knowledge and they are likely to have more sophisticated number sequences already developed.

To elaborate, some participating teachers' common partitioning operations were evoked by their strategy of finding a common denominator between two given fractions. While the teachers brought forth common partitioning operations by themselves, they believed they used an algorithm. They were referring to the algorithm in finding a common denominator for two fractions, which was a procedural strategy that they usually used in fraction addition or subtraction problems. It is plausible that some of the teachers could have already been equipped with procedural knowledge that was associated with (mental) operations for common partitioning operations because they have revisited the content over and over. However, we think this may cause serious problems when the teachers go back to their classrooms because they may implicitly demonstrate common partitioning operations, but outwardly teach an algorithm to their students. This conclusion also has an implication for designing effective professional development program in which teachers could explicitly become aware of the associations they make between the procedural algorithm and the key mathematical operations embedded in it.

Endnotes

[1] Simply saying, levels of units denote the number of units available to a problem solver prior to actual solving activity. For instance, a child with two levels of units might be embarrassed with the division problem "How many fours in twenty?" because the available units for the child are only 'one' and "twenty."

[2] We use the term, *commensurate fractions* (Steffe & Olive, 2010) to describe when one uses drawings of quantities to figure out, in conventional terms, equivalent fractions.

[3] A *co-measurement unit* is defined as a measurement unit for commensurable segments, that is, segments that can be divided by a common unit without remainder (Olive, 1999).

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PEER COACHING: IMPROVING MATHEMATICS TEACHING IN ELEMENTARY SCHOOL MATHEMATICS

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The purpose of this paper is to describe a research project that investigated school improvement in mathematics in two Canadian provinces. Nine schools were selected based on their interest in improving mathematics education at their school, and various contexts such as public and private schools, rural and urban schools, and size of schools. This report will provide researchers, teachers and administrators with strategies and tools to make change to mathematics programs within elementary schools. Further, elementary school administrators can use the findings to use a professional development strategy to improve teaching of mathematics within their school.

Keywords: Elementary School Education, Teacher Education–Inservice/Professional Development; Teacher Knowledge

Introduction

The purpose of this paper is to describe a research project that investigated school improvement in mathematics in two Canadian provinces. Nine schools were selected based on their interest in improving mathematics education at their school, and various contexts such as public and private schools, rural and urban schools, and size of schools. This report will provide researchers, teachers and administrators with strategies and tools to make change to mathematics programs within elementary schools. Further, elementary school administrators can use the findings to help teachers and support staff to use a professional development strategy to improve teaching of mathematics within their school.

Theoretical Framework

Peer coaching is professional development strategy for educators to consult with one another, to discuss and share teaching practices, to observe one another's classrooms, to promote collegiality and support, and to help ensure quality teaching for all students. In peer coaching, usually two teachers share conversations, and reflect on and refine their practice. Their relationship is built on confidentiality and trust in a non-threatening, secure environment in which they learn and grow together; therefore, peer coaching is usually not part of an evaluative system.

The primary goal of peer coaching is to provide positive feedback to instructors, most of who regularly receive negative comments from students on teaching evaluations. Peer coaches also provide support and companionship for their partners (Joyce & Showers, 1982). Finally, peer coaching may improve student learning because good instructors teach their students more effectively (Weimer, 1993). Evidence of the positive effects of such in-service learning on teacher implementation of math education reform and student achievement is accumulating (e.g., Loucks-Horsley et al., 2003; Schifter & Simon, 1992; Smith, 2000).

Method and Data Sources

This project explores the effectiveness of teacher peer coaching teams and describes strategies used to enhance mathematics teaching in elementary schools. To this end, teachers met to determine which dimensions to focus their discussion, negotiate improvement goals, devise strategies to implement the goals, observe teaching and provide feedback.

The method is exploratory case study (Yin, 2009) combining qualitative methods (interview and observations) with quantitative measures (survey and continuum scores, provincial test results). The participants are principals and teachers in elementary schools selected to represent a range of contexts (particularly traditional and reform teachers). Participants completed the Ten Dimension Mathematics Continuum (McDougall, 2004), met in pairs with another teacher, discussed the survey and continuum results, identified two dimensions for discussion and focus, negotiated improvement goals and implemented innovative teaching strategies.

Teachers observed each other while researchers sat in the classroom taking field notes. The teachers were interviewed before, during and after each math lesson to elicit the teacher's intentions and reflections on the lessons observed. The interview guides for teachers and principals were based on guides developed for earlier studies (McDougall, 2004; Ross, McDougall, Hogaboam-Gray, & Le Sage, 2000).

We interviewed the Principal and three teachers in each of the five schools. We selected the teachers from the staff list so that we have one primary teacher, one junior teacher and a resource teacher. The principal and teachers had an opportunity to indicate that they do not want to participate in the project. Each interview was about 45 minutes in length. They were audiotaped and transcribed.

Peer-Coaching Sessions

Teachers formed pairs, and each pair met during a pre-observation conference in which each teacher was given a form containing several initial questions and other questions to assist their observation of the dimension that the teacher who was being observed had identified. The discussion during the conference was guided by these questions, including, what are you planning to do today in the classroom? What did you do in the past in this topic? What would you like me to observe? The teachers met to select the dimensions on which they wanted to focus the discussion, negotiate goals for improvement, and devise strategies to implement goals, observe teaching, and provide feedback.

During the pre-observation conference, the observing teacher asked probing questions so that they could find out more about the teacher's goals for the lesson, which Dimension would be focused on and what parts of their teaching they wanted the observer to pay close attention to. During the lesson, the observer would take notes.

After the lesson, the pair met for a post-observation conference to discuss the lesson and the observing teacher would share their observations on the lesson, specifically on the areas that the teacher asked them to focus on. Judgments about the teacher were withheld unless the teacher asked for suggestions for improvement. During the discussion, the teacher was asked to share their thoughts about the lesson, what they learned from the experience and once feedback was given, prompted to reflect on what they would change/do differently the next time and what area(s) of improvement they would like to focus on during the next session.

Data Analysis

The data analysis included an initial exploratory review of the data and a constant comparison analysis (Miles & Huberman, 1994) of interview transcripts, field notes, observation notes, and feedback from participants. Computer qualitative research software, nVivo8, was used to assist in the analysis of the data. The initial coding scheme was based on the characteristics of mathematics reform and was elaborated based on the emerging themes. The use of codes such as context, diversity, peer coaching, professional development, distributed leadership, and each of the Ten Dimensions of Mathematics Education (McDougall, 2004) helped to further illustrate the richness of the data.

Findings

When asked about the benefits of peer coaching, many teachers revealed that they have more opportunities for reflection and gain valuable feedback from their peer coach, which they found have strengthened their collegiality and increased their personal and professional growth.

Many teachers mentioned the level of comfort that they have with a peer observing them in the classroom. As part of the peer coaching process, teachers interact through coaching, discussing, and sharing thoughts and information with each other. This process creates more informal and formal conversations between teachers and as well as opportunities for collaboration. One teacher stressed the need to discuss and brainstorm ideas with his peer before implementing constructivism and incorporating manipulatives in the class.

Teachers found that, as the observer, they have more insights into what other teachers are doing in their classrooms and benefit from visual understanding of how students react and learn in the class. From the perspective of an observer, a teacher declared, “it was good to just watch the students and how they were learning” (Weeping Willow, T2, Reflection) and another stated that it is “interesting to see where and what others are doing in terms of classroom management and teaching” (Weeping Willow, T4, Reflection).

Teachers also felt that, through discussing and receiving feedback from their coach, they became more oriented and focused in their teaching. For example, one teacher explained, “It is useful to have a process in place to help me focus on specific aspects of my teaching I want to improve—otherwise I am often caught up in ‘juggling’ everything” (White Oak, T8, Reflection) and another stated, “It was a chance to really focus myself on key aspects of what I was trying to do” (White Oak, T8, Reflection).

Peer Coaching – Challenges

When asked about the challenges that the peer coaching model presented, the participants frequently spoke about themes that were prevalent in their everyday teaching. One of the challenges that teachers faced was time. They believed they felt a pressure to do their job well given the amount of time that they had to spend preparing and teaching their lesson, so when asked to engage in peer coaching sessions, teachers found it difficult to schedule the session into their routine.

Some teachers found that it was challenging to get other teachers participating in the process. One teacher shared that recruiting her colleagues to engage in this model of professional development “might take some convincing. I think the newer teachers are more onboard than some of the ones that have been around for awhile” (Alder, T2, Interview, January 24, 2008).

One of the main components of the peer coaching process is having colleagues observe each other teach. This component gave rise to many fears and concerns by the participants. Many teachers had initial concerns about having an observer in the classroom. This observer could have been a member of the research team, another teacher or an administrator.

A colleague at the same school echoes the importance of a well working partnership. She says that, “It is important to be] sensitive when giving feedback. [And you] need to have someone who can accept honest feedback” (White Oak, T4, Reflection).

In addition to the challenge of having a colleague watch their teaching practice, teachers found it a challenge when they realized that their lesson was not going exactly as planned. A teacher said her challenge was “realizing that at times I had not thought certain things all the way through—and suddenly I was on the spot!” (White Oak, T8, Reflection).

Significance of the Study

Many teachers who were being observed mentioned the level of comfort that they have with a peer observing them in the classroom. Some teachers felt comfortable and appreciated having the other set of eyes to give them feedback. Being observed in the classroom is taken as a great opportunity to improve teaching since it is hard to come up with things that need improvement when teachers shut the door and teach in isolation.

Similarly, observers in the classroom gain pedagogical techniques and teaching strategies, such as how to react to weak students, how concepts and skills are introduced and how to be 'hands-on' to assist students' learning. Some teachers even visualize what tomorrow's lessons will look like based on observation. In addition, teachers further understand how learning occurs and how to take interventions with individual students. By observing their peers, teachers realize the different challenges that each age group poses and the different strategies that they should adjust and work with.

The peer coaching process strengthens collegiality and creates opportunity for collaboration. Many teachers consider collegial collaboration as mutual respect. This good rapport allows teachers to explore one another's ideas or opinions, to encourage each other, observe each other, share sources and across-share in different grades.

Many teachers find that timely feedback and constructive suggestions are beneficial to their professional development. Teachers value the feedback that they receive as it helps them focus on specific aspects of their teaching rather than juggling everything. As a consequence, teachers become more open and flexible to try new lessons and to tweak lessons for particular students' needs. Many teachers appreciate feedback that gives them ideas on what could be worked on, other ways to look at a problem and how to vary strategies.

Traditional methods of professional development may be ineffective in promoting teacher's continual learning as well as the transfer of knowledge into practice. Sustaining changes and improvements in teachers' instructional practices is a challenge that educational stakeholders are facing today. Peer coaching may be one answer for teachers who wish to improve their teaching in a supportive, non-threatening environment.

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TO FOIL OR NOT TO FOIL

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Mnemonic devices have been around for thousands of years and are commonly used in mathematics classrooms to help students remember formulas, units, operations, etc. One of the most common mnemonic devices is the acronym, FOIL (First Outer Inner Last). This study takes a closer look at the acronym FOIL, how a group of preservice elementary and middle school mathematics teachers used it, and its possible implications for classroom instruction.

Keywords: Algebra and Algebraic Thinking; Teacher Education–Preservice; Mathematical Knowledge for Teaching; Teacher Knowledge

Introduction

KFC = Kentucky Fried Chicken = Keep Flip Change (Dividing Fractions)

KHDMDCM = King Henry Danced Merrily Drinking Chocolate Milk = Metric Units

ND = Notre Dame or Nice Dog = Numerator before Denominator

PEMDAS = Please Excuse My Dear Aunt Sally = Order of Operations

IPRT = I am PReTty = Simple Interest Formula

May I have a large container of coffee? = Pi to 7 decimals (word lengths are digits) = 3.1415926

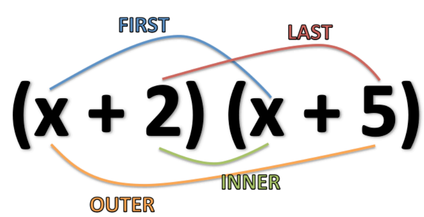
5 tomatoes = 5 to (m)ate oe(s) = 5 2 8 0 feet in a mile

FOIL = First Outer Inner Last = Multiplying a binomial times a binomial

Walk into a freshman level college mathematics classroom and ask the students what is the one thing they remember from high school mathematics. What do you think the result will be? The responses will probably vary, but if your students are anything like mine were, they will overwhelmingly say “FOIL” or “that FOIL thingy.” When I walked into my first college algebra class back in Spring 2003, I wanted to get a better gauge on my students. I didn’t have a pretest prepared for them; I thought I would just verbally quiz them to see where they were and where I needed to start. I left after the first day and thought, where do you start when they answer FOIL? I taught several more entry-level college mathematics classes and each and every time, the resounding answer for the most memorable thing from high school mathematics was FOIL.

First, Outer, Inner, Last, or FOIL, is a mnemonic device to aid in multiplying two binomials together. FOIL is the acronym for the steps:

“First” terms are multiplied together
“Outer” terms are multiplied together
“Inner” terms are multiplied together
“Last” terms are multiplied together



The result is then simplified, if necessary, but most commonly adding the resulting middle terms together.

It is a tool for secondary students to use in order to not forget to distribute all of the terms when multiplying binomials together. It is limited in its use as it only works for the multiplication of two binomials.

Objectives of the Study

The purpose of this study was to explore preservice teachers' use of the mnemonic FOIL (First Outer Inner Last) when answering questions about multiplying binomials and factoring trinomials into binomials and its implications for K–12 classrooms.

Perspectives

The above examples are common mnemonic devices found in the mathematics world. Mnemonics are devices, operations or procedures used to improve memory. Although mnemonics can be found in nearly every content area imaginable, they have more of a "home" in special education. Mnemonics is one of the most highly touted and commonly used intervention strategies in Special Education. Learning disabled (LD) students are the most common to receive this type of intervention strategy because semantic memory difficulty is a common characteristic of LD students (Scruggs & Mastropieri, 1990).

Mnemonics have been around for thousands of years (cf. Yates, 1966) and have seen positive results amongst the Special Education Community, especially in mathematics (Manalo, Bunnell, & Stillman, 2000; Mastropieri & Scruggs, 1989; Scruggs & Mastropieri, 1990). There are several types of mnemonic devices including: Keyword method (i.e., "of" means multiply, see Schoenfeld, 1982), Pegword method, Acronyms (i.e., FOIL, HOMES, PEMDAS), Reconstructive Elaborations, Phonic mnemonics, spelling mnemonics, number-sound mnemonics, and "Yodai" methods (cf., Higbee & Kunihira, 1985; Kilpatrick, 1985).

Other than the occasional response to an article, a majority of the research articles related to mnemonic devices and/or acronyms lie within the Special Education realm. This is problematic given that K–12 classrooms use mnemonic devices on a regular basis beginning early in elementary school with the keyword method. Mnemonic devices, especially acronyms, become more prevalent in the middle and high school classrooms as students are taught little tricks to help them remember algorithms foundational in numeracy and algebra.

The mnemonic device, FOIL, is a common term found in many current algebra textbooks beginning as early as middle school, through beginning college mathematics courses, and can even be found in middle and secondary mathematics methods textbooks for preservice teachers. It is often thought that FOIL is a newer term in the mathematics world, coming along after the "New Math" movement. Contrary, you can a reference and picture of FOIL in Shute's (1956) *Elementary Algebra* book and even further back is Betz's (1931) *Algebra for today, second course* (p. 150) in text and picture reference. As this was not Betz's first book text, it's more than likely the case that FOIL had been written about before this.

Methods

This study was part of a larger concurrent triangulation mixed-methods study in which preservice middle grades teachers' mathematics knowledge for teaching was investigated. Preservice teachers were given questions based on items from 6th–8th grade math content standardized assessments. The questions were broken down into four different subject areas, one of which was Algebra. This study focuses on two particular questions from the Algebra assessment taken by 158 preservice middle grades teachers pursuing Mathematics/Science Specialist degrees and who were enrolled in the middle-grades specialist undergraduate degree program at a large public university. These two items were related to each other and common in current Algebra curricula: the multiplication of two binomials, and the factoring of a trinomial into two binomials.

(1) Multiply the expression: $(3x - 5)(2x - 8)$
Answer: _________
Explanation of Answer: _______
Explanation to someone who didn't understand: __________

(2) Factor $y^2 + 3y - 18$ into two binomials.
Answer: _________
Explanation of Answer: __________
Explanation to someone who didn't understand: ___________

Data were analyzed qualitatively using constant comparative analysis (Denzin & Lincoln, 2000).

Results and Discussion

Results indicate that a majority of preservice teachers were able to correctly answer both questions, but relied heavily on the term FOIL to explain their answer to the first question and as an instructional technique for explaining it to someone who did not understand. A lack of the phrases "distributive property" and "multiplication of each term" among the explanations was noted:

- I just used FOIL to multiply it out and then I added like terms.
- I came to this answer by foiling or multiplying the two parentheses together. By doing this I get 6x^2-24x-10x+40. Then combining like terms I get 6x^2-34x+40, which is my answer.

In the second item, there was a tendency to misuse mathematical vocabulary and procedures in explanations of solutions. Instead of explaining how they factored their solution or talking about various methods of factoring, a majority of students used several variations of FOIL. Again, there was noted a lack of mathematical vocabulary akin to this content, especially the distributive property:

- y is squared so it needs to be in both parentheses. Then use FOIL.
- The solution is (y – 3)(y + 6) because when you FOIL these two factors together it gives you y^2+3y-18. FOIL is just a way of multiplying the two factors together without leaving out a part.

Across item two explanations, the most common method of arriving at a solution was to work the FOIL method backwards:

> I did the FOIL method backwards in a way. I know y^2 is y times y so that will be the first portion of each binomial. Then I thought about the factors of 18 and picked on that when subtracted would equal 3.

FOIL is a process that is supposed to aid students in using the distributive property to multiply two binomials together. However, from these examples, one can see that participants memorized the procedure and may not understand what they are doing. FOIL is taught as early as 6th grade and ingrained into students' minds throughout high school and even into college. There are alternative procedures for teaching the multiplication of binomials that the participants are taught during the middle-grade program at the site of this study, including algebra tiles. However, there was no mention of these methods.

Implications

In a heightened era of assessment and teacher evaluation, it is critical to continue to address the mathematical content needs of preservice and inservice teachers. In the late 1980s, the National Center for Research on Teacher Education found elementary and secondary teachers were unable to explain their reasoning or why the algorithms they used worked (RAND, 2003). Instead, they exhibited a rule-bound sense of understanding, similar to what was exhibited in this study by the preservice middle school mathematics teachers. This rulebound sense of understanding reflects the nature of teaching and the curriculum teachers experienced in elementary and secondary schools (RAND, 2003). Through sustained professional development and preservice teacher mathematics and education coursework, inservice and preservice teachers should have exposure to mathematics and mathematics methods that force them to explain their reasoning and begin to understand how and why algorithms they use work. There should be a strong emphasis placed on the importance of using correct mathematics vocabulary (e.g., distributive property instead of "FOILing") and conceptually understanding that vocabulary (e.g., distributive property).

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PROPORTIONS, RELATIONS, AND PROPORTIONAL RELATIONSHIPS: ONE TEACHER'S NAVIGATION BETWEEN PROFESSIONAL DEVELOPMENT AND PERSONAL KNOWLEDGE

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In this paper, we report an exploratory study considering how one teacher made sense of professional development related to proportions. We consider this from the perspective of knowledge in pieces, paying attention to the organization of the teacher's knowledge as it related to the structure of the content in the PD. Implications for the design of PD are discussed.

Keywords: Teacher Knowledge; Teacher Education–Inservice/Professional Development; Number Concepts

Purpose

Professional development for teachers has led to disappointing results in terms of teacher learning (e.g., Garet et al., 2011). Further, studies of teacher knowledge have proven ineffective for determining teacher effectiveness as measured by student outcomes (e.g., National Mathematic Advisory Panel, 2008). Some researchers have posited that the constructs that matter for teacher knowledge are unique to teaching, such as pedagogical content knowledge (Shulman, 1986), and efforts to measure those constructs and correlate them to student learning have led to somewhat more promising results (e.g., Baumert et al., 2010; Hill, Rowan, & Ball, 2005). This shift in thinking about teacher knowledge opens opportunities for researchers to further examine what teachers need to know and how they develop their knowledge.

In this study, we consider how a teacher understood proportions after completing a professional development (PD) course that included 9 hours (3 sessions) of instruction on proportions. For our analysis, we used a content-mapping strategy to create visual representations of the concepts discussed in PD and the content discussed in a clinical interview with the teacher. Our research sought to explore how this teacher made sense of the content in the PD by comparing.

Theoretical Framework

We rely on the knowledge-in-pieces epistemology (diSessa, 2006). Knowledge in pieces asserts that each of us has a variety of fine-grained understandings that work in concert with each other to make sense of complex situations. Thus, for any given problem situation, we are likely to invoke some number of these pieces to create a more complex, synergistic knowledge. Thus, learning can be seen as (a) developing more fine-grained knowledge pieces, and (b) refining existing pieces so they have more connections between them, allowing a more coherent understanding. Learning occurs when a perturbation causes the learner to reassess an understanding in ways that lead to new understandings or new connections among existing understandings. For example, if a student only understands fractions as *n* pieces of an *m*-sized whole, (e.g., 3 pieces of a 4-piece cake is ¾) that student cannot use that understanding to make sense of 7/4. The student needs to both add a new piece of knowledge about fractions and reassess the existing piece of knowledge to better understand how and when it is appropriate. The development of expertise from this perspective involves building connections and refinements that allow appropriate pieces of knowledge to be invoked in various situations. In our study, we consider the connections between ideas that were discussed in the PD and the ways those did or did not translate to the teacher's understanding of proportional reasoning.

Methods

For this study we interviewed Walt, a 6th grade teacher with six years of teaching experience. He participated in our 14-week PD designed to strengthen teacher's mathematics knowledge by engaging in open-ended exploratory tasks involving mathematical reasoning. Walt was one of 14 participants in the course, which was led by an advanced doctoral student with a strong mathematics background and prior teaching and PD experience.

The three PD sessions that focused on proportions were video recorded with two cameras to capture the speaker(s) and any written work. These were combined using a picture-in-picture technique. A 90-minute clinical interview on proportional reasoning with Walt was conducted and videotaped 10 months after the PD. Walt was asked to engage in a number of proportional tasks that featured sample student reasoning and hypothetical interactions in professional development. This interview was also videotaped using two cameras.

To understand how content was organized and discussed for Walt and in the PD, we created content maps (Empson, Greenstein, Muldonado, & Roschelle, in preparation) from the videos. To do this, we considered relationships among lexical content to compare what mathematical language is explicitly linked within the two situations. Phrases representing concepts or objects became nodes of our maps. When we noted Walt using two words in similar ways, we recorded that connection as a line connecting the nodes. When Walt described a relationship, we recorded the description of the relationship as an annotation to the line (e.g., a proportion *can be* set-up).

In the case of the PD, this mapping showed our analysis of the whole-class discussions about proportions in which Walt participated. For Walt's clinical interview, this mapping showed our observations of Walt's use of language while reasoning about proportional situations.

We analyzed within these maps for notable groupings and absences of connectedness in concepts important for proportional reasoning. We then compared the PD mappings to Walt's interview mapping.

Results

Our analysis uncovered fundamental differences between the ways in which the content was organized in the PD and the ways in which Walt made sense of the content. Due to space limitations, we focus on only two key aspects of knowledge organization in this paper: representations of proportions and the definition of proportion.

Representations for Proportions

In the PD, one key element of all three proportions classes was the use of representations for making sense of proportional situations. These sessions included not only discussion of different representations (e.g., double number lines, graphs, tables, etc.), but also how they could be used to support thinking about the relationships rather than providing illustration of pre-calculated answers. By the third proportion class, the discussion included how the representations were related to each other. In the PD, the representations discussed were generated and introduced by the participants, but the facilitator took the role of engaging the participants in seeing connections between and among the representations as well as considering how the representations of directly proportional relationships, particularly for graphs, varied from those of inverse proportion and linear relationships.

Despite actively engaging in these sessions, Walt seemed not to rely on knowledge of representations in his own proportion problem solving. In our interview, Walt described only three representations: the equation, graphs, and a drawing that could be used as a build up strategy in which a ratio was expressed by a certain number of x's in the numerator to match the value of one quantity and a number of x's in the denominator to match the value of the other quantity. Walt explained that this drawing could be iterated to determine the "missing value" in a proportion.

The limited discussion of graphs in the interview mirrored our observation from PD that Walt demonstrated a general lack of fluency with the representation. For example, in PD he commented that he never learned about slope. In the interview, he drew a proportional relationship with the axes reversed, leading him to note that the graph was not representing his intended idea, but also leaving him without the resources to fix it. In the PD, he instead relied on symbolic notation, which in the interview also, in his words, failed to represent his idea. He did not invoke any knowledge of other representations during the interview.

Definition of Direct Proportion

In the PD, direct proportion was defined in terms of relationships. For example, it was described as division, a constant product, a constant rate, and a relationship. Further, after three weeks of refining the definition, the community definition in PD was that proportions were two equal ratios and that the equality could only be maintained through a multiplicative relationship. The PD also highlighted that a direct proportion could be expressed as $y = kx$ or as $y/x = k$ and that the graph of a proportion is a straight line that goes through the origin.

In our interview with Walt, we found that his definition of proportion seemed split. He had a set of ideas related to "relationships" between numbers and a separate set of ideas related to "proportion." The connection between these was not explicated. His definition for relationships seemed bound by the concrete world. He described them as connected sets of units of certain size. For example, a 2 : 5 ratio was, to him, a 2-unit connected to a 5-unit that he described with statements like, "2 is to 5", "2 is 5" and "2 equals 5, in a way." He was able to conceive of these units increasing (e.g., doubling) or decreasing (e.g., halving) through build-up or iteration strategies and he was able to reason about them in ways that attended to the quantity (Thompson, 1994). He also dismissed the importance of composed unit reasoning, focusing instead on one value in a proportion increasing at a time. In contrast, his definition for proportion was grounded in calculation. He said that proportions were "equal ratios" which he elaborated to be equivalent fractions in which the ratio stayed equal while it increased proportionally. He also talked about proportions in terms of filling in missing values, multiplying, and sometimes dividing. For Walt, contextualized understanding was separated from procedural knowledge of the content.

Discussion

In our analysis of these data, we considered Walt's learning from the perspective of knowledge in pieces. While the PD offered opportunities for teachers to development understanding of the concepts through connection making and through the introduction of new ideas, we found that Walt's activation of proportional knowledge did not generally reflect the PD. It appeared that the definition of proportion and the use of a variety of representations were not evoked when Walt was faced with questions that required him to reason proportionally and make sense of student reasoning. We suggest this is a result of two different issues.

First, the PD offered a socially negotiated opportunity for exploring ideas related to proportions, however, it may not have scaffolded Walt's connection making. While he clearly had considerable understanding about proportions, particularly his understanding about "relationship," those understandings were not supported by use of the representations from the PD. Further, his definition for "proportion" lacked the mathematical focus of the PD. The PD may not have provided the right perturbations to support Walt's connection making. Alternatively, Walt's limitations with graphing may have led him to overlook aspects of the representation discussion that were accessible to him because he perceived the representations to be outside his zone of development.

Second, aspects of the PD remained tacit that perhaps should have been made more explicit. While the facilitator asked many questions to help participants see connections between and among ideas and to promote precise mathematical description, there was no meta-discussion about these moves. Perhaps discussion should have included more insight into the reasons for asking connection-making questions. Further, more opportunity for reflection or concept mapping may have helped fit the knowledge pieces together more. Such metacognitive moves may be necessary for creating a coherent body of understandings.

In short, further research needs to be conducted to understand the relationship between professional development and teachers' knowledge. By understanding how professional development supports or fails to support the development of connected understandings, we situate ourselves to be more successful in supporting teachers in the development.

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STORIES OF MATHEMATICAL PROBLEM SOLVING IN PROFESSIONAL LEARNING COMMUNITIES

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Mathematics teacher educators often engage teachers in solving and discussing mathematics problems relevant to the school curriculum. This paper reports on elementary teachers' discourse as they shared solutions to a mathematics problem in professional learning communities. Teachers employed narrative structures as they shared their mathematical work. A structural analysis of the narratives suggests questions, insights, and conjectures about how teachers learn mathematical knowledge for teaching in and from such discussions. Since narrating is a natural way for teachers to discuss mathematical experiences, it makes sense to understand how narrative functions in such contexts to open opportunities for learning. Implications for future research are suggested.

Keywords: Mathematical Knowledge for Teaching; Teacher Education–Inservice/Professional Development; Teacher Knowledge

Purposes and Perspectives

Mathematics professional developers have employed professional learning communities (PLCs) to embody new understandings of cognition and foster changes in practice (e.g., Arbaugh, 2003; Crespo, 2006). However, there are few studies of teachers' discourse practices in PLCs. Researchers have begun to study important elements of discourse in PLCs, such as how teachers take up ideas from literature (Herbel-Eisenmann et al., 2008), render stories of classrooms (e.g., Crespo, 2006; Horn, 2005), and challenge one another's ideas (e.g., Males et al., 2010). Such work focuses primarily on how teachers talk about pedagogy and teaching practice.

Engaging teachers in solving and discussing math problems is also part of many PDs (e.g., Crespo, 2006; Schifter, 1998). Such work seeks to move teachers toward the deep, flexible knowledge of mathematics that is essential for effective teaching (Ball, Thames, & Phelps, 2008). Experienced teachers are uniquely poised to work toward nuanced understandings of relevant mathematics (Feimen-Nemser, 2001); however, little is known about how teachers engage in discussing mathematics. Teachers in one study talked about mathematics in more tentative, exploratory ways than they talked about teaching (Crespo, 2006). This is important because common challenges with PLCs include fostering inquiry and critique. Another study found that preservice teachers' discussions of mathematics problems influenced their conceptions of mathematics and themselves (Schifter, 1998). This is important because mathematics teaching is intertwined with identity (e.g., Drake, 2006). Understanding how discussions of mathematics problems create opportunities to learn is informative to the design and facilitation PLCs.

This paper reports on data collected when teachers in eight PLCs were sharing solutions to estimation problems. Estimation is crucial for academic mathematics and everyday situations; many contexts constrain the possibility of an exact answer (e.g., measurements are never exact) or do not require exactness (Bell & Bell, 2002; Usiskin, 1986). Estimation is often uncomfortable for learners, who may prefer certainty. It was historically taught as discrete procedures (primarily rounding) with correct answers (Reys, 1986; Trafton, 1986). However, other strategies are often more efficient than rounding, including front-end (focusing on left-most digits), clustering (assigning a common value for several numbers that are close together), compatible numbers (using values that add to a nice number like 10 or 100), and special numbers (rounding to numbers that are easy to compute) (Reys, 1986). Estimating encourages number sense, mental flexibility, and confidence (Bell & Bell, 2002).

The purpose of the analysis was to point to places in teachers' discourse about mathematics problems that potentially inform our understanding of *how* and *why* teachers learn mathematics knowledge particular to the practice of teaching when solving problems and sharing their solutions.

Methods

This paper reports on part of a design research project (Cobb, Zhao, & Dean, 2009) on elementary teacher PD that seeks to foster teachers' mathematical knowledge for teaching and skill with teaching mathematics and mentoring novices (Moss, Boerst, & Ball, 2010). Eight PLCs (five to ten teachers each) meet bi-monthly to investigate records of practice as well as solve and discuss mathematics problems. The purpose of the PD is to engage teachers in collaborative assessment of their own and others' teaching to develop a shared language. Four facilitators with experience as successful elementary teachers, teacher educators, and field instructors lead the groups. The curriculum consists of semester-long units, each focused around a high-leverage content area and a high-leverage teaching practice. Facilitators video- and/or audio-record all sessions and collect teachers' written work.

I analyzed excerpts from a session on computation that included discussion of estimation problems. Teachers solved five problems then shared their work. I report only on discussion of estimating the sum of the following numbers: 392, 1547, 84, and 3198. Facilitators chose the problem to engage teachers in discussing multiple estimation strategies, context, and place value.

I transcribed the segments of the recordings where the teachers discussed the estimation problems. I noted the estimation strategies teachers named (e.g., rounding). I noticed that when teachers shared their work they always employed a basic narrative structure. Therefore, I re-examined the transcripts to identify *estimation stories*, which are the focus of this paper.

Once I had identified estimation stories, I conducted a structural analysis of each one (Riessman, 1993), adapting Labov's (1972) narrative structures. In this paper, I focus on the complicating action and evaluations in the stories. The complicating action of a story—its action or sequence of events—is the only necessary story structure. Evaluation serves as justification for telling the whole story or for elements of the story. It may be present as a separate evaluation section or embedded in other narrative structures.

Results

A diverse array of narratives emerged as the teachers shared their strategies. A few narratives included only complicating actions, as seen in Molly's story. (The stories that follow indicate the complicating action with *italics* and evaluations with **bold**):

> *All I did was round it to either a ten, ten hundreds or a thousands place*. *And then add quickly*. Yeah.

Others were more complex, intertwining evaluation with various other structures, as seen in Estelle's story:

> ***Three hundred and ninety two is closer to four hundred*** *(implying that she rounded to four hundred)*, **even though I could've gone three ninety**. *And then* ***one thousand five hundred and forty-seven, forty-seven is less than fifty*** *so I just kept it at fifteen hundred*. *And eighty-four,* ***I don't know why****, I went to a hundred,* ***'cause it is over fifty in the tens place*** *(laughs)*. **I guess that's why I did it instead of going, yeah. I screwed up.** But any way. *Three thousand one hundred and ninety eight, I went to thirty two hundred,* ***because a hundred and ninety eight is closer to two hundred than it is to thirty one hundred****.* And I came up with fifty one hundred, **if I added that correctly, which I don't even know right now.** So that's what I did.

Estelle embedded evaluation into the complicating action and provided additional evaluation statements. She started by evaluating the number three hundred and ninety two in relationship to four hundred, implying a justification of her decision to round to four hundred, thus embedding evaluation into the complicating action. However, she then suspended the story's action and further evaluated that decision, saying, "I could've gone three ninety."

There were only a few estimation stories that, like Molly's, did not originally include the evaluation structure. In most of these cases, when the facilitator prompted for more information, the storyteller provided evaluation. For example, Molly told the above story and then paused to indicate that the story was complete. Then the facilitator stepped in:

Facilitator: And so, can you talk a little bit about that, what made you decide, in terms of place value, what were you looking at?
Molly: **In third grade I was taught this and so** (laughs)=
Teachers: (laughing, talking over one another)
Molly: =**I went with what I knew. Place value**. *I just looked to the right of the number, the largest number. I went to the right to see if it was five or above. Flip it up, or down, or stay the same.*

Teachers often added evaluation to narratives (both those that were originally evaluated or unevaluated) after being prompted by the facilitator or other teachers, even when the prompt did not ask for an evaluation. Consider the interaction around Jeni's estimation story:

Jeni: *I rounded to the hundreds with every one except the last one* ***because I wanted to leave fifty****. I just went four plus fifteen is nineteen. Plus another one, twenty. Plus thirty* is fifty hundreds. (Pauses, as if she is finished with her story.)
Facilitator: How did you, I didn't catch that.
Jeni: *I rounded to the hundreds with everyone except four, the last one, I just took the hundreds.*
Facilitator: You took the hundreds.
Jeni: Yeah, well, **three ninety two was closer to four hundred**.

In this interaction, facilitator asked Jeni to repeat, then echoed Jeni. However, Jeni responded by providing evaluation.

I noted a number of thematic and topical patterns in the narrative structures as well. For example, in the evaluation structures of the narratives, the most common themes were feelings or aesthetics (e.g., "It just looks a little bit better"), the ease or difficulty of a procedure (e.g., "I was just trying to think about making it easier to do in my head"), judgments about numbers (e.g., "a hundred and ninety eight is closer to two hundred"), recollections of social interactions (e.g., "I heard you say 'estimate'"), and evaluations of the quality of decisions (e.g., "I screwed up").

Discussion

Narrating is a sense-making activity and means for consolidating and transforming learning (Goodson & Gill, 2011; Ochs & Capps, 2001). Therefore, it makes sense that the teachers narrated their work; they were not simply communicating their work to others, but organizing it to make sense of the mathematics and themselves.

A task that is common to all storytellers is that of justifying and convincing, and the disposition to do those things is one that is also important to mathematical practice. Although elegance and economy of syntax are often privileged in the communication of mathematical solutions, more complex narratives make teachers' thinking more visible and therefore open to probing by the facilitator and other teachers. While previous work found that more robust and reflective narratives provide the most opportunities for the storyteller's learning (Goodson & Gill, 2011), it is interesting to think about the tension between the discursive norms for mathematical justification and narrative. This tension has implications regarding the interplay between teachers' ways of knowing and communicating mathematical knowledge for teaching.

This preliminary theoretical and analytical work has only investigated the first session in the project where teachers solved problems and discussed solutions. Further questions remain, including whether teachers tell increasingly robust narratives over time and what specific facilitator moves impact teachers' narratives of their mathematical work.

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PERIODIC MOTION, TECHNOLOGY, FICTIVE MOTION: DEVELOPING APPLETS FOR TEACHING

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This paper discusses the role of technology and embodiment theory for the teaching of graphing trigonometric functions to promote students' understanding of time as a component axis in Cartesian graphs for periodic movements. The embodiment theory will be discussed not only as our framework but also as a possibility in helping teachers to better understand students' thinking about. This is an ongoing research that is occurring in Brazil. We follow a design based research methodology including three cycles, the first one, presented here, shows the development of applets taking in account the notion of fictive motion. We conclude arguing that technology may transform a fictive motion in factive one.

Keywords: High School Education; Technology; Embodiment Theory; Fictive Motion

Introduction and Problem

The purpose of this study is to *discuss the role of technology and embodiment theory for the teaching of graphing trigonometric functions to promote students' understanding of time as a component axis in Cartesian graphs for periodic movements.*

Research on that theme revealed that there are many different problems regarding understanding functions and graphs of trigonometric functions, even among teachers (Quintaneiro, 2010; Weber, 2005; Kendal & Stacey, 1997; Costa, 1997). Weber (2005) findings indicate that the students who were taught in the lecture-based course developed a very limited understanding of these functions, while students who received an instruction based on the process-object perspective developed a deeper understanding. Quintaneiro (2010) investigated the appropriation of formal definition of sine and the sine function among teachers, observing that offering different ways of representing these notions were crucial to enable students' to access formal definitions.

We start this study investigating teachers' meaning production for graphs involving periodic phenomena. The investigation took place in Brazil with 10 mathematics teachers. We began posing the following problem:

> *Sketch a Cartesian graph showing distance x time of a person seated in a capsule/chair of a Ferris Wheel. Knowing that the person seats for 4 turns of the wheel.*

Two responses called our attention because their analysis let us raise a few hypotheses about how teachers thought about trigonometric functions.

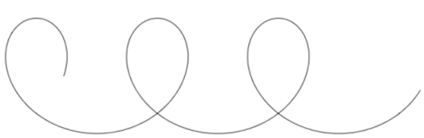

Figure 1

Figure 2

Studies such as, Quintaneiro (2010) and Weber (2005), as well as other research on the teaching or learning trigonometry, do not provide interpretations for these answers. In general, these studies focus on understanding of mathematical objects, by their formal definitions. Moreover, for us it seems that researchers, in this area, often take more into account what is missing in students' or/and teachers' answers and less what they are actually saying, talking or drawing.

Theoretical perspectives lead us to different interpretation. In the next section we briefly present our theoretical lens. After that we present our analysis of these responses, based on the notion of fictive motion. And as a conjecture in this cycle of the larger research, we will present, computer applications, applets that may help in promoting a deep reflection and understanding about periodic phenomena and their graphics.

Theoretical Perspective

We share a premise with others that work with embodiment theory that mind and body are not separable as proposed by Descartes' work. According to Damasio (1996) a human being interacts with the environment as a whole, the interaction is not exclusively of the body or of the mind. Edwards (2011) points out that "embodiment theory offers an answer to the question of how meaning arises and of how thought is related to action, emotion and perception. Embodiment theory proposes that meaning and cognition are deeply rooted in physical, embodied existence" (p. 1). Since we are interested in meaning production for periodic phenomena we assume this theory. Lakoff and Johnson (1999) stated that our conceptual system is directly related to the way we think and act.

For mathematicians a concept is a definition, but to someone who is learning mathematics a concept is something rather different. We adopted Rosch (1999) ideas that concepts are one aspect of the study of categorization, one of the basic functions of living beings, "concepts are open systems by which humans can learn and invent new things" (p. 61).

Due to constraints of space we do not present works on digital representations, to name a few Luis Radford that looks at graphs as semiotics constructions, the use of graphic calculators and sensors as proposed by James Kaput and Roschelle, and Ricardo Nemirovsky.

Ramscar, Matlock, and Boroditsky (2010) argue that more abstract domains can be understood through analogical extensions of domains based on experience. They describe a series of studies that indicate the following: (a) people's understanding about the time comes to thinking about concrete experiences of movement in space, (b) the way people think about the time can also be influenced by a nonliteral type of motion called fictive motion.

We agree with the fact that engaging in reflection about movement is crucial to change the way we think about the time. Concerning the problem of our research, we anticipate that this point seems to be fundamentally important, once we intend to engage study participants in thinking about movements determined by computing experiences that are designed to allow "visualization" of movement through time.

Matlock (2004) indicates that the presence of an actual motion isn't necessary to process such movement, and concludes that the processing of the fictive motion includes mental simulation of movement. Barsalou (2009) indicates that simulation is typically located on the experience and that we conceptualize insight gained in frequent experience with the world body and mind, as captured by multimodal patterns,

Relationship Between Fictive Motion and Charts in Periodical Phenomena

We conjecture that the ideas associated with understanding of graphs, relating to the movements of periodic phenomena, may be related to how we understand the "passage of time" in these movements. We also agree with other works such as Nunez (1999) and Font, Bolite, and Acevedo (2010) that the understanding of time flow utilizes ideas and inferences grounding in experience with the physical world, and, even unconsciously, associated with a horizontal translation of a self or a thing.

On the responses (Figures 1 and 2) the problem of the wheel we found different ways of meaning production to the required graphing in a time vs. distance representation; mainly approaching the graph as if it were only the trajectory. Moreover, we believe this response is related to a cognitive simulation of motion (in the sense the Matlock 2004), when used in a horizontal translation of the replacement time, because the graph was obtained as if it were a "path of bicycle tire" (see Figure 3). The graphs in Figures 3 and 4 show the route given by a particle that is a uniform circular motion, where the center of the circular motion is in a uniform rectilinear motion.

Considering the bold circle rolling on a plane without slipping, then the curve described by a point on this circle is called a cycloid. In this case the rose curve is long cycloid. What we want highlight is that this curve is obtained by the trajectory of a point that undergoes a translation composed with a rotation. In Figure 4, the red curve represents the path of the red chair, and the difference is that in the case of Ferris wheel the rotation can occur in the counterclockwise.

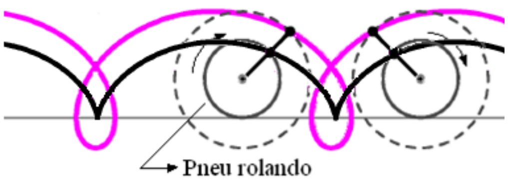

Figure 3

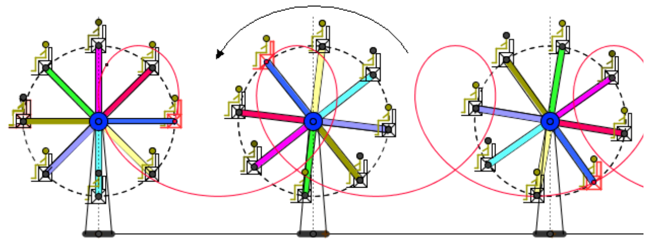

Figure 4

Taking into account the need for a horizontal translation to obtain this type of curve, we conjecture that it is necessary for a teacher/student to make a cognitive simulation, i.e., a person should simulate a horizontal movement of the wheel instead of looking for time flowing, although this is a fictive motion because there is no such movement. This fact fortifies our conjecture that there is a relationship between fictive motion and graphic involving periodic phenomena. Thus, converge with the ideas of Ramscar brought in the previous section, that people can recruit concepts acquired experience of the physical world, horizontal displacement, to give sense of more abstract concepts like time.

Given our theoretical perspective, thinking about different curves and different situations may further the development of different elements for categorization and, consequently, may bring life to a richer set of experiences for those immersed in these type of problems. We then start developing applets, thinking in helping physical and mental simulations of these periodic phenomena.

Developing Applets

Based on our framework, in order to be able to analyze the impact of using these applets we also had to create possibilities for exploration that promoted discussion, gestures, verbalization and sketches. Some questions included arguing about fictive motion, a movement that at least until it was materialized, were not physically there. We developed two applets[1] that may enable the animation of fictive motion "Ferris Wheel" and "Piston," which are still being improved. One can see in the applet "Ferris Wheel," for example, a Ferris wheel, as in Figure 4, moving (horizontal translation) in the time axis. Our hypothesis is that with this translation the technology can transform a fictive motion in a factive one. This fact enables discussion and reflection about cognitive simulation that could be difficult to conceive without technology. These facts characterize these applets as simulators according to Barsalou (2009).

Final Considerations

For us, there is a strong connection between the role of technology and embodiment theory for the teaching of graphing trigonometric functions to promote students' understanding of time as a component axis in Cartesian graphs for periodic movements. Due to space constrains we did not present the different responses that led to rich discussions and the development of other applets. We will bring that for presentation.

We emphasized the problematic of our investigation as well as possible discussions on periodic phenomena, involving the idea of fictive motion and the use of computational resources. It is worth noting how our theoretical perspective supported the considerations that led to the development of applets. The features we have chosen try to explore more about our experiences in the world of motion, namely our repertoire of movements and self movement, the applets do not only display static figures, but took advantage of dynamics representations.

Finally, we observed that this presentation reveals part of a larger research, and the applets are being used with graphic calculators and sensors to promote, students and teachers, to build a repertoire of experiences that may function to further conceptualizations based on situations of periodic phenomena.

Endnote

[1] Available at the following URLs:
https://sites.google.com/site/wellersonquintaneiro/applet
https://sites.google.com/site/wellersonquintaneiro/applet/trigonometria/roda-gigante/roda-gigante-c
https://sites.google.com/site/wellersonquintaneiro/pistao/pistao-c

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A CASE FOR CHINESE LESSON STUDY

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The predominant version of lesson study described in the literature and attempted by U.S. educators is still the Japanese form. Chinese lesson study may be a softer introduction—or even a possible substitute—for U.S. teachers challenged by the cultural and logistical demands of sustained Japanese lesson study.

Keywords: Chinese Lesson Study; Japanese Lesson Study; Professional Development; Reform

The purpose of this article is to describe Chinese lesson study and advocate for its use. The Japanese form is the most common type attempted in the U.S.; unfortunately, current U.S. school climates are generally not conducive to adopting this more rigorous and demanding form of lesson study. Chinese lesson study (Fraser, Allison, Coombes, Case, & Linder, 2006) could be a happy medium for teachers interested in grass-root, practice-based professional development that do not have access to the resources necessary to enact Japanese lesson study. After briefly describing the Japanese and Chinese forms, I argue that Chinese lesson study: (1) will make an excellent form of professional development in and of itself, and (2) may also provide crucial training for those teachers wanting to eventually attempt the more demanding Japanese form.

Japanese Lesson Study

Japanese lesson study has been proposed as a model of effective professional development that could be imported into U.S. schools. Japanese lesson study involves the collaborative planning by teachers of a lesson designed to address specific student-centered goals The lesson is then taught in an actual classroom, with close observation by the other lesson study teachers to verify the effectiveness of the designed lesson in reaching the lesson study goals (Lewis, 2000; Lewis & Tschudia, 1998). Japanese lesson study is an experiment in practice that turns teachers into researchers-in-action. Stigler and Hiebert (1999), the ones who introduced the U.S. to Japanese lesson study, described its 8 sequential parts: (1) defining the problem, (2) planning the lesson, (3) teaching the lesson with observation and debriefing, (4) evaluating the lesson, (5) revising the lesson, (6) teaching the revised lesson to another class, (7) further evaluation and reflection with other teachers, and (8) sharing the results.

Despite being a powerful form of professional development, Japanese lesson study clashes with the current U.S. school climate of (1) time constraints (Darling-Hammond et al., 2009; Wei et al., 2009), (2) teachers as consumers not generators of professional knowledge (Fernandez, Cannon, & Chokshi, 2003; Hiebert, Gallimore, & Stigler, 2004), and (3) the "persistence of privacy" (Little, 1990). Teachers' isolation, often reinforced by entrenching school cultures and norms, is damaging to the profession of teaching (Ball, Lubienski, & Mewborn, 2001) and prevents teachers from becoming reflective practioners (Schön, 1983). Overcoming this isolation culture is very difficult (Grossman, Wineberg, & Woolworth, 2001).

Chinese Lesson Study

Chinese lesson study is another form of lesson study that may provide a solution for U.S. educators wishing to adopt lesson study. Like Japanese lesson study, the Chinese versions respect the central role that researching an actual lesson plays in teachers' professional development. Chinese lesson study is not yet well-known to the U.S. professional development field; the reason for this may be that mainland China has only recently emerged from self-imposed isolation during the Cold War—now, as China embraces the global village, U.S. educators have been able to visit and document Chinese versions of lesson study.

There are two principal forms of Chinese lesson study: (a) the model lesson, and (b) exemplary lesson development.

The first version of Chinese lesson study is when a school or district invites a regionally recognized "master teacher" to teach a mathematics lesson to a random class of students. During the lesson, usually held in an auditorium, the school or district teachers sit and watch the "master teacher" teach the class of students, whom she or he has never interacted with prior to this lesson. As a side note, this practice of teaching on-the-spot an unfamiliar class of students challenges basic U.S. assumptions about teachers needing to "know" their students before being able to effectively teach them. This "fishbowl" arrangement of observing teachers looking in on an actual lesson provides a chance for practicing teachers to: (1) watch a "master" at work, (2) closely observe students, (3) talk quietly with colleagues about issues that arise during the lesson, and (4) witness excellent instruction—See! It is possible to do it! These are not activities afforded to teachers during the act of teaching. Teachers have a chance to step back and reflect on this model lesson from a vantage point not possible during the rapid decision-making of instruction. After the model lesson is concluded, the students are dismissed, and the master teacher has a chance to debrief with the observing teachers. In a way, the observing teachers have become students at the feet of a master.

Exemplary lesson development. The second version of Chinese lesson study is exemplary lesson development—sometimes called "keli": A teacher will prepare his or her best lesson on a particular topic, and teach this lesson to his or her own class, with invited colleagues watching (Huang & Bao, 2006). After the lesson the students are dismissed, and all of the teachers engage in a debriefing discussion. The presenting teacher has an opportunity to talk about the lesson, especially the reasoning behind its design. Afterward, other teachers can make comments, critique, or ask questions of either the teacher or other participants. Often, invited district or regional observers will offer their advice and insight. The debriefing meeting ends with the teachers breaking out into small groups to further discuss the lesson in detail. The presenting teacher then continues to refine the lesson to make it even better, and will eventually teach it again with invited colleagues observing. This cyclic process can be repeated again, with the goal of developing an exemplary lesson that can be shared in a report detailing the research process and product (Huang & Li, 2009).

Keli occurs in the context of a group of teachers and outside educators (school, district, or regional levels) or researchers from nearby universities that form a research group. They identify a central goal for improvement related to the national standards and then move through three phases of (1) demonstrating existing practice by teaching a lesson; (2) reflecting, revising, and re-teaching the updated lesson; and (3) further reflecting, revising, re-teaching, and disseminating their lesson (Huang & Li, 2009). The lesson can either be individually or collectively planned. But the activity of reflectively revising through systematic practice is closely connected to having recognized excellent teachers participate in this process.

Chinese lesson study occurs in the well-established framework of Chinese educational professional development, which includes school and city-based research activities, regional and national teaching competitions, and a rigorous promotional system that identifies these excellent teachers and encourages them to work with less-advanced colleagues (Huang & Li, 2009). The Chinese highly value the role these excellent teachers play in developing knowledge and guiding the research groups. Although knowledgeable others do play a role in Japanese lesson study, the Chinese version emphasizes the role these "experts" play in working closely with groups of teacher who are actively studying how to be better teachers in the context of their own practice (Fernandez, Cannon, & Chokshi, 2003; Huang & Li, 2009).

Both the Chinese and Japanese versions of lesson study share common features that make lesson study such a powerful professional development activity. They both enable the teacher to be a researcher, essential to improving one's practice (Stigler & Hiebert, 1999); they both allow for structured collaboration; they both focus on issues of practice, principally lessons; they both aim at student learning both in the goal orientations and in the collection of measures of student learning demonstrate that evidence is a central measure of the success of lesson study.

Despite their similarities, I believe that Chinese lesson study has a greater chance of being initially adopted by U.S. teachers than the more demanding Japanese form. In particular, Chinese lesson study places great emphasis on experts participating as co-partners in the research process; experts would fit better with current U.S. professional development climates where teachers are not yet used to taking the initiative and being researchers of their own practice. Additionally, the flexibility of the Chinese version means that teachers new to lesson study can adopt the less rigorous types. Attendance at model lessons could be a great way to encourage even the most resistant teachers to consider researching practice. Watching a master teacher successfully teach a random class of students demonstrates that reform teaching is possible; further, deprivitization of practice has begun. As teachers become more accustomed to observing, critiquing, and reflecting on others' practice, they can be encouraged to open their own lessons up for peer scrutiny. This can be done in ways that the teachers feel comfortable, perhaps with only a small group of departmental colleagues (or even just one!). When prepared, a revised lesson can be taught again to another class. Over time, this continual attempt to research one's practice will provide tangible rewards and a sense of confidence in one's teaching.

This article has proposed Chinese lesson study as a softer introduction—or perhaps even a simpler alternative—for schools overwhelmed by the demands of sustained Japanese lesson study. Chinese lesson study offers an intriguing solution; model lessons by recognized expert (or veteran) teachers can begin to deprivatize mathematics teaching practice (Little, 1990). As teachers grow comfortable observing, discussing, critiquing, and reflecting on others' practice, they can begin to participate with experienced colleagues in small study groups to develop their *own* exemplary lessons. As their confidence grows, they can invite other teachers and administrators, even parents, university personnel, and state officials, to observe their successful—and polished—lessons. Additional teachers may see the fruits of such collaboration and wish to join in on the small teacher study groups. In this way, some of the barriers to implementing practice-based research can be overcome, opening up the possibility for more intense Japanese lesson study for the daring. Or teachers may wish to stay with Chinese lesson study; either way, teachers would begin studying localized teaching in an attempt to improve their students' learning.

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INTERDISCIPLINARY ALGEBRA CURRICULUM MODEL

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This presentation proposes a model for designing interdisciplinary curriculum to supplement College Algebra courses. Interdisciplinary curriculum is often implemented in a team-taught format—effective but expensive and complex to implement, limiting the number of students who experience interdisciplinary challenges. This model makes content and learning goals drawn from a partner discipline explicit and limited, so that a mathematics teacher is responsible for a manageable amount of material in class. A sample curriculum module that unites financial mathematics with psychologist Erik Erikson's theory of life stage development is discussed. Reflection on the teaching experiment and on student reactions is guided by research in interdisciplinary curriculum design, mathematical modeling, and universal instructional design.

Keywords: Post-Secondary Education; Curriculum; Modeling

Interdisciplinary education is a high-impact practice for early undergraduates, but implementing it can be a daunting prospect. Interdisciplinary education often requires significant organizational changes such as learning communities, team teaching or longitudinal programs that allow for strong disciplinary grounding and gradual development of integrative skills (Boix Mansilla, Miller, & Gardner, 2000; Lattuca, Voigt, & Fath, 2004; Wentworth & Davis, 2002). Full support for interdisciplinary education is complex, expensive, and it requires an enormous commitment from institutions and instructors.

The disciplinary insulation that makes this work challenging may be particularly strong in mathematics. Mathematics curriculum is often defined more rigidly than in other fields (Grossman & Stodolsky, 1995). Mathematics faculties often have limited professional interactions with faculty in other disciplines (Ewing, 1999). At times, even the philosophical basis of mathematics is held in contrast to the integrative goals of interdisciplinary education (McGivney-Burelle, McGivney, & Wilburne, 2008; Siskin, 2000). These curricular boundaries impede the adoption of deeply interdisciplinary curricula in mathematics classes (Staats, 2007). This presentation proposes a model for designing supplemental interdisciplinary curriculum that addresses the dilemmas of expense, disciplinary expertise, and organizational sustainability.

For the last several years, the author has experimented with interdisciplinary curriculum design in college algebra classes at the University of Minnesota. The outcome is a model for interdisciplinary algebra curriculum design that encompasses five components: (a) an introduction covering content from a partner discipline, (b) an essay written by a specialist in a partner discipline, (c) explicit learning goals for algebra and for content in the partner discipline, (d) discussion and homework questions, and (e) a bibliography of readings in the partner discipline. The five-part model responds to the needs of both instructors and students by conveying disciplinary expectations explicitly and efficiently. This curriculum is assigned as homework without direct lectures, and so can supplement any College Algebra class without displacing large amounts of existing course content.

The introduction presents content from the partner discipline to support both mathematics instructors and their students in integrating the discipline with mathematics. The central essay may be written in any genre. The one discussed in this presentation is a short story that was written to accompany the financial mathematics section of the algebra course. The story poses dilemmas about indebtedness from a financial and a psychological viewpoint. In general, posing interdisciplinary dilemmas through a wide variety of writing genres—fiction, memoir, poetry, expository writing—signals to students their responsibility work outside of the expectations of a single discipline as they develop their integrative solutions to the dilemmas. The introduction and the learning goals section help both instructors and students access information from another discipline. They both inform the instructor and limit the range of new information discussed in class. Discussion and homework sections include scaffolding questions in both disciplines, and interdisciplinary assignments. The bibliography is a support for instructors who wish to conduct further reading in the partner discipline.

Theoretical Foundations for Interdisciplinary Algebra Curriculum Design

This model is based on research in interdisciplinary curriculum design, in the pedagogy of mathematical modeling, and on principles of universal instructional design. Interdisciplinary learning involves creating an explanation or tool that could not be produced through the perspective of a single discipline (Boix Mansilla & Duraisingh, 2007; Klein, 1990; Newell, 2009). Interdisciplinarity may require choices among incompatible worldviews as well as a synthesis of them. Because interdisciplinary assignments require creative integration of ideas, they cannot be judged according to a fixed content. Evaluation may focus on students' analysis of multiple aspects of a problem, moving towards an integrated, generalized understanding (Biggs & Collis, 1982; Invantiskaya, Clark, Montgomery, & Primeau, 2002) or their rigorous grounding in disciplinary content and methods, whether they produce an integrated understanding of the scenario, and the level of student reflection on purposes and limitations of their work (Boix Mansilla & Duraisingh, 2007).

Two of these aspects of interdisciplinary learning—a process orientation to learning and critical reflection—are also important in the pedagogy of mathematics modeling. Through "Model Eliciting Activities," students learn about the broad context of a realistic scenario, then they develop mathematical problem-solving tools, and finally, they reflect on their method, improve and generalize it (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003; Maaß, 2006). This project's goals rely heavily on the correspondences between interdisciplinary education and mathematical modeling: students create integrative tools and critically reflect on their solutions in order to identify weaknesses and to create a more generalized solution or perspective. The design model aims to develop strong novice-level interdisciplinary analytical skills.

Universal instructional design (UID) developed within the field of disability studies to encourage communities and institutions to develop infrastructure and information systems that are accessible to people with disabilities. UID broadened to include the development of curriculum and pedagogies that improve access for people who experience barriers to educational opportunities (McGuire, 2011). The difficulties of implementing authentic interdisciplinary curriculum in a mathematics class suggest that the concept of UID should be broadened again to include the instructor as well. Interdisciplinary instruction can pose an educational barrier for an instructor just as a weakly designed class poses an educational barrier for a student. The mathematics instructor needs to have access to information in partner disciplines that is substantial, reliable, and easily understood. The proposed model reduces knowledge barriers for faculty by making the content and their own responsibilities for knowledge limited and explicit, particularly through the learning goals and the introduction, as recommended by UID.

Indebted: Exploring the Emotional Side of Financial Mathematics

For the last several years, the author has used a short story in class that addresses family financial decision-making written by Gary Peter titled *Indebted*. Peter wrote the short story to accompany the material on financial mathematics that is covered in the algebra class, with the goal of complicating the typical word problems of the textbook. A typical word problem asks:

> A couple wishes to save money for their daughter's college education. They save $500 per quarter in an annuity that pays 3.5% interest for 18 years. How much money have they saved after 18 years?

The story *Indebted* provides an example of a more realistic scenario. The narrator in the story is a young man visiting his grandfather, who suffers from Alzheimer's disease and who lives in a nursing home. The grandfather is distressed because his life's savings are expended on his health care, instead of contributing to his grandson's education as he had hoped. Finally, the young man uses a finely crafted writing pen—a graduation gift from his grandfather—to sign his college loan papers. The story hints at mathematical questions that might arise through the young man's considerations: the question of whether working and saving money is a better option in view of rapidly rising college tuition; and the per capita value of the national debt.

The introduction to the story frames these dilemmas in terms of content covered in many introductory psychology textbooks, Erik Erikson's model of life stage development. This disciplinary context was chosen because it presents a challenge—we may learn more about interdisciplinary teaching and learning by integrating topics that are usually very distinct. In addition, Erickson's theory of identity crises that emerge at distinct points in the life span meshes well with Peter's story of intergenerational relationships. Students complete this module with writing short stories or essays that describe the emotional dimension of financial decision making integrated with calculations of the financial outcomes of these decisions.

Uniting the research on interdisciplinary learning and on mathematical modeling suggests two major themes for assessing students' interdisciplinary algebra solutions: integration of disciplinary content and critical reflection. In the *Indebted* module, for example, students showed knowledge of financial mathematics by describing life trajectories accompanied by calculations of the results of financial decisions. They demonstrated integration of psychological ideas by analyzing these decisions in terms of Erikson's model of life stage development. Students demonstrated critical reflection when they considered the possibility of different outcomes or different interpretations than the ones that they elaborated in their writing.

In a recent small class of algebra students, a preliminary analysis of student writing suggests that students tended to perform disciplinary integration more easily than critical reflection. Students were able to construct stories and essays that posed humanistic dilemmas that effect financial decision-making and most were able to use financial mathematics equations to describe the financial outcomes of these scenarios. Students usually presented a single calculation or a comparison of calculations within their stories. While all students were able to integrate ideas from algebra and psychology, relatively few students reflected on interpretations or scenarios that opposed their interpretation. It is possible that the integration of two disciplines was in itself cognitively and creatively challenging, and that students would have required a more directive assignment to reach the level of critical reflection on their work.

Conclusion

This presentation outlines an approach to interdisciplinary curriculum design that may enable mathematics instructors to supplement a standard college algebra class with novice-level, but authentic interdisciplinary activities. While team-taught classes and linking classes into learning communities are superior ways of promoting interdisciplinary thinking, their expense and complexity preclude the involvement of many students. If future versions of this project prove successful, the curriculum design model will support a single instructor delivery format that increases the number of students who can experience the challenges of interdisciplinary learning.

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EPISTEMIC NETWORK ANALYSIS FOR EXPLORING CONNECTEDNESS IN TEACHER KNOWLEDGE

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While researchers agree that teacher knowledge matters, the nature of the relationship between teacher knowledge and practice remains elusive. To better understand what knowledge matters, researchers have tried to characterize the special knowledge teachers need (e.g., Ball, Thames, & Phelps, 2008). These efforts have made strides in identifying subconstructs of teacher knowledge. Baumert and colleagues (2010) have linked specialized teacher knowledge to student growth (Hill, Rowan, & Ball, 2005). To better understand how to support students' learning, we need to better understand how teacher understanding matters. To this end, we are considering how teacher knowledge is organized rather than simply discussing quantity of knowledge. This builds from prior research on expertise (e.g., Bransford, Brown, & Cocking, 2000) and knowledge in pieces (diSessa, 2006). This poster will present findings of an approach for looking at teacher's coherence, rather than amount, of knowledge.

To examine coherence of knowledge, we rely on Epistemic Network Analysis (ENA; Shaffer et al., 2009). This mixed methods approach is based on social network analysis. It allows us to identify and code for the key understandings a teacher needs to teach a particular topic. Then, through statistical analysis, patterns in the co-occurrences of those understandings emerge, thus highlighting the aspects of teachers' knowledge that are invoked for particular problem situations.

For this pilot effort, we drew data from clinical interviews of three teachers focused on aspects of proportional reasoning that intersect with fraction knowledge (e.g., if there is a ratio of 2:5, is there 2/5 of something in that situation?). Our analysis indicated that ENA was able to differentiate between the teachers in terms of the connections between their understandings. The poster will feature visual mappings of the teachers' understandings from the clinical interviews.

Acknowledgments

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THE IMPACT OF DATA-DRIVEN DECISION MAKING ON MATHEMATICS INSTRUCTION: OPPORTUNITIES FOR PROFESSIONAL LEARNING

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In many districts, the use of student data to inform decision-making is a key practice at virtually all levels. These efforts are premised on the notion that armed with the right data soundly analyzed stakeholders can make well-informed policy and instructional decisions that improve student performance. This study takes up this notion, examining how stakeholders in a large, diverse district in the mid-Atlantic region—including teachers, school leadership and district personnel—use data to make decisions that impact middle school mathematics instruction and to address pressures to improve middle school mathematics achievement. The study also examines how teachers' data use supports or constrains their professional learning.

A growing body of literature has identified factors that shape data-driven educational decision making, such as the questions asked, forms of data, technologies for presenting and analyzing data, and school and district conditions (e.g., Coburn & Talbert, 2006; Ikemoto & Marsh, 2007). This literature, however, has paid little attention to how data collection, analysis and interpretation shape planning, teaching and assessment in subject matter specific ways. In addition, research has not examined how data use can provide teachers with opportunities for professional learning. If the aim of data use is improving student outcomes, then it is reasonable to examine how teachers may be developing understandings and practices specific to mathematics through data use that improve mathematics learning.

Primary data include interviews with stakeholders in mathematics in four middle schools and the district office and video recordings of department and school meetings involving mathematics instruction relevant data. The selected schools contrast along several dimensions, including demographics, adequate yearly progress, and openness to examine data use. Analysis utilizes a grounded theory approach to identify themes within and across the cases and discourse analytic methods to examine opportunities for learning about mathematics teaching and learning in data meetings through teachers' problematizing of their practice (Horn & Little, 2010).

The poster reports on opportunities for teacher learning in "data chats," a type of district mandated teacher meeting typically involving student assessment data (e.g., formative, unit summative and state standardized tests). One case highlights how organizational structures and norms support meaningful use of data for collaboratively reflecting on student learning and instructional practices. Another case illustrates how, when a school is under severe pressure to raise student outcomes, mandated collection and reporting procedures can undermine the professional development goals of data chats, resulting instead in a focus on identifying and labeling students. These and other cases suggest the potential of data use for mathematics teacher learning and the conditions under which learning is cultivated or stymied.

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LEARNING TO TEACH MATH WITH TECHNOLOGY: COLLABORATION BETWEEN MIDDLE SCHOOL MATHEMATICS TEACHERS AND LIBRARIANS

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This study examines the conditions that shape how mathematics teachers and school librarians collaborate in the use of technologies to support student mathematics engagement and learning in four low performing, diverse middle schools and how those collaborations may promote teachers' technological pedagogical content knowledge (TPACK). Research on technology use in mathematics instruction suggests many barriers to technology integration, including lack of access to technologies, inadequate professional development, inexperience with technology, and skepticism about relevance to mathematics (e.g., Guerrero, Walker, & Dugdale, 2004; Wallace, 2004). Furthermore, recent conceptualizations of teacher knowledge, particularly TPACK (Mishra & Koehler, 2006), underscore the complexity of the knowledge needed for thoughtful pedagogy with technology and the challenges of supporting technology integration.

Little research, however, has attended to resources that may already be available in schools, particularly school librarians. Research on librarians suggests that they may be key change agents in technology integration because they are leaders of school media programs, well-versed in newer technologies (e.g., social media, Web 2.0) as well as more traditional technologies, and positioned to spread innovations via their collaborations with teachers (Subramaniam et al., 2012). Also, since research suggests that teachers adopt technology more effectively from collaborating with colleagues, librarians are well suited to be partners in taking up pedagogical innovations. However, the potential of librarians to facilitate the technology integration in math instruction and support TPACK development has not been explored.

We conducted semi-structured interviews with mathematics teachers, librarians, school and district leadership. Utilizing a grounded theory approach, our analysis reveals that while librarians seek deeper collaborations with mathematics teachers, their ability to work with them is limited to isolated events. Math teachers seek guidance and expertise about effectively integrating technologies but not at the perceived expense of instructional time or test preparation. We find several factors influence the nature of collaboration and opportunities for TPACK development, including: institutional structures (e.g., administrative support, scheduling); access to technologies; professional roles; testing pressure; and beliefs and knowledge about mathematics teaching and learning. This work informs professional development for technology integration in mathematics instruction, particularly regarding collaborative teaching between teachers and librarians, as well as school leadership's understanding of how their decisions affect opportunities for professional learning.

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PROBLEM SOLVING ABILITIES AND PERCEPTIONS IN ALTERNATIVE CERTIFICATION MATHEMATICS TEACHERS

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The purpose of this study was to understand alternative certification middle and high school teachers' mathematical problem solving abilities and perceptions. Participants were given a problem solving examination and required to reflect upon their students' and their own problem solving. Findings revealed there was a significant improvement in problem solving abilities for the teachers over the course of the semester. Teachers perceived their students' problem solving abilities as generally weak, but found they also shared some of the same weaknesses in problem solving as did their students.

Keywords: Problem Solving; Teacher Knowledge

Problem solving continues to be of high importance in mathematics education (NCTM, 2000). The objective of this study is to understand the problem solving abilities and perceptions of their own and their students' abilities among New York City Teaching Fellows (NYCTF).

Research Questions

1. What differences were there in problem solving abilities between the beginning and end of the semester in a mathematics content course for NYCTF teachers?
2. What were teacher perceptions of their students' problem solving abilities? Further, what differences in perceptions of student problem solving abilities existed between the beginning and end of the semester in a mathematics content course for NYCTF teachers?
3. What were teacher perceptions of their own problem solving abilities? Further, what differences in perceptions of their own problem solving abilities existed between the beginning and end of the semester in a mathematics content course for NYCTF teachers?

The sample in this study consisted of 34 new teachers in the NYCTF program. Teachers were given a problem solving examination at the beginning and end of the semester, and there was a statistically significant difference found between the pre- and post-tests. Additionally, teachers were required to reflect upon both their students' and their own problem solving at the beginning and end of the semester. It was found that teachers perceived their students' problem solving abilities as generally weak, but found they also shared some of the same weaknesses in problem solving as did their students.

Strong problem solving abilities and skills are essential not just in mathematics, but in other subject areas and life in general. It is important that teacher educators be aware of their pre- and in-service teachers' problem solving perceptions both for the students and the pre- and in-service teachers themselves.

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DEVELOPING TEACHERS' ALGEBRAIC REASONING SKILLS WITH SPREADSHEET TECHNOLOGY IN AN ONLINE EDUCATIONAL ENVIRONMENT

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Keywords: Teacher Education–Inservice/Professional Development; Technology

Framework

Online, technology focused educational experiences provide access to continuing professional education for an extended teacher population, some of whom have no other means of continuing their professional education (Shea & Bidjerano, 2009; Stein, Wanstreet, Glazer, Engle, Harris, Johnston, Simons, & Trinko, 2007). To explore the impact of situating a continuing education program focusing on integrating technology (spreadsheets) into teaching mathematics (algebraic reasoning) in an online context, two research questions guided this study: Is an asynchronous, text-based online continuing professional education learning experience able to support teachers in learning: (1) about algebraic reasoning with spreadsheets as the learning tool, and (2) about teaching algebraic reasoning with spreadsheets as the learning tool?

Study and Results

Using a cadre of 10 teachers participating in a mathematics education course as part of a three-year online MS program focusing on mathematics, science and technology as study participants, course artifacts consisting of prompted reflective essays focusing on using dynamic spreadsheets as a teaching and learning tool and their knowledge of using dynamic spreadsheets as a teaching and learning tool were collected. In addition to these essays, learning products illustrating their spreadsheet Technological Pedagogical Content Knowledge (TPACK) about algebraic reasoning were collected. The analysis of these data sources revealed two primary themes: (1) through their activities in the course, the participants developed their understanding of how to use spreadsheets as tools to develop their own algebraic reasoning skills, and (2) through their activities in the course, the participants developed their understanding of how to facilitate students using spreadsheets in developing algebraic reasoning skills.

A sampling of evidence is provided by comments from two participants. K wrote, "The primary idea this topic emphasized is the universal nature of the average function versus a user created form of the same function." Her understanding of the importance of generalization indicated her developing knowledge of using spreadsheets as tools for developing algebraic reasoning skills. M wrote, "The students can conceptualize the numbers that are input aren't definite, rather can be changed by the user to observe different situations." This comment illustrated his understanding of how students are able to use spreadsheets in algebraic reasoning, through the manipulation of variables and values, to investigate mathematical concepts.

Discussion and Implications

The results of this study demonstrated the capabilities for online continuing education programs to deliver educational experiences that are effective in supporting teachers in developing critical mathematics TPACK with spreadsheets skills. This effectiveness reinforces situating meaningful learning for educators in an online context where deep thinking and higher-order learning are the desired outcomes. The importance of developing algebraic reasoning skills and an understanding of teaching and learning with spreadsheets suggests a need for further research in how best to situate these learning experiences in an online context.

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ANALYSIS OF RESEARCH CONTRIBUTIONS AND FRAMEWORKS FOR MATHEMATICS PROFESSIONAL DEVELOPMENT ONLINE

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Synthesis of Main Descriptive Aspects

In this work we review different scenarios of instruction issued from important research related to teacher professional development. Specifically, we have proposed to account for the work of the professional development of teachers of mathematics through the use of digital platforms and where the aspect of technology integration in classrooms played a primary role in the design of the scenarios of instruction.

In two of the scenarios of instruction reviewed, the use of mathematics technology (Zbiek & Hollebrands, 2008) served as a catalyst to trigger the reflection of teachers on teaching some school mathematics topics (see Sánchez 2010), or on teacher planning or preparing lessons dealing with innovative topics of mathematics education (see Borba et al., 2005–2009). In another study, (Silverman 2011–2012) teachers gained knowledge of mathematical content for teaching. Finally, in the fourth of the works reviewed (Hoyos et al., 2009–2011), teachers made moderate progress in the incorporation of digital technologies into their classrooms, showing different ways of integrating technology into their teaching. These contributions are summarized in the Table 1.

Table 1: Characteristics and Contributions of the Scenarios of Instruction Reviewed

Characteristics / Teams	Online Communication modality	Internet interface used	Type of interaction	Theoretical frame	Main Contribution
GPIMEM (Brazil)	synchronous	Chat room	Between participants and the tutor	Humans-with-media and Lévy contribution on production of knowledge	Defining characteristics of interaction in online modality
Silverman et al. (USA)	asynchronous	forum	Between participants and the tutor	Sociocultural and discursive approach to development of knowledge	Clarification of interaction role in online learning
CICATA (Mexico)	asynchronous	forum	Between teams of participants	Documentary approach	Promotion of reflection in online modality
MAyTE (Mexico)	asynchronous	forum	Between participants and content	PURIA model and development of craft knowledge	Achievements on the incorporation of mathematics technology in teaching practice

Note: More information and bibliographical references will be given during the conference.

KNOWLEDGE AND PERSONAL EFFICACY FOR TEACHING AND THE SOURCES OF TEACHING SELF-EFFICACY FOR MULTIPLICATIVE REASONING

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Teachers' dispositions toward the work of teaching and the content they teach influence how their knowledge is used in practice. This poster session reports quantitative validation work on a new measure of mathematics teacher disposition that was built from Bandura's (1986, 1997) social-cognitive theory and Shulman's (1986) pioneering work on teacher knowledge.

Bandura (1986) theorized self-efficacy as the major source of motivation for human endeavor and identified two dimensions: personal efficacy and outcome expectancy. I focus on personal efficacy beliefs (confidence in one's own ability to do a task) because these beliefs are less stable and may respond to professional development efforts. Shulman (1986) differentiated *subject matter knowledge* from *pedagogical content knowledge*: the knowledge of a subject that is closely tied to the demands of teaching that subject. I evaluated a new model of teaching self-efficacy with two factors: personal efficacy (PE) and (pedagogical content) knowledge efficacy (KE). These factors represent teachers' confidence in their ability to teach and their confidence in their content knowledge for teaching, respectively.

The new measure was adapted from one designed for pre-service science teachers (Roberts & Henson, 2000) by focusing on inservice teachers' multiplicative reasoning (fractions, ratios, and proportions). This content is a major goal of elementary curriculum and a foundation for secondary curriculum, yet is difficult for K–12 teachers and their students. To support the validity argument, I developed measures for the four sources of self-efficacy postulated by theory (Bandura, 1997) that were specific to inservice teaching and multiplicative reasoning by adapting items for the sources of (general) teaching self-efficacy previously piloted by Morris (2010).

Results using a preliminary sample of 266 K–12 mathematics teachers in Texas indicate high reliability for both 7-item subscales ($\alpha_{PE} = .83$, $\alpha_{KE} = .88$). A χ^2 difference test showed that the two-factor confirmatory factor analysis (CFA) model was superior to the undifferentiated single-factor CFA model ($\chi^2 = 146.93$, df = 1, p <.000). All factor loadings in the two-factor model were significant and had R^2 values greater than .28. In multiple regression analyses, the four source measures explained the majority of the variance of the new self-efficacy measures ($R^2_{PE} = 0.67$, $R^2_{KE} = 0.57$). Moreover, a structural equation model with direct paths from the sources to the correlated self-efficacy measures exhibited reasonably good fit ($\chi^2 = 2575.06$, $df = 265$, RMSEA = 0.068, CFI = 0.91, SRMR = 0.056).

The self-efficacy measure will be used as tool to better understand how teachers navigate the professional learning continuum, for example, by assessing dispositional differences among teachers at different grade levels and with different preparation and training experiences.

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MIDDLE SCHOOL MATHEMATICS TEACHERS' JUSTIFICATIONS ABOUT FRACTIONS

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Keywords: Teacher Knowledge; Middle School Education; Reasoning and Proof; Teacher Education–Inservice/Professional Development

Current reforms suggest that the teaching and learning of proof should be integrated into all mathematical domains, not only geometry (National Council of Teachers of Mathematics [NCTM], 2009). In order to implement these reform recommendations regarding proof successfully, mathematics teachers should possess adequate understanding of proof itself. The wealth of studies investigating pre-service and in-service mathematics teachers' conceptions of proof suggests that many teachers view example-based justifications as sufficient mathematical proofs (e.g., Knuth, 2002; Morris, 2002). To date, not enough is known about what middle school (grades 5–9) teachers' justifications look like and how teachers' educational background affects their competencies in constructing justifications about fractions—a topic that is both central to and pervasive in middle school mathematics. To address this research gap, this study investigates what types of justifications about fractions middle school mathematics teachers construct and how their justifications relate to their educational background.

This study involved 56 in-service middle school mathematics teachers from nine public school districts in the northeastern United States. The participanting teachers had varied educational backgrounds in matheamtics (n = 13), mathematics education (n = 8), science (n = 8), and other subjects (n = 27). The primary source of data for this analysis was teachers' written justifications for two problems about fractions. Their justifications about fractions were categorized according to Harel and Sowder's (2007) and Simon and Blume's (1996) taxonomies of proof. The results suggest that many teachers tend to construct symbolic justifications with logical errors. The results also suggest that many teachers experience difficulty generating a correct proof, regardless of whether or not their educational background is in mathematics.

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HOW AND WHY DO MATH TEACHERS BECOME INTERESTED IN EQUITY?

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In this poster, I compare and contrast preservice teachers' beliefs about equity in mathematics education with those of teacher participants of an annual conference about mathematics education and social justice (Creating Balance in an Unjust World www.creatingbalanceconference.org). I aim to better understand the development of interest in equity among math teachers and how it may shape their pedagogical practices. How and why do math educators become interested in issues of equity relating to math instruction and how does it affect their pedagogical approaches? Might teachers' 2,000-hour "apprenticeship of observation" (Lortie, 1975) as students impact their beliefs about teaching mathematics, particularly related to issues of equity? Scholars have investigated various pedagogical approaches that result in more equitable outcomes, such as complex instruction (Boaler, 2002), culturally relevant pedagogy (Ladson Billings, 1997), and comparison of algebraic problem solving methods (Star & Rittle-Johnson, 2009). However, despite preservice teachers' exposure to research, some may not implement these pedagogical practices. Some teachers may be interested in equity and believe that they are implementing such practices, while in reality they are not (Cohen, 1990). Others may not be interested in equity, or more plausibly must focus on a myriad of other more urgent concerns as a preservice teacher, such as classroom management, daily planning, and grading. In this poster I present findings of preservice teachers' weekly journal entries to explore their thoughts regarding equity in mathematics instruction. In comparison I present interview results and attendees' written evaluations of the Creating Balance in an Unjust World conference, in an attempt to understand how and why they became interested in equity in mathematics education. Preliminary results suggest great variation in preservice teachers' interest in equity. Some teachers enter the teacher preparation program with an interest and commitment to equity in math achievement. Some have other interests (not related to equity) and reasons for pursuing mathematics teaching as a career, yet others grow to value equitable pedagogical approaches in the math classroom. Conference attendees demonstrate strong commitment to equity in math education often related to a broader and more general commitment to social justice. Results suggest that prior education and personal experience play an influential role in math educators' interest in equity. Results also raise questions about the ability of a one-semester preservice methods to impact teacher beliefs regarding equity in mathematics achievement. Further research, using qualitative methods such as interviews and participant observation, may help shed light on the reasons and experiences behind math teachers' interest in pursuing equity.

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CO-EVOLVING LEADERSHIP AMONG MIDDLE GRADES MATHEMATICS TEACHERS IN AN AUTONOMOUS PROFESSIONAL DEVELOPMENT GROUP

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Keywords: Teacher Education–Inservice/Professional Development; Middle School Education; Instructional Activities and Practices; Equity and Diversity

Kazemi and Hubbard's (2008)'s concept of *co-evolution* offers understandings of teachers' *transformations of participation*. Here, an autonomous Professional Learning Community (PLC) shifted teacher learning with respect to noticing (Sherin, Jacobs, & Philipp, 2011). Autonomous teacher-led groups contrast with mandated PLC's because teachers take control of purposes and directions (Cochran-Smith & Lytle, 2009). This study relates to PMENA 2012 themes in three ways: (1) conceptualization of students' transitions between middle and high school mathematics, (2) intra-group professional learning with colleagues, and (3) innovative use of research for equitable grading. The study responded to the following research questions:

1. What is the nature of the issues teachers deem important and choose to explore?
2. How do teacher/participants set norms for structure, topics and leaders for each session?
3. What demonstrates changing leadership development among the teacher/participants?

Fourteen mid-career middle grades mathematics teachers from two partner districts of the NSF-sponsored New Jersey Partnership for Excellence in Middle School Mathematics (NJPEMSM) were invited to participate. Seven teachers volunteered.

Teachers designed a structure for the 90 minute sessions, divided into three parts: (1) recap, (2) "cool" mathematics class activities, and (3) selected topics led by rotating leaders from the group. The data consist of 10 hours each of video and audio recordings analyzed using NViVO software beginning with an interaction framework (Eskelson, 2012). Initially, teachers expressed: (a) great value in increased communications with fellow participants; (b) reluctance to share this work with oppositional colleagues—later, several presented their activities at a department meeting; (c) increased confidence to lead discussions with peers; (d) new acceptance of one teacher's introduction of an unconventional, equitable approach to grading; and (e) agreement about the overall worth of these sessions, with intentions to meet in the 2012–2013 school year.

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MATHEMATICS SPECIALISTS "NOTICING": IDENTIFYING THE ROLE OF "NOTICING" IN THE DEVELOPMENT OF STRATEGIC COMPETENCE

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The goal of the current research project is to explore the nature of state-licensed mathematics specialists' and teachers' noticing of student use of representations of mathematical problems.

Imagine mathematical knowledge taking the form a network, the individual nodes representing five representations of a mathematical problem or concept. Following the Lesh Translation Model, these five nodes are real life contexts, manipulative representations, pictures, symbols, and also verbal representations of a problem (Cramer, 2003). Each of these nodes is populated by various manifestations, each separate and distinct from others. Building connections between each of these nodes and sub-nodes creates a complex web of conceptual links between representations of a mathematical context. Strategic competence, defined as the "ability to formulate, represent, and solve mathematical problems" (NRC, 2001, p. 116), is the direct result of a strong, interconnected understanding of a problem context. In the network model, a teacher who "notices" is willing to extend a tenuous link between one representation and another in order to follow student thinking. This conception of noticing, the act of making observations of a classroom and managing choices based on those observations (Mason, 2011), provides a lens to examine the dialogue that occurs during the lesson study debrief.

Four classroom teachers collaboratively planned a lesson introducing subtraction of integers to a class of seventh grade middle schoolers. The students' use of the provided manipulatives exposed their difficulties formulating and representing the mathematical contexts of the given tasks. The lesson debrief revealed that the mathematics specialists and the teachers who created the lesson differed widely in their understanding of manipulatives as representations of mathematical contexts. The specialists, on the whole, showed a qualitatively different type of noticing of students' attempts to link representational nodes than did the lesson study participants. Specialists were more likely to show awareness of links between different representations of the problem. While this is a small case study, the conceptual framework and the important differences between teachers' and specialists' acts of noticing indicate that further study of the role of mathematics specialists in teacher development may be revelatory.

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EXPLORING THE USE OF URBAN STUDENT MATH LEARNING VIDEO TO CHALLENGE INSTITUTIONALIZED EDUCATION STEREOTYPES

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This research examines how video can be used to help professionals transition their own perceptions about the potential of students and the power of student explorations in mathematics. It is known that student behaviors and attitudes in school often affect the education they receive. It is also known that decisions about how to design the curriculum and manage classes can be based on the socio-economic level of a community, the behavior profiles of a school, and stereotypes about student behaviors (Anyon, 1980; Jussim, Robustelli, & Cain, 2009). The question guiding our research is: How can innovative video technology be used to parse out a student's learning behavior from other behavioral attributes to help educators see their students in a new light?

This poster presentation reports on our analysis of videos from the Video Mosaic Collaborative (VMC) that feature urban students engaged in mathematics learning. Our work is part of an initiative at Rutgers University where a team of education researchers and librarians are creating a digital video repository to house 25 years of research on children's mathematics learning. The VMC (www.videomosaic.org) gives the educational community access to students' math learning in classrooms and informal settings, in urban and suburban communities (Agnew, Mills, & Maher, 2010). This project also introduces a new tool, the Analytic, which supports creation of video narratives that concatenate multiple video clips with descriptive text to reveal specific aspects of the full video. Video data is rich and supports development of a variety of analytics, each having a unique lens on the original video. The tool is designed to maintain the relationship between each video clip, the source video in the library, source video transcripts and other metadata pertaining to the original learning context (Agnew, Mills, & Maher, 2010).

Our preliminary work focused on three different groups of urban students with the goal of revealing the intellectual power in students at risk of being judged incapable by their non-math behavior. We used a descriptive model for video data analysis (Davis, Maher, & Martino, 1992) to extract the math learning clips from full videos that contained both learning and distracting behavior. Our poster presentation details the results of our video analyses and showcases our goals of repurposing video for (1) professional development for educational leaders in low SES urban districts, and (2) promoting a transition among teachers and students to embrace a model of high cognitive demand in math classes with low SES, urban (or rural) students.

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CHINESE MATHEMATICS TEACHING REFORM IN FOUR-YEARS TIME: THE DJP MODEL

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Like the United States, China struggles with the professional development of its mathematics teaching corps. We describe how middle school mathematics teachers from a poor rural area of southwest China successfully reformed their practice in only four years time by utilizing a systemic collaborative reflection model called DJP—D (导) refers to guided self-learning, J (讲) to sharing solutions, and P (评) to reflective evaluation. Data in our study include 13 video-taped consecutive classroom teaching observations, semi-structured teacher interviews, and surveys of all 152 students' perceptions of mathematics and the respective pedagogy in their classes. Our findings indicate that since the enactment of the DJP model, the school (historically ranked last on achievement tests), now ranks in the top level on academic testing in its district. Students developed supportive learning communities and their interests in learning mathematics improved significantly during this time frame.

We specifically focus our discussion on strategies and issues related to the learning tasks, the structure of instruction, the grouping of students, and the assessment approach in the DJP model. Teachers purposefully chose learning tasks to engage students in higher-level mathematics thinking (Stein et al., 2000). Similar problems were posed, analyzed and synthesized to find common mathematical structures underlying various situations (Cullen, 2002). The students worked on the problems collaboratively and shared their solutions with the larger class. The DJP model of teaching allows students more opportunity to teach each other, and replaces more traditional Chinese lecture formats. The teacher and the class reflected together on core mathematics content knowledge and related reasoning strategies that facilitated successful solution generation, and together evaluated the performance of each student-group. Small groups were the basic units accountable for learning; ongoing multifaceted assessments focused on the student-groups rather than on specific individuals. Friendly competition between groups and rewards for achievement and progress aided student learning. We also discuss some issues that arose for the teachers during DJP implementation, including development of dynamic discourse and underlying cultural and social assumptions that might support the DJP model as a reform approach in other schools, especially the United States. Our poster describes the DJP model for a North American audience, and provides structured suggestions for how this Chinese professional development can be adapted for American mathematics classes. Such recommendations are significant and timely as the United States seek to improve its mathematics education.

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SUPPORTING TEACHERS IN TECHNOLOGY INTEGRATION IN KENYAN SECONDARY SCHOOLS

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Teaching is a complex practice that requires teachers to drawn upon their content knowledge, pedagogical approaches and strategies, and knowledge about learners in order to support learning. Integrating technology into the teaching and learning practice of a classroom is a strategy that many teachers are drawing upon. When integrated effectively, technology can support student learning and lead to deeper conceptual understanding and procedural fluency (Bransford, Brown, & Cocking, 2000). However, there are factors that serve as barriers to teachers integrating ICT into their classroom practice and some may seem difficult to overcome, especially in developing countries (Ogwu & Ogwu, 2010). It is therefore important for teacher educators and other educational policymakers to understand what factors assist in effective ICT integration and what factors may inhibit this integration.

Researchers have identified a number of common factors that promote effective ICT integration (e.g., McMillan Culp et al., 1999). While most of this research has been carried out in developed countries, Light has applied a framework of common factors to classroom integration of ICT in developing countries (Light, 2010). These researchers have used what they describe as the seven of the "most commonly cited factors" (Light, 2010, p. 41). In an international study examining 174 case studies of innovation pedagogical practices using technology in 28 countries, Kozma (2003) identified seven "meaningful patterns of classroom practice" (p. 6).

In this study, we are using a case study approach to examine how eight secondary mathematics and science teachers moved from no experience in technology use to integrating technology into the teaching and learning practice of their classroom, and to examine what factors assist and what factors inhibit this integration. We are analyzing these teachers' classroom practices in light of research literature in this area, while being open to other factors and patterns emerging within the context in which we are investigating. We are collecting data through questionnaires, individual interviews, and classroom observations.

The research study participants are teachers at two national high schools in Kenya, one boys' school and one girls' school. The participants were involved in a HP Catalyst Initiative project and were each given a HP tablet and participated in several professional development workshops to be exploring ways to integrate technology into their classroom practice. Our poster will present several case studies as examples of what we found through our investigation.

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ELEMENTARY TEACHERS' IMPLEMENTATION OF COMPLEX INSTRUCTION

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As one component of a larger research project, we examined how eight elementary mathematics teachers in an English/Spanish dual language program learned to implement Complex Instruction (CI) (Cohen & Lotan, 1997), an approach to group work with a special emphasis on equity issues (e.g., status). Over a seven-month period, we investigated the dilemmas that these teachers faced as they learned to implement CI. The following data was collected and analyzed using qualitative methods: video recordings of teacher meetings, audio recordings of teacher interviews, and researchers field notes of classroom observations.

One unanticipated finding that surfaced early on was the tension between teachers' willingness to implement components of CI and their orientation towards their district's chosen curriculum. The school had previously used a reform-based curriculum, which had recently been replaced by a nonreform-based curriculum. However, the enacted curriculum is what is implemented in the classroom (Clandinin & Connelly, 1992) and may differ from the intended curriculum. According to Remillard and Bryans (2004), teachers' orientation toward curriculum influences their implementation. They identify three categories of teachers' implementation. *Intermittent and narrow use* teachers use the curriculum minimally, relying on their own routines or other materials. Teachers in the *adopting and adapting* category use the curriculum as a guide for instruction, such as in sequencing topics. Finally, *thorough piloting* involves teachers who use curriculum as their primary guide for instructional activities.

We found that teachers' orientation to the curriculum influenced their implementation of CI. Use by one of the eight teachers was categorized as intermittent and narrow use. This teacher began to implement CI at the onset of the study and began to use CI effectively almost immediately. Often, this teacher relied on the reform-based curriculum that had been replaced. She felt that she had the freedom to do so. Use by four of the teachers was categorized by adopting and adapting. These teachers only began to implement CI when explicitly asked to do so. They did not consistently use CI effectively and did not always recognize possible causes, such as implementing inappropriate group tasks. However, once these teachers found success in implementing parts of CI, they were willing to continue to keep trying to incorporate CI in their instruction. Use by the other three teachers could be categorized as thorough piloting. These teachers were the most hesitant to attempt to implement CI in their classrooms. They often described a desire to use CI, but they felt constrained by many elements of the newly adopted curriculum, such as insufficient time to engage students in CI tasks due to the number of topics that needed to be covered. They also felt pressure to adhere closely to the curriculum, even though they felt it was ineffective in engaging children in significant, meaningful mathematics.

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TRANSITIONING FROM RATIONALIZATION TO CRITICAL REFLECTION THROUGH REFLECTIVE TEACHING CYCLES

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Practice-based professional development provides teachers with learning experiences that are connected to and contextualized in professional practice. Therefore, this type of professional development can also be customized to meet teachers at their point of need in part by the use of *professional learning tasks* (Smith, 2001), which takes teachers' prior knowledge and experiences into consideration. The type of professional learning task I used in this study was the *reflective teaching cycle* (Smith, 2001). The reflective teaching cycle consists of three phases: (a) planning, (b) teaching, and (c) reflecting. The purpose of this study was to examine how a series of cycles influenced two middle school mathematics teachers' selection and implementation of tasks that had the potential to facilitate higher-order thinking. I used a series of seven cycles to engage two seventh-grade mathematics teachers in conversations about mathematics, pedagogy, and higher-order thinking in order to provide students with mathematics lessons that could promote the use of higher-order thinking. During planning and reflection meetings, I recorded the conversations and used thematic analysis to identify, analyze, and report themes within the data.

In this poster, I present results from the study, which illustrate how the cycles helped one teacher transition from *rationalizing* to *critically reflecting* on her practice. Teachers rationalize when they do not, or cannot, view a problem in other ways and possibly see it as "residing within the students rather than in the practice setting itself" (Loughran, 2002, p. 35). Critical reflection involves teachers considering the best way of understanding, changing, or implementing their practice, where 'best' implies "considering implications of practice and weighing them against relevant goals, values, and ethics" (Jay & Johnson, 2002, p. 79). In this study, one teacher transitioned from rationalizing the reasons why she was unable to facilitate higher-order thinking in her classroom to critically reflecting on her practice to understand how it was inhibiting her ability to facilitate higher-order thinking.

The reflective teaching cycles allowed the teacher to make this transition through collaboration and the focus on mathematics and pedagogy. In particular, the teacher was able to hear what her colleague was doing, which helped her consider the implications of her practice and how she could change it. Also, as the facilitator, I was able to ask her questions about specific events in her classroom, which prompted her to think about her practice rather than the students. This research could help teacher educators and professional developers determine the most effective characteristics of the reflective teaching cycle and the types of facilitation that would be most successful at promoting critical reflection.

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TEACHER MODELS OF STUDENT THINKING

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The purpose of this poster is to describe a study, which explores how precalculus teachers using a research-based curriculum develop models of students' mathematical thinking. The study builds on a framework developed by Silverman and Thompson (2008), which describes how teachers develop Mathematical Knowledge for Teaching (MKT). Teachers develop MKT through developing deep connected mathematical understandings which Silverman and Thompson call Key Developmental Understandings (KDU's) (Simon, 2006), and through attending to student thinking in ways that allow them to understand both how students think and how students might interpret new information. The teachers' new MKT is developed through reflection, which occurs when they are faced with perturbations to their current way of thinking. (Dewey, 1910).

In this study we explore teachers' current models of student thinking about key precalculus ideas including functions and rate of change, and try different approaches to challenge teachers to reflect on student thinking in ways that will support the teachers in developing more complete and useful models of student thinking. In interviews, teachers are asked to review student work and video clips of their own classroom interaction with students, and then to interpret how the students might be thinking. Initially, it appeared that teachers and researchers had different interpretations of what is meant by attending to student thinking. Questions were developed that we hoped would help support teachers in knowing what it might mean to attend to student thinking, and suggest a different possibility for considering student thinking. Teachers were asked several different types of questions. They were given different possibilities, based on research, for how students might be thinking, and asked to evaluate each possibility. They were asked to compare student mathematical thinking to their own mathematical thinking, and to consider how these ways of thinking might influence how students interpreted classroom activities. They were asked what meaning they hoped students would make of the mathematics and what meaning they thought students were making of the mathematics. Finally, they were asked how they might in a specific situation have explored further what a student was thinking.

We will report progress in understanding teachers' models of student thinking and how these models might develop, and in understanding how to promote teacher reflection on student thinking and mathematics.

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RECIPROCAL FUNDS OF KNOWLEDGE IN PRE-K MATHEMATICS

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Families' observations of children's counting and number expertise typically occurs in everyday contexts whereas teachers see this expertise in the context of school. The differences in these everyday and school based contexts has been shown to leave children with the view that there are two kinds of mathematics—one for school and one for everything else (Presmeg, 2007). We hypothesize that if this dissonance begins early, it will become entrenched early. Thus, for mathematics pedagogy to be effective, teachers need to transition from a learned focus on what and how mathematics is traditionally taught in school to connecting to the mathematical practices children engage in outside school (Wager, 2012). Further, there must be a shared understanding between teachers and families of both mathematics contexts. Identifying ways for families and teachers to inform each other and bridge that divide is at the heart of the concept we call "reciprocal funds of knowledge."

Drawing on scholarship in funds of knowledge, early childhood, and early mathematics, a two-year professional development program was designed in an effort to support pre-K teachers as they worked with children and families to build the bridge between home and school mathematics. As part of the professional development teachers engaged in multiple practices with families including: (a) home visits, (b) two-way reflective conferences, (c) creating and sharing artifacts/learning stories, and (d) developing family math activities. Each activity contributed to a rich understanding of what children know, how they know it, where they learned it, and perhaps most importantly—what teachers can do with that knowledge in order to support children's learning. This study was designed to respond to the following questions:

1. How does reciprocal engagement with families contribute to teachers' understanding of a child's numeracy skills?
2. What various resources do families provide to support children's numeracy development?
3. In what ways do teachers modify their instruction based on learning about family knowledge, resources, and practices?

Through their engagement with families, teachers learned the types of numeracy activities families engaged in with their children, the wide range of resources provided by families, and the different competencies children demonstrated depending on the context in which they counted. By accessing the knowledge and resources that reside in children's homes teachers planned for instruction that better supported children's mathematical understanding. Without drawing on this knowledge from home, we are only accessing part of children's understanding. Further, by engaging in the practices supported by this professional development, connections between families and schools are improved.

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THE ROLE OF FRAMING IN ADVANCING ARGUMENTATION

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Keywords: Algebra and Algebraic Thinking; Instructional Activities and Practices; Teacher Education–Inservice/Professional Development; Reasoning and Proof

We use ethnography of argumentation to analyze part of a lesson from an algebra course for elementary teachers preparing to be mathematics specialists (Krummheuer, 1995, 2009). During this part of the lesson, the instructor and teachers discussed the behavior of the quantity *r/s* as *s* increases. As they did so, they established two qualitatively different argumentations. Following Krummheuer (1995, 2009), we use the notion of framing to explain the qualitatively different backings, and hence arguments, participants established that supported the claims that they made. Like Stephan and Rasmussen (2002), we highlight the types of argumentative supports that the classroom participants offer that advance the argument at hand. Whereas Stephan and Rasmussen illustrate how the absence of these supports points to collective shifts in the classroom participants' understandings, we focus our discussion on the important role that these argumentative supports play in promoting individual learning opportunities.

Participants established an *inductive backing* to support the ensuing warrants in the form of backings that collectively served a generalizing role. One of the teachers initiated this process by introducing arbitrary examples for the quantity, *r/s* as *s* increases, using a partitive division frame to support her claim. The instructor took a lead role in facilitating this part of the discussion by representing the teacher's examples and, in collaboration with the teachers, generated additional examples. The instructor and the teachers established this framing by collectively generating examples situated within the context of sharing donuts. As they engaged in this discussion, teachers had opportunities to reason quite sensibly about the behavior of the quantity, *r/s*.

We also provide an example of *authoritative backings*, argumentative supports that are irrefutable. This type of backing was socially constituted when the instructor, along with one of the teachers, referred to a meaningful context (steepness of a hill) that provided argumentative support for another argument. This teacher gave a different explanation for the behavior of *r/s* as *s* increases. The structure of the argument was different from the previous one. First, participants established an inductive warrant. The instructor provided a backing (and challenge) that unequivocally validated the teachers' argumentative supports. In this case, the backing and, more generally, the argument are indisputable because the backing was couched exclusively in a contextual situation. In fact the framing and the backing seemed to be one and the same.

In sum, teachers had opportunities to make connections about *r/s* albeit the connections they made were constrained and/or enabled by the backings they and the instructor established.

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TWO MODELS OF PROFESSIONAL DEVELOPMENT USING EDUCATIVE CURRICULUM MATERIALS

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This poster shares two models of professional development in which teachers wrote curriculum materials to promote their own learning and change their instruction. In one project, teachers from a K–5 school wrote materials for lessons they deemed critical for students' success in mathematics. In the other project, middle and high school teachers from six schools prepared materials for topics in algebra that students find difficult to learn. These materials are referred to as educative curriculum materials (ECM) (Davis & Krajcik, 2005).

Keywords: Teacher Education–Inservice/Professional Development

Hill, Rowan, and Ball (2005) note that teachers' mathematical knowledge affects the effectiveness of their teaching. Moreover, how they hold that knowledge affects their ability to use it in teaching. Thus, our research goals were to characterize and articulate the nature of content knowledge for teaching and to understand how teachers developed this knowledge.

Both professional development models focused on content knowledge for teaching through teachers' creation of curricular materials, including pedagogical suggestions, to appropriately prepare themselves for successful implementation of their own goals and ideas. Pedagogical content knowledge is developed through planning, classroom instruction, and reflective collaboration (Van Driel & Berry, 2012). The professional development provided for teacher learning that involves developing and integrating one's knowledge base about content, teaching, and learning. To be reflective practitioners, teachers must participate in discourse on teaching and apply that knowledge to make instructional decisions. ECM set up these opportunities.

Preparing ECM that contain accurate mathematics, coherently portray the trajectory of the content, include appropriate technology, and effectively use technology is a difficult mission. We provided guidelines for teachers to use in writing their ECM in an effort to assist them in developing a clear purpose for their teaching and to provide multiple opportunities for students to explain their ideas. These included: (1) learning how to anticipate and interpret students' thinking and responses; (2) broadening their repertoire of instructional strategies and developing their knowledge base; (3) connecting topics during the year; (4) making pedagogical judgments visible; and () attending to ideas underlying the tasks rather than merely guiding actions.

By participating in the construction of ECM, teachers deepened their mathematical content knowledge, developed and included processes such as problem solving, reasoning, and communication into their instructional materials and instruction, made their teaching public, and reflected on their knowledge of mathematics and their teaching of mathematics.

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Chapter 6

Teacher Education and Knowledge–Preservice

Research Reports

Brief Research Reports

Poster Presentations

Van Zoest, L. R., Lo, J.-J., & Kratky, J. L. (Eds.). (2012). *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Kalamazoo, MI: Western Michigan University.

PROSPECTIVE TEACHERS' TRANSITION FROM THINKING ARITHMETICALLY TO THINKING ALGEBRAICALLY ABOUT EVEN AND ODD NUMBERS

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We present findings from a study of prospective middle school teachers' reasoning as they transitioned from thinking arithmetically to thinking algebraically about even and odd numbers. Teachers were asked to make sense of and use two representations of even and odd numbers to model them and to make connections between the representations. Analysis of a whole-class discussion indicates that although teachers easily represented even and odd numbers using an algebraic generalization, they grappled to make sense of a given geometric model. As teachers worked to make sense of the geometric model, they transitioned back and forth among three ways of interpreting the model (two of which were incorrect; one of which was correct).

Keywords: Algebra and Algebraic Thinking; Classroom Discourse; Number Concepts and Operations; Teacher Education–Preservice

Purpose of the Study

Standards for K–12 mathematics emphasize the use of multiple representations (e.g., written words, diagrams, symbolic expressions, physical models, graphs) for making sense of and communicating mathematical ideas (Common Core State Standards Initiative [CCSSI], 2010; National Council of Teachers of Mathematics [NCTM], 2000). Scholars argue that examining different representations can make mathematics more meaningful by illuminating different aspects of a mathematical idea or relationship (Cuoco, 2001; NCTM, 2000). Representations may be particularly important in the middle grades as students make the transition from thinking *arithmetically*—for example, working with specific even and odd numbers—to thinking *algebraically*—for example, making sense of even and odd numbers as sets of numbers that can be generalized (CCSSI, 2010; NCTM, 2000). The difficulties that many students have in making this transition are well documented (e.g., Chazan, & Yerushalmy, 2003; Smith, 2003), and scholars suggest that considering both visual and numerical modes of generalizing may facilitate this transition by helping students understand the nature of variable and familiarizing them with the structure of algebraic expressions (Lannin, 2003; Rivera & Becker, 2009; Thornton, 2001).

Considering different mathematical representations has also proven to be beneficial for teachers. Research shows that making sense of and using different representations can strengthen teachers' content knowledge by requiring them to make connections among representations and can strengthen teachers' pedagogical content knowledge by providing them greater access to student thinking as students interpret different representations (Herbel-Eisenmann & Phillips, 2005; Izsák & Sherin, 2003). Thus, one way to strengthen teachers' ability to support middle school students' transition from arithmetic thinking to algebraic thinking is to provide them with opportunities to reason about and use different representations to make sense of mathematical ideas. In particular, the mathematics content courses taken during their teacher preparation programs might be a promising context for such learning experiences.

This research report presents findings from a study that investigated the ways in which a class of prospective middle school teachers (PMSTs) reasoned about different representations of even and odd numbers during their work on a number theory unit in a mathematics content course for PMSTs. The context of even and odd numbers was chosen for two reasons: (a) it is central to the number theory ideas that are studied in the middle grades (CCSSI, 2010; Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006); and (b) it is accessible, yet still presents some challenge, for teachers (Smith, 2011).

Theoretical Perspective

Much of the mathematics education literature in recent years conceptualizes the learning of mathematics as a collective act. The view that learning is a social endeavor is rooted in theories of cognitive development based on foundational ideas by Jean Piaget and Lev Vygotsky. Piaget argued that social interactions are central to knowledge development because they influence individuals' attempts to resolve conflicts between their perspective and the perspectives of others (Brown & Palincsar, 1989; Rogoff, 1998). Vygotsky also emphasized social interaction as an essential element of cognitive development. He asserted that there is fluidity between self and others, and cognitive exchanges at this boundary mitigate the process through which knowledge development occurs. Sociocultural theories of mathematics learning integrate the perspectives of both Piaget and Vygotsky and maintain that one cannot examine individual students' reasoning and cognitive development without considering how it is influenced by the social context or examine the social context without considering how individual students' reasoning influences that context (Cobb, 2001; Cobb & Yackel, 1996).

Particularly relevant to the present study is the notion that representation can be conceptualized as a collective act occurring within a specific social and mathematical context. Collective representation involves "negotiating individually constructed representations in the shared space of a group or classroom as well as the teacher's role in facilitating these interactions" (Stylianou, 2010, p. 327). This study investigated the extent to which considering different representations of even and odd numbers was helpful in PMSTs' collective transition from thinking arithmetically to thinking algebraically.

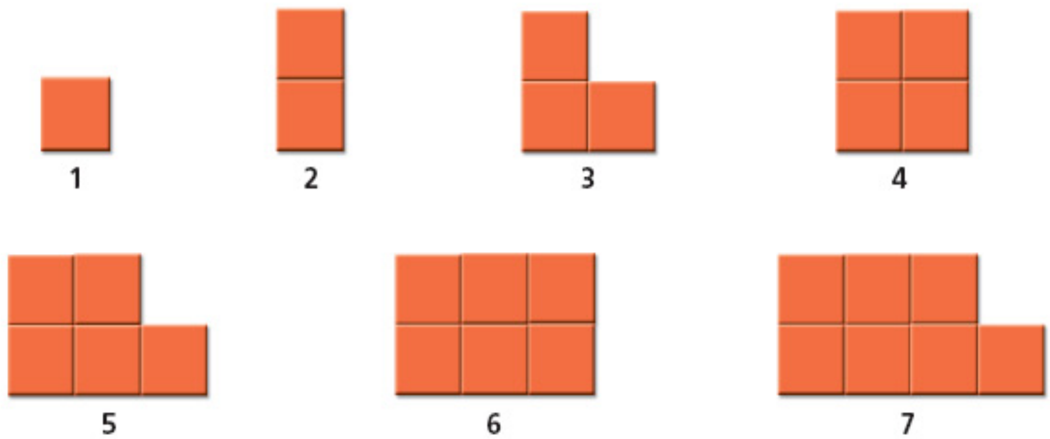
Methods

The study was conducted during Spring 2011 at a four-year, public university in the southeastern United States. The 22 participants (14 female; 8 male) were PMSTs enrolled in a required 16-week mathematics content course for prospective middle and secondary school teachers. (Since secondary school teachers at this university are certified to teach both middle and secondary school, all participants were considered prospective middle school teachers.)

The course met once a week for two hours and thirty minutes and focused on three content strands central to the middle grades: (a) number and operations; (b) algebra and functions; and (c) geometry and measurement (CCSSI, 2010; NCTM, 2000). Throughout the course PMSTs were asked to reason about and make connections between and among representations of mathematical ideas across these three content strands. The instructor (and first author) also immersed PMSTs in the processes of mathematical inquiry (CCSSI, 2010; NCTM, 2000) and modeled a type of teaching whose goal was learning with understanding (Carpenter & Lehrer, 1999). PMSTs were encouraged and expected to regularly engage in discussions with their peers as they shared their thinking about problems and solution strategies, reasoned about and made connections between mathematical ideas, made and evaluated conjectures, and developed and revised mathematical arguments.

Data collection occurred during a five-week unit on number theory that included considering numeric, algebraic, and geometric representations of even and odd numbers and exploring how to add and multiply even and odd numbers using these different representations. The primary data source was transcripts of video of the whole-class discussions that the second author filmed. Secondary data sources included: (a) the second author's field notes; (b) written work produced by PMSTs during each class; and (c) audio recordings of weekly meetings between the first and second authors in which they reflected on each class meeting, identified and discussed any ideas that the PMSTs seemed to be struggling with, and discussed the instructor's plans for the next class meeting.

The study reported herein focuses on the first idea that PMSTs grappled with during the data collection, which occurred during the second class meeting of the unit (the fifth class meeting of the semester): explaining how the different components ('2', 'n', and '+1') of the algebraic generalizations of even and odd numbers ('2n' and '2n+1', respectively) were represented geometrically in "Tilo's model" (shown in Figure 1). In particular, PMSTs had difficulty making connections between: (a) the '2' in the algebraic generalizations of both even numbers and odd numbers and the maximum number of tiles in each column of Tilo’s model; (b) the ‘n’ in the algebraic generalizations of both even and odd numbers and the total number of ‘complete’ columns (i.e., columns that contain two tiles); and (c) the ‘+1’ in the algebraic generalization of odd numbers and the one extra tile that makes an ‘incomplete’ column. (Although 2n-1 is also a valid generalization of the set of odd numbers, the instructor chose to focus on the 2n+1 generalization in order to be consistent with the source of Tilo’s model, the *Connected Mathematics Project* curriculum.)



(Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006, p. 25)

Figure 1: Tilo’s model

The transcript of the class meeting that is the focus of this study was analyzed to identify the ways in which PMSTs used the different representations of even and odd numbers to reason about the meaning of the components of the algebraic generalizations of even and odd numbers. Three milestones in PMSTs’ thinking about the variable ‘n’ (shown in Figure 2) emerged from the data: (a) claiming that ‘n’ represented *one tile*; (b) claiming that ‘n’ represented *one complete column*; and (c) claiming that ‘n’ represented the *total number of complete columns*.

Thinking Arithmetically		***Thinking Algebraically***
Milestone a ‘n’ represents one tile (incorrect)	Milestone b ‘n’ represents one complete column (incorrect)	Milestone c ‘n’ represents the total number of complete columns (correct)

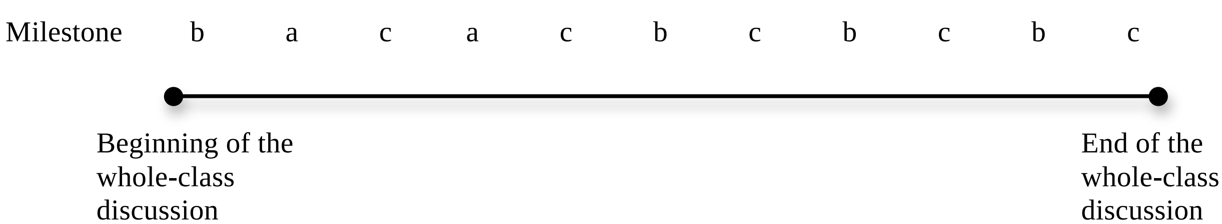
Figure 2: Milestones in PMSTs’ transition from arithmetic to algebraic thinking

Conceptualizing ‘n’ as representing *one tile* (Milestone a) would result in a different algebraic expression for each natural number (e.g., 2=(2n), 3=(3n), 4=(4n), 5=(5n), 6=(6n), where n=1 tile) and reflects arithmetic thinking since it treats each number *individually* rather than as belonging to a *set* of numbers with shared characteristics. Conceptualizing ‘n’ as representing *one complete column* (Milestone b) also results in a different algebraic expression for each natural number (e.g., 2=2(n), 3=2(n)+1, 4=2(2n), 5=2(2n)+1, 6=2(3n) where n=1 complete column); however, it recognizes that even numbers have the characteristic of being able to put the tiles into groups of two with no tiles left over, and that odd numbers have the characteristic of being able to be put the tiles into groups of two with one tile left over. Thus, Milestone b is considered to be a more algebraic way of thinking because it suggests that there are some shared characteristics among numbers within the same *set*. However, this way of thinking is not completely algebraic because it does not allow for a single algebraic generalization to represent even numbers and another algebraic expression to represent odd numbers. Conceptualizing ‘n’ as representing *the total number of complete columns* (Milestone c) reflects algebraic thinking by recognizing the shared characteristic among even numbers as being able to put the tiles into groups of two with no tiles left over, the shared characteristic among odd numbers as being able to put the tiles into groups of two with one tile left over, and results in one algebraic expression for the *set* of even numbers (2n) and another algebraic expression for the *set* of odd numbers (2n+1).

After identifying and coming to agreement on the extent to which the three milestones reflected arithmetic and algebraic thinking, the authors reanalyzed the transcripts to trace PMSTs’ ideas as they moved back and forth between the three milestones in their transition from thinking *arithmetically* about *individual* even and odd numbers to thinking *algebraically* about *generalizations* of the *set* of even numbers and the *set* of odd numbers.

Results

The transition from thinking arithmetically about *individual* even and odd numbers to thinking algebraically as *sets* of numbers that can be generalized was challenging for this group of PMSTs and did not occur in one direction (from thinking arithmetically to thinking algebraically). Instead, in their efforts to make connections between the geometric and algebraic representations, PMSTs’ understanding of the meaning of the variable ‘n’ in the algebraic generalizations of even and odd numbers (‘2n’ and ‘2n+1’, respectively) shifted back and forth between the three milestones: (a) ‘n’ representing *one tile*; (b) ‘n’ representing *one complete column*; and (c) ‘n’ representing the *total number of complete columns*. As shown in Figure 3, all three milestones were considered at least twice during the whole-class discussion. That Milestone a was abandoned fairly early in the discussion suggests that PMSTs realized that putting tiles into groups of two to make complete columns helped to make some distinction between the *set* of even and the *set* of odd numbers. However, the movement back and forth between Milestones b and c indicates that PMSTs were not convinced that grouping the complete columns together would allow them to clearly distinguish the *set* of even numbers from the *set* of odd numbers and connect to the algebraic generalizations they had identified. Thus, although the discussion ended on the (correct) Milestone c, PMSTs’ understanding of ‘n’ as representing the *total number of complete columns* may have remained fragile. Field notes from the subsequent class, in which PMSTs began operating on even and odd numbers, also reflected this instability.



Milestone a: ‘n’ representing *one tile*
Milestone b: ‘n’ representing *one complete column*
Milestone c: ‘n’ representing the *total number of complete columns*

Figure 3: PMSTs’ transition among the three milestones

The excerpts that follow are illustrative of the struggle that PMSTs had as they collectively made sense of the different representations and transitioned from thinking arithmetically to thinking algebraically about even and odd numbers. In the first section, PMSTs work to make sense of the variable ‘n’ in the algebraic generalizations and to find commonality within the *set* of even numbers and within the *set* of odd numbers – thus making the transition toward thinking more algebraically, and moving away from Milestone a. In the second section, PMSTs focus on making explicit connections between the features of Tilo’s model and the specific components of the algebraic generalizations and move back and forth between Milestones b and c. All names used in the excerpts that follow are pseudonyms.

Finding Commonality Within the Sets of Even and Odd Numbers

Prior to introducing Tilo’s model, the instructor asked PMSTs to describe the set of even numbers. PMSTs easily identified commonalities across all numbers in this set. For example, they described even numbers as being *divisible by 2, having no remainder after dividing by 2, being a multiple of 2, being an integer, and including 0*. Despite these ways of describing the set of even numbers, PMSTs struggled to make sense of Tilo’s model (shown in Figure 1) when the instructor introduced it. In particular, PMSTs struggled to make connections between individual representations of even and odd numbers in Tilo’s model and the algebraic generalizations of even and odd numbers that they identified (Even=2n and Odd=2n+1, where n is a whole number). The source of difficulty was in making sense of the meaning of the variable ‘n’ in both the algebraic and geometric representations of even and odd numbers. This difficulty is illustrated in the following excerpt where the PMSTs grapple with whether ‘n’ represents *one tile*, *one complete column*, or *the total number of complete columns* in Tilo’s model:

Uberto: Every, every n is a column.
Olive: Yeah, every n is a column.
Uberto: [Tilo’s] saying there [are] no sets- there’s no columns and then there’s one extra.
Kaila: Every two n would have to be a column.
Uberto: Yeah. Every two n is a column.
Instructor: Every two n is a column, ok. Help me see where that is here. How do you see that?
Uberto: Actually isn’t it just the n? It is just the n because, look at two. If you plug it [into] the two n equation, you have one [complete] column, and therefore two tiles.

In this excerpt, Uberto and Olive initially state that every ‘n’ is a column – suggesting that ‘n’ represents each column. This understanding of the variable is not consistent with the algebraic generalizations of the sets of even or odd numbers. Instead, even and odd numbers were being represented differently and treated *individually* rather than as belonging to a *set* of numbers with common attributes (e.g., 2=2(n), 3=2(n)+1, 4=2(2n), 5=2(2n)+1, 6=2(3n) where n=1 column). Kaila argues that every ‘2n’ would be a column – suggesting that ‘n’ represents an individual tile. Again, this understanding of the variable would result in different expressions for even and odd numbers (e.g., 2=(2n), 3=(3n), 4=(4n), 5=(5n), 6=(6n), where n=1 tile). While both of these ways of making sense of even numbers suggest that the PMSTs were thinking more arithmetically than algebraically, there seems to be a qualitative difference in Uberto and Olive’s thinking as compared to Kaila’s thinking. Whereas the expressions aligned with Kaila’s thinking showed no commonality within the sets of even or odd numbers, expressions for Uberto and Olive’s thinking suggested that the components of the algebraic expressions for even and odd numbers (i.e., ‘2’, ‘n’, ‘+1’) needed to be considered. This became more evident after Uberto responded to Kaila’s idea about the meaning of ‘n’ by drawing the PMSTs’ attention to the number of tiles in a column compared to the total number of columns. As such, Uberto was extending the PMSTs’ reasoning beyond thinking about individual even and odd numbers to beginning to find commonality within these two sets of numbers.

Explicitly Connecting the Geometric and Algebraic Models

As the PMSTs continued to reason collectively about the meaning of the variable ‘n’ in the algebraic generalizations of even and odd numbers, they frequently used the geometric representations to make further distinctions between the components of the algebraic expression. This movement back and forth between representations is reflected in Bobbie's comment:

> How about we just define n and say for two, you have two tiles on top of each other and that's a column, that's one [complete] column. So if you define n as the [total] number of [complete] columns, if you put [it] in the formula, two n, two times one, it'd give you two. So we have seven, right, to get eight, you would add another tile on top of the one that's out we would end up with one, two, three, four columns, n representing the number of [complete] columns, you do two times four, it would give you eight. So we should just have- n should just represent the [total] number of [complete] columns, and it'll give you the number of- um even number.

In this excerpt, Bobbie was thinking algebraically and making an argument for why 'n' represents the *total number of complete columns* rather than representing *one complete column*. By comparing specific even and odd numbers in the geometric representation to the algebraic generalization of even and odd numbers, she was able to determine that the '2' represents the number of tiles in a column, and that the 'n' represents the total number of complete columns. Furthermore, Bobbie identified a difference between columns that are complete (by explicitly stating, "you have two tiles on top of each other and that's a column, that's one [complete] column") and those that are not complete for an odd number (by stating, "you would have to add another tile on top of the one that's out.") As such, Bobbie's way of thinking was consistent with the algebraic generalizations of even numbers and odd numbers and allowed for even numbers to be expressed as '2n' and odd numbers to be expressed as '2n+1'. Isabel clarified the meaning of the components even further by noting:

> I think what [Bobbie and others] are trying to say is that two is a constant; there are two tiles in a column, so that doesn't change. So that the only number that changes is n, and it says how many [total complete columns]. So if you have like eleven [complete columns], it'd be two times eleven which would put you at twenty-two tiles.

Despite PMSTs' transition toward thinking about even and odd numbers algebraically (the discussion ended with Milestone c), as the class continued to make connections between the different representations, the meaning of the variable 'n' in the algebraic generalization was often revisited. The pattern of Milestones c, b, c, b, c, b, c (shown in Figure 3)—shifting back and forth between Milestones b and c—suggests that PMSTs' understanding of the meaning of 'n' in the algebraic expressions for even and odd numbers remained unstable.

Discussion

This study examined PMSTs' reasoning as they transitioned from thinking arithmetically to thinking algebraically about even and odd numbers while making connections between algebraic and geometric representations of even and odd numbers. Flexibility in connecting multiple representations is a critical aspect of mathematical understanding (e.g., Lesh, Post, & Behr, 1987; NCTM, 2000), and the results of this study suggest that this is challenging for teachers. The geometric model (Tilo's model) seemed to be a catalyst for helping the PMSTs in this study reason about the different components of the algebraic generalizations of even and odd numbers (i.e., the '2', the 'n', and the '+1') as they transitioned from thinking about *individual* even and odd numbers to thinking about the *set* of even and the *set* of odd numbers. Furthermore, although PMSTs easily generated algebraic representations of even and odd numbers—that included the variable 'n'—they had difficulty making sense of what 'n' represents when asked to *connect* the algebraic representations to the geometric one. Thus, the introduction of the geometric model revealed PMSTs' conceptions in ways that working exclusively with the algebraic model may not have.

The results of this study also suggest that the transition from thinking about *individual* numbers to thinking about *sets* of numbers is not uni-directional, but rather involves several conceptual milestones—some of which are incorrect—that teachers move back and forth between as they made sense of generalizations of even and odd numbers. It is also important to note that teachers reverted back to the incorrect conceptual milestones throughout the whole-class discussion – suggesting that their misconceptions were rather stable. Thus, the results of this study indicate that mathematics teacher educators face a serious challenge: how to support prospective or practicing middle school teachers in making sense of and connecting multiple representations of mathematical ideas. Future work will investigate how PMSTs used these different representations to add and multiply even and odd numbers and how the mathematics teacher educator who was the instructor of this course supported them in making sense of the representations.

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DEVELOPING THE MATHEMATICS EDUCATION OF ENGLISH LEARNERS SCALE (MEELS)

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In this paper, we describe the initial stage of reliability and validity testing for the Mathematics Education of English Learners Scale (MEELS), which is designed to measure preservice teachers' beliefs about the mathematics education of English learners. To address the content validity, we consulted with experts within the field of mathematics education to assess the relevance and representation of specific items. We used Principal Component Factor Analysis to determine construct validity. Finally, we tested reliability of the factors using Cronbach's coefficient alpha. After sharing the findings from these analyses, we describe the next stages in the development of MEELS.

Keywords: Beliefs; Measurement; Equity and Diversity; Teacher Education–Preservice

The growing population of English Learners (ELs) in schools across the country has made it necessary to prepare all teachers, including mathematics teachers, to work with EL students in their classrooms (Costa, McPhail, Smith, & Brisk, 2005; Lucas & Grinberg, 2008). Given that beliefs are "lenses that affect one's view of some aspect of the world or as dispositions toward action" (Philip, 2007, p. 259), it is important for educators to know what preservice teachers (PSTs) believe about the mathematics education of ELs, especially when they assume a deficit perspective. As a first step in challenging these beliefs, the goal of our study is to develop a valid and reliable instrument, called the Mathematics Education of English Learners Scale (MEELS), which will measure the beliefs that PSTs have about issues related to the teaching and learning of mathematics to ELs. In this paper, we will describe the first stage analysis for validity and reliability that was conducted after 334 PSTs responded to MEELS.

Literature Review

A review of the literature revealed that there are a number of surveys in the areas of PSTs' or teachers' beliefs about diversity and multiculturalism, language attitudes, and inclusion of ELs. One such survey is the Cultural Diversity Awareness Inventory (CDAI; Henry, 1986), which consisted of 28 statements. The CDAI was developed to measure general cultural awareness that educators had about young children from culturally diverse backgrounds. The inventory was based on the understanding of "culture" that included five areas: (1) values and beliefs, (2) communication, (3) social relationships, (4) food and diet, and (5) dress. Despite the lack of information regarding the validity and reliability of the CDAI, it has been used by other researchers (e.g., Davis & Turner, 1993; Larke, 1990) to assess the cultural sensitivity of elementary PSTs.

Pohan and Aguilar (2001) designed two measures to elicit the personal and professional beliefs that educators had about diversity. In their survey, they extend traditional definitions of diversity of race and/or ethnicity to include other marginalized groups based on social class, gender, religion and sexual orientation. The researchers conjectured that there were some beliefs in the personal and professional spheres that could be in conflict. For example, an educator might believe that bilingualism was good in the current diverse society; however, they may not approve of public money being spent in maintaining bilingual programs. The reliability and validity of both scales were extensively tested with samples that included PSTs, graduate students, and practicing educators from rural and urban schools.

Byrnes and Kiger (1994) designed the Language Attitudes to Teaching Scale (LATS) that contained 13 items about the beliefs that teachers had about EL students. A factor analysis yielded 3 factors tied to the politics of language, intolerance of EL students and language support. The researchers carried out face and construct validity, and the subscale reliabilities, using Cronbach's alpha, ranged from 0.60 to 0.72.

Discussion about best practices for EL students in the content area have consistently advocated for inclusion of these students in mainstream classes along with the use of Sheltered Instruction practices (Echevarria, Vogt, & Short, 2008). This aspect has motivated researchers to examine the beliefs that teachers have towards the inclusion of ELs in their content classes. For example, Reeves (2006) investigated the beliefs that secondary teachers had about having EL students in their classroom through a self-designed survey. Walker, Shafer, and Iliams (2004) used a survey to assess mainstream teachers attitudes towards EL students and how these attitudes varied across schools where there were few ELs, a rapidly growing EL population, and ones where ELs were predominant. In both studies, only the face validity of the instrument was discussed.

All of the surveys reviewed, only a sample of which were discussed in this section, measured teachers' and/or PSTs' beliefs about cultural diversity, their attitudes towards linguistic diversity, and inclusion of EL students in their classroom. The instruments were broad in scope and encompassed teachers and PSTs in all subject areas. Reliability and validity were done only in two instruments (Byrnes & Kiger, 1994; Pohan & Aguilar, 2001). In all other cases, researchers developed surveys to examine the impact of an intervention, but validity and reliability of the instruments were either not discussed or the researchers only used participants' feedback to determine the clarity of the questions (face validity). Absent from these descriptions were other forms of validity like content, construct and criterion, and measures of reliability such as internal consistency reliability and test-retest reliability. Thus we need an instrument that will capture the different dimensions of the construct, have items that are consistent on each dimension, and that remains stable over time.

Beyond the dearth of statistically demonstrated valid and reliable surveys, we also did not find a survey that measured discipline specific beliefs with respect to the mathematics education of ELs. According to Cooney Shealy, and Arvold (1998) and Philip (2007), beliefs are tied to the context and this could be very different for different situations. In mathematics, as compared to other subjects, it is possible that the PSTs may assume that language plays a minimal role in the teaching and learning of mathematics, despite evidence to the contrary (e.g., Moschkovich, 2010). Teachers who assume that language plays a minimal role in the teaching and learning of mathematics then may be less likely to adjust their teaching in order to accommodate ELs. In fact misconceptions about language lead to a high proportion of ELs being labeled with a learning disability when they can converse in English, but struggle with the content, which involves academic English (Gandára & Contreras, 2009). With these considerations, the goal of our study was to design a valid and reliable measure that examined the beliefs that PSTs had about the mathematics education of ELs.

Theoretical Perspective

The overall framing of MEELS and the items in particular were guided by non-deficit views about ELs and their communities (Civil, 2007; Moschkovich, 2010). According to Moschkovich, "deficit models stem from assumptions about learners and their communities based on race, ethnicity, SES (socio-economic status), and other characteristics assumed to be related in simple, and typically negative ways to cognition and learning in general" (p. 11). Non-deficit models, on the other hand, assume that the EL students are part of different Discourse communities and have valuable resources that are assets which teachers can use to develop the students' knowledge (Civil, 2007; Moschkovich, 2010). For example, mathematical algorithms that students may have studied in other countries could be welcomed by the teacher, viewed as a resource, and shared with other students in the class. Further, the communities and parents of the ELs are also seen to have valuable knowledge that could be utilized in the classroom (Civil, 2007). Thus in framing and later scoring the items we assumed, for example, that bilingualism was an asset rather than a hindrance to EL students and that parents from all communities fundamentally cared about the intellectual development of their children, even if this was not visible to the teacher or did not adhere to a preconception of what that caring should look like (e.g., attending parent-teacher conference, volunteering in the classroom, etc.). Overall, the items for MEELS were drawn from recommendations from research, interactions with other mathematics educators, and other diversity surveys [refer to

McLeman & Fernandes (under review) for a review of the literature that guided the item development of MEELS].

Methods

Sample

MEELS was administered to 334 PSTs, about 75% of whom were located at one university in the southeast of the United States. Of the 330 responses we analyzed (4 were determined to be outliers), about 86.1% were female and about 84.8% self-reported their race as "White, not of Hispanic origin." Furthermore, a little more than 70% of the participants wanted to teach grades K–5, while about the same percentage of participants had no prior teaching experience. Close to 72% had been exposed to issues involving ELs in prior coursework and about 77% had been involved in some type of field experience during their teacher preparation program. Finally, about 92% of the participants self-reported that they were not fluent in a language other than English, though about 85% did report that they had some experience in learning a second language.

Instrument

The first iteration of MEELS consisted of two sections: the first was comprised of 8 demographic items while the second had 26 items related to the teaching and learning of mathematics to ELs. Each item was measured on a 5-point Likert-type scale: Strongly Disagree (1), Disagree (2), Undecided (3), Agree (4), and Strongly Agree (5). At the end of the survey, participants were asked to answer three open-ended questions to ascertain the readability and clarity of the instrument. Content validity was determined by sending an initial set of items to 10 mathematics education experts in the area of ELs. Based on their feedback, we modified some of the items before the first pilot of MEELS. For example, the language in several items was modified and additional items were included.

MEELS was administered online through SurveyShare (http://www.surveyshare.com) and thereafter the data were downloaded to SPSS 17 for analysis. Using our theoretical perspective as a basis, we reverse scored certain items that would reflect deficit beliefs based on our reading of the literature. For example, the item "Some EL's home culture negatively impacts their math learning" was reverse coded with a PST strongly agreeing scoring a 1 and strongly disagreeing scoring a 5. Since beliefs can only be inferred, it was conjectured that a PST who believed that an EL's home culture could negatively impact their mathematical learning would be less open to seeing certain ELs' home culture as resource in the classroom. In total, 14 of the 26 items on the survey were reverse coded; these are shown with an r in Table 1. Note that there are 23 items in Table 1 as three items were dropped in later analysis. One of the dropped items was reverse coded.

Statistical Analyses

Our main goal was to establish construct validity of MEELS through factor analysis and examine reliability of the resulting subscales associated with the factors (also referred to as internal consistency). We first scanned the data for outliers using Mahalanobis distance. Next, we examined the coefficient matrix, which summarizes the interrelations between the set of items. We performed Bartlett's test of sphericity to ensure that the correlation matrix was not an identity matrix, which would indicate that there was no relationship between the items. We examined the Kaiser-Meyer-Olkin (KMO) statistic that indicates if the sample size was adequate relative to the 26 items in the instrument. In addition to the overall KMO, we also examined the anti-image correlation matrix for a Measure of Sampling Adequacy (MSA) for the individual items. Pett, Lackey, and Sullivan (2003) recommended that the individual MSA (numbers along the diagonal of the anti-image correlation matrix) should be greater than 0.60 to ensure the presence of underlying factors.

After this preliminary analysis, we proceeded to identify clusters of inter-correlated items, usually referred to as factors, which would indicate the various dimensions related to our construct of the mathematics education of ELs. It is important to note that in developing the items we had some conjectures based on the literature about what these dimensions might be. We used exploratory factor analysis as opposed to confirmatory factor analysis because we were uncertain about the dimensions, and in the process also wanted to ensure construct validity. There are different factor extraction methods in SPSS 17, though Principal Component Analysis (PCA) and Principal Axis Factoring (PAF) are most widely used (Pett, Lackey, & Sullivan, 2003). In trying to determine the best method for our purposes, we relied on the advice of Pett, Lackey, and Sullivan to start with a preliminary solution using PCA, refine the solution by examining the items that load on the various factors (where *load* refers to the correlation between an item and factor), and then develop a preliminary solution. This solution is then compared to the PAF solution on the same matrix and the final solution is "one that is the best fit and that makes the most intuitive sense" (Pett, Lackey, & Sullivan, 2003, p. 115). Once we obtained our factors we worked out the internal consistency of the items that made up a particular factor using Cronbach's coefficient alphas.

Results

The Mahalanobis distance for multivariate data ($p < .001$) (Stevens, 1992; Tabachnick & Fidell, 1996) revealed four outliers that were dropped from the subsequent analysis. Thus the total number of responses examined was 330. Bartlett's test of sphericity was significant (chi-square = 1951.35, df = 253, $p < 0.001$) indicating that there were relationships between the items. The Kaiser-Meyer-Olkin of 0.854 was greater than 0.7 suggesting a sufficient sample size (Kaiser, 1974). The diagonal of the anti-image correlation matrix yielded the Measures of Sampling Adequacy (MSA) which ranged from 0.54 to 0.91, with most values greater than 0.7 and the off-diagonal absolute values were small, thus suggesting that the matrix was factorable (Pett, Lackey, & Sullivan, 2003).

Factor Analysis

Initial PCA suggested 7 factors that satisfied Kaiser's rule (Kaiser, 1974) with eigenvalues greater than 1. Note that our goal in factor analysis was to reduce the number of variables (items in the instrument) into a smaller number of factors that would account for as much of the variation between the individual variables. Towards this end, we chose factors with the largest eigenvalues that would explain more of the variance than an individual item. In exploratory factor analysis, once the initial solution is obtained, variables generally tend to load highly on a factor and have small loadings on others. By rotating the factor axes, the loading of a variable is increased on one of the extracted factors and is minimized on the other factors (also called a simple structure). Thus rotation increases the interpretability of the factor as a group of items load highly on it. In trying to determine the number of factors to retain after rotation, Pett, Lackey, and Sullivan (2003) suggested retaining the fewest number of factors that explained at least 50% of the variance and that the factors make intuitive sense in the given context. With this in mind, we sought to have at least 3 items load on a factor with a loading greater than 0.3, all the factors account for at least 50% of the variance, and that the resulting item-factor correlation matrix achieve simple structure. A closer examination of the rotated matrix with loadings that were more than 0.30, suggested factor 6 comprised of items 18, 23, and 33, and items 23 and 29 loaded on factor 7. Items 33 and 29 loaded only on factors 6 and 7, respectively, while items 18 and 23 also loaded on factors 4 and 5. Since items 29 and 33 only loaded on 6 and 7, we decided to rerun the analysis without these items. A subsequent PCA with Varimax rotation yielded 6 factors with eigenvalues more than 1 and several items that loaded on multiple factors. To make it easier for interpretation, we decided to look at items that loaded 0.40 or more on a factor. Still there were 5 items that loaded on multiple factors and after dropping these items one at a time in further analyses, a decision was made to eliminate item 26. A rerun of the PCA with Varimax rotation, in this case, extracted 5 factors with eigenvalues more than 1. The factors accounted for about 52% of the variance in these items. Table 1 displays the 23 items with loadings greater than 0.4 that loaded on 5 factors in an almost simple structure (three items loaded on multiple factors). After we obtained this initial solution, we followed Pett, Lackey, and Sullivan's (2003) suggestion and ran the PAF with Varimax rotation (with items 26, 29, and 33 dropped). This yielded 4 items that did not load (>0.40) on any of the 5 factors; one of the factors had only 2 items, and another factor that was difficult to interpret. Thus we retained the 5 factor solution obtained with PCA and Varimax rotation as it was the one that most aligned with our extraction criteria (modifying the loadings from >0.30 to >0.40). For the three items that loaded on multiple factors, we decided to associate them with the factor for which a higher loading was demonstrated. Note that a consideration of loadings more than 0.45 would have yielded a simple structure for the item-factor loading matrix shown in Table 1. Note that since we approximated simple structure with the simpler orthogonal rotation (Varimax), we avoided using oblique rotation in the above analysis.

Table 1: Factor Loadings

	Items	**T**	**LSC**	**F**	**LM**	**C**
17	Open to teaching ELs math.	0.52				
22	Open to integrating EL's background in math.	0.62				
27	Adjust the language on math problems.	0.65				
28	Focus on the language skills.	0.67				
30	Accept EL's non-verbal communication.	0.70				
31	Accept alternative math algorithms.	0.55				
36	Open to using alternative math assessments.	0.57		(0.41)		
11	*Fluency* in more than one language.		0.65			
12r	English as only language of instruction		0.75			
13r	More important for beginning ELs to learn English.		0.45			
14	Open to use of native language.	(0.44)	0.56			
15r	Native language use hampers learning English.		0.50			
16	State math tests in different languages.		0.50			
18r	Rich math discussions.			0.61		
32r	Teach ELs and non-ELs in the same way.			0.62		
34r	Accommodations are unfair.			0.53		
35r	Same standards for ELs and non-ELs.			0.79		
19r	Math ideal for transition of beginning ELs.				0.71	
20r	Conversational language.			(0.40)	0.64	
21r	Math is not language intensive.				0.77	
23r	EL's home culture.					0.57
24r	Parents.					0.72
25r	Some ethnicities better at math.					0.71

T=Teaching, LSC=Language in school context, F=Fairness, LM=Language and Mathematics, C=Culture.

Based on the items that loaded on the 5 factors, and paying particular attention to the items that displayed higher loadings, we labeled the factors—Beliefs about teaching (T; 7 items), Beliefs about language in the school context (LSC; 6 items), Beliefs about fairness (F; 4 items), Beliefs about the interconnection of language and mathematics (LM; 3 items) and Beliefs about culture (C; 3 items). The alpha coefficients are given in Table 2.

Table 2: Cronbach's Coefficient Alphas

	Alpha
Teaching	.79
Language in School Context	.73
Fairness	.66
Language and Mathematics	.59
Culture	.48

Discussion

The development of a survey and establishing validity and reliability is an evolving process. In this first iteration, we determined content and construct validity along with internal consistency reliability. For the content validity we consulted with experts within the field of mathematics education to assess the relevance and representation of the items we included in our instrument. We used Principal Component Factor Analysis (PCA) along with Varimax rotation to determine the construct validity of the underlying factors that impact PSTs' beliefs about the mathematics education of ELs. To determine reliability of the factor subscales, we used Cronbach's coefficient alpha. These analyses point to the next steps in the refinement of MEELS.

Next Steps

Cronbach's coefficient alphas for the two subscales LM and C are lower than 0.6, which is poor (Burns & Burns, 2008) and will need to be addressed in the next phase of refinement and testing of MEELS. The presence of only 3 items in each of these subscales could be a possible reason for the low alphas. Thus one course of action would be to increase the number of items in this category. However the inclusion of additional items will need to take into the consideration the connection that issues of culture and education have to outside influences such as political ideology. Furthermore, how a participant interprets particular words in the items is of importance to consider. Therefore, we intend to interview a group of PSTs to ascertain their interpretations of particular items. By doing so, we will be able to refine the wording of certain items that were dropped and/or add additional items. In either case, the goal will be to load the items on to the specific factors we have (and not to generate new factors).

The interviews with the PSTs will also help us reframe items 36, 14 and 20 that loaded on multiple factors. In each case, deleting the item from the factor to which it was assigned (based on the higher load) would reduce the alpha. Therefore, a decision was made to retain each item and refine the wording so that it was more likely to load on the assigned factor. Further, by removing item 20 the LM factor would consist of only two items and not meet our criteria for retaining factors. Given the goal of MEELS is to examine PSTs' beliefs about the mathematics education of ELs, having a subscale related to the beliefs about the interconnection of language and mathematics is important.

In the next iteration of MEELS, once the items are refined, we intend to also carry out Confirmatory Factor Analysis (CFA) and test-retest reliability. Confirmatory Factor Analysis (CFA) allows us to test the five factors that we have. The test-retest reliability ensures that the instrument has temporal stability. Specifically we will administer MEELS to a group of PSTs at differing points in the semester (but no longer than a few weeks of each other) to determine if there is a high correlation between their scores at each point. The purpose of administering MEELS at multiple points within a short period of time is to speak to stability in the given constructs.

Finally, our long-term goal is to establish predictive validity of the MEELS. This would require tracking the PSTs into their teaching careers to establish a relationship between the PSTs performance on the MEELS and their teaching of ELs in their mathematics classes. The latter would require the development of observational instruments that could measure the performance of the teacher in implementing best practices for teaching ELs. Currently, we are not aware of such an instrument.

Implications

The Mathematics Education of English Learners Scale (MEELS) is a powerful and necessary instrument for teacher preparation programs. At the end of our validation and reliability process, we will have developed a measure that can be used to ascertain the beliefs of PSTs regarding the mathematics education of ELs on a large scale, an undertaking that we have not come across in our review of the literature. Additionally, among other things, MEELS can be shared with individuals in the field so that the effectiveness of particular interventions with ELs can be measured.

While still in the early stages of reliability and validity testing for our instrument, we have generated some important implications for the field of mathematics education, in general, and teacher education, specifically. Though the content validity of MEELS was determined and only a few (about 4%) participants reported any vague or confusing questions, our initial analysis revealed that not all of the items we developed mapped onto our initial dimensions. In other words, some of the items in MEELS did not describe the beliefs with which the wording appeared. Thus the use of surveys that are not shown to be statistically validated and reliable may be problematic, as they may not always be measuring what they appear to be on the surface.

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THE STAGES OF EARLY ARITHMETIC LEARNING: A CONTEXT FOR LEARNING TO PROFESSIONALLY NOTICE

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The goal of this study is to develop the professional noticing abilities of preservice elementary teachers in the context of the Stages of Early Arithmetic Learning. In their mathematics methods course, the preservice elementary teachers participated in a researcher-developed multi-session module that progressively nests the three interrelated components of professional noticing—attending, interpreting, and deciding. A pre- and post-assessment was administered to measure their change in the three components of professional noticing. The preservice elementary teachers demonstrated significant growth in all three components.

Keywords: Elementary School Education; Learning Trajectories; Number Concepts and Operations; Teacher Education–Preservice

Introduction and Literature

This study examines the extent to which an innovative learning experience focused on the professional noticing of children's early numeracy develops Preservice Elementary Teachers' (PSETs) capacity to attend to, interpret, and make effective instructional decisions related to the mathematical thinking of children. The study relies upon a module, *Noticing Numeracy Now (N^3)*, developed by the researchers and based on literature in the areas of professional noticing (Jacobs, Lamb, & Philipp, 2010) and the Stages of Early Arithmetic Learning (SEAL) (Steffe, von Glasersfeld, Richards, & Cobb, 1983; Steffe, Cobb, & von Glasersfeld, 1988; Steffe, 1992). Specifically, we intend to investigate the following research question: To what extent can teacher educators facilitate the development of PSETs' capacity to professionally notice children's mathematical thinking?

Professional Noticing

Professional noticing is an ability to recognize and act on key indicators significant to one's profession. The literature exploring the impact of professional noticing in mathematics teaching grew significantly in recent years. Sherin and van Es (2009) examined teacher video clubs as a tool for analyzing the club participants' classrooms and found that using focused noticing as a lens for learning about teaching was productive beyond the video club, impacting the teachers' instructional practices. Star and Strickland (2008) demonstrated improvements in preservice teachers' ability to attend to the salient features of a secondary mathematics classroom. Numerous professional development modules incorporated the use of video to focus observers' attention on children's mathematical thinking (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Schifter, Bastable, & Russell, 2000; Seago, Mumme, & Branca, 2004). Sherin, Jacobs, and Philipp's recently edited volume (2011) contributed to the compounding evidence of both the need and the value of professional noticing to effective mathematics teaching.

Recent evidence shows that teachers' attention to children's mathematical thinking can positively affect student learning (Carpenter et al, 1999; Kersting et al, 2010); however, such attention is just one component skill of *professional noticing* of children's mathematical thinking as defined by Jacobs, Lamb, and Philipp (2010). They conceptualized professional noticing as "a set of three interrelated skills: *attending* to children's strategies, *interpreting* children's understandings, and *deciding* how to respond on the basis of children's understandings" (p. 172). Their research examined the professional noticing of preservice teachers as well as three groups of in-service teachers all having 12–14 years of teaching experience but varying degrees of professional development. The results indicate that teaching experience alone does not develop all three components of professional noticing. Teachers with 12–14 years experience, but no sustained professional development, aligned more closely with preservice teachers on this construct, especially on the deciding component. Star and Strickland (2008) further contend that developing the skill of professional noticing must be an early focus of teacher preparation programs given the importance of skillful, nuanced observation in learning to teach.

Practice-based Teacher Preparation

Preparing for the profession of teaching requires opportunities to practice one's teaching. Current trends in teacher preparation focus on practice-based teacher preparation but with varying degrees of guidance on the particulars of what should be practiced and where practice should take place. Grossman, Compton, Igra, Ronfeldt, Shahan, and Williamson (2009) found that teacher preparation practices focused heavily on the preactive aspects of teaching, such as lesson and unit planning, and less on interactive or reflective aspects. The interactive and reflective aspects include the nearly invisible decisions based on professional noticing that must be made 'in the moment' of teaching. We expect novice teachers to observe the professional noticing in real-time classrooms, but without explicit guidance, novice teachers may not observe it. Videos of teaching/learning exchanges are representations of practice, one of three pedagogies of practice proposed by Grossman and her colleagues. Such representations can potentially provide powerful settings for learning the practice of teaching and in doing so can provide a scaffold for subsequent practice in actual classroom-based contexts.

Stages of Early Arithmetic Learning

There is consensus that the term "numeracy" is grounded in an understanding of fundamental and foundational aspects of number and operations (Mulligan, Bobis, & Francis, 1999; National Council of Teachers of Mathematics, 2006) and is a significant content strand for PSET exploration. Steffe and his colleagues have provided a useful model for structuring the mathematics of the N^3 module. Born of longitudinal teaching experiments, the Stages of Early Arithmetic Learning (SEAL) hypothesize a progression for the development of quantitative understanding (Steffe, von Glasersfeld, Richards, & Cobb, 1983; Steffe, Cobb, & von Glasersfeld, 1988; Steffe, 1992; Wright, Martland, & Stafford, 2006). This progression includes the following levels: Emergent, Perceptual, Figurative, Initial Number Sequence, Intermediate Number Sequence, and Facile. Given the supporting methodology, SEAL is exemplary of "learning trajectories built upon natural developmental progressions identified in empirically based models of children's thinking and learning" (Clements, 2007, p. 45).

Methodology

Participants

The participants in this study were preservice elementary teachers (PSETs) enrolled in one of three participating universities; two are regional universities and one is a "Research Very High" university (Carnegie Foundation for the Advancement of Teaching). All universities are public institutions and the participants (n = 94) represent a cross-section of the general population of a state in the east-central United States. Participants were enrolled in elementary mathematics methods courses at their respective universities and the module was a component of this course.

Module Description

The module consists of multiple in-class sessions during which the three components of professional noticing are developed in the context of the Stages of Early Arithmetic Learning. The three components of professional noticing, attending, interpreting, and deciding, are nested through the module (Boerst, Sleep, Ball, & Bass, 2011). The first two sessions focus on the development of attending only. Subsequent sessions further develop attending while progressively layering in interpreting and deciding. Integrated with the nested development of professional noticing, the Stages of Early Arithmetic Learning gradually unfold through video clip representations of practice. The researchers of the N^3 project intentionally chose video around early number sense for two reasons: (1) video is a representation of practice that provides opportunities to explicitly attend to and discuss salient features of children's mathematical thinking that can go unnoticed by novices in a real-time classroom setting, and (2) early number sense is an area of mathematics with which PSETs are generally comfortable so the mathematics itself would not be a barrier to the examination of children's mathematical thinking.

The video clips are diagnostic interviews with children conducted by teacher educators or a former PSET. A significant number of video clips are from one author's dissertation research of quantitative mental imagery (Thomas, 2010). PSETs are asked to respond to the videos in various ways, including writing about what they attend to in the video and engaging in discussion with a partner, with a small group, and in whole class discussions led by the instructor. At the beginning of the module, the discussion prompts are more general, such as asking them to observe the physical actions and verbal exchanges taking place. As the module continues, the PSETs learn to focus on the salient features of students' mathematical actions and words. In addition to these salient features, PSETs' attention is drawn to teacher moves and the mathematics of the tasks. As the sessions progress, PSETs learn to interpret the salient features in terms of SEAL and finally they learn to make decisions about next steps, either diagnostic or instructional. Between the sessions, the PSETs have homework including articles to read and videos to watch. The culminating experience is an assignment that requires the PSETs to conduct at least one diagnostic interview with a child. The interview assignment varies across the universities. Some are assigned immediately following the module, and others are assigned much later in the semester, dependent upon each university's field placement schedules.

Professional Noticing Measures

A pre- and post-assessment designed to measure a PSET's ability to apply professional noticing to a video clip representation of practice was developed. The pre-assessment was administered within a week of the start of the semester at all participating institutions. The professional noticing measure consists of a brief video clip (25 seconds) in which the interviewer poses a partially screened task that goes beyond finger range. The task is a comparison task, where the difference between two sets is unknown (Carpenter et al, 1999). After viewing the video twice, PSETs were asked to respond to three prompts, each related to one of the three aspects of professional noticing—attending, interpreting, and deciding. The three prompts, drawn from the work of Jacobs, Lamb, and Philipp (2010) are: (1) Please describe in detail what *you think* this child did in response to this problem, (2) Please explain what you learned about this child's understanding of mathematics, and (3) Pretend that you are the teacher of this child. What problems or questions might you pose next? Provide a rationale for your answer. In subsequent semesters, the italicized words were removed from the prompt to emphasize addressing the factual evidence of the video clip, not assumptions. The post-assessment task and protocol for delivery was identical to the pre-assessment. Administration of the post-assessment occurred within the last two weeks of the semester.

Construction of Noticing Benchmarks

The research team reviewed the professional noticing video segment and identified key response features for the attending prompt. We examined PSET attending responses from a single institution to identify emerging themes (Glaser & Strauss, 1967). Themes that emerged from this analysis included:

- Identifying key, salient activity (i.e., " . . . the child counted the bears, and then counted up to the amount of shells on his fingers")
- Identifying additional activity (i.e., " . . . he then looked to see how many fingers he had up")
- Operational presumptions (i.e., ". . . he subtracted 11-7")

- Purporting evidence that did not occur in the segment (i.e., " . . . the child counted back from 11 to 7")
- Cognitive interpretations (i.e., ". . . the child lacks a sense of cardinality")

We created the themes based on the emerging patterns in order to co-construct a set of initial common benchmarks. The identification of emergent themes played a significant role during this initial drafting process as the characteristics of PSET attending responses coupled with researcher-identified key features suggested four distinct response types (elaborate, salient, limited, subordinate), ranging from a score of 4 to a score of 1. We scored our calibration data using these initial benchmarks. We replicated this process for the construction of the interpreting benchmarks and the deciding benchmarks. It is important to note that the coupling of emergent themes with researcher-identified key features resulted in three response categories, for the interpreting and deciding benchmarks, one fewer than the four natural levels that resulted for the attending benchmarks.

Scoring and Statistical Tests

Once the scoring benchmarks for each of the professional noticing questions were established, the research team scored all data. Each researcher scored one set (grouped by university) of data and a second researcher scored it again. Scores were compared and any discrepancies were discussed between the two researchers for a consensus. A third researcher was used in any case where a consensus was not reached.

An example of a time when a third researcher was needed consisted of a response in the attending question of the survey that stated the following: "I've learned that children, especially at younger ages, use their fingers a lot to help them count. It takes children a while to think mathematically in their head and not rely on objects to count." The first researcher felt this response deemed level 2 because the PSET recognized the progression of children's thinking from more concrete to more mental, however the second researcher felt it was too generic for this question and it should score a level 0 score. The second researcher felt if the response was specifically directed toward the student in the video, instead of children in general, then they would have agreed with the first researcher's score. When the third researcher scored this question, they also felt it was too generic and a score of 0 was given for this response.

Only participants with six scores consisting of the pre-assessment and post-assessment scores for each of the three components of professional noticing at the conclusion of the scoring process remained in the data set. Any participant who did not have six was removed, resulting in $n = 94$. *T*-test analyses were conducted to determine if significant changes occurred between the pre-assessment and post-assessment scores. This information was determined for the entire group as well as between the individual universities.

Results and Discussion

A one-way ANOVA was conducted to determine if there were overall statistically significant differences between the pre- and post- total scores for the professional noticing measure. Additionally, pre- and post-scores on each of the three individual professional noticing questions were tested for significance using a one-way ANOVA. The descriptive statistics, stratified by university as well as totals, are reported in Table 1 below. Also, Table 2 below shows the average gain in score from pre- to post-assessment, stratified by university as well as total participants.

Table 1: Descriptive Statistics of Professional Noticing Measures by University

			Attending		Interpreting		Deciding	
		N	*M*	*SD*	*M*	*SD*	*M*	*SD*
University A	Pre-Test	37	2.14	.79	1.59	.797	1.54	.61
	Post-Test	37	2.43	.87	2.05	.84	2.22	.79
University B	Pre-Test	23	2.39	.99	1.82	.89	2.04	.56
	Post-Test	23	3.09	1.04	2.43	.73	2.70	.56
University C	Pre-Test	34	2.38	1.10	1.76	.78	1.97	.67
	Post-Test	34	3.00	1.10	2.15	.89	2.47	.75
All Participants	Pre-Test	94	2.29	.96	1.71	.81	1.82	.66
	Post-Test	94	2.80	1.03	2.18	.84	2.43	.74

Table 2: Average Gains of Professional Noticing Measures by University

	Average Change			
	Attending	*Interpreting*	*Deciding*	*Total*
University A	.29	.46	.68	1.43
University B	.70	.61	.66	1.97
University C	.62	.39	.50	1.51
All Participants	.51	.47	.61	1.59

The means show growth on all three professional noticing measures between the pre-assessment and post-assessment measures at all universities. The increase was found to be significant ($F = 63.169$, $p < .001$) following tests to determine whether there is a statistically significant difference between the total pre- and post- scores of professional noticing. Interaction between the total scores and each university was not found to be significant ($F = .493$, $p = .612$). This is a positive result in that each university in the study is showing gains consistent with other universities.

A one-way ANOVA was conducted for each of the three questions to determine whether statistically significant gains were found for each component of professional noticing (attending, interpreting, deciding). The results of the ANOVA are found in Table 3 below. The smaller *F*-value found for the attending component can be attributed to the larger scale for that question (4-point scale) when compared to the interpreting and deciding questions (3-point scale).

Table 3: Results of ANOVA Comparing Pre- and Post-Assessments of All Universities

	Scale	*N*	*F*	*p*
Attending	1-4	94	61.43	< .001
Interpreting	1-3	94	1075.92	< .001
Deciding	1-3	94	1014.84	< .001

The interpreting component demonstrated the smallest change in growth overall ($\Delta = .47$), while the deciding component found the largest change in growth ($\Delta = .61$). Although significance was found between the pre- and post-scores for interpreting, this was the most difficult benchmark to construct due to the diversity of the responses, so the researchers are optimistic that a suitable benchmark has been created to adequately measure the PSETs interpreting skills. PSETs reported that the five-session module was repetitive in instruction for attending and interpreting and did not include enough instruction on the deciding component. They believed they needed further instructions on how to determine the next steps in mathematical questioning and instructional tasks to advance students into a higher SEAL stage. The five-session module was reduced to a four-session module (to reduce length and repetitiveness) and a stronger emphasis on instructional deciding was included. Future analyses with the new module and further refinement of the scoring benchmarks could provide more accurate assessments for growth in the three areas of professional noticing.

A statistically significant difference was not found for the attending component ($F = .519$, $p = .597$) and the deciding component ($F = 2.187$, $p = .118$) when comparing the three components of professional noticing within the three universities in the study. However, a statistically significant difference was found ($p < .05$) between the universities for the interpreting component ($F = 3.962$, $p = .022$). This lack of a statistically significant difference between the universities in attending and deciding is a positive result because it informs the researchers that the PSETs in the study are equally distributed in those components. In order to examine the statistically significant difference in the interpreting component, a Tukey's post-hoc analysis was conducted. This test suggested that University A's scores were the cause of the significance when compared within the overall data. This seems realistic considering University A's scores in the interpreting component were lower than University B and C. However, an independent means t-test was conducted for each of the professional noticing components in efforts to further investigate the scores from the individual universities.

The data was stratified by university and University A had statistically significant differences at $p < .05$ between pre- and post-assessments for interpreting ($t = -2.217$, $p = .033$) and at $p < .01$ for deciding ($t = –4.104$, $p = .000$). Attending was not statistically significant ($t = –1.571$, $p = .125$). University B had statistically significant differences at $p < .05$ for attending ($t = –2.729$, $p = .012$) and interpreting ($t = –2.440$, $p = .023$), and at $p < .01$ for deciding ($t = –4.035$, $p = .001$). University 2 had similar results with statistical significance at $p < .05$ for attending ($t = –2.670$, $p = .012$) and interpreting ($t = -2.196$, $p = .035$), and significance at $p < .01$ for deciding ($t = –3.253$, $p = .003$). Despite the fact that University A was statistically significantly lower overall than Universities B and C in interpreting based on the ANOVA test, the *t*-tests still reveal significance between pre- and post-assessments in interpreting for University A. The lack of significance in the attending component for University A is not surprising either, considering the PSETs from University A demonstrated the smallest amount of growth when compared to Universities B and C.

It should be noted that in cases where students scored high on the pre-assessment, they would not be expected to show growth. For example, in the attending component, 37% of students (12 of 37) from University A scored at least a 3 on the pre-assessment indicating they attended to the most salient features of the video assessment. University B and C had similar results with 39% (9 of 23) and 38% (13 of 34), respectively on the pre-assessment. In the interpreting component, 19% (7 of 37) from University A, 30% (7 of 23) from University B, and 21% (7 of 34) from University C all scored the highest possible score on the pre-assessment. The results in the deciding component were similar with 5% (2 of 37) from University A, 17% (4 of 23) from University B, and 21% (7 of 34) from University C all scoring perfect scores in the pre-assessment for instructional deciding, thus not allowing those students to show growth in that component. The researchers see the limited upper range as an opportunity to further refine the scoring benchmarks to include additional range, allowing for the sector of PSETs who scored perfect pre-assessment scores to show growth.

Final Remarks

In summary, preliminary findings at three sites suggest the efficacy of a researcher-developed module aimed at promoting professional noticing capacities among PSETs in the area of early number and operation. The development of such capacities among aspiring teachers at multiple sites bodes well for scaled establishment of responsive teaching practices within teacher preparation programs. Towards this end, subsequent module implementation and measurement will occur in teacher preparation programs at two additional institutions (for a total of five implementation sites); moreover, two different institutions have been identified to serve as non-implementing comparison sites.

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PROSPECTIVE TEACHERS' PERCEPTIONS, BELIEFS, AND DISPOSTIONS TOWARD STUDENTS' FAMILY, COMMUNITY, AND CULTURE

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Research has repeatedly documented that teachers are underprepared to teach mathematics effectively in diverse classrooms. We believe critical aspect of learning to be an effective mathematics teacher for diverse learners is developing knowledge, dispositions, and practices that support capitalizing on children's cultural, linguistic, and community-based knowledge and experiences in mathematics instruction. This study examined beginning perceptions, beliefs, and dispositions of prospective teachers (PSTs) toward students' family, community, and culture. Results indicate that PSTs hold a range of beliefs based on how they see the resources and supports available to students in the home and community, how they compare and contrast themselves with their students and the students with each other, and how they see the nature of the relationships that can and should be formed by teachers with students and families.

Keywords: Equity and Diversity; Teacher Beliefs; Teacher Education–Preservice

Research has repeatedly documented that teachers are underprepared to teach mathematics effectively in diverse classrooms (Kitchen, 2005; Sleeter, 2001). While there is significant research related to preparing teachers to work in diverse classrooms, little of it addresses the specific challenges and resources of learning to teach *mathematics* to diverse learners (for exceptions see Aguirre, 2009; Foote, 2009; Kitchen, 2005; Moschkovich & Nelson-Barber, 2009). One component of learning to be an effective mathematics teacher for diverse learners is developing *knowledge*, *dispositions*, and *practices* that support capitalizing on children's cultural, linguistic, and community-based knowledge and experiences. Also relevant are dispositions and practices that support eliciting and incorporating this knowledge into mathematics instruction (e.g., Civil, 2007; Ladson-Billings, 1994; Leonard, 2008). Research documents that historically underrepresented groups benefit from instruction that draws upon their diverse cultural, linguistic, home, and community-based knowledge (Ladson-Billings, 1994; Lipka et al., 2005; Silver & Stein, 1996). This research has argued that teachers need to understand how children's *funds of knowledge*—the knowledge, skills, and experiences found in children's homes and communities—can support children's mathematical learning (Civil, 2002; González, Andrade, Civil, & Moll, 2001). And yet there exists a gap in lived experiences between the largely White middle-class teachers and their ever more diverse students (Howard, 1999; Wiggins & Follo, 1999) that may influence their ability to do just that.

This study is part of a larger research project entitled Teachers Empowered to Advance CHange in Mathematics (TEACH MATH). The overall goal of this project is to transform preK–8 mathematics teacher preparation so that new generations of teachers will be equipped with powerful tools and strategies to increase mathematics learning and achievement in our nation's increasingly diverse public schools. In this paper we examine the beliefs, perceptions, and dispositions that prospective teachers (PSTs) bring to the mathematics methods class. In preparing PSTs, it is important for mathematics teacher educators (MTEs) to understand the range of PSTs' beliefs, perceptions, and dispositions so that MTEs can support PSTs in valuing what students bring to school rather than looking at homes and communities from a deficit perspective. Insights from this paper may serve to shape how MTEs interact with PSTs around issues of supporting the development of effective mathematics instruction (Ball & Tyson, 2011).

In previous work (Turner et al., 2012) we presented a hypothetical trajectory for PST learning. Drawing on Mason (2008), one phase of this trajectory includes a focus on initial practices of *attention* and *awareness. Attention* refers to *what* teachers attend to, including what they notice (as well as what they fail to notice), and the depth and detail of their attention. Mason argued that a key role of teacher educators is to direct PSTs' attention to relevant practices, theories, and ideas that can guide their decisions and actions when teaching. One particular form of attending to specifics Mason refers to as *discerning details.* In cases we will discuss in this paper, PSTs are discerning details when they attend to the specifics of children's home and community contexts.

Awareness refers to understandings, insights, knowledge, and beliefs about teaching and learning mathematics. While attention refers to *what* teachers notice, awareness refers to *how* teachers interpret what they notice (Mason, 2008). Mason described the role of the teacher educator as one of educating awareness, both so teachers are more cognizant of their own knowledge and beliefs and to help teachers develop core types of awareness that support effective mathematics teaching. What PSTs attend to and how they interpret or assign meaning is influenced by the beliefs and dispositions they bring to observing and working with students.

We are attempting to support PSTs in developing a positive stance toward students and their families and communities. We hope that in accessing PSTs' notions, we can confront deficit thinking in the mathematics methods classroom so that PSTs are indeed better prepared to teach the diverse students they will meet. We nonetheless have seen (Turner et al., 2012) that PSTs' awareness is often inconsistent with a positive stance. Mason (2008) referred to such inconsistencies as *fragmented awareness*. As the findings from Turner and colleagues demonstrated, fragmented awareness was often evident in PSTs' comments about the role that families and communities play in supporting children's mathematics learning. Although PSTs spoke about families as capable of encouraging children's learning and supporting the development of mathematics skills (i.e., families as resources), at the same time, they framed certain characteristics of some families and communities as deterministic and detrimental to children's learning, particularly a lack of English proficiency and a low socioeconomic status (i.e., deficit-based view of families/communities). In this paper, we examine more specifically both early positive and deficit notions of children's families and communities. One research question for the larger project is: What is the nature of PSTs knowledge and beliefs related to integrating children's mathematical thinking and children's cultural, linguistic and community-based funds of knowledge in mathematics instruction? In this study we are focusing on PSTs' initial self-positioning by asking: What is the nature of PSTs' knowledge and beliefs with regard to children's family, culture, and community?

Methods

This study draws on interview data from 17 participants interviewed either at the beginning or the end of the semester (or in some cases at both junctures) [9 pre and post; 4 each pre or post] while they were enrolled in an elementary or middle school mathematics methods course. Participants were selected from a group of approximately 200 elementary and middle school PSTs enrolled in mathematics methods courses at six university sites that represent a diverse range of teaching contexts (i.e., urban; a mixture of urban, suburban, and rural; suburban; and borderlands). These PSTs participated both in pre- and post-course surveys. The surveys included 18 Likert-type items, six short answer responses, and between two and four instructional scenarios (pre-survey only). Follow-up interviews to the survey were conducted with 17 PSTs at three of the universities (one on the east coast, one in the western region, and one in a borderlands region in the southwest). During the interview PSTs were asked to clarify and expand on their answers to a number of the survey items. The interviews were audio-recorded and transcribed for analysis. These interviews serve as the data source for this paper.

Three pairs of researchers worked on the initial coding of data, using a set of codes that emerged from the research question. These codes included such things as PSTs themselves as teachers, students as learners, and the role of language and culture in the learning of mathematics. HyperResearcher, a coding software, was used in the coding process. Each interview was coded twice by pairs of researchers. These pairs then met to discuss discrepancies in coding until there was agreement on the coding of each interview passage. We define a passage to be a number of lines of interview text that addressed one of the ideas we were examining. Codes were then collapsed and refined, and the data were reorganized by these new categories. Different pairs of researchers began work on subsections of the data. One of these subsections included data that had been coded in the first round with codes that pertained to family, culture, and community both generally and as they pertained to the teaching and learning of mathematics. This subsection of data is the one that was analyzed for this study. These data were then coded using an open-coding scheme (Bogdan & Biklen, 2006). After an initial pass through this subset of data, codes were collapsed under three themes - all from a perspective of learning mathematics. Coding of passages with more than one of the three themes was allowed.

Results

The three themes that emerged from the data were: resources and influences of home and community, sameness versus difference, and relationships between teacher and student/parent. In the sections that follow, we discuss the results in relation to each of these themes in detail.

Resources and Influences

There were 90 passages that were coded with the resources and influences code, making this theme the most prominent of the four themes with slightly more passages being coded with this theme than with the other three combined. This theme encompassed responses that pertained to how students' backgrounds, families, and communities serve as resources or influence the teaching and learning of mathematics. The discussion of resources and influences was not always positively oriented and this collection of passages also included instances discussing a lack of resources or negative influences on students. There were two orientations that we noticed in examining these passages. In one orientation, PSTs positioned mathematics as a school activity, and in the other they discussed how teachers need to understand, value, access, and search for ways to build on the knowledge that children bring to school from their homes and communities.

Mathematics as a school activity. Within this orientation toward school mathematics, PSTs (a) indicated reasons why children may not be receiving support at home, (b) characterized the resources available in the home and/or community as those that can or should be mustered in the service of supporting the teaching of mathematics in school, or (c) discussed ways in which the school needs to take responsibility for supporting children who they characterize as not receiving support at home. About half of the passages expressed an orientation toward mathematics as a school activity.

In discussing reasons why they believe children may not be being supported at home, PSTs often in some way indicted the home environment. PSTs indicated that there may be no one at home who can help the student. Sometimes a deeper understanding of the situation was evident as some PSTs also said that this could be due to parents working multiple jobs. "If the parents are working two jobs and aren't home when the students come home [they won't be able to] reinforce that the students should really spend time doing [homework] and learning from school."

PSTs put significant emphasis on the importance of the home in mathematics learning and often explicitly named practices that SHOULD be going on in the home in order to support the learning of school mathematics, saying such things as, "Being successful in [math] depends even further on what's going on at home." "If they're not getting anything [i.e., help with homework] at home and they're just getting it at school it's going to be harder for them." PSTs also suggested that parents should take responsibility for relating activities that occur in the home or community to mathematics. "They could go to the grocery store and have their child work on math. They could have their kid sit next to them while they're doing their bills."

In discussing the importance of home support for school learning, PSTs mentioned that some homes have fewer material resources for parents to draw on, and therefore may be less able to help. "In a very poor community, then you might not have all of the availability that others from say a better socio-economic class would really have." In the following passage, the PST did not talk about deficits in the home, but rather mentioned that the help and support at home might be inconsistent with school mathematics teaching and thus contribute to underperformance.

If the student doesn't understand math most of the time they go to their parents for help. And if their

> parents are teaching them a different way than what the teachers are teaching . . . the teachers need to understand that.

One final focus within this orientation was raised only a few times and so represents the view of a minority of participants. In these passages, PSTs mentioned what schools should do in order to support students whose families are not able to provide support at home. Within these passages, although there is an inherent belief that the home is not providing support, the focus is shifted from merely recounting these failings to elaborating on what the school should do to make up for this perceived lack of support. For example, one PST said,

> Some children don't have the time they need at home with their parents to help them and what not, so in the classroom you may need to give the student more time or give them work. . . . I guess it's more taking into consideration the individual child's home environment, how to mesh what you do in the classroom with what goes on at home.

Teachers need to build on knowledge students bring from home. The second orientation within the category of resources and influences focused on the need for teachers to understand, value, access, and search for ways to build on the knowledge that children bring to school from their homes and communities. About half of the passages were coded as reflecting this more positive orientation toward families. Within these passages PSTs indicated understanding that all children have experiences at home that may be leveraged in the mathematics classroom if the teacher accesses them. PSTs said such things as, "the ideas for what your context of your problem would be based on family issues or community issues," and "I think [mathematics] is going to be more meaningful if they investigate things that are happening in their lives or their community." PSTs also indicated that it is the responsibility of the teacher to access these practices. One PST noted,

> I think teachers, let's say after they do the family conferences, could find something that all students have in common, or that the families have in common, and implement that into the curriculum. And I think they should let students kind of draw on that to solve mathematics.

PSTs mentioned specific activities that might serve this purpose. "They cook with their grandmother, and they calculate measurements and then things like that they can use that to do math problems in the classroom."

In addition, this code was used to capture the few times in which PSTs discussed how school mathematics could influence what was happening at home, suggesting that the link between home and school is bi-directional, with home practices being available for use in school and school mathematics practices being available for use at home. One PST suggested that while studying measurement, families could support children in applying measurement concepts and skills at home by engaging in a "home improvement project."

Sameness and Difference

A second theme that emerged from participants' responses is one of sameness and difference. Within this theme participants compared and/or contrasted such things as culture, race and ethnicity, home and family life and resources, communities, and mathematical skills, abilities, and performance. The comparisons/contrasts made by the PSTs were of two orientations: (a) those that made comparisons among various groups of students and their families, and (b) those that made comparisons between PSTs and the students and families about whom they were speaking. There were 50 passages that were coded with the sameness/difference code.

Comparisons among students and families. Within this orientation, PSTs compared students and/or their families and family-community circumstances in one or more of the following ways: (a) differences in support or differences in circumstances that students have at home, or (b) differences in the knowledge bases that students brought to the classroom.

In some cases the differences in home circumstances were looked on positively by the PSTs. For example, one PST noted, "It doesn't really matter whether or not the parents are educated or not, and know the information themselves. . . . They can help the student find the information they need with the resources in the community." In other cases, PSTs focused on situations where family resources were limited or family circumstances did not support family involvement in schooling. In these cases some of the PSTs saw these situations as negatively impacting success in mathematics as with the PST who said, "If they're not getting anything at home and they're just getting it at school it's going to be harder for them." Other PSTs, on the other hand, saw them as instances in which the school needed to be more involved. "I think that, where students may not get it at home, it is important for teachers to bring it in."

In most instances PSTs saw the different knowledge that students brought to the classroom as something that teachers could learn about, and should be aware of. In addition, some explicitly discussed how they might build on this knowledge in instruction. In a couple of instances PSTs discussed differences in computational or problem solving strategies that parents might employ when helping children compared to what they might be learning in school. In discussing how she would react to a student who brought to the classroom a division algorithm learned at home, one PST said she would put the example on the board and ask, "Has anybody else seen this before? And go from there. That could be your starting point."

Comparisons between PSTs and students. PSTs at times made some type of comparison between themselves and their circumstances and the circumstances of the students they worked with. One major focus was the significance of help and support for the learning of mathematics. PSTs presented the view that their families had helped them with homework and emphasized how important it was for families to be supportive in this way in order for children to be successful with mathematics learning. A small number of PSTs shared that for different reasons they had not had help with mathematics at home, suggesting by contrast that if students don't have help at home it isn't necessarily something that interferes with academic success.

Other PSTs discussed their own experiences interacting with parents or university teachers around non-school type mathematics. They suggested that these informal encounters with mathematics were likely to be available to all students. "Doing fractions and I would think a lot about cooking because I like to bake with my mom and so I can draw on my baking experience to help me with math because then I can visualize what's a fourth or what's four ounces."

Relationships

A third theme that emerged from participants' responses was how they have, intend to, and/or need to build knowledge about and/or relationships with students, their families, communities, and culture. There were 25 passages that were coded with the relationships code. There were three major orientations from which PSTs discussed these relationships: (a) specific ways in which teachers could be involved with parents or make connections to the family or community of the students, (b) relationships with students and their families/communities in terms of a general orientation that they had toward parents, and (c) challenges involved in getting to know the communities to which students belonged as well as difficulties incorporating community knowledge into mathematics teaching.

How teachers can connect with families. The most prevalent views on relationships with families and communities fell into this category, with some PSTs having more than one idea as to how to forge these connections. One idea presented by several PSTs related to drawing on home/community knowledge in instruction. Another idea discussed by others was communicating with parents in specific ways. Some of these included home visits, student interviews, email or phone communication with parents, questionnaires sent home, and family conferences where families are invited to share information with the teacher. (This stands in contrast to parent-teacher conferences where the parents come to learn from the teacher how their child is performing in the school setting.) One PST suggested that school-based events were sometimes difficult especially for parents from non-dominant groups, and mentioned that she would like to organize a fun family activity: "I would see myself planning, like, a family picnic and having all the students come with whoever they want, their parents, brothers and sisters."

A general orientation toward parents. PSTs also discussed in general terms their orientations toward families and their expectations or plans for connecting with them. These included such ideas as "keep[ing] in contact,"; "talk[ing] to parents as much as possible;" and "know[ing] our students." Some PSTs understood that there were things that they could learn from parents. One said, "I think as much communication as possible [is important], and also as much listening. Parents are going to have ideas; they're going to know their kid way better than you are." Some of these passages, however, indicated a deficit orientation toward families or a lack of faith that all families are prepared to support student success. One PST, for example, said, "[Parents] play a big role. It just depends on whether it is a positive one, or a negative one."

Challenges in developing relationships with parents. A few PSTs discussed challenges they might face or difficulties they might confront in interacting with or drawing on community resources. The lack of familiarity with and understanding of students' home and community contexts due to having grown up in a different environment or being new to the school community was one challenge that was articulated.

Discussion and Implications

PSTs bring beliefs and dispositions to the methods class that inform (a) what they attend to, and (b) the awareness they bring to the interpretation of that attending. In the above results we see PSTs struggling with multiple perspectives on students' family, culture, and community. While we see considerable positive thinking about the resources families and communities offer children, we also see considerable deficit thinking from PSTs. These negative views can be hidden and subtle—but it's important for MTEs to recognize that, as our data indicate, many PSTs hold these negative views. We see that these views are based on how PSTs are attending to and interpreting evidence they gather from working with and talking to children and cooperating teachers.

Some of the deficit thinking may be due to the fact that PSTs have a limited perspective on the role of parents. Some of it is because PSTs hold a deterministic view of parents and families, thinking that a non-English home language or a lower socio-economic status means that children necessarily lack resources in their homes and communities. MTEs need to support PSTs in broadening their perspectives to see that parents can educate their children in other ways than helping with homework. For example, instead of "feeling sorry" for the child who helps her mother pay bills because her mother doesn't read English, we can support the PST to consider that this child has an opportunity to apply mathematics in a real-world context.

At the same time, we are encouraged by the number of PSTs who saw families and communities as full of resources that the teacher could draw on in the classroom. This indicates that at least some PSTs have adopted a cultural affirmation approach to difference (Brenner, 1998). Not only did some PSTs recognize the resources present in children's lives, they also articulated how they could bring this knowledge into the classroom. These are instances in which PSTs are discerning details about children's lives outside of school when the attend to the specifics of those children's experiences (Mason, 2008).

A prevalent idea held by PSTs is that responsibility for success in mathematics rests in the home as opposed to the notion that the responsibility rests in the school. PSTs need support in seeing that they can be active agents of change, supporting students' academic growth and taking responsibility for their learning, instead of assigning that responsibility to parents. PSTs can be supported to see that drawing on resources children bring to school will assist them in this effort. Drawing on those resources involves becoming a culture broker (Gay, 2010). This involves getting to know students and their circumstances, competencies, experiences, and interests. It also involves making connections with parents and families to tap into the vast and useful knowledge they have of their child. From the Relationships findings, it seems clear that many PSTs are not only attending to the need to establish these kinds of connections, but also considering practices that might help them in meeting this need. However, the tendency of PSTs to think in terms of Sameness and Difference tends to construct children and their families in ways that limit the kinds of connections and relationships PSTs can build. Here, again, we see the fragile and often conflicting ways in which PSTs are making sense of the roles of teachers, children, families, and schools in teaching and learning mathematics.

The results of this study can shape how MTEs approach PSTs and the views they bring to the methods classroom. It is important to be aware of the views that PSTs bring to their mathematics teacher preparation and to be aware that in the early stages of this preparation, the awareness of many is fragmented (Mason, 2008) and includes deficit thinking. An encouraging note is that many PSTs do bring positive views of families and communities that provide a balance for the negative views and moreover can serve as an entry point to challenging those negative views.

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A GENERAL THEORY OF INSTRUCTION TO ASSIST THE PROFESSIONAL DEVELOPMENT OF BEGINNING MATHEMATICS TEACHERS

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In this paper we discuss how a general theory of instruction called Concept-Focused Instruction (CFI) can assist in the professional development of prospective teachers. CFI provides prospective teachers with a foundation and logical decision-making process for selecting, designing, and teaching mathematics. First, we provide a theoretical and practical context for having a theory of instruction. This leads into a description of the theory. The background and three core principles are provided, in particular in the context of the university methods course. The paper concludes with an overview of the findings and areas for future development.

Keywords: Teacher Education–Preservice; Teacher Beliefs; Teacher Knowledge

Introduction

Researchers in mathematics education have shown considerable interest in the professional development of prospective mathematics teachers. Ball and her colleagues (Ball, Lubiennski, & Mewborn, 2001) have contributed for over a decade to this knowledge base by contemplating the specific knowledge, skills, and dispositions that prospective mathematics teachers need to learn while they are in their university program. For example, prospective mathematics teachers need to learn how to elicit student talk; how to lead a discussion by eliciting and asking good questions; how to interpret students' thinking so they can learn to develop and support student learning. In a more general sense, Darling-Hammond and her colleagues have suggested the importance of framing the university experience more towards the sensitive nature of learning and the effects of teaching. In particular, Darling-Hammond (1998) states there is a need to prepare prospective mathematics teachers "with greater understanding of complex situations rather than seeking to control them with simplistic formulas or cookie-cutter routines" (p. 10).

According to Sowder (2007), "successful professional development programs remains the greatest challenge we educators face… our goal should be to prepare them [prospective teachers] for future learning, in part because at the university we can focus only on learning-for-practice (and not enough of that), and we know they have much more to learn *in* practice while teaching" (p. 213). Prospective teachers need opportunities to understand and use effectively and creatively fundamental mathematical content and concepts; teach content through the perspectives and methods of inquiry and problem solving; integrate education theory with actual teaching practice; and integrate mathematics teaching experiences with research on how people learn mathematics.

Hiebert, Morris, Berk, and Jansen (2007) address this challenge, of preparing the beginning teacher, by proposing a framework composed of four competencies: (a) setting learning goals for students, (b) assessing whether the goals are being achieved during the lesson, (c) specifying hypotheses for why the lesson did or did not work well, and (d) using the hypotheses to revise the lesson. They claim that teaching prospective teachers the knowledge, skills, and dispositions in each of these areas will allow the future teachers to have a deliberate, systematic path to analyzing cause-effect relationships between teaching and learning; and therefore, becoming an effective teacher over time.

The mathematics methods course typically lays the foundation for developing the knowledge, skills, and dispositions for mathematics teaching. The challenge there, as established by mathematics educators, is the prospective teachers often interpret the university-based course as disconnected to what actually happens in classroom experiences in schools. Frequently the preservice teachers expect they will learn *how* to teach mathematics but instead feel like they are presented with a menagerie of instructional theories (Mewborn, 1999, 2000). In addition, the seemingly discrete instructional techniques and approaches presented in the methods class create the impression that the mathematics methods course is theoretical and impractical. This perception is then reinforced during field experiences where classroom management issues and the cooperating teachers' views seem incompatible with information presented by the mathematics methods instructor Ebby (2000). Over three years ago, this disconnect between the methods course and the student teaching classroom experience, caused us to consider an alternative approach for the mathematics methods university course. Our alternative, related to what Heibert and colleagues describe above, was to frame the course using a theory of instruction.

According to Jerome Bruner (1966), a theory of instruction "sets forth rules concerning the most effective way of achieving knowledge or skill…a theory of instruction, in short, is concerned with how what one wishes to teach can best be learned, with improving rather than describing learning" (p. 40). Further, Jerome Bruner says a viable theory of instruction (1) identifies the experiences that are compatible with the way students learn, (2) explains the structure of the knowledge within a discipline, (3) identifies the most effective instructional sequences, and (4) addresses appropriate pacing and motivational strategies. Others have described a similar set of principles, each focusing on the application of knowledge and guidance on how to help students learn (Reigeluth, 1999). In summary, instructional theories are created as a set of principles and guidelines. They are not rigid sets of rules that must be followed at all cost but are guidelines that help the practitioner.

One could argue that every teacher employs a theory of instruction when they select, design, and teach mathematical content, albeit typically an implicit theory of instruction. If prospective teachers could begin their professional development with an explicit theory of instruction that meets the above criteria as defined by Bruner, the new teacher could leave the university experience with a means to logically select, design, and teach mathematics. As Bruner (1966) states, the viable theory of instruction must identify with the experiences that are compatible with the way students learn mathematics, explains the structure of mathematical knowledge, identifies an effective instructional sequence, and addresses appropriate pacing and motivational strategies. If prospective teachers experience this during their professional development at the university, would they be more likely to connect their university coursework experiences with the public classroom?

In this paper, we wish to share the evolution of a general theory of instruction that was developed for prospective mathematics teachers. The theory of instruction, called Concept-Focused Instruction (CFI), has been used the last three years in the university mathematics methods course. Each year, the theory and how it is being implemented in the methods course has been refined. Preliminary findings indicate the prospective teachers from 2011-2012 demonstrate (1) an improved understanding of mathematical concepts, (2) lesson plans and teaching practices that are learner-centered, and (3) more in-depth reflections and conversations about student learning and understanding in the classroom.

In the next section of this paper, the theory of instruction is shared by first describing the process that led the authors of this paper to consider this alternative approach for teaching the methods course. Concept-Focused Instruction (CFI) is then defined, including an overview of the progress that led up to the current version. Afterwards, a brief overview of the research findings showing how CFI has had a positive impact with the training of prospective mathematics teachers is described. The paper ends with a discussion of current thoughts, challenges and questions.

Concept-Focused Instruction

Background and Context

Over four years ago the authors of this paper began a conversation about the relationship between the methods course experiences and the student teaching, or internship, experience. This conversation was consistent with the established research by Mewborn (1999), that is, the prospective teachers did not employ much of the knowledge and skills learned in the course. The prospective teachers seemed to view the knowledge they gained in professional education classes as rules or prescriptions to apply to classrooms, and because these rules were not consistent with what they experienced, they saw it as a disconnect between the university coursework and classroom experience. The views of the cooperating teacher became the student teachers' dominant practice.

This encouraged us to look at the traditional teacher-training model, in particular for the mathematics methods course. The course typically addresses the instructional blocks framed in one of these three elements (Figure 1).

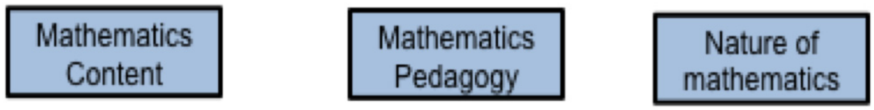

Figure 1: The traditional mathematics methods course elements

What we found is that the prospective teachers seemed to interpret these elements as separate and distinct. Mathematics content was what was learned in their mathematics coursework. The nature of mathematics was considered a philosophical idea that did not relate to what they were there to learn, and pedagogy was the sole purpose of the methods course. Because mathematics educators have established that the teacher's mathematical knowledge for teaching is directly related to how a teacher teaches and how well students learn (Hill, Rowan, & Ball, 2005) as well as their beliefs about the nature of mathematics (Borko & Putman, 1996; Philipp, 2007), our initial attempt was to find ways to connect the course elements.

Even when we purposefully tried to connect these elements, our preservice teachers did not connect the blocks. At the conclusion, preservice teachers still interpreted the set of information as discrete elements and believed the sole purpose of this course was to teach them *how* to teach mathematics. Then during the student teaching experience, when the prospective teacher would try to apply the knowledge learned in the methods course, the issue of classroom management would become prevalent. This resulted in their not thinking at all about the methods course material. Instead the teachers found themselves in a figurative sea of educational ideas, lesson plans and activities which in turn made the task of planning and implementing unfamiliar, yet recommended practices, daunting. They resorted to the educational practices of the cooperating teacher, which were often incompatible with the methods course knowledge, skills, and dispositions. In our conversations, the preservice teachers would recall the knowledge discussed in the methods about teaching and student learning, but did not have the mindset to access and apply this information in their practice. This finding led us to reconsider again the methods course elements and towards the idea of finding a clear and decisive format that prospective teachers could make sense of and use to make effective instructional decisions that focused on student learning. In other words, we aimed to identify an alternative approach for teaching prospective teachers about the nature of mathematics, mathematics content, and pedagogy so they would begin to (1) see the connections between mathematics and mathematical knowledge, (2) understand how students learn mathematics, and (3) teach for conceptual understanding.

Reframing the Methods Course Elements

Instead of looking at the three blocks of information as separate, we began to focus our attention on finding a common idea across the blocks. The common idea was conceptual understanding. For the nature of mathematics, in the methods course, the aim was to develop a view of mathematics being a dynamic, conceptualizing process. In terms of mathematical content, the emphasis was on conceptually understanding the mathematics the prospective teachers may teach. Finally, in the pedagogy block, a key element was how to teach and assess conceptual understanding. It was hypothesized that focusing on conceptual understanding would allow us to naturally address the nature of mathematics, pedagogy, and content topics.

To place an emphasis on conceptual understanding, we first had to clarify this idea for a prospective teacher. We established all mathematical concepts have three attributes: a macroscopic, model, and symbolic attribute (Hitt, 2005; Hitt & Townsend, 2007). The macroscopic attribute defines the context for the mathematical concept, in other words, a situation that enables individuals to visualize an application of the concept. The model attribute is the tangible representation of the macroscopic attribute. Finally, the symbolic attribute embodies the definitions and formulas associated with the concept. Figure 2 below shows this relationship.

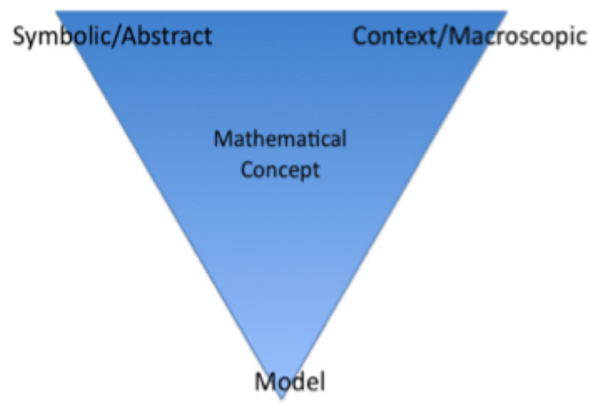

Figure 2: Attributes of mathematical concepts

For example, when thinking about adding fractions, it connects two concepts: fractions and addition. The macroscopic attribute could be a word problem that applies these concepts (Bob and Sue ordered 2 identical-sized pizzas, one cheese and one pepperoni. Bob ate ¾ of a pizza and Sue ate 1/8 of a pizza. How much pizza did they eat together?). The model would be a drawing of the pizzas, and the symbolic would include the vocabulary and algorithms for adding and finding common denominators. See Figure 3.

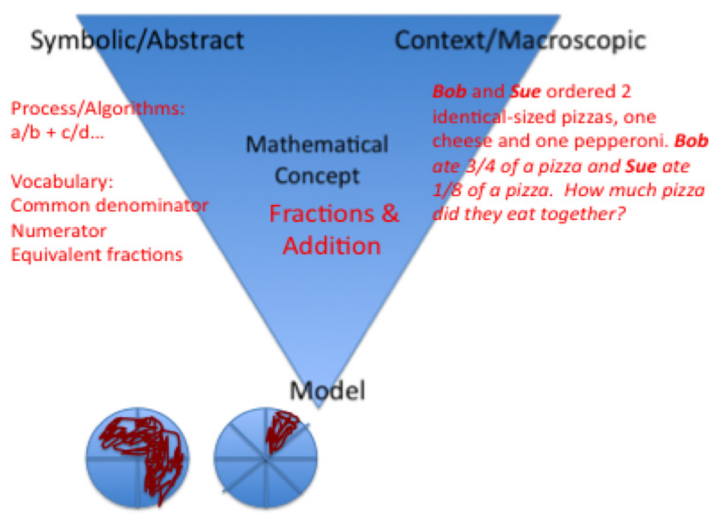

Figure 3: Three attributes describing the concepts Fractions and Addition

The methods course shifted from addressing the nature of mathematics, pedagogy, and content to a focus on identifying and analyzing macroscopic, model, and symbolic attributes of mathematical concepts first as a learner, then as a teacher. Within 2 years, we could establish Concept-Focused Instruction (CFI) as a theory of instruction. Readers may wish to refer to earlier publications that explain and justify CFI as a theory of instruction (Forrest & Hitt, 2010). In this paper, the focus will be on explaining how the theory of instruction has provided a systematic and deliberate framework for training prospective teachers, and specifically how it has been a better format for the university mathematics methods course.

Concept-Focused Instruction (CFI) Principles

Concept-Focused Instruction (CFI) is based on three core principles. The core principles are (1) mathematics is a conceptualizing process; (2) when individuals can explicitly reflect on the three attributes of mathematical concepts (macroscopic, model and symbolic) and can relate them to one another, they achieve conceptual understanding; and (3) in order to teach conceptually, teachers need to provide instruction that addresses the three attributes of mathematical concepts beginning with the macroscopic and model relationship (macroscopic, model and symbolic). These three related core principles frame the methods course design and delivery. Our hypothesis is that CFI enables prospective teachers to make better sense of teaching and learning mathematics as well as apply fundamental perspectives and methods that support education theory and the research on how people learn mathematics. It is not a novel idea to train prospective teachers by focusing on mathematics and developing understanding. Sowder (2007) describes successful implementations of such programs, for example, Cognitively Guided Instruction (CGI). CFI extends this line of work.

The focus of the next section is on the core principles of Concept-Focused Instruction (CFI). Each principle clarifies further the notion of conceptual understanding. Each principle can be interpreted as a basic hierarchical process building up and clarifying further the thoughts around conceptual understanding.

Core principle #1. Mathematics is a conceptualizing process. It has been shown by researchers who have studied mathematics teachers' epistemological beliefs that teachers who view mathematics as a dynamic conceptualizing process will be more inclined to use approaches for teaching mathematics. Additionally, students who understand mathematics as a conceptualizing process will likely have a more accurate perspective on what it means to understand and learn mathematics.

In the methods course, prospective teachers begin as mathematicians analyzing and solving a collection of mathematical tasks. The outcome is that the prospective teachers experience mathematics as a conceptualizing process. Discussion focuses on the process and thinking they used to solved the tasks. The instructor role models the tenets of Concept-Focused Instruction (CFI), highlighting each attribute of the concept to allow the prospective teachers to derive at Core Principle #2.

Core principle #2. When individuals can explicitly reflect on the three attributes of mathematical concepts (macroscopic, model and symbolic) and can relate them to one another, they achieve conceptual understanding. This principle helps us address two key ideas for teaching and learning mathematics. First, it provides a fairly simple explanation for mathematical understanding. The prospective teachers analyze a variety of mathematical topics in terms of the macroscopic, model, and symbolic attributes. That is they make and/or find possible macroscopic observations that allow a mental picture of the concept to be formed, create models that represent the phenomena, and then state the symbolic terms and formulas that are applicable. As a result, not only do the prospective teachers develop a better understanding of the mathematical content they are expected to teach. Through this process they begin to conceptualize and integrate complex mathematical ideas into their thinking. They also start to realize and discuss reasons why individuals may have a superficial and restricted understanding of mathematical content. Finally, they realize how critical the model attribute is when discussing mathematics. This sets up the third principle, which focuses on teaching mathematics.

Core principle #3. In order to effectively teach mathematical concepts, teachers need to provide instruction that addresses the macroscopic, model and symbolic attributes of concepts beginning with the macroscopic and model relationship. Specifically teachers provide a macroscopic experience that allows an opportunity to visualize the mathematics being taught. The students then create models to represent the macroscopic experience, and the teacher is then instructed to use the students' models to diagnose whether the students have a working model. Once a working model has been established, students are prepared to learn the symbolic attributes.

During the phase of instruction where the teacher is diagnosing the student models, the prospective teachers are trained to ask critical questions such as "How does your model explain this situation?" "Show me where your model addresses this particular idea?" etc. In addition, prospective teachers are instructed that once students have generated models, and while instruction is focused on students learning the mathematics, the instruction will be highlighting either the macroscopic attribute of the mathematics, that is the visual, or instruction will be highlighting the symbolic attribute. Highlighting refers to what is being made loud and clear during the instruction.

Looking back again at our diagram of the three attributes of a concept (the figure is repeated below for convenience), one can see how the model attribute becomes the pivotal attribute. Once students produce a model of the macroscopic experience, depending on whether this model is a workable model, the prospective teacher thinks about emphasizing either the macroscopic-model relationship or the model-symbolic relationship. This allows the prospective teacher to focus his/her instruction on a specific outcome.

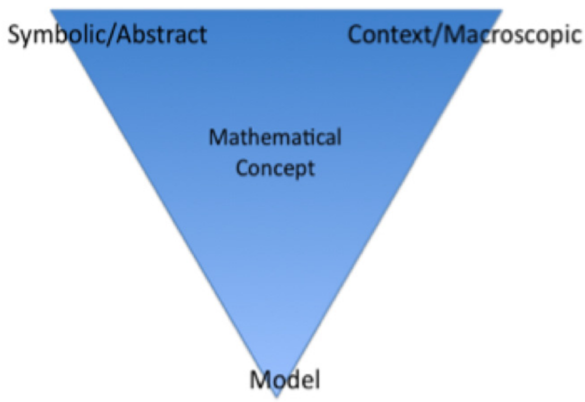

Figure 4: Attributes of mathematical concepts

In summary, using the three core principles of Concept-Focused Instruction (CFI) to frame the university methods course has resulted in our prospective teachers having better understanding and practice in the teaching and learning of mathematics. Without explicitly teaching the knowledge, skills, and dispositions related to the nature of mathematics, mathematics pedagogy and mathematical content, our preservice teachers have naturally acquired these skills in a manner that makes sense to them. They now comprehend how the nature of mathematics, pedagogy, and content are related when it comes to teaching students mathematics.

Concept-Focused Instruction is beginning to show promise as a means for developing prospective mathematics teachers, but it is still a work in progress. Each year, empirical evidence is gathered to create a case documenting each of the prospective teachers experience in the university mathematics methods course through the internship. Each case includes documentation showing the prospective teachers' mathematical content knowledge, beliefs about teaching and learning, the prospective teachers' views of mathematics and mathematical knowledge, and use of inquiry and mathematical processes in planning and delivery. In addition, documents ranging from lesson plans, supervised observations and evaluations are part of each prospective teacher's case.

Conclusion

Concept-Focused Instruction (CFI) is a theory of instruction that has been successfully used to develop prospective teachers' understanding of mathematics, as well as their knowledge, skill, and dispositions for teaching and learning mathematics. However, the authors are just beginning to establish a comprehensive research design to test the specific impact of CFI on preservice teachers. For example, the authors have identified one issue that they feel warrants further investigation. It is unclear ***how*** Concept-Focused Instruction (CFI) shapes prospective teachers' perspectives on mathematics or their ability to teach lessons effectively. In order to determine when and how preservice teachers integrate the CFI core principles into their thinking, it is critical to collect and analyze more qualitative data from artifacts such as journal reflections, class assignments and digitally recorded teaching presentations.

In closing, we want to summarize (1) the purpose of integrating Concept-Focused Instruction (CFI) into mathematics methods instruction, and (2) clarify how CFI fits within the current practices in the field of preservice mathematics teacher education. First, a theory of instruction, specifically CFI, can help prospective teachers simplify and visualize the connections between how mathematics is done, the way that students intuitively learn mathematics and effective approaches for teaching mathematics. When prospective teachers are just beginning their professional development, they tend to see these ideas as separate because as a student of mathematics, they have no explicit experiences with connecting these elements. We have found that once the above connections make sense to the prospective teacher, they are better able to understand and discuss many of the current ideas found in the mathematics education literature. In addition, we have realized until the connections make sense to the preservice teacher, the preservice teacher will fail to integrate the more constructivist ideas in their instruction.

Second, Concept-Focused Instruction (CFI) is not a replacement for instructional approaches discussed in methods courses, such as problem-based or inquiry-based instruction. CFI merely provides a foundation for prospective teachers to build upon. Once this foundation is established, the instructional approaches discussed in the method course make more sense to them. Concept-Focused Instruction merely provides preservice teachers with a basic understanding of how to think about mathematics learning and teaching, which then assists them in understanding and then designing and implementing learner-centered instruction.

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NORTH CAROLINA NAEP: APPLYING THE NATIONAL ASSESSMENT OF EDUCATIONAL PROGRESS WITHIN PRESERVICE TEACHER EDUCATION

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This study developed twelve instructional modules based on the National Assessment of Educational Progress for mathematics content and methods courses for preservice elementary and middle school teachers and examined their impact on PSTs' mathematical content knowledge and self-efficacy beliefs about teaching mathematics. The modules help preservice teachers: (1) improve their mathematical content knowledge, (2) learn how to use effective methods to teach mathematics; and (3) become aware of uses of NAEP. Mathematical content knowledge was measured by instruments from the Learning Mathematics for Teaching project and mathematics teaching efficacy beliefs were measured by the Mathematics Teaching Efficacy Beliefs Instrument or the Yackel Beliefs Survey. Modules were found to increase PSTs' mathematical content knowledge for teaching and improve their teaching efficacy beliefs.

Keywords: Teacher Education–Preservice; Teacher Knowledge; Teacher Beliefs

National reports (Project Kaleidoscope, 2006; National Academies, 2003, 2008; Ball, Ferrini-Mundy, Kilpatrick, Milgram, Schmid, & Schaar, 2005) pointed out the urgent need to improve the quality of science, technology, engineering, and mathematics (STEM) education programs. The National Academies (2006, cited in PKAL, 2006) advised increasing "America's talent pool by vastly improving K–12 science and mathematics education" (p. 4-2). Such progress rests on improved mathematical education of teachers. Morris (2006) stated, "Preservice teachers rarely exit their mathematics teacher education program as experts" (p. 471). The question therefore becomes, how do mathematics teacher educators help PSTs better develop their expertise, especially during their early careers? It is difficult to address the learning needs of preservice teachers (PSTs) due to the large number of concepts, skills, and strategies that must be acquired at a high level of competence to teach successfully. This fact must be kept closely in view by faculty of mathematics content and methods (MC&M) courses for teachers. We must carefully examine the goals of these courses and ask: (1) *Are we giving students enough experiences in the areas we expect them to master?* (2) *Do they have sufficient opportunities to consider problems from both students' and teachers' viewpoints?* and (3) *Do they have sufficient opportunities to examine both student work and student achievement data?*

It is insufficient to discuss problem solving, development of student conceptions, and assessment of mathematical learning, in abstraction. Preservice teacher education must have a strong student focus and be rooted in authentic classroom data. Novices need specific experiences in how to analyze student work, assess student understanding, and in scoring student work with various rubrics. Previous researchers (Morris, 2006; Osana, Lacroix, Tucker, & Desrosiers, 2006) reported the benefits of asking PSTs to analyze mathematics teaching episodes from real practice. Morris (2006) described a study where PSTs' abilities to analyze videotaped teaching episodes differed markedly on the basis of whether they were told beforehand that a lesson was unsuccessful or whether they had to make this determination independently. Morris's study suggests that what PSTs *focus on and attend to*, in analyzing teaching and student performance, is linked to what experiences and guidance they receive from teacher educators.

Current efforts to improve mathematics teaching recognize the importance of helping teachers (a) gain depth in their mathematical content knowledge, (b) master specialized content-related strategies needed to help children learn mathematics, and (c) learn pedagogical and assessment practices to improve the quality of teaching and student learning (Hill, Rowan, & Ball, 2005; Ball & Hill, 2004; Ball & Bass, 2000). We believe that the analysis of sound mathematical tasks, discussion of explanations for mathematics procedures and concepts, careful analysis of student work, and discussion of assessment practices should be a focus of preservice mathematics content and methods courses and that these experiences should occur across the teacher education program. The North Carolina NAEP project found preservice teachers to be especially receptive to NAEP-related instruction that was clearly linked to examples of student work, analysis of that work, and examination of student performance data.

Purpose

The goal of the *North Carolina NAEP: Improving Mathematics Content and Methods Courses* project was to modify materials from *Learning from NAEP: Professional Development for Teachers* (Brown & Clark, 2006) for use in preservice mathematics content and methods courses, to expand the materials to include more recent NAEP assessment results, and to include, and focus on, more mathematics content. While NAEP produces a vast amount of data concerning students' learning and achievement, this wealth of data is not always used effectively within preservice teacher education courses to help PSTs become aware of what these data show. The project addressed three research questions, two of which are addressed in this paper:

1. How can mathematics content and methods (MC&M) courses for preservice undergraduates and graduate students be improved through the integration of NAEP?
2. How can NAEP-related materials be used in MC&M courses to help beginning teachers see the connections among the following areas: (a) teachers' content knowledge, (b) student understanding, (c) classroom assessment practices, (d) analysis of student performance data, and (e) use of NAEP data to address issues of equity?

The results described here explain the ways in which the modules appeared to influence preservice teachers' mathematical content knowledge and how they seemed to influence their mathematics teaching efficacy beliefs. Goals and outcomes of the project include:

1. The improvement of MC&M courses for elementary and middle school PSTs to produce teachers knowledgeable about mathematics content and pedagogy and knowledgeable about difficulties students have in learning mathematics topics.
2. The improvement of MC&M courses to produce teachers able to use NAEP and other assessment data to consider issues of equity and to modify teaching to address them.
3. Improving teachers' knowledge of various assessment strategies including designing and using problem solving rubrics.
4. The development of (a) multimedia materials that illustrate critical mathematics concepts, NAEP-related problems, examples of student errors, statistics concerning student achievement on NAEP problems, and activity sheets providing guidance for group analysis of this information within MC&M courses; and (b) a project website.
5. Enhancing instruction and communication between institutions within the North Carolina Community College system and the University of North Carolina system.

The project team involved mathematicians and mathematics educators from the following universities and community colleges: Appalachian State University, the University of North Carolina Charlotte, the University of North Carolina Wilmington, Forsyth Technical Community College, Mayland Community College, and Wilkes Community College. *The UNC Teacher Preparation Program Effectiveness Report*

(Henry, Thompson, Bastian, Fortner, Kershaw, Marcus, & Zulli, 2011) concluded that the following programs were outperforming their reference group comparisons in these areas:

> Appalachian State University – Elementary Program: *elementary mathematics*
> University of North Carolina Wilmington – Middle School and Secondary Mathematics Programs: *middle school and secondary mathematics*
> University of North Carolina Charlotte– Middle School Programs: *middle school mathematics* (Henry et al., 2011, p. 11)

Thus, we were able to integrate, within the project modules, instructional practices that have been deemed effective in improving teacher preparation at the participating institutions.

The project team included two practicing teachers, one from the elementary level (Ms. Anderson) and one from middle school (Mr. Schmal). These teachers helped the team link the NAEP assessment results to the realities of daily classroom practice through their contributions to the writing teams and through their commentaries concerning how NAEP data mirror the types of student work and difficulties that they observe in their classrooms. Twelve modules were produced, four for each of these levels: elementary, middle school, and community college. The community college modules are directed at undergraduate mathematics courses frequented by preservice teachers. The modules employ a variety of instructional approaches including: using active learning strategies; conducting analyses of NAEP results; conducting analyses of student work; developing understanding of the mathematics contained in NAEP problems; and developing awareness of NAEP rubrics and procedures for assessing student work. The elementary modules address the areas of: *fraction number sense, addition and subtraction, early algebra,* and *geometry*. The middle school modules include the topics: *proportional reasoning, geometric and spatial reasoning, linear growth and rates of change,* and *data analysis*. The community college modules cover the topics of*: algebra, probability, proportional reasoning,* and *spatial reasoning*. The study results suggest that seeing their college mathematics and mathematics education instructors model more inquiry-based pedagogical strategies positively influences and broadens PSTs' vision of effective mathematics instruction.

The modules are flexible allowing for inclusion within different course structures and time allotments. Each module contains:

1. Purpose
   a. Specification of the mathematical concept(s) addressed
   b. Specification of pedagogical approaches
2. Overview
   a. Module goals
   b. Module activities
3. Background and context notes
   a. Includes research brief concerning math concept and relevant pedagogical issues
   b. Includes discussion of common student errors based on research
   c. Examines NAEP student performance data in context of relevant research
4. Preparing to teach the module (Instructor notes not covered elsewhere)
5. Introductory PowerPoint presentation for the module (to be presented to PSTs)
   a. Specification of the mathematical concept(s) being addressed
   b. Specification of pedagogical issues addressed
6. *Teaching the Module* plan for the university/college instructor that provides:
   a. Goals and objectives of the module
   b. Time required for module
   c. What mathematics is addressed and grade band(s)
   d. How NAEP resources will be used
   e. Materials required
   f. NCTM Principles or Process Standards addressed
   g. NAEP Content Strand Emphasized

h. Description of class activities (Activities should include samples NAEP problems, student work, and performance data)
i. Directions for conducting class activities
j. Student activity sheets
k. Discussion guide
7. References

Each module includes recommended readings and Teaching Notes to help instructors implement the activities. The instructor can access PowerPoint presentations as well as a Moodle-based course shell that includes the instructional materials. Example modules are available on the project website http://ncnaep.rcoe.appstate.edu/. Online and face-to-face professional development opportunities are planned for 2013–15.

Method

Instructional modules based on NAEP data were developed and implemented in mathematics content and methods courses aimed at preservice elementary and middle school teachers who were enrolled at three universities and two community colleges from fall 2008 to spring 2010. Thus far, roughly 750 PSTs have been impacted by the project. External evaluators analyzed three data sources: (1) PSTs' performance on a mathematics knowledge assessment (LMT); (2) PSTs' responses to mathematics teaching beliefs questionnaires, the *Mathematics Teaching Efficacy Beliefs Instrument* (MTEBI) (Enochs, Smith, & Huinker, 2000) or the *Yackel Belief Survey* (Quillen, 2004); and (3) faculty responses to the *NC NAEP* workshop surveys and their reported use of *NC NAEP* modules. The sample included PSTs who were given a mathematics content knowledge test and a teaching self-efficacy questionnaire at the beginning and end of the semester. The project included sections of experimental courses at each university as well as control sections that did not use the project modules. The elementary preservice teachers were given the *LMT: Grades 4–8 Geometry* test or the *LMT: Elementary School Number Concepts and Operations–2004* test, dependent upon their specific course enrollment. The middle school preservice teachers were administered the *LMT: Middle School Number Concepts and Operations* test, as were the community college students. It is significant to note that the LMT tests do not just measure mathematics content knowledge; they measure a teacher's knowledge of mathematics for teaching. Thus, this data helps us learn not only how the PSTs think about mathematics concepts but it also helps us to form a picture of how they interpret possible student responses to certain mathematical scenarios. To ensure comparability across all data collected, the analysis included only students in each group, (e.g., course specific, higher education setting, and grade-level focus) for which evaluators were able to match both the pre-post scores on each instrument. After linking students with their pre and post scores, the evaluators conducted a range of statistical tests including significance and regular multiple regression tests, as well as item response analyses.

Key Findings

The results from the *NC NAEP Project* suggest that the National Assessment of Education Progress can inform instructional practice. This report shares findings based on the evaluators' analysis—using a mix of data from several instruments (i.e., *Mathematics Teaching Efficacy Beliefs Instrument* (MTEBI), the *Yackel Belief Survey* (YBS), and the *Learning Mathematics for Teaching instrument* (LMT)) used at different stages of the project, and with populations of students from 2 to 4-year undergraduates. Three main project components evaluated are discussed below: (1) PSTs' learning in terms of content knowledge and mathematics teaching efficacy beliefs; (2) learning by the college /university faculty impacted by the project; and (3) the quality of the modules and accompanying materials.

Impact on Pre-Service Teachers' Content Knowledge

Finding #1. Project faculty were differentially effective in increasing PSTs' mathematics content knowledge for teaching as measured with an LMT test. Because positive growth varied strongly across students linked to different project faculty, the results suggest overall positive effects. The difference in faculty effectiveness is significant at $p \leq 0.01$. Over half of the project faculty had students who gained, on average, a standard deviation or more in post-test scores.

Finding #2. All PSTs showed improvement in their knowledge of mathematics, with elementary PSTs making the largest gains. Treatment elementary PSTs showed statistically higher gains in their pre-post knowledge of mathematics for teaching as compared to a control.

Elementary Pre-Service Teachers

The average number of correct items on the baseline LMT test for all pre-service elementary teachers was 10, while the average number correct on the post-test was 12.4. There was little variation in the test scores between students in the control and treatment groups—the average pre-test score was 10 for both groups, and the average post-test scores were 12 and 12.50 respectively.

The post-LMT test gain for elementary pre-service teachers corresponds roughly to a 2 to 3-item increase, per student, in the raw number of correct items. Considering that the assessment was not designed exclusively to match the curriculum of all or any particular course, this gain is a promising finding. The standard deviation of the pre and post-test scores for students in the treatment group were .892 and .626, respectively, making this gain a third standard deviation in size, and statistically significant at $p < .001$.

Pre-service teachers pre and post-LMT scores in spring 2010 were higher in the number of overall correct items by 2 than compared to teacher's pre-post scores in fall 2009. An analysis of matched items from the pre-post LMT showed that spring 2010 students performed significantly better on items designed to measure operations content knowledge than other students. Correct responses to items 5 and 8 were positively and strongly predictive of teachers' knowledge of mathematics, and higher raw scores than other matched items. These results suggest that as the project proceeded and the project modules were more effectively integrated within the courses, the benefits became more pronounced.

Middle School Pre-Service Teachers

The number of middle schools pre-service teachers for which there were matched pre-post test scores for was considerably smaller than the sets for the community college and elementary pre-service teachers. This data is reported for comparison purposes, but we suggest the results be viewed with caution because of the relatively high standard error compared to other sub groups. Analysis of the pre-post LMT scores for pre-service middle school teachers revealed a statistically significant increase in post-test scores at the 5 percent level overall, and in both fall 2009 and spring 2010.

Finding #3. Preliminary results indicate a high degree of correlation $r = 0.78$ between PSTs' personal beliefs of math and math instruction and their performance on the LMT. The relationship between knowledge of mathematics teaching was strongest with the personal efficacy sub-scale.

Impact on Pre-Service Teachers' Self-Efficacy

Finding #4. Evidence suggests that the *NC NAEP* modules used in different contexts influence PSTs self-awareness and confidence in their personal efficacy for mathematics and mathematics instruction. We examined the relative effects of instructional environment (2 or 4-year university/college) and instructor with different characteristics on PST outcomes. Four unique pair-wise comparisons of PST beliefs were conducted: (a) 4-year university PSTs relative to community college PSTs, (b) elementary school PSTs relative to middle school PSTs; (c) elementary pre-service treatment teachers relative to elementary pre-service control teachers; and (d) fall 2009 PSTs relative to spring 2010 PSTs.

Among the aggregate results on the MTEBI, PSTs responses indicate changes in nearly every personal efficacy item, with significantly positive changes ($p \leq 0.001$) on three items: (e.g., "I know how to teach mathematics concepts effectively"). In nearly every dimension (e.g., subject matter-knowledge, pedagogy and subject-specific pedagogy), PSTs from spring 2010 had markedly higher self-efficacy beliefs toward mathematics, with elementary PSTs showing dramatic shifts—both experimental and control. Both elementary pre-service and middle school PSTs from 4-year universities, showed a statistically significant differential ($p \leq 0.001$ and $p \leq 0.05$, respectively) intra-group in their post-mathematics efficacy scores—indicating changes in their mathematics content and PCK.

Finding #5. At the end of the semester, PSTs, in general, noted changes in their attitudes that they can positively influence student learning—their belief in outcomes expectancy. Elementary and middle school PSTs from spring 2010 showed significant changes ($p \leq 0.05$) in their outcomes expectancy beliefs when compared to candidates from Fall 2009—both experimental and control.

Impacts on Faculty Instructional Practice

Finding #6. Project faculty found value in using project resources to improve their own instructional effectiveness. *NC NAEP* modules forged connections between mathematics content, instructional practice and assessment, to help faculty better prepare PSTs.

Finding #7. Non-project faculty found the *NC NAEP* materials very useful. They reported that the training and materials helped them form a personal action plan for using *NC NAEP* resources and illustrate concrete resources and strategies for improving students' preparedness.

Finding #8. *NC NAEP* resources helped to challenge participating faculty's preconceptions concerning their teaching of mathematics content and methods. Participating faculty recognized the importance of changing their practice to evaluate their students' preparedness and to emulate authentic classroom activities.

Conclusions

The *NC NAEP* modules had important effects on the development of PSTs' mathematical content knowledge, and improved their personal efficacy, outcomes expectancy, and attitudes toward mathematics. The modules provided a meaningful format from which to draw situated, authentic resources to support critical thinking and reflection about mathematics instruction. These modules have the potential to play a critical role in the preparation of high-quality, well-prepared teachers of mathematics. However, some limitations of the evaluation should be mentioned. While the LMT is a reliable measure of mathematical content knowledge for in-service teachers, its use with PSTs is preliminary (Gleason, 2010). To address this concern, we plan for future evaluations of the modules to validate individual LMT results using either an assessment of PSTs use of mathematics content during situated classroom teaching or using interviews where students explain their thinking and solution process to the test items. Gleason also suggests that the reliability of the LMT with preservice teachers is strengthened through the use of multiple instruments.

In addition to the findings discussed above, the project team submitted the modules to an external mathematics educator from the Pennsylvania State University for evaluation. The evaluator provided detailed feedback concerning the quality and utility of the modules, noting:

> Overall, I can see that the project has worked hard to integrate NAEP items, results, and student work into modules that are intended to be disseminated for use in PST education. Developing meaningful activities for PSTs is challenging; developing accompanying facilitator notes adds levels of difficulty to the task. The developers of the materials are to be commended for their work thus far, particularly in the area of integrating NAEP into your materials.

Some of the evaluator's suggested revisions include: (a) limiting the scope of some of the modules, particularly the community college materials, to make them easier for instructors to implement; (b) addressing the issues associated with making referenced video clips more accessible; and (c) changing some of the facilitator notes to provide more guidance to instructors concerning how to best implement the modules. We are making revisions to the modules based on this helpful feedback.

Another indicator of the quality of the project modules is their adoption for use in selected courses within a new multi-university program in North Carolina for an *Add-on Certificate in Elementary Mathematics* (which is similar to a K–5 mathematics specialist). Evaluation of the pilot add-on certificate program showed improvements in mathematics content knowledge of the teachers enrolled as compared to a control group.

This project provides an example of how different communities of professionals can contribute to the effectiveness of mathematics teacher preparation programs. Mathematicians and mathematics educators, including instructors from various levels of higher education (doctoral institutions, 4-year institutions, and community colleges), and classroom teachers all have a crucial role to play in the development and implementation of authentic instructional resources in mathematics teacher education. Such cooperation among professionals and institutions facilitates transitions across the continuum, including the transition that students make as they progress through a teacher education program and emerge as an effective beginning mathematics teacher.

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USING A KNOWLEDGE-IN-PIECES APPROACH TO EXPLORE THE ILLUSION OF PROPORTIONALITY IN COVARIANCE SITUATIONS

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The non-normative application of proportional strategies in non-proportional covariance situations is widespread and documented in studies conducted in many countries and with participants across a wide range of ages. In the present study, we found that preservice middle-grades teachers have many of the same problems with proportional reasoning as those reported with other populations. We employed diSessa's (1993) knowledge-in-pieces perspective to track how pre-service teachers used knowledge resources before and after a unit on proportional reasoning in their methods course. Past research has often characterized this phenomenon as the result of intuitive or impulsive responses to familiar missing-value problem presentations. Our data show that even a detailed understanding of the relationship between covarying quantities by no means guarantees the normative use of the proportion equation.

Keywords: Teacher Knowledge; Teacher Education–Preservice; Rational Number

Purpose

Meno's slave famously told Socrates that a square of double area is obtained by doubling the side length of the given square. The inappropriate application of proportional reasoning strategies in non-proportional covariance situations (termed here *non-normative*) is widespread and documented in studies from many countries and among participants with a wide range of ages (for a review, see Van Dooren, De Bock, Janssens, & Verschaffel, 2008). However, knowledge of the root psychological sources of this tendency remain tentative and pedagogical remedies unsatisfactory (see the first issue of *Mathematical Teaching and Learning 12*).

In the present study, we found that preservice middle-grades teachers have many of the same problems with proportional reasoning as those reported with other populations. To gain further insight into these difficulties, we employed diSessa's (1993) knowledge-in-pieces perspective to track how pre-service teachers used knowledge resources before and after a unit on proportional and non-proportional reasoning in a methods course. We report two cases in which participants judged non-proportional relationships to be proportional even though they demonstrated clear understanding of the non-proportional covariance relationship.

Perspectives

Teachers and the Illusion of Proportionality

Fruedenthal (1983, p. 267) wrote, "Linearity is such a suggestive property of relations that one readily yields to the seduction to deal with each numerical relation as though it were linear." Teachers, even those certified for mathematics, are not exempt from its lure. Riley (2010) found that in a sample of 80 preservice elementary teachers, less than 50% solved *constant difference* and *inverse proportion* problems correctly. We illustrate these terms with example tasks from other studies with teachers. We used similar tasks in the present study.

Constant difference. Cramer, Post, and Currier (1993) report that 32 out of 33 preservice elementary teachers solved the following task using the proportion $9/3 = x/15$. "Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?" The relationship between these runners' laps is not proportional: They remain an equal distance apart: $9 - 3 = x - 15$.

Inverse proportion. Fisher (1988) reported that 12 out of 20 inservice secondary mathematics teachers did not solve the following problem correctly; "If it takes 9 workers 5 hours to mow a certain lawn, how long would it take 6 workers to mow the same lawn?" The common error was to assume a direct proportion such as $9/5 = x/6$. Instead, the relationship between men and minutes is inversely proportional: The amount of work is constant: $9(5) = 6x$.

Knowledge-in-Pieces and Conceptual Change

Prior work on this problem has been conducted within a traditional knowledge-as-theory perspective (Özdemir & Clark, 2007). Knowledge is understood as a unified and coherent structure and conceptual change is characterized as a paradigm shift: old structures are replaced with new ones. Instead, we use diSessa's (1993) knowledge-in-pieces epistemology that models knowledge as networks of loosely connected knowledge resources that are each highly sensitive to context. Conceptual change from this perspective is characterized as the piecemeal construction and reorganization of knowledge resources (e.g., diSessa & Sherrin, 1998) as learners gradually navigate the continuum from novice to expert.

Within the larger knowledge-in-pieces perspective, diSessa and Sherin (1998) proposed the *coordination class* as an empirically verifiable alternative to the black-box notion of concept in traditional work on conceptual change. A coordination class is made up of *readout strategies* by which one acquires information ("sees") in knowledge-use situations. The *causal net* of a coordination class is made up of the syllogism-like ways of inferring new information not directly available from readout. For example, someone might use the equation $F=ma$ to obtain information about acceleration from a situation that specified only force and mass. Different contexts require different readout strategies and result in different kinds of inferences in the causal net. *Coordination* in an expert's coordination class means (a) that one *integrates* all of the relevant information in a particular context, and (b) that inferences are *aligned* or consistent across the range of applicable contexts.

Under this analytic approach, knowledge and its use are not carefully distinguished, and knowledge per se is empirically linked at a fine grain size to contextual differences across situations of knowledge use. Wagner (2006) applied diSessa's framework and found that conceptual change happened as the participant's knowledge, "came to account for (rather than overlook) contextual differences" (p. 6). We follow Wagner in specifying terms for more precisely discussing the contextual differences that problem solvers encounter in activity. The *type* of a problem is defined by appealing to normative or expert judgment. The *aspects* of a problem are defined separately for each problem solver by obtaining empirical evidence for what features or details are perceived as relevant. The term *context* is broader, including type, aspect, and the cover story for the problem. For example, although the problems reported by Crammer et al. (1993) and Fisher (1988) are of a different type (one describes a constant difference and the other describes an inverse proportion), a problem solver might likely read out a similar aspect in both situations: the women run and the workers mow at equal rates.

Methods

Participants and Context

This study took place in the context of an 18-week content/methods course on number and operations for preservice middle-grades mathematics teachers that met for two 75-minute sessions each week. Students in the course ($N = 28$) were in their third year of college and had taken (at least) a course in introductory calculus prior to the study. Two weeks of the course were devoted to a unit on proportional reasoning. Students worked on tasks in small groups and participated in whole-class discussions. The instructors selected tasks and orchestrated discussion to focus on the big ideas for the unit such as: proportional situations are those in which covarying quantities maintain a constant multiplicative relationship (ratio).

Problem Situations

We gave students problem situations describing both proportional and non-proportional covariation during the unit and on assessments. The present study focuses on the following four problems that are variants of problems used in previous studies (e.g., Cramer et al., 1993; Fisher, 1988). The first two problems describe situations of constant difference. The last two are non-linear as well as being non-proportional. The Work Problem describes an inverse proportion situation, and the Interest Problem describes (an approximation to) an exponential situation.

The Running Problem (used in the pretest, posttest, and interviews). Determine whether the following problem is a *mathematically valid* illustration of the proportion A/B = C/D: Bob and Marty run laps together because they run at the same pace. Today, Marty started running before Bob came out of the locker room. Marty had run A laps by the time Bob had run B laps. How many laps C had Marty run by the time that Bob had run D laps?

The Combine Problem (used in the methods course unit on proportion). Two combines harvest grain at the same rate. The first combine starts harvesting 10 minutes before the second combine. After 20 minutes of operation, the second combine harvests 400 lbs of grain and the first harvests 600 lbs of grain. How many pounds will the second combine harvest by the time the first has harvested 1000 pounds of grain?

The Work Problem (used in the pretest, posttest, and interviews). Determine whether the following problem is a *mathematically valid* illustration of the proportion A/B = C/D: If A men paint the outside of a house in B minutes, then how many minutes D would it take C men to paint the same house, if all the men work at the same rate?

The Interest Problem (used in the methods course unit on proportion). Karl has a savings account that pays interest monthly at a rate of 5%. Three months ago, there was $300 in his account. If he did not withdraw any money from the account, how much is there now?

Data Collection

Each class session was video recorded using two cameras. One camera was stationary and positioned to capture the activity in the whole classroom. The second camera was hand-held and tracked the primary instructor (second author) and the written work of the students' with whom he was interacting. Students also took a pre-test and post-test. The primary data for this study were from video taped interviews (running 60 to 90 minutes) conducted with four pairs of students from the class. During the interviews, each pair was presented with a sequence of tasks and asked to solve the task together while verbally explaining their reasoning. The interviewer encouraged the students to talk freely and occasionally asked clarifying questions.

Analysis

Video data from each class was summarized, and student and instructor comments were time-stamped and supplemented with screen-shots to facilitate review. The interview data were fully transcribed. We watched all of the interview data and wrote summaries comparing each of the students' pre- and post-unit interviews. Then we reviewed the classroom video data for each student and looked for interactions or written work that might inform the observed changes. Finally, we reviewed students' written work on the pre-test and post-test, on the course midterm and final exam, and on the proportional reasoning homework assignments.

Results

We found that students who correctly explained relationships between quantities that were not proportional in the Running and Work problems (given above) still tried to set up and use proportion equations. To understand how this could take place, we examined the knowledge resources that students used to determine whether or not proportions could be set up to solve these problems. Our analysis led us to focus on (a) the consistent use of one knowledge resource (the necessary correspondence between quantities and their position in a proportion equation) by all interviewed students across all tasks, (b) the readout and use of the *same rate/pace* aspect of the tasks which varied among students, and (c) the various aspects of the non-proportional Interest and Combine problems (given above) that students found relevant for inferring proportionality.

First Interview, Work Problem

In this section, we use data from the Work Problem to describe a common aspect of the problems perceived as proportional—correspondence between quantities. *(Note on transcription:* Pauses are indicated by ellipses and interruptions by the em dash; overtalk is within double-slashes and action within square brackets; there are no deletions in the transcript provided.)

Lisa and Tess. These students immediately agreed that the Work Problem was proportional, and Lisa notably recognized the aspect of *same rate*.

Tess: I think [the proportion is] accurate because you have the number of men on top over //*Lisa*: and they're going at the same rate// the number of minutes and the same rate so you have your second number of men over your second number of minutes.

Alice and Clara. After reading the problem, Alice said, "So I guess like you would start off by setting it up like this," and wrote the proportion A/B = C/D. Then Clara used her pencil to point to A and then B while saying, "This amount of men paint a house in this many minutes." Alice touched her pencil to B at the same time as Clara, and then Alice said, "This many men [pointing to C] and minutes [pointing to D]."

Discussion. Across all pairs, students began almost all interview tasks by carefully reading out the correspondence between the quantities represented by variables and the position of these variables in the presumptive proportion equation. This is evident in Tess's initial comments ("... so you have your second number of men over your second number of minutes"). When asked about similar behavior during the second interview, Lisa said, "Proportion has to correspond." Alice and Clara also read out the correspondence between initial and final, men and minutes in the Work Problem, and this shared activity and its result had a shared interpretation: the proportion equation was applicable. Several other students we interviewed noted that a failure of correspondence would rule out the proportion equation in their view.

First Interview, Running Problem

In this section, we report how different interview pairs read out the aspect of *same pace* in the Running Problem, and how they made very different use of it.

Lisa and Tess. Tess established the correspondence between the quantities as she read the problem out loud by pointing to each variable in the proportion as she read its description. After a 30-second pause, Lisa began.

Lisa: Hmm. I mean if they keep on at same pace, //*Tess*: Right// isn't that going to be the same difference between the two //*Tess*: Right// so it would be an equal proportionality I would think ... kind of like equal fractions //*Tess*: Yeah// equivalent fractions?

They tried a numerical example, then both agreed the proportion was valid.

Interviewer: So run me through your reasoning one more time.

Tess: Okay. We just plugged in numbers to make sure that it was accurate and valid. So we said that if Marty had run 4 laps by the time that Bob had run 2 laps, we're looking for how many laps C, Marty ran by the time that Bob had run 8 laps. So we put 4 over 2 equals X over 8. And so our proportion, we're going to do 4 times 8, which is 32 equals 2 times X, just 2X, and then we're going to divide by 2. So we get 16 equals X and [rewriting the proportion] it's going to be 4 over 2 equals 16 over 8 and that is correct because you can simplify ... that's 2, 2 [writes 2=2]. So yes it is valid.

Alice and Clara. After working with a numerical example, the students made a discovery.

Alice: So we're saying by the time Marty had ran 4 laps Bob would have ran 2 //*Clara*: Yes// because I feel like if they run it in the same pace—

Clara: —then it should have been only 1 more lap.

Within a few minutes, they agreed that the proportion equation was not valid.

Interviewer: Why are you questioning its validity?

Clara: Because if they're running the same pace ... if Bob had run 1 more lap than Marty should have run 1 more lap. He just started earlier but they're running at the same pace, so the same speed of 1 lap should just be in Marty's 1 more lap. If Marty went and started and he ran 1 lap and then Bob came and started and ran another lap, Marty is still running so Marty would have run 2 laps by the time Bob ran 1. And then when they run another lap, Marty would have run 3 laps by the time Bob ran 2, but with this proportion it's saying that Marty would have run 4 laps by the time Bob ran 2 because it's doubling.

Discussion. It is clear that the *same pace* aspect of the problem was quite salient for Lisa. She interrupted Tess on the Work Problem to point out a similar aspect of that problem, *same rate*. In the Running Problem, she used the *same pace* information to conclude (normatively) "the same difference." However, the information about *same difference* served as a warrant for Lisa's use of the proportion equation: "so it would be an equal proportionality, I would think." Clara made a different inference, "they're running at the same pace, so the same speed of 1 lap should just be in Marty's 1 more lap." These data are evidence of differences in these students' causal nets. That is, although both pairs had access to the same information about the problem, awareness of a constant difference led them to different conclusions about the applicability of the proportion equation.

The data presented thus far admit the alternative hypothesis that Lisa and Tess did not fully understand the problem situation or were simply failing to be attentive and careful. Certainly, it makes intuitive sense that students who possess or develop the kind of quantitative understanding that Alice and Clara displayed of the Running Problem would not apply the proportion equation in non-proportional situations. The unit on proportional reasoning in the methods course was designed to help students develop just this kind of quantitative understanding for several types of non-proportional covariance that are frequently viewed as proportional.

The Proportion Unit

The first and second interviews bracketed the two-week unit on proportional reasoning that included work distinguishing proportional and non-proportional relationships. The Interest Problem and the Combine Problem received prominent attention as explicit examples of non-proportional situations throughout the unit. Students in the methods course witnessed clear, normative explanations from their peers and had opportunities to work on these and similar tasks in class and on homework (including inverse proportion tasks). They also received feedback on their work from the instructors in class and on homework.

The unit appeared to have little effect on students' tendency to use the proportion equation on the non-proportional Work and Running problems. (These specific problems were not discussed during the unit.) Table 1 shows how many students maintained or changed their responses for the Work and Running problems on the posttest ($N = 27$, one student was absent for the posttest) and serves to contextualize the data from the second interview described in the next section. We used McNemar's (1947) test for matched data and found that there was no statistically significant change in students' responses on these items ($p_{\text{Work}} = 1.00$, $p_{\text{Running}} > .450$). Students were almost entirely agreed on the Work Problem, but the consensus response was non-normative. By contrast, the proportion of normative responses on the Running Problem was not much different than that expected by chance.

Table 1: Student Pretest and Posttest Responses on Two Non-Proportional Problems

	Posttest Response			
	Work Problem		*Running Problem*	
Pretest Response	Proportional	Non-prop.	Proportional	Non-prop.
Proportional	25	1	9 2	
Non-prop.	1	0	5	11

The Second Interview

In this section, we report two cases where students clearly understood quantitative relationships that were not proportional in the problem but still endorsed the proportion equation.

Clara. (Alice did not participate in this interview because of a family obligation.) Clara began the Work Problem with confidence and set out to justify her use of the proportion equation by using a numerical example: 2/15 = 4/? . Then she paused for 35 seconds.

Clara: It is proportional, but it's going to be a proportion going down this way if the men were increasing [draws a line with negative slope] because if more men are working on the house then it's going to take fewer minutes. But if it takes 2 men 15 minutes then it's going to take one man 7.5 minutes so it is going to be proportional. I mean ... not 7.5, that doesn't make sense ... takes one man 30 minutes. It's proportional, it's just a decreasing proportion.

After working for 3 more minutes, Clara rejected her graph but not the proportion equation. A changed tone of voice suggested decreased confidence, but she remained adamant.

Clara: What I'm thinking is that as the amount of people painting the house increases the amount of time it takes to paint the house would decrease. I'm just not sure how to illustrate that. It would be a proportion, I feel it's proportional ... I just don't know how to represent it proportionally [in a graph].

Lisa and Tess. Both Lisa and Tess immediately decided that the Running Problem was proportional.

Interviewer: How do you know it's proportional?
Tess: Because if they run at the same pace this says they run laps together because they run at the same pace. Even if Marty starts before Bob, however many ... they're going to run at the same pace. So it's going to in ... like the amount difference is going to stay the same the whole time because they're running at the same pace.
Lisa: Mm-hmm [yes]. It's a constant increase …
Tess: Right, if Marty starts he runs two laps, and Bob starts. So by the time that he runs 4, by the time Marty runs 4 laps, Bob will have run 2 laps ... then 6, 4 ... 8, 6 ... so the same amount of increase every time.
Lisa: Yeah, and if you know the lap difference between the two then you can give me any value of laps and I can figure out where they are //*Tess*: Right// without having to go step by step like you do with interest problems.
Tess: Right.
Interviewer: So that's what tells you it's proportional?
Lisa: Yeah, because you're not factoring in time or anything. You're like ... I mean if you were to ask at twenty minutes then you'd have to factor in the time difference.

A little later in the interview, during the Work Problem, Tess continued in the same vein.

Tess: Right, or like on interest problems like every month there's five percent interest. She starts with this amount; then she puts this amount in each month, and then you have to also think about your

> interest and how that's affecting your amount. But this is just a same rate you know, like there's nothing adding in.

Discussion. In both cases, the students provide normative reasoning about how the quantities in the situation covary using specific numbers. Clara said, "If it takes 2 men 15 minutes ... [it] takes one man 30 minutes." Tess gave a sequence of example values for the consecutive pairs of laps run; "Marty runs 4 laps, Bob will have run 2 laps ... then 6, 4 ... 8, 6" The students clearly demonstrated accurate knowledge of the relationships among the quantities in the problem situations, yet in both cases the students endorsed the proportion equation.

The data from Clara contrast with her work during the first interview on the Running Problem, where she used similar quantitative understanding to normatively reject the proportion equation. In this case, contextual differences led to non-normative knowledge use. Clara read out the aspect of *more men, fewer minutes* and interpreted the Work Problem as a negative proportion problem like those discussed in class and on the homework.

In the case of Tess and Lisa, some of the data are consistent with the first interview. For example, Tess read out the aspect of *same pace*, and used this to judge that "the amount difference is going to stay the same the whole time." Lisa apparently agreed, saying, "It's a constant increase." But rather than moving directly from the aspect of *constant increase* to the proportion equation (that is, in parallel to how Lisa moved from the aspect of *same difference* to the proportion equation during the first interview), this pair referenced their course experiences with the Interest and Combine problems.

Lisa and Tess read out and used aspects of the Combine and Interest problems that would likely go unnoticed by experts. The Running Problem had a functional aspect for Lisa, "You can give me any value of laps and I can figure out where they are." By contrast, Tess and Lisa's work on the Interest Problem was iterative; they used each monthly total to compute the next. In the class discussion of the Interest Problem, Tess said, "You need to know all of the previous months to find the next month." Several other aspects of the non-proportional problems discussed in class distinguished them from the interview problems. For example, Lisa read out the time interval of 20 minutes specified in the Combine problem and used this as a warrant for non-proportionality. She said that the Running Problem would not be proportional "if you were to ask at twenty minutes." Moreover, both students agreed that to solve the Interest and Combine problems, one had to "factoring in" an extra quantity like time or interest. In the Running and Work problems, Tess observed, "there's nothing adding in." Tess's and Lisa's knowledge resources were evidently refined and reorganized to incorporate their experiences in the course but their judgments remained non-normative.

Implications and Conclusion

The quantitative results of this study provide a partial replication with preservice middle-grades teachers of prior results with elementary and secondary teachers (e.g., Cramer et al., 1993; Fisher, 1988; Riley, 2010) showing that teachers face some of the same challenges as children when reasoning about situations involving non-proportional covariance. Recent research (e.g., Van Dooren, De Bock, Vleugels & Verschaffel, 2010) suggests that thinking about problems rather than answering reflexively is necessary if students are to avoid applying the proportion equation in non-proportional situations. The qualitative data reported here warrant a much stronger claim: Even a detailed understanding of the relationship between covarying quantities may not be sufficient for the normative use of proportionality. These results suggest a sobering assessment of the pedagogical challenge faced by teacher educators. Preservice teachers may need a broad and coordinated collection of fine-grained knowledge resources developed in a wide variety of contexts in order to distinguish between covariance relationships and to apply the proportion equation appropriately across contexts and problem situations.

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TOWARDS EXPERT CURRICULUM USE: DEVELOPING A MEASURE OF PRE-SERVICE TEACHERS' CURRICULAR KNOWLEDGE

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The overall purpose of this study is to develop a measure of elementary mathematics teachers' curricular knowledge (Shulman, 1986) and curriculum use practices. In this paper, we present the first step in this larger effort—the piloting of one set of questions that document pre-service teachers' (PSTs') knowledge and practices for reading, evaluating, and adapting a Standards-based curriculum lesson. We present the range of responses elicited from 34 PSTs related to the goals and purposes of the lesson, the strengths and weaknesses of the lesson, and possible changes to the lesson. These survey questions and our findings about the range of PSTs' responses to the questions are intended to help researchers further develop the constructs of curriculum use and curricular knowledge.

Keywords: Curriculum; Elementary School Education, Teacher Education–Preservice, Teacher Knowledge

Introduction

The overall purpose of this study is to develop a measure of elementary mathematics teachers' curricular knowledge (Shulman, 1986) in order to document the development of this knowledge as PSTs move through elementary mathematics methods and then into student and novice teaching. Mathematics curriculum materials are ubiquitous and often mandated in elementary classrooms, yet the field of mathematics education has few tools for developing and measuring teachers' knowledge related to using these materials in productive ways. In this paper, we present the first step in this larger effort—the piloting of one set of questions that document PSTs' knowledge and practices for reading, evaluating, and adapting a *Standards*-based curriculum lesson.

Theoretical Framework

We understand these practices—reading, evaluating, and adapting curriculum materials—to be part of a larger construct of expert curriculum use that incorporates many of the aspects of curricular knowledge described by Shulman (1986). In our work, we have begun to develop a conjectured learning trajectory describing teachers' curriculum use practices from initial curriculum use (beginning of the methods course) to expert curriculum use. Our definition of expert curriculum use draws from a substantial body of work that has been conducted in the past several years, including the work of Remillard (2005; Remillard and Bryans, 2004), Brown (2009), Sherin and Drake (2009), and Taylor (2010).

Taken as a set, this work suggests that the teachers' curriculum use is a dynamic, interpretive, and interactive process in which both teachers and materials contribute resources in the design and enactment of instruction. "Expert" curriculum users seem to have (1) curriculum vision—an understanding of the goals of the curriculum, as well as strategies for using the curriculum materials to reach those goals (Cirillo & Drake, in revision); (2) particular strategies for reading, evaluating, and adapting curriculum materials in productive ways (Sherin & Drake, 2009); (3) practices for using curriculum materials to accomplish instructional goals (Brown, 2009); and (4) strategies for "systematically" adapting curriculum materials to meet the needs of students (Taylor, 2010). The portion of the curriculum use survey that we describe in this paper focuses on the second set of practices—reading, evaluating, and adapting curriculum materials. Our ultimate goal is to develop a measure that reflects all of these components of expert curriculum use, as well as additional features of curricular knowledge as described by Shulman (1986).

Methods and Data Sources

Participants

We piloted the survey with 34 PSTs enrolled in a small liberal arts university located in the Mid-West. Thirty-one participants were female; three were male. Seven took the survey at the beginning of a semester-long elementary mathematics methods course that included a focus on the use of *Standards*-based curriculum materials, and the remaining PSTs took the survey at the end of the course. For this study, responses from the beginning and end of the semester were combined into a single set of responses in order to identify and describe the range of PST responses.

Description of Lesson

The curriculum use questions focus on a first-grade lesson from *Math Trailblazers* (University of Illinois at Chicago, 2008a). The lesson begins by presenting several numbers (e.g., 125) to students and asking them what those numbers mean. In the materials, anticipated student responses are listed (e.g., 5 groups of 25, 12 groups of 10 with 5 left over). During student exploration time, students consider the number 172 and represent it in any way they choose. Next, they share their representations with a partner, and then a whole group discussion occurs.

Data Collection and Analysis

The survey consists of 18 questions. For the purposes of this study, we selected six questions:

<u>Reading</u>

1. As a teacher, what would be your specific goal(s) for your students' learning with this lesson?
2. On page 37 in the first bullet point under the assessment heading, the lesson plan states, "Even though counting by ones is an inefficient strategy, it works if done carefully." What does that mean?

<u>Evaluating</u>

3. Does this lesson have multiple entry points? In other words, is the task accessible to a wide-range of learners? Explain.
4. When thinking about student learning, what are the strengths and weaknesses of this lesson?

<u>Adapting</u>

5. If you would make changes to this lesson, what would they be?
6. Another pair of students represented 172 with 6 groups of 25 and had 22 left over. What would you say to or ask these students after they have shared their solution?

The first two questions were designed to measure PSTs' *reading* of the curriculum materials by asking them to explain the learning goals and meaning of a selected phrase from the materials. The third and fourth questions addressed PSTs' *evaluation* of the materials by asking them to assess the lesson against the concept of "multiple entry points" discussed in class and determine the strengths and weaknesses of the lesson. The last two questions allowed PSTs to describe the ways in which they might *adapt* the lesson after having read and evaluated the curriculum materials and respond to a particular solution strategy.

Each survey question was analyzed separately through a process of open and emergent coding (Strauss & Corbin, 1998). For each question, a set of codes was generated that illustrated the type of survey responses. Codes will be presented in our results section. Our goal at this point in the development of the survey is to capture the range of possible PST responses to each item in order to further refine the survey.

Results

In this section, we present results from the six survey questions in sections related to each of reading, evaluating, and adapting curriculum materials.

Reading Curriculum Materials

PSTs' goals (Question 1) for teaching the 172 Lesson were categorized using three primary codes. Responses were categorized as *procedural* if they focused on counting/grouping; as *conceptual* if they focused on the understanding/meaning of number and/or place value; and as with *connections* if there was explicit mention of making connections across multiple strategies and/or representations. Responses could be any combination or all of the above three codes, which led to six types of goal responses. Table 1 summarizes the response to Question 1.

Table 1: Responses to Question 1 (Goal Question)

Number of PSTs	Type of Response	Example
7	Procedural	Counting and grouping objects that are greater than 100.
11	Procedural with Connections	Students will be able to represent numbers greater then 100 using manipulatives and words. Students will be able to group objects by ones, tens, and hundreds.
3	Conceptual	My goals for this lesson would be for the students to understand what 3 digit numbers mean and to be able to talk about and explain them.
1	Conceptual with Connections	To have students talk about the meaning of a number, represent a number by a picture, and use different objects to represent a number.
5	Procedural and Conceptual	As the teacher, my specific goal for this lesson would be that the students group numbers between 101 and 199 in a way that shows that they understand place value.
6	Procedural and Conceptual with Connections	Understanding place value, hundreds, tens, and ones. Being able to break numbers into parts and recognize they belong to a whole representing numbers with pictures or symbols grouping and counting objects by ones, tens and hundreds.

In the curriculum materials, the goals were listed as the following:

- Representing numbers greater than 100 using manipulatives, pictures, symbols, and words
- Grouping and counting objects by ones, tens, and hundreds. (University of Illinois at Chicago, 2008b, p. 33)

Using our coding scheme, the above goals would be categorized as procedural with connections, which makes these results particularly interesting to us. We conjecture that the curriculum authors had a conceptual purpose in mind when writing these goals, but that purpose was not explicit in the materials. Identifying a conceptual goal for the lesson required a significant amount of interpretive work while reading the lesson, and we found that many of the PSTs (15/33) did engage in that work.

For Question 2, PSTs were asked to interpret the following statement: *Even though counting by ones is an inefficient strategy, it works if done carefully*, there was a wide range of responses. Each response was coded with one or more of the following codes—*better ways* if the PST stated there were "better ways" to count or represent 172 than counting by ones; *specific limitation(s)* if the PST stated one or more limitations of counting by ones; *another strategy* if the PST suggested another strategy that students should use; *it works if careful* if the PST stated the strategy works, but students need to be careful; *count by ones* if the PST discussed that it is expected that some students will counts by ones to represent 172; and *acceptable strategy* if the PST seemingly took a stance for students who wanted to use that strategy. Table 2, along with some text after the table, summarizes the responses to this question.

Table 2: Responses to Question 2 (Interpreting a Phrase)

Number of PSTs	Type of Response	Example
16	Specific Limitations	This means that it takes longer to count three digit numbers by ones, and is more prone to mistakes because of the tediousness of the strategy.
7	Better Ways	Counting by ones is not the quickest way to assess a large number of items. It does work, but there are better ways to do it.
6	Counts by ones	It is expected that some students will count by ones even though the number is so large. When it states, "it works if done carefully," I think they are saying that if is ok that students do that.
4	Acceptable Strategy	The purpose of this lesson is counting 172, not grouping 172. If the student's method is counting by ones and they are getting the correct answer, then they are completing the lesson. From this foundation, you can build with them an understanding of grouping and they may change their method of counting as they grow older.

Of the 16 PSTs who stated one or more specific limitations of the counting by one strategy, five mentioned another strategy (e.g., grouping larger numbers) students could use and six mentioned that the strategy works if it is done carefully. Of the seven PSTs who stated that there are better ways to count 172, three also mentioned specific limitations of the strategy while another mentioned the strategy was OK to use. Of the six PSTs who thought it would be expected for students to count by ones, one PST also said that it was okay to do. In this set of findings, the role of PSTs' beliefs is clear, particularly their beliefs about how children learn mathematics. Many PSTs elaborated the aspect of the statement that counting by ones is an inefficient strategy by describing in one or more ways how counting by ones is inefficient. Others perceived the statement as saying that some students will need to count by ones to solve the problem and/or that strategy is acceptable.

Evaluating Curriculum Materials

Responses to the third question about whether or not the 172 Lesson had multiple entry points were first sorted into yes or no categories. Eight PSTs did not think the lesson had multiple entry points and all eight stated that this was because only one number was given for students, although some of the eight noted that this number could be adjusted by the teacher, as in the example response in Table 3. Twenty-six of the 34 PSTs thought that the lesson *did* have multiple entry points. For those 26 responses, a set of codes was developed to describe PSTs' reasoning. A response was coded as *student develops/uses own strategy* if the PSTs discussed that the students could develop or use their own strategy to represent 172; *number can be changed* if the PST thought the number could be changed to meet the range of learners in the classroom; and *count by ones* if the PSTs discussed the idea that if students could count by ones to represent 172, then the lesson had multiple entry points. Table 3 summarizes these responses.

Table 3: Responses to Question 3 (Multiple Entry Points)

Number of PSTs	Y/N	Type of Response	Example
8	No	Only one number choice	There is only one number choice provided for the students to work with and it is in the high range of the 100's. Thus, I would provide additional number choices of one slightly above 100 like 112 and another number choice in the middle (e.g.132) to provide access to a greater range of ability levels.
14	Yes	Student develops/uses own strategy	There are multiple ways to draw 172 beans, but there really isn't a clear "solution.” They already know there are 172 beans and have to draw them. The only different will be how they drew it.
4	Yes	Number can be changed	For slower or higher learners, you could adjust the number of beans to an easier or more difficult number, and give more or less support to the students as needed.
3	Yes	Students develops/uses own strategy and Number can be changed	I think the lesson has multiple entry points. The lesson doesn’t specifically give a way to illustrate the number. I think students could represent it in a lot of different ways. Because of the number, the lesson may be harder for lower level students. I would adjust the number for a different range of learners.
2	Yes	Count by Ones	As long as students know the number about 100 and can count by ones, then yes it is accessible to wide-range learners.

All but three responses could be sorted into our above codes. Of those three responses, one talked about targeting multiple learning styles, another mentioned the teacher being able to ask students to count in a certain way, and the third suggested that the lesson did a good job of providing manipulatives. Another noteworthy finding was that four PSTs suggested alternative number choices as in the first example above. Many PSTs focused on the idea of offering multiple choices as a way to provide multiple entry points for students, or PSTs focused on the idea that students could develop their own strategies. Three PSTs thought a combination of those two ideas provided multiple entry points.

PSTs were asked to list the strengths and weaknesses of the 172 Lesson in Question 4. For this response, we developed a set of 12 codes that categorized ideas listed as strengths or weaknesses. In Table 4, we list the code and how many times it was mentioned as a strength and weakness.

Although the PSTs listed a wide range of strengths and weaknesses, we can identify some important themes in looking across their evaluations of the lesson. First, the PSTs focused a great deal on the students’ role in the lesson, with “Student-directed” and “Multiple Strategies” as the most common strengths. The most commonly noted weaknesses were in the structure of the lesson (e.g., the lack of an opening routine, the use of worksheets) and that the lesson was perceived as too challenging for some students and not challenging enough for others. Finally, many aspects of the lesson that were viewed as a strength by some PSTs were also viewed as a weakness by other PSTs, suggesting that PSTs vary widely in their evaluations of lessons.

Table 4: Responses to Question 4 (Strengths and Weaknesses)

Code	Frequency as Strength	Frequency as Weakness
Student-directed	14	4
Multiple Strategies	15	1
Lesson Structure	3	10
Interactions	10	2
Too challenging or not challenging enough	0	11
Concrete	8	1 (Abstract)
Differentiation – lack of/can be/cannot be	7	2
Assessment	4	4
Number Choice	1	3
Teacher-directed	2	2 (lack of)
Affective (e.g., enjoyable, comfortable)	3	0
Connection to real-life	1	2 (lack of)
Connection to more advanced mathematics	1	0

Adapting Curriculum Materials

Eleven PSTs would not make any changes to the given lesson. The remaining PST responses fell into three categories—would provide multiple number choices, would change aspects of the lesson that did not affect the overall approach in lesson, and would practice a model beforehand or give example. Table 5 summarizes responses to Question 5.

Table 5: Responses to Question 5 (What changes would you make?)

Number of PSTs	Type of Response
11	Would make no changes
11	Would provide multiple number choices
7	Would change aspects of the lesson (e.g., add opening routine, count something more meaningful to students) that did not affect overall approach in the lesson
4	Would practice a model beforehand or give example

It was not surprising to us that the most frequent adaptation (N = 11) was to provide multiple number choices. The result can be explained, in part, by the fact that these PSTs were in a methods course in which they had many opportunities to observe and reflect on lessons that provided multiple number choices for students. Seven other PSTs thought they would change aspects of the lesson that did not affect the overall approach in the lesson and four wanted to provide a model or example before the students began to work.

Question six pertained to how PSTs might question students as they engaged with the 172 Lesson: *Another pair of students represented 172 with 6 groups of 25 and had 22 left over. What would you say to or ask these students after they have shared their solution?* Responses were categorized according to which aspect of the solution PSTs questioned. In a few instances, a PST questioned multiple aspects of the solution. Table 6 summarizes the foci of PSTs questions.

Table 6: Responses to Question 6 (Questioning students about solution strategy)

Number of PSTs	Foci of Question
8	Questioned if there was another way to group or represent the leftover 22.
7	Questioned students as to why they used groups of 25.
4	Questioned if there was another way to group or represent 172 due to having 22 leftover.
4	Asked another type of question about the strategy (e.g., any patterns)
3	Questioned students as to why they used groups of 25 and if there is another way to group or represent the leftover 22.
2	Questioned if there was another way to group or represent 172.
2	Questioned if there was a way to consolidate the groups of 25.
3	Other

Most interesting to us was how PSTs reacted to the leftover 22 given that the curriculum materials listed the grouping by 25 strategy as an anticipated student response and the connection to money. Nine PSTs (second and sixth rows) asked if there was another way to group or represent the leftover 22 even with the other groups of 25 as in this response: "I would say, "how can we group the leftover 22 in an organized way? How could we split those 22 beans into 5 groups?" Another four PSTs wanted students to regroup the 172 entirely due to the leftover 22 as in the following response: "Could you have made less groups of more beans in order to not have so many left over?" Two others wanted students to consolidate their groups of 25.

Quite opposite from the disdain for the grouping by 25 strategy, was the response from one PST, who fell in the other category, as he/she made the connection (not the students) to money in their response. That response is given below:

> That's another great idea as we know that just like in a dollar there are 4 quarters (25), right? (relating it to a real life situation) then break it down further like 2 quarters in 50 cents, so there would be 22 ones left over (if you are thinking in those terms).

The other responses seemed not to value or dis-value the grouping of 25 strategy, as PSTs are just asking students to explain why they grouped by 25 or if there was another way to group or represent 172. The intent of these responses may be to support or extend student thinking.

Implications

This study is a step towards understanding expert curriculum use. These survey questions and our findings about the range of PSTs' responses to the questions can help researchers further develop the constructs of curriculum use and curricular knowledge, through an understanding of ways in which PSTs read, evaluate, and adapt curriculum materials. At the same time, these findings might support mathematics teacher educators in designing learning experiences for PSTs that contribute to the development of PSTs' curriculum use practices. This study also contributes to the field by providing a measure for documenting growth in teachers' curricular knowledge—an important knowledge base for teaching first identified by Shulman (1986). Ultimately, this measure, and others like it, can be used to understand PSTs' growth in knowledge and practices as they progress through teacher education courses and programs.

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COGNITIVE OBSTACLES AND MATHEMATICAL IDEAS RELATED TO MAKING CONNECTIONS AMONG REPRESENTATIONS

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This study investigates cognitive difficulties and mathematical ideas that are related to making connections among representations. A three-week intervention was designed and implemented to help prospective secondary mathematics teachers develop understanding of big ideas that are critical to connection of representations in algebra. This study finds that most participants had difficulties with the concept of variable, the Cartesian connection, and its related idea, graph as a locus of points, and held incomplete concept definitions and concept images of conic curves. During the intervention, however, many showed signs of progress. Many who were initially dependent on memorized forms of algebraic formulas made efforts to consider aspects of the Cartesian connection to make sense of their work.

Keywords: Teacher Education–Preservice; Teacher Knowledge; Algebra and Algebraic Thinking; Advanced Mathematical Thinking

Objectives

The purpose of this study was to investigate cognitive obstacles and mathematical ideas related to making connections among representations and how prospective teachers' thinking progressed while they worked on tasks that were designed to bring out these cognitive issues. The importance of representations in mathematical understanding is well documented (Brenner, Mayer, Moseley, et al., 1997; Knuth, 2000; Moschkovich, Schoenfeld, & Acavi, 1993), and students' learning of mathematics through representations is recommended in the Principles and Standards for School Mathematics (NCTM, 2000) and the Common Core State Standards (CCSI, 2010). However, not much is known about cognitive obstacles or ideas that are involved in making connections among representations or how to help students develop mathematical understanding through connection of representations (Bosse, Adu-Gyamfi, & Chetham, 2012; Even, 1998) except that even mathematically capable individuals have compartmentalized understanding of mathematical representations (Gagatsis & Shiakalli, 2004; Hitt, 1998; Vinner, 1989). Making connections among representations is one of the big ideas in mathematics education (Lacampagne, Blair, & Kaput, 1995; Knuth, 2000) and more research is needed to understand this complex, yet critical issue.

This study is a component of a larger project intended to develop and study a mathematics curriculum for prospective secondary mathematics teachers. This report specifically deals with a three-week unit designed to promote understanding of algebra through the connection of representations. Twenty prospective secondary mathematics teachers participated in this study. Qualitative research methods were employed in order to delve into cognitive difficulties and ideas that are related to the connection of representations and to document how student understanding evolved during the unit.

Theoretical Framework

In the mathematics education community, representations are regarded as critical tools for mathematical communications and problem solving (Brenner et al., 1997; Goldin & Shteingold, 2001; Hiebert & Carpenter, 1992; Hollar & Norwood, 1999; Mousoulides & Gagatsis, 2004; Thompson, 1994). However, it has known that mere representation of a mathematical concept in a certain mode of representation is not enough for mathematical understanding. In order to understand mathematical concepts or to be successful in problem solving, learners need to be able to connect representations. They need to be able to not only recognize ideas embedded in various representations but also convert a representation in one form to another and translate ideas from one representation to another within and across various representations (Borko & Eisenhart, 1992; Dufour-Janvier, Bednatz, & Belanger, 1987; Even, 1998; Hitt, 1998; Gagatsis & Shiakalli, 2004; Knuth, 2000; Lesh, Behr, & Post, 1987).

Many researchers have investigated how learners attempt to connect various representations of functions, a concept critical to much of algebra and higher mathematics. For example, Knuth (2000)) showed that the Cartesian connection—"a point is on the graph of the line L if and only if its coordinates satisfy the equation of L" (Moschkovich et al., 1993, p. 73)—is related to students' problem solving ability where they must connect symbolic and graphical representations of linear functions. Hitt (1998), Williams (1998), and Hansson (2005) have shown that teachers' inability to use subconcepts, such as variables and domain/range, affected their problem solving. Following up on the ideas posed by Knuth (2000) and other researchers cited above, a primary focus of this research was to examine how learners' understanding of critical mathematical ideas (that are related to connections among representations), such as the Cartesian connection, interacts with their abilities to connect representations and/or their problem solving ability.

In order to address this issue, we designed a unit focusing on representations in algebra. The unit, indeed the entire course, was developed and implemented along with the educational and philosophical principles of *realistic mathematics education* (Freudenthal, 1991; Gravemeijer, 1999) and *constructivist learning theory* (Cobb, Yackel, & Wood, 1992). As such, mathematical understanding in the course and hence in this study was regarded as social construction (or reconstruction) of ideas through mathematical communications and collaborations while learners are actively engaged with "realistic" (Gravemeijer, 1999) tasks.

For this study, we restrictively use the term, representations, in regard to only four types of representations in algebra—algebraic, spatial, numeric, and verbal representations. We also extended the notion of the Cartesian connection to "a point is on the graph of the mathematical relation, $R(x, y) = 0$ if and only if its coordinates satisfy $R(x, y) = 0$" in order to accommodate this idea to other mathematical relations with two variables, conic curves in this case.

The constructs of *concept definition* and *concept image* (Tall & Vinner, 1981; Vinner, 1991; Vinner & Dreyfus, 1989) are utilized as a tool to understand participants' cognitive structures relating to conic curves. According to Vinner and others, an individual's understanding of a concept is related to her verbal definition of the concept (concept definition) and her non-verbal image of the concept (concept image). When her concept definition and her concept images are aligned with the *formal concept definition* (the one accepted by the mathematics community), she can solve non-routine problems or prove theorems by successfully consulting both the definition and the image (Vinner, 1991). As with other studies concerning learners' understanding of mathematical concepts (Bangi, 2006; Bingolbali & Monaghan, 2008), this construct aided us to understand the mathematical thinking and understanding of the participants.

Methods

Research Setting

This research was conducted in an inquiry-based classroom during the winter quarter, 2009, when a three-week teaching unit focusing on Algebra was implemented. 20 prospective secondary mathematics teachers (PSMTs), mostly juniors and seniors majoring in mathematics, participated in this study. The Algebra unit started with the opening activity, a discussion of symbolic and spatial meanings of the solution of the algebraic equation $x^2 = 2$, followed by the major task, an interpretation of the historical work by Omar Khayyam (1048–1131). PSMTs were asked to figure out how Omar Khayyam's geometric approach to the solution of a cubic equation made sense, i.e., how an intersection point of the parabola $py = x^2$ and the circle $x^2 + y^2 = qx$, with p and q positive integers, determines the solution of the algebraic equation $x^3 + p^2x = p^2q$. This task involved three subtasks: (a) graphing the circle represented by algebraic representation $x^2 + y^2 = qx$ and proving why the formal concept definition of circle defines the equation of circle, (b) deriving the equation of a parabola based on the concept definition of parabola, and (c) explaining how the solution of $x^3 + p^2x = p^2q$ is represented spatially. During the Omar Khayyam task, two quick writes were administered to examine PSMTs' understanding of representations, the one examining PSMTs' abilities to connect the concept definition and the algebraic representation of circle, and the other examining their abilities to connect algebraic and geometric representations.

All sessions were videotaped using two video cameras. PSMTs' responses to quick-writes and group posters that they prepared for the presentations were also collected.

Methods of Analysis

For the analysis, we adopted and modified an analytic method by Powell et al. (2003), designed for research using video data. At first, we viewed the video recordings and prepared a brief, written record of the video content. The written record at this stage included rough transcriptions of some episodes and focused on mathematical activities, situations, and meanings (Powell et al., p. 416). We also roughly viewed PSMTs' quickwrites to get a sense of their responses. We then developed a priori codes, based on the research framework, research questions, and the problematic areas of the mathematical investigations that were found from the viewing of video recordings, quickwrites, and posters (Miles & Huberman, 1994). With these observational codes, we rewatched the video recordings and reexamined the written data. At this stage, we revised the codes by identifying more codes and constantly comparing with the existing codes (Glaser & Strauss, 1967), and identified *critical events*—significant moments showing learners' cognitive difficulties, conceptual leaps from previous understanding, or intuitive mistakes (Powell et al., 2003). At the next stage, we prepared word-to-word transcripts of the portion of the video data, including critical events and other episodes that "provided evidence for important theoretic or analytic matters to our guiding research questions" (Powell et al., 2003, p. 423). The new transcription data, combined with the written data, then, put through another phase of analysis for the accuracy and the consistency of the results.

Results

The Case of Circle

Most PSMTs knew the definition of circle correctly, as "the collection of points equidistant from a point". However, their algebraic concept image of circle, $(x - a)^2 + (y - b)^2 = r^2$, and the spatial concept image of circle were compartmentalized or existed without proper understanding of the roles of variables x and y or constant r.

Only one out of five groups (each group had 4 PSMTs) successfully transferred the algebraic representation, $x^2 + y^2 = qx$, to its graphical representation, the circle with the center $(q/2, 0)$ and the radius $q/2$. Three groups transferred the equation, $x^2 + y^2 = qx$, to a circle centered at the origin, with radius labeled $\sqrt{qx}$, by taking the left part of the equation, $x^2 + y^2 =$, as a process of drawing a circle with center (0, 0) and qx as the square of the radius, without paying attention to the variable x. In subsequent class discussions, they also showed lack of understanding of the Cartesian connection. Although these groups had an understanding that $x = 0$ and $y = 0$ satisfy the equation, $x^2 + y^2 = qx$, they were unable to translate this idea to the graphical representation in that they did not recognize that the graph of $x^2 + y^2 = qx$ had to pass through the origin. One group transferred $x^2 + y^2 = qx$ to a bow-tie figure passing through the origin (see Figure 1). Although this group's graph passes through the origin, they did so because they believed that their "radius" $\sqrt{qx}$ approached 0 as x approached 0. Understanding of the concept of variables in graphical representations or the Cartesian connection was absent in most of these students.

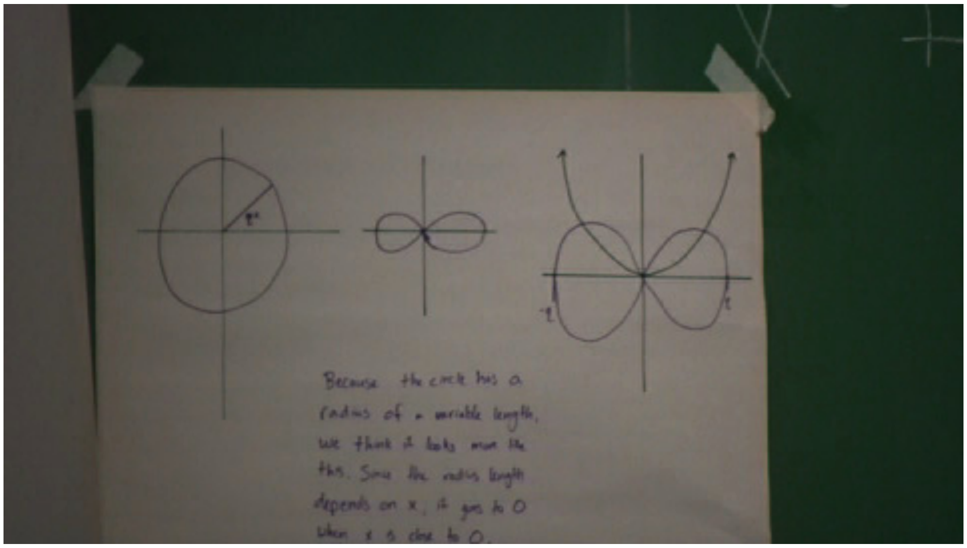

Figure 1: Representation of circle and parabola

The Case of Parabola

Only one out of 20 PSMTs knew the concept definition of parabola—the locus of points equidistant from a point, called focus, and a line, called directrix. When they were asked to find the equation of parabola with the focus $(0, f)$ and the directrix $y = -f$ given, the vast majority had no clue how to start. For them, a parabola was given by the equation $y = ax^2$ or $y = a(x - h)^2 + k$ where the only meaningful information they could draw from these expressions were the vertex (h, k) and the coefficient a, and their discussion was mainly about the concavity of the parabola, how wide the parabola was depending on the value of a, or how to shift $y = ax^2$ to $y = a(x - h)^2 + k$ using vertical and horizontal translations. Only after being reminded by the instructors that their job was to derive the equation of parabola using the definition of parabola and that they could name a random point that is equidistant from the focus and the directrix as (x, y) in the Cartesian plane, did they attempt to interpret the definition of parabola to come up with some kind of algebraic equation. Even then, a lengthy discussion within their group and with the instructors was required to reach to the algebraic representation, $x^2 = 4fy$. Although they seemed to understand that x, y represent variables in an algebraic relationship, $r(x, y) = 0$, they did not understand that (x, y) could represent varying coordinates in the geometric context. Most of them also did not understand or use the idea that a graph of a mathematical relation is a locus of points whose coordinates satisfy the relation even if they had heard the definition multiple times from their classmate and instructors. Many of them had difficulties in translating mathematical concepts, such as distance and equivalence, from verbal/geometric representations to algebraic representations.

Connecting the Solution of the Cubic to the Intersection Point of the Circle and Parabola

For PSMTs, finding the relationship between the real solution of the cubic $x^3 + p^2x = p^2q$ and the point of intersection of the parabola and the circle was difficult as well. Two groups falsely claimed that the solution of the cubic equation was the distance between the point of intersection and the origin or the distance between the point of intersection and the focus of the parabola (The actual solution of the cubic equation was the x-coordinate of the non-origin intersection point of two graphs). The examination of PSMTs' quickwrites also suggested that they fell short in understanding the relationship between algebraic and graphical representations. In one of the two quickwrites, only 5 out of 18 PSMTs specified that the solution of a system of linear equations, $2x + y = 10$, $x + 2y = 8$, was the x, y coordinates of the point of intersection $(4, 2)$, discriminating the point $(4, 2)$ on the plane from its coordinates $x = 4$ and $y = 2$, another evidence for their lack of understanding of the Cartesian connection. For them, splitting a single object, a point (x, y) in the Cartesian plane, into two objects, its coordinates x and y, or using x as a distance between the point (x, y) and the y-axis in their descriptions was very challenging.

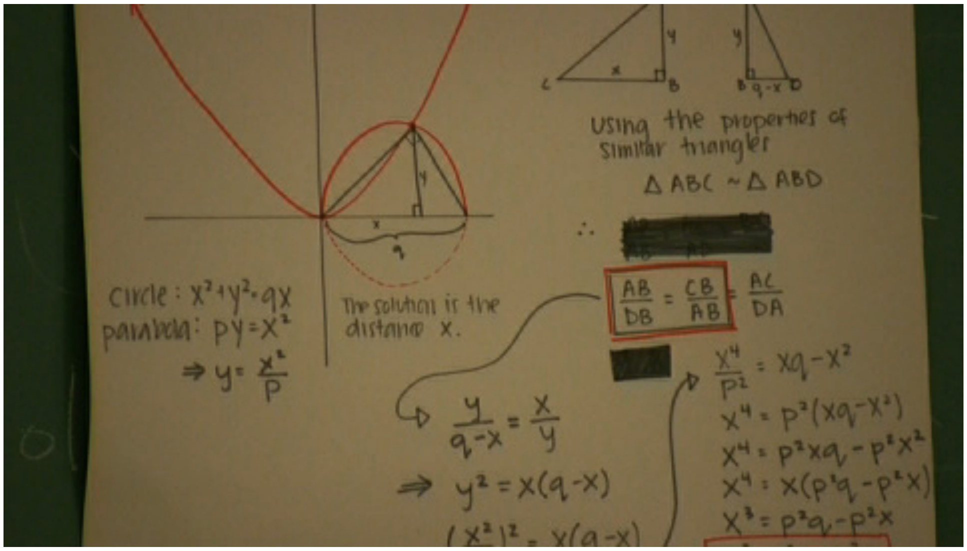

Figure 2: Graphical to algebraic transfer

Toward the end of the Omar Khayyam task, however, there were many signs that PSMTs were making progress in connecting representations. For example, one group showed the relationship between the algebraic representation of a circle $x^2 + y^2 = qx$ and the graph of circle with center (q/2, 0), by incorporating a concept in geometry—the proportionality of similar triangles—into the Cartesian coordinate system (Figure 2). This group first showed that $y/(q - x) = x/y$ and then derived the equation $y^2 = x(q - x)$ by cross-multiplying, which then can be transformed into $x^2 + y^2 = qx$. Their work, finding an analytic expression of circle using geometric properties on the Cartesian plane, as Descartes did, was remarkable progress, compared to their initial work on the Omar Khayyam task. In the beginning, they mostly focused on formulas and algorithms of which they did not make much sense. Two other groups also came up with explanations how Omar Khayyam could have represented his solution and idea in his period. The importance of verbal, spatial, and algebraic representations in understanding algebra was embedded in the groups' work.

Conclusion

This study found that without interventions prospective secondary mathematics teachers were largely dependant on memorized formulas or algorithms rather than focusing on meanings and ideas in connecting algebraic equations and their Cartesian graphs. Their problem solving was handicapped by incomplete concept images and definitions of circles and parabolas, similar to learners in the other studies on the concept of function (Even, 1993, 1998; Vinner, 1991; Vinner & Dreyfus, 1989; Williams, 1998). In the case of circle, although most of them acknowledged the formal definition of a circle, they were unable to prove why the definition gives the equation of circle. In the case of parabola, most of them had no knowledge of the definition of parabola or the subconcepts of focus and directrix. Even after they were reminded of the definition repeatedly by their classmate and by the instructors, they had difficulties in translating the ideas in the definition to an algebraic representation.

This study also affirms that the Cartesian connection, the idea that connects algebraic and graphical representations of a line (Moschkovich et al., 1993; Knuth, 2000), is a critical idea to connect representations in the concept of conic curves. The concept of variable and a graph as a locus of points—an extended idea of the Cartesian connection with the concept of variable intertwined—were also identified as critical ideas in making connections of representations.

Through the intervention, prospective teachers made some progress. During the three-weeks of instruction, we saw some positive changes in prospective teachers' mathematical thinking and behaviors and progress in understanding Omar Khayyam's solution to cubic equations. Many prospective teachers who initially were dependent on memorized forms of algebraic formulas made efforts to consider aspects of the Cartesian connection to make sense of their own work. Their exposure to these critical ideas and concepts also helped them analyze students' thinking. In the subsequent activities that dealt with students' understanding of algebra (which will be documented in a later article) they tried to relate many of these same issues that they experienced to students' cognitive difficulties and understandings in algebra.

An implication of this study is that teacher education programs might have to pay special attention to these critical ideas so that their graduates can better help their future students understand connections among algebraic equations and their Cartesian representations. This study shows one of those examples. Using a historical task of Omar Khayyam in accordance with the principles of realistic mathematics education (Freudenthal, 1991; Gravemeijer, 1999), we provided prospective teachers opportunities to reconstruct these critical ideas that are essential for their own understanding of algebra and for their future instruction. Further, by providing subsequent tasks with which prospective teachers could discuss student mathematical thinking around the same issues that they had experienced, we tried to provide them opportunities to develop pedagogical content knowledge along with subject matter knowledge.

Connection among representations is identified as one of the "big ideas" (Lacampagne, Blair, & Kaput, 1995; Schifter & Fosnot, 1993) in algebra by many researchers (Knuth, 2000). Understanding the role of the Cartesian connection and the idea that a graph is a locus of points that satisfy a relation in sense making about the connections between symbolic and graphical forms is a crucial piece of pedagogical content knowledge that secondary teachers need to be familiar with. Only when a teacher is aware of the importance of this big idea and hold the understanding of the big idea, can she teach for the big idea (Schifter & Fosnot, 1993; Schifter, Russel, & Bastable, 1999).

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TEACHER IDENTITY AND TENSIONS OF LEARNING TO LEVERAGE STUDENT THINKING IN MATH TEACHING

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In mathematics teaching, high-leverage practices include eliciting, analyzing, and responding to students' ideas and reasoning (Kazemi, Franke, & Lampert, 2009). While a focus on student thinking in professional development can be a powerful mechanism for linking pedagogy, mathematics, and student thinking, it is less clear how to support teacher candidates (TCs) to leverage student thinking in their mathematics teaching. Findings suggest developing an understanding of self as mathematics teacher relates to TCs' capacity for leveraging student thinking as a pedagogical practice and for making sense of their relations to the complex dynamics in schools. The tensions TCs experienced when leveraging student thinking in the actual socio-political contexts and constraints of schooling and the influence of their identity on their representations of practice lead to implications for mathematics teacher education.

Keywords: Teacher Identity; Teacher Education–Preservice, Instructional Activities and Practices

There is emerging consensus in education research of importance of teachers' attention to "high-leverage practices" (e.g., Ball & Forzani, 2009). In mathematics teaching, high-leverage practices include eliciting, analyzing, and responding to students' ideas and reasoning (Kazemi, Franke, & Lampert, 2009). By focusing teachers' attention on students' reasoning and then leveraging their thinking in instruction, teachers validate students' mathematical thinking and support all students in making connections, solving authentic problems, and participating in the discipline (Windschitl, Thompson, & Braaten, 2011).

A focus on student thinking in teacher professional development has also been shown to be a powerful mechanism for linking pedagogy, mathematics, and student thinking (e.g., Franke & Kazemi, 2001). As teachers struggled to make sense of students' thinking, they elaborated on how problems were posed, asked each other questions, and learned about their students and their practice. Student work thus promoted collective inquiry into relations between teaching and learning and supported teachers' experimentation in their classrooms (Franke & Kazemi, 2001).

Objectives or Purpose of Study

While research suggests that elementary teacher preparation should support teacher candidates (TCs) in development of practices such as leveraging student thinking, research with TCs is less widely documented. Attention to student thinking in teacher education and inquiry into instructional practices has been hypothesized as a way to support TCs' understanding of teaching and student learning as well as their enactment of practices of leveraging student thinking (Windschitl et al., 2011). I contend that to support TCs in leveraging student thinking, math teacher education must prepare TCs to enact these practices *while* navigating the many social, political, and institutional dynamics in math classrooms and schools that complicate and regulate teacher practice and construct images of being a math teacher.

The following response from Laura, a teacher candidate in an intensive elementary masters' certification program, underlines this point. Laura is a White female in her late-twenties from a middle-income background. While her age, ethnicity, and socioeconomic class are consistent with what educational research labels as the "typical preservice teacher" (Lowenstein, 2009, p. 166), the following discussion is not meant to present TCs as learners who have deficient resources or experiences from which to build on when it comes to learning about issues of mathematics learning or teaching for equity. The goal, instead, is to illustrate the complexity of Laura's positioning as an elementary mathematics teacher and how she understands it.

Laura participated in a seminar on critical self-examination and mathematics teaching in context. In a written response, she described how her self-understanding as a mathematics teacher and her understandings of her students are influenced by the way her institution frames student learning, students' abilities, and mathematics:

> All the time, we refer to students as "above," "on," or "below." In many ways, attaching these labels makes life easier, but it also attaches stigmas about the student's ability and desire to learn…. [In my math class] the students work in separate classrooms and with different teachers to learn the 3rd grade objectives and to practice problem solving…. My impression is that the expectations of their [problem-solving] abilities are low, and as a result, they are given rather simple problems. (May 3, 2010, Positioning statement)

She identified how labels, such as "above-grade-level" and "below-grade-level," which are determined by students' test scores, come to characterize the students' capacity to learn and motivations. The institutional policies of grouping shape teachers' expectations, define mathematics as skills or problem solving, limit students' opportunities to learn, as well as restricted her own teaching practices:

> When I suggested some word problems from the 3rd grade math textbook, I was told the numbers in the problems were too difficult and had to be changed. I recognize that for some of my students, they would really struggle using fractions with different denominators; however, I also know that many of my students could handle the challenge. Since they are not challenged to solve more difficult problems (that are still considered 3rd grade level), they do not have the opportunity to make as much progress as students in other classes who are challenged. (May 3, 2010, Positioning statement)

Teaching mathematics is fundamentally about teaching within this complexity. The consequences of leaving teachers unprepared are damaging, not only for those teachers, but also for students, as teachers who are unprepared for the realities of schooling may be unable to enact ambitious mathematics teaching practices. New teachers need to understand the dynamics mathematics teaching in their school contexts that Laura mentions, and their own relationships to these dynamics. Understandings of self and relations to these dynamics structure teacher identity.

Theoretical Framework

Teacher identity is a complex construct that is conceptualized in a multitude of ways in educational research. To explore teacher identity and teacher practice, I extended Judith Butler's (1999) premise of the performativity of gender and gender identity and sought to conceptualize identity as an agentic act as much as possible within the discourses that concurrently frame identity as a process. TCs' mathematics teacher identities are shaped by the political, social, and institutional *discourses* that provide systems of categories, terms, and beliefs that organize and structure ways of thinking and acting in relation to mathematics, teaching, and learning (St. Pierre, 2000). Prevailing discourses about mathematics teaching and learning that influence TCs' understandings of being a mathematics teacher may include: institutional discourses around curriculum and testing (Brown & McNamara, 2005); social discourses around race, class, and abilities (de Freitas & Zolkower, 2009); discourses about mathematics as skills or as practices of "making sense" (Fuson, Kalchman, & Bransford, 2005); or, discourses of teacher as "savior."

Discourses, as theorized in poststructural feminist discussions of education, constrain and enable what teachers do, say, and even conceive of as appropriate (Britzman, 1993; Walshaw, 1999). For example, test-driven school cultures in the U.S. create institutional discourses about teaching and students, including notions of fixed student abilities and one-sided positive notions about standardization (Apple, 2004). These discourses, norms, and the way they position individuals (e.g., Davies & Harre, 1990) are understood to structure teacher identities, influence how teachers position students, and *interfere* with teachers' authentic teaching and learning relationships with children in their schools (Olson & Craig, 2009). In becoming a math teacher, Laura, for example, wondered how to navigate the ability grouping in her classroom, contest teachers' low expectations, and work within competing discourses of mathematics.

Teacher identities have been shown to shape why people teach (Britzman, 1993), how they understand the mathematics they teach (Gellert, 2000), and how they learn to teach (Horn, Nolen, Ward, & Campbell, 2008). TCs' experiences as students and with mathematics also manifest in practices of attending to student thinking in math teaching as obstacles or resources (Neumayer DePiper & Edwards, 2009). While there are a variety of influences on new teacher practice and how TCs attend to student thinking, the relations between teacher identity and teacher practice need to be better understood and identity work needs to be fostered in teacher education (Ponte & Chapman, 2008). Building from and responding to research on teacher identity, my theoretical stance is that TCs need to understand, on one hand, that how they position themselves as mathematics teachers and are positioned by others (e.g., instructors, mentor teachers, school administration, students, etc.) is shaped by social and political discourses about mathematics and teaching, and, on the other hand, that they can use this understanding to (re)author their positions toward mathematics teaching and learning. Developing TCs' understandings of classroom dynamics may support them in negotiating their multiple positions as elementary mathematics teachers and enacting teaching practices that best support all students in learning with understanding.

Modes of Inquiry

I used design-based research, situating my stance on identity and identity work in action both to support these TCs and to theorize about identity work in mathematics teacher education. The following questions guided this study: For TCs participating in a seminar on critical self-examination and mathematics teaching in context, how are they understanding themselves as mathematics teachers and teaching in context? How are those understandings shifting?

Study Context

The study took place during the last semester of a 15-month master's certification program. Ten female TCs, between the ages of 25 and 35, volunteered to participate in the eight seminar sessions and the study. Eight TCs self-identified as White, one as an immigrant from Argentina, and one as African-American.

Seminar Design

In the seminar design, I operationalized critical pedagogy (e.g., Kumashiro, 2000) and feminist poststructuralist notions of identity in mathematics teacher education. I sought to engage TCs in determining what discourses were present and in understanding how they operate in order to think differently about *themselves* as mathematics teachers and their relations to these dynamics. The objectives of the seminar focused on: identifying and examining the many prevailing social and political discourses that shape mathematics, mathematics teaching, and their positioning; analyzing the implications; rethinking these in relation to self; and problematizing teaching in relation to them. Seminar activities included case analysis, group discussions, and reflective writing prompts about their positioning and teaching experiences.

Data and Analysis

The data included seminar video and transcripts, interviews, and TCs' written work. I used discourse analysis to analyze TCs' identity and understandings of teaching. I analyzed identity as positioning, repositioning (Davies & Harre, 1990), and being reflexive about positioning (Mauthner & Doucet, 2003). Understanding of mathematics teaching was conceptualized as problematizing, that is how TCs took up particular conversational routines that open opportunities for learning about practice (Horn & Little, 2010) and how they negotiated attention to principles of teaching, instructional strategies, and specific practices (Pollock, 2008). In each case study analysis (Yin, 2003), I followed each TCs' discursive participation across seminar sessions. In line-by-line analysis of seminar transcripts, I identified an emergent theme in relation to each TC's self-understanding and followed each TC's discursive participation about herself and her teaching in relation to this theme and TCs' shifts in participation.

Analysis and Results

The following analysis presents TCs' participation in one analytic episode and serves to illustrate TCs' representations of their practice and one of the main findings of this study.

Examining Practices of Grading and Evaluating Student Understanding

In preparation for Session 4 of the seminar, TCs analyzed an artifact of their teaching as situated within the discourses and messages that we had discussed. They submitted this analysis in writing, including their intentions and specific interactions with students, and shared their artifacts and analysis with the group during Session 4. Candice began by describing challenges with judging students' work and tensions between being supportive of "where students are" and also wanting "to get [her students] where [they] need to go" (group discussion, May 18, 2010). Analysis suggests that she felt that the pressure to maintain pace with the curriculum and the standardization of grading (on a scale of 1 to 10) were institutional constraints on her teaching and in tension with an emphasis on attending to students' individual understandings and abilities:

> I don't want a kid to always get a [score of] 1. Like, you know, it's this big thing if you're not a "1 kid." You want to be a "10 boy" or a "10 girl." But if they're not doing 10 work, but they're trying their—like, it's a 10 for them, but it's not a 10 according to our rubric. And I actually had a conference about that because I was giving out too many 10's. And like, "Do you really think this is 10 work? Like on a BCR, you have to do this, this, and this." And I was like, "Okay, that was his 10 and I, he put forth effort, he sat next to me and he tried, so I don't give him a 10 on his best try? So, why would he try again?" (discussion, May 18, 2010)

Candice's comment serves as evidence that she recognized how she was positioned by both the pressures of standardized grading systems and by her desire to attend to and leverage students' thinking and self-confidence. She evidenced problematizing of grading students in her context and repositioning herself by contesting standardized grading and supporting individual learning.

In response, Brooke did not take up Candice's concerns. Rather, she suggested Candice use effort grades, specifically a "star, smiley face, and check" system to "affirm [students'] effort." Brooke stated that this communicated her goals to students: "I want [students] to know that they need to keep going and what they're doing isn't perfect yet. They should just keep working as hard as they can, just like an on-grade or above-grade level student should be pushed the exact same way" (group discussion, May 18, 2010). Analysis suggests that Brooke did not problematize how grades, either for achievement or academic behaviors such as effort, position students, but emphasized individual student responsibility.

Sarah then described how she communicated with students about their writing in language arts by creating facilitated opportunities for her students to read her comments and ask her questions. Sarah articulated a guiding principle, "You have to give them opportunity," emphasizing students' and teachers' responsibilities in grading and assessment. Although her mentor did not approve of her practice of taking class time for individual writing conferences, Sarah stated that she felt that it supported her students' understandings of their grades and their writing and repositioned students in relation to the pressures of grading. In this manner, analysis suggests that she took up some of tensions Candice felt about grading and supporting students' progress while also contesting the pressures of test-based accountability.

When Sarah described how all students "scored an A or a B on these assignments" because there were "multiple points to catch them," Laura questioned what those students' grades meant. Laura felt that there were differences in the understandings evidenced between the students who had received support and those who had completed the assignment independently but that these differences, both in her classroom and Sarah's, were glossed over in the grading system and when supporting students' learning. Laura described an incident with a particular student who had "great math assignments" after working with the assistant teacher and the tensions she felt:

> But it's so hard because then, like, there were other kids that were sort of normally on like his level, that were still like maybe getting low grades, and his grade was like up here, but you knew that she

really like pulled that from him. And, I don't know. Grades are weird, right, but I always just wonder. (group discussion, May 18, 2010)

Laura felt a conflict between supporting her students, standardized grading, and seeking to understand students' independent thinking and skills. In discussion, she identified how grades influenced multiple stakeholders, including parents, next-years' teachers, and administrators, and understandings of students' abilities and progress. Thus, analysis suggests that Laura struggled to make sense of how to attend to students' thinking and differences in understanding in the actual socio-political contexts and institutional constraints of schooling.

How Personal Theories of Teaching and Learning and TC Identity Related to TCs' Analysis of Student Work and Representations of Practice

Across seminar sessions, TCs experienced tensions when seeking to leverage student thinking as a pedagogical practice in their teaching contexts, particularly TCs who demonstrated a belief in and capacity for these practice in methods courses. While the focus and framing of these tensions were different, TCs' personal theories of teaching and learning *and* their identity influenced the ways in which these tensions emerged in their analysis of student work and representations of practice.

Figure 1 presents a framework outlining three stances of how TCs' individual theories of teaching and learning and TC identity relate to analysis of student work and representations of practice. Windschitl et al. (2011) found that in guided discussion about student work, some TCs took an intellectual stance of teaching and learning as problematic and others took a stance of teaching and learning as unproblematic. They suggest that TCs' theories of teaching and learning were underlying their analysis of student work, representations of practice, and the dilemmas they felt in their classroom. Building on the analysis and frameworks of Windschitl et al. (2011), my analysis highlights another dimension, specifically that TCs' understanding of their positioning also influenced framing of practice and the questions they brought to peers.

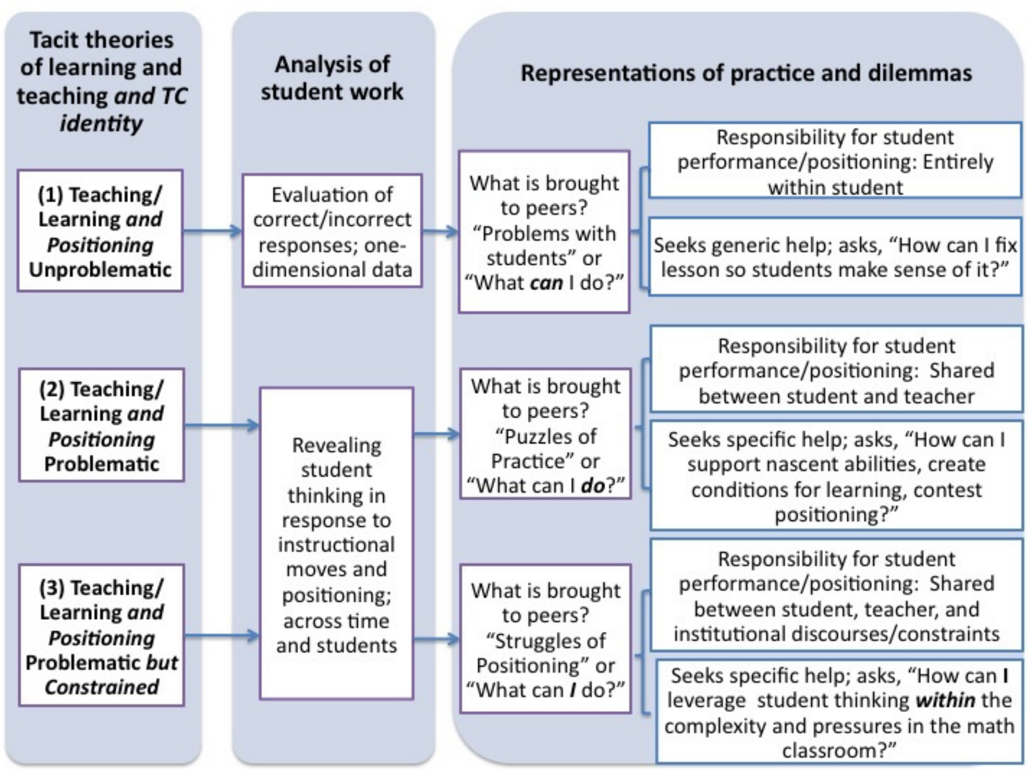

Figure 1: How personal theories of teaching and learning and TC identity related to TCs' analysis of student work and representations of practice

Brooke's participation across sessions is representative of the first intellectual stance emergent in TCs' engagement: Teaching/Learning and Positioning as Unproblematic (Figure 1). In analysis of student work, she focused on correct/incorrect responses and how there were *students* who did and did not get it, would and would not get it, and did not understand their placement in what she called the "math class hierarchy." Her comments suggest beliefs about learning mathematics as a process of skill acquisition as well as the ways in which she did not see students' positioning, her influence on students' positioning, or their positioning by grading practices as problematic.

Analysis of TCs' tensions with mathematics teaching and their representations of practice identified three iterations of the question, "What can I do?" which is also referenced by Pollock et al. (2010) in discussions of teaching teachers about race. Analysis suggests Brooke brought the question, "What *can* I do?" to her peers, questioning what she *could* do, as student performance and positioning were problems of *students* needing to understand the mathematics and take responsible for their performance and positioning. She sought generic help, not specific to students' understanding or thinking. Her positioning, in particular how she understood math abilities and students' responsibilities, framed her analysis of student work, and she struggled to interrogate her positioning towards mathematics, the framing of abilities in mathematics, and how her school framed learning and progress. This case highlights the challenges TCs have with working at the intersections of these issues and their positioning. By not seeing her positioning or her students' positioning as problematic or examining herself in relation to these issues, she struggled to problematize practice and remained focused on fixing students and lessons.

Sarah and Candice's participation is representative of an intellectual stance where both teaching and learning and their positioning were problematic, the second stance articulated in Figure 1. Sarah and Candice both analyzed student work to reveal student thinking and emphasized sense-making in her interactions with students. In analysis of student work and when discussing instruction, they identified their positioning as an intern, how students were positioned, and the discourses of test-based accountability and tensions, where standardized assessments mattered more than what was learned from students' work. In representations of practice and dilemmas, Candice and Sarah asked specific questions of practice; that is, "What can I *do*?" They engaged in puzzles of practice and demonstrated a shared responsibility for working with students to address their positioning and context. Sarah, for example, saw her positioning as problematic and asked questions about supporting students' abilities, creating opportunities to learn, and contesting discourses of test-based accountability, her positioning and her students' positioning. Understanding positioning as problematic and seeking to contest it coincided with bringing puzzles of practice and specific questions about practice to peers.

Laura also represented teaching and learning as problematic, but analysis suggests that she sought to work within how she was positioned and the complexity of institutional pressures in her mathematics classroom. As evidenced in her analysis of student work and representations of dilemmas of practice, her participation is representative of a third stance, Teaching/Learning and Positioning as Problematic but Constrained (Figure 1). During her mathematics methods course and across the seminar, Laura demonstrated a strong belief in and the capacity for attending to student thinking as a critical instructional practice and also that responsibility for student performance was shared between student, teacher, and institution, but analysis suggests that her understandings of herself left her feeling as if her options for her practice in her school context were constrained. Thus, while she problematized practice and brought these puzzles of practice to the group, analysis suggests that she felt limited and struggled with her positioning in context.

As this study and seminar were framed in relation to sociopolitical discourses of teaching, Laura's discursive participation suggests that through this attention to what is framing her positioning, more tensions about enacting this work in context emerged for her. Analysis suggests that she was asking, "What can *I* do?" The self-understanding and awareness that she evidenced may have helped her better understand how teaching context was a critical element of her practice and her positioning and how she could assert agency in choosing a teaching context, but she remained uncertain about how to engage in the instructional practices that were important to her and encouraged by her methods courses. While Windschitl et al. (2011) suggests that TCs' enactments or lack of enactment of ambitious practices related to TCs' beliefs or knowledge, it was Laura's concern about who she was and what she could do, specifically, her understandings of herself and her identity that influenced her representations of practice.

Discussion and Conclusions

This study emphasizes the importance of attending to the many tensions TCs feel in enacting practices of leveraging student thinking in context. As Windschitl and colleagues (2011) suggest that unsophisticated views of teaching and learning underlie analysis of student work, this analysis finds that through attention to the sociopolitical dynamics of teaching, TCs articulate other tensions in their representations of practice and the ways in which they understood enacting practices such as leveraging student thinking. The framework outlining the different stances and TC representations of practice and dilemmas suggests that TCs were asking different questions about themselves, their agency, and instruction. In response, math teacher education needs to respond to the questions and tensions TCs feel about their positioning and enacting specific practices in classrooms. Specific implications are for more activities in math teacher education that help TCs contests one-sided positive notions of standardization and the category systems and grading systems prevalent in math classrooms and for continued discussions about TCs' complex positioning. In seeking to support TCs in their enactment of high-leverage pedagogical practices in context, mathematics teacher education needs to focus on the complexity of math teaching contexts and TCs' understandings of these dynamics as related to practice.

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A TENSION IN TEACHING "MATH FOR TEACHERS": MANAGING COGNITIVE AND AFFECTIVE GOALS

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This paper presents partial results of a completed study which investigated the experience of teaching mathematics content courses to preservice elementary teachers. Interviews with ten mathematics instructors who teach these courses revealed several major tensions, including one that arises as instructors strive to set priorities and balance their affective and cognitive goals for their students. An analysis of three of the instructors' experiences of this particular tension will provide insight into the factors that contribute to it and how it is managed.

Keywords: Teacher Education–Preservice; Affect, Emotion, Beliefs, and Attitudes; Mathematical Knowledge for Teaching

Background

Concern over the mathematics preparation of elementary school teachers has led to increasing calls for prospective teachers to take specialized mathematics content courses, i.e. Math for Teachers (MFT) courses, during their undergraduate programs (Greenburg & Walsh, 2008; Conference Board, 2010). These courses are usually taught by instructors in mathematics departments, and some recent studies have begun to call into question whether these instructors are equipped to meet the needs of MFT students, particularly with respect to affect (Hart & Swars, 2009).

Although there has been some research done into teaching styles of post-secondary mathematics instructors generally (Strickland, 2008), and into the difficulties they face in implementing reform approaches (Wagner, Speer, & Rossa, 2007), there seems to be little information about mathematics instructors in the context of teaching MFT courses. The original study upon which this paper is based sought to address this gap in the literature. Interviews with ten mathematics instructors who teach MFT courses at various post-secondary institutions in British Columbia were analyzed in order to answer questions, including: What are the major tensions they experience? What factors contribute to these tensions and how are they managed?

Given space limitations, this paper will discuss only one of six major tensions revealed in the full study, specifically, the tension related to instructors' efforts to balance their affective and cognitive goals for their students.

Supporting Literature

The research reported in this paper is informed by prior research into the cognitive and affective needs of prospective elementary school teachers (with respect to mathematics), as well as literature on tensions.

Cognitive and Affective Needs

With respect to the students in MFT courses, there is evidence to support concerns that they have poor understanding of the elementary school mathematics topics; Ball's (1990) study of 252 preservice teachers revealed "understandings that tended to be rule-bound and thin" (p. 449). Regarding their beliefs, the elementary preservice teachers in the group "tended to see mathematics as a body of rules and facts, a set of procedures to be followed step by step, and they considered rules as explanations" (p. 464). Preservice elementary teachers often suffer from mathematics anxiety (Hembree, 1990), and some are only enrolled to "fulfill a requirement rather [*sic*] to learn more mathematics" (Kessel & Ma, 2001, p. 477).

Although it is clear that MFT students have much mathematics to learn, and often come in with negative attitudes and beliefs, the literature does not provide specific advice on whether cognitive skills or affect should take precedence in teacher preparation. In fact, there is considerable literature engaged in debate over this issue. While some researchers make a case for the priority of strong mathematics knowledge, pointing out that such knowledge can both boost confidence and make teacher practice (i.e., the implementation of teachers' pedagogical beliefs) more effective (Schwartz & Riedesel, 1994; Goulding, Rowland, & Barber, 2002), a large number advocate for an emphasis on teachers' beliefs in specialized mathematics content (and methods) courses (Kessel & Ma, 2001; Liljedahl, Rolka, & Roesken, 2007), observing that beliefs will affect both students' learning in preservice mathematics courses and their later teaching. Still others promote the view that students' knowledge *and* beliefs need to be challenged in teacher education programs (Borko et al., 1992). This debate in the literature is reflected in the tension experienced by the MFT instructors described in this study.

Tensions

Tensions, often expressed as "dilemmas," have been recognized as an integral part of teaching practice, dating back at least to the early 1980s. In their seminal work, Berlak and Berlak (1981) examined the complex and sometimes contradictory behaviors of teachers in responding to the curriculum within socio-cultural contexts. Their use of the language of dilemmas was taken further by Lampert (1985), who emphasized the personal and practical aspects of dilemmas.

Lampert (1985) observes that tensions in teaching are often "managed" rather than resolved. She characterizes teachers as "dilemma managers" who find ways to cope with conflict between equally undesirable (or desirable but incompatible) options without necessarily coming to a resolution. Faced with a teaching dilemma, the teacher must take action, finding a way to respond to the particular situation, even while the "argument with oneself" (p. 182) that characterizes the dilemma remains. For Lampert, the ongoing internal struggles presented by the tensions arise from and contribute to the developing identity of the teacher, and as such have value in themselves. Furthermore, she comments: "Our understanding of the work of teaching might be enhanced if we explored what teachers do when they choose to endure and make use of conflict" (p. 194).

More recently, Berry (2007) focused on "tensions" in a self-study that examined her own efforts to improve her practice as she made the transition from *teacher* to *teacher educator*, finding that the notion "captured well the feelings of internal turmoil experienced by teacher educators as they found themselves pulled in different directions by competing pedagogical demands in their work and the difficulties they experienced as they learnt to recognize and manage these demands" (p. 119). In the study presented here, the instructors are similarly experiencing a transition between teaching future *users* of mathematics and teaching future *teachers* of mathematics, a situation which influences the tensions they experience.

Methodology

Data for this study were gathered through interviews with ten participants, five male and five female, all instructors in mathematics departments at post-secondary institutions who teach MFT courses. Theoretical sampling (Creswell, 2008) was used to achieve variety in type of post-secondary institution represented, as well as varying degrees of experience in teaching MFT. The ten instructors represented nine different institutions, and their experience teaching the MFT course ranged from novice to 20 years.

The one-hour long interviews were semi-structured, beginning with a set of core questions but allowing for variations and additional questions to be asked as needed. Such an open-ended ("clinical") approach is advocated by Ginsburg (1981) in situations where discovery or identification/description of a phenomenon is the objective. The questions sought to elicit the instructors' conceptions of the MFT course by asking them to examine their goals, describe the approaches they take, compare the teaching of MFT with teaching of other mathematics courses, and reflect on the challenges and the successes they experience.

The interviews were audio-recorded, transcribed, and analyzed using constant comparative analysis (Creswell, 2008). An iterative coding process (Charmaz, 2006) was employed in order to allow concept codes and themes to be identified. Very few new codes emerged after the tenth interview, suggesting saturation of the data. Specific concept codes, including "priorities," "wishes," "doubts," "barriers," and "resistance," helped to locate instances of instructor tensions in the transcripts.

Analysis of the tensions was further facilitated by techniques of discourse analysis (Rowland, 1995) and considerations of positioning (Harré & van Langenhove, 1999). The former helped to locate hesitation, uncertainties and inconsistencies, while the latter supported understanding of contexts and factors contributing to the tensions identified.

Results

One of the major tensions that emerged through the coding and thematic analysis involved instructors' struggles with managing their cognitive and affective goals for their students in the MFT course. Each of the instructors experienced this tension differently, however, due to space limitations, only the cases of Bob, Maria and Alice (all pseudonyms) will be described here. These three cases will suffice to exemplify the diversity and scope of the views expressed.

Bob

Bob's experience of this tension emerged in the contrast between his comments at the beginning of his interview with respect to his primary goals in his MFT course and his later reflections on the ultimate outcomes for his students.

Bob reported that his main emphasis is on building deep conceptual understanding in his students, although affective considerations are also very important to him. His course "focuses on a very sound fundamental ability to appreciate [mathematics], in a theoretical way, why things work," along with having "a secondary by-product of what you do in the classroom is to get the students to enjoy it." The cognitive and affective are closely related for Bob. From his responses to prompts about dealing with students' anxiety, he expresses the view that his students' anxieties are at least in part caused by, and at the same time the cause of, their lack of arithmetic skills. It is his hope that helping his students learn about the structure of mathematics will solidify their understanding, giving them confidence, competence, and enjoyment.

Later in the interview, commenting on what his students leave the course with, Bob observed that his students "have improved most in their technical abilities," along with having gained some problem-solving skills, although these need to continue to be developed. But he is ultimately disappointed, both in his hopes to build deep theoretical understanding, and in his hopes to increase his students' appreciation for, and love of, mathematics.

> *Bob*: In terms of appreciating some of the more subtle aspects of the theory, I think that's another thing that they could do better, if they had better basic arithmetic skills, coming in. So ... yeah, in terms of what I produce, I guess, in terms of the other goal, for love of math? Unfortunately, the course is so packed, that in some ways, I think they do get a little bit beaten by the end, and they're just tired.

He does see some success in improving their technical skills, although their depth of understanding still falls short, but admits with regret that he is less than successful (by his own standards) in terms of affective aims. He is trying to cover too much, to the extent that his students are overwhelmed.

A closer look at this passage, with particular attention to pronoun use, offers some further insights. In the first sentence, he ostensibly places the responsibility on the students, "*they* could do better, if *they* had better basic arithmetic skills." However, as Bob is aware, the prerequisites for the course are not controlled by the students, or by him, but are set by the larger community. Whether it is the fault of this community or the students themselves, he sees the lack of student skills coming into the course as an impediment to his ability to realize his goals for his students.

He then switches to consider what he ("I") produces. Having already mentioned that students have increased their technical abilities, he moves to his "secondary" goal, improving affect. The results here are "unfortunate"; he describes his students as "beaten" and "tired"—not at all what he desired. The phrase, "the course is so packed," is telling. It is offered as an explanation for the students' states of exhaustion;

there is too much material in too little time. Bob's use of the passive voice here suggests that he is not in control of the course content; with it he positions himself as unable to remedy this "unfortunate" situation. The course, as he believes he is expected to deliver it by his institution, demands too much of the students.

Bob is not a new instructor of the MFT course, and so has likely lived with this problem for some time. He is stuck in this dilemma. On one side he has students who are unprepared for the level of mathematics he believes they need in order to "appreciate" the mathematics (both in a cognitive and in an affective sense). On the other side, he has a prescribed curriculum he is expected to "cover." He feels a strong responsibility as a mathematics instructor, seeing himself as being charged with "delivering the content" (Bob's words). From Bob's perspective, the situation could be improved if the students were stronger coming in, but this is not within his immediate power to change. So he manages the tension by adhering to his practice in his other mathematics classes—he focuses on the content, despite his dissatisfaction with the outcomes.

Maria

Similar to Bob, Maria expressed a strong intention to improve students' mathematical understanding, emphasizing cognitive goals. However, Maria's use of the past tense in describing these goals in her interview, even though she was teaching the course at the time, suggested she was having second thoughts about her priorities.

Maria was a first-time instructor of the course at the time of the interview, and was surprised by the needs of her students, not only their weak mathematics skills, but their mathematics anxiety and the barrier to learning it presented.

> *Maria*: So my goal was, primarily, sort of more content, and I [...] knew that there would be some issues of, let's describe it as "math phobia" or anxiety, with math. I just [was] still surprised to see it so strong at this level, that it overrides their learning, that it blocks their learning! That's what I discovered, and it surprised me that it would be this strong.

She went into the course expecting that she would be teaching mathematics and would need to deal with math anxiety, but at a certain point she realized that, at least for some of her students, the affective issues would need to be addressed before they could learn the mathematics. Maria commented that she believed she had lost about a third of her students, and was not sure how to get them back on track.

> *Maria*: For this group of students at this point, content? Forget it. I need an attitude change. I need [their] perception of math to change. And I can't reach it anymore. It was very high, you know, it was a good high in the beginning of the course, because of what I did, free, sort of, problem-solving, open discussion, everybody let's just ... [there was a] fuzzy, cozy atmosphere. But the topic does get difficult, yeah?

Maria seemed to feel that she had missed an opportunity. For this particular group of students, she did not believe it would be possible for them to progress without an attitude change, and this change was not possible to attain "anymore." She spoke nostalgically about a time at the beginning of her course when her approach was different: there was "open discussion," "free" problem solving, and a friendly atmosphere. She seemed to take responsibility for these initial positive feelings; it was good because of "what [she] did," but something changed; her approach changed, and in this excerpt the reason offered for the change was the "topic," i.e., the mathematics, which gets more difficult as the course progresses.

Like Bob, during her interview Maria expressed a sense of obligation to complete the prescribed mathematics content for the course, a disposition that appeared to be in tension with her goals to both promote deep conceptual understanding and address her students' affective needs. Her desire to "cover the content" influenced her choice of teaching methods, leading her to reduce in-class activities, such as open discussions of readings and problem-solving sessions, methods that she described as effective, but not time-efficient. At the same time, she reported that it troubled her that she was leaving students behind, students who would continue to suffer from negative attitudes to math and continue to have weak skills.

An additional consideration for Maria that contributes to this tension is a perception that the MFT course has the potential, if not the responsibility, to act as a filter. Early in her interview, comments with respect to the importance of deep content knowledge for mathematics teachers (not cited here) revealed a strong commitment to ensuring that she does her part in the preparation of future elementary teachers; i.e., if the mathematics skills of the prospective teachers are too weak, they should not be permitted to go on to become teachers.

Maria was far from resigned to living with this tension. At the time of the interview she was still seeking to understand her students better and find methods that would be more effective for them, to find a way to change their attitudes so that the mathematics could be learned.

Alice

In contrast to Bob and Maria, Alice was less concerned about building mathematics knowledge and much more concerned about affect. This also created tension for her, although this was not evident early in her interview. Alice's emphasis was on helping her students see the "fun" of mathematics. She described the many ways she tries to address her students' anxieties and to build their confidence, including striving for a very relaxed classroom atmosphere where questions are encouraged and student interaction and exploration of concepts is the norm.

The tension between her affective and cognitive goals for her students did not emerge until she considered whether her students will be prepared to go on to be teachers of mathematics.

> *Alice*: That's a very good question. That, that's a very deep question. Because we don't teach so much math in that class, you know. We don't drill them on whether they can do those fractions. We kind of believe they have the elementary math, that's how we let them in [...]. But how much above it should they be? You see they always say that you should be significantly above what you want to teach, because then you have the big picture, you see the troubles and all that. I don't know that much about that. [...] Many at least will not be afraid to go for it. But I still think there are people who will be afraid. I still think I let people go in there being afraid.

In this passage, she begins with the admission that improving students' mathematics proficiency is not a major objective in her course. This is followed by a justification that students are presumed to come into the course with sufficient mathematics skills, however the hedge "we kind of believe" and other comments in her interview suggest that she realizes those skills are often lacking. She then considers that even if they could do the arithmetic, perhaps that would not be enough, that teachers of mathematics should have a *deeper* understanding of the subject. She even provides reasons for why this deeper understanding might be helpful, but then quickly dismisses this as education theory, something she is not an expert in. She looks to her goal of improving attitudes next, to see if "at least" her students will no longer be afraid of mathematics, but sadly admits that even in this respect, some of her students are not ready.

A careful parsing of this passage reveals some of the different forces contributing to the tensions that Alice operates under. As she thinks aloud, her pronoun use changes from "we" to "they" to "I." "We" likely represents her institution as she describes what does not happen in the course: there is not much math and no skill drill. Even if she disagrees, the objectives for the course are set by her institution. In the phrase "they always say...," the "they" seems to point to education experts, or at least to those who have an informed opinion, but she disassociates herself from this group, switching to the pronoun "I," and denying any expertise in deciding what students need. Ultimately, responsibility for the content and objectives of the course is deferred to others, her institution and/or the community.

Alice seems to believe that the goals for improving students' attitudes and diminishing their anxiety are important, and this is consistent with the reported aims of her institution. As a result of this local orientation, she does not express the same concern as Bob and Maria with respect to "covering" the course content. But there is still a tension here as she contemplates what she achieves with the course, and what future teachers might need both in terms of mathematics proficiency, which she does not address to a great extent, and attitudes towards mathematics, which she tries to address, but feels she does not entirely succeed in. She deals with this tension by deferring authority for deciding these priorities to others at her institution and within the teacher education system, but remains concerned with the implications for future teachers.

Discussion

Bob, Maria and Alice experience the tensions differently, but all struggle with finding the balance between building students' mathematics proficiency/understanding and fostering positive attitudes, within the parameters set for the course. Bob hopes to improve affect through building cognitive skills (one of the views reflected in the literature), but his affective aims are sabotaged by an emphasis on content that he sees as too much for his students to absorb given their skills coming into the course. He opts in favor of covering the course content, fulfilling his perceived obligation as a post-secondary mathematics instructor, even though this means the students leave the course far less excited about mathematics than he would like. Maria's comments revealed a growing awareness that her cognitive aims cannot be attained, at least for some of her students, until affective barriers have been removed (reflecting the other side of the affective/cognitive debate). She, too, sticks to the course curriculum, even though students are left behind, largely to try to ensure that students who do not have a certain level of understanding will not become teachers before they are ready. For Alice, whose emphasis is already primarily on the affective, there is an uneasiness that what her course provides may not be enough to meet either one of her students' affective or cognitive needs, at least for some of her students.

Both Bob and Maria seem to manage this tension between cognitive and affective aims by staying true to the course syllabus and "covering the material," even though they are unhappy with the consequences. There are indications within the broader study that this commitment to the prescribed course content is a prevalent norm amongst post-secondary mathematics instructors. This may not be surprising given that the traditional calculus-stream mathematics courses that they generally teach tend to be sequential in nature with topics in one course building on knowledge of earlier course content. It is unclear whether instructors are consciously aware of this norm or have considered its appropriateness in the context of MFT courses.

Exceptionally, Alice does not appear to adopt this norm in her MFT course, at least in part because the mandate from her institution is a focus on affective goals. Yet this does not free her from the experience of tension. She manages her situation by deferring to the authority of others at her institution, but she is left uneasy with the mathematics proficiency of some of her students in the context of their role as future teachers, while not being wholly satisfied that her affective goals are being reached. Her comments point to additional factors that contribute to this tension, including the often weak mathematics skills of students coming into the MFT course, also observed by Bob, and the sense that one of the roles of the course is to act as a filter to prevent those with poor mathematics skills from becoming elementary school teachers, also expressed by Maria.

Both of these concerns point to larger problems within the system of teacher preparation, problems with defining the level of mathematics proficiency elementary teachers need, and with clearly defining the role of MFT courses in their preparation.

Conclusion

This tension is not easily resolved. As illustrated by the case of Alice, it is certainly not simply a matter of refocusing priorities on affective rather than cognitive goals—both are important in the development of future teachers of mathematics. Rather than attempt to resolve the tension, in the spirit of Lampert (1985), we consider instead what can be learned from it.

The study by Hart and Swars (2009) suggests that approaches of MFT instructors may negatively impact student affect. This study counters that even when instructors are concerned about students' attitudes and beliefs, their ability to respond to the affective needs may be constrained by normative commitments to course syllabi, beliefs about the level of mathematics proficiency needed by future teachers, and understandings of the role of the MFT course. Maria's comments about reducing in-class activities in order to get through the material suggest that these factors may also be barriers to instructors' adoption of more reform-oriented approaches. Furthermore, a perceived mismatch between students' prior mathematics preparation and course expectations is also implicated.

The question then becomes, how can mathematics instructors best be supported by the mathematics education community as they strive to manage both cognitive and affective aims for MFT courses? This study suggests that identifying norms in post-secondary mathematics instruction that may differ from those in teacher education may bring to light preconceptions that inhibit instructors' transition from teaching future *users* of mathematics to teaching future *teachers*. Furthermore, ongoing research into the knowledge, beliefs and attitudes needed by teachers of mathematics would assist in clearer articulation of goals for MFT courses and a better understanding of their place in the process of teacher preparation. Finally, although the debate between the priority of affective versus cognitive goals in the literature is exemplified within the cases of these instructors, it is worthy of note that the research literature does not play a direct role in informing these instructors' efforts to deal with their tensions. This is even more evident in the larger study. Although the literature to date offers no clear resolution, closer contact with the mathematics education community might expose these instructors to new strategies or alternate perspectives, placing them in a better position to manage their tensions.

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A RESEARCH SYNTHESIS OF PRESERVICE TEACHERS' KNOWLEDGE OF MULTIPLYING AND DIVIDING FRACTIONS

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This paper describes a synthesis conducted to determine what research says regarding preservice teachers' understanding of fractions and identify the gaps in their existing knowledge basis. Specifically, this paper will address a smaller portion of the synthesis and report the findings from fraction multiplication and division topics. Results indicated that preservice teachers' understanding of fraction multiplication and division is limited and largely based on rote procedures. Implications for teacher education programs and future research studies are provided.

Keywords: Teacher Education–Preservice; Rational Numbers

Objectives

Elementary teachers need a "solid understanding of mathematics so that they can teach it as a coherent, reasoned activity and communicate its elegance and power" (Conference Board of the Mathematical Sciences (CBMS), 2001, p. xi). However, research studies on preservice teachers' mathematics knowledge have shown that many possess a limited knowledge of mathematics in key content areas such as number. For example, Thanheiser (2009) found that only 3 of 15 preservice teachers held a conception of place value that allowed them to explain how and why the subtraction algorithms with three-digit numbers work. The National Mathematics Panel affirmed the "proficiency with fractions" as a major goal for K–8 mathematics education because "such proficiency is foundational for algebra and, at the present time, seems to be severely underdeveloped" (p. xvii). Therefore, developing such proficiency in preservice elementary teachers is a critical task for mathematics educators. As the authors of *The Mathematical Education of Teachers* suggest, "The key to turning even poorly prepared prospective elementary teachers into mathematical thinkers is to work from what they *do* know" (CBMS, 2001, p. 17). There is still much to be learned about preservice teachers' (PST) conceptions in a wider array of topics and how we might use such knowledge in designing mathematics courses for PSTs. In this paper, we discuss the main findings from a research synthesis of existing studies on preservice elementary teachers' fraction knowledge to identify critical directions for future research specifically in the area of fraction multiplication and division.

Theoretical Framework

Shulman (1986) proposed three categories of content knowledge for teachers: (a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge. For Shulman, subject matter content knowledge includes knowing a variety of ways in which "the basic concepts and principles of the discipline are organized to incorporate its facts" and "truth or falsehood, validity or invalidity, are established" (p. 9). Pedagogical content knowledge refers to the knowledge of useful forms of representations (e.g., analogies, illustrations, explanations) of subject-matter ideas that make it understandable to others, and an understanding of the conceptions and pre-conceptions students bring to the learning processes. The third type of knowledge, curricular knowledge, includes knowledge of a "full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances" (p. 10).

Shulman's ideas on pedagogical content knowledge sparked a huge interest in knowledge for teaching, eliciting over a thousand studies throughout a number of content areas with a large number of these studies focusing on teachers' knowledge of mathematics (e.g., Davis & Simmt, 2006; Ball, Thames, & Phelps, 2008; Hiebert, 1986; Ma, 1999). Deborah Ball and her colleagues introduced the term "mathematical knowledge for teaching" (e.g., Hill, Ball, & Schilling, 2008), which focused on the work that teachers do when teaching mathematics.

Building on Shulman's (1986) categories of knowledge, Ball, Thames, and Phelps (2008) introduced a framework for mathematical knowledge for teaching. This framework broke subject matter knowledge into three categories: common content knowledge (CCK), the mathematical knowledge that should be known by everyone; specialized content knowledge (SCK), the knowledge of mathematics content that is specific to the work of teachers; and horizon content knowledge, which involves understanding how different mathematical topics are related. Pedagogical content knowledge was similarly broken into: knowledge of content and students (KCS), which dealt with understanding how students relate to different topics; knowledge of content and teaching (KCT), which involves the sequencing of topics and the use of representations; and knowledge of the curriculum as a whole. While a number of different frameworks look at mathematical knowledge for teaching, this framework proposed by Ball and her colleagues is typically looked at by groups focusing on what teachers know about mathematics and served as a framework for our study as well.

Background and Research Questions

This work was initiated at a PME-NA working group in 2009 and 2010. The members of the working group all taught specially designed mathematics courses for elementary school teachers in the United States and sought to improve their practice by building on PSTs' current knowledge. The working group was formed with a goal of summarizing the prior research addressing PSTs' content knowledge and its development with the idea that we could both improve our teaching and design further research to extend what we know about PSTs' mathematical knowledge. We broke into smaller groups by content area, (whole-numbers, fractions, decimals, geometry, and algebra), and attempted to synthesize the current research in each of these fields.

This paper reports a synthesis of the research that has been done to this point on fraction multiplication and division. These are the areas that came up most frequently in the literature. Our goals for the research synthesis were to: (1) to identify what we already know about preservice elementary teachers' knowledge of fractions in both the domains of subject-matter and pedagogical content knowledge, and (2) to identify the knowledge gap in the existing knowledge basis to help guide future research endeavours.

Methods

The first step of conducting this research synthesis was to identify the existing literature. Initially, we decided to restrict our search to only published research journal articles to maintain the quality of the findings. However, recognizing the time lag required for publication, we decided to expand the search to also include proceeding papers published in 2007, 2008 and 2009, and dissertations published since 2005. With key words such as "preservice teachers," "preservice elementary teachers," "fraction," "fraction concepts," "fraction operations," "fraction multiplication," "fraction division," and "rational numbers," we searched the ERIC, Google Scholar, Dissertation Abstract and Rational Number Reasoning databases (*gismo.fi.ncsu.edu/database)*. We also manually searched through the recent PMENA and PME proceedings because we were not sure about the time lag for a proceeding paper to be included in the above databases. This search yielded 42 journal and proceeding articles and 3 dissertations between 1988 and 2011.

The second step required the research team to locate these papers and skim through them to determine if they had a research question focusing on preservice elementary teachers' fraction knowledge. Fifteen papers were rejected because they did not meet this criterion. For example, some papers were about curriculum sequence or instructional activities that would facilitate preservice elementary teachers' learning of fractions, while others were about preservice elementary teachers' teaching of fractions. So the synthesis we report on in this paper is based on 30 papers and dissertations.

Careful readings of these documents were carried out during the third step. To assist the comparison across these documents, an entry in a synthesis table was filled with information such as “research questions,” “research design,” “descriptions of participants,” “content foci,” “data collection,” “data analysis,” “findings,” and “implications” for each one.

We soon noticed the following main trends in our synthesis table. First, all but two papers were published in two distinct time periods: 1988 to 1994, and 2005 to 2011, with 21 articles falling in the latter time period. This shows a growing interest in preservice elementary teachers’ fraction knowledge. Second, the majority of the papers focused on preservice teachers’ subject matter content knowledge, with a handful on pedagogical content knowledge such as their ability to construct valid fraction story problems and representations, and their ability to predict students’ errors, and none focused on preservice elementary teachers’ fraction curricular knowledge. Third, the majority of the studies (17 of them) focused on preservice elementary teachers’ fraction knowledge in a single sub-concept, for example, fraction multiplication or fraction division, with six studies focused on two or more fraction sub-concepts. Only four focused on the development of fraction knowledge. In the following section, we will report the main findings about preservice elementary teachers’ fraction knowledge related to multiplication and division.

Results

We organized our findings around the major themes that we found in the research. These dealt with PSTs’ common content knowledge of fraction procedures, their specialized content knowledge of being able to write story problems modeling situations with fractions, their knowledge of content and students, in relation to common student errors in regards to fractions, and of different instructional interventions designed to help improve this knowledge.

Preservice Teachers’ Understanding

Research illustrates that preservice elementary teachers are most uncertain about dividing fractions, followed by subtracting, multiplying, and then adding fractions (Newton, 2008). This becomes problematic especially when the ability to represent an operation with diagrams and story contexts has been identified as an important type of specialized mathematics knowledge for teaching (Ball, Thames, & Phelps, 2008). Such ability even becomes more important in the context of the Core State Standards of Mathematics implementation. Grade 5 and 6 students are expected to solve real word problems involving fraction multiplication and division through visual fraction models (e.g., a tape diagram, number line diagram, or area model) and equations to represent them.

However, studies have shown that the majority of preservice elementary teachers do not have a strong ability to represent fraction multiplication and division with story problems (Ball, 1990; Luo, 2009; Simon, 1993). While preservice teachers have an easier time on writing story problems when one of the two numbers involved in multiplication and division computation is a whole number, many of them were not able to do the same when mixed fractions are involved or when both given numbers are fractions. For example, Luo (2009) found that preservice teachers struggled to provide an appropriate context and representation given a symbolic expression of fraction multiplication. In agreement with Goodson-Epsy (2009), Luo concluded that whole number by fraction multiplication is easier than problems with two fractions. Of the 127 preservice early childhood and elementary education students in Luo’s study, only 27% could construct a valid word problem to represent $1/2 \times 1/3$, while 58% could construct a valid word problem to represent $1\ 2/3 \times 4$. In addition, Luo found that the majority of the preservice teachers used a “multiplication as repeated addition” construct which can be problematic when working with non-whole numbers.

Findings from Ball (1990) and Simon (1993) indicated that many preservice teachers were unable to generate a valid story problem for fraction division problems such as $1\ 3/4 \div 1/2$ or $3/4 \div 1/4$. Many of them wrote story problems that were actually for multiplication of fraction by either the given fractions or by the reciprocal of the divisor. It also appeared that preservice teachers who attempted to use measurement division contexts were more successful than those that used partitive division. The field has just began to examine preservice teachers’ proficiency of using diagrams to represent fraction multiplication and division. One study that compares Taiwanese and U.S. preservice elementary teachers' fraction knowledge contains a multiple-choice item asking them to choose the diagram that can not be used to model $3/4 \times 4/5$ or $4/5 \times 3/4$ (Luo, Lo, & Leu, 2011). The finding suggested that the majority of preservice elementary teachers from both populations were unable to identify the correct answer which simply showed a diagram of 3/4 and a diagram of 4/5 with a multiplication sign listed in between.

Preservice Teachers' Knowledge of Student Errors

When teachers enter the classroom, they need to have an understanding of mathematics content as well as student thinking (Ball, Thames, & Phelps, 2008). By understanding how students think, teachers can establish classrooms where discussions focus on the validity of students' responses. Knowing how preservice teachers interpret student responses before they enter a classroom can provide a foundation for the types of activities needed in teacher education programs.

In the context of fraction division, research has shown that preservice teachers' analyze student responses at a surface level (Son & Crespo, 2009; Tirosh, 2000). For example, when shown a correct student's method that included dividing the numerators and denominators, preservice teachers argued that the method works but only because the answer matched to what they got by inverting the second fraction and multiplying (Son & Crespo, 2009). The participants in this study did not delve deeper into the concepts underlying the methods and their beliefs about teaching and learning strongly correlated with their responses toward the non-traditional algorithm.

Other research has shown that similar conceptions hold true when preservice teachers analyze students' incorrect methods for dividing fractions (Tirosh, 2000). Tirosh found that participants were able to identify common problems that students would have, but they generally attributed these errors to students not understanding the algorithm for dividing fractions. Thus, preservice teachers had some understanding of the types of difficulties students may have, but were not able to justify why those methods are incorrect (Tirosh, 2000).

Improving Preservice Teachers' Understanding

Several recent studies have examined the effects of special instructional strategies on preservice teachers' procedural and conceptual knowledge for fraction: for example, use of manipulatives (Green, Piel, & Flowers, 2008); web-based instruction (Lin, 2010) and problem posing (Toluk-Ucar, 2009). All three studies used the experimental design with control and experimental groups. All showed significant better improvement by the preservice teachers in the experimental groups.

For example, in a study conducted by Green, Piel, and Flowers (2008) only 15% of the preservice teachers in the experimental group were able to illustrate the fraction division $1\ 1/2 \div 3/4$ during the pre-test. After working with manipulatives for four weeks, 66% of them were able to do so. These studies also pointed out that it was more difficult for preservice teachers to illustrate fraction multiplication and division situations than it was for them to write story problems. This pattern remained true after the treatment. For example, while the percent of preservice teachers in the experimental group who were able to write story problems for fraction division increased from 2% to 88% between the pre- and post-test in the study by Toluk-Ucar (2009); the corresponding result for drawing diagram for fraction division were 2% and 80%.

Discussion

Research regarding preservice teachers' mathematical content knowledge illustrates that they have a rule-based conception of fraction multiplication and division. Misconceptions result from overgeneralized rules from other number systems, such as multiplication always makes bigger, or result from not understanding algorithms for multiplying and dividing fractions. Other difficulties preservice teachers have with fractions stem from not having a conceptual understanding of the mathematics. Thus, when asked to provide contextualized situations, they tend to create situations not related to the original problem or are unable to generate a situation at all.

Preservice teachers' conception of fraction multiplication is based off of the part-whole meaning of fractions. Studies suggest that in the context of multiplication, preservice teachers are more successful when the problem contains fractions less than one and whole numbers (Goodson-Epsy, 2009; Luo, 2009). Thus, more experiences with fractions greater than one as well as more problems not incorporating whole numbers are needed in teacher preparation programs.

Preservice teachers' conception of fraction division is largely focused on the sharing meaning of division. In addition, fraction division understandings are procedurally or algorithmically based. As a result, preservice teachers have difficulty with interpreting fraction division situations and struggle with representing the situation with an appropriate context.

Recent reports have begun to document the ways in which preservice teachers' develop an understanding of fractions (Tobias, 2012). Tobias (2012) found that their fraction understanding does not develop linearly in that knowledge of one topic may not be fully developed before they start to learn another. Thus, classroom instruction may need to focus on multiple fraction concepts before preservice teachers can develop an understanding of one idea.

Conclusion

By understanding preservice teachers' knowledge of fraction multiplication and division, future studies and improvements in teacher education programs can start to investigate the ways in which preservice teachers overcome their misconceptions to develop the mathematical understandings needed to be an effective teacher. Virtually every study suggests that strong teacher education programs and improvements to teacher education courses are needed, however little has been done to document the types of experiences preservice teachers need. With current recommendations suggesting that mathematics teacher education programs design instruction around what preservice teachers do know, and with the majority of studies focusing on fraction division, research is needed regarding preservice teachers' understanding of other fraction topics. Though there has been a recent increase in the number of publications pertaining to preservice teachers' knowledge of fractions, this is still not enough for teacher educators to have an adequate understanding of how preservice teachers think.

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WRITING PROBLEMS TO BUILD ON CHILDREN'S THINKING: TASKS THAT SUPPORT PRE-SERVICE TEACHER TRANSITIONS

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A critical practice in teaching elementary mathematics is posing problems that build on children's mathematical thinking. As such, teacher educators must provide pre-service teachers (PSTs) with a set of learning experiences to support PSTs in this practice. In this study, we present our analyses of PSTs' responses to a sequence of three methods course activities that engaged them in increasingly complex tasks requiring the PSTs to write problems in response to authentic student work.

Keywords: Children's Thinking; Teacher Education–Preservice; Mathematical Knowledge for Teaching

Introduction

Research suggests that a critical practice in teaching elementary mathematics is posing problems that build on children's mathematical thinking (Carpenter et al., 1999). An implication of this research is that teacher educators must provide pre-service teachers (PSTs) with a set of learning experiences to support PSTs in engaging in this critical practice. However, as a field, we know little about the design, enactment, or sequencing of these kinds of experiences. In this study, we present our analyses of PSTs' responses to a sequence of activities that engaged them in increasingly complex tasks requiring the PSTs to write problems in response to student work.

Theoretical Frame

Shulman (1986) suggested three types of knowledge are important for teaching - subject matter knowledge, pedagogical content knowledge, and curricular knowledge. Since that article, Ball and colleagues (Ball, Hill, et al., 2005; Hill, Sleep, et al., 2007) have built on Shulman's work and have provided the Mathematical Knowledge for Teaching (MKT) framework further defining subject matter knowledge (SMK) and pedagogical content knowledge (PCK) and identifying subsets of these knowledge bases. We are grounding this study in two subsets of PCK—knowledge of content and students, "knowledge that teachers possess about how students learn content" (Hill, Sleep, et al., 2007, p. 133); and knowledge of content and teaching, "mathematical knowledge of the design of instruction, includes how to choose examples and representations, and how to guide student discussions toward accurate mathematical ideas" (Hill, Sleep, et al., 2007, p. 133). These subsets of the PCK construct are useful as we are asking PSTs to think about how students solved particular problems and then use that knowledge of students to design subsequent instruction.

Also relevant to this study is the *professional noticing of children's mathematical thinking* construct (Jacobs, Lamb, & Philip, 2010). Three interrelated skills: *attending* to children's strategies, *interpreting* children's understandings, and deciding how to *respond* on the basis of children's understandings comprise the construct. Within our methods course, we ask PSTs in several instances to talk about what they notice in student work (via video clips and written) and discuss what they think students do or do not understand. Finally, we ask PSTs to use what they know about students to generate a next problem.

Methods

Data were collected from thirty-three, first semester, senior level PSTs (32 female, 1 male) enrolled in an elementary mathematics methods course taught by the first author in fall 2011. The data include PST responses to three different activities, each of which are from a set of methods course materials written by the second and third authors (Drake, Land, et al., 2011). The activities were designed to scaffold and support PSTs as they developed the capacity to make sense of student strategies and to write appropriate subsequent tasks for students. Each of the activities is set in the context of actual classrooms. The three activities were posed over the first six weeks of the course and were sequenced in order to provide PSTs with various experiences analyzing and writing effective tasks based on student thinking. The first activity (Natalie's Class the Next Day) was designed to give PSTs the opportunity to notice and analyze how an experienced teacher used her students' current knowledge of division with fractional remainders to design a subsequent story problem and number choices. The second activity (Counting Sequences) required the PSTs to write an opening number routine (ONR) and problem, including number choices to address a class-wide addition misconception. The third activity (Fishbowl Problem) asked PSTs to analyze 14 students' multiplication strategies and write a subsequent problem with number choices to address the wide range of learners. We organized the activities to form a trajectory along several dimensions – moving from noticing an expert teacher's task design to having PSTs design tasks themselves, moving from designing a task to address a single misconception to writing a task that addressed a wide range of student understandings, and moving from PSTs noticing an expert teacher's number choice to selecting numbers for a pre-written task to writing an entirely new task.

Data Analysis and Results

Natalie's Class the Next Day

Prior to completing the Natalie's Class the Next Day task, PSTs watched a video with transcript of Natalie and her 2nd grade class as they solved two partitive division story problems:

> <u>Problem #1</u> Trisha and Allie are sharing ______ chocolate chip cookies. If they are shared equally, how many will each of them get?
>
> 2 4 5 8 9 12 13
> 30 31 50 51 66 67 83
>
> <u>Problem #2</u> Trisha, Allie, Lance, and Kathy are sharing brownies. If they are sharing _____ brownies equally, how many will each person get?
>
> 4 5 8 9 16 17 20
> 32 33 44 45 48 49 50

Multiple number choices are given to provide for differentiation. Students were to choose the row of number choices "just right" for them. In Problem #1, even numbers were posed followed by the next consecutive number (with the exception of 2 and 83). In Problem #2, multiples of four were posed followed by the next consecutive number. The use of next consecutive numbers was intended to provide a scaffold in that students could use what they knew about one number choice to help with the next. Included in the video are examples of student work, teacher/student interactions, and a sharing session where students explain their various strategies. After discussing the video, PSTs are asked to complete the following activity:

> The next day, Natalie posed the following problem. Solve the problem for a few of the number choices. Then, answer the questions below.
>
> There are ____ miniature candy bars. Dustin, Jose, Sam, and Joe are going to share the candy bars. If they split up the candy bars equally, how many will each of them get?
>
> 11 17 22 35 48
> 65 83 75 99 104

1. Why do you think Natalie posed this particular problem next?
2. What do you notice about the number choices in this problem compared to the number choices given the day before?

Analysis: Natalie's class the next day. As we examined responses from the PSTs, we focused on their responses to question two. The next problem Natalie posed is also a partitive division problem and extends the second problem from the day prior in sharing a set of objects among four people. We analyzed PSTs' responses according to their noticing of three aspects of Natalie's number choices: (1) the numbers in both rows are both *larger* numbers than the day before; (2) the numbers are more *complex* in that students had to think not only about sharing remainders of zero and one, but also two and three as well; (3) the next consecutive number *scaffold* that had been used the day before has now been removed. Two authors independently coded the PSTs' responses for evidence of these three facets with 94.9% agreement (94/99).

Here is a sample response from one of the PSTs, Jaceylyn (all names are pseudonyms):

> The first thing that stood out to me about these number choices was that they were generally larger than the ones offered on the previous day. Next, when I actually started working with them, I found that these number choices granted me with quite different answers than the day before. On the previous day the answers had either been whole numbers, or sometimes involved a half as well, but today the answers came out with remainders of ¾ or ¼.

This response was coded as identifying larger numbers as well as more complex numbers.

Results: Natalie's class the next day. We examined the 33 responses to the Natalie's Class the Next Day activity in two ways: (1) how many of the facets of the number choices were identified by each PST, and (2) number of PSTs that identified each facet. The results are presented in Tables 1 and 2. We interpreted this data through the lens of MKT, specifically as indication of knowledge of content and teaching; knowledge of how to choose examples and design instruction. From the data one can see that ~48% of the PST identified either zero or one facet of the number choices, ~42% identified two of the three facets and only a small percentage (~9%) were able to identify all three. We posited that it might be more likely for PSTs to notice the larger numbers and the lack of scaffolds than recognize the complexity of the numbers, as the first two required less developed knowledge of content and teaching.

Table 1: Number of Facets Identified

# Correctly identified	# PSTs
0	7
1	9
2	14
3	3

Table 2: Percentage of Each Facet Identified

Facet	# PSTs	% PSTs
Larger numbers	13	39.4
More complex numbers	18	54.5
No scaffolds	15	45.5

Counting Sequences

The Counting Sequences activity begins with the PSTs watching a video with transcript of Jenny's first grade class. For her ONR, Jenny poses the following counting sequences to her students that focus on base-ten concepts: 30, 40, 50, ______, ______, ______. 44, 54, 64, ______, ______, ______. 57, 67, 77, ______, ______, ______. 157, 167, 177, ______, ______, ______. Jenny's students are able to solve the tasks by counting by 10. Students also notice the units place remains the same and the tens place number increases by one each time. They are able to solve the sequence that "crosses over" from a 2-digit number to a 3-digit number. The video ends with Jenny posing a story problem about a paleontologist:

> A paleontologist had _____ dinosaur bones. He found some more. Now, the paleontologist has _____ dinosaur bones. How many bones did he find?
> (10, 70), (20, 84), (26, 126), (15, 65), (60, 150), (42, 53)

The activity provides PSTs with a description of student work from the Paleontologist Problem:

> Most of the children solved the paleontologist problem by using a hundreds chart, but many counted by ones when counting up to the second number instead of counting by tens. Some children did count by tens. For 20 and 84, the children who were counting by tens either counted by ones from 20 to 84, or counted by tens to 80, then counted 4 more. Nobody solved for 42 and 53 (Drake, Land et al., 2011).

Following this description, the story problem Jenny used the next day (without number choices) is given, "Today, the paleontologist is looking for fossils. He already had ____ fossils in his collection. He found some more. Now, the paleontologist has ____ fossils. How many fossils did he find?" The counting sequences activity was then posed for the PSTs to complete:

> Now that you have seen the Counting Sequences video (and its transcript), consider these questions related to students' solutions to the Paleontologist Problem.
>
> 1. What is the disconnect between how students counted in the opening routine and the counting strategies they used when solving the problem?
> 2. Why do you think the disconnect exists?
> 3. Considering this disconnect, generate two artifacts for the next day's lesson: an opening number routine and number choices for the Paleontologist problem given below. Briefly justify your choices.

In this activity, we were interested to see if: (1) the PSTs could recognize many children did not see their counting by tens strategy in the sequence activities as applicable in solving the join-change unknown story problem; (2) they could posit reason(s) for the disconnect; (3) they could design an ONR to address the reason(s) stated in 2; and (4) they could select appropriate number choices for the next day's problem. We believe this task was a natural progression from the previous task, as it required PSTs to interpret and respond to a general mathematical misconception within a class of children.

Analysis: Counting sequences. Prior to analyzing this data set, the authors collaboratively examined several responses to this activity from a previous course and through open and emergent coding (Strauss & Corbin, 1998) established a series of codes and operational definitions for: (1) identifying the disconnect explicitly and accurately (yes/no/no response); (2) number of reasons given for the disconnect (0, 1, 2 or more); (3) identifying the degree to which the ONR addressed the reason(s) given (low, medium, high); and (4) classification of the types of number choices we believed were appropriate for Jenny's students (count by 10s from a decade number as given in Jenny's original Paleontologist Problem, count by 10s from a non-decade number, count by 10s and 1s). We operationalize the degree to which the ONR addressed the reasons for the disconnect by examining the approaches the PSTs took in selecting the type of task, structure and/or number choices for their ONR. PSTs who used the same approach as Jenny, or used approaches that did not connect to their reason, were ranked low. PSTs who attempted at least one new type, structure or number choice related to their reason, and did so in a manner we *believed* could be effective, were rated medium. PSTs who made significant changes (more than one new approach) related to their reason, and did so in a way we were *confident* could be effective, were rated high. Reliability percentages for each of the four categories are as follows: Disconnect: 90.9%; Reasons: 75.8%; Degree: 78.8%; Number Choices: 87.1%. We discussed disagreements and reached consensus on the final codes.

Chelsea's response follows. The numbers correlate with the questions given above:

> 1) When students were counting in the counting sequences opening routine, they were counting by tens and realized that the second digit of the number was remaining the same. However, once they tried solving the problem, the students began counting by ones, and it threw them

off to try counting larger numbers by ones.

2) When counting by tens, the second digit of the number remains the same. It creates a pattern and makes it easy to continue in an almost rhythmic-like pattern of repeating “10, 20, 30, 40, 50...” and so on. However, when counting by ones, the second number changes along with the first number and this can be very confusing for kids if they are counting “10, 11, 12, 13, 14, 15, 16, …”

3) Opening Number Routine – Fill in the blanks with the missing numbers.

10, _____, _____, _____, 50, 60, _____, _____, 90, _____
5, _____, 25, 35, _____, _____, 65, 75, _____, _____, 105
100, _____, _____, 130, _____, 150, _____, _____, 180, _____, _____

Problem for the next day
Today, the paleontologist is looking for fossils. He already had ____ fossils in his collection. He found some more. Now, the paleontologist has ____ fossils. How many fossils did he find?
[10, 30]   [5, 25]   [100, 175]   [3, 43]

I chose these numbers because I started out with simpler numbers that they could easily apply their counting sequence strategy to a word problem (10, 20, 30). I then moved on to [5, 25] because starting at 5 and counting by tens is slightly more difficult. Next I did [100, 175] because starting at 100 is difficult, and they also have to count by 5’s once they get to 70. Finally, I placed the hardest number choice last because the students have to count by tens, but they are starting at 3, which will throw them off to see a 3 as the last digit, and they will really need to understand the process of counting by tens to get from 3 to 43.

The above example was coded as (1) *yes* to identifying the disconnect; (2) *0* for not identifying a reason for the disconnect; (3) as *low* for the degree in which he/she addressed the disconnect as it is the same approach used by Jenny; and (4) as having *counting by tens from a decade number, counting by tens from a non-decade, and counting by tens and ones* in the number choices.

Results: Counting sequences. Of the 24 PSTs who attempted to identify the disconnect within Jenny’s class (9 no response), 18 were able to accurately do so (75%). 18 of those 24 (75%) were able to posit at least one reason why the disconnect may have occurred. When it came time, however, to design an opening number routine that would address the disconnect, more than 50% of PSTs simply posed “more of the same” approaches Jenny used. Ten PSTs (30.3%) made an attempt to try something different, but only five PSTs (15.2%) were able to do so in a way we felt confident would afford the children multiple opportunities to make the connection between skip counting by 10s in patterns and using skip counting by 10s as a strategy for solving join-change unknown addition problems. The number choices data were more encouraging. There was a high percentage of PSTs (42.4%, 14/33) who included at least two of the three appropriate number choices or all three of the appropriate number choices (48.5%, 16/33) in the next day’s problem. Three PSTs included only one of the appropriate number choice types. One emerging pattern from these data is our PSTs seem to be able to understand and identify student thinking, but often struggle using this information to effectively address it.

Table 3: Counting Sequences Results

Disconnect			Reasons			Degree			Number Choices		
Y	N	NR	0	1	2+	L	M	H	10sD	10sND	10s1s
18	6	9	15	9	9	18	10	5	25	29	24

Fishbowl Problem

The Fishbowl Problem is set in the context of Molly’s 2nd/3rd mixed age classroom. This task was built around PSTs’ examination of examples of student work from 14 children in Molly’s class in response to the following multiplication problem:

> Sam had ____ fish bowls. He had _____ fish in each bowl. How many fish did Sam have? Molly presented 4 pairs of number choices for her students to pick from: A: (2, 10), (5, 10); B: (4, 20), (8, 20); C: (3, 11), (6, 11); D: (4, 12), 8, 12).

The task for the PSTs was as follows:

1. First, consider Molly's learning goals—what are they?
2. Next, look at the student work on the following pages. What do you find interesting? What evidence can you identify that students are or are not making progress toward the learning goal(s)?
3. Write a problem for the next day along with a rationale. What do you think will be an appropriate problem that will meet the range of needs in Molly's classroom? Reference at least three students or group of students specifically in your rationale.

We believed this activity was an appropriate next task for the PSTs' development as it required them to analyze and make sense of several children's thinking, to write a story problem appropriate for the entire class and simultaneously attend to specific strategies and learning goals when writing number choices. This activity is very similar to the work of teaching and required PSTs to use many different knowledge bases to effectively complete the activity. Molly had different goals for different groups of children in her room. For some children she wanted to see if they were able to skip count by multiples of ten. For others, she wanted to see if they could notice and use the doubling relationship between the pairs of numbers she had chosen for them to solve. She included number choices like 11 and 12, to see if any children would solve using the distributive property and their knowledge of tens.

Analysis: Fishbowl problem. In our analysis of the children's work, we classified their approaches into one of four categories: (1) direct modeling: children in this group either could not solve any of the multiplication tasks, or did so by directly modeling the solution with drawings; (2) skip counting: children in this group skip-counted by 10s and/or multiples of 10; (3) repeated addition/break apart by place: children in this category solved tasks by writing the multiplication problems as repeated addition and then broke the 2-digit numbers like 11 and 12 apart by place value and added the 10s and 1s separately; and (4) doubling: the children in this group also used repeated addition to solve the first number choice in the pair, but were also able to recognize the relationship between doubling the number of groups and doubling the product.

Similar to our analysis of the Counting Sequences Activity, the authors first collaboratively examined several responses to this activity from a previous course and established a series of codes and operational definitions for writing an appropriate story problem (yes/no) and demonstrating understanding of children's strategies (yes/no). In our analysis of the PSTs' number choices, we coded their responses in terms of addressing current student understanding and in terms of addressing Molly's learning goals. As we coded the responses in terms of addressing students' current understanding, we first looked for evidence in the rationale that the PSTs were *attempting* to choose numbers for specific individual's (or groups of children's) strategy. If we found evidence, we then examined the number choices they selected in order to determine if they had *successfully* done so. We coded their number choices in terms of learning goals in a similar manner. If PSTs explicitly mentioned a learning goal in their rationale, we coded it as an *attempt.* If an attempt was made, we then determined if the number choices were appropriate. If so, we coded it as a *success*. Reliability percentages were calculated for each category and ranged from 73.5% – 93.9%. Consensus was reached on all disagreements.

Samantha's response follows as an example. For space purposes only her problem is shared:

> Olivia has ____ drawers. She has ____ pencils in each drawer. How many pencils does Olivia have?
>
> I want to address the same goals, but have structured them so some are easier than her first examples, some the same difficulty, and some harder.

Group A (3, 10) (4,10) (6, 10) (11, 10) *Here I want to practice going over 100 to provide some extra challenge. I also wanted those struggling to recognize the relationship between the 3 and 10 and the 4 and 10.*

Group B (1, 20) (2, 20) (4, 20) (8, 20) *Here I want the students to start on the 20s to focus on the relationship between the first and second number in the problems, but also the first numbers over the sequence.*

Group C (4, 11) (5, 11) (7, 11) (5, 12) *Here I want students to apply their knowledge of counting by 10s and then adding 1s to solving the problem. Hopefully having the second number switch to 12 will have these extend that knowledge.*

Group D (2, 11) (2, 12) (2, 13) (2, 14) *I wanted the students who've really gotten a hang of this 10s and 1s concept to apply it and to see patterns by keeping the 2 consistent.*

In this case, the above problem was coded not attempting, and thus, not successful, in addressing specific student's strategies. However, it was coded as attempting and successful in choosing numbers for specific learning goals.

Results: Fishbowl problem. The data supports the preliminary result from the Counting Sequences activity. We can see by this stage in our sequence a vast majority of the PSTs (31/33, 93.9%) made sense of the student work provided and were able to write an appropriate story problem type (28/33, 84.8%). When it comes to writing number choices for the next story problem however, it becomes evident that: (1) PSTs have difficulty in addressing multiple groups of student thinking simultaneously; (2) when PSTs do attempt to write specific number choices to address or further student thinking, they are not often successful in doing so (9/17, 52.9%; 8/26, ~31%; 7/15, ~47%; 5/12, ~42%); and (3) PSTs have difficulty writing number choices that attend to both student thinking and learning goals.

Table 4: Fishbowl Problem Results

		Number Choices for Students								Number Choices Learning Goals					
		Low/Direct Model		Skip Count		Repeat+ /BABP		Doubling		Skip by 10s		Doubling		Distributive Property	
Story?	Understand?	A?	S?	A?	S?	A?	S?	A?	S?	A?	S?	A?	S?	A?	S?
28	31	17	9	26	8	15	7	12	5	22	11	17	12	14	10

Discussion

As we interpret the results from this sequence of activities through the work of Jacobs and her colleagues (2010), we conclude PSTs have become more adept at *attending* to and *interpreting* student thinking. The activities however, have not helped the PSTs to make similar progress in *responding* to student thinking. One possible reason for this result is that our sequence of tasks does not provide enough educative supports to develop PSTs' ability to respond appropriately to student thinking. We have not explicitly attended to the question, "What makes a number choice appropriate or inappropriate to support/extend a student's current way of thinking?" An activity that presents an example of student thinking and requires PSTs to select and justify an appropriate number choice from a list of possibilities might help to develop PSTs' ability to interpret, evaluate and write appropriate number choices. These conclusions can be explained in terms of the construct of MKT. Our sequence of activities appears to support the development of PSTs' knowledge of content and students. Through repeated exposure to authentic student work (both video and written), PSTs have improved in their ability to make sense of and evaluate students' thinking strategies in a variety of mathematical contexts. This knowledge base is paramount in *attending* to and *interpreting* student thinking. PSTs' knowledge of content and teaching however has not shown similar improvement. Though the PSTs have demonstrated an ability to interpret student thinking and "diagnose" mathematical inconsistencies, they have not yet developed the appropriate content knowledge base to respond effectively in "prescribing" the next treatment.

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REPRESENTATIONS OF ALGEBRA INSTRUCTION IN PRESERVICE TEACHER EDUCATION

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This study examined preservice teachers' reactions to and reflections on a set of five animations as representations of algebra instruction. The results showed that most comments in the preservice teachers' reflections were related to: (a) student learning and motivation, or (b) the classroom and instructional environment. Few comments in the reflections focused on the preservice teachers' reactions to the realistic nature of the animations. The data suggest that animations are a useful alternative to video. Not only do they allow for a condensed format that is free of distractions and focuses on a specific scenario to be analyzed, this study suggests that animations can also promote productive reflection without prior scaffolding.

Keywords: Teacher Education–Preservice

Purpose of the Study

This study examined preservice teachers' reactions to and reflections on animations as representations of algebra instruction. It revolved around the implementation of animations with preservice teachers in a methods course and focused on their responses to the classroom dynamics portrayed in the animations. Since animations are a relatively new medium to represent classroom instruction, we explored preservice teachers' reactions to the animations themselves as well as the extent and quality of the preservice teachers' discussions relative to what they noticed in the animations. More specifically, this study addressed the following research question: What are preservice teachers prompted to discuss after viewing the animations when implemented without prior prompts or scaffolding?

Theoretical Framework

Star and Strickland (2008) define noticing as what preservice teachers identify and deem important or noteworthy when viewing a classroom scenario. Santagata and Guarino (2011) identify what they consider "fundamental skills for reflecting and learning from teaching" as "the ability: (a) to attend to important elements of instruction, (b) to reason about these elements in integrated ways, and (c) to propose alternative instructional strategies" (p. 134). While noticing is important, preservice teachers must also be able to reflect on what they have noticed before they are able to apply what has been observed.

Providing representations of classroom scenarios prompts preservice teachers to consider and discuss the dynamics portrayed thus helping to bridge theory with practice (Harrington & Garrison, 1992). Video is commonly utilized to provide these representations along with guidance from provided prompts or scaffolding to facilitate what is noticed and reflected upon. However, Stockero (2008) points out that preservice teachers "need to move away from a dependence on external prompting if they are to continue to ground their analyses in evidence after a university course is over" (p. 377).

While videos have been a common medium in teacher education, animations are relatively new but provide unique benefits. Although animations represent practice, they are scripted in a manner that allows for a condensed format in terms of both focus and duration. Animations, because they are manufactured representations of instruction, can be pared down to specifically what one wants the viewer to consider in a time-efficient manner. This reduction in complexity removes many of the distractions that naturally accompany video. In fact, Moore-Russo and Viglietti (2011) found in their study of teachers' reflections on animations of geometry instruction that the lack of complexity did not impact the preservice teachers' impressions of the animations as realistic representations of teaching.

Methodology

Participants

Data for this study came from two groups of preservice teachers enrolled in a methods course for secondary mathematics teachers in the fall 2010 and fall 2011 semesters at a large research university in the northeastern United States. All of the preservice teachers held mathematics degrees and were enrolled in a certification program through a graduate school of education. The fall 2010 group of preservice teachers consisted of five females and three males. The fall 2011 group of preservice teachers consisted of four females and four males. Five of the eight preservice teachers in the fall 2010 section and six of the eight preservice teachers in the fall 2011 section were concurrently enrolled in their field observation components.

Setting

The methods course is a three-credit hour, graduate-level course with a primary goal of preparing those with mathematics degrees to be secondary school mathematics teachers. This course is designed to address the aspects of pedagogical content knowledge including: knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum (Ball, Thames & Phelps, 2008). The course covers a variety of topics including classroom management, lesson planning, and implications of educational theories (e.g., social constructivism) on the practice of teaching. There is a particular emphasis on the National Council of Teachers of Mathematics' (NCTM's) (2000) vision for mathematics education.

About half way through each semester, the preservice teachers were assigned to view five animations over a period of five weeks, one per week. The instructional intentions were to use the animations as a common reference for collaborative reflection, discussion, and analysis. They provided an opportunity for teachers to engage in reflective discourse grounded in a shared experience, one that represented different secondary algebra classrooms.

The animations used cartoon figures to represent episodes of instruction in various secondary algebra classrooms. The cartoon figures were each colored, and their names corresponded to their colors. For example, one student was Blue, another was Red. All cartoon characters were gender neutral.

Since the preservice teachers were not able to go as a group to observe a classroom, the animations served as condensed, "shared" experiences for the methods class. The animations were from ThEMaT (Thought Experiments in Mathematics Teaching), an NSF-funded program at the University of Maryland, which has developed animations as representations of classroom scenarios to be used in teacher education (for more information on the animations representing algebra instruction that were used, go to http://www.education.umd.edu/MathEd/ThEMaT.html).

The animations were chosen as representations of teaching for two reasons. First, they portrayed secondary classroom instruction—all from different classrooms with different teachers (as noted by the varying voices used for the teachers in the animations). Second, they have been found to be effective in eliciting discussion among experienced teachers often compelling them to project themselves into the stories (Herbst & Chazan, 2006).

Data Collection and Analysis

After viewing each animation, the preservice teachers were required to make three entries (two in response to classmates) to an asynchronous, electronic discussion forum over the course of a week as a part of their homework assignment. The instructor did not contribute to the online discussion boards nor was any feedback to the entries provided. Students were informed that they were to be graded on their entries, considering both the content of their entries and the personal knowledge growth evidenced from the entries over the course of the semester.

Inductive content analysis was used to identify the themes emerging from the discussion board data. This consisted of multiple readings followed by the recording of each researcher's thoughts utilizing theoretical memoing (Glaser, 1998). Once this stage was completed, the researchers revisited the discussion board entries jointly and compared their memos, forming categories. The researchers noted that most notions raised in the memos were mentioned by both researchers and often on more than one animation.

The researchers jointly revisited each animation in light of the identified categories until it was clear that all ideas in the entries would be able to be coded. The research team then jointly conducted a "horizontal" pass through the discussion boards agreeing on themes that emerged from the categories that ran across all five animations. After this, the researchers then independently coded all 248 of the preservice teachers' discussion board entries. Each individual discussion board entry served as the unit of analysis. Each unit of analysis was assigned as many codes as were deemed applicable. The overall percentage of agreement for coding was above 90% for both 2010 and 2011 sets of data. The Cohen's kappa values were at or above 0.80 for each data set, well above the generally accepted 0.60, which represents good agreement (Landis & Koch, 1977; Altman, 1991). The two independent coders reached consensus by means of discussion for each entry that had been assigned a different code. The research team began analysis at this stage.

Results

During the analysis, several themes evolved from the discussion forums. In reviewing and coding the discussion board entries over five animations and two methods courses, six recurrent themes were identified: (a) *supporting student learning and motivation*, (b*) classroom and instructional environment*, (c) *mathematics in the animations*, (d) *reflections on past observations and one's own future practices,* (e) *reality of the classroom*, and (f) *reactions to the animations*.

Table 1 presents the frequency of comments relating to each identified theme, along with the percentage, or relative frequency, for each category. Examining the compiled data, the greatest amount of time was spent in discussing *supporting student learning and motivation* while the least amount of time was spent in discussing the *reactions to the animations*. The *supporting student learning and motivation* theme included categories focused on student learning such as understanding as the goal of instruction, providing positive feedback for students' efforts, building on what the students know, responding to students' questions, and preventing or correcting student misconceptions.

Table 1: Presence of Themes in Discussion Forum

Theme	Frequency	Relative Frequency
Supporting Student Learning and Motivation	461	42
Classroom and Instructional Environment	303	27
Mathematics in the Animations	168	15
Reflections on Past Observations and One's Own Future Practices	81	7
Reality of the Classroom	59	5
Reactions to the Animations	39	4

The theme with the second highest frequency of comments was the *classroom and instructional environment* theme. This theme contained categories that were concerned with the teachers' instructional obligations, preparedness and flexibility, attitude, and the classroom environment in general. Comments in this theme were often related to what a teacher should do such as support and explain ideas presented and provide more direct instruction when it is clear that guiding is not productive. Comments also captured the nature of the classroom environment as one that should be safe and respectful where students are not afraid to ask questions.

The *mathematics in the animations* theme included comments revolving around the mathematics presented in the animations and the use of technology such as calculators. This theme also captured the multiplicity of mathematics in that mathematical concepts can often be represented or considered in multiple ways and often there is more than one solution method that may be applied when solving a problem.

The final three themes that presented were not as frequently noted. The *reality of the classroom* theme focused on the challenges faced in the classroom environment such as time constraints and pressure to cover a certain amount of material or to cover it in a certain way in order to satisfy state assessment requirements. Comments in this theme also addressed the fact that not all students will perform optimally and give forth their best efforts. The *reflections on past observations and one's own future practices* theme included comments that related to individual preservice teachers' reflections on what they believe they will encounter in their classroom and the decisions they may make in response. This theme also included categories that captured the preservice teachers' reflections that compare what was presented in the animation to what they experienced throughout their academic career and in their field observations. The *reactions to the animations* theme captured comments and discussion that focused solely on the nature of the animations themselves, such as comments related to the depiction of the characters in the animations or to the animations as a particular snapshot in time where the viewer does not know what came before or transpired after that featured episode.

Discussion

The data show that the preservice teachers were most likely to post entries on the discussion boards that fell under the *supporting student learning and motivation* theme. This theme had 13 categories; the most frequently evidenced category related to *understanding* as the goal of instruction and issues around promoting deeper student thinking. The *understanding* category represented 21% of the comments under this theme. The following entries exemplify the comments made by the preservice teachers that were assigned to the *understanding* category:

> *The teacher asked a lot of questions to see where the students understanding was, and [did] not move on until they all showed understanding.*

> *The students that have to explain it are forced to have a complete understanding of the material ... This shows a class that is much more focused on a conceptual understanding of what is going on than a simple procedure.*

The second most prominent category was *constructivism*, which was assigned to 16% of the comments under the *supporting student learning and motivation* theme. Comments that mentioned building on what the students know and allowing students' ideas to guide the direction of instruction were assigned to this category. The following entries exemplify the comments made by the preservice teachers under the *constructivism* category:

> *I loved the way that the teacher was able to step back and allow the students to explain their solutions and their reasoning.*

> *I think "tricking" the students into thinking that they are coming up with all of the work is a pretty good strategy to use. In reality, they are actually coming up with most of the work.*

Other categories under the *supporting student learning and motivation* theme also were evidenced in 10% or more of the preservice teachers' comments. The *affective concerns* category, which was assigned comments related to providing positive feedback to students and acknowledging their efforts, represented 13% of the comments under this theme. The *responding to students' questions* and the *addressing student misconceptions* categories each represented 10% of the comments under this theme. The following entries exemplify the comments made by the preservice teachers under the *affective concerns* and *addressing student misconceptions* categories, respectively:

> *Showing them that we care will give them more confidence and make them more successful math students.*

> *Common mistakes, such as when half of the class gets the same wrong answer, need to be addressed, and correct answers need to be shown.*

The prominence of comments assigned to these categories provide evidence that even without direct, prior prompting or scaffolding, the preservice teachers were able to view the animations with a student-centered focus.

The theme that ranked in second place in terms of the number of discussion board entries assigned to categories under it was *classroom and instructional environment.* Under this theme, many comments made it clear that even though the preservice teachers were student-focused, they also were concerned with the teacher's role in the classroom. This theme had nine categories. The most frequently evidenced category related to *teachers' instructional obligations*, which included such things as providing clear, direct explanations. This category represented 21% of the comments under the *classroom and instructional environment* theme. The following entries exemplify the comments under this category:

> *[T]he teacher simply explains that "it isn't appropriate to use [negation] here" and that "time can't be negative anyway." Her first comment is vague at best and does not help the students.*

> *I felt the teacher was ineffective in their [sic] explanation…nothing was being summarized.*

The preservice teachers also posted a number of discussion board entries regarding how: (a) the material was presented, including the pace of the instruction and how the board and other resources are used in instruction (16%); (b) how teachers need to be prepared with the necessary knowledge of content and pedagogy that allows them to have flexibility in their teaching (16%); and (c) how the classroom environment should be a safe and respectful one where students are encouraged to ask questions and share (15%). The following three entries exemplify the comments under *instructional presentation*, *instructor's preparedness and flexibility*, and *safe/respectful environment* categories, respectively:

> *I really liked the use of the balance as an analogy for solving the problem. Many people talk about viewing an algebraic equation as a scale, but I liked how this teacher actually drew out a scale… It gave the equation a very tangible representation.*

> *… the teacher seemed like the type of teacher who is comfortable with one way to solve a problem [who] does not want to open up to the possibilities of other correct options. He is not confident in his understanding of the other ways of looking at the problem…*

> *I believe that it is all about creating an environment in your class where it is okay to be wrong… if I can get as many students as possible to feel comfortable doing this, the better my classroom will be.*

The theme with the third highest frequency was the *mathematics in the animations* theme. While all five of the animations focused on the mathematics inherent in a secondary algebra classroom, the greatest amount (42%) of entries under this theme were assigned to the *multiplicity of mathematics* category. Under this category, preservice teachers' comments frequently dealt with how there are multiple solution methods available for any given mathematical problem and that most, if not all, mathematical concepts may be thought of in multiple ways. The *mathematics in the animations* theme only had two other categories: the *general mathematics* category, which contained general comments about the mathematics in the animations, and the *technology in mathematics* category, which included comments about the use of technology as a tool to help solve mathematical problems or represent mathematical concepts. These categories held 40% and 18%, respectively, of the comments under the *mathematics in the animations* theme. Almost all of the entries assigned to the *technology in mathematics* category were for the fourth animation, which specifically dealt with the utilization of a graphing calculator. The following entries exemplify the comments made by the preservice teachers under the *multiplicity of mathematics* and *technology in mathematics* categories, respectively:

> *… although some students approached the problem differently all of the methods are correct and completely acceptable so its ok to choose the approach that makes the most sense to you as an individual …*

I think it is interesting though, how they were supposed to be doing it on the calculator. We include these technologies in classrooms so that students can get experience using the tools they will have in the workplace.

Reflections on past observations and one's own future practices was the fourth most common theme. Although some of the preservice teachers reflected on their own academic careers as students, most of the entries that were coded under this theme (44%) related to the *future-oriented reflections* category where the preservice teachers pondered what it would be like in their own classrooms and how they would handle situations that arose. The *reflections on past observations and one's own future practices* theme contained two other categories: the *past observations* category, which contained comments that made comparisons between the events in the animation and the preservice teacher's experiences as a student or in field observations, and the *other sources* category, which included comments revolving around comparisons between the animations and what the preservice teachers had been exposed to in courses, readings, and outside educational resources. These categories held 33% and 22%, respectively, of the comments assigned to the *reflections on past observations and one's own future practices* theme. The following entry exemplifies the comments made by the preservice teachers under the *future-oriented reflections* category:

I have always wonder [sic] how and what I should do with students who miss class, do I cross my fingers and hope they can just catch on and move along with the class? Or should I take them aside and help them learn the material they have missed?

The theme with the fifth highest frequency was the *reality of the classroom* theme. This theme also had only three categories. The most frequently evidenced category related to *students are different*, which included such things as recognizing that students often think about things differently and learn differently. Thus, teachers need to be aware that differentiated instruction may be required in order to satisfy the needs of all students. This category represented 46% of the comments under this theme. Other preservice teachers' comments often related to time constraints in the classroom and pressure to cover a certain amount of material or pressure to cover the material in a certain way to satisfy state assessments. This category, *pressure and constraints*, represented 42% of the comments under the *reality of the classroom* theme. The remaining category, *students do not perform optimally*, included such things as students' inability to recall information, minimal student effort, and poor student attitudes. This category represented 12% of the comments under this theme. The following entry exemplifies the comments made by the preservice teachers assigned to the *students are different* category:

There was [sic] definitely different levels of learners in this class, as well as different styles of learners.

Finally, the last theme contained entries that addressed the preservice teachers' *reactions to the animations*. The *animations as snapshots* category represented 38% of the comments under this theme. This category contained comments related to the realization that what transpired before or after the particular episode could have impacted the teacher's actions. By not having knowledge of prior or subsequent events, the preservice teachers recognized that their interpretation of the teacher's actions in the animations could be inaccurate. The *reactions to the animations* theme contained three other categories. The *characters in animation* category contained comments related to the depiction of the cartoon characters in the animations, in particular that the characters were identified without the use of gender, ethnicity, etc., and represented 31% of the comments under this theme. The remaining two categories were the *verifying reality of animations* category which suggested that the animations reflected what actually occurs in classrooms and the *questioning reality of animations* category which suggested that the animations do not reflect what actually occurs in classrooms. These categories held 26% and 5%, respectively, of the comments assigned to the *reactions to the animations* theme. The following entry exemplifies the comments made by the preservice teachers under the *animations as snapshots* category:

I'm not saying this is necessarily what I think, but there's a world of information not presented to us while watching a 2:49 clip.

In light of the fact that the *reactions to the animations* theme presented the least amount of comments overall, this provides evidence that the preservice teachers seemed to think that the animations were reasonable representations of teaching, which corresponds with Moore-Russo and Viglietti's (2011) study findings.

Overall, animations are a useful alternative both to group classroom observations and to video. While reducing the complexity of the classroom, they still provide a sense of the temporality of events and represent the dynamic, often ill-structured, nature of classroom instruction in a time-efficient manner. They focus on what teacher educators might want preservice teachers to notice, reflect upon, and discuss with their peers. In summary, this study suggests that unlike videos where research suggests that guidance is needed to direct preservice teachers' focus to particular events for analysis (Santagata & Angelici, 2010), animations seem not to need instructor scaffolding in order for preservice teachers to be able to productively notice and reflect on the presented classroom instruction.

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ASSESSING PRE-SERVICE ELEMENTARY TEACHERS' DISPOSITION TOWARD MATHEMATICS USING OPEN-ENDED METAPHOR

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This qualitative study examines (N = 33) pre-service elementary teachers' disposition toward mathematics. The theoretical framework of the study encompasses a sociotransformative constructivist perspective which best supports research using a positioning theory lens to view human experiences, dispositions and relationships developed through discursive practices. In order to examine discursive practices, we used an open-ended metaphor approach that allowed researchers to identify characteristics of disposition by opening up a "fresh space of truth-telling" and "powerful use of language and image" (Hagstron et al., 2007, p. 27). Through the use of linguistic and deconstruction analysis as well as meaning coding and meaning interpretation techniques (Kvale & Brinkmann, 2009), we found that the participants had a higher than expected productive disposition toward mathematics.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Teacher Beliefs; Teacher Education–Preservice

Purpose of the Study

The purpose of this study is to ascertain affective disposition of pre-service elementary teachers ($N = 33$) towards mathematics and gain a better understanding of the nature of this disposition. The participants were asked to develop their own novel metaphor for mathematics. They were provided with the topic "mathematics" and asked to choose their own vehicle to describe mathematics as well as to provide a rationale for their choice. By eliciting the rationale, we were able to determine which salient aspects of their chosen vehicle are to be ascribed to mathematics.

For purposes of this study, disposition is defined as a habitual inclination on a continuum of feelings, thoughts and behaviors toward mathematics. In this study, disposition toward mathematics is considered by the degree of its productivity. The guiding research questions are: (1) What are the most frequent affective characteristics of pre-service teachers' disposition toward mathematics? (2) To what degree is pre-service teachers' disposition toward mathematics productive?

Theoretical Framework

The theoretical framework of the study is grounded in sociotransformative constructivism and is guided by disposition and positioning theory. Positioning theory, as a framework, guides the study of social phenomena i.e. productive and/or nonproductive dispositions. To further focus and narrow the conceptual framework, the affective domain of disposition theory is used as the basis for examination and evaluation of teacher dispositions. Teacher and student affective disposition characteristics influence mathematical knowledge constructed; thus the importance of identifying and measuring affective dispositional characteristics is in the significance of its role in the learning process (Beyers, 2011). The positioning triad, Figure 1, represents the interactive components of the positioning process and the essence of the theoretical framework applicable to this study.

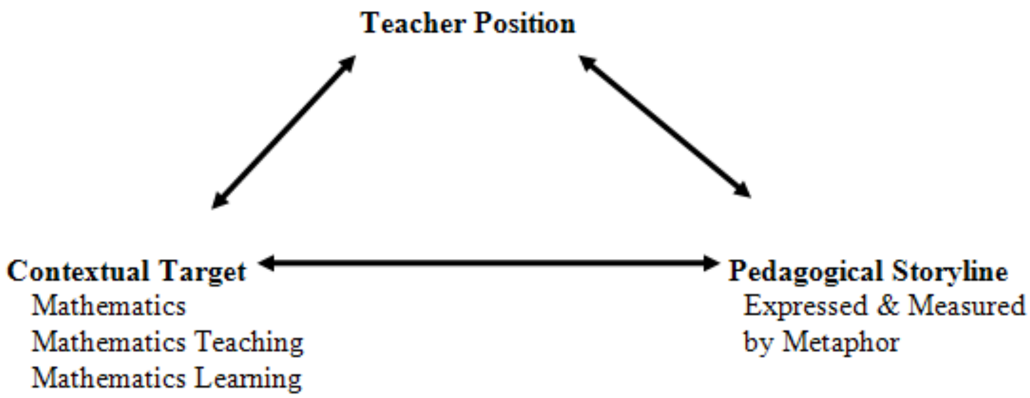

Figure 1: Positioning Triad

Source: (van Langenhove & Harré, 1999, p.18): modified to the framework of the teacher disposition study

Method of Inquiry

Metaphor Methodology in Studying Disposition

One way to gather self-reported assessment of disposition towards mathematics is through the use of a metaphor as a prompt. Metaphors are a vehicle that is easily understood by participants and encourages freedom of expression. In order to distinctly identify characteristics of disposition, the format of metaphors provides an indirect prompt allowing for free expression of feelings, emotions, and thoughts – positive or negative. For this study of disposition towards mathematics, participants were asked to develop a novel conceptual metaphor in response to mathematics (contextual target), such as "*Mathematics is like...*"

Participants

The thirty-three participants in this study were selected based on their membership in an undergraduate course titled *Teaching Mathematics in Elementary School*, offered at a university located in a binational/bicultural, border region of west Texas. All participants are considered to be senior education majors. Based on ethnicity, gender, and language, the sample of thirty-three participants would be considered a homogeneous sampling of pre-service educators.

The participants (self-reported) are predominantly female [94%]. A majority of the participants identified themselves as Hispanics [88%] and as bi-lingual English-Spanish speakers [79%]. Ethnicity and language abilities mirror the demographics of the region. Two participants indicated they are mono-lingual Spanish speakers and four participants indicated they are mono-lingual English speakers. 42% of participants indicated that Spanish was their primary language spoken. Of participants reporting years spent in schools in the United States, the mean number of years attending American schools is 13.7 years, with a range of 3.5 years to 19 years.

Data Collection and Analysis

Participants were asked to respond to an open-ended metaphor "*Mathematics is like ... Explain why.*" To maintain anonymity, each participant was assigned an alphanumeric code. Participants' responses to the open-ended metaphor were coded through linguistic and deconstruction analysis (Kvale & Brinkmann, 2009) and rated by two independent raters with regard to the degree of productivity of the pre-service teachers' disposition toward mathematics. Degree of productivity was measured using a 1–5 scale (1 – lowest productivity, 5 – highest productivity). Spearman's *rho* was calculated to report inter-rater reliability. The value of the reliability coefficient *.686 ($p<.001$)* indicates adequate inter-rater agreement. The participants' responses were also assessed by affective categories that were addressed in the metaphor. The categories are: nature of mathematics, usefulness, worthwhileness, and sensibleness of mathematics, as well as attitude, anxiety, and self-concept. Table 1 provides examples of affective dispositional statements characteristic of categories and levels of intensity.

Table 1: Affective Example Descriptors and Levels of Intensity

Rating Level		Example of *Mathematics is like…*	
		High Intensity	**Low Intensity**
1	**Highly Non-Productive**	*Mathematics is like a dule (dull- inserted) hair brush, because it makes me want to pull out my hair"*	*"Mathematics is like the unknown; not everyone knows math"*
2	**Non-Productive**	*"Mathematics is like a foreign language. Just when I feel like I am grasping it, I am shown something different"*	*"Mathematics is like something you are just not born for. I believe that every person has gifts and defects. Some are born to dance, some can write poetry, and I just believe that even though I can do some math, I wasn't born to be great at it"*
3	**Neutral**	*"Mathematics is like an adventure, because it always takes your mind to exploring different possibilities!"*	*"Mathematics is like a daily routine. There is always problems to resolve and you have to resolve them using the best strategy that you know and have on your hands, you learn from it and use it in future problems."*
4	**Productive**	*"Mathematics is like breathing. You do it every day, sometimes without even thinking."*	*"Mathematics is like second nature. I don't have as much of a difficult time, if I feel confident about it."*
5	**Highly Productive**	*"Mathematics is like air/oxygen. It's all around us and we can't seem to live without it."*	*"Mathematics is like the gateway to a better life. People who understand and do math well have more opportunities and better choices to make in life"*

Results and Discussion

The mean value for elementary pre-service teachers' disposition was at the level *3.41*, which was surprising to the researchers, considering a commonly held belief of elementary teachers' lack of content knowledge and avoidance of teaching and learning mathematics. Researchers believe that the "surprising" effect was due to the nature of the assessment instrument. An open-ended metaphor, as opposed to a closed-ended Likert-type scale question, allowed participants express themselves authentically and not locate their response to a predetermined scale.

The coding process was based on meaning coding and meaning interpretation techniques (Kvale & Brinkmann, 2009). Let's consider the following response from a Hispanic female pre-service elementary teacher (Survey Code # PS7): *"Mathematics is like something you are just not born for. I believe that every person has gifts and defects. Some are born to dance, some can write poetry, and I just believe that even though I can do some math, I wasn't born to be great at it."* Both raters assigned a level of "*2*" (nonproductive) on the (1–5) scale of productivity. This response reflected the stereotypical view of the existence of the "math gene phenomenon" that she refers to as *just not born for*. At the same time, the respondent recognizes the multiple intelligences (*gifts)* as positive characteristics, whereas the absence of "math gene" as a deficit (*defect).* Her final statement suggests a lack of perseverance toward learning success in mathematics by stating *that even though I can do some math, I wasn't born to be great at it.* The first portion of this final statement (*I can do some math)* contains an inclination toward self-efficacy, which deterred the raters from assigning a level of "*1*" (highly nonproductive) to this response. With regard to the affective categories, both raters agreed on the fact that this participant's response overwhelmingly reflected self-concept.

Let's consider another response from a Hispanic female (Survey Code # PS4): "Mathematics is like the gateway to a better life. People who understand and do math well have more opportunities and better choices to make in life." Taking into consideration a high inter-rater reliability coefficient, this response was independently rated at level "4" (productive) by one and "5" (highly productive) by the other. High productivity of this disposition was obvious to both raters because of the worthwhileness of mathematics (the gateway to a better life) as well as its usefulness (more opportunities and better choices). She also believes in sensibleness of mathematics and her self-concept (people who understand and do math well).

Conclusion

Disposition towards mathematics is under-researched in educational arenas, specifically with regard to pre-service elementary teachers' positioning toward the subject they feel less prepared to teach and therefore, avoid any encounter with it. The major intent of the proposed study is to address the gap in the research utilizing the lenses of positioning theory and disposition construct. Findings of this study, specifically a higher than expected productive disposition toward mathematics, may inform mathematics teacher educators in the development of learning cultures that recognize and address the positioning of pre-service teachers, and further guide them in the development of productive mathematical disposition.

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STUDYING ABSTRACT ALGEBRA TO TEACH HIGH SCHOOL ALGEBRA

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This study explores the transition from studying abstract algebra to teaching high school algebra in the context of an abstract algebra course designed for pre-service teachers. It addresses the question of how teachers develop mathematical knowledge for teaching using a case study approach. This research provides an empirical understanding of the role of mathematical content courses in teacher education programs. A preliminary finding highlights the importance of mathematical practices in the development of mathematical knowledge for teaching. By beginning to establish a research base on how teachers develop mathematical knowledge for teaching at the secondary level, teacher educators can begin to systematically and strategically incorporate these learning opportunities into teacher preparation programs.

Keywords: Mathematical Knowledge for Teaching; Post-Secondary Education; Teacher Education–Preservice, Teacher Knowledge

The transition from thinking about mathematics as a student to thinking about mathematics as a teacher marks a critical place in the trajectory of learning how to teach secondary mathematics. Pre-service teachers must develop mathematical knowledge for teaching (MKT) (Ball, Thames, & Phelps, 2008), which includes learning to use their mathematical content knowledge in ways that are distinct from how other educated adults use mathematics. Research has shown the importance of MKT in furthering student mathematical achievement (Ball, Lubienski, & Mewborn, 2001; Ball et al., 2008; Ma, 1999). However, the way in which secondary teachers successfully develop this knowledge remains an open question.

Developing Mathematical Knowledge for Teaching

Research shows that math majors or teaching licenses do not guarantee a teacher has sufficient mathematical knowledge for teaching (e.g., Ball et al., 2001; Monk, 1994). The German COACTIV study provides evidence that strong mathematical content knowledge (MCK) can support development of pedagogical content knowledge (PCK) in secondary teachers (Baumert et al., 2009). COACTIV uses a natural experimental design to demonstrate not only the importance of developing MKT (which includes MCK and PCK) in pre-service teachers, but also the important role that strong MKT plays in student achievement. There are many different, but complementary, conceptions of MKT (Hill, Sleep, Lewis, & Ball, 2007). This study takes the perspective that mathematical knowledge for teaching encompasses two domains: MCK and PCK (Ball et al., 2008; Baumert et al., 2009; Tatto & Senk, 2011), with strong MCK being a prerequisite for strong PCK (Baumert et al., 2009). MCK can be subdivided into knowledge of secondary and tertiary mathematics (common content knowledge) and specialized content knowledge (SCK). SCK is mathematical knowledge directly related to the practice of teaching (Ball et al., 2008). Both MCK and PCK are necessary parts of a teacher's knowledge.

Many mathematicians and mathematics educators have theorized about how pre-service teachers can develop the mathematical content knowledge they need to teach (Cuoco, 2001; Ferrini-Mundy & Findell, 2010; Hill, 2003; Stanley & Sundström, 2007; Usiskin, 2001; Wu, 2006; Zazkis, 1999). While each takes a slightly different view, they all agree that a traditional math major does not adequately prepare secondary math teachers. They emphasize the lack of connection between typical undergraduate mathematics courses and the mathematics taught in high school. However, these papers take a theoretical perspective and do not support their conclusions through empirical results. Therefore, evidence is still needed to address the question of how pre-service secondary teachers transition to developing MKT.

Mathematics Content Courses for Teachers

There are many possible settings for pre-service secondary teachers to learn and develop MKT. These include high school math courses, undergraduate math courses, math methods courses, and field experiences (Ball et al., 2001; G. Hill, 2003; R. Hill & Senk, 2004; Peterson & Williams, 2008). Learning in undergraduate math courses often suffers from what Cuoco (2001) calls the "vertical disconnect" (p. 3). Pre-service teachers are not able to see how higher level mathematics necessarily connects with the content they will be asked to teach. Learning through field experiences is idiosyncratic, as it is based entirely on the cooperating teacher attending directly to developing MKT (Peterson & Williams, 2008). While mathematics learning may occur in methods courses, focusing specifically on mathematical content is often beyond their scope. Since none of these settings are intentionally designed to emphasize content learning for teaching, MCK learning opportunities vary greatly. Mathematical content courses designed for pre-service secondary teachers potentially offer a focused opportunity to develop MCK.

R. Hill and Senk (2004) describe their process of developing a capstone course in mathematics for secondary teaching, which emphasizes secondary mathematics from an advanced perspective and aims to help pre-service teachers see the depth of content in the high school curriculum. Another style of course focuses on undergraduate mathematics while making explicit links to the high school curriculum. G. Hill (2003) describes how a pre-service teacher used ideas from an abstract algebra course to enhance student understanding of complex numbers in her field placement. Regardless of their form or content, math classes designed for pre-service teachers offer an excellent opportunity to examine the development of MCK.

Math content courses for pre-service teachers provide a possible site for developing MKT in general, and mathematical content knowledge in particular. This study examines one such course to address the following research question: How does a mathematical content course for pre-service secondary teachers influence their mathematical knowledge for teaching?

Methods

This study considers an abstract algebra course that makes explicit connections to high school algebra, a case of teaching undergraduate content specifically to pre-service teachers. Through written MKT measures, course observations, artifacts, and interviews with the professor and students, I will consider both the opportunities for developing MKT and learning outcomes. This case study provides insight into the effectiveness of a specific context in which MKT might develop by allowing for detailed qualitative analysis of opportunities for learning (such as homework problems focusing on "unpacking" a math concept [Ball et al., 2008]) as well as quantitative and qualitative measures of pre-service teacher MKT. The course took place at an east coast university within a teacher preparation program emphasizing mathematical content courses for pre-service teachers.

Data Sources

In order to identify opportunities to develop MKT in the class, I observed each class session and took detailed field notes. I collected artifacts such as the syllabus, assignments, and student work samples. I also interviewed the professor to discuss course goals and plans (if any) for creating opportunities for students to develop MKT. To document the student learning, all participants completed written pre- and post-measures of MCK focusing on algebra and geometry topics. To supplement the written measures, I interviewed participants to investigate their beliefs about teaching and learning math and to unpack their mathematical thinking. These semi-structured interviews help capture both MCK and PCK through the use of a think aloud protocol. Incorporating a range of measures helps form a more complete picture of the role of the course in developing MKT (Hill et al., 2007).

Data Analysis

To understand the opportunities participants have to develop MKT, data were analyzed qualitatively using both MCK and PCK lenses. For example, I coded field notes by identifying the math content (e.g.

algebra, geometry) and the level of content (secondary or tertiary). I categorized each learning segment as an opportunity to develop common or specialized content knowledge. I coded for connections between math and teaching (e.g., how to respond to a student error). I take these two lenses because while the course will primarily provide opportunities to develop MCK, it may also provide some opportunities to develop PCK.

After unpacking the opportunities provided for learning, I looked in greater detail at the changes in MKT that occurred after participation in the course. Data analysis was both quantitative and qualitative. The written MKT measure gave insight into the impact of participation in the math class. Analysis considered changes from pretest to posttest while controlling for factors such as previous math learning opportunities. Qualitative analysis of pre- and post-interviews used line-by-line coding to look for changes in the ways participants talk about and use mathematics. I coded for dimensions such as conceptual understanding and procedural fluency (Kilpatrick, Swafford, & Findell, 2001), as well as using mathematical practices, making connections between high school and college math, and making connections to the practice of teaching. Finally, I coded these interviews for links to the opportunities to learn identified in the analysis of the course itself. This helps clarify how students built on the opportunities to enhance their own mathematical knowledge for teaching.

Preliminary Findings

This research investigates the transition from thinking about mathematics as a student to thinking about mathematics as a teacher by considering how a math content course influences the development of mathematical knowledge for teaching. Participants in the course entered with varying levels of MKT and differing use of mathematical practices. As the class progressed, participants improved their ability to be mathematically precise, particularly in their use of language, construct rigorous mathematical proofs, communicate mathematical ideas, and persevere in problem solving. These improvements can be clearly linked to course emphasis on proof and communication. The professor also emphasized the importance of clear communication when teaching secondary math. To help participants develop these mathematical practices, the professor uses challenging mathematical tasks and a problem-based approach to teaching. This attention to mathematical practices suggests that they play a fundamental role in developing mathematical knowledge for teaching.

Implications and Directions for Future Research

This careful investigation of a mathematics class focused on developing mathematical knowledge for teaching offers an important insight into secondary teacher preparation. The ability to use mathematical practices plays a crucial role in the process of developing mathematical knowledge for teaching. This indicates that content courses should make practices an explicit part of the learning experience. By beginning to establish a research base on how teachers develop MKT at the secondary level, teacher educators can begin to more systematically and strategically incorporate these learning opportunities into teacher preparation programs. Since strong mathematical knowledge is a prerequisite for successful mathematics teaching, attention to its development, along with other aspects of teacher preparation, may improve the quality of mathematics teaching in this country.

While this study presents one case of focused mathematical content instruction, it is clearly only one of many possible cases. Future research might focus on the variety of ways in which teachers develop this knowledge. Other research will take the important step of following teachers with this type of preparation into the classroom, documenting their mathematical quality of instruction and connecting it to student learning. This research represents a first step toward a firmer understanding of where and how mathematical knowledge for teaching is developed.

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THE CONTINUUM OF PRE-SERVICE TEACHERS' MKT OF MATH STRINGS

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Twenty pre-service teachers in a mathematics methods class interpreted patterns embedded in math strings and responded to questions about representing the patterns and anticipating student's solutions. The pre-service teachers' responses fall along a continuum of specialized content knowledge and pedagogical content knowledge; furthermore, their knowledge varies according to whether they are asked to focus on the problems versus representations. These results highlight the types of mathematical content that methods courses must emphasize (along with possible contexts for introducing them) to help pre-service teachers support their students in developing key mathematical practices.

Keywords: Mathematical Knowledge for Teaching; Number Concepts and Operations; Teacher Education–Preservice; Teacher Knowledge

Together with the process standards put forth by the National Council of Teachers of Mathematics (NCTM) and the recent standards for mathematical practice, which are part of the Common Core State Standards for Mathematics (CCSSM), students face increasing pressure to make sense of and interact with the mathematics they learn, rather than memorize procedures. Three standards in particular require that students identify connections among problems (*Look for and make use of structure*, CCSSM; *Connections*, NCTM), model or represent relationships among problems using materials and drawings (*Model with Mathematics*, CCSSM; *Representation*, NCTM), and effectively communicate about mathematical relationships (*Attend to Precision*, CCSSM; *Communication*, NCTM) (National Council for Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). These mathematical practices are important for teachers to emulate and emphasize in mathematics instruction with students. However, it is challenging for pre-service teachers (PSTs) to learn how to implement student-centered instruction, especially given their own direct-instruction experiences as students.

One instructional method that emphasizes the process standards and that can help teachers make the shift to student-centered teaching is math strings (also referred to as number strings or minilessons) (Fosnot & Dolk, 2001; Parrish, 2011). Math strings are "a structured series of computation problems that are related in such a way as to develop and highlight number relationships and operations" (Fosnot & Dolk, 2001, pp. 105–106). For example, a teacher might choose a series of problems to highlight the distributive property (i.e., $2 \times 5 = 10$; $10 \times 5 = 50$; $12 \times 5 = 60$). During a math string lesson, the teacher puts the first of a series of problems on the board and asks students to solve the problem (usually mentally). Students then share their answers and solution methods. The teacher asks clarifying and prompting questions to ensure that all students understand each other's methods and to help students evaluate the similarities and differences among methods. Further, the teacher represents the students' thinking using models (such as empty number lines, arrays, and branching notation) so that students can reflect on their methods. This process then continues for the rest of the problems.

Although the structure of math strings is centered on patterns among computational problems and ways of solving problems, there is little research on how teachers—especially PSTs who may not be comfortable with multiple ways of thinking about a problem—make sense of the problems. Identifying the continuum of responses that PSTs make in their thinking about and use of math strings will provide teacher educators with insight into challenges and possible scaffolds for helping new teachers understand mathematics in relation to the process standards.

Theoretical Framework

The Mathematical Knowledge for Teaching (MKT) framework, described by Ball, Thames, and Phelps (2008) and which evolved from Shulman's (1986) seminal speech on the types of knowledge that teachers need to be successful, provides a useful lens for identifying the types of knowledge that PSTs need to develop in order to make sense of math strings and successfully foster productive practices in the mathematics classroom. For the purposes of this investigation around the use of math strings, two categories of knowledge are of particular importance: specialized content knowledge (SCK) and pedagogical content knowledge (PCK), which includes knowledge of content and students and knowledge of content and teaching.

Specialized content knowledge involves content knowledge that pertains specifically to mathematics teaching, like knowing how to capture students' strategies using multiple representations or explain mathematical relationships. Pedagogical content knowledge encompasses knowing how students might solve problems, identifying what difficulties they might have, deciding which examples to use in instruction, and weighing the pros and cons of using different materials or representations (Ball, Thames, & Phelps, 2008). In terms of math strings, teachers must move beyond focusing solely on answers and be able to determine the relationship among problems in the math string, including the strategy the string promotes. Further, they need to know for what problems the strategy works. All of these areas draw on teachers' SCK. Regarding PCK, when choosing additional problems for the math string, teachers must pick numbers that their students can handle but which also help draw their attention to the underlying pattern among the problems. Finally, they must understand what strategies students might use to solve the problems and how to use representations to highlight the relationship among problems (high PCK) and not just the problems themselves (low PCK).

The research question guiding this pilot study was as follows: What are pre-service teachers' levels of SCK and PCK regarding and as reflected in their analysis of math strings?

Methods

Participants and Setting

This study took place at a large Midwestern university. All 20 pre-service teachers (all female) from one section of a mathematics methods course participated. The PSTs were in their senior year and most were set to student teach in the semester following the course.

Materials and Data Collection

The pre-service teachers completed two worksheets, which required them to analyze several math strings, create problems to continue the strings, and draw representations to highlight the patterns of the strings. They completed one worksheet before and one after two class periods focused on the use of math strings. Due to the short period of instruction, though, the two worksheets are not likely to reflect significant learning.

Data Analysis

One set of questions from each worksheet was chosen for analysis (see Table 1). These two sets were chosen because they drew on different content knowledge (multiplication versus subtraction) and were likely to elicit different representations. First, I categorized each question according to the type of knowledge it targeted (see Table 1). Then, I categorized PSTs' responses according to whether they were correct, incorrect, or partially correct. I further coded PSTs' responses in each category according to what part of the problem they represented, and then I looked for themes to explain the differences in responses and knowledge of the math strings.

Table 1: Math String Questions and Knowledge Categories

Math String Question	Category
1a. Solve. $3 \times 4 =$ $6 \times 4 =$ $8 \times 6 =$ $8 \times 3 =$ $8 \times 12 =$ What relationship do you notice?	SCK
1b. What strategy or strategies is the math string promoting?	SCK
1c. How might a student solve the last problem (8×12) if he understands the pattern?	PCK
1d. Draw a picture/representation that you could use to help students see the relationships among these problems.	PCK
1e. Write 2 more problems that fit the math string pattern and explain how they fit.	PCK
2a. Solve: $35 - 10 =$ $34 - 9 =$ $37 - 12 =$ $31 - 6 =$ What do you notice? Why does this happen?	SCK
2b. Draw a representation that you could use to help students see why this happens.	PCK
2c. Does this strategy work for addition? Why or why not?	SCK

Results

SCK for Math String Construction

In general, responses that indicated low SCK focused on the patterns of answers for the problems. For problems 1a and 1b, 8 out of 20 PSTs (40%) focused solely on the answer. Common responses included identifying that all of the products are divisible by 12, are factors of 96, or that the answers are double previous answers. The latter response is an important part of a complete answer but fails to identify the pattern among the problems as well. An additional 5 PSTs (25%) focused on patterns in the problems *and* in the answers *without* explicitly connecting the two, moving them along the SCK continuum. For example, one person identified that the answers were all divisible by 12, and in the problems "4 is half of 8 and 3 is half of 6." Finally, 7 PSTs (35%) demonstrated high SCK. Their responses highlighted the relationship between the problems *and* the answers: "When the first number [in the problem] doubled, the answer doubled."

A similar pattern emerged for question 2a. Pre-service teachers with low SCK focused on the answer, indicating that the problems "each are a different way to get to the number 25." The PSTs with medium SCK focused on the relationship among the problems but were vague: "The number being subtracted from went up, so did the number being subtracted." As with the previous problem, PSTs with high SCK made an explicit connection between the relationships among the problems and what that means in terms of the answers: "From the original problem if the first number went up or down a certain way, the second number would too, so the difference stayed the same."

PCK in Relation to Math String Construction

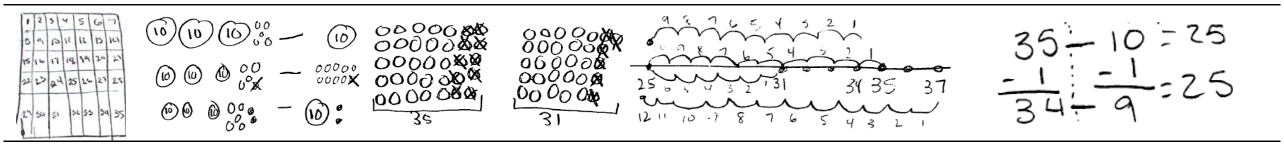
The pre-service teachers' responses to questions 1c and 1e were frequently based on their previous answers. For instance, if they identified that all answers were multiples of twelve in 1a (low SCK), they also thought the student would count by multiples of twelve to find the answer, and they suggested problems for 1e that resulted in a multiple of 12 (low PCK). However, 7 PSTs showed deeper PCK than they did for SCK. One pre-service teacher who said all answers were multiples of 12 on 1a provided much more detail when asked to think about how a student might solve the problem: "The students would double 48 to get the number 96. The student would realize that when one of the numbers in the previous problems is doubled, the answer is doubled."

PCK for Representations

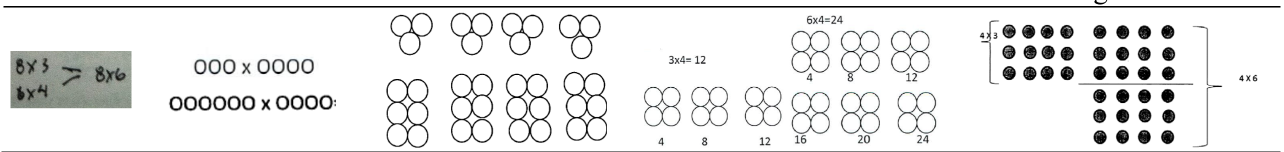
PSTs used a variety of representations for questions 1c and 2b along the PCK continuum. At the lowest level of PCK, pre-service teachers represented one problem instead of the relationship among problems or the relationship between the problem and representation was unclear (see Table 2a). Moving up the continuum, some PSTs substituted pictures for the numerals in the written problem (see Table 2b). Others correctly represented the answers but did not capture the relationship between problems in their representations. Responses in the middle of the continuum had the potential for highlighting relationships, but relationships were not explicit (see Table 2c). Towards the upper end of PCK, pre-service teachers represented the connection among problems (although sometimes focusing on secondary aspects) (see Table 2d & e).

Table 2: Pre-service Teachers' Representations Along a Continuum of PCK

a	b	c	d	e



Low PCK -- High PCK



Discussion

Overall, the math strings context required pre-service teachers to reason about multiple dimensions of SCK and PCK. The results of this study indicate that pre-service teachers have varying levels of SCK and PCK in relation to math strings, and they range in their ability to describe relationships among problems and represent these relationships. However, when asked to interpret the patterns in relation to students, they demonstrated deeper understanding. Future research should investigate the extent to which presenting content through the lens of their future students helps pre-service teachers internalize the mathematical practices they must promote in the classroom. Also, the use of math strings needs to be investigated further, not only in regard to benefits for pre-service teachers and their MKT but also in relation to students' learning.

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THE IMPACT OF PROGRAM AND COURSE ON PRESERVICE MIDDLE SCHOOL TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING

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This report follows from a study investigating factors that influence preservice middle school teachers' mathematical knowledge for teaching (MKT). Two such factors, with which there has been speculation, include the preservice teachers' certification program type and the college-level mathematics courses they take. Findings include significant statistical differences in MKT for preservice teachers in secondary preparation programs versus all others and for preservice teachers who took axiomatic geometry, the calculus sequence, differential equations, discrete mathematics, and probability versus other courses. I discuss the implications of these findings for mathematics teacher educators.

Keywords: Teacher Education–Preservice; Mathematical Knowledge for Teaching

Navigating the transitions of the professional learning continuum from the beginning preservice teacher to the teacher leader is very important. Preservice teachers are very impressionable as they are learning how to teach and master their subject. In doing so they need the vigilance of teacher educators.

For example, recently the National Governors Association (NGA) and the Council of Chief State School Officers (CCSSO) sponsored an initiative to establish a set of mathematics standards to "ensure that we maintain America's competitive edge, so that all of our students are well prepared with the skills and knowledge necessary to compete with not only their peers here at home, but with students from around the world." (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

There are two areas of middle school teacher education that should be investigated that can significantly influence preservice teachers' mathematical knowledge: (a) the type of certification program they're in, and (b) which college-level mathematics courses they have taken.

As preservice teachers are trained, one would assume that formal mathematics courses should be one important way to increase their content knowledge. However, studies have shown that simply taking college-level mathematics courses does not contribute to the knowledge needed to teach elementary and secondary school (e.g., Ball, 1991; Begle, 1979; Borko et al., 1992). Ball (1991) found that the courses do not provide the preservice teachers with "the opportunity to revisit or extend their understandings of arithmetic, algebra, or geometry, the subjects they will be teaching" (p. 24). Thus, the ways in which college mathematics courses affect U.S. preservice teachers' mathematics content knowledge also need further study.

Research Questions

1. How can preservice teachers' choice of certification program (i.e., elementary, secondary, or middle grades) affect their mathematical content knowledge?
2. How can preservice teachers' choice of college-level mathematics courses affect their mathematical content knowledge?

Perspectives

Three areas in mathematics education give perspective on this study: (a) the idea of mathematical knowledge for teaching, (b) the different types of middle school teacher certification programs, and (c) what constitutes a college-level mathematics course.

Mathematical Knowledge for Teaching (MKT)

Ball, Thames, and Phelps (2008) described MKT as “the mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students” (p. 399). Their perspective on MKT reflects my view of the areas of mathematical knowledge that preservice teachers need to foster. The MT21 items tap into three types of MKT.

First, Ball et al. (2008) interpreted *common content knowledge* (CCK) as the mathematical knowledge and skill that is used in situations other than teaching. Preservice teachers need to know well the actual mathematics that they were taught in their K–12 education, especially the mathematics at the grade level in which they will teach.

Second, *specialized content knowledge* (SCK) is the mathematical knowledge and skill that is used in teaching situations (Ball et al., 2008). It is the knowledge of mathematics above and beyond what the students need to know. This knowledge is a deeper, specialized, and more abstract version of the content than what is taught to the students.

Third, Ball et al. (2008) described *knowledge of content and students* (KCS) as being a combined knowledge of knowing mathematics and knowing students. This knowledge includes anticipating what students will think about mathematics, including their difficulties and misconceptions.

Certification Programs

With regard to certification programs, I used Schmidt et al.’s (2007) categories. First, middle school teachers of mathematics could be trained in elementary preparation programs and later teach in the (upper) elementary and middle grades. Second, middle school teachers could be prepared in secondary programs and teach sixth through twelfth grades. Third, middle school teachers could be trained in special middle grades programs that focus on sixth- through eighth-grade mathematics. Fourth, I noticed that some preservice teachers certified to teach at all three levels, elementary, middle, and secondary.

Because of the varied nature of these training programs, the required content courses and levels of instruction among these programs differed. Schmidt et al. (2011) showed that preservice teachers who were prepared in secondary programs were those who outperformed those preservice teachers who were prepared in elementary or special middle grades programs on the MT21 items.

College Mathematics Courses

My view of what constitutes a college-level course is also in line with what Schmidt et al. (2007) have designated. They stated that some courses could be considered as a topic of a course in some programs. In this study, I am assuming that the each course is taught similarly among the universities surveyed in the MT21 study.

Methodology

Sample

The sample that I will be using for this study is 381 preservice middle school mathematics teachers in the last year of their preparation program from 12 universities in eight states.

Data Collection

The data that used for this study came from the MT21 study (Schmidt et al., 2007). The data corpus consisted of the preservice teachers’ responses to 18 mathematics items (MT21 items) along with their responses to questions on their background information (e.g., gender, age, certification type, and number of mathematics courses taken).

Methods

To answer my first research question about the differences in the MKT of those in different certification programs, I performed the ANOVA statistical procedure using the preservice teachers’ certification program with their respective averages on all five mathematical domains together and separately.

Then, to answer my second research question about the differences in the MKT of those taking certain college-level mathematics courses, I performed an independent samples *t*-test for each mathematics course, which compared the means, of performance on all domains together and separately, between those who did not take the course and those who did.

Results

Certification Program and MKT

I found that those who were prepared in secondary programs or all levels in fact significantly outperformed those trained in elementary or solely middle school programs. They also significantly outperformed all other preservice teachers in each of the domains separately.

College-Level Mathematics Courses and MKT

The most influential courses on preservice teachers' knowledge of all domains surveyed were: axiomatic geometry, the calculus courses, differential equations, discrete mathematics, and probability because their mean differences were statistically significant in overall domain knowledge between those who did not take those courses versus those who took those courses.

Axiomatic geometry, abstract algebra, and the calculus courses are particularly effective because those who took those courses scored significantly higher means in at least three of the five domains than those who did not.

Other courses in which at least one domain had significantly higher means included: differential geometry for the function domain, topology for the number domain, linear algebra for the data domain, theory of complex functions for the data domain, and differential equations for the data domain.

The courses in analytic/coordinate geometry, non-Euclidean geometry, number theory, functional analysis, statistics, and history of mathematics did not have a significant difference on a preservice teachers' MKT. Additionally, one would think that the number courses would help with number knowledge, the algebra courses would help the algebra knowledge, etc., but for the most part they did not.

More particularly, certain courses had statistically significant higher means for those who did not take the course. Those who did not take differential geometry had a significantly higher knowledge of functions, likewise with abstract algebra and algebra knowledge, and theory of complex functions and data knowledge.

Discussion

In light of these results, we must consider whether or not we should require all our preservice middle school mathematics teachers to go through secondary programs. Experts caution against this action by saying that if we impose such a requirement, we will be left with few middle school mathematics teachers because the mathematics courses would be too demanding (Center for Research in Mathematics and Science Education, 2010). Our next recourse is with the courses taken by the preservice teachers.

With regard to mathematics courses, it seems that there are some courses that can be very beneficial for domain knowledge if one looks at significant differences in mean performance on a mathematical knowledge test. Thus, for example, those who take axiomatic geometry, the calculus courses, differential equations, discrete mathematics, and probability are likely to be better at middle school mathematics than if they did not take those courses.

It seems, however, that there are some courses that can have a negative impact on certain areas of mathematical knowledge at this level if preservice teachers take them. Similarly, Begle (1979) found that some teachers who took a certain number of courses past calculus had a negative effect on student performance. Ball and Bass (2000) attributed this negative effect on the compression of knowledge comes with increasingly advanced mathematical work. This compression is likely to interfere with the unpacking of content that teachers need to do, either for their students or for a test.

Therefore, we must consider the idea to require that preservice middle school teachers take more of the courses that will benefit their MKT. However, here is one area in which I believe we can help preservice teachers. If we ensure they take courses like calculus, geometry, and probability, their MKT should be strengthened. The other kinds of mathematics courses they should take are ones that allow them to revisit and extend their knowledge of arithmetic, algebra, and geometry—the very subjects they will be teaching (Ball, 1991).

Conclusion

As we consider the continuum of the evolution of preservice teachers' MKT as they become ready to teach, we find that there is much work to do. With recommendations for preservice teachers to take courses that increase their MKT the most, be it a combination of college-level mathematics and content-level mathematics, we will help preservice teachers know the mathematics and ensure that they can successfully implement curricula based on the Common Core State Standards.

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SUPPORTING TEACHER RETENTION AND DEVELOPMENT THROUGH TEACHING POSSIBLE SELVES

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It is a challenge of pre-service teacher education programs to prepare beginning teachers for many of the difficulties they will face in the teaching profession inside and outside of the classroom. The possible selves literature from psychology provides a lens for understanding how to help teachers sustain action towards successful classroom practice. In this paper, I will describe how possible selves is distinct from a similar construct, teacher vision, and outline recommendations for teacher education based in the possible selves literature that are absent from the teacher vision literature.

Keywords: Teacher Education–Preservice; Teacher Beliefs; Metacognition

Introduction

Enabling teachers to succeed is a challenge that faces the educational system. Teaching itself is a difficult, complex task that requires a large amount of expertise (Lampert, 2001; Lampert et al., 2010). Teachers are situated in a difficult environment that poses many challenges. One of the ways that the difficulties of the teaching profession manifest themselves is through teacher turnover and attrition. Teacher turnover is highest for beginning and highly experienced (usually retiring) teachers and attrition is found to be the highest during the first five years of teaching (Singer & Willet, 1988; Murnane, 1984). In particular, public schools serving high-poverty and/or low-achieving students are more likely to have higher turnover (Hanushek et al., 2004; Smith & Ingersoll, 2004). Additionally, it is often the best teachers that are likely to leave teaching (Murnane, Singer, & Willett, 1989).

The goal of teacher education programs can be thought of as helping teachers transition from pre-service to in-service teachers by preparing them have a long and successful career. An often overlooked aspect of pre-service teacher education is developing an identity as a teacher. In particular, a large part of the transition to becoming a successful classroom teacher includes developing a teaching identity and the practices associated with that identity. In contexts outside of teacher education, such as weight loss and academic achievement, identity-based interventions have been able to affect self-regulatory behavior (Oyserman et al., 2006; Murru & Martin Ginis, 2010). Given the difficulty associated with persisting in the teaching profession, this literature can help us discover new ways to improve teacher education and ease the transition from pre-service to in-service teacher. In the following section, I will review the literature on possible selves and identity-based interventions and discuss how this literature can supplement current research on teacher vision.

Theoretical Background

Possible selves is one way psychologists view the self-concept. Possible selves are a type of self-knowledge that "pertains to how individuals think about their potential and their future" (Markus & Nurius, 1986, p. 954). As with the dynamic self-concept (Markus & Wurf, 1987), individuals are thought to have multiple possible selves. Some possible selves relate to the people we aspire to be, sometimes referred to as positive possible selves. Other possible selves represent our feared selves, or negative possible selves. Possible selves link identity and motivation together because goals and fears are represented as part of the self-concept. Using this construct can also help us understand changes in actions or practices because “development can be seen as a process of acquiring and then achieving or resisting certain possible selves” (p. 955), which intimately relates possible selves with motivation and self-regulatory processes.

The construct of teacher vision is very similar to *teaching possible selves*. Hammerness (2006) defines teacher vision as “a set of vivid and concrete images of practice” and as “images of ideal classroom practice” (p. 1). She also explains how vision affects teaching practice:

> Vision shapes the way that they feel about their teaching, their students and their school and helps to explain the changes they make in their classrooms, the choices they make in their teaching, and even the decisions they make about their futures as teachers. (Hammerness, 2006, p. 2)

In this way, vision guides the decisions teachers make.

A teacher’s vision is similar to teaching possible selves in many ways. First, both talk about future, possibly achievable selves or roles. Both constructs are seen to have a large impact on the actions that individuals take. However, teacher vision differs by only representing the positive or ideal teaching practice, while possible selves can be both positive and negative. Additionally, an individual is thought to have multiple possible selves, beyond just positive and negative, whereas teacher vision is conceptualized as a singular coherent view of teaching practice. For example, teaching possible selves could include images of practice that are vastly different depending on the context they are working in, but teacher vision would not explain these “inconsistencies.”

Oyserman et al. (2006) assert that possible selves are more likely to be achieved (i.e. sustain self-regulatory action) if they are connected to social identity, linked to concrete strategies to achieve them, cognitively accessible, and balanced between positive and negative. Possible selves that are incongruent with important social identities are less likely to be created and maintained (Oyserman et al., 2006), making possible selves are more achievable when they are connected to social identity. Social identity may include group membership based on race or ethnicity, but may also include professional affiliation. Individuals are also more likely to apply behavior towards a possible self when they have strategies to achieve them, and when those strategies are cued by the context (i.e. the possible selves are cognitively accessible). Finally, both positive and negative possible selves are important for obtaining desired outcomes, but recent research indicates that having balanced possible selves may not be optimal in all contexts. Oyserman and Destin (2011) show that the fit between context and valence of possible selves plays an important role in planned behavior. They argue that in success-prone contexts, positive possible selves and in failure-prone contexts, negative possible selves are more likely to lead to desired self-regulatory behaviors than positive possible selves in failure-prone contexts.

With respect to pre-service teacher education, Hammerness (2006) suggests that teachers have opportunities to surface their visions initially and continually reexamine their vision throughout their teacher preparation program. Pre-service teachers’ visions are somewhat vague, disconnected from the subject matter they will teach, and inconsistent. Commonly, teachers begin pre-service teacher education programs with a vision for instruction that is largely traditional and based on their experience in schools. In the reform context, teacher preparation not only needs to help teachers clarify their existing vision, but it should also challenge teacher vision and provide new possibilities of ideal teaching practice. Hammerness (2006) also argues for the importance of equipping teachers with tools and strategies to manage the gap between their current teaching practice and ability and their vision.

Implications for Teacher Education

Three main differences between the implications from the possible selves literature and the teacher vision literature are discussed below in terms of recommendations for teacher educators.

Congruence with Social Identity

In the possible selves literature we see the importance of relating teaching possible selves to social identity. With teacher vision we do not see explicit mention of ideas related to shared identity. With respect to teaching, various social identities may come into play. Racial and ethnic social identities of teachers may seem relevant to teaching depending on the student and teacher populations found in the school. Social identities related to the teaching profession, such as associating with reform teachers or teachers concerned with social justice. Each of these social identities may be congruent or incongruent with teaching possible selves.

Ideally teaching possible selves would be congruent with all social identities that are relevant in teaching. In the case of teaching possible selves being incongruent with a teacher's racial and ethnic social identities, Oyserman et al. (2006) suggests trying to increase the congruence between possible selves and social identity instead of attempting to disconnect teachers from social identity. Teacher educators may wish to be aware of the different social identities pre-service teachers identify with and attempt to show congruence between these identities and the teaching possible selves they are developing.

Teacher educators can also attempt to relate teaching possible selves and social identities about the teaching profession. Teacher education may already support these links because pre-service teachers often take the same classes and share similar visions. Teacher educators can also help pre-service teachers develop professional social identities in tandem with teaching possible selves by exposing them to professional organizations. Even mentor-student teacher relationships could be leveraged to develop affiliation with reform or other teaching groups.

Incorporating Both Positive and Negative Possible Selves

One significant difference in implications is in considering positive and negative possible selves. In the case of possible selves, a balance (or a context-dependent emphasis) of negative and positive possible selves is ideal. In the case of teacher vision, we only see teacher educators developing ideal visions, or positive possible selves. Because teaching can be considered a failure-prone context, negative teaching possible selves are a useful tool in teacher education.

Pre-service teachers have pre-existing ideas of what good and bad instruction looks like from their experience as a student, which are not likely to give pre-service teachers an example of a reform vision of teaching. Mathematics teacher educators often take part of their mission as helping teachers become familiar with reform teaching and possibly developing a vision of teaching that is consistent with reform principles. Developing new visions is also supported by the possible selves literature, but it may not go far enough.

Teachers are rarely given opportunities to clarify negative or feared visions of teaching. For example, teachers may initially have a positive vision of teaching that incorporates elements teacher educators would hope would not be part of the vision after a teacher preparation program. One could imagine that teachers' vision might include using clear lectures to change student ideas. Without explicitly considering feared visions of teaching, this element may stay associated with positive visions of teaching.

Two mechanisms might be able to explain how developing negative teaching possible selves could lead to more successful teaching practice. Having a clear negative vision of teaching may change teacher noticing (Sherin 2001, 2007) by enabling teachers to notice elements of teaching practice in their negative vision. Teachers may also interpret these practices as something they do not want to do. Additionally, teacher reflection on practice may be influenced. A teacher may focus reflections on elements that are explicit in their visions, whether positive or negative.

Strategies for Achieving Possible Selves

Both constructs examine the need for concrete strategies linked to possible selves or visions, and in the case of teachers this is often conceptualized in terms of strategies to allow teachers to move from their current teaching toward their ideal practice. However, the teacher vision literature does not emphasize that the strategies should be explicitly tied to the vision and be made cognitively accessible, as with the possible selves literature. In addition, making sure that teachers are aware of the entire trajectory of growth toward their vision is not emphasized.

As teacher educators, we often focus only on progress that we would like our teachers to learn by the end of the program, rather than the progress we would like them to make through their beginning years of teaching. Making sure that teachers have concrete strategies for achieving their possible selves could include learning progressions for teachers or progressions of teaching practices that extends beyond teacher preparation programs. These learning progressions should be explicitly tied to teacher vision as well.

Strategies for achieving teacher vision could be seen as strategies for responding to challenges and dilemmas teachers face. As was mentioned in the beginning of the paper, teachers face numerous challenges in the classroom as well as outside the classroom, particularly in low-achieving schools. Having strategies to achieve possible selves can be seen as having strategies for overcoming the specific obstacles that arise when attempting to enact teacher vision.

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EXPLORING CHANGES IN PRESERVICE TEACHERS' CONCEPTIONS WITHIN THE CONTEXT OF MATHEMATICS EXPERIENCES

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This paper reports findings from the Mathematics Experiences and Conceptions Surveys (MECS) implemented at four universities across the United States. Preservice elementary education teachers enrolled in mathematics methods coursework completed pre/post surveys designed to understand the evolution of conceptions towards mathematics teaching and learning. Our overall aim was to determine what, if any, changes occur in conceptions and whether or not particular experiences during the course help explain such change. ANOVAs indicate significant differences in specific aspects of conceptions. These differences are explained using multiple regression analyses, in part, by mathematics methods and field experiences scales within MECS.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Teacher Beliefs; Teacher Education–Preservice

Objectives of the Study

This paper reports preliminary findings from the Mathematics Experiences and Conceptions Surveys (MECS) implemented at four universities across the United States during the fall 2011 semester. Preservice elementary education teachers (PSTs) enrolled in mathematics methods coursework completed pre/post surveys designed to understand the evolution of attitudes, beliefs, and dispositions towards mathematics teaching and learning. The overall aim of this research was to determine what, if any, changes occur in attitudes, beliefs, and dispositions over the duration of a mathematics methods course and whether or not particular experiences during the methods course help explain such changes. While a number of surveys exist that measure beliefs and attitudes, none are contextualized to various points in teacher education, or account for experiences within mathematics teacher preparation to the extent found in the MECS (see Welder, Hodges, & Jong, 2011, for a more detailed account of MECS instrumentation). Consequently, our focus adds to the knowledge base in mathematics teacher education by observing factors within teacher education that influence dispositions, beliefs, and attitudes towards mathematics teaching and learning. This knowledge might then be used to leverage learning opportunities within mathematics teacher preparation.

Perspectives

Decades of mathematics education research suggests the strong role beliefs and attitudes play in influencing the instructional practices teachers use within the classroom (Ball & Cohen, 1999; Ernest, 1989; Richardson, 1996; Wilkins, 2008), and students' opportunities to engage in significant mathematical thinking (Fennema et al., 1996; Staub & Stern, 2002). Given the significant role beliefs and attitudes play, researchers and teacher educators often look for opportunities to bring into focus conceptions of mathematics within methods courses and other important experiences within the continuum of teacher education (e.g., Charalambous, Panaoura, & Philippou, 2009; Quinn, 1997). We use the term *conceptions* as an umbrella to represent three central and interrelated subconstructs: dispositions, beliefs, and attitudes (cf. Welder, Hodges, & Jong, 2011).

Prior research indicates mathematics methods coursework may in fact shift attitudes and beliefs about mathematics teaching and learning to better align with aspects of reform-oriented recommendations in mathematics education (Connor, Edenfield, Gleason, & Ersoz, 2011; Philipp et al., 2007; Roscoe & Sriraman, 2011). Perhaps the most detailed account of experiences in teacher education and their relationship to beliefs about mathematics teaching and learning comes from Philipp et al. (2007), who observed that PSTs with field experiences that focused more on children's mathematical thinking developed more sophisticated beliefs about mathematics teaching and learning than PSTs whose field experiences did not focus on children's mathematical thinking. From our perspective, this is only one, albeit critical, aspect often present in mathematics methods coursework. Taylor and Ronau's (2006) investigation of mathematics methods syllabi indicates a wide array of goals and activities present in methods courses, with varying degrees of empirical evidence supporting such emphases. Consequently, we seek to better understand the relationship between particular mathematical experiences in elementary teacher preparation and the evolution of beliefs, attitudes, and dispositions towards mathematics teaching and learning.

Methods

The data presented here include ninety-one PSTs enrolled in mathematics methods courses at four universities in the Eastern United States during fall 2011. MECS-M1, a survey designed to be taken at the beginning of mathematics methods coursework, was administered during the first week of class in each of the mathematics methods courses and MECS-M2, a survey designed to be taken at the end of mathematics methods coursework, was administered during the final week of each of the same courses.

MECS-M1 is designed to measure constructs related to preservice teachers' past K–12 experiences in mathematics, entering beliefs about mathematics, dispositions toward teaching mathematics, and attitudes toward mathematics. MECS-M2 is designed to measure constructs related to preservice teachers' fieldwork experiences in mathematics, experiences in the mathematics methods coursework, beliefs about mathematics, dispositions toward teaching mathematics, and attitudes toward mathematics. The two instruments, which primarily consist of five-point scale Likert items and a few open-ended questions, were created with similar constructs to avoid a form of single-method bias and to measure growth after a mathematics methods course and over time. Exploratory factor analyses were completed to examine psychometric properties of the instruments. Reliability was examined in terms of the instruments' internal consistency using Cronbach's alpha. Table 1 provides alpha levels for MECS-M1 on beliefs, attitudes, and dispositions. Alpha levels for mathematics methods course experiences and field experiences are reported from MECS-M2. While other MECS subscales exist, we focus explicitly on the subscales in these analyses.

Table 1: Reliability of Subscales

Construct	*α-level*	*Instrument*
Attitudes toward Mathematics	.875	MECS-M1
Beliefs about Mathematics	.732	
Dispositions toward Teaching Mathematics	.870	
Dispositions: Instruction	.659	
Mathematics Methods Course Experiences	.938	MECS-M2
Mathematics Methods: Students	.850	
Mathematics Methods: Materials	.693	
Mathematics Methods: Pedagogy	.748	
Field Experiences	.825	

To determine change in attitudes, beliefs, and dispositions from MECS-M1 to MECS-M2, an analysis of variance (ANOVA) was conducted on each of the three subscales. To examine whether significant relationships existed among the subscales in Table 1, Pearson correlations were used (but not included here) to help determine which explanatory variables to include in the multiple regression models. Then we developed multiple regression models to examine which factors could explain the variation in attitudes, beliefs, and dispositions towards instruction.

Results

There was a statistically significant difference between *attitudes_pre* and *attitudes_post* as determined by an ANOVA (F (3,86) = 1.03, p = .001). There was not a statistically significant difference between *beliefs_pre* and *beliefs_post* (F (3,86) = 1.25, p = .266). The dispositions subscale was disaggregated to three separate scales related to dispositions, but our analysis focused on dispositions towards instruction (*disp_inst*) because it is an indication of PSTs' stance on reform-oriented practices. However, there was not a statistically significant difference between *disp_inst_pre* and *disp_inst_post* (F (3,86) = 0.15, p = .704).

Using the significant correlations and our understanding of the literature on mathematics methods courses and field experiences, multiple regression models for attitudes_post, beliefs_post, and disp_inst_post were developed. The overall regression of *attitudes_post* on *attitudes_pre*, *field_exp*, and *MM_mat* was statistically significant [R^2 = 0.59, F (3,87) = 44.93, p < 0.001]. The three factors accounted for 59% of the variance in PSTs' attitudes following the mathematics methods course, but 52% of that variance was accounted by *attitudes_pre*. The overall regression of *beliefs_post* on *beliefs_pre* and *MM_mat* was statistically significant [R^2 = 0.30, F (2,88) = 20.27, p < 0.001]. The two factors accounted for 30% of the variance in PSTs' beliefs following the methods course, but 23% of that variance was accounted by *beliefs_pre*. The overall regression of *disp_inst_post* on *disp_inst_pre*, *field_exp*, and *MM_mat* was statistically significant [R^2 = 0.42, F (3,87) = 20.96, p < 0.001]. The three factors accounted for 42% of the variance in PSTs' instructional dispositions following the methods course with 20% of that variance accounted by *disp_inst_pre.* The mathematics methods materials (*MM_mat*) subscale included items about resources that promote the development of students' mathematical thinking; thus, it is not surprising to see that it accounted for significant changes in PSTs' conceptions toward mathematics teaching and learning. We acknowledge that this subscale had a lower reliability and that further analyses are needed to accurately detect whether the relationship is important and remove noise in the results.

Discussion

Beliefs are felt more intensely and tend to be more cognitive than attitudes (Philipp, 2007); thus, it is anticipated that changes in attitudes might occur over a shorter time interval than beliefs or dispositions. Dispositions are tendencies to act in specified ways and take on particular positions (Bourdieu, 1986), such as how PSTs position themselves in relation to reform-oriented recommendations in mathematics education. Given the shortened timeframe between MECS-M1 and MECS-M2 (one semester), we were encouraged by a significant growth in attitudes, while not surprised that beliefs and dispositions toward instruction changed little over the course of the semester. Clearly, it is critical that participants continue to complete future iterations of MECS instrumentation to determine if in fact beliefs and dispositions change longitudinally within teacher education and whether or not attitudes serve as a precursor to changes in beliefs and dispositions. A longitudinal approach will allow us to capture the evolution of PSTs' conceptions as they transition along the continuum of teacher education. In the future, we plan to continue gathering MECS data on participants as they enter into the early induction phase of teaching to further examine conceptions during such an important period.

While the variation in attitudes, beliefs, and dispositions towards instruction apparent in the regression models is strongly tied to entering attitudes, beliefs, and dispositions, it is hopeful that we are beginning to understand the relationship between mathematics methods course experiences, field experiences, and their connection to conceptions. In particular, significant relationships between PSTs that perceive an emphasis on materials for mathematics instruction focused on developing students' mathematical thinking and the constructs of attitudes and dispositions is an important first step in understanding what types of experiences bring conceptions into focus. The implications of such work is two-fold: (a) MECS instrumentation has proven useful for understanding changes in conceptions in relation to benchmark mathematical experiences within teacher education programs; and (b) MECS might well serve as a useful tool for programmatic assessment, comparisons between institutions, and in the context of design experiments within teacher education.

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TURNING BELIEFS INTO ACTIONS: A TEACHER'S ROLE IN THE CLASSROOM TO SUPPORT DEVELOPING MATHEMATICAL LEARNING

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Although teacher beliefs have a positive impact on student learning and achievement, it is the teacher's practice that makes the difference. Student communication through dialogue is an effective strategy to encourage mathematical understanding. Qualitative data from teacher interviews and classroom observations of a Grade One teacher uncover how this teacher viewed her role in the mathematics classroom and how she enacted upon these beliefs in her practice. Findings show that the teacher viewed her role as a guide, created a positive learning environment and encouraged student dialogue by modeling mathematical communication and empowering students to participate in classroom activities.

Keywords: Classroom Discourse; Elementary School Education; Instructional Activities and Practices; Teacher Beliefs

Objectives

In the reform-based classroom, teachers move away from the traditional role of knowledge-provider and transform their classroom into a student-centered environment. Teachers, especially those who have been teaching for many years or those who were taught using traditional methods, may not know what the current role of the teacher should be and how they fit into today's student-centered learning environment (Ball, 1996; Raymond, 1997; Tzur, Martine, Heinz, & Kinzel, 2001). The purpose of this study is descriptive: it focuses on a teacher's beliefs about her role as a teacher of mathematics and how she acts on these beliefs in the classroom.

Perspectives

Teacher perception can affect student learning and student achievement (Bruce & Ross, 2008). Teachers who believe that their teaching practices affect student learning and achievement positively are more willing to implement new teaching strategies, take risks with their instructional practices and work to achieve student goals (Bruce & Ross, 2008). Research has shown that this type of teacher contributes to an increase in student achievement (e.g., Bandura, 1997; Brouwers & Tomic, 2001; Henson, 2002; Ross, Bruce, & Hogaboam-Gray, 2006; Tschannen-Moran & Hoy, 2001). These teachers also affect student perceptions of their own abilities (Ross, 1998).

In the reformed classroom, the role of the teacher is very important. Instead of solely being the knowledge provider, they are now seen to be guides helping navigate their students in the discovery of knowledge (Zack & Graves, 2001). Students need to create their own understanding by linking new ideas to previous personal experiences, thus, teachers cannot impose their own comprehension of ideas onto their students. As guides, they can take a step back and encourage students to reach back to their prior knowledge to form the connections for themselves (Alagic, 2003).

Teachers must also facilitate student thinking by encouraging dialogue. They should ask students to talk through their ideas, sort through confusions and explain their understanding of a concept (Alagic, 2003). If students have difficulty with this, teachers should be able to give suitable prompts or even model the appropriate behaviour themselves (Jansen, 2006). Students should also be expected to listen to other students and reflect upon the comments that are shared. Teachers can encourage students by asking probing questions and leading students to look for patterns (Reys, Suydam, Lindquist, & Smith, 1998).

The teacher needs to develop a sense of community within the walls of their classroom, so that the students feel safe and able to explore freely. Without this climate, students will not reach out to the teacher to ask for help nor will they be able to completely immerse themselves in the variety of tasks presented to them (Zack, 1993). If the students do not feel comfortable in the classroom or in classroom discussions, they will not be as willing to take risks and try different approaches to solving problems (Pape, Bell, & Yetkin, 2003).

Method

This paper is built upon work from the School Improvement in Mathematics study (McDougall, Jao, Yan, Kwan, 2011) for teachers wishing to improve their mathematics teaching practices through the lens of the Ten Dimensions of Mathematics Education conceptual framework (McDougall, 2004). In this study, teachers engaged in a peer coaching model of professional development to collaboratively improve self-selected elements of their teaching practice (McDougall et al., 2011). Alice was chosen as a case study (Stake, 1995) about teacher beliefs and practices because of her enthusiasm to improve her professional practice, willingness to reflect on her intentions as an educator and openness to having researchers in the classroom.

Alice is a Grade One teacher with 25 years of teaching experience at the Kindergarten, Grade One and Grade Two levels. All of her teaching experience has taken place in Canada, first starting in Central Canada and later moving to Western Canada.

Data Collection and Analysis

The School Improvement in Mathematics study took place between September 2006 and June 2008. Data was collected for the study through teacher interviews, peer coaching sessions and classroom observations. Alice was interviewed at the beginning of the study to determine her background as en educator, beliefs about teaching, goals as an educator and ideas of success for her students. Peer coaching sessions involved classroom observations and pre- and post-lesson interviews. Classroom observations allowed the researchers to see the types of teaching strategies that Alice used. Following each classroom observation, Alice was interviewed and questions were asked about teaching strategies observed and her rationale for using these strategies. A final interview asked questions about her teaching practices and final reflections as a participant of the School Improvement in Mathematics study.

There were six sets of interviews in total. All interviews were audio-recorded and transcribed. These transcripts were then coded using a series of coding cycles. An open coding (Strauss & Corbin, 1990) format was used for this study. Initial data was coded using two general categories (teacher beliefs and teacher practices) and subcategories were chosen based on emerging themes from subsequent data.

Findings

The data indicate that Alice's beliefs about her role as a teacher of mathematics is closely aligned with her practices. The findings show that Alice is cognizant of her role as a teacher in the classroom and has come to realize the effect that she has on her students. Her belief is that she is a role-model to her students. She believes that her students will pick up on any cues that she gives regarding how to interact with materials and learn a concept. She enacts on her beliefs by using modeling to demonstrate appropriate classroom behaviour and foster mathematical understanding among her students.

Alice uses modeling to support student learning in two different ways. Sometimes, she models an approach to solving the problem in hopes that her approach will be enough of a push to get them to think creatively and to come up with their own ideas. Other times, she asks the students to act as the modeler. If Alice sees that a student has come up their own connection or strategy, she will often ask the student to share their idea with the rest of the class. Alice states that, by having one student share with the entire group, "you may have some more encouragement to get other kids to give their ideas" (Teacher interview, January 23, 2008). By sharing their ideas, students also have the chance to clarify their learning. By getting the students to express their ideas, the students will have to organize and clarify their thoughts and this creates a deeper understanding of the material. The students will have had to truly learn the mathematics concepts to be able to share and voice their ideas.

Giving students a voice is a teaching strategy that Alice often applies to help her students learn mathematics concepts. By empowering students and giving students a chance to have their voice to be heard, students take ownership of their learning, are eager to learn and help make mathematics meaningful. Alice believes that it is important to empower her students and to give them a feeling that “I contributed to that, that was my contribution” (Teacher interview, January 23, 2008). Sharing also gives students a chance to hear the ideas of their peers. These ideas may be ones that the students have not considered themselves. Activities where student tell stories, act out scenarios and incorporate their own drawings into the activity are all examples of how Alice involves her students and gives them ownership of their learning.

Alice creates an environment in her classroom where students can concentrate on learning mathematics. She has developed a level of respect amongst the students by positively reinforcing good behaviour and modeling appreciation statements. Alice encourages her students by giving them praise or doing something as simple as giving them a handshake for doing good work. Alice also models behaviour for the students in the class to follow. She will often applaud a student after they have shared their idea and towards the end of a lesson the students often applaud their peers without being prompted to do so. The students feel safe to experiment and take risks to create their mathematical understanding. She also wants her students to view mathematics and the learning of mathematics as something fun.

Conclusion

This case study is one example of how a veteran teacher views her role in the mathematics classroom to meet reform-based trends. Although teacher beliefs and perception has a positive impact on student learning and achievement (Bandura, 1997), it is the teacher’s practice that makes the difference. The case of Alice shows that she chose to use teaching strategies to align with her beliefs that she is a role-model for her students.

Alice’s Grade One students are in an early stage in their mathematical education. They are still learning the language of mathematics and need help to communicate their understanding. Alice’s actions echo the work of Zack and Graves (2001). She acts as a guide by modeling mathematical communication to encourage her students to begin to communicate their own emerging mathematical understanding. In parallel with Osterman (2000), by creating a safe environment, Alice’s students feel comfortable to participate in classroom activities. Additionally, the opportunity that Alice provides for her students to talk through their understanding supports their mathematical learning (Jansen, 2006).

In conjunction with her ongoing professional development to improve her teaching practices, this study allowed Alice to reflect on her role and how she could best facilitate student learning (McDougall et al., 2011). This case study allows others in the field to learn from Alice’s story to further improve their own mathematics programs.

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LEARNING TO LEAD MATHEMATICALLY PRODUCTIVE DISCUSSIONS

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Five practices are thought to help preservice teachers learn to facilitate mathematically productive discussions in elementary classrooms: anticipating student responses, monitoring student work, selecting responses to share, purposefully sequencing the sharing of those responses, and connecting responses to other mathematical ideas (Stein, Engle, Smith, & Hughes, 2008). This paper describes a self-study in which the authors encouraged teachers to use the five strategies as a framework for instruction. Data from one representative participant show that she developed the ability to anticipate student responses and identify key ideas to highlight during the discussion, but did not enact effective monitoring, sequencing, or connecting responses to other mathematical ideas. We conclude that specific changes to our instructional practices might help our students strengthen their understanding and use of the five strategies.

Keywords: Teacher Education–Preservice; Teacher Knowledge; Instructional Activities and Practices; Classroom Discourse

Mathematics teachers in the United States typically follow a conventional lesson structure that includes showing students how to solve a particular type of problem, providing time for individual practice, and then checking students' answers in a short discussion (Stigler et al., 1999). Though the *Common Core State Standards for Mathematical Practice* recommend that students engage with mathematics through problem solving, modeling, and reasoning (Common Core State Standards Initiative, 2010) many teachers continue to rely on the conventional script. Achieving the Common Core's vision for instruction can be especially challenging for preservice and novice teachers who have not yet developed strong pedagogical content knowledge (Shulman, 1986) or specialized mathematical knowledge for teaching (Ball, Hill, & Bass, 2005). As instructors of preservice teachers in a graduate-level math methods class, we model and encourage problem-based teaching but recognize the difficulties many preservice teachers face when attempting to implement this kind of instruction during school-based fieldwork. Inspired by Stein, Engle, Smith, and Hughes' (2008) belief that preservice teachers can learn explicit practices for facilitating productive mathematical discussions, we integrated the practices into our own teaching and for the first time in our math methods courses, we asked our preservice teachers to use the strategies when planning and teaching three focus lessons.

Learning to Teach Mathematics

To address predictable difficulties that arise as preservice teachers learn to implement discussion-based teaching, Stein et al. (2008) developed a model to explicitly teach novice practitioners how to lead productive discussions about challenging tasks. By emphasizing components of this type of teaching that can be planned in advance, the model is designed decrease the difficult and sometimes intimidating improvisational nature of leading such discussions. Briefly, the five practices are

> (1) anticipating likely student responses to cognitively demanding mathematical tasks, (2) monitoring students' responses to the tasks during the explore phase, (3) selecting particular students to present their mathematical responses during the discuss-and-summarize phase, (4) purposefully sequencing the student responses that will be displayed, and (5) helping the class make mathematical connections between different students' responses and between students' responses and the key ideas. (p. 321)

We selected this frame because it unpacks the strategies that are implicit in the instructional examples of master teachers that we share with our students (i.e., Ball, 1993; Burns, 2002; Lambert, 2001; Toliver, 1995). When our students read descriptions or watched videos of master teachers at work, the gulf between what they observed and the reality of their own math teaching was wide. With the intent of examining that divide, our guiding question for this study was, in what ways did our preservice teachers describe using the five practices? We hoped the answer to that question would help us understand, transform, and reform our own practice to ultimately improve our students' teaching.

Methodology

The participants in this qualitative study were graduate students in the second of four semesters in an elementary education licensure and masters degree program. Of the 22 students enrolled in a math methods course taught by one of the authors, 12 (11 women and one man) gave consent for analysis of their written work. Participants' ages ranged from approximately 22 to 55, and their ethnicities include white (10), Asian American (1), and Arab American (1).

The data for this study consist of assignments submitted for the course, including in-class writing prompts, summaries of three lesson plans, and reflective narratives. Due to limitations in the program model that did not allow us to observe preservice teachers in the classroom, we relied on written descriptions of their teaching as evidence of their understanding and use of the target strategies. To ensure participant anonymity, papers were relabeled with pseudonyms.

Data were coded using the five target strategies as categories and mapped to show how (or if) individuals changed over time. From the coded data, we drew conclusions about how the participants envisioned and used the practices at several points in the semester. The limitations of this method are potential researcher bias, as we hoped that the students would describe using and identifying with the five practices. To minimize the bias in analysis, we analyzed the data separately and compared our findings, grounding our conclusions in the participants' comments.

We used faculty self-study (Samaras & Freese, 2006) as a methodology for examining our own teaching practices. Through this self-study cycle of planning and revising a lesson plan assignment, we aimed to improve our own understanding of how our preservice teachers conceptualized and incorporated the five strategies into their practice. Analysis of our data will inform the next self-study cycle, during which we will modify our teaching based on what we learned in this initial cycle.

Findings

The findings described below are from an analysis of one representative participant, Tara, who was selected because her descriptions were typical of most respondents.

Anticipating Likely Responses to Cognitively Demanding Tasks

One month into the semester, Tara incorrectly listed this strategy as the third step in the sequence, "so that you can be ready to answer or use them to pose new questions to the class." Her first lesson plan reflected inaccurate anticipations about students' engagement with the task, the parts that would need scaffolding, and how to keep the lesson moving, but included little mention of math content. In the second lesson plan, she anticipated that students would have difficulty understanding the game she had selected and planned to scaffold for related mathematical concepts, including providing multiple representations of the target content and making explicit connections to prior activities. She anticipated cognitive dissonance in the opening activity and actually hoped that the students would be "a little bit confused." In the third lesson plan, she noted that students needed certain mathematical understanding in order to complete the task. She also reflected that she noticed steady improvement in her ability to anticipate students' responses during the course of the semester.

Monitoring Students' Responses to the Tasks

In all three lesson plans, Tara envisioned herself helping students as they worked on the task and "asking leading questions." By the third lesson, she acknowledged that it would be okay for students to make mistakes and "search for the answer themselves." She monitored by discussing students' strategies with them, assisting those having difficulty, and making anecdotal records of their proficiency with the task. She planned to ask questions like, "How did you know…?" or "How did you find…" to encourage them to explain their thinking. Early in the semester, she described this strategy as "use the information to choose students you want to share their ideas with the class," but her lesson plan summaries do not include evidence that she did that while monitoring.

Selecting Students to Present Responses During Discussion and Summary

Tara first described this practice as "monitor and listen" and explained the purpose is to "use the information to choose students you want to share their ideas with the class." There was no evidence in her first lesson plan that she purposefully selected students to share ideas, but students did "share different strategies and explained the steps they used in those strategies." In the second lesson, she planned to share comments from her anecdotal records, and she anticipated asking students to "share a problem they solved, a strategy they used, or a problem they found difficult." The extent to which this happened was unclear. In the third lesson, she planned to "showcase" certain responses and anticipated asking students to share approaches that linked to prior strategies, were new, or were "surprises." She reflected that in reality, the discussion was rushed, seemed jumbled, and that the students were restless.

Purposefully Sequencing Student Responses

Tara developed an awareness of this strategy late in the semester. Her first two assignments do not mention the sequencing of responses. In the second lesson plan, she wanted to save the big idea—how addition and subtraction are related—until the end of the discussion, as a kind of finale. It was not clear in her reflection if this happened as planned. Her third lesson plan included a detailed sequence for sharing: "difficulties first, surprises next, then those who discovered new or interesting strategies…" She noted that a summary discussion did take place, with a few students sharing fact families and others asking questions about doubles.

Making Connections Between Responses and Key Ideas

In Tara's final reflection of the semester, she wrote that one of her ongoing goals is to include more real-life connections to the math curriculum to increase student interest in math. She also thought it was important to link new strategies to students' prior knowledge. In the third lesson plan, she reflected that she would have liked to link the content to a number line representation or to ponder how to solve more difficult problems using fact families. Those connections were made in the lesson reflection rather than in the lesson implementation.

Discussion and Conclusion

Tara grew in her use of one of the practices (anticipating responses) during the course of the semester and described heightened awareness of the others. She moved from anticipating management-related issues to recognizing mathematical content and representations that students would need to understand or might find difficult. She even hoped that students would not understand the problem right away in her third lesson, showing that she was becoming confident in her ability to choose and manage stimulating and challenging tasks.

Nevertheless, leading a mathematically productive discussion remained an elusive goal for Tara. She included plans for discussions in her second and third lessons, but described only one such conversation about fact families. Her focus on helping students and asking leading question during the monitoring phase probably made it difficult to select students to share their responses because she did not have a detailed understanding of the ideas they might share. She stated the importance of connecting a lesson to students' prior knowledge or to real-world problems, but did not plan to illuminate those connections in a summary discussion. Rather, she used those features to engage students early in the lesson.

Though Tara's descriptions did not show that she led mathematically productive discussions as envisioned by Stein et al. (2008), she began to lay the groundwork for such instruction. Based on these findings, we hope to modify our teaching so that students interact with these strategies in more and deeper ways than occurred during this cycle. First, we will continue to model the practices in class while preservice teachers engage in mathematical problem-solving as learners. Second, we will revise the lesson plan template to highlight the importance of planning for each of the five areas, and emphasize only one or two strategies for each of the three lesson plans. Third, we will ask the students to engage explicitly with the practices during their fieldwork. As a precursor to writing their lesson plans, students will analyze their cooperating teachers' use of the five practices, highlighting evidence of their usage or offering suggestions of ways in which they could have been incorporated into the lesson. Lastly, before students teach their lessons, we would like them to observe instruction about a similar topic to help them anticipate student responses, recognize mathematically productive ideas, and envision themselves leading a summary discussion. True to the self-study model, we hope that our next cycle of reflection will further improve our instruction and also make our students better teachers.

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A SURVEY OF TEXTS AND CORRESPONDING ASSIGNMENTS USED IN SECONDARY MATHEMATICS METHODS COURSES

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This paper presents preliminary findings of a study aimed at summarizing the curriculum of a set of secondary mathematics methods courses at universities in the United States. The analysis began with the identification of texts and related assignments from a survey of secondary mathematics methods syllabi. The findings indicate that a variety of topics are addressed in the texts and that the accompanying assignments represent varying levels of engagement with the texts.

Keywords: Teacher Education–Preservice; Curriculum Analysis; Instructional Activities and Practices

Background

Broad recommendations have been made about the knowledge needed for teaching mathematics, as well as desired characteristics of mathematics teacher education programs (National Research Council, 2001; Conference Board of Mathematical Sciences, 2012). However, we do not yet have research-based recommendations regarding the mathematics pedagogy content of these programs.

> We do not understand well enough how mathematics and teaching, as inter-related objects, come to produce and constitute each other in teacher education practice. We lack adequate knowledge about what and how this happens inside a teacher education program, and then across ranging or contrasting programs, contexts and conditions. (Adler et al., 2005, p. 378)

Before we can design research studies to investigate which aspects of current secondary mathematics teacher education courses are productive, we must first unearth what mathematics teacher educators are currently doing in secondary teacher education programs across the United States, specifically in their methods courses where mathematics pedagogy is the focus. In a recent study of MMC syllabi in the United States, Taylor and Ronau (2006) examined the assessments and stated goals and objectives in mathematics methods course syllabi and found considerable variability among them. The purpose of this paper is to present preliminary results from a study of syllabi that focused on the texts and accompanying assignments in secondary mathematics methods courses (MMCs). We began our exploration of secondary MMC syllabi with the following questions: How does the focus of texts used in secondary MMCs align with standards outlined by NCTM? How are students asked to engage with these texts in the courses?

Studying Intended Curriculum via Analysis of Syllabi

In attempting to gain insight into what is happening in secondary MMCs in the United States we examined the *intended curriculum* of the courses (Remillard, 2005). We agree with Gorski (2009) who also conducted a syllabus analysis and pointed out that "as a teacher educator, I often have diverted from an official course design once the classroom door was closed. As teacher educators, we bring our philosophies, strengths, and limitations into our teaching. Therefore, I cannot claim to have discerned what, exactly, occurred in any particular course by examining its syllabus" (p. 309). Thus we are not attempting to characterize the teaching or the experiences that took place in these courses. Instead we are seeking to gain a picture of the landscape of texts and how they are used in secondary MMCs as they are represented in the syllabi.

Methods

The preliminary analysis presented here included the syllabi from secondary mathematics methods courses from 17 universities across the United States. Some of the collected syllabi contained full reading lists and others did not. In cases in which the reading list was not included, the authors contacted the course instructors to request these lists. We began by identifying the specific texts (e.g., books, articles, curricula) that appeared in more than one syllabus (n = 30 texts). The National Council of Teachers of Mathematics' (NCTM) *Principles and Standards for School Mathematics (PSSM)* (2000) was referenced in at least one syllabus from each university; therefore, we used the NCTM content and process standards as a frame to conduct an analysis of the texts. The topics of the readings were coded as corresponding to one or more areas within this framework (e.g., Problem Solving). Texts were coded only when the major focus represented one of the content or process standards. For example, Driscoll's (1996) book *Fostering Algebraic Thinking: A Guide for Teachers Grades 6–10* was coded with the content standard *algebra* because that was the primary focus of the book. After identifying and coding the content of the texts we then identified and coded the assignments corresponding to the readings in order to determine how students were being asked to engage with these texts (e.g., read, reflect, analyze). Following Larnell and Smith's (2011) recent analysis of verbs in curriculum standards to "describe the character of mental work that students must carry out" (p. 96), we used Bloom's Taxonomy in an attempt to capture the cognitive demand suggested in the assignments.

Findings

Our analysis revealed that the universities used a variety of types of texts in their courses, including secondary MMC textbooks, journal articles, and curricular documents. This included 32 specific texts which are the focus of this analysis.

Content Areas of Focus

Of the texts that did attend to specific content areas the greatest number focused on Number and Operation (see Table 1). We found that not all texts were related to specific content standards, for example several texts addressed the topic of assessment but not in a particular content area.

Table 1: Number of Texts Focused on Specific Content Standards (n = 30)

	PSSM Content Standards				
	Number and Operations	Algebra	Geometry	Measurement	Data Analysis and Probability
Number of Texts	8	6	5	4	3

This finding is particularly interesting given that *PSSM* recommends that Algebra should receive the most instructional attention in the secondary grades while Number and Operations should receive considerably less. Also noteworthy is the fact that three of the texts, *PSSM,* the *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics* (NCTM, 2006), and Brahier's (2005) MMC textbook, addressed all five content standards.

Process Areas of Focus

Problem Solving appeared the greatest number of times in the texts (see Table 2).

Table 2: Number of Texts with Emphases on Specific Process Standards (*n* = 30)

	PSSM Process Standards				
	Problem Solving	Reasoning and Proof	Communication	Connections	Representations
Number of Texts	7	5	5	2	3

The relative lack of focus on Connections represents a mismatch with NCTM's recommendation for incorporating the process standards into high school: "With the experience proposed here in making connections and solving problems from a wide range of contexts, students will learn to adapt flexibly to the changing needs of the workplace" (NCTM, 2000, p. 288).

Assignments Related to Readings

All levels of Bloom's Taxonomy were represented in the assignments (see Table 3).

Table 3: Number of Universities with Assignments at Each Level and Examples

	Bloom's Taxonomy Level					
	Knowledge	Comprehend	Apply	Analyze	Synthesize	Evaluate
Number of Universities	17	15	9	10	7	14
Example	Read…	Summarize…	Write a related problem…	Analyze readings…	Draw connections…	Reflect critically…

All of the universities had required assignments in their syllabi in at least two of the levels. The two universities with only two levels had assignments at the *knowledge* and *evaluate* levels, indicating a broad range of expectations in their syllabi. Fourteen of the 17 universities had at least one assignment at the *synthesize* and/or *evaluate* level.

Conclusion

This paper presents the findings of a preliminary analysis of secondary mathematics methods course syllabi with attention on the course texts and accompanying assignments. The evidence suggests that outside of *PSSM* there are not commonly accepted texts used in the secondary MMCs at these universities. Despite the fact that all institutions made use of *PSSM*, the relative emphasis of particular content and process standards did not mirror NCTM's recommendations. A similar statement could be made about the content standards of the *Common Core State Standards for Mathematics* (2010) which also has a decreased focus on number in the secondary grades and an increased focus on Algebra. Our preliminary findings of assignments indicate that secondary MMCs are asking students to engage with their texts on a variety of levels. This finding differs somewhat from previously held notions of the role of texts in university courses. That is, "the traditional form of the textbook is largely one that assumes and perpetrates a 'received knowledge, passive consumption' pedagogical model" (Issitt, 2004, p. 689).

Discussion

Our use of syllabi limits the conclusions we can draw about the content of methods courses in general, but our findings do suggest several important areas for future discussion and research. First, the sheer number of different readings employed by the 17 universities provides evidence of the wide range of text resources available to secondary mathematics teacher educators for use with preservice teachers. This is perhaps both a blessing and a curse for new secondary mathematics teacher educators. Discussion of appropriate texts has the potential to benefit new mathematics educators who are often asked to teach courses (e.g., methods) in which they have no experience. Second, the discussion of how secondary mathematics teacher educators are making use of their readings could be helpful both within and outside of the mathematics education community. Our findings may indicate that secondary MMCs are making use of texts in novel ways that merit further inquiry, discussion, and dissemination.

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AN INVESTIGATION OF FACTORS INFLUENCING PRESERVICE MATHEMATICS TEACHERS' PREFERENCES TO TEACH IN URBAN SETTINGS

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Policymakers and educators have recently focused on the challenges associated with staffing qualified and effective mathematics teachers, especially in urban schools serving predominantly low-income and minority students. Social experiences and working conditions have been linked to teacher turnover and job satisfaction and have been theorized to influence teacher recruitment and retention in urban settings. However, very little is known about how prospective teachers' social experiences and perceptions of working conditions may influence where they choose to teach. The purpose of this study is to describe the background and perceptions of a sample of New Jersey mathematics and STEM-area teacher candidates enrolled in traditional and alternate route programs, and to investigate the factors that impact the candidates' preference to teach (or not teach) in urban districts.

Keywords: Equity and Diversity; Educational Policy; Teacher Education–Preservice

Purpose

The issue of teacher quality has risen to the top of the school reform agenda; research has found that the quality of a child's teacher is the most important in-school predictor of his or her achievement (Wright et al., 1997). Consequently, teacher supply and the initial distribution of teachers into classrooms are important educational policy issues. Of particular concern is the challenge of staffing the nation's schools with qualified mathematics teachers, especially in urban schools serving predominantly low-income and minority students (Ingersoll et al., 2010).

To better understand and address the challenges faced by urban districts in recruiting and retaining quality teachers, it is beneficial to consider not only the perspectives of those already in the teacher workforce, but also the beliefs and viewpoints of preservice teachers. In this study, survey responses were analyzed to investigate the factors that impact teacher candidates' preference to teach in urban schools.

Perspectives

A review of the literature illustrates several factors that contribute to the difficulties involved in the recruitment and retention of mathematics teachers, and more generally of teachers in the STEM content areas (science, technology, mathematics, and engineering). Some researchers argue that the levels and configurations of teacher pay structures contribute to an inadequate supply of math and science teacher candidates, as individuals with strong STEM content-area backgrounds frequently have other career options available offering more generous levels of compensation (Ballou & Podgursky, 1997). Others focus on factors related to heightened demand for qualified teachers; particularly the demand that is a result of high non-retiring turnover, often due to job dissatisfaction. The potential factors that influence teachers' career satisfaction, and by extension, their decisions to remain or leave teaching, can be understood by applying social learning theory, first utilized by in Krumboltz (1979) and later Chapman (1983) as a theoretical framework to review the literature on attrition. Through the social learning theory lens, social learning experiences could represent a relevant and valuable mechanism in addressing the quality of the mathematics teacher workforce. Social experiences, and their impact on one's perception of the environment/working conditions, have been linked to teacher effectiveness and to attracting and retaining sufficient numbers of qualified individuals to the profession despite its relatively low pay (Johnson et al., 2004).

Teacher shortages are frequently more pronounced in urban, high-needs settings. Haberman (1988) suggested that most traditionally-prepared teachers desire to teach in suburban environments, a preference

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that may reflect the area in which those teachers grew up (Boyd et al., 2005). Even within urban districts, there can be an inequitable distribution of qualified teachers where teachers were found to transfer out of schools with relatively higher percentages of poor minority students into schools with generally less poor and higher performing students (Hanushek et al., 2004; Liu, Rosenstein, Swan, & Khalil, 2008). Accordingly, many researchers view student demographics, especially socioeconomic status and race/ ethnicity, as characteristics of schools that can lead to teacher turnover (Grissmer & Kirby, 1997).

The lack of retention of newly hired teachers in high-poverty schools can be often related to organizational factors, frequently referred to as "working conditions." Working conditions include unsupportive leadership style, inadequate salary/benefits, insufficient resources and support, and poor hiring practices (Johnson et al., 2004). Working conditions have been linked to teacher effectiveness and to the attraction and retention of qualified teachers into the profession despite its relatively low pay, especially regarding teachers in STEM fields (Johnson et al., 2004). While working conditions have been linked to teacher retention, relatively few researchers have examined how prospective teachers weigh the working conditions of schools in their considerations of where to seek employment.

This study takes an organizational approach (Ingersoll, 2001) to examine how social and professional backgrounds, perceived working conditions, and facets of teacher preparation might influence mathematics and STEM teacher candidates' preference of the type of district they want to teach in. To date, there is a lack of consensus in the literature around the factors that predict teachers' decisions to remain in or exit instructional contexts, which factors may be more or less important to those decisions, and how those factors might be related to teacher attributes or school characteristics (Ingersoll, 2001; Useem & Neild, 2005).

Data and Methods

The research reported here is part of a larger study investigating the challenges in recruiting and hiring teacher candidates in New Jersey. Data for this study were collected via survey from 696 teacher candidates enrolled in traditional and alternate route teacher education programs. Survey items were designed to measure the possible relationships between teacher candidates' perspectives and their future career plans. Survey items were constructed from the National Center for Education Statistics School and Staffing Survey and the NYC Teacher Pathways Project Survey, with additional items tailored to the specific New Jersey context.

For the analyses presented in this document, analytic samples of 199 STEM-area teacher candidates and 116 mathematics teacher candidates were utilized. We included teacher candidates who intended to pursue any STEM-area certification because these candidates may qualify for and potentially be motivated to become mathematics teachers. The samples consist of approximately 56% to 61% female teachers, and of these prospective teachers, about 75% are White, 8% are Black, 9% are of Asian origin, and about 8% are Hispanic. Of the total analytic samples, almost half were enrolled in teacher education colleges and/or programs, about 37% were completing New Jersey's alternate route programs, and 14% were from TFA or TNTP.

The outcome variable of interest was whether or not teacher candidates' indicated a preference to teach in urban schools. Approximately 39% of the participants in our samples indicated this preference. A binary logistic regression model was utilized to examine the impact of personal and professional characteristics, social experiences and preferences of perceived working conditions on teacher candidates' preference to teach in urban schools. Our final model included 11 predictor variables and the single nominally scaled dependent variable.

Results

Logistic regression models were run for both the STEM-area sample and the mathematics sample. Based on these data, we found that teacher candidates' perceived working conditions, previous experiences, and preparation route are significantly associated with their preference to teach in urban schools. The strongest predictor for both samples, in terms of statistical significance, was a scale that emerged from principal component analysis of survey items related to candidates' preferred school characteristics. This scale, which we refer to as the "High-Needs School factor" measures the tendency of a candidate to prefer to work in schools with a diverse population (student and teacher) and with many students of poverty, English Language Learners, and low achieving students. Teacher candidates scoring high on this scale were significantly more likely ($p < .001$ for STEM, $p = .002$ for math) to want to teach in urban schools. In addition, for both analytic samples, study participants who either attended high school or grew up in urban settings were significantly more likely ($p < .001$ for STEM, $p = .005$ for math) to want to teach in urban schools, when compared to their counterparts from suburban backgrounds. Teacher candidates who completed their field experience in an urban school were also significantly more likely ($p = .001$ for STEM, $p = .002$ for math) to prefer to teach in urban schools.

Based on our chosen level of significance ($\alpha = .05$), the other statistically significant predictors for the STEM sample included a flag indicating candidates enrolled in either Teach for America (TFA) or The New Teacher Project (TNTP), an indicator for candidates who were enrolled in teacher preparation programs located in a low-SES districts (identified in this study as District Factor Groups A or B), and a flag for "first-career late starting teachers," or teacher candidates who were 25 to 29 years old when the survey was administered. The only other statistically significant predictor for the mathematics sample was whether a participant was enrolled in TFA or TNTP. Lastly, for both samples, non-White teacher candidates were not significantly more likely to want to teach in urban settings as compared to their White counterparts, after controlling for the other analytic variables.

Discussion and Implications

These results suggest that teacher candidates' perceptions, experiences, and characteristics are important considerations when conceptualizing teacher recruitment policies. Just as Chapman (1984) utilized social learning theory to empirically examine factors influencing teacher career satisfaction and retention, this study found those factors were all positively associated with candidates' preferences to teach in an urban district.

As previously noted, the only factor measuring perceptions of preferred environmental factors/working conditions that was a significant predictor of teacher candidates' preference to teach in urban districts was the High-Needs School factor. In other words, factors tapping into some of the aforementioned working conditions of interest to Liu and his colleagues were not significant predictors in our analyses. This could be due to the high demand for STEM-certified teachers; thus, candidates may feel as if they can choose a position with optimal non-monetary benefits (or perceived working conditions). On the other hand, many STEM-certified candidates may be beneficiaries of pipeline subsidies (such as NOYCE, AmeriCorp, loan forgiveness) that require candidates to work in high-needs districts, resulting in their proclivity for such school characteristics.

Teacher preparation programs are a major investment at the local, state, and national levels. Urban teacher preparation programs are designed to recruit teachers prepared for the rigors of teaching in an urban environment, and to ensure a good fit in such an environment (Haberman, 2005; Ladson-Billings, 1994). Our findings provide evidence that the completion of an urban field experience, as well as the locale of the teacher producing institution, are important influences on teacher candidates' preference for urban teaching. Thus, we contend that teacher preparation programs could make better use of their surrounding urban areas to assist teacher candidates in the facilitation of practical field experiences.

For both of our analytic samples, another significant predictor of candidates' preference to teach in an urban district was whether they grew up or attended high school in an urban area. This is consistent with a finding of Boyd et al. (2005) that 34% of their teachers began their first teaching job near their high school, 61% chose a school within 15 miles of their home, and 85% stayed within 40 miles of their hometown. Taken together, these results support community-based Grow Your Own (GYO) models for teacher pipelines, to produce more committed, culturally aware teachers recruited from the local community (Skinner et al., 2011). Local recruits are more likely to remain in the workforce for longer periods of time, and know more about the students, parents, schools, and the communities where they teach.

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ASSESSING ELEMENTARY TEACHER CANDIDATES' KNOWLEDGE FOR TEACHING MATHEMATICS

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The purpose of this study was to describe the application of the Mathematical Knowledge for Teaching-Think Aloud Rubric (MKT-TAR) to assess and quantify elementary teacher candidates' (ETCs') common content knowledge and specialized content knowledge for teaching mathematics. The rubric was used to score over 100 think aloud podcasts created by teacher candidates at various stages of their academic coursework. The results indicated that the MKT-TAR was an effective tool to characterize the quality of ETCs' mathematical knowledge for teaching.

Keywords: Mathematical Knowledge for Teaching; Teacher Education–Preservice; Assessment and Evaluation; Technology

Purpose of the Study

The examination of inservice and preservice elementary teachers' mathematical knowledge for teaching (MKT) has been a focus for many years (Ball, Thames, & Phelps, 2008). Researchers have used a variety of tools to examine MKT including written protocols and interviews (Thanheiser, 2009; Ma, 1999; Tirosh, 2000). To extend this research, the authors chose to focus on how elementary teacher candidates (ETCs) would demonstrate MKT through think-aloud podcasts that simultaneously captured their voice and written work. The authors sought to capture and understand ETCs' MKT as it related to not only the types of strategies they used, but also their use of mathematical language and the depth of their explanations specifically related to whole number subtraction and the division of fractions. In order to describe the nature and quality of ETCs' MKT the authors developed the Mathematical Knowledge for Teaching-Think Aloud Rubric (MKT-TAR). Therefore, the purpose of this brief report is to document the development and refinement of this rubric and provide specific examples of how the rubric was applied to ETCs' think-aloud podcasts.

Theoretical Framework

Building on a practice-based theory of mathematical knowledge for teaching, Ball and her colleagues (e.g., Ball, Thames, & Phelps, 2008) identified and defined subsets of subject matter knowledge and pedagogical content knowledge. Relevant to the development of the MKT-TAR are the two distinguishable domains of subject matter knowledge. The first domain is associated with the knowledge of mathematics that is commonly used in settings other than the classroom. This common content knowledge (CCK) encompasses "being able to do particular calculations, knowing the definition of a concept, or making a simple representation" (Thames & Ball, 2010, p. 223). The second domain is comprised of the mathematical knowledge that is used in teaching tasks. This specialized content knowledge (SCK) involves knowing different representations of mathematical procedures and concepts, using appropriate mathematical vocabulary and providing robust mathematical explanations of common rules and procedures.

Methods

The participants for the study and rubric development were ETCs attending a Midwestern university who were at various stages of their academic program. During the initial pilot of the rubric, over 100 teacher candidates were active participants in the project. Similar to the tasks from Ma (1999), participants were asked to create think-aloud podcasts based on the following situations. "Present the following problems as though you were explaining them to a student: $452 - 286$ and $1\frac{3}{4} \div \frac{1}{2}$."

Rubric Development

Similar to coding methods established by Miles and Huberman (1994), the development of the MKT-TAR began with a list of common types of teaching tasks derived from research. These common tasks included using and choosing representations, attending to and using math language, defining terms mathematically and accessibly, and giving explanations (Thames & Ball, 2010). The researchers analyzed a subset of think-aloud podcasts in order to collapse and cluster these teaching tasks into the domains of CCK and SCK.

During this initial development of the MKT-TAR, CCK was further subdivided into three categories. These categories included *Computational Accuracy,* use of *Representations,* and *Strategies Used*. The second domain, SCK, was divided into the categories of use of appropriate mathematical *Vocabulary* and *Depth of Explanation*. In the initial iteration of the MKT-TAR all domains utilized a 3-point scale.

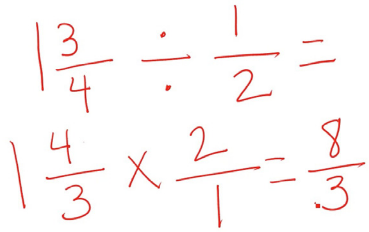
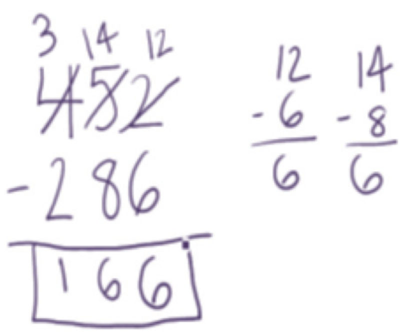
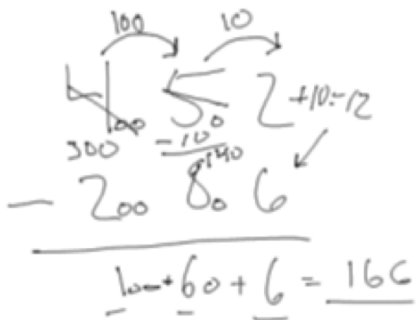
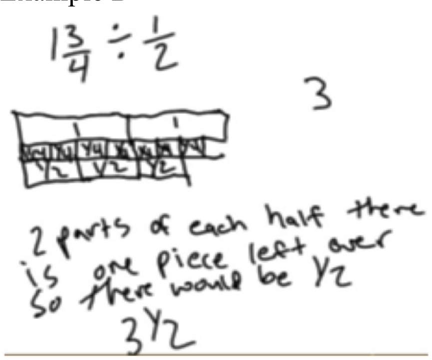
The process of developing the MKT-TAR was cyclical. The researchers used the initial version of the MKT-TAR to assess five think-aloud podcasts for each task in order to discuss scoring procedures and further refine the rubric. The MKT-TAR was then modified to address discrepancies in scores. One such discrepancy was the use of a 3-point scale for *Depth of Explanation* where it became apparent that there existed two distinct situations that were coded at a Level 1. The first situation was that an ETC utilized a coherent explanation to demonstrate their CCK (e.g., they could do the procedure), but did not explain why the procedure worked. The second situation involved an ETC providing evidence of CCK and limited understanding of SCK. That is, they were able to address the need for regrouping in multi-digit subtraction by discussing place value concepts, but their explanation contained minor errors such as “borrow a group of ten” or “you can’t take six from two”. To account for these distinct situations, the second iteration of the MKT-TAR utilized a four-point scale for coding *Depth of Explanation*. To determine inter-rater reliability, a second iteration of the MKT-TAR was used by two researchers to individually code twenty think-aloud podcasts for each task. Cohen’s Kappa correlation coefficients of .682, 1.00, .674, and .896 ($p < 0.001$) were calculated in each of the four domains (Computational Accuracy, Representations, Vocabulary, and Depth of Explanation).

Results

For the purposes of this brief report, we will discuss results related to the SCK subdomain, *Depth of Explanation*. Table 1 highlights specific think-aloud examples that were coded according to the 4-point scale for this subdomain. This table includes a final screenshot of the ECT’s work as well as a partial transcript of their explanation where emphasis has been added to areas of concern (italics) and areas of understanding (underscored).

Example A, exemplifies a Level 0 because the ETC said, “you invert and multiply” and then proceeded to “invert” both fractions. This example shows a novice level of CCK and a limited understanding of SCK. A Level 1 score was given when participants provided a coherent explanation of the procedure, but provided little to no evidence of the mathematical meaning underlying the procedure as shown in Example B. At this level, statements like “borrow a one from the tens place” or “invert and multiply” exemplify common phrases for think-alouds that received scores of 1. In both cases, ETCs document a more practiced CCK than a Level 0, but still exhibit a limited understanding of SCK. A Level 2 indicates that the explanation provided mathematical meaning to the procedure, but further clarification was needed. As shown in Example C, phrases such as “we can’t take away six from two” or “we need to borrow a ten from the tens place in order to subtract” do indicate an attempt to explain the need to regroup however, the explanation could be more refined by saying “we can’t take away six ones from two ones” or “we need to regroup a group of ten into ten ones” which are phrases consistent with a Level 3 score. In Example D, a Level 3 indicates an explanation was both coherent and provided mathematical meaning to the procedure. At this level the ETC demonstrated a practiced CCK and deep SCK by showing multiple representations to solve the division of fractions task.

Table 1: Examples of Depth of Knowledge Coding

Example A 	**Level 0: Inappropriate explanation that distracts from understanding** “Ok now we will be doing fractions. The problem is one and three-fourths divided by one-half. With fractions *you invert and multiply*, so that would be one and *four-thirds times two over one*. Four times two is eight. Three times one. Over three. And that is all you have for fractions.”
Example B 	**Level 1: Explanation provides little to no evidence of the mathematical meaning underlying the procedure** **“**So what we want is we want the *numbers on top to be larger than the numbers on the bottom so we can subtract them*. Looking at the two and the six, the two is smaller than the six, so what we need to do is *borrow from the number before it*. So what we’re going to do is borrow from the five we’re going to make that a four and then we’re going to borrow what we’re adding onto the two and <u>we’re going to make that a twelve by adding ten</u>. Now the twelve is larger than the six, and that’s what we want, so we can subtract that. So we can do *twelve minus six* and that’s going to give us six, so we can put that right there…”
Example C 	**Level 2: Explanation provides mathematical meaning to the procedure but needs further development** **“**Now to begin to help us to understand place values, I might tell them to add little zeros in here, in the hundreds place and in the tens place. So that they can get a better idea when we start to borrow what kind of numbers they are dealing with… *And since we can’t take away six from two*, we’ll need to begin borrowing so we’ll tell the student that we’ll need to <u>borrow ten from the tens place</u> here. So fifty, we’ll take ten over here and that means we’ll have to subtract ten from fifty and now we’ll have forty. And then we add the ten to the two, plus ten is twelve. So now we have six, twelve minus six…”
Example D 	**Level 3: Explanation is coherent and provides mathematical meaning to the procedure** **“**I’m going to do the problem one, one and three-fourths divided by one-half. I’m going to do fraction tiles so we’ll start off by doing, that’s one whole and that’s one whole. Then do the one and three-fourths and then <u>how many halves can I get from the three-fourths</u>. So then, oops that’s one-half another half, then another half. So then it’s three and since there is two parts of each half there is one piece left over so there would be one-half left so the answer would be *three-halves*.”

Discussion

Analysis of the data demonstrated that the MKT-TAR was able to capture distinct levels of both CCK and SCK for teaching. Extending on the work of Hasenbank and Hodgson (2007) who examined levels of procedural knowledge, these domains could be further classified into novice versus practiced CCK and limited versus deep SCK. Therefore, further research will focus on determining what instructional materials are best suited for moving ETCs towards practiced CCK and deep SCK.

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NAVIGATING THROUGH PRE-SERVICE TEACHERS' PROFESSIONAL VISION IN FIELD EXPERIENCES

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This study investigates the aspects of learning/teaching that pre-service teachers examine at distinct time periods while taking on different roles. Pre-service teacher participants (N=76) reported on what they noticed from past teachers' and current field teachers' teaching as well as their own teaching in the field setting around loosely structured themes. Results showed that participants mainly focused on the factors of teachers and tasks. Their attention to the student factor was weak. Also, they tended to avoid making critical reflections when they were in the field compared to their reflection on previous learning experiences. It would be desirable to consider more structured observation/reflection activities in collaboration between the teacher education program and the field setting to support pre-service teachers' appreciation and development of the full range of teaching and learning of mathematics.

Keywords: Teacher Education–Preservice; Instructional Activities and Practices

Objectives

One of the common features of teacher preparation programs around the world is the field experience component that aims to provide pre-service teachers with firsthand experience in classrooms and to support their smooth transitions into classroom teaching. While the specifics of field experiences vary, one common characteristic is that pre-service teachers spend substantive time in the field setting '*observing*' their cooperating teachers' teaching and other events happening in the field. Recent research in mathematics education has drawn attention to effective professional development, realizing the importance of understanding what and how teachers are attending to in their classroom events (e.g., Mason, 2002; Jacobs et al., 2007; Sherin, Jacobs, & Philipp, 2011). While the main body of research focuses mostly on in-service teachers' professional growth, little is reported on how these ideas can be incorporated in teacher preparation programs. In this regard, this study focuses on what pre-service teachers attend to during their field observation and how they interpret what they observed. The purpose of this study is to share the aspects of learning and teaching which a group of pre-service teachers perceived while taking on distinct roles (e.g., as students in past experiences, as teacher candidates in the university course, as student teachers in field settings), how their patterns of noticing changed, and, ultimately, to open further discussion on how to utilize field experiences to support the development of pre-service teachers' professional vision. Specifically, this study examines the following questions:

1. What events do pre-service teachers recall from their past mathematics learning experience?
2. What events from the field settings do pre-service teachers pay attention to?
3. Do the noted past/present events influence pre-service teachers' planning and implementation of mathematics lessons? What are pre-service teachers' justifications of incorporating or not incorporating what they noticed into their actual teaching?

Theoretical Framework

It is the role of teacher educators to consider ways to engage teachers/teacher candidates in authentic aspects of practice so that they can learn to utilize teaching practice as a source of inquiry and a way to develop a professional vision. Mason (2002) frames this professional vision as developing the sensitivity to "notice" things in the beliefs of "teaching as disciplined" inquiry. Mason (2011) states the discipline of noticing as a collection of techniques for (a) *pre-paring* to notice in the moment, and (b) *post-paring* by reflecting on the recent past to select what to notice in order to act freshly rather than habitually. Similarly, Endsley (2000) defined situation awareness, which is the term that embodies a theory of noticing, as involving three factors: (a) perception of meaningful elements in an environment, (b) comprehension of their meaning, and (c) projection of their status in the near future. Most of the other studies also characterize teacher noticing as consisting of three aspects: (a) attending to noteworthy events, (b) reasoning about such events, and (c) making informed teaching decisions on the basis of the analysis of these observations (van Es, 2011). One of the key considerations provided by the previous research is its continuous, cyclic sequence, which emphasizes the component of reflective practice. Informed by these previous works, this study offers similar professional development contexts to pre-service teachers by providing the following opportunities: (a) noticing critical events in their past/current learning experiences from the observer's perspective; (b) incorporating/implementing the noticed areas into their teaching plan; and (c) reflecting upon their teaching.

Methodology

Context

This study was conducted in multiple sections of a K–8 mathematics methods course in a Midwestern university in the USA. A total of 76 pre-service teachers engaged in the following phases throughout the semester: (a) reflection on what their past teachers taught by reporting episodic events, (b) reflection on what their current field cooperating teachers taught by reporting the event they noticed, (c) creating/implementing a lesson in the field based on what they had noticed, and (d) reflection on the results of their own teaching implemented in the field. To help encourage participants' engagement, participants were asked to report on their experience around a loosely structured format of observation: (a) what should be *lessened*, (b) what should be *expanded*, (c) what should be *altered*, and (d) what should be *dropped.* In each phase, written episodic memories, called 'LEAD' reports, were collected as data. In the LEAD report, participants described their learning/teaching experiences and their tentative disposition (e.g., Lessened/Expanded/Altered/Dropped) along with personal justifications. The table below shows the phases of study.

Table 1: Phases of Study

Phase	Phase 1	Phase 2	Phase 3
What is the teaching/learning context involved?	Past learning experiences	Current learning experience	Lesson planning/ Post-teaching reflection for future teaching
Whose teaching is the LEAD report based on?	Past mathematics teachers	Current field cooperating teacher	Self
What is the rationale of each LEAD phase?	* Identifying/noticing the satisfaction/dissatisfaction with others' teaching practice that potentially produce the motivation to move toward the ideal state * Getting a more profound level of reflection under the frame of LEAD * Learning to see important features of classroom practice		* Changing perspective from 'information receiver'/ 'observer' to a more active leading role * Connecting with what is noticed from others' teaching practice with own teaching practice * Finding important results that influence future teaching

Data Source/Analysis

Multiple and primarily qualitative methods were utilized. Major data sources were participants' personal narratives (LEAD reports) documented via an online depository (Moodle). Qualitative data obtained from participants were analyzed with aspects of a double-coding procedure suggested by Miles and Huberman (1994). Multiple people jointly developed a coding scheme and coded the data. The interrater reliability, about 83-85%, was calculated as the number of agreements divided by the number of sample items coded.

Results

Initially, the researcher and a research assistant reviewed all entries and identified emerging themes, independently focusing on significant features participants revealed. Then, the themes each person identified were compared and combined. Since each participant demonstrated multiple thoughts on the same topic over time, demonstrating multiple dimensions of their thinking of the given examples and related strategies, the coding process focused on whether the specific theme was present or absent in each participant's statements. The final coding scheme used and the frequencies are shown in the following table.

Table 2: Coding Scheme and Frequencies

	Phase 1	Phase 2	Phase 3
A. Class environment/set-up *(A1. Materials; A2. Curriculum/Textbook)*	19.3%	14.1%	20.7%
B. Teacher *(B1. Teacher preparation; B2. Classroom management; B3. Communication with students; B4. Modes of representation used; B5. Teacher knowledge/attitude; B6. Grouping strategies; B7. Flexibility/pacing)*	24.2%	41.2%	45.3%
C. Students *(C1. Understanding; C2. Emotions/Attitude; C3. Engagement; C4. Strengths/Abilities)*	10.3%	2.5%	4.4%
D. Tasks *(D1. Types of tasks; D2. Types of instruction)*	36.5%	34.5%	28.1%
E. Assessment *(E1. Types of assessment; E2. Grading/Purpose of assessment)*	9.7%	7.7%	1.5%

Notes: (1) Frequencies are calculated based on the total number of meaningful entries of events described in each phase [Phase 1: 518 entries, Phase 2: 1,252 entries, Phase 3: 135 entries] (2) Frequencies for each sub-category are calculated, but not reported in this proposal due to the limited space.

The frequencies of participants' disposition on their learning/teaching experiences in the LEAD report along with personal justifications are shown in the following table.

Table 3: Frequencies of Participants' Disposition

Phase 1					Phase 2					Phase 3				
L	E	A	D	N	L	E	A	D	N	L	E	A	D	N
20.3%	28.7%	27.2%	17.0%	6.8%	3.4%	26.1%	20.0%	3.2%	47.3%	3.6%	53.3%	42.4%	0.7%	0.0%

Notes: L (Lessen), E (Expand), A (Alter), D (Drop), N (no clear dispositions/justifications)

Some noteworthy findings from this study are listed below:

- In all phases, participants mostly attended to the themes related to "Teacher" and "Tasks". Participants gave more attention to the theme of "Teacher" in Phases 2 and 3 during the field experiences compared to Phase 1.
- Overall, participants demonstrated weak attention to the theme of "Student". In Phases 2 and 3, it became much weaker than Phase 1.
- Overall, two dispositions, *Expand, Alter*, were dominant.
- In Phase 1, there were relatively high number of reports related to *Lessen* and *Drop*, representing negative past learning experiences.
- In Phase 2, almost half of the reports did not indicate specific dispositions. These reports only described what participants observed without judgment.

Possible Conclusions

This study showed that what participants attended to in the field setting is somewhat unbalanced, missing many important aspects in classroom teaching (e.g., less noticing on students). Also, it was shown that many participants did not provide any reflective dispositions on what they noticed in the field setting, even though a loosely designed reflection format (LEAD report) was provided. This may imply that pre-service teachers considered field experiences as the context to learn solely from those in authority (e.g., cooperating teachers) rather than as the opportunity to actively generate their own ideas and thoughts. If teacher education programs continue to provide field experiences as a context to help pre-service teachers have holistic views and critically reflect on teaching/learning in the classroom, it would be important to know what pre-service teachers observe, think, and do in the field setting in order to create conditions that are conducive to both creating a holistic view and reflecting critically. To do so, it would be desirable to consider more structured observation/reflection activities in collaboration between the teacher education program and the field setting. It is hoped that this study brings attention to the creative ways teacher educators can support pre-service teachers' appreciation and development of the full range of teaching and learning of mathematics.

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PARALLEL TRANSITIONS: CHALLENGES FACED BY NEW MATHEMATICS TEACHERS AND NEW MATHEMATICS TEACHER EDUCATORS

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In this paper we share initial findings from an investigation of the similarities in challenges faced by new K–12 mathematics teachers with those faced by new mathematics teacher educators. Data about the challenges faced by new mathematics teacher educators were collected through a survey, and data about the challenges faced by new K–12 mathematics teachers were collected by reviewing relevant literature. Themes from both sources were identified and compared. One goal of this project is to improve the preparation of both groups of teachers by applying principles of successful preparation and professional development from one domain to the other.

Keywords: Teacher Education–Preservice

Purpose of the Study

For decades researchers have investigated the experiences of pre- and in-service K–12 schoolteachers, often focusing on the struggles of new teachers (e.g., Moir, 2009). Within this body of research, there has also been a dedicated effort to understand the experiences of new mathematics schoolteachers (e.g., Sowder, 2007). In contrast, little is known about the experiences of new mathematics teacher educators (Goos, 2009). New mathematics teacher educators face unique challenges, as they are expected to strike a balance between the demands of their work, including their responsibilities for preparing pre-service teachers. Whereas others (e.g., Golde & Walker, 2006) have looked to these challenges to help understand how doctoral education could provide better preparation for careers in academe, the aspect of our research project reported here looks for similarities between the challenges faced by two groups of new teachers (fewer than four years of teaching experience): those teaching mathematics education courses at universities and those teaching K–12 mathematics. Here we focus our attention on addressing the following questions: (1) How prepared do new mathematics teacher educators feel to begin teaching pre-service teachers? and (2) How are the transitional challenges faced by new mathematics teacher educators similar to those of new K–12 mathematics teachers?

Methods of Inquiry

Participants

A National Science Foundation (NSF)-funded program to mentor recent graduates of mathematics education doctoral programs, within their first four years of post-doctoral work, began in 2010. From a national pool of applicants, 46 participants were selected. All program participants were invited to complete our survey with a resulting sample of n = 40 participants (87% response rate).

Instrumentation

The research team constructed a pool of survey items based on existing literature on the responsibilities, expectations, and challenges often faced by new faculty as they transition into a career in academe. A pilot test of the survey instrument was conducted with six mathematics education faculty members who have served as mentors for new mathematics teacher educators. These faculty members were chosen for the pilot because they have knowledge about the challenges faced by new mathematics teacher educators. Based on responses to the pilot test, the research team revised the survey instrument. The final survey instrument, consisting of 27 questions partitioned into demographics and open response items, focused on new mathematics teacher educators' beliefs about their doctoral program experiences regarding: (1) their transition to becoming faculty, and (2) the preparation, support, and resources they received for their teaching responsibilities.

Data Collection and Analysis

New mathematics teacher educators were contacted via email with an explanation of the research study and its purpose. A link to the online survey was provided through email. Participants were informed that participation was voluntary and their responses would be aggregated, so that no one individual could be readily identified. Additionally, participants were informed that their candid responses would be used for research purposes only, and their participation in the survey would be an indication of consent for their data to be used.

The survey data were analyzed using a two-tiered analytic method. We first analyzed the responses to identify patterns and commonalities in describing the struggles, challenges, successes, resources, and reflections on what did and did not prepare the participants well for their work. Two team members read the open response questions independently, used open coding to code responses, and then compared their codes. Differences were discussed and consensus was reached on each code. The constant comparative method was then used to compare these codes across questions to reduce similar codes to categories (Glaser & Strauss, 1967). These categories were then compared across all questions to finalize descriptive themes (Bogdan & Biklen, 1998).

Results

Initial Findings: Challenges for New Mathematics Teacher Educators

Mentoring. While institutional support was fairly common—58% of participants said their institutions offered seminars on assessing student learning, and 65% said their colleagues openly shared student assessment materials—the results for individual mentoring were not as strong. Thirty percent were assigned an official mentor at the start of their academic position with whom they have been steadily working. Twenty-three percent were assigned an official mentor but have not been working with them or have found their interactions ineffective. Forty percent were not assigned an official mentor but have found their own informal mentor. One representative quote spoke to the need for strong research mentoring at the graduate level: "I felt unprepared for independent research in mathematics education. I feel that my program needed more active, publishing faculty members in mathematics education to mentor me and guide me through the early years of my career." Participants also indicated difficulties in striking a balance between research, teaching, and service and sought advice from their faculty mentors, such as by "Talking with them about their jobs, how they're managing their time, etc." Many participants indicated that they found it quite challenging to transition from being a graduate student to a faculty member, recommending the importance of "Having a strong mentor or mentors who will help you navigate through your first few years. There are so many 'opportunities' that come your way. It's good to have some trustworthy advice regarding what to steer away from."

Lack of teaching experience. Participants felt they had good knowledge of mathematics education literature, which satisfies Jaworski's (2008) call for "strong knowledge of the professional and research literature relating to the learning and teaching of mathematics" (p. 1). However, participants felt less prepared for their roles as teachers. Responses included leaving a doctoral program without a comprehensive tool kit for teaching. For example, one participant stated, "My program did not include any training on teaching—how to lead a discussion, construct a syllabus, or create assignments. Some faculty may have discussed this in passing. I didn't teach at all in graduate school." Another representative response: "I did not graduate with any resources for teaching. I had taught one course during one semester (a methods course), but had nothing from which to create courses on my own." Furthermore, one participant did not feel well versed in K–12 mathematics:

> My program didn't help me understand the world of K–12 math education in general, such as NCTM Principles & Standards, state curriculum standards, teacher certification requirements, education associations like AMTE, math ed conferences/other professional development opportunities, etc. I felt like it was assumed that we knew these things as a first year professor.

These findings, while similar to those cited by Prewitt (2006) and Golde (2006), are unfortunately out of sync with Jaworski's suggestion that new mathematics teacher educators know "pedagogy related to mathematics [and] mathematical didactics in transforming mathematics into activity for learners in classrooms" (2008, p. 1). This may be due in part to the variation in programs for preparing future mathematics teacher educators as many new faculty do not have K–12 or post-secondary teaching experience.

Initial Findings: Challenges for New K–12 Mathematics Teachers

Researchers have investigated the challenges faced by beginning K–12 teachers, typically using surveys and interviews to learn from the teachers themselves. Similar to the challenges reported by new mathematics teacher educators (as described above), new K–12 teachers describe difficulty in establishing relationships that provide community, guidance, and mentoring as a challenging aspect of career induction (e.g., Chappell, Choppin, & Salls, 2004; Roehrig & Pressley, 2002; Rust, 1994). Another theme that emerged from our investigation of the literature is the "disconnect" that beginning K–12 teachers perceive between their preparation programs and the "realities" of classroom teaching. This perception is often accompanied by a suggestion that teacher preparation programs include more frequent and more authentic teaching experiences (e.g., Cady, Meier, & Lubinski, 2006; Gleason, 2011; Luft, Roehrig, & Patterson, 2003; Olson & Osborne, 1991; Rust, 1994). Both of these challenges resonate with those reported by participating new mathematics teacher educators.

Discussion

We, the authors of this brief report and new mathematics teacher educators ourselves, took as the premise of this study the recognition that there is value in reflecting on our own experiences transitioning into careers in mathematics teacher education. By comparing our transitions with what we know about the struggles that new K–12 teachers face (based on our own experiences, those reported by our students, and those reported in literature), we think we can become more thoughtful about how we prepare both groups. The research reported in this paper is ongoing. We are currently collecting more data in the form of surveys from additional cohorts of new mathematics teacher educators, and we are continuing to examine reports from other research projects that have sought to investigate the experiences of new K–12 mathematics teachers. Nevertheless, our initial findings as shared here confirm our hunch that becoming a new teacher in any setting—higher education or K–12—is not easy and is far from automatic. We are confirming, in other words, that we must help all new teachers, no matter where they fall on the continuum, to navigate that transition. Experiences from one domain may well apply to the other. For example, much is known about characteristics of effective professional development for K–12 mathematics teaching. How can those principles inform professional development of new faculty members who work in mathematics teacher preparation? Can promising aspects of the student teaching experience, so common in K–12 teaching preparation, be replicated in the preparation of mathematics teacher educators? Similarly, some participants reported that their universities were able to provide effective mentoring and had successful structures in place. Can K–12 teachers benefit from similar models? The next step of our research project considers these and other questions. By viewing our work as not so different from the population we serve (pre-service teachers), we may be able to be more empathetic in our personal interactions and more thoughtful and responsive in our teacher preparation courses and programs.

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PRE-SERVICE TEACHERS' ABILITIES TO INTEGRATE A SOCIAL JUSTICE PEDAGOGY IN MATHEMATICS

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The purpose of this study was to examine pre-service educators' abilities to integrate social justice *into mathematics. The study examined the change in their abilities to describe and create a social justice lesson over the course of a semester-long study of mathematics taught through a critical, social justice lens. Lessons were coded based on their social justice connection (real world, cultural integration, change agency) as well as the context in which mathematics was used (connected versus critical).*

Keywords: Teacher Education–Preservice; Equity and Diversity

Purpose of the Study

What is the purpose of teaching mathematics using a social justice framework? Adams, Bell and Griffin (2003) define social justice as the "full and equal participation of all groups in a society that is mutually shaped to meet their needs" (p. 1). In education, social justice themes are enacted to enhance students' learning and their life chances by challenging the inequities of school and society (Michelli & Keiser, 2005).

The literature provides multiple definitions of social justice mathematics. This study operationalizes social justice mathematics as a teaching and learning environment where students are introduced to the various issues of equity, diversity, and social injustices as they increase and strengthen their mathematical content knowledge. Additionally, students learn to use mathematics to identify and examine social issues with the intent to enact change. As such, students can begin to see math as a dynamic base of knowledge that can be used to "to create meaning and make sense of human and social experiences" (Gutstein, 2006, p. 4).

With the growing movement of social justice mathematics, various studies have been conducted with in-service educators (Gutiérrez, 2007); however, fewer studies have been conducted with pre-service educators, marking their abilities to integrate social justice (Muller, 2008). This study adds to the emerging literature on teaching a social justice pedagogy by addressing the following research question: What are elementary pre-service teachers' abilities to integrate social justice into mathematics?

Theoretical Framework

One opportunity for teachers to learn about being critical educators is during their teacher education programs. Therefore, it is important to understand how pre-service educators' integrate social justice pedagogy into the math classroom. Cochran-Smith (2004) writes, "In most of their pre-service programs, the role of the teacher as an agent for change is not emphasized" (p. 29). Price and Ball (1998) found that prospective teachers struggled "with learning to see classrooms from new perspectives, to reconsider the roles of teachers and students, issues of power and opportunity, and knowledge itself" (p. 262).

One key aspect of creating a socially just mathematics classroom is to include issues of inequities found in our local and global society. Teachers can integrate information about social injustices in order for students to begin breaking down stereotypes found in society and promoting change of inequities. Gutiérrez (2007) refers to this integration as critical math, one that "acknowledges the positioning of students as members of a society rife with issues of power and domination and which furthermore, takes students' cultural identities and builds mathematics around them in ways that address social and political issues in society" (p. 40).

Overview of the Study

This study was placed into the context of three elementary math methods sections ($n = 66$) which met for two hours weekly over the course of 15 weeks. Each week, the instructor modeled social justice connections that could be integrated into elementary math. During the course of the semester, pre-service educators were administered a pre- and post- survey regarding their understanding of and ability to enact a social justice curriculum in mathematics. As part of a final assessment, pre-service teachers developed a concept plan, a set of lessons focused on one particular topic, which incorporated an aspect of social justice.

A grounded theory approach was utilized to find themes within the pre-service educators' surveys and concept plans through text-based coding. This type of coding is defined as "a progressive process of sorting and defining those scraps of collected data that are applicable to your research purpose" (Glesne, 1999, p. 135). The coding of data was done in such a way as to be open to themes that emerged from the data, while at the same time working within a framework of codes that were consistent with the research question.

Results

Abilities Survey

A question on the pre- and post-survey asked pre-service educators to describe an activity or lesson that could be implemented into the math classroom that addresses social justice. On the pre-survey, 23 of the 66 pre-service educators commented that they were unsure of how to complete this integration. Sixteen others included the aspect of differentiating a lesson to accommodate all learners. The remaining pre-service educators included a connection to social justice into their initial activity. Some of these lessons incorporated diversity into math problem, such as this pre-service educator who wrote, "Including Hmong cultural traditions in math problems would illustrate a diverse approach." It is unclear, though, how fully infused these diverse cultural traditions would be into the problems. Mills (2008) contends we must move away from superficial treatments of diversity; this should also be done in the math classroom where diversity is not an add-on to questions but rather an integral part of a problem.

By the post-survey, all pre-service educators described a social justice math lesson that they would implement. In order to fully analyze the pre-service educators' abilities, this survey item was coded into three hierarchical categories as operationally defined in Table 1 below.

Table 1: Social Justice Survey Coding

Describe a mathematics activity or lesson you could implement into the elementary/middle mathematics classroom that addresses a social justice issue.		
Code & Operational Definition	**Pre-Survey Example**	**Post-Survey Example**
Social Justice as Real World Responses indicated a lesson that addressed a real-life situation	You could relate a lesson to school funding and student population. Determining how much money is invested for each student.	Analyze what students eat in terms of processed food and different nutritional levels—create graphs and statistical examples out of this.
Social Justice as Cultural Integration Responses indicated a lesson that addressed the culture and identity of students or a cultural awareness	I believe having a lesson based on different forms of money, for example the U.S. dollar, peso, euro and bartering. Discuss the different forms of payment then have the children set up a market and use the forms of payment to buy "cookies, pencils, drawings" and other objects for sale.	A teacher could have students create their own survey questions and ask peers about their culture, family life, and favorite ethnic food. Data can be collected and a bar graph or pie chart can be created to show similarities and differences within their own classroom community.

Code & Operational Definition	**Pre-Survey Example**	**Post-Survey Example**
Social Justice as Change Agency Responses indicated a lesson that addressed a social inequity and/or a way to enact change	Students could find out how much it costs to purchase one net (malaria nets) for children in Africa. They could be given a budget and work out how many nets they could buy and how many children's lives would be saved.	Students could compare wages of child labor to that of adult wages. Students can find the injustice in this and graph data on a bar graph. Students can then discuss plans on how to pass laws against child labor.

Social Justice Concept Plan

As part of a final assessment, pre-service educators developed a concept plan, which required the incorporation of social justice. These educators completed the concept plan individually or in small groups of two or three. The concepts plans were initially sorted into one of three main codes as shown in Table 1 above. Each concept plan was further coded based the utilization of mathematics. The mathematics usage fell into one of two categories: *Connected* versus *Critical*. A total of 29 concepts plans into one of the six possible themes as given in Table 2.

Table 2: Concept Plan Operational Definitions

	Connected Mathematics	**Critical Mathematics**
Social Justice as Real World	Concept plan integrates a real world issue yet does not enhance the student understanding of the concept through mathematics. (N = 10)	Concept plan integrates a real world issue and critically engages students in understanding the concept through mathematics. (N = 3)
Social Justice as Cultural Integration	Concept plan integrates a cultural concept yet does not enhance student understanding of the concept through mathematics. (N = 5)	Concept plan integrates a cultural concept and critically engages students in understanding the concept through mathematics. (N = 1)
Social Justice as Change Agency	Concept plan integrates a social justice inequity yet does not enhance student understanding of the concept through mathematics only sometimes leading students to enact change in their society or world. (N = 4)	Concept plan integrates a social justice inequity and critically engages students in understanding the concept through mathematics often leading students to enact change in their society or world. (N = 6)

For the purposes of this brief research report, we will further clarify the coding of the *Social Justice as Change Agency* concept plans. Pre-service educators who created a concept plan that was coded under *Social Justice as Change Agency* needed to have chosen a topic that reflected an inequity of societal, national, or global implication, as well as actively encourage critical analysis of that inequity, with the even greater intent of creating change.

Two pre-service educators created a *Social Justice and Connected Math* lesson centered on creating a global community dinner. These pre-service educators began their lesson by reading the book *If the World Were a Village* by D. Smith followed by a list of probing questions they would have their students discuss regarding the inequities expressed in the book, such as "What statistics were the most interesting? How does this book change the way you think about the world?" These questions led to the activity of creating a community dinner for a village of 100 people. Fifth grade students would use multiplication concepts to predict how much food might be needed and how much the dinner would cost. This concept plan was coded as connected math versus critical math because the math was not used to examine or analyze an inequity.

Pre-service educators created a *Social Justice and Critical Math* lesson centered on the injustice of child labor. Linking the lesson to Social Studies, these pre-service educators provided a historical context for child labor in the United States as well as current working conditions. They expanded their lesson to include child labor at a global scale citing "in Sub-Saharan Africa around one in three children are engaged in child labor, representing 69 million children. In South Asia, another 44 million are engaged in child labor." To incorporate mathematics, students would determine what fraction of the workers are children in areas such as Sub-Saharan Africa or South Asia. This lesson critically incorporated mathematics as the fractions were utilized to analyze the inequity. As an extension to social justice as a change agency, this lesson would "have students formally present their findings through graphs and charts, and explain their thoughts and opinions on child labor laws."

Conclusion

After participating in their mathematics methods class focused on social justice, all pre-service educators were able to create a social justice lesson plan. Their abilities to integrate social justice in the mathematics class ranged from using social justice to connect to the real world, promote cultural integration, or serve as an agent of change. Depending on the usage of mathematics, this integration was further defined to be connected or critical. Further research is needed to determine how this ability as a pre-service educator translates into the classroom as an in-service educator.

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THE EQUAL SIGN: AN OPERATIONAL TENDENCY DOES NOT MEAN AN OPERATIONAL VIEW

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Literature concerning pre-service teachers' (PSTs') content knowledge is not as developed as one might want (Thanheiser & Roy, 2011); this includes the equal sign. This paper considers one PST, Betsy, who has a strong tendency toward an operational view—that is, as an operator where the answer immediately follows the equal sign. Betsy also holds some relational ideas where she treats expressions on opposite sides of the equal sign as equivalent. Betsy serves as an example of a PST with a strong operational view of the equal sign. Her example also demonstrates that a PST with a strong operational view may hold some relational understanding. Reworking or modification of the current framework for equal sign understanding may be useful in light of her case.

Keywords: Teacher Education–Preservice; Algebra and Algebraic Thinking; Mathematical Knowledge for Teaching; Teacher Knowledge

Rationale and Background Literature

Consider one pre-service teacher's (PST's) solution to the box problem (see Figure 1).

Betsy: So 54 and 37 is 91 so that would go in the box, and then how that I know that it is 91 is that you do a check by taking 91 and subtracting one of the numbers… and I don't know what the 55 would mean.

What is the number that goes in the box? How do you know? 37+54= $\square$ +55

Figure 1: The box problem

This PST has gotten an incorrect answer by making use of the equal sign as an operator instead of an equivalence relation. She would like to teach children, but, based on her solution, she would not be ready to help children develop an understanding of the equal sign as a relation.

Equivalence underlies many aspects of mathematics (e.g., algebra, geometry). This suggests that a strong understanding of the equal sign would aid in a deep understanding of mathematics as suggested necessary by organizations and researchers within the mathematics education field (e.g., National Council of Teachers of Mathematics, 2000; Kilpatrick, Swafford, & Findell, 2001). Past researchers have focused on four views of the equal sign: operational, substitutive, basic relational, and full relational (e.g., Jones & Pratt, 2012). Some studies (e.g., Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005) lump the two relational views together and focus only on the distinction between operational and relational. A student holding an operational view uses the equal sign as an arithmetic operator or a place indicator where the answer immediately follows. A student holding a relational view knows that the expressions on each side of the equal sign are the same. Using this framework, Knuth et al. (2005) categorized students as holding an operational or relational understanding; students were credited with a relational understanding if a relational definition was given regardless of the presence of an operational definition. They found the majority of middle school students (n = 373) held an operational view. Those with a relational view tended to be more successful on problems involving the equal sign, such as the box problem (Knuth et al., 2005). Thus, the equal sign may be one way to help bridge ideas from arithmetic to algebra, aiding students in making it through the algebra "gateway" (Moses & Cobb, 2001). In order to aid in this process, teachers must have a relational view themselves.

Although there is a lot of literature concerning PSTs, a recent working group found that literature specific to PSTs' content knowledge is not as developed as one might want (Thanheiser & Roy, 2011). Stephens (2006) found the majority of PSTs in her study, all half way through their program, were able to interpret student solutions involving a relational use of the equal sign. More recently, Kinach (2011) found PSTs in a methods course self-reported a move from an operational view to a relational view, even if the operational "do something" meant to "solve for a variable" (p. 1420). The literature appears to focus on what PSTs are capable of developing, but does not identify what PSTs understand about the equal sign as they begin their first courses. This paper begins exploring this gap by examining one PST with a strong tendency toward an operational view of the equal sign.

The goal of this paper is two-fold. First, this paper will explicitly demonstrate that PSTs with a strong tendency toward an operational view exist by describing what one PST with this view does and says. Second, this PST will serve to demonstrate that a reworking of the current framework may be useful in further analyzing PSTs' knowledge.

Methods

One PST, Betsy (a pseudonym), near the end of her first content course (including place value and whole number operations) was interviewed for 70 minutes. The interview was videotaped, transcribed, and analyzed in light of the four views of the equal sign. Her responses to the first, second, and fourth interview questions (Figure 2) are the focus of this analysis.

Betsy's math background most recently includes calculus, in which she received a D. During the time of the interview, she was taking a trigonometry course in addition to the content course. Betsy identifies herself as strong in basic math and algebra, but weak in math beyond algebra.

1. Consider the following symbol:=
   What is the name of this symbol? What does it mean? Can it mean anything else? (If yes, what?)
   Give one example of how you might use this symbol. Can you give a different example of how you might use this symbol? If so, give the example and describe what makes it different. If not, state why not?
2. Box Problem: What is the number that goes in the box? How do you know?
   37+54= +55
4. Can you think of alternate solution strategies for the box task

Figure 2: The first, second, and fourth interview questions

Results and Discussion

The results will be broken into two larger sections: (1) Betsy's dominant operational view, and (2) Betsy's relational understandings.

Betsy's Dominant Operational View

When Betsy was asked what the equal sign meant, it was unclear whether she was thinking about the equal sign relationally or operationally. "Uhh, it's the symbol, I don't know, *is* is one of the meanings for it… or equivalent to." She was then asked for examples using the equal sign. In Betsy's first example, she writes *7+7=14* and says, "so [the equal sign] can be that the answering an equation or putting seven together." She implies that the equal sign produces an answer and that it is the operator to put the two numbers together, the embodiment of an operational view. Betsy also discusses the use of the equal sign in a formula (Figure 3). "Let's see, so like uhh area, let's say the area of a rectangle, for example, is the height times the width equals area." Betsy's written version of this formula is different from how formulas are typically presented ($A = l \cdot w$). In her verbalization of the formula, she both begins and ends with the area. She also labels the formula with the area and then leaves the area as the answer on the right of the equal sign suggesting a tendency toward an operational view.

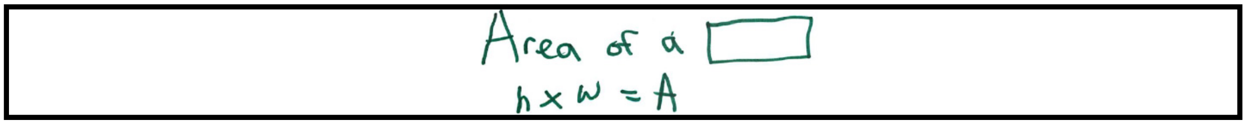

Figure 3: Betsy's work for the formula for the area of a rectangle

The operational tendency here coincides with the quotation at the start of this paper. In this initial quotation, Betsy uses the equal sign as an operator giving her the result of the left side (37+54) instead of making both sides equal. Further, Betsy's confusion with the 55 on the right hand side demonstrates that she is not considering the equal sign as a relation but as an operator. Thus, Betsy's operational view appears dominant.

Bits of Relational Understanding

Later in the interview, Betsy was prompted for alternate solutions to the box task where she demonstrates that she has access to a relational view. During this discussion, Betsy was able to express her confusion with the 55 as well as discuss a second interpretation of the question.

Betsy: Well, I'm not sure what you want me to do with the 55. Is this supposed to mean that this answer plus 55 is supposed to equal this side, umm, that's what I'm confused about… Like, if it's an equation, both side have to be equal and this side [left] equals 91 and this side [right] equals 91 plus 55 and those are not the same… a different way to interpret that… is to try to make them satisfy the other side…this side [right] you can do a bunch of things with because you have a blank spot. So if we wanted them to be equal to each other, we could take umm 91 minus 55 umm and put that in the spot, box

Betsy demonstrates the existence of both her operational—confusion with the 55—and relational—awareness that *37+54* and *91+55* does not hold the same value and is not equal—views. Additionally, she was successful at finding a number to put in the box that would make both sides equal by trying to make them "satisfy the other side" and she views the blank spot as something "you can do a bunch of things with."

The idea of satisfying the "other side" and that many things can go in the blank are in contrast to a third solution strategy that Betsy offers for the box problem.

Betsy: I wonder if maybe you could do a add something to one side type of thing, type of issue like algebra… I have no idea if it would work, but if we take the same thing and say 37 plus 54 equals and then just say *x* instead of the box, it's our unknown plus 55. If we wanted to do this algebraic way, we would want to find this out by isolating the *x*, so the first step would be to subtract 55 from both sides… and this is another rule in math, in algebra, that as long as you do it to one side, you can do it to the other if it's an [equation]

Interviewer: So why, why is that?

Betsy: Umm, well because this 55 minus 55 is umm is zero and so as long as you're not adding anything to the problem itself, the problem doesn't change… so we'll see how this works, so here's *x* and so now what we have to do is we have to come up with what this is… 36 which is what we got there

In this third strategy, Betsy offers an algebraic solution. Although she is not confident in this strategy, she notes at the end that the solution matched her second strategy. Betsy had to solve for *x* using pre-defined rules which she recalled but could not justify, and her notion of "many options" for what could go in the box becomes a single option for what *x* could be.

Betsy is a PST in her first content course with a strong tendency toward an operational view of the equal sign. Her discussion, examples, and solution strategies, give insight to how a PST with this view may think about and understand the equal sign. Nevertheless, Betsy does have some relational understanding available, and it is possible that activities building on those ideas will help her transition to a relational understanding of the equal sign.

Conclusions, Limitations, and Future Research

Betsy serves as an example of the knowledge a PST with a strong tendency toward an operational view of the equal sign may be able to communicate. Further, Betsy demonstrates that even PSTs who tend toward an operational view may hold some relational understanding of the equal sign and, when put in the right situation, be able to draw on, expand on, and use that view. This demonstrates that PSTs may not fit cleanly within the confines of the currently defined framework of equal sign understanding. A more refined framework, such as one including a clear transitional stage where students *tend* to draw on an operational view but are *able* to draw on a relational view, may better serve teacher educators in understanding the knowledge of PSTs.

There are at least two limitations to this study. First, Betsy is just one PST. Second, Betsy may not be a typical PST. In the future, more PSTs' knowledge should be studied to determine the proportion of PSTs with a tendency toward an operational view as well as create situations where their operational view is not sufficient. Four suggestions for future research follow.

1. Modify the current framework of equal sign understanding to aid in understanding what PSTs understand about the equal sign.
2. Analyze the proportion of PSTs with a strong tendency toward an operational view. Is it large enough that teacher educators should address the equal sign explicitly in classes?
3. Investigate whether it is common for PSTs demonstrating a strong tendency toward an operational view of the equal sign also have a relational understanding, which they can draw upon given the right situation.
4. Identify problem types/experiences aiding in development of a relational understanding.

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PRE-SERVICE TEACHERS' PERSPECTIVES OF 0.999… = 1

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This research highlights pre-service teachers' perspectives of repeating digits and specifically investigates the equality $0.\overline{9} = 1$. *Previous literature and reports indicate that some prospective teachers have misconceptions about this relationship. The researcher argues that understanding equivalence and magnitude of repeating digits is a key component for all teachers who are tasked with teaching these concepts to students in elementary and middle grades. Presented in this paper are pre-service teachers who demonstrate a fragile and undeveloped notion of single- digit repetends. The data presents evidence from prospective elementary and middle school teachers' journals that reveals their belief is* $0.\overline{9} \neq 1$. *Questions are raised and concerns are voiced about the implication of these findings.*

Keywords: Mathematical Knowledge for Teaching; Teacher Education–Preservice

Introduction

The concept of 0.999… = 1 has merited the attention of mathematicians and mathematics educators in the form of proof and students' understanding; however, the contexts of these studies have been confined to mathematics students (in calculus) or fields of study heavily laden with mathematics that use concepts such as limits. Only recently has concept of students' understanding of single-digit repetends been investigated using both prospective and in-service teachers. The Burroughs and Yopp (2010) study of five prospective teachers finds "deep-rooted" misconceptions about repeating digits that they claim originated from elementary school concepts such as whole number, fractions, and decimals. The Yopp, Burroughs, and Lindaman (2011) study of in-service teachers demonstrates the participants' misconception of 0.999…. Furthermore, they argue single repeating digits in equivalence statements, such as 0.333… = 1/3 and 0.666… = 2/3, are common in the middle school curriculum specifically in fifth grade and they should not be dismissed as trivial. They are careful to assume that their study does not generalize to others.

Justification

In light of the Common Core State Standards Initiative (2010), teachers are expected to help elementary students develop a full understanding of non-terminating decimals. Specifically, the standards require a student to develop computational fluency and number sense of decimals and fractions. For example in fifth grade, students are to understand 1/3 can be conceptualized as 1 divided by 3. In eighth grade, students are to comprehend the relationship between rational numbers and repeating decimals. Ni and Zhou (2005) point out that a typical middle school student does not view the distinction of rational numbers as important. They argue that the students' perceptions are skewed by concrete discrete numbers. The researcher suspects prospective elementary school teachers exhibit behavior similar to middle school students who do not acknowledge the significance of the 0.999… = 1.

Purpose and Research Questions

The purpose of this study was to investigate the understanding and the relevance of single-digit repetends in pre-service teachers. What knowledge do pre-service teachers possess about 0.999… and its relationship to one? How do they value that knowledge?

Methodology

Participants

This research was conducted at a small institution in the United States. The participants were undergraduate students seeking Bachelor of Science degrees in elementary education. After the successful completion of six hours of collegiate mathematics courses, these students enroll in a Numbers and Operations course. The Numbers and Operations course is designed to focus on mathematical content knowledge and pedagogical content knowledge specific to elementary mathematics. The course includes a deep understanding of fractions, decimals, and relationships that exist between them. Data were collected from 36 volunteers in multiple sections of the described course. Most were juniors in their programs of study and only five were males.

Data

The students are required to learn how to represent repeating decimals as fractions and vice versa. The instructor had embedded journal prompts throughout the course as a means to promote deep and reflective thinking about mathematics. As reported from the instructor, the topic of the journal prompts was usually mathematical in nature and required the students to reflect on the activities in class. At the time of data collection, interviews with the instructor suggest that the participants had been exposed to the statement, 0.999…= 1, without specific attention to its meaning or origin. Furthermore, the instructor reports that the statement was made when guiding the class through an activity that demonstrates how to turn a repeating decimal into a fraction and vice versa. The data for this study were collected from the routine journal prompt that the instructor assigned. The journal prompt is given below.

Are you convinced that 0.999…= 1? Write a few sentences that explain your thoughts.
Show two different ways to explain 0.999…= 1 to a future middle school student.

Results and Analysis

The researcher used thematic coding as the basis for the data analysis of the students' writing (Merriam, 2001). Thematic coding involved first coding categorically, based on evolving data, and then was defined in terms of broader properties that depict a continuum of dimensions. This type of technique allowed for open data exploration. A few of the common categories in coding included: equal due to rounding, decontextualized, and external authority. Thirty-two students (out of 36) reported in their journals that the two representations were not equal. Below typical responses and reasoning are given to signify the cognitive difficultly experienced by the participants.

Equal Due to Rounding

The three words most frequently included in students' responses were *rounding*, *estimation*, and *approximation.* Journal segments with these identifying descriptors demonstrated the students' internal conflicts stating that obviously these two quantities are equal by rounding. The researcher posits that students are unaware that rounding numerical values is a different subject matter than equivalent magnitudes. Furthermore, many students went on to say the person making the calculations could use as many decimal places as they deemed necessary and then estimate appropriately. For example, one student wrote, "Even though the number is 0.99999…., the only way I would call it 1 would be if I were asked to round the number. But logically, I would still call it 0.999999." Another student scripted, "It's true because basically you would round up to the next number. The only thing I don't understand is when do we [*sic*] know to round up."

Decontextualized

Other students referenced the context or lack of context for the given prompt. The students wrote about their desire for a concrete tangible model to use in making connections to an abstract idea. For these students magnitude of numbers has greater meaning in context when measuring specific concrete items no matter how large or how small the quantity. One student actually calls the two numbers *practically equivalent* citing the insignificantly small difference between them. The researcher argues that for this student she is not making claims about the mathematical magnitude or numeric value but rather the use of numbers in the context of everyday-life examples. Another student wrote, "This concept is very hard for me to wrap my brain around because it is not very visibly concrete. I feel as though at some point there is still a fraction of a point that is missing and contributing to inequivalency [*sic*]." Yet, another compared the subject matter to science writing, "I believe no, I feel like math is a set of numbers where as if you were doing science then it could be a little different. In science you estimate."

External Authority

Students defer the topic to an external authority, usually the course instructor, to inform them of the content they are required to know rather than engaging in the complexity of mathematics and self-discovery. They show resistance to engage in reflective thought about the nontrivial concept. According to the instructor, these participants have worked with decimals and fractions separately before they examined relationships between the two representations. To exacerbate this point students are not seeking understanding for themselves but rather an explanation that they can reiterate to their future students. One student's journal revealed, "I do not think it (0.999…) equals 1. How would you explain this to students? I am unclear on this reasoning." In this example, the student admits that she believes the statement is not true. Additionally, she does not seek clarification but rather for a recallable argument that she could use to convince her own students at some future time.

Mathematical Understanding

Only four students demonstrated sound mathematical reasoning or provided evidence that the statement $0.\overline{9} = 1$ was a fact that they personally believed to be true. These students had convinced themselves without creating any formal mathematical proof. All of the students who made such a statement reported taking action to investigate this claim outside of class. One student's journal revealed his interesting thought process.

> I did not believe you [the instructor]. I did a quick on-line search and revealed several proofs…. After some thought, I realized that if .999….. And 1 were not equal, I should be able to do things with them that I am not able to do, like find the difference, or their average, or name a number between them. It's funny how I was convinced more by the things that I could not do rather than the mathematical proofs. (journal entry student #5)

Although this student's statement addresses their personal conviction, it does not address how to discuss and teach such topics to fifth graders.

Conclusions and Implications

The purpose of the study was to demonstrate the variations of prospective teachers' understanding of 0.999… = 1. Participants revealed a gross lack of understanding and focused on the concrete, finite, tangible numbers rather than on the magnitude that the numbers represent. These results have broader implications. First, the researcher encourages others to consider the consequences, for middle school students if these results were true for most pre-service teachers. Will these pre-service teachers pass this deficiency to their students? Further research is warranted to determine the general state of teachers' understanding and how this knowledge is disseminated in the classroom. Secondly, these results demonstrate the participants' misconceptions of the equal sign. They are missing the precision that accompanies the representation. How do the misinterpretations of the symbol impact the foundational knowledge base in algebra?

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TRANSITION TO TEACHING: BELIEFS AND OTHER INFLUENCES ON PRACTICE

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We present the cases of 2 pairs of teachers who entered their teacher education programs with similar beliefs but whose teaching practices were substantially different by the end of their second year of teaching. We describe their beliefs and the evolution of their teaching practices across 2 years of a teacher education program and 2 years of teaching in their own classrooms. We then identify factors that contributed to the differences in these teachers' practices.

Keywords: Teacher Beliefs, Teacher Education–Preservice

The goal of this paper is to describe two pairs of elementary school teachers who started the teacher education program with similar belief profiles but whose teaching practices diverged significantly over time. We use Ernest's classification of beliefs to describe the teachers' beliefs and practices as they navigated the transition from preservice to inservice teacher, and we posit the factors that led to or inhibited change over time.

Theoretical Framework

The study of beliefs and practices in mathematics teaching has been a topic of scholarly interest for over 3 decades now with studies designed to articulate teachers' beliefs, identify and explain links or gaps between beliefs and practice, and posit means of changing beliefs. Ernest (1989) provided a scheme for making sense of relationships between teachers' beliefs and instructional practice, arguing that teachers' views of the nature of mathematics influence their models of learning and teaching mathematics. These models are then mitigated by the contexts in which the teachers work and lead to enacted models of learning and teaching. Ernest classified teachers' beliefs about mathematics as instrumentalist, Platonist, or problem solving. He described teachers' views of the teacher's role as instructor, explainer, or facilitator. Related to a teacher's view of teaching is a view of learning as active or passive. Ernest suggested that there are logical links among a teacher's beliefs about mathematics, views of learning, and enacted classroom practices. For instance, he posited that a teacher with a Platonist view of mathematics would likely act as an explainer, viewing learning as the passive reception of knowledge. Ernest noted that teachers are socialized by the context in which they work by the expectations of students, parents, fellow teachers, administrators, and the larger community. Further, a teacher's actions in the classroom are affected by the level of consciousness about the beliefs held and the degree of reflection on what happens in the classroom. When teachers are aware of their views, can justify them, identify viable alternatives, make deliberate choices, and actively reconcile conflicting beliefs and practices, they are more likely to adopt a problem solving orientation to mathematics, a facilitator role in teaching, and an active view of learning

Methods

The data corpus for this study included data collected on 15 participants across a 4-year period from their first year in a teacher education program through the end of their second year of teaching. For purposes of this paper we analyzed a subset of the data from 4 participants.

Participants

We identified 2 pairs of participants who entered the teacher education program with similar belief profiles but whose teaching practices differed markedly. They had taken one mathematics content course for elementary education majors prior to the study. During the study, they completed 2 mathematics methods courses for elementary education majors, the first of which included a mathematics-specific field experience. During the second and third semesters they participated in 4-week field experiences in local schools; the fourth semester was a traditional student teaching experience.

Data Collection and Analysis

Data on participants' initial beliefs were collected using the Integrating Mathematics and Pedagogy (IMAP) web-based beliefs survey (Ambrose, Philipp, Chauvot, & Clement, 2003). In addition, participants were interviewed once per semester for 4 years. Each participant was observed once during an early field experience, twice during student teaching and 4 times during each of the first two years of teaching. Data from the IMAP instrument were analyzed using the protocol provided by the developers. We then classified the teachers' beliefs about mathematics, the teachers' role, and learning based on IMAP data and interview data according to Ernest's scheme (1989). We further analyzed classroom observation data to characterize the teachers' practice according to Ernest's frame as well.

Findings

Comparing Laura and Jennifer

Laura and Jennifer began the teacher education program with beliefs about mathematics consistent with Ernest's description of an instrumentalist. Both equated finding a correct answer with fully "understanding" the mathematics, although both believed that children had the ability to construct novel approaches to problems that differ from adults' thinking. They believed that students should be actively engaged in learning and should have opportunities to create and express their own ideas. Their beliefs about active learning diverged, however, in their end goals. Laura wanted to understand students' thinking and strategies, while Jennifer wanted to make mathematics fun by letting students be "creative." Laura and Jennifer interpreted their teacher education experiences in different ways. For instance, during their first methods course, they had a chance to work one-on-one with an elementary school pupil weekly for 2 months on problem solving tasks. Because of her interest in making sense of students' thinking, Laura learned how to scaffold her pupil's learning process without giving away the answers to problems. Jennifer, on the other hand, was heavily influenced by her desire to make mathematics fun for her pupil, so she tried to provide her pupil with easier problems so that he would enjoy their time together.

For student teaching Laura was placed with a mentor teacher with an instructor orientation toward teaching and adhered to her mentor's expectations, although she occasionally used tasks that put her more in the role of a facilitator. Once she had her own classroom, however, she functioned more as a facilitator and used tasks that enabled her students to be active learners. As she created her own teaching style, she was able to reconcile her previous beliefs about teaching for the right answer with her new beliefs about how students learned and to mediate the context in which she was teaching. By her second year of teaching, Laura actively disregarded some school policies and used her textbook as more of a guide as she took on a facilitator role. Her orientation toward mathematics tended more toward problem solving, and her students were actively engaged intellectually in constructing mathematical ideas. Jennifer's beliefs led her to adopt an instructor orientation to teaching during student teaching and her first year of teaching. She frequently engaged students in using manipulatives and hands on activities, but this portion of the lesson generally had a playful, rather than mathematical, orientation. Jennifer was still developing her time management and classroom management skills, and this seemed to lead her to a more directive instructional style. She rarely asked students to talk about what they were doing; when she did, she asked for short answers to procedural questions. In her second year of teaching, Jennifer gained more control of her classroom, and she took on more of an explainer role, occasionally asking students to explain their thinking but still looking for a "correct" answer.

Although these two teachers began with similar beliefs, they became dramatically different teachers. Laura changed her belief about mathematics learning as a result of her experiences and used what she learned to become a confident facilitator of pupils' mathematical learning. Jennifer was strongly influenced by her view of mathematics as difficult and boring and her desire for students to have fun while learning. She filtered what she learned through these beliefs and selected pieces of what she learned to give her tools to implement these beliefs.

Comparing Jayne and Alex

Both Jayne and Alex initially exhibited a Platonist of view of mathematics, and they subscribed to an explainer view of the teacher's role, even though both of them valued listening to children. When working one-on-one with children during their first mathematics methods course, they both drew on their belief that it is important for teachers to listen to children to learn how to use children's mathematical thinking to alter their instruction. Thus, they both began to enact more of a facilitator orientation toward the teacher's role.

As they began their teaching careers Jayne and Alex shared a commitment to respecting and listening to children. Jayne maintained her facilitator role, engaged her students as active learners, modified curriculum materials to suit her students' needs, and focused on students' understanding, reflecting a Platonist view of mathematics. Jayne asked a lot of higher order questions and appeared to be moving toward a problem solving view of mathematics by the end of her second year of teaching. In contrast, Alex tended toward more of an explainer role, demonstrating procedures for students to emulate and often using manipulatives to illustrate the procedures. Alex's teaching practice was heavily influenced by the context of his school district, which required teachers to adhere tightly to a district level curriculum guide. This conflicted with Alex's views, and he was very articulate about this conflict between what he wanted to do as a teacher, what he was required to do by his district, and his decision to honor the wishes of his school administrators. He was likely moved in the direction of adopting an explainer teaching mode because he believed he needed to model mathematics first for his students.

Comparing Jayne and Laura

Although Jayne and Laura began their teacher education program with different beliefs, their teaching practices were similar by the end of their second year of teaching. Both exhibited internal authority and the capacity to make decisions that were in the best interests of their students despite administrative pressure to do otherwise. Jayne's beliefs about the importance of students as active learners and her role as a facilitator enabled her to resist administrative pressure, close her door, and teach the way she wanted to teach. After trying to follow an explainer model of teaching with passive learning and close adherence to the textbook, Laura concluded that her students were not learning mathematics. Thus, she crafted her own style of teaching, blending some of what was expected of her with some of what she learned in her teacher education program. Jayne and Laura both cared deeply about their students' mathematical learning, which seemed to empower them to internalize authority and craft lessons that actively engaged students in making sense of mathematics.

Discussion

Beliefs and Context Act as Filters

Both Alex and Jennifer used their beliefs as a filter when interpreting their methods courses. Jennifer believed that mathematics was boring and therefore looked for ideas that would help her make mathematics fun. In contrast, Jayne filtered her experiences through her belief in the importance of encouraging and building on students' thinking and adopted practices from her methods course that supported this belief. For Alex, context was a filter that affected his teaching practice. He believed that it was important to satisfy the demands of his administrators by teaching to the test, which led to demands on his instructional time that precluded a more facilitative style of teaching.

Teacher Education Makes a Difference

Some literature has reported that teacher education programs have minimal impact on preservice teachers' future teaching styles. Laura, in particular, provides a contrast to these findings as her beliefs and subsequent practice were clearly affected by her experiences in the program. All of the preservice teachers came to realize that mathematics can and does make sense and that the use of particular teaching practices and materials can help pupils make sense of mathematics. During the field experience, pre-service teachers were surprised by students' creative mathematical thinking and recognized that each student's thinking was different. From this they learned the importance of focusing on students' mathematical thinking and listening to children. Moreover, they began to see their role as facilitating student's mathematical thinking.

Reflection

As noted by Ernest (1989), reflection plays a key role in the ways that teachers enact their beliefs in the classroom. Jennifer was not inclined to reflect on her beliefs, and in fact was not able to articulate her beliefs clearly, suggesting that she may not have been aware of them. Therefore, she did not make deliberate choices about teaching actions. In contrast, Alex was able to articulate his beliefs as well as his concerns about the context in which he was teaching and was able to explicitly reconcile the conflicts between his beliefs and practices. Laura, on the other hand, became aware of her beliefs and the influence on her practices during her teacher education program. As she articulated her beliefs, she realized that they resulted in teaching practices that she did not condone, and she made a deliberate effort to change her teaching practices. Jayne was also very reflective, and although her beliefs did not change much, she was able to articulate her beliefs clearly from the start of her program, and she made deliberate decisions about instruction in light of her beliefs.

Conclusion

The data from this study provides further evidence of the importance of teachers' beliefs in shaping their teaching practices, although those practices can be mitigated by school contexts. A key finding of this study is that the preservice teachers' ability to articulate their beliefs was central to their deliberate and informed decision-making regarding teaching practices. This finding suggests that teacher education programs need to continue to work to help preservice teachers become aware of their beliefs, as they are an important filter of new experiences.

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MAKING THE TRANSITION TO REINVENTION TEACHING

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The Common Core State Standards for Mathematics (CCSS-M) place heavy emphasis on eight Mathematical practices consistent with development of social and sociomathematical norms. Present teacher preparation prepares candidates to work on these norms with students, but their field experiences too often acculturate them to traditional teaching practices. This research begins to develop a way to induct teacher candidates into the planning and reflecting practices of a master teacher who is using reinvention teaching to encourage children to re-invent mathematical ideas. The master teacher includes the candidates in her thinking about the sequence of activities children will take part in, candidates observe the lessons while focusing on student learning, and then together they reflect on what happened as students engaged in the activities and on the implications for the next day's lesson.

Keywords: Teacher Education–Preservice; Learning Trajectories; Instructional Activities and Practices

Purpose

The purpose of this research is to develop a model for mentoring undergraduate elementary teacher candidates into what we are calling the *reinvention approach to mathematics teaching*. The reinvention approach to teaching is a blend of standards-based instruction and Realistic Mathematics Education (RME) instructional design theory. In an RME approach, instructional materials are designed so that students re-invent important mathematical concepts. In the reinvention approach to mathematics teaching, the teacher's challenge is to create a standards-based environment that encourages students to actually do this *re-invention*. A reinvention approach to teaching requires highly specialized knowledge about mathematics teaching and learning and can be associated with a number of sophisticated planning and classroom practices. Our purpose in this research is to learn how to provide opportunities for new teachers to appropriate these practices as they engage in a variety of teaching and mentoring activities.

Theoretical Perspectives

In developing a model for understanding mathematics classrooms that take a reinvention approach, we ground our work in the *emergent perspective* (Cobb & Yackel, 1996). The emergent perspective is a theoretical position stating that learning is both a social and individual accomplishment simultaneously, with neither taking primacy over the other. An individual's cognitive reorganizations are made as she participates in and contributes to the emerging mathematical practices of the community of which she is a member. The framework that emerges as a result of this perspective coordinates both social and psychological perspectives of learning.

Table 1: Emergent Perspective

Social Perspective	Psychological Perspective
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activities

The framework above has been used to orient both our analysis of classroom interactions as well as our discussions with teachers who are attempting to incorporate a reinvention approach in their teaching practice. One of the main components of our teaching approach is that the teacher and students interactively establish the *social norms* that are productive for engaging in the mathematical discourse necessary for the reinvention of ideas. During discussions, the teacher is charged with facilitating turn-taking, encouraging students to explain their thinking, prompting students to ask questions when they do not understand, and asking for agreement or disagreement. In addition, the teacher and students negotiate productive *sociomathematical norms* when they talk about the *nature* of the mathematical arguments they are giving. Are the solutions different? Which ones are more sophisticated and efficient? Have students given an adequate explanation and what is our class criterion for acceptable? (Yackel & Cobb, 1996)

It is our contention that most approaches to "standards-based" or "inquiry math" can often be equated with establishing the social norms and the sociomathematical norms described above. While these are inarguably foundational to establishing classroom environments based on mathematical justification and argumentation, in order to guide students in reinvention of mathematical ideas, careful consideration must be given to the mathematical activity of the learner and the instructional tasks that support the students' learning. To this end, we have found that instructional materials developed using the ***Realistic Mathematic Education*** (RME) instructional design theory can be used to create mathematics instruction that guides students, together with their teacher, to re-invent important mathematical concepts. In an RME design experiment, the researcher begins by first reading research on students' cognitive development of the mathematics and uses these findings to design a hypothetical learning trajectory. This trajectory is comprised of the mathematical learning goals of the sequence along with the means of supporting it, including whatever *classroom mathematical practices* might emerge and whatever discourse, notation, gestures, tools and tasks can support these practices. These are tested in classrooms and revised as needed. Well-designed RME sequences are those that have gone through several iterations of such development.

Akyuz (2010) analyzed the *planning* (Table 2) and *teaching* practices (Table 3) of an expert reinvention teacher over the course of approximately six weeks. She documented the teacher's pedagogical practices in two broad categories, each consisting of five practices and associated teacher actions. The first category comprises the teacher's planning practices below.

Table 2: Planning Practices of Reinvention Teachers Using RME Sequences

1. Preparation. The teacher familiarizes herself with the mathematical content to be learned, students' preconceptions, and the conceptual mathematical ideas to be learned. The teacher also works through the instructional activities to unpack the intent of the tasks and to create a hypothetical image of the variety of pathways that can emerge as a result of the diversity of her students' reasoning.
2. Reflection (looking back). This includes a daily reflection on a) student strategies, b) the big ideas for the following day, c) the status of the social and sociomathematical norms, and d) student thinking. These allow the teacher to determine the status of the classroom mathematical practices.
3. Anticipation (looking forward). The teacher attempts to anticipate the best ways to introduce tasks the next day, works out problems to anticipate possible student thinking, and uses anticipated student thinking to imagine discussion topics.
4. Assessment. The teacher creates and implements formative assessments to ascertain the daily evolution of the classroom mathematical practices (i.e., learning) as well as summative assessments to document the finalized learning of the students.
5. Revising. The teacher revises instruction based upon the formative and/or summative assessments.

Not only does the expert reinvention teacher engage in planning practices outside the classroom, she also adopts practices *during* classroom instruction. This second category of pedagogical practices comprises the teacher's ***teaching*** practices:

Table 3: Teaching Practices of Reinvention Teachers Using RME Sequences

1. Creating and Sustaining Social Norms. This involves creating and sustaining environments that are conducive to children's reinvention in whole class and small group settings.
2. Facilitating Genuine Mathematical Discourse. This is characterized by introducing mathematical vocabulary when students have invented an idea, asking questions that promote higher thinking, restating students' solutions in clearer or more advanced ways, and using solutions to engineer a summary discussion during which mathematical practices are established.
3. Supporting the Development of Sociomathematical norms. These involve establishing the criteria for what counts as an acceptable mathematical argument, and as an efficient or elegant solution.
4. Capitalizing on Students' Imagery to Create Inscriptions and Notations. This is directly related to the rich imagery and tool development that are part of the RME instructional design.

As can be seen from above, the reinvention teacher has highly specialized knowledge and skills. The question is, what mentoring experiences best help novice teachers to appropriate the practices of reinvention teachers? Unfortunately, the most prevalent model for teaching practicums involves placing novice teachers in classrooms with experienced school mathematics teachers for 30–45 classroom hours. Under the best circumstances, novice teachers become acculturated into the practices of good *traditional* teachers. With this model, there is little hope that school mathematics teaching traditions will change. Even among the small number of pre-service teachers who have host teachers that are proficient at standards-based teaching, their host teachers typically have little, if any, training as a mentor. The primary goal of this project is to develop a model for apprenticing future teachers into a re-invention teaching tradition.

Modes of Inquiry

In order for future teachers to become experts in the reinvention approach to teaching, we argue novice teachers must be engaged in activities that focus them on the planning and classroom practices that are unique to this teaching method. During Summer 2012, we will begin the first step in our program designed to develop elementary school math reinvention teachers. Eight to twelve pre-service teachers (PSTs) who have already successfully completed at least two math courses for elementary school teachers will be enrolled in a new course designed to deepen their mathematical content knowledge and begin their acculturation into the practices of the reinvention teacher. The course will consist of an intensive two week field experience led by a master reinvention teacher, who is also a member of the research team and was the teacher Akyuz (2010) studied. The master teacher will teach a unit on ratio and proportion to a group of students entering 7th grade during the 2012–13 school year. PSTs will meet with the expert teacher both before and after the class and will be present during teaching in order to participate in Planning Practices and observe the Teaching Practices of the teacher. PSTs will have opportunities to interact intensively with the expert teacher as well as reflect on the actions and decision-making processes that accompany this approach. The research team will use the *cognitive apprenticeship* model (Collins, Brown, & Newman, 1989) as the framework for designing mentoring activities for the PSTs. The interactions and reasoning of the novice teachers will be documented.

Discussion

Standard-based teaching has been gaining momentum in this country with most didactic discussion involving the establishment of social environments conducive to problem solving. The Common Core State Standards place heavy emphasis on eight Mathematical practices that should be encouraged in students. These practices are consistent with social and sociomathematical norms for standards-based classrooms. Less emphasis, however, has focused on the nature of the problem solving that is to take place among students. This is where the reinvention approach provides more detailed practices for teachers. The goal of the instructional materials used in our summer program is for students to re-invent mathematics in a logical and meaningful way with guidance from a well-researched instruction theory (Gravemeijer &

Stephan, 2002, Yackel and Cobb, 1996). Teaching practices associated with the reinvention approach are sophisticated and, we posit, require specialized mentoring experiences for teachers. We expect our data to show that placing a cohort of pre-service teachers in a classroom situation in which they will co-plan mathematics lessons with an expert reinvention teacher enhances their ability to listen to students with more than an evaluative ear. We expect, through various mentoring experiences and observations of the expert teacher in action, that pre-service teachers will learn to use student thinking as the basis for their decisions and that co-planning will provide opportunities for the teacher candidates to construct deeper content knowledge. Our brief report will outline the mentoring experiences used, illustrate our findings and discuss implications for future teacher preparation.

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TRANSITIONS IN PROSPECTIVE MATHEMATICS TEACHER NOTICING

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Teacher noticing is key to student-centered instruction, but it cannot be assumed that teachers automatically know how to notice productively. This study engaged prospective mathematics teachers in targeted activities, including research-like analysis of unedited classroom video and group discussions of their analysis, for the purpose of helping them notice mathematically important moments during a lesson. The data revealed several transitions in the participants' noticing, including shifts in who and what they noticed, as well as in the specificity of their noticing. These shifts, as well as initial conjectures about what facilitated them and directions for future research, are discussed.

Keywords: Teacher Education–Preservice; High School Education; Middle School Education

Student-centered instruction requires teachers to carefully attend to, assess the potential of, and respond to student ideas during instruction. To support such instruction, one important transition that prospective teachers (PTs) need to make is in how they view classroom instruction—transitioning from a student to a teacher perspective. School-based field experiences might be a venue for supporting this transition, but it has been found that many PTs cannot meaningfully make sense of classroom interactions (Masingila & Doerr, 2002) and that the goals of such experiences are often ill-defined, with little connection to mathematical content or student understanding (Leatham & Peterson, 2010). Thus, meaningful learning from such experiences requires substantial teacher educator involvement (Oliveira & Hannula, 2008).

Teacher noticing has emerged as central to student-centered mathematics instruction (e.g., Sherin, Jacobs, & Philipp, 2011). Novice teachers, however, often fail to notice or act upon instances that experienced educators intuitively recognize and respond to (Peterson & Leatham, 2010). Fortunately, noticing is a skill that can be learned, with video cases found to be one effective method of supporting such learning (e.g., Sherin & van Es, 2005; Stockero, 2008). Many interventions use video clips that have been purposefully selected by experienced teacher educators, eliminating the opportunity for teachers to determine which instances are worthy of analysis. This is problematic since teachers need to learn to recognize instances that can be capitalized on to support mathematical learning.

This paper reports on findings from the first iteration of a design experiment in which prospective mathematics teachers engaged in research-like analysis of unedited videos of mathematics instruction during an early field experience. The purpose was to support their ability to notice, analyze and consider how to capitalize upon important mathematical moments that occur during instruction. We discuss transitions in the PTs' noticing that resulted from this work.

Theoretical Perspectives

Teacher noticing is defined in a variety of ways in the literature (e.g., Sherin et al., 2011). We follow Sherin and van Es (2005) in defining noticing to include three components—noticing, analyzing and deciding. We limit our study of noticing to *mathematical noticing.* This stems from a perspective that not all instructional events are equally important to notice in order to support mathematical learning. For instance, not all student thinking is equally valuable in supporting the goals of a lesson and, thus, should not be pursued in similar ways. We aim to focus teacher noticing on high-leverage instances of student mathematical thinking that provide rich opportunities for developing important mathematical ideas—those instances that have been conceptualized as Mathematically Important Pedagogical Opportunities (MIPOs) (e.g., Leatham, Peterson, Stockero, & Van Zoest, 2011).

Methodology

The participants were four secondary mathematics PTs; each was paired with an experienced mathematics teacher at a local school during a 14-week field experience. Each week, one PT video recorded a mathematics lesson. The instructional portions of each video were left unedited for the PTs' analysis. The PTs used the Studiocode video analysis software (SportsTec, 2011) to individually code the classroom video, tagging "mathematically important moments (MIM) that a teacher needs to notice during a lesson" and writing a brief explanation of their reasoning. The PI and a graduate research assistant (GRA) independently coded the same video and then met to discuss their own and the PTs' coding and decide which instances to discuss in a weekly meeting with the PTs. The PTs coded eight videos and met with the PI ten times.

Data for this analysis included the PTs' coded video timelines and video recordings of the weekly meetings. Building from coding frameworks used in previous research (Stockero, 2008; van Es, 2011), each MIM that a PT identified was coded by the PI and GRA for agent (who was noticed), topic (what was noticed) and specificity (whether noticing was mathematical or non-mathematical/general or specific). The meeting videos were used to help with the coding if a written comment was unclear. During this process, the researchers met regularly to discuss, refine and verify the coding. The codes were analyzed to characterize shifts in the PTs' noticing.

Results and Discussion

The data revealed that the PTs' noticing transitioned within all three coding categories. These shifts in noticing, as well as initial conjectures about what facilitated them, are discussed below.

Agent

The agent coding analysis revealed two shifts in noticing: (a) from student groups to individual students, and (b) from teacher moves to teacher-student interactions. In Video 1, when the PTs focused on students, it was largely on groups of students (76%), rather than individuals. For example,*"Students struggle to give a definition for [quadrilateral]"* (PT3) was coded as group-focused, while *"[The student] was able to see the connection, which led to finding an equation"* (PT4) was coded as an individual focus. The shift began in Video 2 and 3, with only about 50% of comments focused on groups of students. By the last three videos, the percent of student-centered comments focused on groups of students ranged from 25–31%. The cause of this shift is difficult to determine with certainty, but we conjecture that it may be the result of targeted facilitator moves that included replaying the video during meetings to focus on specific student comments and detailed discussions about student ideas. This shift is significant in that it supports the careful listening necessary to teach in a way that is responsive to student thinking.

The second shift—from teacher to teacher-student interactions—provided evidence that the PTs were starting to think about how teacher actions support student learning, rather than on teacher moves independent of students. When focused on the teacher in the early videos, the PTs considered how the teacher action was likely to support learning less than half the time. Starting with Video 5, a shift occurred, with over 70% of teacher-focused comments considering teacher-student interactions. The timing of this shift suggests that it was at least partly facilitated by an assignment in which the PTs read a portion of a paper about MIPOs (Leatham et al., 2011) that defined such moments as the intersection of important mathematics, student thinking, and pedagogical opportunity. In particular, the pedagogical opportunity component seemed to prompt the PTs to think about how a teacher might be responsive to student ideas.

Topic

Several noteworthy shifts in topic were noted. First, the PTs transitioned from noticing when students gave correct answers, to noticing incomplete answers (e.g., *"His definition was close but missing an important part—only 1 pair of parallel sides,"* PT4), incorrect answers, and student questions and confusion. They also learned to listen more carefully, as they noticed the teacher making incorrect assumptions about what a student had said (e.g., *"I think there might be some miscommunication here. I* *heard the student say 3, but the teacher heard 30,"* PT1) and described specific student thinking (e.g., *"[The student] gives an answer that [the teacher] finds interesting and never heard before. Instead of the simple splitting [of] either a or b, he used both addition and subtraction,*" PT2). Together, these shifts indicate that the PTs became more attentive to a range of student comments and to the details of them, along with what they might indicate about student understanding—important foci for student-centered instruction.

Directly related to the agent shift discussed above, a third shift was a decrease in claims about the understanding of groups of students, as the PTs became more focused on individuals. A fourth shift was from focusing on the mathematics itself (e.g., *"Order of vertices is important,"* PT4), to focusing on students' understanding of it. Finally, affective noticing, such as student participation, decreased as the PTs became more attentive to issues directly related to students' mathematical learning.

These findings are tentative because we conjecture that the topic of noticing may be more context-specific—dependent on what takes place during a lesson—than agent or specificity. It is also unclear what triggered the shifts in topic. They may be related to facilitator moves during the group discussion, or to other activities, such as asking the PTs to label each coded instance. Additional data are needed to better understand and verify these shifts and their causes.

Specificity

The PTs became both more focused on mathematical instances and more specific in their noticing. Despite the fact that the PTs were given instructions to code mathematically important moments (MIMs), some of their early noticing was focused on non-mathematical instances. For example, one PT noticed that the teacher incorporated exercise into her lesson. The shift to mathematical noticing was fairly easy to facilitate by pushing the PTs to discuss the mathematics in each coded instance; thus, no non-mathematical noticing was documented after Video 2.

A second shift, from general to specific mathematical noticing, is important because it indicates that the PTs were engaged in more detailed analysis, rather than making general observations about a lesson. For example, *"The student gave the wrong answer to the teacher's question"* (PT3) is general in nature, while*"The student asks a question about the placement of negative signs and the order [of the points] in finding slopes"* (PT1) is specific. In the first three videos, about 30% of noticing was specific. Collectively, the PTs' noticing became more specific than general in Video 4, but individuals shifted at different times. Two of the PTs transitioned to primarily specific noticing (over 66% of coding) beginning with Video 4, while the others did so in Video 6. In the last two videos, over 90% of all instances were specific in nature.

An analysis of the facilitator moves gives some indication of what triggered this shift. Throughout the weekly discussions, the facilitator intentionally focused on specific student comments and asked what it was that the teacher had to notice, often replaying a video segment so that the PTs could listen to what was being said. This is the likely cause of the earlier PT shifts. The later shifts, however, were likely also influenced by reading the MIPO paper, which highlighted student thinking about important mathematics as criteria for such an instance.

Conclusions

The results indicate that it is possible to facilitate transitions in mathematics teacher noticing, even early in a teacher education program. The data revealed that the PTs in this study became more focused on individual students and how teacher-student interactions affect learning, and better able to attend to specific details of mathematically important instances that surfaced during a lesson. These transitions are significant because teacher noticing of important mathematical ideas is central to student-centered instruction.

The intervention in this study included many elements that are often missing from field experiences: structured analysis, substantial mathematics teacher educator support, and clear mathematical goals. Together, these elements appear to have been effective in supporting desired transitions in noticing. Although the results are promising, more work is needed to fully understand the transitions, as well as what activities supported them. Understanding the details of transitions in noticing has the potential to inform interventions for supporting mathematics teachers in a range of contexts to more productively notice during instruction.

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POSITIVE TURNING POINTS FOR GIRLS IN MATHEMATICS: DO THEY STAND THE TEST OF TIME?

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The purpose of this study is to examine the equity issue in mathematics from perspectives not traditionally included in equity claims. This study offers a close up view of personal experiences that female preservice teachers have encountered in their own journey as students of mathematics as well as how they make sense of their experiences, especially as they learn to teach. Different themes that arise in this issue of mathematics equity were examined in a study conducted by Stoehr and Carter (2011). This paper extends the previous study by examining and discussing the data-derived theme that centers on girls who experienced positive turning points in mathematics.

Keywords: Gender; Teacher Education–Preservice

Literature Review and Theoretical Framework

Girls enter the mathematics classroom with just as much potential to excel as boys (Boaler, 2008; Huebner, 2009). However, the stories that some girls tell about their mathematical experiences in the classroom suggest that they do not believe that this is true (Stoehr & Carter, 2011). This can lead to girls bowing out or not pursuing higher-level mathematics classes in high school and college, which are required for the more lucrative careers in science, technology, mathematics, and engineering (Else-Quest, Hyde, & Linn, 2010).

Sometimes girls experience a positive turning point in their mathematics journey that puts them on a path to believing they can do well in mathematics. This generally occurs because of the efforts of caring and helpful mathematics teachers. It is these positive moments in time that can lead to a substantial change in how girls view their mathematical abilities, as teachers empower students to succeed (Drake, 2006). These events exemplify the belief in the power of education to truly change a person's life. Turning point stories in education give credit to educators being able to make a real difference in a student's life (Yair, 2009).

Teachers involved in the day-to-day school lives of students have the capacity to effect compelling turning points (Yair, 2009). Often this occurs in a student's educational career when a previously failing student is met with a new teacher. Suddenly someone new sees merit in the failing student and views the student as capable. The belief that the new teacher has in the student can lead to the student believing in herself. Trusting a new teacher is powerful and can lead to a turning point in a student's educational journey (Bryk & Schneider, 2002).

Lemke (2002) makes several valid points regarding the power of turning points events. He argues that fundamental changes in attitudes or habits of reasoning cannot happen on short timescales. What has to be evaluated is whether the turning point event fades away or gets erased by events that occur afterwards. The question to ponder is whether these turning points are in reality just pleasant anecdotes that occur rarely and are unable to be planned for and put into practice. Or are turning points that occur in the lives of students significant enough to lead to personal transformation and empowerment? (Yair, 2009). It is these events that must be examined in an effort to evaluate if the change has a longer-term agenda.

It is also the influence that one person has over another that is important to consider when looking at positive turning point events. If pre-service teachers who have experienced a positive turning point event due to the efforts of a teacher they had in their K–12 years, can identify and emulate in their own teaching the characteristics of the positive turning point teacher, then perhaps strides can be made to break down the barriers that often prevent girls from believing they are capable mathematic students.

Methods

Participants and Setting

This preliminary study focused on women who experienced positive turning points in mathematics during their K–12 school years. This theme was derived from a larger qualitative study currently being conducted at a Research I University in the Southwestern United States. One hundred forty-nine narrative stories were prepared by a diverse group of female elementary pre-service teachers primarily in their early twenties. Twenty-one narratives revolved around the theme of positive turning points in mathematics.

Data Collection

The participants all wrote a mathematics narrative as part of a requirement for an introductory teacher education course. In these narratives, termed "Well-Remembered Events," the preservice teacher candidates were asked to describe and analyze a particularly salient mathematics event from their own experiences as students in K–12. This genre of personal narrative was derived from Carter's (1994) work on well-remembered events as windows into the understandings preservice teachers have of teaching. The task consisted of a 2–3 page paper organized around the following parts (1) the selection of a particularly salient mathematics event from one's past experiences in mathematics as a K–12 student; (2) a detailed description of the event; (3) an explanation of why the mathematics event was memorable; and (4) a statement of what impact this turning point experience might have on the writer's understanding of what it means to be a teacher and how she perceives it will affect her future teaching of mathematics.

Findings

This paper will briefly touch on the narrative writings of the stories told by the participants that related to a positive turning point experience in mathematics they recalled during their K–12 years. The research to date includes five main patterns and themes. They are as follows:

Theme 1: Thank You for Caring About Me

Nine participants wrote about how a caring and understanding teacher helped them work through the struggles they were having in mathematics. They described the impact it made on their mathematical performance. One participant who was struggling to learn how to add and subtract negative numbers in seventh grade wrote:

> I was too afraid to speak up and ask questions because I did not want to be the only student in the class who did not understand this concept. Mrs. Brown must have realized I was struggling and asked me to stay after school with her for a little while. Mrs. Brown was willing to give up her time after school to make sure that I understood the topic. We went through each problem and to this day, I can remember the feeling of accomplishment that I finally understood not only how to add and subtract negative numbers, but I also understood the concept of it. This gave me a ray of hope for my future math career.

Theme 2: My Teacher Believed in Me

> Panic attack was the first thing that popped into my head when Mr. Granger informed my AP Calculus class that we were to have our first test at the end of the next class period. I stayed after class and explained that I was very overwhelmed with the fast pace. Mr. Granger told me I should attempt the homework study guide to the best of my ability, and come into his classroom at lunch for help. After going over the homework the next day, he assured me I had nothing to worry about. When Mr. Granger passed out the tests, I went into panic-mode. He reassured me that I knew what I was doing. He had me take slow breaths to calm myself down. After putting a smiley face on each page, I dove right into the test and received my first A in the course. By believing in me Mr. Granger helped me to be more confident in my knowledge of the subject.

This participant's response is shared and reflected in the writings of three other women who wrote about how having their mathematics teacher believe in them changed how they viewed their mathematical abilities. One participant said, “Teachers have the ability to make students believe in themselves like no other person can.”

Theme 3: My Teacher Showed Me How To “Do Math”

Four participants recalled successful mathematics experiences as a result of a teacher’s ability to show them how to do a particular mathematics problem. This led to positive mathematical feelings. One participant talked about how her sixth grade teacher drew pictures so that she could visually see the math. She wrote:

> I was working on mathematical story problems and I was feeling anxious and worried because I knew I was not going to get the assignment finished before the bell rang.
>
> As Mr. West was walking through mathematical story problems and I was feeling anxious and worried because I knew I was not going to get the assignment the rows of desks to check on our progress, he noticed that I was having some trouble. I remember him asking me to walk back to his desk with him. It was there that Mr. West taught me one of the ways that I could go about solving story problems. We read the problem together and then he asked me to draw a picture of what I thought was occurring in the problem. From there we walked through each step of the problem, each with little drawings. I will never forget him saying to me that I was a concrete thinker that needed to see things for them to make sense. I remember feeling more confident with math after this point in the school year.

Theme 4: Math Does Not Need to be Scary

Two preservice teachers recalled mathematics stories that revolved around mathematics being scary and something they could not master. With the help of a teacher they were able to see mathematics in a positive and accessible manner. One participant wrote about a third grade mathematics experience:

> It was after lunch that we worked on mathematics. I dreaded it during recess wondering what was wrong with me and why I could not comprehend it. In fact I felt like a failure. Long division was getting the best of me. The student teacher, Mrs. Allen, noticed how I struggled with it. She asked me to come back to the classroom after lunch instead of going to recess. I remember sitting there looking at the board as we worked on long division problems and being so terrified. She helped me to understand that long division was simply backwards multiplication. More importantly, she allowed a young girl with glasses to see that mathematics is not a scary subject.

Theme 5: A Calm and Approachable Mathematics Teacher Makes a Difference

Two participants wrote about how having a teacher who was calm and who offered extra support helped them to see they could be successful in mathematics. As one participant said, “My teacher gave me the courage and strength to succeed in mathematics. If it wasn’t for her, I would still be the student sitting in the third row of the class in fear of asking for help.”

Discussion

The preliminary results of this study revealed promising findings in relation to positive turning point events in mathematics. All participants reported a surge of confidence and a more positive attitude in mathematics as a result of a teacher’s supportive efforts. The amount of detail these preservice teachers can remember years after the turning point experience took place exemplifies the power that these experiences can have on girls and how they view mathematics. Allowing girls to see themselves as capable mathematics students through the eyes of a positive turning point teacher offers a glimpse of how important and significant these experiences can be.

All participants in the study wrote about how significant it was that their teachers believed in them and gave them opportunities to be successful mathematics students in their class. Teachers empower students in these magical types of moments (Yair, 2009). This pilot study seems to suggest that positive turning point events in mathematics can make an impact on some individuals who experience them.

Conclusion

There is great optimism for girls in mathematics (Huebner, 2009). This pilot study suggests that girls who have struggled in mathematics can be positively affected by teachers who believe they can be successful mathematics students. The participants in the study identified characteristics that are important for mathematics teachers to possess in order to successfully meet the learning needs of their students as well as have confidence in their students and offer them the chance to succeed in their class. Identifying positive factors and teaching characteristics that this study uncovered is a step in the right direction. Teachers must embrace the critical role of encouraging girls in mathematics (Gavin & Reis, 2003). It is important for mathematics teacher "to help girls to find the spark" that can lead to success in mathematics (James, 2009, p. 158).

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USING INTERVIEWS TO MOTIVATE TEACHERS TO LEARN MATHEMATICS AND REFLECT ON THEIR LEARNING

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Helping preservice elementary school teachers (PSTs) recognize that they have something useful to learn from university mathematics courses remains a constant challenge. We found that an initial content interview with the PSTs—deeply probing a mathematics concept—motivates PSTs to learn by (a) showing them the limits of their own understanding, (b) helping them recognize that knowing more than just procedures is worthwhile, and (c) helping them recognize that participating in mathematical activities during the course helps them learn. In addition, viewing the video of their initial interviews at the end of the course to reflect on their knowledge at the beginning of the course helps PSTs appreciate the content they learned and, thus, motivates them for future courses.

Background/Rationale

Preservice teachers (PSTs) often approach mathematics assuming that knowing how to apply procedures is synonymous with understanding (Graeber, 1999), a view that affects PSTs' views not only of children but also of themselves. With recent calls for a focus on having students in the United States develop conceptual understanding (Common Core State Standards Initiative, 2010; Kilpatrick, Swafford, & Findell, 2001), U.S. mathematics teacher educators need to focus on how to help their PSTs develop such knowledge. Although most PSTs and teachers can execute algorithms, many struggle when asked to explain them conceptually (Ball, 1988/1989; Ma, 1999; Thanheiser, 2009). PSTs often do not know that there is a rationale behind the procedures and thus equate procedural proficiency with understanding. Thus PSTs often approach mathematics courses for elementary school teachers in a perfunctory manner (Philipp et al., 2007), assuming that if they already know something they need not relearn it, and if they do not know something, they certainly cannot imagine having to teach it to elementary school students. We need, therefore, to help PSTs recognize the knowledge they are lacking and the value in learning mathematics beyond procedures.

Another challenge in teacher education relates to the "I-knew-it-all-along effect," or *hindsight bias* (Kahneman, 2011), which is the tendency of people after they learn something to lose their ability to recall what they knew or believed *before* they learned. The hindsight bias is problematic for preservice mathematics education because PSTs who forget how little conceptual understanding they held at the beginning of a course may not recognize what they learned, leading to their undervaluing the learning experience and failing to recognize the difference between knowing only procedures and understanding the underlying concepts. Consequently, we need to help PSTs establish a baseline of knowledge at the beginning of the course to which they can compare their learning while they move through the course.

Theoretical Framework

If we want PSTs to value their content courses for elementary school teachers and grapple more deeply with the content, then we believe that we need to first help PSTs

1. Recognize the limitations of their own understanding.
2. Recognize that knowing more than mathematical procedures is worthwhile.
3. See that engaging in mathematical sense making during the content course helps them learn the mathematics they need.

In addition we need to help PSTs

4. Reflect on what they have learned in their content courses.

Our assumption is that if PSTs realize #1–3, they will be motivated to learn mathematics in our content courses and that through #4, reflection, they will recognize what was learned. Our hypothesis is that these four goals can be realized by including particular kinds of experiences at the beginning and end of the course, combined with experiencing a sense-making approach throughout the class. In this study we conducted a videotaped content interview with each PST at the beginning of the course and then arranged for the PSTs to watch and reflect upon their interviews during a second interview at the end of the course.

Because interviews provide opportunities for instructors to ask probing follow-up questions to assess the depth of a PST's understanding, the instructor can give immediate and personal feedback and assessment. Furthermore, interviews enable both the instructor and PST to develop awareness of what the PST knows and does not know. By becoming aware of their own understandings, PSTs begin to develop intellectual integrity (Chamberlin & Powers, 2007). Furthermore, the PSTs are also aware that their interviews have made public to their teacher what they do and do not understand, thereby encouraging PSTs to more openly address their knowledge gaps during class, rather than concealing them from their instructor, their classmates, and even themselves.

Viewing the videotaped interview at the end of the course enables PSTs to (a) compare their knowledge at the end of the course to their knowledge at the beginning, and (b) reflect on their learning and the experiences that engendered the learning.

Methods

Participants in the study were 23 PSTs enrolled in a 10-week mathematics course for preservice elementary school teachers at a large state university. The course met for two 110-minute class sessions per week. The class was focused on whole numbers and operations.

The data analyzed were drawn from two 10–15-minute videotaped individual interviews with the 23 PSTs conducted at the beginning and at the end of the course and student reflections collected at the beginning of the course (immediately after the first interview), in the middle of the course (week 6), and at the end of the course (before the last interview). The first and last reflections were collected via web surveys and the middle one via e-mail. Sample reflection questions are listed in Figure 1. Because little is known about how PSTs experience such an interview, we analyzed the data using a grounded theoretical approach with *open coding* (Strauss & Corbin, 1990). In reading all PST responses, we identified four themes. Next, two coders coded all the data by theme and then met to resolve disagreements through discussion. We share those themes in this paper.

Sample reflection question from the middle of the course (collected via e-mail)

- Thinking back to the interview at the beginning of class, how do you think that interview affected your learning in this class? Please try to be as specific as you can in this response.
  Sample end-of-course question, posed after PSTs had viewed their initial interviews
- Watch your first interview and give your general reactions to the interview; we are very interested in your reactions, so feel free to share.

Figure 1: Sample reflection questions

The instructor probed the students' understanding of regrouping in the context of addition and subtraction using Thanheiser's (2009) interview protocol and framework. This framework was then used to categorize the PSTs' conceptions in the two interviews. Each interview was independently coded by two coders with 86% agreement. All disagreements were resolved through discussion.

Results and Discussion

Consistent with previous research (Thanheiser, 2009), at the beginning of the course, 17 PSTs held an incorrect conception of number and 6 held a correct conception. At the end of the course, 21 PSTs held a correct conception; only 2 PSTs held an incorrect conception. In this section we (a) briefly review the themes that emerged from the PSTs' reflections on their interview experiences, and (b) discuss the themes that emerged from PSTs' viewing their initial interviews at the end of the term and reflecting on viewing the interviews.

Four themes emerged throughout the interview reflections (see Table 1). For an in-depth description of the themes, see Thanheiser et al. (2012). These themes, taken together, indicate that PSTs feel motivated to learn mathematics in their content courses as a result of the initial interview and the reflections on that experience.

Table 1: Themes Identified in 23 PSTs' Reflections

Theme	# of PSTs
1. PSTs are not typically asked to think about or explain mathematics beyond procedures.	5
2. The PSTs recognized the limitations of their own understanding, the value of knowing more than the procedures, or both.	15
3. The PSTs stated that they were motivated to learn the mathematics of the course.	21
4. The PSTs reflected on their own knowledge and learning.	19

At the end of the course, when PSTs watched their interviews from the beginning of the course and reflected on that experience, two themes emerged (see Table 2).

Table 2: Two Themes Identified in the 23 PSTs' Reflections After Watching the Interviews

Theme	# of PSTs
Theme A. Recognition of their limited mathematics knowledge at the beginning of the course (akin to Theme 2 above but in stronger language). PSTs explicitly acknowledged feeling embarrassment, shock, or surprise at their own lack of knowledge and noted how confusing and unorganized their explanations had been.	18
Theme B. Recognition of what they learned in the course. PSTs expressed pride in their correct explanations and feelings of accomplishment and recognition or pride at what they had learned in the course.	13

Kendra, for example, reflected on viewing her interview:

> I was shocked at the level of understanding of math I had. I am surprised that it did not occur to me to think about the ones that were being carried over were tens or hundreds, not just one. My understanding of numbers was the idea that the number represented one value. For example, I did not know that the one carried to the tens place was not a 1 but a group of ten ones. Overall, I am surprised by my thinking at the time.

The notion of experiencing shock at their own understanding was prevalent in many reflections. A characteristic of elementary-mathematics learners is that after learning an idea, one had difficulty imagining not knowing it. Consider, for example, the fact that $345 = 300 + 40 + 5$. This fact, once understood, seems obvious; however, it is a fact with which many people struggle. A result of this phenomenon is that PSTs often leave their content courses thinking that they learned nothing because now that they do understand the mathematics, not knowing it seems unimaginable. The interview (and the fact that it is recorded so that PSTs can watch it later) enables PSTs to remember what it is that they knew and did not know at the beginning of the course and thus to appreciate what they learned in the course. This knowledge, in turn, can serve as a motivational tool for future courses.

Several students commented that, in addition to being shocked, they recognized what they had learned. Aubrie, for example, stated, "After watching my first interview, I was really shocked, to say the least. I couldn't help but laugh at myself at how little I actually knew, and now, realizing how far I have come. If I had only knew [*sic*] then what I know now."

As a result of participating in an initial content interview and viewing that interview at the end of the term, PSTs were motivated to attend carefully to the activities in the content course and to engage with them more deeply than they otherwise would have. In addition, PSTs were aware of and proud of what they had learned. Thus PSTs valued their course/learning and were motivated for future courses.

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EXAMINING CONNECTIONS BETWEEN MATHEMATICAL KNOWLEDGE FOR TEACHING AND CONCEPTIONS ABOUT MATHEMATICS TEACHING AND LEARNING

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This study explores connections between preservice elementary teachers' mathematical knowledge for teaching (MKT) and their conceptions about the teaching and learning of mathematics. Researchers administered the MKT measures developed by the Learning Mathematics for Teaching Project and used the Mathematics Experiences and Conceptions Surveys (MECS) to measure attitudes, beliefs, and dispositions. Preliminary findings from data collected at two universities show significant gains in MKT and attitudes about teaching mathematics over the duration of an elementary mathematics methods course. Regression models indicate that factors within the course did not account for a significant portion of the growth in MKT. However, changes in MKT, along with course and fieldwork factors, helped explain a significant portion of the preservice teachers' changes in attitudes.

Keywords: Mathematical Knowledge for Teaching; Teacher Education–Preservice; Teacher Beliefs

Purpose of the Study

This paper reports preliminary findings from an exploration of elementary teachers' development of Mathematical Knowledge for Teaching (MKT) and the connection between MKT and conceptions (defined as attitudes, beliefs, and dispositions) about mathematics teaching and learning. During the fall 2011 semester, preservice elementary teachers (PSTs) enrolled in mathematics methods courses at two U.S. universities completed pre- and post-iterations of the Learning Mathematics for Teaching (LMT) Project's MKT measures (Hill, Schilling, & Ball, 2004) and the Mathematics Experiences and Conceptions Surveys (MECS) (see Welder, Hodges, & Jong (2011) for details regarding MECS instrumentation). The purpose of this study was to examine changes in MKT over the duration of a mathematics methods course and to explore potential relationships between PSTs' MKT, experiences afforded in mathematics methods courses and related fieldwork, and the attitudes, beliefs, and dispositions PSTs hold towards mathematics teaching and learning. This investigation is part of a larger research agenda to understand how teacher educators can foster student achievement by facilitating the development of teachers' MKT and conceptions about mathematics.

Perspectives

The process of learning to teach mathematics is multifaceted and includes a number of domains known to influence the instructional practices teachers employ and thus the learning opportunities provided to their students. In this study, we focus on two domains: MKT and conceptions about mathematics teaching and learning. We argue that the development of PSTs in each of these domains is a critical aspect of mathematics teacher education.

Over the years, research has identified content knowledge as a significant factor in the pedagogical decisions of mathematics teachers and the achievement of their students (Graeber, 1999; Hill et al., 2008). However, the work of the LMT project has confirmed that the content knowledge needed for teaching mathematics requires more than general mathematical ability (Ball, Thames, & Phelps, 2008; Hill et al., 2004). The daily tasks of teachers, such as interpreting a child's work or developing an alternative explanation, require skills beyond those needed to perform a mathematical procedure. These responsibilities require *unpacked* mathematical knowledge, in addition to pedagogical knowledge, demanding teachers to know *more* and *different* mathematics than their non-teaching counterparts. For years, LMT researchers have worked to conceptualize and measure the various domains of what they have coined Mathematical Knowledge for Teaching (Ball et al., 2008); our study is grounded in their conceptual framework. Studies have indicated that MKT plays an important role in the practices employed by mathematics teachers and have identified a strong relationship between teachers' MKT and the mathematical quality of their instruction (Hill et al., 2008). Therefore, we argue that teacher educators can support PSTs by attending to the development of their MKT. This initiative aims to understand of how MKT is affected by factors within teacher education.

In addition to MKT, Ball, Lubienski, and Mewborn (2001) suggest that efforts to improve teacher quality should consider the ways in which teachers think about the processes of teaching and learning. Supporting research has shown that teachers' beliefs and attitudes heavily influence their instructional practices (Ball & Cohen, 1999; Wilkins, 2008) and the opportunities their students have to engage in significant mathematical thinking (Staub & Stern, 2002). For example, Wilkins (2008) found a strong link between beliefs, attitudes, and the extent to which teachers employ reform practices in their classrooms. In the current study, we use the term *conceptions* to encompass beliefs, attitudes, and dispositions, three central and interrelated subconstructs of how one thinks about the teaching and learning of mathematics (cf. Welder et al., 2011). Although researchers vary in their definitions of these subconstructs, consensus exists on the critical importance of aligning PSTs' conceptions with effective mathematics teaching practices within teacher education (Metzger & Wu, 2008).

The preparation of PSTs must be multidimensional to support the development of robust MKT and a set of beliefs, attitudes, and dispositions towards mathematics that will foster student learning. However, little is known about how these vital domains of learning to teach mathematics co-evolve over time. Furthermore, there is a dearth of research explicitly investigating relationships between the domains. In one study, Philipp et al. (2007) found connections between teachers' knowledge and beliefs about mathematics to have significant impact on instructional practices. These results are promising and offer support to our hypothesis that teachers' knowledge and conceptions about teaching and learning are interrelated.

Methods

Participants and Context

The results of this study are based on data collected on 59 elementary PSTs within two universities in the Eastern United States. Participants were enrolled in an elementary mathematics methods course as part of a teacher education program for initial licensure during the fall semester of 2011. Participants at University A were first-semester graduate students enrolled in a non-traditional, two-year elementary certification program. These students complete one three-credit mathematics methods course and a co-requisite field experience requiring them to observe ten hours of elementary mathematics instruction. Participants at University B were undergraduates, enrolled in a traditional, four-year elementary certification program where students complete one three-credit mathematics methods course during the semester prior to full-time student teaching. The co-requisite field experience for this course places PSTs in a classroom full-time for five weeks, where they teach a minimum of twelve mathematics lessons.

Data Analysis

The mathematics methods course served as a common point of interest for the researchers to examine PSTs' MKT and conceptions. Data were collected at the beginning and end of the course using the LMT Project's MKT measures and the MECS, under development by Welder et al. (2011). PSTs completed computer-adaptive pre- and post-administrations of the 2001 Number Concepts and Operations MKT measure using the LMT online assessment system. The MKT measures have undergone a rigorous validation process with the application of item response theory and are used extensively in mathematics education research (Hill et al., 2004). Concurrent with MKT administrations, participants completed two iterations of the MECS, being developed to study the evolution of PSTs' conceptions over time. MECS instruments correspond with significant benchmarks in teacher education programs. Participants in this study completed MECS-M1 and MECS-M2, designed to be pre- and post-surveys for mathematics methods courses and co-requisite fieldwork experiences. The subscales within MECS-M1 and MECS-M2 have been found to be highly reliable (Welder et al., 2011) and the surveys continue to undergo rigorous validation. We began our analyses by conducting an analysis of variance (ANOVA) to examine changes in MKT, attitudes, beliefs, and dispositions over the course of the semester. We then created multiple regression models to examine factors accounting for the variance noted in MKT and attitudes about mathematics teaching.

Results

An ANOVA of the aggregate data showed a statistically significant difference in PSTs' MKT scores from pre to post ($F(2, 56) = 6.24$, $p = .015$). However, ANOVA results of the disaggregate data indicated a significant gain in MKT scores for University A ($F(1, 40) = 5.97$, $p = .019$) but not University B ($F(1, 15) = 1.97$, $p = .179$). A significant change in attitudes was also noted for University A ($F(1, 40) = 11.69$, $p = .001$) but not for University B. No significant changes were found in beliefs or dispositions for either group. This could be due to the small sample sizes included in these preliminary analyses.

To examine changes in MKT scores and attitudes, we developed regression models in search of explanatory variables for University A. The overall regression of *MKT_post* on *MKT_pre* and *field_exp* was statistically significant [$R^2 = 0.24$, $F(2, 39) = 5.32$, $p < 0.027$]. The two factors accounted for 24% of the variance in PSTs' change in MKT scores following the methods course, but 14% of that variance was accounted for by initial MKT scores. It is surprising that field experiences accounted for the remaining 10%, considering the minimal fieldwork requirements at University A. However, since their methods course is graduate-level and tends to be taken in the program's first semester, this is commonly the PSTs' first elementary classroom experience in years. Thus, the influence of this fieldwork may be explained by the PSTs' lack of recent exposure to elementary education and a potentially more advanced level of maturity.

The overall regression of *attitudes_change* on *MKT_change*, *MM_mat,* and *field_exp* was also statistically significant [$R^2 = 0.37$, $F(3, 38) = 4.31$, $p < 0.045$] for University A. The three factors together explained 37% of the variance in PSTs' change in attitudes towards mathematics following the mathematics methods course. Change in MKT scores accounted for 15% of this variance, methods course materials accounted for another 15%, and field experiences accounted for an additional 7%. It is not surprising that the three variables entered into the model explained a significant portion of PSTs' positive change in attitudes given that attitudes typically have a positive or negative orientation and can shift more easily than beliefs (Philipp, 2007).

Conclusion

To capture a more holistic picture of the process of learning to teach, it is critical that research co-examines the development of content knowledge and conceptions about teaching and learning. Our results suggest that teacher education can significantly influence PSTs' MKT and attitudes about teaching, simultaneously, over the course of only one semester. Interestingly, when the data from the two universities was disaggregated, significant changes were noted only for the PSTs in the graduate certification program. We realize that our analyses focused on relationships; hence, no causal claims can be made. Additional research will be necessary to explore the variables accounting for this difference and examine the co-development of these constructs. We suspect a number of factors, including the context of the program, course instructor(s), and maturity level of the students, might be influential.

While small sample sizes were a clear limitation in this preliminary study, we are beginning to find evidence that links MKT and conceptions about mathematics teaching and learning. It is promising that, for University A, MKT accounted for a significant portion of PSTs' changes in attitudes alongside the mathematics methods course and fieldwork experiences. Continued research is needed to validate our findings with larger data sets and to longitudinally examine the development of PSTs. This study is a small, first step towards an understanding of how teacher educators can facilitate the interrelated development of MKT and conceptions about the teaching and learning of mathematics.

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PROSPECTIVE ELEMENTARY TEACHERS' JUSTIFICATIONS FOR THEIR NONSTANDARD MENTAL COMPUTATION STRATEGIES

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We report on the justifications that prospective elementary teachers offered for their nonstandard mental computation strategies. Seven participants were interviewed before and after a whole-number unit in a mathematics course designed to promote number sense development. We investigated the sense that the participants made of their new non-standard strategies (one invalid) for addition and subtraction by examining their justifications.

Keywords: Number Concepts and Operations; Teacher Education–Preservice

We briefly describe the number sense of prospective elementary teachers, focusing on how they reason in mental computation. Mental computation is closely linked to number sense (Sowder, 1992). Teachers need good number sense in order to support students' inquiry learning because a teacher whose understanding of numbers and operations is bound to standard algorithms is not equipped to make sense of children's often-unorthodox solution strategies (Ball, 1990; Ma, 1999; Sowder, 1992; Yang, Reys, & Reys, 2009). Yet, studies of preservice elementary teachers have found that this population tends to exhibit poor number sense even after having completed required college mathematics courses and that their understanding is tied to the standard algorithms (Tsao, 2005; Yang, 2007; Yang, Reys, & Reys, 2009).

Good number sense is exhibited in the use of a variety of computational strategies, which are selected based on the particular numbers at hand, rather than using an automatic procedure for the given operation. Inflexibility manifests as overreliance on the mental analogues of the standard paper-and-pencil algorithms. Flexible individuals use strategies that often stray far from standard, as in reformulating computations or rounding and compensating (Carraher, Carraher, & Schliemann, 1987; Heirdsfield & Cooper, 2002; Markovits & Sowder, 1994; Sowder, 1992; Yang, Reys, & Reys, 2009). Our focus is on investigating prospective teachers' understanding of the computational strategies they use, so that we can better design instruction.

Setting

We report on the second iteration of a classroom teaching experiment (CTE) in a mathematics course for prospective elementary teachers. Topics in the curriculum include quantitative reasoning, place value, meanings for operations, children's thinking, meanings for algorithms, representations of rational numbers, and operations involving fractions. In the CTE, intended to foster number sense, we identified in the curriculum particular opportunities to engage students in activities such as authentic mental computation, as well as to facilitate rich discussions concerning students' strategies and ways of reasoning. Over time, a shared set of strategies was established via mathematical argumentation. These strategies were given agreed-upon names, and the class maintained a list with examples of each.

Methods

This study took place at a large, urban university in the Southwestern U.S. The participants in were students enrolled in a first mathematics content course for prospective elementary teachers, belonging to a four-course sequence. The second author was the instructor of the course.

Seven of the students, all female undergraduates participated in pre/post mental computation interviews. Participants were asked to solve story problems for each of the four basic operations. These problems were presented verbally and written. For example, participants were asked, *If Bobo buys an oboe for $49 and then sells it for $125, how much is his profit?* The participants were asked to solve each story problem mentally and to describe the steps that they had performed. They were not allowed to do any written work during these interviews. Whenever a participant used a strategy that departed from the mental analogue of the standard algorithm, she was asked to justify this strategy (e.g., “Why does that work?”).

For each problem posed, the participant’s strategy was coded, using a previously developed scheme (Whitacre, 2007). Once the strategy coding had been completed, we identified the justifications that had been given for each nonstandard strategy. Through a process of open coding, we generated mathematical idea codes to describe the ideas that students had used to justify their strategies. The set of codes and operational definitions was refined through a process of constant comparative analysis (Strauss & Corbin, 1998). Our sensitivity to ideas of interest and our descriptions were informed by the literature, especially Carraher et al. (1987), Heirdsfield and Cooper (2002), and Thanheiser (2010).

Results

In this paper, we focus on addition and subtraction. We report on the nonstandard strategies that participants used in the interviews, together with the ideas that they offered as justification. We give examples and discuss nuances in the participants’ reasoning. There were four distinct nonstandard addition strategies used, and all of these were valid. (See Table 1.)

Table 1: Participants’ Nonstandard Addition Strategies and Justifications

Strategy	Mathematical Ideas used in Justification
Left to Right Separation. Added place-value wise from left to right. Decomposed the addends, added ones, tens, and hundreds separately, and then combined.	*Tens and Ones.* Gave a decomposition justification, describing the addends and their sum as composed of tens and ones and hundreds.
Leveling. Altered the problem such that part of one addend was taken and given to the other prior to finding their sum.	*Addition is Associative*. Expressed, either formally or informally, the associative property of addition.
Single Compensation. Rounded one of the two addends up or down, then compensated by adding or subtracting appropriately.	*Inverse Operations.* Reasoned about compensation in terms of the inverse operation: addition undoes subtraction; subtraction undoes addition.
Double Compensation. Rounded both addends prior to computing. Typically, one was rounded up and the other down. Added and then compensated by adding or subtracting, taking into account the net effect of her two rounding moves.	*Inverse Operations.* As with Single Compensation but compensation applied to the balance of the two rounding moves. *Compensation Balance.* Adding an amount to one addend and subtracting from the other resulted in a net change to sum.

Example 1

In her second interview, Val used Single Compensation to find the sum of 38 and 99. She quickly answered that this was 137. Val explained her reasoning to the interviewer:

Val: Um, I made ninety-nine into a hundred by adding one. Uh, and then a hundred plus thirty-eight equals one thirty-eight. And I just subtracted the one and got one thirty-seven.
Interviewer: So, the one that you added, where did it come from?
Val: Well, I just added ‘cause I know I can add one to make it a hundred. And so, if I add one to make it a hundred, I know I have to subtract one from the final answer to make it even.

Val thought about rounding in terms of adding one to the sum. In contrast to a Leveling strategy, the one that she added originated outside of the given computation. Val reasoned that because she had added 1 to 99 before computing, she needed to subtract 1 from the sum of 100 and 38 “to make it even.”

There were four distinct nonstandard subtraction strategies used. Three of these were valid, and one was invalid. (See Table 2.) We offer two specific examples of participants' reasoning.

Table 2: Participants' Nonstandard Subtraction Arguments and Justifications

Strategy	Mathematical Ideas used in Justification
Aggregation. Either began with the (a) subtrahend and added onto it in convenient chunks until the minuend was reached or (b) minuend and subtracted off the subtrahend in convenient chunks.	*Difference as Distance Between.* Described the difference between the minuend and subtrahend as a distance between number-locations.
Minuend Compensation. Rounded the minuend prior to computing and found the difference between the subtrahend and rounded minuend. Compensated appropriately for rounding.	*Inverse Operations.* Reasoned about compensation in terms of the inverse operation: addition undoes subtraction; subtraction undoes addition.
Invalid Subtrahend Compensation. Rounded the subtrahend and found the difference between the minuend and rounded subtrahend and then subtracted from or added to the difference to compensate for change to the subtrahend.	*Inverse Operations.* Reasoned about compensation in terms of the inverse operation: addition undoes subtraction; subtraction undoes addition.
Valid Subtrahend Compensation. Rounded the subtrahend and found the difference between the minuend and rounded subtrahend and then added to the difference or subtracted from the difference to compensate.	*Compensating for Effect.* Reasoned about compensation on the basis of the effect of a rounding step, as opposed to the action itself. Understood how increasing or decreasing the subtrahend decreases or increases the difference.

Example 2

Trina and Natalie were both presented with the following story problem: *If Bobo buys an oboe for $49 and then sells it for $125, how much is his profit?* Trina solved the problem by Valid Subtrahend Compensation. She added 1 to 49 to make 50. She knew that 125 – 50 was 75. Then she added 1 to 75 to get her answer of 76. Trina's justification was contextualized. She said that by changing 49 to 50, she had "pretended [Bobo] used more money than he did." Because Bobo had actually purchased the oboe for $1 less than $50, this meant that his profit was $1 greater than $75.

Natalie also rounded 49 up to 50. She subtracted 50 from 125, obtaining 75. She then decided that she needed to subtract 1 from 75 to compensate for the fact that she had added 1 to 49 initially. So, she arrived at an answer of $74. Natalie explained, "I'm technically adding a one to the problem. But then I have to subtract the one again in order to correct what I did originally." Natalie's approach is an example of Invalid Subtrahend Compensation. In her view, she had added 1 "to the problem," and she had to remove that 1 in the end.

Discussion and Conclusion

As MTEs, we find it important to consider students' justifications for their nonstandard strategies because these reveal their understanding of key mathematical ideas. Understanding the reasoning of students like Natalie and Trina helps to inform the design of instruction to support prospective elementary teachers' mental computation strategies and number sense development. In the case of Natalie and Trina's subtraction strategies, the key idea lacking concerns the distinct roles played by the subtrahend and minuend and, as a result, how each affects the difference. When students understand difference as distance between number locations, they apply aggregation strategies or understand compensation in terms of maintaining that distance.

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PROSPECTIVE TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING ALGEBRA AND GEOMETRY

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Relatively little research exists on prospective secondary mathematics teachers (PST-Ms) MKT and how it is developed. This paper discusses preliminary findings from a project investigating the characteristics of PST-Ms' MKT in the domains of algebra and geometry, and how participating in lesson study with mentor teachers correlates with change in their MKT.

Background

MKT is the "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Ball, Thames & Phelps, 2008, p. 399). PST-Ms may develop MKT through participating in lesson studies with mentor teachers, which involve planning, enacting, debriefing, revising, and re-teaching a research lesson in a mentor's classroom. The research questions that guided this study were: *What are the characteristics of PST-M's MKT in the domains of algebra and geometry? To what extent does engaging lesson study with mentor teachers influence the development of PST-M's MKT?* This poster presents findings to address the first question.

Methods

Thirty-five students, completing two university-based mathematics pedagogy courses (A and B) prior to student teaching placement, participated in the study. Course A focused on MKT for algebra and Course B focused on MKT for geometry. There were two sections (a treatment and a control group). The treatment group completed two mentor-guided lesson study cycles during Course B while the control group completed two Plan-Teach-Reflect assignments in field placements. All 35 students completed two written MKT exams in both Course A and Course B (4 exams), which included items from the Knowledge for Algebra Teaching measures.[1] Other sources of data from the lesson study cycles included reflective logs and audio-recorded lesson study meetings. This poster reports only results from analysis of exam responses.

Results

The results suggest that there are no significant differences between the content knowledge and specialized content knowledge portions of PST-Ms' algebra and geometry MKT. There were significant differences in the dispersion of the scores, suggesting that PST-Ms' geometry MKT may be slightly stronger. However, differences between the MKT measured in the items may account for the differences in the dispersion. Two potentially significant findings did emerge: (1) PST-Ms common content knowledge for algebra appears to be stronger than for geometry; and (2) lesson study may support the development of PST-Ms MKT, particularly aspects of pedagogical content knowledge.

Endnote

[1] The Knowledge of Algebra Teaching (KAT) assessment was developed by R. E. Floden, J. Ferrini-Mundy, S. Senk, M. Reckase, R. McCrory, with grant support from NSF (REC 0337595). Information about the KAT assessment is available at www.educ.msu.edu/kat.

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PRE-SERVICE TEACHERS' INTERPRETATIONS OF THE COMMON CORE STANDARDS: A COMPARATIVE STUDY ALONG THREE DIMENSIONS

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This report describes the second phase of an ongoing study designed to analyze the ways in which pre-service teachers (PSTs) interpret the *Common Core State Standards for Mathematics* (CCSSI, 2010). During phase I, our goal was to identify which of the eight math practices were most comprehensible to PSTs and those which subjects found most difficult to interpret and implement (see Bowers & Fredenberg, 2011). The methodology for both studies involved asking PSTs to create scripts depicting tutoring scenarios that target any two of eight math practices. When analyzing the results from study 1, researchers coded students' scripts on a scale of 1–4, with 1 representing the least sophisticated interpretation of the practice, and 4 representing a well-defined interpretation. Results from phase one of the study revealed three critical findings:

- The two most targeted (and most understandable) practices were 1 (Persevere) and 2 (Argumentation), while the two least targeted (and least understood) were 7 (Patterns) and 8 (Structure).
- A statistical analysis of the PSTs' descriptions indicated that the average rating of 1.8 (on a scale of 1–4) revealed no significant differences between the levels of sophistication among descriptions of all eight practices. This suggests that PSTs had equal difficulty describing all of the practices without any formal introduction to their intent and wording.
- Many of the PSTs' descriptions of the math practices reflected everyday interpretations of words that have specific meaning in the mathematics education community.

Target Dimensions for Phase II

Based on the findings from Phase 1, we have identified three dimensions that need to be addressed in order to improve PSTs' understanding of the math practices: (1) Communication (specifically the role of argumentation and precision); (2) Classroom Roles (specifically the student practices and the teacher moves that supported the emergence of those practices); and (3) Enculturation into mathematics education community (specifically, the ways in which various everyday terms are interpreted by students and teachers learning the CCSS as well).

Subjects, Setting, and Data Collection

The subjects in this study were, once again, all undergraduate students enrolled in a capstone mathematics class for prospective middle school teachers at a large southwestern university. After discussing the importance of teacher questioning and the intent of the CCSS, subjects were asked again to write scripted videos that featured a teacher and student solving a proportional reasoning problem and again asked to target two of the eight practices. Once the videos were created, subjects were asked to watch three randomly assigned videos and describe which practices they believed the authors were targeting. These ratings, along with those from the previous year, will be analyzed along the three dimensions described above. Results from this round will be compared with the results from phase I of this study to determine if the revised treatment approach was more effective for promoting PSTs' appreciation of the practices described in the CCSS-M.

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PRE-SERVICE ELEMENTARY TEACHERS' DEVELOPING UNDERSTANDING OF ELEMENTARY NUMBER THEORY

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Keywords: Teacher Education–Preservice; Teacher Knowledge

In order to be effective, teachers must develop a deep understanding of the mathematics that they will teach their future students (Ball, Thames, & Phelps, 2008). Moreover, the mathematical knowledge needed for teaching is shown to be positively associated with elementary students' mathematics achievement (Hill, Rowan, & Ball, 2005). Research shows, however, that many pre-service and in-service elementary teachers struggle to make sense of the mathematics that they must teach (Ma, 1999; Ball, 1990). One branch of mathematics that is important for all teachers to understand is elementary number theory, which includes topics in divisibility, prime factorization, factors, and multiples. Teacher preparation is critical in this area because number theory is closely connected to topics in number and operations and has recently taken a more prominent role in elementary curricula and standards documents.

Unfortunately, little is currently known about pre-service teachers' understanding of number theory. Only a handful of studies (Zazkis & Campbell, 1996a, 1996b) have begun to identify pre-service teachers' knowledge of divisibility and prime factorization, and even fewer have provided detail about how such knowledge changes during teacher preparation coursework. However, if teacher educators are to help pre-service teachers learn with understanding, they must first determine how pre-service teachers come to make sense of such topics.

This poster presentation will detail the results of a mixed methods study whose goal was to describe pre-service elementary teachers' developing understanding of number theory during a three-week instructional unit within a mathematics education content course. Using elements of the *action-process-object-schema* theory of understanding (Dubinsky, 1991), qualitative analysis of six individual clinical pre- and post-interviews identified participants transitioning from concrete to abstract levels of understanding. Additionally, statistical analysis of fifty-nine participants' pre- and post-test results revealed changes in their achievement across questions requiring procedural and conceptual understanding of number theory. Using both analyses, the presentation will describe participants' transitions between levels of mathematical understanding of number theory topics, as well as report on the details and structure of the research project.

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LEARNING TO TEACH TEACHERS: MAKING A TRANSITION

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As I transition from mathematics teacher to mathematics teacher educator, I wonder how I can support the mathematical development of future teachers. Like Ball (1988), I wanted to accomplish two objectives at once: "learning about prospective teachers and developing strategies for working with them" (p. 40). Toward these ends, I began a study where I interview several students aspiring to become elementary school teachers, while they are still in their pre-service preparation program. This poster focuses on the preliminary findings with one particular student. The question guiding this particular study is: how does a pre-service teacher apply his/her knowledge of proportional reasoning and fractions to a sequence of contextual problems? In the poster, I answer this question and determine how those answers guide my own development as a teacher of future teachers.

I set up a sequence of four interviews with each participant. The first interview allowed me to learn about the student's background, and to pose five word problems. The middle two interviews involved the participant responding to ten word problems in each interview. In the three interviews, participants identified proportional relationships, using whole numbers, unit fractions, and composite fractions respectively in the three interviews. The final interview allowed the participant to reflect on his/her experiences. The interviews match Steffe and Thompson's (2000) description of exploratory teaching, where my goal as teacher-researcher is to "become thoroughly acquainted, at an experimental level, with students' ways and means of operating in whatever domain of mathematical concepts and operations are of interest" (p. 274).

Stephanie (a pseudonym) possessed the content knowledge necessary to answer all of the word problems correctly. To achieve her goal to "make things [solutions] clearer," Stephanie began each response with the same two activities, eventually writing out as detailed a solution as she could create. During the interviews, Stephanie recognized the need to acquire additional knowledge of some form to identify different solutions or representations to the same question, as in Ball (1993). I believe Stephanie wrote as complete a solution as she could create, so that she could reduce, or eliminate, any confusion a student might have answering a similar problem. However, Stephanie recognizes that students could accurately answer one problem in a variety of ways that she is not familiar with. As I develop as a teacher educator, I feel the desire to balance two concerns in my own work with future teachers: supporting pre-service teachers' own mathematical knowledge and translating mathematical knowledge to useful explanations for younger students. In my future work with Stephanie, my intention is to support the translation of her established content knowledge into a specialized knowledge base meaningful for her future teaching of mathematics.

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CURRICULUM ANALYSIS OF SECONDARY MATHEMATICS TEACHER PREPARATION IN RUSSIA: TRANSITIONING TO NEW STANDARDS

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The contemporary social, economic, and political changes that have occurred in the Russian Federation as a result of the breakdown of the Soviet Union have imposed new reforms in education during the last two decades. Currently, Russia is in the process of transitioning from a traditional teacher preparation model to a restructured progressive model led by new standards, which abide by the commitment to the European Bologna Declaration in 2003. These significant changes are reshaping the higher education system and curricula in a radical, and perhaps, contentious manner. The underlying principle of this research is to offer a comprehensive analysis of the changes that have occurred regarding secondary mathematics teacher preparation in Russia, due to these developments, as a way to advocate for reforms to improve teacher preparation programs. Teacher training curricula and degree plans from several Russian pedagogical universities were used to support the analysis.

The findings uncovered that the system of secondary mathematics teacher preparation in Russia within the framework of the new model includes the following distinctive characteristics: greater importance is placed on general educational goals over professional goals at the first tier of the teacher preparation; inclusion of a research component in the content of the teacher preparation, which becomes increasingly important with transition from the Baccalaureate to the Master's program; and a distinction between professional accreditation of graduates of Baccalaureate and Master's programs with regard to grade level teaching assignment (Stefanova, 2010).

From a comparative perspective, the secondary mathematics teacher preparation in Russia is more extensive than it is in the U.S., with higher emphasis placed on the content and pedagogical content preparation of teachers along with increasing attention to its research component. The comparison between the Master's programs in Mathematics Education in Russia and the U.S. shows that there is a significant difference in the professional specialization component of the Russian curriculum compared to the U.S. curriculum. Secondary Mathematics teacher training programs in Russia, particularly at the upper secondary school level, also offer double content majors, which is not the case in most of the U.S. teacher preparation programs (Ministry of Education and Science of Russian Federation, 2000). Learning about secondary mathematics teacher preparation in Russia presents diverse view on teacher education and pedagogical approaches that are used in other countries. With this diversity, mathematics teacher educators could reflect on their own practices and re-evaluate strategies that can be incorporated in their own teacher preparation system. However, further studies in exploring the impact of different teacher preparation curriculum on teacher quality characteristics, such as teacher knowledge, are needed.

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EXPERIENCES FILTERED BY BELIEFS: THE EFFECTS OF A PROFESSIONAL DEVELOPMENT SCHOOL SECONDARY MATHEMATICS METHODS COURSE

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In a Professional Development School (PDS) setting, the preparation of mathematics teachers is a collaborative effort between university faculty, high school faculty, and the pre-service teachers (PST). All PDS participants can experience transitions through their work with PSTs as they explore research-based best practices in mathematics teaching. In this study, the impact of one such collaborative effort on the beliefs and practices of the methods course co-instructor (a high school faculty member) and one PST and his collaborating teacher (CT) were explored.

This study was framed by the view that teachers' beliefs are sensible systems (Green, 1971; Leatham, 2006) through which they filter research and suggestions to employ reform-oriented teaching practices as envisioned by NCTM. Classroom practices were characterized through the use of research-based classroom discourse practices (Hufferd-Ackles, Fuson, & Sherin, 2004; Smith & Stein, 2011). Data gathered in this qualitative case study included a pre and post belief survey, interviews, classroom teaching observations, and methods class observations and assignments. This report is drawn from the findings from the first year of a two-year study.

Findings indicate that participants with differing beliefs were impacted in different ways. The methods course co-instructor began the collaboration with stable beliefs that were more reform-oriented than the other participants and classroom practices that clearly displayed those beliefs. She was conflicted as she compared her practice to the research-based practices described in course readings and said "I found myself really evaluating myself" (Interview 1). Though her beliefs measure did not change, she implemented new pedagogical tools in order to more closely align with research-based practices. The CT's original measure of beliefs was less reform-oriented than the co-instructor's measure by a statistically significant amount. His beliefs were found to be context specific based on comments such as ". . . [the role of student discourse] might be constantly changing depending on the topic that we're talking about" (Interview 5). Although he did not, as requested, read the course readings or participate in methods discussions, he was open to new ideas brought in by the PST. He did not express conflict between research and practice. However, his beliefs measured substantially more reform-oriented on his post-survey than on his pre-survey. The PST also exhibited a shift in the measure of his beliefs toward more reform-oriented beliefs. His beliefs were in transition throughout the year as he struggled to construct a vision of reform-oriented practices within the application of day-to-day teaching.

Exposing these PDS participants to best practices during a methods course and through classroom instruction had varying impacts. This indicates the need for future investigation into the trajectories along which teachers at all levels move as they make changes in their practice and for understanding of how beliefs interact with these trajectories.

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WHO TEACHES MATHEMATICS CONTENT COURSES FOR PROSPECTIVE ELEMENTARY TEACHERS? RESULTS OF A NATIONAL SURVEY

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The mathematical experiences that prospective elementary teachers have during their teacher preparation are vitally important. However, there is evidence that many prospective teachers do not receive adequate experiences from their teacher education programs in order to develop deep, conceptual knowledge of the mathematics that they will teach (Ball & Bass, 2000; Greenberg & Walsh, 2008). Teachers of mathematics content courses for prospective elementary teachers play an important role in helping prospective teachers acquire the knowledge they need for teaching, however, not much research has been done on this important group of educators.

This research reports on a national survey of higher education institutions in the United States to answer the question, "Who teaches mathematics content courses for prospective elementary teachers, and what are these instructors' academic and teaching backgrounds." We surveyed 1,926 institutions and a faculty member from each of 825 institutions (42.8%) participated in the survey. The survey results demonstrate that the majority of institutions are not meeting the recommendations of the CBMS (2001), the NCTM (2005), and the NCTQ (2008) for prospective elementary teachers to take at least nine credits of mathematics content designed for them. Additionally, most instructors for these courses do not have elementary teaching experience and have likely not had opportunities to think deeply about the important ideas in elementary mathematics, and most institutions do not provide training and/or support for these instructors. If nothing changes with the preparation and professional development of these instructors, the cycle of unprepared prospective teachers whose college experience has little effect on their mathematical understanding (CBMS, 2001) will continue.

In order to change this situation, we first suggest that all institutions preparing elementary teachers offer and require at least nine credits of mathematics content courses designed for this population, and prepare and support the instructors who teach these courses. We also suggest that there be collaboration among instructors. Institutions with multiple instructors can form communities of practice at their sites. Instructors who are alone in teaching these courses at their institutions can seek out instructors at other institutions for support. Professional organizations, such as the Association of Mathematics Teacher Educators (AMTE), offer resources through membership, conference sessions, and pre-conference workshops. Finally, our survey results, combined with other research on prospective elementary teachers' achievement, may help the mathematics education community develop standards for the teacher educators who teach mathematics content courses for this population.

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GENERATING A PEER MENTORING CULTURE THROUGH MATHEMATICS CAMPS

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One of the hallmarks of a preservice teachers' transition into the profession is their aptitude and desire for sharing ideas and overcoming difficulties faced by the community of teachers. Teacher educators strive to design programs that address the many facets necessary to foster preservice teachers through the transition from student to professional educator. In many cases preservice teachers move according to a prescribed program of study in which they learn from professors, engage in tutoring and/or field experiences with students, and apply their knowledge in an internship with an experienced mentor teacher. During the program of study there is typically cooperation with peers for group work on designing lessons, units, pedagogical strategies and portfolio development (Davis & Honan, 1998; Freidus, 1998), but there is not a significant and prolonged collaboration among preservice teachers in solving the immediate problems presented by issues such as disengaged students, struggling students, or students with behavior problems. What would it mean for a program to increase expectations and transition from peer collaboration to peer mentoring by enacting teaching in mathematics camps?

In this poster I will describe an *exploratory evaluation* study (Patton, 1987) of a mathematics teacher program at a university in Southeast Asia. The program in the study incorporated mathematics camps with public school children from grades K–12 each semester. Preservice teachers in the program were expected to design, implement, assess, discuss, and redesign mathematics camp curriculum and processes. The mathematics camps were enacted for many different grade levels and schools in the local community. The preservice teachers applied their learning from mentors and their experiences from each camp to become more effective at engaging students in enjoyable learning of mathematics. First-year preservice teachers are mentored by the second- and third-year preservice teachers. In the first year they learn about the camp structure and participate as students in a large camp experience. Second-year preservice teachers are mentored by the third-year preservice teachers in the design, implementation, assessment, discussion, and redesign process. The second-year students work alongside the third year to enact the camps in several schools each academic semester. Third-year students are in authority at the camps and must ensure their success. They may consult their fourth- and fifth-year elders and their professors on issues that arise, but during the camp, they must make the decision they feel is best and then reflect on that decision with others later.

The resulting descriptions from the study suggest the program has a rich and endearing peer mentoring culture that has been established through the teaching of mathematics camps with local school students. These descriptions reveal that through the peer mentoring and the teaching during the mathematics camps preservice teachers have a strong bond to others within the profession and the mathematics unit. Preservice teachers also appear to operate with increasing pride and efficacy in their teaching of mathematics. Lastly, there is evidence of significant development of the preservice teachers social, organization, and leadership skills as they progress through the program.

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EXAMINING PRESERVICE TEACHERS' PERSPECTIVES ON WRITING IN THE MATHEMATICS CLASSROOM

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Secondary mathematics preservice teachers (PSTs) are typically well grounded in content knowledge; however, the development of their ability to teach mathematics is not an automatic outgrowth of their content knowledge, as it may be impeded by issues such as the influence of past experiences and an inability to recall their own developing conceptions of mathematics.. Our research objective was to examine how engagement in two activities: *writing to learn mathematics* (WTLM) and *reflection,* can broaden perspectives on what it means to teach and learn mathematics . Through writing, learners can monitor strategies used in problem solving (Countryman, 1992), while reflection engages existing understandings of mathematics and supports the development of new understandings about teaching and learning (Shoffner, 2009).

Data for this study consisted of PSTs' in-class WTLM activities and discussions and reflective writings posted in a wiki during a six-week mathematics methods course. The PSTs reflected on their experiences with WTLM and answered specific prompts about their conceptions of the use of writing as tool for teaching and learning in mathematics. We analyzed these reflections to examine teachers' growing perspectives on the use of this tool.

Initially, PSTs explained that, in their experiences, writing in mathematics was limited to geometry or proof. Many PSTs could not recall writing in mathematics in high school, and felt that it was "kept to a minimum." Those that did recall writing in mathematics considered it "pointless" or "tedious." Inexperience with writing influenced PST's ideas about how students might view WTLM and often led to initial reluctance toward using it in their future classroom. Many felt that writing in mathematics classroom would "distract" from the "real" mathematics.

Following their engagement in multiple WTLM tasks and reflective discussions on the course wiki pages, we were able to identify several changes in PSTs' perceptions on the use and role of writing in a mathematics classroom. Many PSTs now mentioned teacher benefits in their final posts, lauding WTLM's usefulness for knowing how different students think about mathematics. One suggested that "verbalizing helps make more connections to the concepts and may be helpful in clearing up any discrepancy that the student might have." The PSTs' reflections to their own responses to the WTLM prompts seemed to frame their content knowledge in a pedagogical context. Upon completing the WTLM tasks, some PSTs focused their reflections on the students that they would be teaching and some issues that they may need to consider. After considering how students might view a problem they worked on, the PSTs began to consider pedagogical decisions that they might make to build on students' thinking.

This research suggests that many PSTs are willing to accept WTLM as a useful tool to help students understand mathematics after engaging in WTLM tasks themselves. Teachers must be able to understand and explain mathematics in multiple ways in order to reach all students. By diversifying and expanding approaches to mathematics, teachers may support multiple learning styles and address diverse learners.

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PRESERVICE TEACHERS' PERCEPTIONS WHERE INSTRUCTION IN COLLEGE COURSES EMPHASIZES CONNECTIONS TO MIDDLE GRADES MATHEMATICS

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The Conference Board of the Mathematical Sciences (CBMS) (2001, 2012) and Wu (2011) suggest that instruction in mathematics courses for preservice secondary mathematics teachers (PSMTs) should emphasize a connection between the mathematics they are learning and that which they will teach. In this poster we will present the results of a survey given to PSMTs in which they identified college mathematics courses that specifically addressed the content in the Common Core State Standards for Mathematics (CCSSM) (2010) to be taught in grades 7–9.

Keywords: Teacher Education–Preservice; Mathematical Knowledge for Teaching

One CBMS recommendation is that in their preparation PSMTs should have the opportunity to develop, "Knowledge of the mathematical understandings and skills that students acquire in their elementary and middle school experiences, and how they affect learning in high school" (CBMS, 2001, p. 39). For more than a decade, there has been increased interest from the mathematics community to address this issue. With respect to the need to deeply understand K–12 mathematics, Wu (2011) notes that, "Because of the teacher preparation programs' failure to teach content knowledge relevant to K–12 classrooms, the vast majority of preservice teachers do not acquire a correct understanding of K–12 mathematics while in college" (p. 9).

To understand PSMTs experiences in mathematics coursework, a survey was developed to investigate in what college mathematics courses the instruction received mentioned that the mathematical ideas being taught pertain to them as future teachers of mathematics in grades 7-12. Twenty-three key mathematical content standards in the 7th grade, 8th grade, and parts of the Algebra strand of the CCSSM were chosen for the survey.

Programs of study for PSMTs at four institutions of higher education in four different states were examined and compared to the suggestions by CBMS (2001). After considering titles of courses or content included in courses, the comprehensive list of courses developed reflected the required content in the proposed courses of study for PSMTs in each of these universities.

Methods courses were identified at nine institutions of higher education and instructors were sent an electronic survey link to provide to their students. In total, 33 PSMTs at five institutions responded. Among the data to be shared is that from the courses available, PSMTs identified *Calculus or Advanced Calculus*, *Probability and Statistics*, *Linear Algebra*, *Abstract Algebra*, and *Discrete Mathematics/Computer Science*, in order, as the courses where the content was at some point taught with regard to their future as a secondary mathematics teacher.

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USING CLASSROOM PHOTOGRAPHS TO SUPPORT PRESERVICE TEACHERS' NARRATIVES OF MATHEMATICS TEACHING

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Keywords: Teacher Education–Preservice; Instructional Activities and Practices; Elementary School Education; Design Experiments

On this poster we explore classroom photos as tools for eliciting and supporting pre-service teachers' (PSTs') narratives of math teaching and learning. Because narratives are the primary means for making sense of and learning from life experiences, they are useful in supporting teacher candidates' sensemaking and transitions from college students-to preservice teacher-to teacher intern-to beginning teacher. Our work with photographs was inspired by Carter and colleagues' (1988) use of photographs as prompts to research expert-novice teacher knowledge and by Chazan and Herbst's (2006) use of cartoon representations of mathematics teaching in their TheMat Project. It brings into relation research on representations of practice as a means to improve teachers' attention to and reflection on critical features of mathematics instruction (e.g., Herbst & Chazan, 2006; VanEs & Sherin, 2006) and the teacher learning literature that conceptualizes narrative as a sensemaking activity (e.g., Carter, 1988; Drake, 2006; Lloyd, 2006).

In our case, groups of four PSTs were each given one of six different sets of approximately twenty classroom photos and asked to create a flip book, including thought and speech bubbles and captions. Each set of pictures represented classroom scenes from the launch, explore, and debrief sections of a math lesson. After constructing a flipbook that organized the set of photos visually, preservice teachers were then asked to write a plausible mathematics teaching story to describe their flip book. We provide an analysis of these artifacts using an emerging analytic frame participants in this study used to describe and interpret classroom events in terms of teaching—*moments, perspectives, and settings*—within and across classroom photographs while constructing mathematics lesson stories. We describe how participants depicted and narrated teaching *moments*, meaning that they spent significant time unpacking instructional moments of varying time scales in the lesson. They noticed *perspectives*, often narrating the story from different points of view (e.g., students or third-person). They noted the *settings*, meaning that they discussed details of physical materials and space and the role these play in the lesson. We argue that this expanded frame is important, and that photographs and narratives have particular affordances for supporting its development during and beyond teacher preparation.

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ANALYZING THE EVOLUTION OF THE FIELD OF MATHEMATICS EDUCATION: A HISTORICAL PERSPECTIVE

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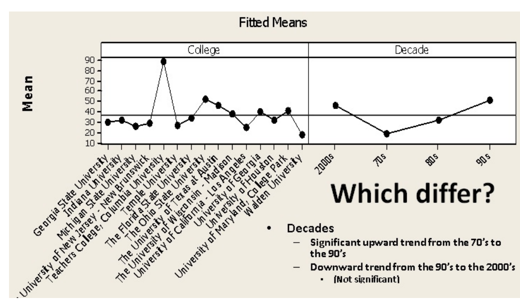
Keywords: Post-Secondary Education; Teacher Education; Technology

Due to the interdisciplinary nature of mathematics education which intersects areas of mathematics, learning theory, educational psychology and others; research efforts have struggled to document the development and evolution of the field (Stanic & Kilpatrick, 2003). Several reviews of doctoral programs and their history of producing mathematics education doctorates have been compiled which have positioned this research effort to develop a systemic, data-driven way to analyze a progression of the field over the course of the past century (Reys & Kilpatrick, 2001; Reys, Glasgow, Tuescher, & Nevels, 2008). This on-going research project aims to dynamically illustrate the evolution of mathematics education since 1906 and demonstrates the impact of academic institutions, and certain influential figures that have helped to mentor and develop future generations of mathematics educators. Some doctoral programs in mathematics education have produced a significantly greater number of leaders who have been vital in advancing teaching, research, curriculum efforts; have shaped educational policies, and contributed to the success of the discipline through committees and professional organizations. Utilizing an innovative and extensive data-driven approach (see Table 1), some of the primary research objectives and initial findings of this project have been illustrated.

Table 1: Goals and Results

<table>
<tr><th>Goals</th><th>Initial Results</th></tr>
<tr><td>❖ Highlight longitudinal comparisons of doctorate granting institutions
❖ Illustrate levels of impact by institutions through cluster statistical sampling techniques in developing mathematics education doctorates
❖ Note historical issues of equity and opportunity in mathematics education
❖ Provide a teaching resource with historical perspective in the preparation of future mathematics educators</td><td>
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MATHEMATICAL KNOWLEDGE FOR TEACHING: THE CASES OF TWO PRE-SERVICE TEACHERS WITH ENGINEERING BACKGROUNDS

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Reports of studies documenting sources of secondary mathematics teachers' pedagogical decision making or the nature of their Mathematical Knowledge for Teaching (MKT) are rare (Goos, 2010). Efforts towards establishing a framework for capturing and measuring MKT for teaching high school Algebra are currently ongoing (McCrory et al., 2010). In this work I adapted the Knowledge for Algebra Teaching (KAT) to examine pre-service teachers' mathematical knowledge pertaining to the framework's tasks of teaching as manifested during episodes of bridging, decompressing, and trimming when making instructional decisions in an attempt to answer the following question: *What factors do pre-service teachers consider when judging students' mathematical work and thinking?*

The proposed poster offers an analysis of two specific cases studied as they assessed instances of children's geometric work and/or their questions concerning geometry content. I examined factors they considered when conceptualizing instructional plans in the presence of these episodes. The two focus participants were from engineering backgrounds (computer engineering and industrial engineering). The decision to focus on these two participants was deliberate. On one hand, initiatives to recruit individuals from non-mathematical backgrounds into the profession of teaching are now widespread. It is crucial to gain a better understanding of the orientations they bring into teaching and teacher education so to design environments conducive to their learning. On the other hand, an examination of the two cases offer valuable insight into how expert practitioners in one field, engineering in this instance, who are also novice to teaching, might attempt to make sense of tasks of practice.

At the time of data collection, both participants were enrolled in a Master of Education program. Each participant completed written surveys which included case based illustrations of children's work for analysis. Each participant was also interviewed. The goal of the surveys and the accompanying interviews was to capture knowledge sources from which they drew when assessing students' work, factors most influential on their thinking about ways to design instruction based on students' ideas/questions, and resources they found most valuable when navigating these demands.

The two pre-service teachers differed in their methods of judging student work based on their level of understanding of geometry. For both the participants, their analysis of student work was influenced by curriculum. The interviews with the participants revealed that while one participant attempted to use learning progressions to analyze student work and plan instruction, the other participant did not perceive learning progressions as useful while making in the moment pedagogical decisions.

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ASSESSING PROSPECTIVE ELEMENTARY TEACHERS' ABILITY TO MAKE LINKS BETWEEN TEACHING AND LEARNING

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Teacher education represents a key milestone in the professional learning continuum, as it is during this time that teacher candidates must transition from student to teacher. Part of successfully making this transition is gaining an analytical approach to teaching. One model for *the systematic analysis of teaching* (Hiebert, Morris, Berk, & Jansen, 2007) proposes four analytic skills which support continuous teacher learning from practice: specifying learning goals, gathering and analyzing evidence of student learning, constructing hypotheses which link the effects of teaching to students' learning, and revising lessons. By using these skills, PTs can understand teaching in terms of students' mathematical learning outcomes, a critical transition in their development as teachers.

The ability to construct hypotheses linking teaching and learning (the third skill in this model) is especially critical to improving instruction over time because it can lead to productive lesson revisions and build general knowledge about effective teaching (Hiebert et al., 2007). However, little current research has examined how PTs construct these hypotheses. This study represents a first look at PTs' initial ability to link teaching and learning in a mathematics lesson. PTs read a transcript of a lesson and were prompted to describe what aspects of the lesson and teacher actions might have led to student learning outcomes. Data were analyzed based on theoretical criteria for useful cause-effect hypotheses as outlined by Hiebert et al. (2007). These criteria include whether hypotheses: (a) were focused on the mathematical learning goal; (b) cited or followed from evidence of student learning; (c) included specific detail; (d) were appropriately cautious; (e) made a link between teaching and learning; and (f) were based on pedagogical principles.

Results indicate that while PTs demonstrate some skill in hypothesizing before instruction, this skill requires further development along specific dimensions. Across the sample, PTs cited student understanding of least some aspect of the mathematical goal in approximately half of their hypotheses. PTs were less likely to mention mathematical aspects of the teacher's actions (cited in approximately one-third of hypotheses). Unfortunately, even when describing students' understanding of the learning goal, PTs did not always cite evidence to support their claims. Only about one-third of hypotheses included supporting evidence. Over 60% of PTs' hypotheses were inappropriately vague or general about lesson features or outcomes. Finally, PTs were rarely cautious in their claims, with approximately 80% of hypotheses stated as fact.

In terms of making links, wide variation existed between individual PTs, with some PTs consistently making links between teaching and learning and others making few or no links. Although nearly all PTs drew on pedagogical principles when constructing at least some of their hypotheses, the fact that they often cited beliefs that were not in alignment with a conceptual learning goal made some of their hypotheses problematic. Interventions aiming to improve PTs' ability to write hypotheses linking teaching and learning could be built upon this description of their baseline skills.

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PEDAGOGICAL CONTENT KNOWLEDGE FOR TEACHER EDUCATING

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Keywords: Post-Secondary Education; Teacher Knowledge

Research concerning the practice of mathematics teacher educating, and knowledge bases used in such a practice is lacking (Speer, Smith, & Horvath, 2010). As a result, a shared knowledge base regarding mathematics teacher educators is yet to be developed. To address this need our study was conceptualized to investigate: (1) What knowledge domains do mathematics teacher educators draw from and use when providing content specific pedagogical experiences for teachers? and (2) How do these knowledge domains influence the activities of mathematics teacher educators as they design and implement pedagogical experiences for teachers? This presentation will report preliminary results from the pilot study.

Data were collected using in-depth interviews, classroom and professional development session observations, and existing documents prepared and used by the participants when organizing activities around teacher educating. Participants included faculty members in mathematics and science education, and school mathematics leaders involved in a PD program.

Data analysis consisted of three levels. At the first level of analysis, a case profile for each of the participants was developed. The framework for analysis of each case included the domains of knowledge referenced by the participants use of mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008), Shulman's (1986) categories of pedagogical content knowledge, knowledge of algebra for teaching (Ferrini-Mundy, McCrory, & Floden, in revision), and constructs identified as crucial to educating adult learners (Marquardt & Waddill, 2004; Mezirow, 1981). Participants' comments, both during the interviews as well as observation sessions were coded according to indicators of knowledge identified by each construct. At the second level of analysis we conducted a cross examination of all cases as a means to generate a comprehensive list of knowledge domains expressed to be used, or used in practice by the participants. This cross analysis allowed us to isolate unique aspects of knowledge for teacher educating in various fields and according to other variables of concern. The final analysis included careful categorization of the referenced and enacted knowledge domains by mathematics teacher educators.

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Chapter 7

Teaching and Classroom Practice

Research Reports

Brief Research Reports

Poster Presentations

THE MAKING OF A MEANING MAKER: AN ENGLISH LEARNER'S PARTICIPATION IN MATHEMATICS

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This paper focuses on meaning making as a process that can transform English learners' participation in mathematics. Using selected video transcripts from a one year long project in a fourth grade bilingual classroom, the paper documents the development of participation along a continuum of meaning making, from preparing students to transform their participation to creating meaning eliciting tasks to showcasing students in meaning making interactions. Findings suggest that the problem of participation in mathematics by English learners—and other students as well—is a problem rooted in the absence of meaning making practices in mathematics classrooms.

Keywords: Meaning Making; Participation; English Learners; Elementary School Education

Purpose

The purpose of this paper is to contribute an understanding of English learners' participation in mathematical discussions as a deliberate meaning-making process that is initiated by the teacher, interpreted by an English learner, and reinterpreted by peers. The paper documents the development of meaning making practices with a focus on an English learner's participation in a Latino classroom. The focus on this student serves to illustrate the intricacies of meaning making and how it transforms and is transformed by participants. Through this analysis, the paper highlights the importance of re-envisioning student participation in mathematics not as an individual act but rather as a continuum of meaning making moments. Along this continuum, participation emerges as a meaning-oriented, other-oriented process.

Theoretical Foundation

The issue of who participates in mathematics has been explored from an equity perspective (Gutiérrez, 2010; Khisty & Chval, 2002), a status and domination perspective (Cohen & Lotan, 1995, Pierson Bishop, 2012, JRME), a racial perspective (Martin, 2007; Stinson, 2010), and a sociocultural perspective (Moschkovich, 2002), among others. This paper focuses on a meaning-making perspective to understand English learners' participation in mathematics. I view meaning making as foundational to student participation in mathematics. The critical need for bringing meaning making into teaching and learning mathematics has been highlighted in research (Schoenfeld, 1991). A view of participation as rooted in meaning making affords a critique of participation as individual and behavioristic and shifts our attention to participation as a continuum of various forms of participation, none of which can stand alone in a significant manner. As Sfard (1998) explains her metaphor of learning as participation, "[f]rom a lone entrepreneur, the learner turns into an integral part of a team" (p. 6).

I start this theoretical foundation with a definition of meaning making as "the translation of one sign into another system of signs" (Jakobson, 1985, p. 251). By translation I refer to the interpretive and continual process of recreating signs (e.g., 9 is also 5 and 4), reconstructing images (e.g., the area of an irregular shape is also the sum of the areas of regular shapes), social representations (Gorgorió & de Abreu, 2009) (e.g., an English learner can solve problem *a* but not problem *b*), and task adaptations (e.g., 272 divided by 8 can be transformed into an open ended task). The expression of meaning is only possible through signs (Radford, Schubring, & Seeger, 2011). However, signs alone refer to objects that are meaningless (e.g., a 9 remains as a 9, or an English learner can remain as someone who can only solve problem *a*) unless participants translate them along a continuum of possible meanings (Otte, 2011). The active translations and interpretations of signs—meaning making—characterize rich participation in mathematics. For example, the number 9 is a sign that, by itself, has no meaning. It requires a student's interpretation as “1 less than 10,” or as “5+4,” or as “3 × 3,” etc. An open-ended task in this framework is a powerful sign in that it invites multiple interpretations from students. A teacher who uses open-ended tasks is primarily interested in the meaning that students will make.

Methods (Participants, Context, Data Collection, Analysis)

Data collection consisted of video taping of the problem of the day in a bilingual Latino fourth grade classroom, four times a week, for an entire school year. At the time of the data collection, the teacher was in her first year of practice. At the beginning of the project, the problem of the day consisted of low-level mathematics tasks, often copied from the school’s adopted textbook. However, she was interested in increasing student participation but did not know how to make it happen. Students, including Sam the English learner who is the focus of this paper, came to her classroom with a history of traditional mathematics instruction from the previous three years. Students were used to the Initiation-Response-Evaluation pattern of participation (Mehan, 1985) found in most U.S. classrooms. For example, Sam would only participate when called on by the teacher, and his ideas would not travel outside the teacher-student interaction. Another form of data collection consisted of reflection and co-planning sessions with the teacher, usually at the end of the school day, and on weekends. The problem of the day used in this analysis was developed during one of these reflection and co-planning sessions.

To analyze data, I selected interactions from video recordings produced at the beginning of the school year, when students were showing resistance to the teacher’s new plan for participation. I also selected interactions toward the end of the school year to illustrate how students were relearning new ways of participation. I transcribed these selected videos. Then I applied the ideas from the theoretical foundation to document how participation emerged as a continuum of meaning making.

Results

Results are intended to illustrate the arduous process of making meaning makers. Years of being denied opportunities to participate in meaning making result in this arduous process of reminding, convincing, and preparing students to participate as meaning makers. Results illustrate moments of tension as students learned to transition from older ways into new ways. An important meditational tool in this evolution of participation was the transformation of mathematical tasks and the roles, norms for participation, and expectations that these tasks required.

Preparing Students to Transform their Participation

At the beginning of the school year, students were used to working in mathematics individually. The prevalent participation pattern was between individual students and the teacher, with the rest of the group acting as passive spectators. How comfortable students were in this participation structure became apparent as the teacher introduced the new norms of participation. These new norms met students’ strong resistance. Their reaction evokes Sherin, Louis, and Mendez’s (2000) comment about how hard it is to make students participate in mathematical discussions. Students were initially participating *for* the teacher, as evidenced by the kinds of questions they were asking the teacher:

“Ms. R, do we take out our math journal?”
“Do we need to copy that?”

Students were also challenging the new rules of participation with expressions of resistance:

“What does this have to do with math?”
“Ms. R, why are we doing all this? Can you explain it to me?”
“Ms. R, are we learning anything here?”

In the midst of these complaints, the teacher and I also noted that students were noticing something about the tasks. For example, when the teacher was explaining the new rules of participation, hearing the teacher say “problem of the day,” three students suggested:

"Why can't there be a problem of the week? Or the month?"
"Yeah, like a really, really hard good question."
"Yeah. That would be good."

Translating the Mathematical Task

In our initial conversations, the teacher and I had a clear goal: We wanted students to participate by responding to each other's contributions instead of responding only to the teacher. We were confident that strong, open-ended mathematical tasks—or as students suggested, "like a really, really hard good question"—could be our ally in achieving our goal. The problem I use in this paper was originally conceived as a single version that the teacher and I translated into two related versions. One version was a measurement division problem; the other was a partitive division (see Table 1).

Table 1: Measurement and Partitive Division Tasks

Measurement Division (size of group known; number of groups unknown)	Partitive Division (size of groups unknown; number of groups known)
1. Ortega Elementary has 272 students. For the fire drill, students were placed in rows of 34 students. How many rows did they form?	2. Sanchez Elementary has 272 students. For the fire drill, students were placed in 8 rows. How many students were in each row?

We split the group in half and each group received one of these two problems. On the back of their problem, we included a discussion question: Explain which school has a better fire drill, Ortega or Sanchez. The teacher instructed students to save this question for a final discussion. One student, however, could not resist the curiosity of reading the question, and he exclaimed:

Rolando: "But this doesn't make any sense!"
Teacher: (touches Rolando on the shoulder): "But it'll make sense whenever you hear what the other problem (points to other group) is about."
Rolando: "Oh!"

Rolando was not finding meaning in the question. For him, the question was a sign disconnected from other signs, therefore lacking meaning. The teacher, however, reminded Rolando that meaning was about to emerge as soon as all participants translated this question along a continuum of ideas. The teacher's reply to Rolando is consistent with a view of meaning as other-oriented (Dominguez, López Leiva, & Khisty, 2012), and it may be difficult to emerge in the sole act of Rolando reading the question.

Making an English Learner a Meaning Maker

The teacher asked each group to read their version of the problem to the other group to establish a common understanding that they had been working on different problems. Some students claimed that the problems were the same, only that the words were different. The teacher helped students with this initial translation of each other's signs (the problem) by emphasizing that one problem was asking for the number of groups, whereas the other was asking for the number of student in each group. But the important part of this interaction came next, when Sam, an English learner, raised his hand to participate, in English. What follows is the process of the teacher and peers collectively making Sam a meaning maker.

Sam: "Ms. Ramos, I did a picture of eight, eight…"
Teacher: "Can you stand up and show what you did?"
(Sam stands up, holding his notebook high up for the other group to see) (See Figure 1.)
Teacher: "Can you explain it to them?"
Sam: "I put 8 rows, 8, 8 lines, and I did it like this, it's the, it's the, it's the rows, and then I put 34 in each row, the number, and then I knew that 34 × 8 is 272."
Teacher: "OK, go ahead and show them what you did (signals to go to the other group's table) 'cause I don't think they can see that when you are standing way over here."

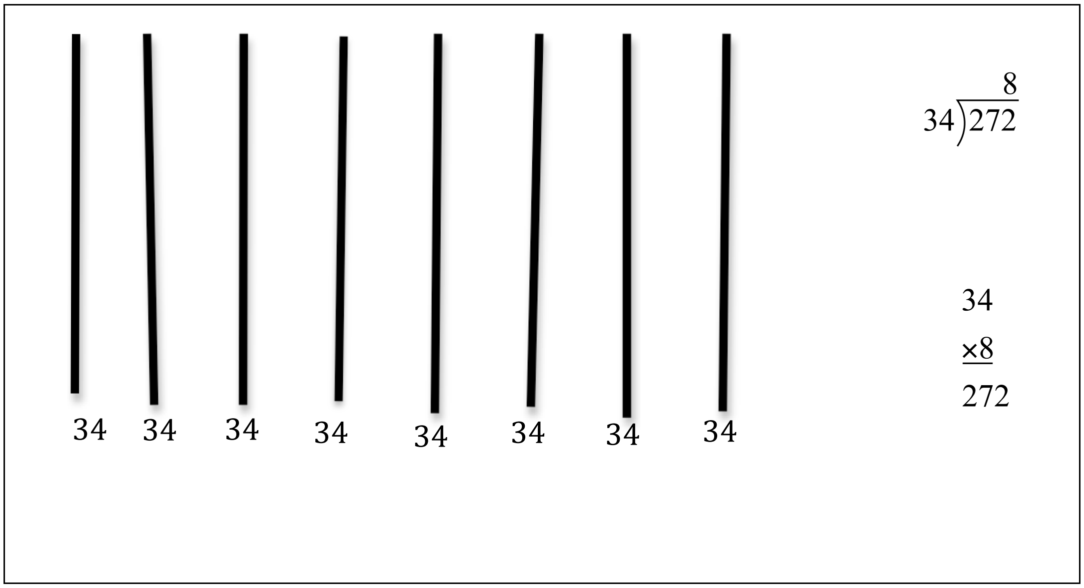

Figure 1: Sam's strategy

When the teacher asked Sam to stand up, she is translating Sam as capable of explaining his strategy to others. By asking Sam to explain his strategy, she is establishing the conditions for the creation of a continuum of meanings, as meaning depends on others to grow. Sam takes up the teacher's translation of him, but his standing up and showing his work has not yet created a continuum of meanings. The teacher seems to be interested in seeing this continuum, as she asks Sam to explain what he did to his audience. In the act of explaining, Sam may find the possibility of refining his translation of the problem (his strategy) for himself and for others. Although Sam accepts the teacher's new translation of him as a capable explainer, he performs this new role from where he is standing distance. The teacher wants to ensure that everyone hears Sam's translation, so she asks Sam to approach the group. Here, the teacher is interested in Sam's audience and their ability to interpret Sam's translation. The teacher is establishing the conditions for making every student a participant in the process of making meaning. She picked Sam as the carrier of a sign to be translated first by him and then to be interpreted by his peers. In the following section, Sam will encounter multiple interpretations of his work as he interacts with peers. The continuum of meanings at this point begins to take shape.

Students Interpret an English Learner's Strategy

Sam walks toward the other group. At the same time that he starts walking, Joel, the same student who, in the beginning of the year, was questioning the new way of participating in mathematics, initiates a student-student interaction focused on Sam's mathematical strategy.

Joel: "What was that for?" (pointing to Sam's drawing)
Sam: "The rows!" (points to his 8 rows)
Joel: "Why did you need it? Why did you do that?"
Sam: "So I can show my strategies. So, some people do believe me. Like this!" (points to Joel's strategy, who drew 8 circles on his notebook instead of 8 lines like Sam did) (See Figure 2)
Joel: "Oh, Oh, every line means a row?"
Sam: "Yeah!" (turns palm of his hand up, as if indicating the obviousness of his meaning)
Joel: "Oh!"
Student: "Duh, dude!" (teasing Joel)
(Sam walks around the table, holding his notebook by his chest so peers can see).

Jennie: “Wait, don’t go so fast! Wait, why did you put 34 in each row? The way you put that.”
Sam: “Because it says right there in the question, that there’s 34 students in one row.” Jennifer: “Oh.”
Liz: “Ah, hey! Wait! How did you know that 34x8 is 272?”
Sam: “Because I did this and then I did this.” (points to his division, then to his multiplication, and finally to his drawing)

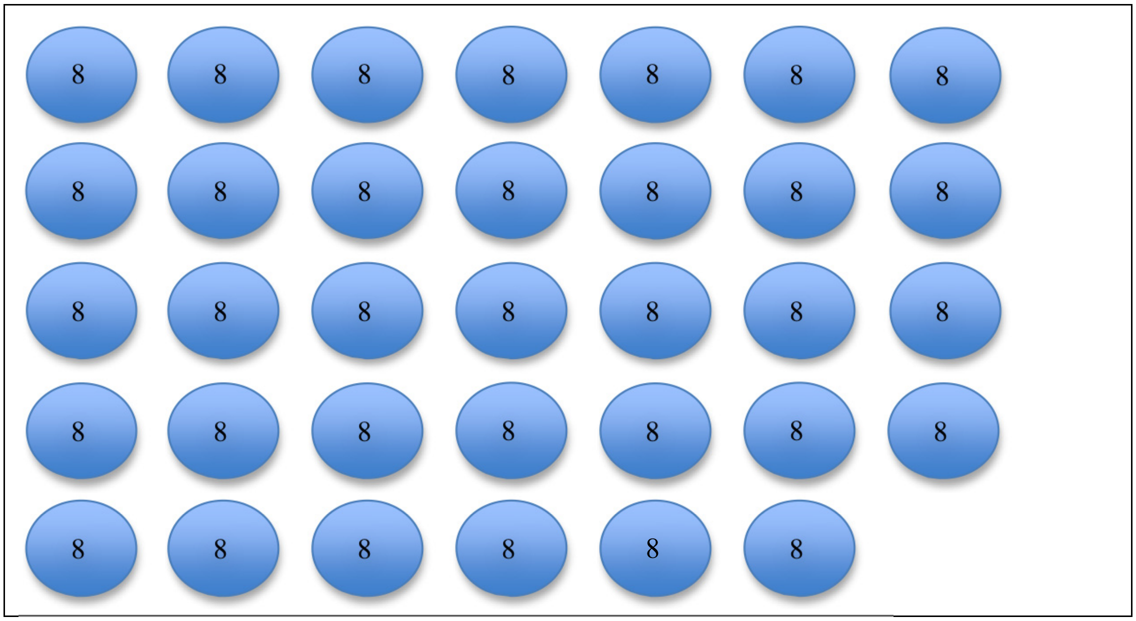

Figure 2: Joel’s strategy

In this interaction, peers are translating Sam as a meaning maker. First, Joel demonstrates interest in Sam’s strategy. Sam responds by placing his strategy (his own translation) along his peer’s strategy (Joel’s own translation), thus establishing a student-generated continuum of meanings. Sam’s strategy in itself reflects his sensitivity to this emerging continuum of meanings, as he represented his strategy in different ways. For example, his response to Joel, “So, some people do believe me” is consistent with the multimodality of his strategy, which includes iconic representations (8 vertical lines representing 8 rows) and also symbolic representations (multiplication and division). Sam has in other words recreated signs for multiple interpreters (so some people would believe him) and has maintained these translations of signs meaningfully connected.

At this point in the interaction, Sam does not need the teacher’s prompts any more. Instead, he shows his strategy to peers spontaneously. At the same time, peers continue asking him questions, in what exemplifies the rarest form of whole group discussion in mathematics, one in which students themselves mediate and control their own participation. The teacher was sitting in one corner of the room, enjoying the unfolding of this discussion. It is important to note that Jennie’s question is a request for an explanation (*why*) and Liz’s question is a request for a justification (*how did you know*). The caliber of these questions, and the fact that they are the building blocks in a discussion for learning mathematics (NCTM, 2000) demonstrate that as a whole, Sam’s peers have helped the teacher to effectively construct an English learner as a meaning maker. And in this collective act, all participants constructed themselves as meaning makers as well.

Discussion: The Making of Meaning Makers

Teachers, students, tasks, and instructional strategies, all contribute to make different kinds of meaning makers. Prior to the interaction reported in this paper, particularly at the beginning of the school year, students were making meaning for and by themselves, sharing it occasionally with the teacher. As one student frankly put it,

Josh: "But this is the first time we've done it in groups. We've always doing it by ourselves."

Similarly, when the teacher asked whether all students had found a group to be in, two students replied:

Mandy: "No! Not all of us."
Joel: "It's more like, per person."

I chose the case of Sam, an English learner, for several reasons. First, Sam illustrates how status in classrooms can shape who gets to participate (Cohen & Lotan, 1995). Second, when tasks offer students the possibility of making meaning in multiple ways, students like Sam can be motivated to share with others their understandings and strategies. Third, the fact that the teacher selected Sam as the representative of his group, was a powerful instructional strategy that served to achieve multiple goals: sharing meanings, creating a continuum of ideas, and translating low status students as active and capable participants in the process of making mathematical meaning.

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LEARNING TRAJECTORIES AS A TOOL FOR MATHEMATICS LESSON PLANNING

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In this paper, I examine the utility of a mathematics learning trajectory as a tool to support teachers' attention to students' mathematical thinking. I present findings from one second grade teacher's use of a learning trajectory as she planned a sequence of three mathematics lessons. Findings suggest learning trajectories support teachers in choosing appropriate tasks and learning goals, and in anticipating students' likely approaches and difficulties. Learning trajectories, as representations of student thinking, provide teachers with a means of evaluating evidence of student learning of intended goals and afford them with a range of instructional moves based on their students' current conceptions.

Keywords: Learning Trajectories; Instructional Activities and Practices

Attention to student thinking has been identified as a critical tool to initiate changes in teachers' knowledge for teaching and improvements in classroom instruction (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Franke, Carpenter, Levi, & Fennema, 2001; Kazemi & Franke, 2004; Sherin & van Es, 2009). Moreover, an emerging hypothesis in the field is that the construct of a *learning trajectory* (LT) has the potential to support teachers in making sense of and use student thinking to improve teaching and learning. The authors of the Common Core State Standards (2010) emphasized the use of research-based LTs in the development of the new standards and committed to using research and evidence of student learning to inform future revisions. Daro, Mosher, and Corcoran (2011) state that LTs serve "as a basis for informing teachers about the (sometimes wide) range of student understanding they are likely to encounter, and the kinds of pedagogical responses that are likely to help students move along" (p. 12). However, little is known about how teachers come to learn about LTs and appropriate them into their instruction. In this study, I identify the ways in which one elementary teacher used a LT to support attention to her students' mathematical thinking during instruction. In particular, the teacher's use of the LT as she planned her math lessons, identified learning goals, and anticipated likely student responses will be highlighted.

Background

Learning trajectories utilize research on student learning to describe probable pathways of learning over time. Researchers that have studied the implications of LTs for teachers have found that LTs provide a framework for making instructional decisions (Bardsley, 2006; Sztajn, Wilson, Confrey, & Edgington, 2011b; Wilson, 2009) and afford teachers with a means to focus on their students' mathematical thinking (Clements, Sarama, Spitler, Lange, & Wolfe, 2011; Edgington, Sztajn, & Wilson, 2011; McKool, 2009; Mojica, 2010). As teachers increasingly attend to student thinking in lesson planning and instruction, research must consider the role of LTs in supporting teachers' complex work. Research has yet to address how teachers adjust their lesson planning and instruction when they have information about the progression of more sophisticated levels of thinking inherent in LTs, or how teachers use evidence of student thinking to inform future instruction in light of LTs. This study contributes to the research on teachers' uses of LTs to support attention to student thinking in planning for mathematics instruction.

For the purposes of this paper, I am reporting one teacher's use of a LT in lesson planning through three different processes: identification of learning goals, choice of instructional tasks, and anticipation of students' work. Often considered a core routine of teaching, lesson planning refers to the time teachers spend preparing for instruction before students enter the classroom. Grossman and colleagues (2005) refer to this as the "preactive" aspect of practice, where teachers focus on lesson planning, unit planning, or even planning for classroom management.

Studies on what teachers attend to in planning their lessons indicated that teachers focus on ideas such as content, activities or tasks, materials, textbooks, routines, as well as students' needs and backgrounds (Fernandez & Cannon, 2005; McCutcheon, 1980; Yinger, 1980; Zahorik, 1975). In his 1975 study of teacher planning, Zahorik showed that teachers attended to content more often than objectives, followed by activities. In a study of 12 elementary school teachers, McCutcheon (1980) found that teachers used their textbook as a main source for activities and depended heavily upon suggestions from the teachers' guide. In a later study, Brown (1988) examined the lesson planning practices of 12 middle school teachers in various content areas. She found that teachers relied heavily on curriculum materials, building their lessons off of objectives expressly stated in the curriculum resources.

More recently, Superfine (2008) studied three teachers' lesson planning with respect to a specific mathematics curriculum. Her study revealed two planning problems: difficulty anticipating student work, misconceptions, and potential errors for a given task, and understanding the treatment of the content in the curriculum. She concluded that the conceptions teachers hold with respect to the teaching and learning of mathematics as well as years of experience mediated their management of the planning problems.

Conceptual Framework

In light of reform efforts to improve the teaching and learning of mathematics, one may question what *should* be the focus of planning when instruction attends to students' mathematical thinking. Teachers must consider how to construct lessons that address specific learning goals and allow teachers to gather evidence of their students' understanding towards the chosen goals. Moreover, as student learning progresses over time, teachers must be able to consider how to build students' current conceptions to reach intended learning goals.

The conceptual framework for this study draws upon the work of Hiebert, Morris, Berk, and Jansen (2007) as well as that of Stein, Engle, Smith, and Hughes (2008). Hiebert et al. (2007) proposed a framework for competencies necessary to analyze teaching with the goal of improving on instruction. Stein et al. (2008) presented five practices to support productive mathematical discourse structured around students' responses to mathematical tasks. These two frameworks were chosen because of their emphasis on student thinking as a central feature. During lesson planning, teachers not only *choose intended learning goals*, but they decompose learning goals into smaller sub-concepts that comprise larger goals (Hiebert et al., 2007). In *considering mathematical tasks* proposed in a lesson, teachers use their own content knowledge as well as their knowledge of how students are likely to approach the task to *anticipate students' responses* and likely areas of difficulty. In this way, teachers can consider how students' responses, both correct and incorrect, can lead to the intended learning goals (Stein et al., 2008). By comparing evidence of student learning to the intended learning goals, teachers can determine what aspects of their instruction helped or hindered their students' understandings (Hiebert et al., 2007). Once instruction has been evaluated, careful planning is important so that teachers consider new learning goals and instructional tasks that build on students' current conceptions and move students to more complex mathematical understanding.

Attending to student thinking can support teachers as they engage in lesson planning. This attention allows teachers to acknowledge their students' current conceptions and design lessons that build on prior knowledge. Furthermore, as teachers consider evidence of students' thinking, they can more explicitly connect students' conceptions to important mathematical ideas. As representations of student thinking, learning trajectories are tools which help advance teachers ability to make sense of this evidence and use it to develop instruction that addresses their students' existing conceptions and moves learning forward.

Method

This study seeks to understand how teachers use the construct of a LT to support attention to students' mathematical thinking and addresses the following research question: *In what ways do teachers use LTs during lesson planning to choose learning goals and instructional tasks, and anticipate students' approaches to intended instructional tasks?* A qualitative approach is appropriate in order to understand participants' created meaning of their use of one particular LT in mathematics instruction. Specifically, case studies allow the researcher to uncover and examine significant interactions that are characteristic of the phenomenon under study as well as provide concrete and contextual knowledge as evident in the end product (Merriam, 1998).

Context

Learning Trajectory-Based Instruction (LTBI) is a research project with a strong professional development component for elementary school teachers (Sztajn, Wilson, Confrey, & Edgington, 2011a). The project was motivated by the adoption of the Common Core State Standards (2010) and current research on learning trajectories in mathematics education (Battista, 2004; Clements & Sarama, 2009; Confrey et al., 2009), with the goals of examining the ways in which teachers learn about learning trajectories and use them in their classrooms to define the concept of learning trajectory-based instruction.

As the context for the professional development, teachers learned about one LT: the equipartitioning LT (EPLT). Based on a synthesis of research and clinical interviews, Confrey and her colleagues developed the EPLT that describes how children use their informal knowledge of fair sharing as a resource to build an understanding of partitive division that unifies ratio reasoning and fractions (Confrey, in press).

The EPLT begins with experiences of fairly sharing collections of items or single wholes. In equipartitioning, students must learn to coordinate three criteria: (1) creating equal sized groups or parts, (2) creating the correct number of groups or parts, and (3) exhaust the entire collection or whole. As students enact strategies to complete these tasks, they gain proficiency in mathematical reasoning practices such as justification and naming (e.g., as a count, fraction, or ratio) and begin to develop understandings of fundamental mathematical properties that later influence the ways that they fairly share multiple wholes (Confrey, Maloney, Wilson, & Nguyen, 2010). The trajectory describes how these strategies, practices, and properties ultimately unify as a generalization of partitive division that relates ratio reasoning and fractions. Important to the trajectory are not only the levels of sophistication of reasoning but parameters associated with the tasks, including the number of wholes and number of sharers. Beginning with equipartitioning collections, the next task parameters address equipartitioning single wholes (rectangles and circles), building on primitive splits such as halves and powers of two, to eventually include arbitrary integer splits. The upper levels of the trajectory address tasks that involve multiple wholes and multiple sharers when the number of wholes is both less than and greater than the number of sharers.

Participants

LTBI is a four-year project and, in its first year, involved 22 kindergarten through fifth grade teachers from one elementary school in a mid-sized urban school district in the Southeastern United States. Project participants were offered the opportunity to continue working with the research team in some respect in the second year of the project. The second grade team, consisting of five teachers, expressed an interest in developing a set of equipartitioning lessons based on the EPLT. The findings presented here are from the analysis of one particular case, Bianca. Bianca is a Hispanic female and had been teaching second grade for five years. For her, LTs represented "a continuum of learning where there are key stopping points and also major misconceptions."

Data Sources and Analysis

The primary sources of data for this study are transcripts from grade level planning meetings, pre-lesson questionnaires, classroom observations of teachers' instruction, and transcripts of teacher interviews. Each data cycle began with a grade level lesson planning meeting, followed by individual classroom observations, and concluded with individual post-lesson interviews. Prior to each lesson, each participant completed a pre-lesson questionnaire to obtain information about the teachers' learning goals and any adaptations they may have made to the lesson plan. Observations took place in each participant's classroom during her regularly scheduled math instructional time and were video recorded. Following each lesson, a semi-structured interview was conducted with the participant to discuss the teacher's perceptions of what learning took place as well as evidence of that learning, and how the teacher used that evidence to inform future learning goals.

Data were analyzed using ATLAS.ti (2012) qualitative data analysis software. Evidence from the grade level meetings and pre-lesson questionnaires were used to examine the ways in which teachers used the EPLT to select learning goals and tasks, and anticipate students' responses. Evidence from post-lesson interviews and grade level meetings were considered to determine the ways in which the EPLT was used to reflect on the impact of instruction on student learning, evaluate evidence of student learning, and to inform future instruction. Using a constant comparative method to build and refine categories (Strauss & Corbin, 1998), open coding and pre-determined codes were utilized. The findings reported here focus on the use of the EPLT to choose learning goals, select tasks, and anticipate likely student responses for three sequenced lessons.

Results

Lesson #1: Sharing a Collection for 2, 4, and 3 Friends

During the first grade-level planning meeting, Bianca initiated the discussion of ideas for their first equipartitioning lesson by suggesting they use an activity they created and tried out the previous year where students engaged in fairly sharing a collection of 24 counters among 2, 4, and then 3 friends, justify their work, and name the resulting shares. In considering her own students, Bianca stated that she was interested in knowing if her students knew the three criteria for equipartitioning and thought that sharing collections would be a reasonable place to start since it is low on the trajectory. She stated, "I mean, for me my objective is to see do they know the three criteria. Which you can see, but you can't fully see. Because they may, with a two split, they may or may not do that right, even if they don't know all three [criteria]." After observing her students' work in the first few weeks of school, she used the nature of the task parameters of the EPLT as a justification for adapting the lesson to increasing the size of the collection from 24 to 36 counters: "that's why I went to the higher number because I know that, you know, that's going to increase the level of difficulty."

Bianca anticipated how her students would engage with the first lesson, expecting them to use dealing strategies along with number facts and doubles facts to help them determine fair shares of collections. She used the EPLT to also anticipate obstacles her students might have by saying, "I think one difficulty will definitely be naming the shares. I know that it is a more difficult task on the learning trajectory and they haven't had many experiences doing so."

After the first lesson, Bianca confirmed the difficulties her students had with naming a share and stated, "I feel as if the naming is the hardest part…Because when we teach fractions explicitly, I feel like they get to the wholes and they get to the actual sharing of things. But I feel as if we'd be doing our kids a disservice if we didn't hit on what they are most needing. Which I, from my class, I definitely think the naming thing. I definitely would not recommend sharing collections with higher numbers. I know that shouldn't be your next step, because that is what I thought."

Lesson #2: Sharing a Rectangle and a Collection for 2 Friends

Bianca used what she knew about her students' understanding and the structure of the LT to consider possible follow-up activities. She recognized that a next instructional step could be to change the task parameters but still focus on naming. Based on her teaching experience, she hypothesized that naming would be easier with a whole so in the second grade-level planning meeting, she suggested starting with sharing a rectangle for two and naming the resulting share to help scaffold students' ability to name the resulting share from equipartitioning a collection for two. She also recognized from the first lesson that students readily made connections to doubles facts, so that could potentially scaffold students' ability to name 2-splits. "What if we went, this is, I'm just throwing this out there, this could be, you know. But what if we went to wholes and just worked on halving to see if a name came out of that? And then we went back to doubles with collections and see, saw if the, you know if the vernacular, if the vocabulary came out with a whole, if they would transfer it then to collections."

The group agreed to begin the lesson by having students share two different sized rectangles fairly for two people and discuss naming the share as "one half," or "one out of two pieces" with respect to the different sized rectangles. Then, students would work to fairly share small collections as represented in drawings of arrays of 6, 8, and 10 counters and name the resulting share for each collection. Bianca identified "naming a share as 'half'" for both rectangular wholes and for collections as the goal for the lesson. She anticipated that the structure of the arrays in the second activity would help her students make a connection between "half" of the rectangle and "half" of the collection of counters. After the second lesson, Bianca considered that a possible follow up activity could be to increase the size of the collection to 12 or 14, but still share for two people, or move to sharing small collections or single wholes for four or three people and continue to focus on naming the resulting share. She attended to the interactions between the proficiency levels of the EPLT (*strategies* for sharing a collection or whole and *naming* the resulting share) and the task parameters (changing the size of the collection or the number of people sharing) to consider follow up activities for her students.

Lesson #3: Sharing a Rectangle for 2, 4, and 8 Friends

During the third lesson planning meeting, the teachers agreed to use a task they called "the wrapping paper task" where they used the context of fairly sharing wrapping paper to wrap holiday gifts. Bianca again attended to the task parameters as a way to address naming with her students. Her specific learning goals were for students to "share a whole fairly for 2, 4, and 8 people. Students will focus on how they might name the share in relation to the whole, for example each person got 'one of 8 pieces.'" She also considered that because her focus was on naming, which is higher in the trajectory, that keeping the task parameters lower would allow her students to focus more easily on the name rather than on the strategy for equipartitioning. When asked why she chose 2, 4 and 8, she stated, "Yeah, so I wanted to keep with repeated halving just knowing the trajectory. You know, I know that that's easier and since naming is a little bit harder, I didn't want the sharing to be as diff—too difficult for them … I wanted them to be able to feel successful sharing so that they could focus on what do we call what we've just shared."

Bianca anticipated that her students would make connections from the previous lesson to sharing the wrapping paper for two people. Because she purposefully chose powers of two, she also predicted that her students would use a repeated halving strategy: "I hope that a few of them notice the repeated halving and give their own language and explanation as we go from 2 to 4 to 8." Bianca also used the EPLT to expect different mathematical names such as "one out of four, or one part of the four whole parts, or one part out of eight parts, etc."

After the third lesson, Bianca again considered the task parameters in relation to her students' learning. She believed they were successful with equipartitioning a rectangle and naming the share for 2, 4, and 8 people and considering changing the task parameter from rectangles to circles for a follow up lesson:

> Well, I think—I mean I would like to see—I would probably do something the same, maybe with circles. And still focus on naming because we've kind of gotten there. But I know we're taking it down a—I would still do two, four and eight, but let's do circles and see can we still name them, but are our shares—but, like, at the side be like, "okay, what's going to happen when we get a circle? Can we share it?" Because here, they were successful sharing it and so they could be successful naming it, all right so now we're just going to take a circle and we're going to try to share it fairly.

She recognized that in changing the task parameter to a circle, she would first need to investigate if her students could use successful strategies to equipartition a circle and then move higher up the trajectory to address naming the resulting shares.

Discussion

Overall, Bianca coordinated the proficiency levels and task parameters of the EPLT to design tasks that focused on the learning goals that she chose for her students related to naming. She was able to bring together the existing curriculum with aspects of the trajectory, such as relating doubles facts with sharing for two people and naming fractional parts of halves and fourths. Bianca was also able to use the structure of the EPLT to consider possible follow up activities based on her students' learning. In considering the planning problems identified by Superfine (2008), Bianca offers evidence that the structure of LTs can support teachers in anticipating student work and in making connections between mathematical content and existing curricula.

For each lesson, Bianca was very clear about the specific learning goals she chose and used information from the trajectory to anticipate how her students would engage with tasks she selected. In light of the LT, Bianca was able to anticipate a variety of students' approaches and difficulties and consider how these related to her students' current conceptions and the intended learning goals. The LT was a tool to gauge the appropriateness of the tasks based on the understandings her students exhibited with the intention of moving students towards more sophisticated conceptions.

The tasks and goals she chose were in service of the larger, long-term goal of understanding the relationship between equipartitioning collections and wholes to naming the resulting share using a fractional name in relation to the collection or whole. Moreover, Bianca used open tasks that provided evidence of her students' thinking with respect to multiple proficiency levels of the EPLT, supporting the engagement of students with various conceptions of development along the trajectory (Sztajn, Confrey, Wilson, & Edgington, in press). Her ability to coordinate the levels of the EPLT with the task parameters supported her in considering follow-up activities based on the learning she observed from her students.

Because LTs describe concepts from less formal to more sophisticated ideas, LTs can aid teachers in selecting appropriate learning goals and provide information about what sub-goals are associated with larger conceptual goals. Hiebert and colleagues (2007) contended that learning goals are the basis for gauging the effectiveness of particular instructional activities and for measuring evidence of student learning. LTs afford teachers with information about likely strategies, misconceptions and important milestones that teachers can then anticipate as they plan instructional activities. Anticipating students' approaches to a task prior to instruction allows teachers to begin to think about how students' work relates to the intended mathematical goals (Stein et al., 2008). The knowledge of student learning inherent in LTs provides teachers with more detail as they compare evidence of student learning to learning goals, and gives them a repertoire of instructional moves based on the understandings their students' exhibit. Bianca was able to use the LT to better understand her students' mathematical thinking and target her lesson planning to her students' needs. Considering the fact that the Common Core State Standards were developed using LTs, it is important for the field to continue to explore the utility of LTs as a tool to aid teachers in attending to their students' mathematical thinking not only in lesson planning, but also during classroom instruction.

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EVALUATING TEACHERS' DECISIONS IN POSING A PROOF PROBLEM

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When Geometry teachers pose proof problems to students, it is the teacher who provides the givens and the statement to be proven; we hypothesize that teachers of geometry recognize this to be the norm. This study examined teachers' decision-making in regards to the posing of a proof problem, and whether recognition of this norm accounted for the decision made. Results of a multinomial regression indicated that the more participants recognized that norm of posing proof problems, the less likely they were to select an action that breached the norm.

Keywords: Instructional Activities and Practices; Reasoning and Proof; Teacher Beliefs; Research Methods

Background and Objectives

During the early 1970s, teachers' decision-making became a focus of educational research through parallel investigations led by Alan Bishop, Lee Shulman, and Richard Shavelson (Borko, Roberts, & Shavelson, 2008). Each initiative viewed the examination of teacher decision-making as a means to better understand teaching. For Shulman (1986), this research on teachers' decision-making exemplified how research on teaching had brought attention to teachers' cognition to a field that had up to then only considered teachers' characteristics and behaviors. Accordingly, much of this early research posited individual resources such as beliefs, goals, knowledge, or schemas as resources for decision-making (Schoenfeld, 2010). But another of the paradigms for the study of teaching that Shulman (1986) described, the classroom ecology paradigm spearheaded by Doyle, had undertaken to improve the study of teaching by attending carefully to its activity structures. Contributing to this approach, Herbst and Chazan (2011) have addressed teachers' decision-making by proposing that teachers draw upon resources of a different kind to justify their pedagogical moves. "Combined with the personal assets (including knowledge, skills, and beliefs) that an individual teacher brings with them to that position and that role, [instructional norms and professional obligations] can help explain teacher action and decision-making" (p. 417). As described by Herbst and Chazan, professional obligations are resources of the profession that regulate the position of a mathematics teacher while instructional norms are resources embedded within the various activity structures in which the teacher plays a role. Thus, in this perspective, the justification of a teacher's decision depends not solely on the individual teacher's personal resources but also on their recognition of those norms and obligations (a recognition that could be tacit). Yet, what remains unclear is the degree to which an individual teacher's resources and their recognition of instructional norms account for the decision that a teacher makes in the moment. The purpose of the current study is to examine this phenomenon in a specific decision-making context.

Personal Resources for Decision-Making

In their review of early literature on teachers' decision-making, Shavelson and Stern (1981) suggest that teachers "make judgments and decisions, and carry them out on the basis of their psychological model of reality" (p. 461), which, in turn, is composed of various beliefs such as those concerning pedagogy and the subject matter. Shulman and Elstein (1975) also suggested that personal resources of the teacher influence judgments. Shulman (1987) later restated this relationship in terms of particular types of knowledge professional teachers hold, and how such knowledge influences teachers' decision-making. In the past several years, Deborah Ball and colleagues have expanded this idea to describe their conception of Mathematical Knowledge for Teaching (MKT) (Ball, Thames, & Phelps, 2008). Like Shulman (1987), Ball et al. (2008) suggest that teachers' in-the-moment decisions require "…coordination between the mathematics at stake and the instructional options and purposes at play" (p. 401).

The literature suggests that one clear resource teachers use in making pedagogical decisions appears to be their pedagogical content knowledge (PCK) (e.g., Ogletree, 2007). Yet, Bishop's (1976) account of decision making made a strong argument for the importance of teaching experience in the development of the schemas that may be associated with decision making. Osam and Balbay (2004) provide additional evidence for this, finding that surveyed novice teachers were more concerned with technical details of a lesson in their decisions while experienced teachers were more concerned with the way students behaved during the lesson.

Another potential resource that may influence teachers' decisions is their degree of autonomy to make decisions about their instructional practices. Behm and Lloyd (2007) observed that while different student teachers were provided different resources, the degree of autonomy afforded those student teachers was a critical indicator of what they were able to do with the materials at hand. Examining decision-making in science classrooms, Gess-Newsome and Lederman (1995) found that teachers' autonomy was a highly influential factor in the types of instructional decisions made. With these considerations in mind, we consider teacher autonomy, along with PCK and teaching experience, to be critical personal resources of teachers in their decision-making.

A Professional Resource for Decision-Making

Aho et al. (2010) note that more than teachers' own personal resources influence their decision-making. Rather, "teaching is influenced by the surrounding society, culture and traditions" (p. 400). Teachers interviewed by Aho et al. noted that some pedagogical decisions they made were agreed upon with school colleagues. Further, these types of collective decisions over time work their way into the routines of the teacher. We argue that while such routines are operationalized by individual teachers, their genesis are social in origin and therefore may be more characteristic of actions normative of a group than of particular individuals: In this case we are interested in the obligations that bind a professional group and the norms of the activities in which they play a role. Herbst et al. (2009) provide an example of one such type of norm. Observing similarities in how proof was facilitated across different teachers' classrooms, Herbst et al. note that "these similarities can be expressed by a common system of implicit norms regulating the events on the surface" (p. 266). Such norms appear to influence teacher decisions in the classrooms particularly shaping the division of labor, or who does what, when the situation is one of doing proofs.

Situational norms and professional obligations on the one hand and individual resources of teacher autonomy, experience, and knowledge are thus two kinds of constructs that might account for the decisions teachers make (e.g., Ball et al., 2008; Bishop, 1976; Gess-Newsome & Lederman, 1995). Given these various resources, it is prudent to investigate the degree to which they influence teachers' decision-making. We focus on the instructional situation of doing proofs (Herbst et al., 2009), and on a particular norm of doing proofs (*when posing proof problems, the teacher provides students with the given information and the statement to be proved*). With this situation-specific focus, we sought to answer the following research question:

> *To what degree do teachers' recognition of an instructional norm account for their decision-making in posing a proof problem, and to what degree do the individual resources of PCK, teaching experience, and perceived teaching autonomy contribute to their decision.*

Methods

Sample and Measures

Data were collected from 55 secondary mathematics teachers (grades 8 to 12) in a Midwestern state. The sample included 43.6% male and 56.4% female teachers. Participants were sampled from a wide range of districts, both urban and rural, and of varying levels of socio-economic status. For example, some participating teachers taught in schools with 4% of the population eligible for free and reduced lunch, while others came from schools where 59% of students were eligible. Of the 55 sampled participants, 44 (80%) completed all assessments that we include in the current analysis, and represent our effective sample.

Participants were invited to complete a series of assessments on an online platform (LessonSketch.org), of which we include data from four of the assessments. LessonSketch allowed for the incorporation of multimedia survey instruments in which participants viewed and answered questions concerning representations of teaching, of which was particularly useful in assessing teachers' situation-specific decision-making.

Dependent Variable

We assessed participants' decision-making based on their multiple-choice responses to a representation of teaching. Participants were presented with a cartoon-based, two-frame teaching representation, preceded with a brief overview of the lesson as one taking place in a high school geometry class in which the teacher was going to assign a proof problem. The representation depicted the teacher drawing a diagram and reviewing with students that to write a proof they would need a set of givens and a statement to prove. Participants were then presented with four potential actions that could follow and were asked to select which they would be most likely to do next following the scenario. Each action was a single-frame depiction representing either compliance or breach with the normative action: *when posing a proof problem, the teacher provides the givens and prove statement to students*. Participants were asked "which action would you be most likely to take in the teaching scenario?" and then to "please explain your reasoning for choosing this action."

Choice A depicted a breach of the norm where the teacher instructs students they will have a discussion to decide, as a class, what the givens and the prove statement will be. Choice B is another breach of the norm where the teacher asks students to work individually, decide what the givens and prove statement are, and then do the proof. The students would later compare their proofs to their peers'. Choice C is a breach where the teacher provides the prove statement, but instructs the class that they will discuss, as a class, what givens they will need to do the proof successfully. Choice D is compliant with the norm where the teacher provides both the givens and "prove" statement, and then asks an initial question for class discussion on how to do the proof. Responses were well distributed with 22.7% selecting Choice A, 18.2% selecting Choice B, 31.8% selecting Choice C, and 27.3% selecting Choice D.

Independent Variables

We included four independent variables in our analysis. The independent variable of interest (*normativity*) was a score representing participants' endorsement of the norm: *when posing a proof problem, the teacher provides the givens and prove statement to students*. The variable, described in detail below, was designed as an indicator of participants' recognition of an instructional norm. Specifically, scores for *normativity* were interpreted to assess the degree to which individual participants recognized the identified norm in the situation 'doing proofs' in Geometry instruction.

We assessed this recognition with a 10-item survey that presented participants with explicit statements regarding the norm of focus. A sample item and available responses is presented in Figure 1. Items were written to assess participants' view of how appropriate it was for the professional group of Geometry teachers to provide students with the givens and prove statement in posing proof problems. Interpretation of the items was validated through cognitive interviews prior to collection of the current data, with results suggesting that items were interpreted as intended. Additionally, we calculated an alpha coefficient of .89, suggesting the items had sufficient reliability as well as validity. Participant responses were averaged into a composite score, *normativity* (M = 3.46, SD = 1.08), for inclusion in the present analysis. Higher scores represented a greater recognition of the norm, and vice versa.

When starting a proof problem, how appropriate is it for teachers to have students decide upon the 'prove' statement (conclusion)?

1	2	3	4	5	6
Very Inappropriate	Inappropriate	Somewhat Inappropriate	Somewhat Appropriate	Appropriate	Very Appropriate

Figure 1: Sample item assessing participants' recognition of the norm

The next independent variable included was a measure of participants' perceived autonomy in their mathematics teaching (*autonomy*). As noted in our literature review, teachers' autonomy has the potential to regulate the effect of teacher beliefs, and therefore represented a useful factor for investigation. Items measuring *autonomy* were adapted from multiple sources to focus both on the content of mathematics and the role of teacher (Deci & Ryan, 2011; Kosko & Wilkins, in review; NCES, 1998; Reeve et al., 2003). Teachers were asked to rate their agreement with statements such as the following: *I am discouraged from teaching mathematics in the way I would like to* (reverse-coded sample item). Available responses were on a 6-point Likert-scale (1–Strongly Disagree; 2–Disagree; 3–Somewhat Disagree; 4–Somewhat Agree; 5–Agree; 6–Strongly Agree). Items showed sufficient reliability (α = .89) and responses were averaged into the composite score *autonomy* (M = 4.62, SD = .81).

The third variable included for analysis was years of teaching experience (*Years*). Teachers in the sample taught an average of 13 years (SD = 7.30). While Bishop (1976) noted that it was schemas developed through experience that influenced teachers' decision-making, we used *Years* as an indicator of having more sophisticated forms of such schemas.

The final independent variable included was a measure of pedagogical content knowledge in geometry (*PCKG*). The assessment included 10 items covering various Geometry topics and addressing teachers' *knowledge of content and teaching* and *knowledge of content and students* (see Ball et al., 2008 for a detailed description of these domains of mathematical knowledge for teaching). Items were validated through cognitive interviews before collection of the present data. Item analysis of present data showed biserial correlations of .30 or higher, and a Cronbach's alpha coefficient of .70, suggesting the construct had sufficient reliability. Scores were based on percentage of items answered correctly, with a possible range of 0 to 1 (M = 0.46, SD = .23). Herbst and Kosko (2012) provide more details on the development of this instrument.

Analysis and Results

We used multinomial logistic regression (MLR) to examine participants' decision-making. Specifically, participants were asked to select one of four potential actions following a depicted teaching scenario. While one of these actions was considered compliant with the norm and the other three breaches of the norm, we did not consider one action as necessarily better than any other. Further, the participant choices could not be ordered in any natural way. Therefore, the responses represent nominal data suitable for an MLR. MLR is a form of logistic regression which uses one category (one of the choices available) as a reference outcome, and creates separate logistic regression comparisons between the reference outcome and each other classification (see Hosmer & Lemeshow, 2000 for a detailed description).

The model examined in the current analysis is presented in the equation below. The outcome of reference is the normative action, Choice D, and is designated by 0 in the equation. Each alternative choice (breaches of the norm in Choices A, B, and C) are represented in variable m, such that we have three distinct regression equations; one for each comparison. So, we evaluated the degree to which each independent variable contributes to participants choosing Action A rather than Action D, Action B instead of Action D, and Action C instead of Action D.

$$g_m(x) = ln\left[\frac{P(Y = m \mid x)}{P(Y = 0 \mid x)}\right]$$
$$= \beta_{m0} + \beta_{m1}(normativity)_1 + \beta_{m2}(autonomy)_2 + \beta_{m3}(Years)_3 + \beta_{m4}(PCKG)_4$$

Customary in performing MLR is an initial checking of model fit, both for the model as a whole as well as for particular variables within it. While the model represented in the above equation had overall model fit ($\chi^2 = 39.34$ (df = 12), $p < .01$), the variable PCKG was found to not have a statistically significant relationship with participants' choices ($\chi^2 = 3.91$ (df = 3), $p = .271$). This initial finding suggests that there was little relationship between participants' *PCKG* scores and their chosen action following the scenario and, therefore, *PCKG* should be considered for removal in the analysis to provide a more parsimonious model. The standard errors associated with *PCKG* were also high (above 2.0), suggesting potential collinearity. Also, a separate MLR with only *PCKG* in the model still suggested no statistical relationship with choice of action. However, an examination of the descriptive statistics suggest that while participants selecting the normative action tended to have higher *PCKG* scores, there was a large degree of variance in these scores, further justifying the removal of *PCKG*. The new model, which includes *normativity*, *autonomy*, and *Years* as predictors, was found to have good overall fit ($\chi^2 = 35.43$ (df = 9), $p < .001$), with no need for further simplification of the model.

Results from the MLR analysis are presented in Table 1, with coefficients represented in logits. In each model comparison, *normativity* scores were found to be a statistically significant predictor of choice at the .10 level, when accounting for participants' perceived teaching autonomy and years of teaching experience. Using the conversion:

$$\frac{e^{\beta_{m0}+\beta_{m1}(normativity)_3+\beta_{m2}(autonomy)_2+\beta_{m3}(Years)_3}}{1+e^{(normativity)_3+\beta_{m2}(autonomy)_2+\beta_{m3}(Years)_3}}$$

we can determine the probability that a participant in our sample would select a particular choice rather than the normative action represented in Choice D. For example, a participant with an average *autonomy* score ($M = 4.62$) and years of teaching ($M = 13$) for the sample, a low *normativity* score of 1.00 would suggest such a participant is 99.9% more likely to select Choice A over Choice D. However, if a similar participant had a high *normativity* score of 6.00, there is a practically zero probability that they would select Choice A over Choice D. These and similar calculations are illustrated, for convenience, in Figure 2.

Table 1: Results from Multinomial Logistic Regression.

Comparison		β (logits)	S.E.	Wald Statistic
Choice A | Choice D	Intercept	12.75	5.27	5.13**
	normativity	-5.35	1.83	8.56**
	autonomy	1.59	1.00	2.55
	Years	-.29	.11	6.52**
Choice B | Choice D	Intercept	-4.17	4.90	.72
	normativity	-1.51	.79	3.68*
	autonomy	2.21	.98	5.15**
	Years	-.10	.08	1.71
Choice C | Choice D	Intercept	2.20	3.13	.49
	normativity	-1.08	.60	3.20*
	autonomy	.57	.57	1.00
	Years	-.04	.06	.48

**p < .10. **p < .05.*

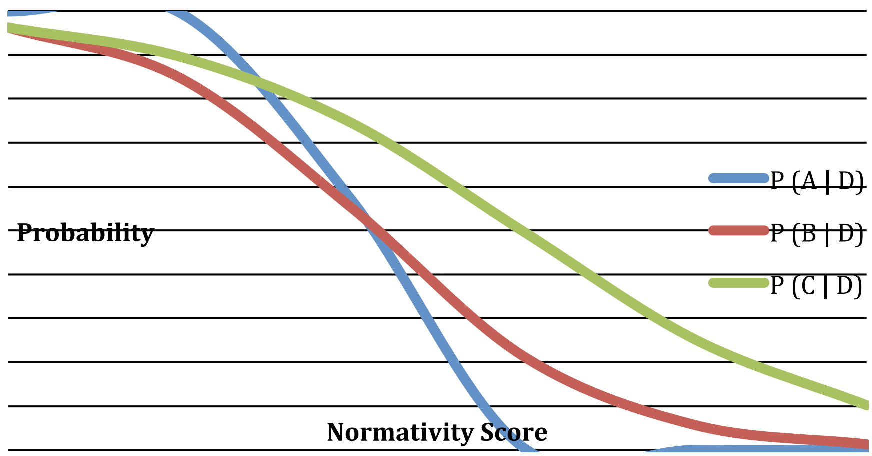

Figure 2: Effect of normativity score on probability of selecting an option other than choice D, with average *autonomy* and *Years* for the sample

These findings indicate that, for each comparison, the degree to which participants recognized the norm was a consistent determiner of how likely they were to select Choice D or an alternative. Additionally, it appears that the more participants recognized the norm, the more likely they were to select the normative action, Choice D, instead of an action that included a breach of the norm. Further, while perceived autonomy and years of teaching experience did influence whether participants would choose one action over another for some comparisons, *normativity* consistently did so and generally at larger magnitudes.

Discussion and Conclusion

The findings from our analysis are preliminary, in that they represent the decision-making regarding the teaching norm of focus for only one particular teaching scenario. Yet, examination of participants' choices suggests that participants who recognize the norm tend to act according to that norm. Additionally, participants' perceived teaching autonomy influenced decision-making in a manner that contrasted *normativity*. Specifically, a higher perception of autonomy was shown to increase the likelihood a participant chose Action B over the normative action, while a higher *normativity* score decreased the likelihood those participants would choose Action B over the normative action (see Table 1). This statistical conflict between *autonomy* and *normativity* is representative of what Pepitone (1989) described as the conflict between rights and obligations. Pepitone noted that "the reaction to the violation of an obligation may be tempered by an internalized right that is in opposition to the obligation, perhaps the very same right claimed and exercised by the 'violator'" (p. 14). Applied to the context of this study, participants' 'violation' or breach of the norm through selecting Action B may have been tempered by their sense of autonomy, which in turn can represent any number of internalized beliefs about mathematics teaching and learning.

The conflict between *autonomy* and *normativity* discovered in the present analysis suggests that for the particular scenario examined, *normativity* wins the conflict. While *autonomy* was shown to have a larger logit size for P(Action B | Action D), *normativity* consistently predicted the decision-making patterns for all actions relative to Action D (the normative action). While this pattern may vary given differing scenarios and options for decisions, the main claim from our analysis suggests that participants' recognition of situational norms in teaching are an important influence in their pedagogical decision-making. Therefore, if we wish to better understand teachers' decision-making, more attention should be given to the characteristics of the situations in which teachers act, as well as to the resources of individual teachers.

Endnote

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FROM RECURSIVE PATTERN TO CORRESPONDENCE RULE: DEVELOPING STUDENTS' ABILITIES TO ENGAGE IN FUNCTIONAL THINKING

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Third- through fifth-grade students participating in a classroom teaching experiment investigating the impact of an Early Algebra Learning Progression completed pre- and post-assessment items addressing their abilities to engage in functional thinking. We found that after a sustained early algebra intervention, students grew in their abilities to shift from recursive to covariational thinking about linear functions and to represent correspondence rules in both words and variables.

Keywords: Algebra and Algebraic Thinking; Elementary School Education

Algebra has historically served as a gateway course to higher mathematics that—due to high failure rates—has been closed for many students. More recent initiatives have identified algebra as playing a central role throughout mathematics education and re-framed it as a longitudinal strand of thinking across grades K-12 rather than as an isolated eighth- or ninth-grade topic (e.g., Common Core State Standards Initiative, 2010; National Council of Teachers of Mathematics, 2000). This is not to be interpreted as a call to shift traditional algebra instruction to earlier grades, but rather as one to introduce elementary school students to algebraic thinking in the context of age-appropriate activities.

In response to this call, we drew from research findings, curricular resources, and standards documents in the area of early algebra to develop an Early Algebra Learning Progression [EALP] organized around five "big ideas": (1) Generalized Arithmetic; (2) Equations, Expressions, Equality, and Inequality; (3) Functional Thinking; (4) Proportional Reasoning; and (5) Variable.

We conducted a one-year classroom-based study in grades 3–5 to gather efficacy data regarding the impact of EALP-based classroom experiences on elementary students' developing understandings of these big ideas. The focus of this paper will be our findings regarding the development of students' functional thinking. We will share pre/post assessment data and representative excerpts from student work and briefly discuss the classroom intervention we believe contributed to the growth we observed.

Theoretical Perspective

Functional thinking has been identified as one of the key strands of algebraic thinking (Kaput, 2008) and one of the core content domains in early algebra research (Blanton, Levi, Crites, & Dougherty, 2011). Blanton et al. (2011) characterize functional thinking as "generalizing relationships between covarying quantities, expressing those relationships in words, symbols, tables, or graphs, and reasoning with these various representations to analyze function behavior" (p. 47).

Elementary curricula often include a focus on simple patterning activities in which the change in only one variable is observed. However, an exclusive focus on this type of activity is suggested to hinder the development of students' reasoning about how two or more quantities vary simultaneously (Blanton & Kaput, 2004). A deeper understanding of change and the modeling of behavior in real world phenomena requires students to look beyond recursive patterns and consider covariation in their study of functions (Blanton et al., 2011). Confrey and Smith (1994) furthermore argue that a covariational approach establishes a good foundation for the development of correspondence rules.

Evidence exists that children in elementary grades are in fact capable of reasoning about covarying quantities and developing correspondence rules. Blanton and Kaput (2004) found that students could construct and reason with function tables and identify covariational relationships and primitive correspondence rules as early as first grade, while older elementary students could successfully transition from using natural language to using symbols to represent correspondence rules more formally. Based on these findings, Blanton and Kaput argue that elementary curricula should go beyond recursive pattern finding to include a focus on relationships between variables.

Martinez and Brizuela (2006) likewise found that third-grade students could successfully reason with linear function tables but sometimes struggled to make the transition from focusing on recursive patterns to identifying general correspondence rules that would apply to all cases. They identified "hybrid" approaches, in which students examined the relationship between input and output as required in a covariational approach while simultaneously relying on a recursive pattern. For example, one student observed in her table that "the number that you add to get from the input to the output is always one more than it was in the previous row" (p. 292). This is a limited approach in terms of generalizing and making far predictions, but it does indicate progress in considering the relationship between two variables.

The study of functions in the elementary grades can lay the foundation for success in later grades. Teachers can nurture students' functional thinking by helping them develop algebraic habits of mind that encourage building patterns, making conjectures, generalizing, and justifying mathematical relationships (Blanton & Kaput, 2011; Moss, Beatty, Barkin, & Shillolo, 2008; Moss & McNab, 2010). Martinez and Brizuela (2006) call for carefully designed interventions that consider the relationship between "what students *know* and what we want them to *learn"* (p. 293). In this study, we aimed to move beyond the patterning experiences elementary curricula and standards documents (e.g., National Council of Teachers of Mathematics, 2000) propose students should have and push students to consider covariational relationships and develop correspondence rules. Specifically, this paper addresses the following research question:

How does the functional thinking of grades 3–5 students who have had a year-long focus on early algebra (including functions) compare to that of students who have had more traditional arithmetic-based experiences? Specifically, how does student performance compare across the following aspects of functional thinking:

a) Constructing a function table?
b) Identifying a recursive pattern?
c) Identifying a covariational relationship?
d) Representing a correspondence rule in words and symbols?
e) Making "far" data predictions?

Method

Participants

Participants included approximately 300 students from two elementary schools in southeastern Massachusetts. The school district in which these schools reside is largely white (91%) and middle class, with 17% of students qualifying for free or reduced lunch. Six classrooms (two from each of grades 3–5 and all from one school) served as experimental sites and 10 classrooms (four grade 3, four grade 4, and two grade 5, from both schools) served as control sites.

Classroom Intervention

Students in the experimental condition participated in an EALP-based classroom teaching experiment [CTE] for approximately one hour each week for one school year. A member of our research team—a former elementary school teacher—served as the teacher during these interventions. A typical one-hour lesson consisted of a "jumpstart" at the beginning of class to review previously discussed concepts, followed by group work centered on research-based tasks aligned with our EALP. These tasks were designed to encourage students to reason algebraically in a variety of ways and justify their thinking to themselves and their classmates.

The last five weeks of the CTE focused on functional thinking, in particular, problem situations in which students investigated linear patterns and relationships. In most of these tasks, students were presented with a scenario in words or pictures and were asked to record and organize data, identify and describe recursive patterns and covariational relationships, express correspondence rules in words and symbols, and make near and far predictions. Multiple representations—verbal, pictorial, tabular, graphical, and symbolic—were typically generated from a given problem context. Students were encouraged to discuss connections among representations (e.g., to identify the meaning of the slope and intercept in a symbolic correspondence rule by referring to the function table or by referring back to the original problem context).

Students in the control condition participated in their usual classroom activity with their regular classroom teachers. District-wide, all classroom teachers used "Growing with Mathematics" (Iron, 2003) curriculum materials. This curriculum does not include a focus on early algebra.

Data Collection

A pretest and (identical) posttest were designed to measure students' understandings of algebraic topics identified across the five "big ideas" of the EALP. The majority of tasks were research-based, adapted from tasks used in our or others' prior studies. In total, 290 students completed the pretest (117 experimental, 173 control) and 293 students completed the posttest (126 experimental, 167 control). We also conducted individual interviews with ten students (6 experimental, 4 control) across grades 3–5 at the conclusion of the study to gain deeper insight into their thinking about a subset of the assessment tasks.[1]

From the pre/post assessment, we will focus in this paper on one task—the *Brady task* (see Figure 1)—that investigated students' functional thinking around a situation involving linear growth.

Data Analysis

Each part of the *Brady task* was first scored dichotomously (i.e., correct or incorrect). For all but part a (which required no explanation), student strategies were also coded.

For parts b, c, and d, student responses were categorized according to the type of relationship described: *recursive*, *covariational*, or *functional*. For example, the most prevalent response to part b was for students to provide a description of a recursive pattern (e.g., "The people column goes up by 2s."). We anticipated students would respond in this way, given the focus of typical elementary curricula, and thus designed parts c and d to try to push students beyond recursive thinking. Part c required students to consider the relationship between two variables, thus requiring either a description of a covariational relationship (e.g., "When the number of tables goes up by 1, the numbers of people goes up by 2") or a correspondence relationship (e.g., "The number of people is 2 more than 2 times the number of tables"). In part d, students were expected to describe the correspondence relationship symbolically (e.g., "$2n + 2 = p$ where n = number of tables and p = number of people").

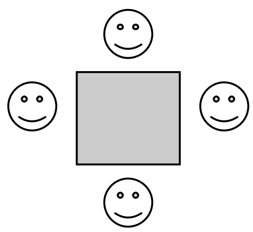
Student responses to part e were coded according to the strategy used to determine the number of people who could sit at 10 tables: *drawing* indicated that students drew 10 tables and counted the number of people who could be seated, *recursive* indicated that students extended the pattern found in the table in part a to 10 tables, and *functional* indicated that students used the correspondence relationship between the two variables to find the solution (i.e., $2 \times 10 + 2 = 22$ people). Student responses to part e that included no work or explanation were placed into an *answer only* category.

Across all of the items, responses that students left blank, or for which they responded "I don't know" were grouped into a *no response* category, and responses that were not sufficiently frequent to constitute their own codes were placed into an *other* category.

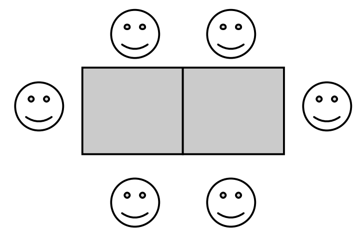
To assess reliability of the coding procedure, a second member of the research team coded a randomly selected 20% sample of the data. Initial agreement between coders was at least 74% for each item. All differences in scoring were discussed by the coders and resolved.

Brady is having his friends over for a birthday party. He wants to make sure he has a seat for everyone. He has square tables.

He can seat 4 people at one square table in the following way:



If he joins another square table to the first one, he can seat 6 people:



a) If Brady keeps joining square tables in this way, how many people can sit at 3 tables? 4 tables? 5 tables? Record your responses in the table below and fill in any missing information:

Number of tables	Number of people
1	
2	
3	
4	
5	
6	
7	

b) Do you see any patterns in the table? Describe them.

c) Find a relationship between the number of tables and the number of people who can sit at the tables. Describe your relationship in words.

d) Describe your relationship using variables. What do your variables represent?

e) If Brady has 10 tables, how many people can he seat? Show how you got your answer.

Figure 1: The *Brady Task*

Results and Discussion

In this section, we report pre/post results from the *Brady Task* and offer representative excerpts from the written assessment to illustrate particular categories of responses.

Completing a Table (Part a)

In the first part of the *Brady task*, students were asked to complete a function table using the given description of the problem situation and accompanying pictures. Third- and fourth-grade students struggled with this task at pretest (see Table 1), while fifth-grade students were already fairly successful prior to the intervention. Grades 3–4 experimental students made significant improvements over the course of the intervention and outperformed control students at posttest.

Table 1: Proportion of Students Who Successfully Completed the Table in Response to Part a

	Grade 3		Grade 4		Grade 5	
	Pre	Post	Pre	Post	Pre	Post
Control	.379	.524	.449	.716	.816	.946
Experimental	.359	.868*	.512	.932*	.857	.955

*Experimental group outperformed control group at posttest ($p < 0.01$).

These findings are consistent with Blanton and Kaput's (2004) and Martinez and Brizuela's (2006) assertions that provided the appropriate experiences, elementary students can learn to construct function tables to represent covarying data. While those students with an arithmetic-based curriculum could successfully construct tables by fifth grade, those students with early algebra experiences could do so sooner.

Recognizing and Describing a Pattern (Part b)

Students were next asked to identify any patterns they saw in the table. This is a task with which we expected students to be fairly successful as only the identification of a recursive pattern was required. One third-grade student stated at pretest, for example, "You count by 2's every time." Most students took this recursive approach; however, some students identified a covariational relationship. A fourth-grade student, for example, wrote "plus 1 table = plus 2 more people" at pretest, indicating attention to the relationship between two variables. One fifth-grade student in the experimental group wrote "× 2 + 2" at posttest, indicating he or she was attending to the functional relationship between the number of tables and the number of people. Table 2 shows the proportion of students who provided a correct pattern or relationship to describe the data in the table. Overall posttest differences were only marginally significant at grade 3.

Table 2: Proportion of Students Who Provided a Correct Recursive, Covariational, or Functional Table Description in Response to Part b

	Grade 3				Grade 4				Grade 5			
	Control		Experimental		Control		Experimental		Control		Experimental	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Recursive	.182	.333	.205	.579	.275	.418	.415	.341	.579	.541	.476	.386
Covariational	.015	.079	.103	.211	.130	.254	.098	.296	.237	.351	.214	.409
Functional	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.023
Total correct	.197	.412	.308	.790*	.405	.672	.513	.637	.816	.892	.690	.818

*Experimental group outperformed control group at posttest ($p < 0.01$).

That students had some initial success with this task—especially by grade 5—is not surprising given the fact that elementary curricula typically focus on identifying recursive patterns in their work with number sequences and data tables. It is interesting to note, however, that an increasing proportion of students provided descriptions of the covariational relationship involved. Note that by fifth grade, more students in the experimental group were providing such responses than were identifying recursive patterns, suggesting the CTE successfully encouraged them to consider relationships between variables.

Expressing a Functional Relationship Using Words and Variables (Parts c and d)

Students were next explicitly asked to move beyond recursive patterning to consider the covariational or functional relationship between the number of tables and the number of people. We initially anticipated this would be very difficult for students given the lack of focus on these concepts in typical elementary curricula. Table 3 shows the proportion of students who provided a correct description of the covariational or functional relationship in words (Part c) and the functional relationship in symbols (Part d).

Table 3: Proportion of Students Who Provided a Correct Covariational or Functional Relationship in Words (in Response to Part c) and Symbols (in Response to Part d)

	Grade 3				Grade 4				Grade 5			
	Control		Experimental		Control		Experimental		Control		Experimental	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Relationship in words (part c)												
Covariational	.015	.079	.026	.237	.101	.164	.146	.136	.316	.487	.119	.182
Functional	.000	.000	.000	.079	.000	.000	.000	.273	.026	.027	.000	.341
Total correct	.015	.079	.026	.316*	.101	.164	.146	.409*	.342	.514	.119	.523
Relationship in symbols (part d)												
Functional	.000	.000	.000	.158*	.000	.000	.000	.295*	.000	.000	.000	.455*

*Experimental group outperformed control group at posttest ($p < 0.01$).

As Table 3 shows, students struggled with these tasks at pretest. Only one student wrote a correct correspondence (i.e., functional) rule in words at that time. All other correct responses to part c at pretest involved describing in words the covariational relationship between the number of tables and the number of people. A fifth-grade student wrote, for example, “The rule is if you add a table two more people can sit.” No students wrote a correct symbolic functional rule in response to part d at pretest. Experimental students improved in this area quite a bit over the course of the intervention, with over 30% of fourth graders and almost half of fifth graders producing correct symbolic rules at posttest. For example,

$A \times 2 + 2 = B$; A for the number of tables, B for the number of people (grade 3)

$x \cdot 2 + 2 = y$; x represents the number of tables, y represents the number of people who sit at the tables (grade 4)

$p \times 2 + 2 = m$; p = # of tables, m = # of people (grade 5)

We attribute this performance to experimental students’ ongoing experience working with variables in a variety of contexts and to the connections continuously made among various representations and the original problem context in the CTE.

Making a “Far” Prediction (Part e)

Finally, students were asked how many people Brady could seat at his party if he had ten tables. As described in the data analysis section, students took three main approaches: drawing ten tables and counting the number of people who could be seated, extending the pattern found in the table in part a to ten tables, or using the functional relationship between the two variables. See Table 4 for the proportion of students who correctly used each approach. “Answer only” refers to students who only answered “22,” with no work shown to indicate strategy use.

Table 4: Proportion of Students Who Correctly Applied a Drawing, Recursive, Functional, or "Answer Only" Strategy to Make a "Far" Prediction in Response to Part e

	Grade 3				Grade 4				Grade 5			
	Con		Exp		Con		Exp		Con		Exp	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Drawing	.167	.143	.077	.132	.203	.284	.220	.296	.368	.324	.333	.091
Recursive	.076	.191	.128	.237	.188	.224	.220	.205	.237	.487	.286	.296
Functional	.000	.000	.000	.079	.000	.000	.024	.273	.026	.027	.024	.364
Answer only	.030	.079	.128	.105	.073	.119	.073	.000	.053	.054	.048	.068
Other	.000	.000	.000	.000	.000	.045	.000	.023	.000	.054	.000	.046
Total correct	.243	.413	.333	.553	.464	.672	.537	.797	.684	.946	.691	.865

Students in both control and experimental conditions showed improvement with this task, but there were no significant posttest differences between groups in terms of correctness. This is not entirely surprising given that the "far" prediction—to 10 tables—is not actually that far. Thus drawing and recursive strategies are not all that inefficient. In subsequent administrations of this task, we plan to ask students how many people could sit at 100 tables. Note, however, the experimental group's increasing use of a function rule to help them solve this problem. We again attribute this difference to the CTE's focus on moving beyond recursive patterning to consider covariational relationships and correspondence rules.

Conclusion

Experimental students showed significant improvement in this study in their abilities to construct function tables (in grades 3 and 4), identify patterns or relationships in tables (in grade 3), and represent a functional rule verbally (in grades 3 and 4) and symbolically (in all grades). These findings support the work of others (e.g., Blanton & Kaput, 2004; Martinez & Brizuela, 2006) who assert that elementary students are capable of sophisticated functional thinking and call into question the lack of focus on relationships between co-varying quantities in many elementary curricula and recent standards documents.

Endnote

[1] Due to limited space, we do not discuss interviews here but will share representative excerpts in our presentation.

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CULTURALLY-MATHEMATICALLY RELEVANT PEDAGOGY (CMRP): FOSTERING URBAN ENGLISH LANGUAGE LEARNERS' MULTIPLICATIVE REASONING

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In this paper we articulate an approach, termed culturally-mathematically relevant pedagogy (CMRP), for fostering urban English language learners' mathematical progression. CMRP integrates three aspects, the use of (1) adaptive teaching to build on students' funds of knowledge for mathematics, (2) tasks that make sense to students given their current mathematical conceptions, and (3) manipulatives and representations that, for the students, meaningfully signify quantities linked to numbers and operations used in a task. To situate CMRP, we use a continuum of conceptual transitions in multiplicative reasoning, which are critical for supporting students' development of algebraic reasoning.

Keywords: Culturally Relevant Pedagogy; Learning Trajectories; Number Concepts and Operations; Instructional Activities and Practices (Adaptive Teaching)

In this theoretical paper, we argue for the need to expand pedagogical perspectives so they integrate, build on, and are relevant to *both* cultural and mathematical aspects of students' prior experiences and knowledge. Our expansion draws on the core notion of *Culturally Relevant Pedagogy (CRP)* (Ladson-Billings, 1995). This notion has greatly enhanced sensitivity to issues of congruency between student experiences of educative processes at their home/community and in schools. The thrust to augment such congruency for diverse student populations was proposed in place of deficit views of students from underserved populations (e.g., lack of family support), which essentially blame the victims for their poor achievements (e.g., Sleeter, 1997).

For English Language Learners (ELLs), CRP entails addressing the complex interactions between their mathematical communication and understandings, while engaging them in work on cognitively challenging tasks (Moschkovich, 2002). CRP encourages ELLs' use of multiple languages as it enables more complex mathematical activity than when using just English (Moschkovich, 2007). We argue that sensitivity to students' cultures and languages is necessary but insufficient to foster their mathematical progressions. This is supported by research findings suggesting that language fluency does not fully account for differences in ELLs' mathematical proficiency (e.g., Abedi et al., 2006). Our thesis is that to foster students' mathematics learning pedagogy has to be relevant to *both* their cultural and mathematical resources.

To realize mathematics teaching for equity an integrated, *Culturally-Mathematically Relevant Pedagogy (CMRP)* is needed. We suggest that being ***M***athematically relevant entails: (1) Using adaptive teaching to build on students' funds of knowledge *for mathematics*, (2) Using tasks that make sense to students given *their* prior conceptions, and (3) Using manipulatives and representations that, *for the students*, meaningfully signify quantities linked to numbers and operations used in a task. Below, we elaborate on each of these three aspects of CMRP.

Adaptive Teaching

By adaptive teaching we refer to pedagogical methods that are tailored, *every mathematics lesson*, to students' resources—conceptions and experiences they have and bring to a learning situation, termed *funds of knowledge* (Moll et al., 1992). These resources *afford and constrain* advances to rigorous mathematical understandings expected of each young person. We set out to identify students' resources, relevant for lesson goals, and design instruction (tasks, activities, materials) that methodically builds on these resources as a means to engender learning of the intended mathematics. We emphasize that adaptive teaching is *not just* student-centered, or standard- (content) informed, or activity- (problem) based. Rather, it integrates all of these into what Tzur (2010) has termed the "*Teaching Triad*" (Figure 1), which stresses the need to analyze, begin at, and build on developmental continua in student available conceptions. This adaptive approach, depicted by the Teaching Triad, is rooted in Simon's (1995) idea of a teaching cycle revolving around hypothetical learning trajectories (HLT).

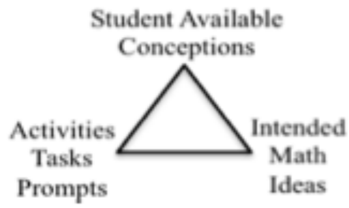

Figure 1: The Teaching Triad

Adaptive teaching, as depicted by the Teaching Triad, includes three principal activities. The first, which ties teaching to students' resources, is the ongoing analysis of students' available mathematical conceptions: goals they might set, activities they might use, contexts familiar to them, objects they might operate on, and effects of their activities that they might notice. The second principal activity is deciphering the intended mathematics into underlying, goal-directed activities. The third principal activity is articulating paths between students' extant thinking and the mathematics they are to learn. The teacher (a) hypothesizes how students' activities and reflections may bring forth the intended learning and (b) designs specific task sequences that can promote the advancement sought (Simon & Tzur, 2004). These three principal activities address five key points proposed by Bransford, Brown, and Cocking (1999): engaging students in tasks, tailoring interaction to students' sense of tasks, motivating students' pursuit of goal, adjusting tasks to students' level of frustration, and possibly modeling execution of a new activity.

Adaptive teaching is consistent with the notion of co-learning (Jaworski, 2001, Tzur, 2004). A teacher selects and uses tasks, guides goal setting, and orients student reflections, while students set their goals, initiate activities, notice effects, and abstract new mathematical relationships. This co-learning approach supports emergence of norms that can increase learning opportunities for ELLs. One critical norm is the constant need to explain various solutions to one another. Another is the expectation to collaboratively solve challenging mathematical tasks, including posing tasks to peers and sharing solution strategies (Boaler, 2006). Such collaboration is coupled with appreciation for diversity and the expectation that every student will participate and share different solutions (errors included!) while not limiting others' participation. Another norm includes awareness that there are many viewpoints, and promote respect for those viewpoints that differ from one's own (Moschkovich & Nelsen-Barber, 2009).

Using Tasks that Make Sense to Students Given *Their* Current Knowledge

CMRP, by engaging every student in solving tasks that are constantly challenging her or his available mathematical thinking, can provide the backbone for equitable lesson/unit design (Simon, 2011). Essential to CMRP is the distinction *between task features and children's thinking*. That is, we contend that the structure of a task as seen by an adult does not, in and of itself, determine the way a child makes sense of and acts to solve it. Rather, implementing a task integrates both—a child's current conceptions and sense making (goal-directed actions, units acted upon), and *task features* designed by adults to promote learning of intended mathematics (Tzur & Lambert, 2011). The need for such a combination is seen in the following example.

Consider the task: Enrique has 3 boxes of cookies. Each box contains 5 cookies. How many cookies does Enrique have in all? Students could solve this task in various ways, such as: (1) using a single cube to signify a cookie and counting, one-by-one, the number of cubes in 3 towers of 5 cubes each to arrive at 15 cookies, (2) counting by 5's, using each finger to signify a tower of cubes (which signifies a box of cookies), and stopping at 15 because 3 fingers have been accrued, and (3) multiplying 3 by 5 to arrive at (a retrieved fact of) 15 cookies.

In the first method, the child needs to count single items—units of 1, but she can use other items, in this case cubes, to stand for these units. In the second method, the child can consider each group of 5 cookies as a "thing," a composite unit. She can continue to accumulate those 5s, anticipating she would be finished when she has accumulated three such units of 5s. In the third method, the child can use a known operation, in this case multiplication, to determine an amount of cookies. As these different methods suggest, although an adult might see a structure in the task and intend for it to elicit a particular operation, the child solving the task engages in operations rooted in her existing conceptions. When implementing mathematical tasks using CMRP, it is necessary to integrate task features with children's sense of the task (e.g., by allowing children to solve the problem in whichever method and language that suit their current ways of making sense of the task). Whereas in this example task features are designed to accentuate coordination of two types of units, a child's solution (e.g., method 1) may include no such coordination.

Using Manipulatives and Representations that Signify Meaningful Quantities for Students

To be adaptive to students' available conceptions, teaching needs to recognize how, when solving tasks, *students* make sense of and use manipulatives and representations, which can signify different quantities for different students. For example, the student who used method (1) above to determine the amount of cookies seemed to be engaging in a one-by-one operation on figural (semi-concrete) objects. That student was able to use a physical object, in this case a cube, to represent an unseen physical object (a cookie). For the student who used method (2), however, a tower (and a finger) represented a composite unit—a box consisting of 5 cookies.

A key feature of adaptive teaching for promoting abstraction of mathematical ideas is therefore identification of how students currently use and interpret particular manipulatives and representations. In turn, teachers need to strategically and gradually promote students' shift from operating on concrete objects, to operating on figural objects (e.g., finger replaces a tower that replaces a box), to operating on abstract (imagined) objects. Students might begin counting only what is visible to them. Next, students could use a different physical object and then figural objects to represent unseen physical objects (e.g., drawing a small square to stand for a box of cookies). Eventually, students could work in the abstract (e.g., using their number sequence to *simultaneously* count the boxes and the cookies—one is 5, two is 10, and three is 15).

Situating CMRP in a Developmental Continuum of Multiplicative Reasoning

In this section, we illustrate a CMRP approach by focusing on multiplicative reasoning—a mathematical prerequisite for advancing to algebraic reasoning (e.g., Mason, 2008; Smith & Thompson, 2008). Rather than focusing on particular topics for students to acquire (e.g., "multiplication of 2-digit by 1-digit numbers"), we target essential ways of reasoning that can underlie students' development of more advanced understandings.

Composite Unit: A Key Distinction between Multiplicative and Additive Reasoning

A key construct we use to distinguish multiplicative from additive reasoning is *number as a composite unit* (Steffe, 1992). When number is conceived of as composite unit, children can anticipate *decomposing* units into "nested" sub-units. For example, a child can decompose 7 into 5+2 because, *for her*, 5 and 2 are "nested" within 7. Key to additive reasoning is that the referent unit is *preserved* (Schwartz, 1991): 11 *cookies* – 7 *cookies* = 4 *cookies*. In contrast, multiplicative reasoning requires a conceptual transformation—a coordination of operations *on* composite units (Behr, 1994). Consider placing 3 cookies into each of 4 boxes; 3 is one composite unit (*cookies per box*) and 4 is another (*boxes*).

Multiplicative reasoning entails distributing one unit over items of another (e.g., 3 cookies into each box) and finding the total via a *coordinated (double) count* (Steffe, 1992): 1 (box) is *1-2-3* (cookies), 2 (boxes) are *4-5-6* (cookies), and so on. C*oordinated* counting involves keeping track of the composite units while accruing the total of 1s based on the size of the distributed composite unit (3 cookies-per-box). As this example indicates, in multiplicative reasoning the referent unit is *transformed* (Schwartz, 1991), and the product has to be conceptualized as a unit of units of units (Steffe & Cobb, 1998): here, "6 cookies" is a unit composed of 3 units (boxes) of 2 units (cookies-per-box). The *simultaneous count* of two composite units and the resulting *unit transformation* constitute the conceptual advance from additive reasoning. Children have difficulty developing multiplicative reasoning, and impoverished conception of number as a composite unit might contribute to that difficulty (Tzur et al., 2010). For example, a student like the one who used method (1) above seemed unable to conceive of a composite unit as a "thing" and thus must have counted individual items to determine the product.

Scheme-and-Task Continuum to Support Students' Multiplicative Reasoning

To foster students' progression in multiplicative reasoning, our CMRP approach follows a sequence of six schemes—goal-directed ways of acting and reasoning (von Glasersfeld, 1995). Below, examples follow each scheme to illustrate tasks on which students work during instruction, to help them advance from prior scheme to the next. A task corresponds to but is not identical with a scheme. For fostering each scheme, we begin by engaging students in operating on tangible objects (e.g., putting cubes together to make a tower). We then proceed to tasks in which students produce the composite units and cover them before figuring out the total. This leads to students' use of figural (substitute) items. Initially these figural items may be physical objects such as fingers. Subsequently students may draw schematic diagrams of the objects. At first, students may draw all single items (1s) in a composite unit. For example, students may draw each of 3 cookies in 5 boxes. Next, we promote a shift to the drawing of only the composite units. For example, students may draw only the 5 boxes and write the number 3 in each box to represent the number of cookies in each box. Finally, we support students' use of numbers to replace the individual items. For example, students may record the numbers 3, 6, 9, 12, 15 to stand for the 5 boxes each containing 3 cookies. To make tasks relevant to children, *we ask them* about and select units and contexts that fit with *their daily experiences* (e.g., shirts and buttons, families and family members, pets and legs, packages and food items, etc.).

For each task we also select numbers that allow gradual progress from "easy" ones (e.g., composite units of 2, 5, or 10 items) to intermediate (units of 3 or 4) and larger/difficult ones (6, 7, 8, and 9, and beyond). The point is quite simple: students are better positioned to construct a new scheme by operating on composite units with which they are familiar. Once the new scheme is evolving, changing to more challenging numbers fosters repeated use and recognition of the invariant nature of that new way of reasoning across any similar situation.

We initiate transition from additive to multiplicative reasoning by fostering students' construction of a ***Multiplicative Double Counting (mDC)*** scheme—the simultaneous counting of composite units and 1s described above. Once students construct mDC, they can anticipate that a total of 1s (say, 24 buttons) is itself a composite unit made of another composite unit (4 shirts), each a composite unit itself (6 buttons). They may solve tasks such as "Enrique has 3 boxes of cookies. Each box contains 5 cookies. How many cookies does Enrique have in all?" Tasks supporting ***mDC*** require students to determine a total number of 1s such that the amount is a composite unit made up of groups of composite units. A solution entails not only figuring out the number of 1s but also and most importantly justifying why this *must* be the total. Typically, we engage students in working on such tasks in pairs, and their justifications often serve to determine which of two different answers is the correct one (including checking their answers by counting the tangible objects).

Next, students can construct a ***Same-Unit Coordination (SUC)*** scheme, in which they learn to apply their additive operations to sets (units) of composite units. They may solve tasks such as, "You had 5 vases, each containing 10 flowers. Now you have 9 vases, each containing 10 flowers. How many *vases* did you gain?" Tasks supporting ***SUC*** require students to reason about a *set* of composite units as made of sub-units, each of which is a set of composite units, without losing sight of their being composed of a particular numerosity. Initially, they may respond to such tasks by mistakenly attempting to count, or calculate, the total of 1s in the set. Gradually, with teaching emphasis on the unit being asked about in a task, they shift their focus to operations on the composite units. Such operations can, for example, support students' learning of adding/subtracting units of 10 (e.g., 90–50 means subtracting five 10s from nine 10s, hence four 10s which are forty 1s).

Next, we promote students' construction of a ***Unit Differentiation and Selection (UDS)*** scheme (McClintock et al., 2011), in which they further separate operations on 1s and composite units. They may solve tasks such as, "You have 7 boxes, each containing 5 pencils. I have 4 boxes, each containing 5 pencils. How are our collections similar? Different? How many more pencils do you have?" Tasks supporting ***UDS*** foster students to conceive of two sets of composite units in terms of sub-units that constitute each, and operate multiplicatively *on the difference* between the two sets. Such operations can, for example, support students' learning of the distributive property of multiplication over addition (e.g., to solve the above problem they can either find the totals first, 35 and 20, and then subtract to obtain 15, or find the difference of 7-4 first, and then multiply it by 5 to obtain 15).

Then, we promote students construction of a ***Mixed-Unit Coordination (MUC)*** scheme (Tzur et al., 2009), in which they operate on two collections—one consisting of composite units and the other of 1s. They may solve tasks such as, "You have 7 boxes, each containing 3 candies. I'll give you 6 more *candies*. If you put these 6 candies into boxes containing 3 each, how many *boxes* will you have in all?" Tasks supporting ***MUC*** foster students' selecting the distributed composite unit (e.g., 3 cookies-per-box), imposing that unit on the singletons (e.g., 6 candies) to yield 2 composite units (e.g., boxes), and then adding these to the initial set (e.g., 7+2=9 boxes). Such operations can, for example, support students' learning of algebra-like mathematics that involves both additive and multiplicative operations on different-but-related units (e.g., 3*10+15=4*10+1*5=45, and later solving for the number 'x' that will make the equation 3x+15=45 a true statement).

From ***MUC*** we proceed to fostering students' construction of a ***Quotitive Division (QD)*** scheme as an inverse operation to multiplicative double counting. They double count to solve tasks like: "I counted 60 legs of chairs in the class. Each chair has 4 legs. How many chairs are in the class?" Tasks supporting ***QD*** require students to operate on a total of 1s and use the size of each composite unit to determine (keep track of) how many such units can be made. Such operations can, for example, support students' learning to use a single structure (equation) to represent, and solve, multiplicative situations by identifying which two of three quantities are given, which needs to be figured out, and what operation is required (e.g., 4x?=60 leads to dividing 60÷4, whereas 4x15=? leads to multiplying the two numbers).

Later, we move on to fostering students' construction of a ***Partitive Division (PD)*** scheme, in which they operate on a total of 1s by distributing it equally into a given number of composite units. They solve tasks such as "Our class has 24 students. We want to place them into 3 groups. How many students will be in each group?" Tasks supporting ***PD*** require students to figure out the size of each distributed composite unit, given the amount of composite units and the total of 1s. Such operations can, for example, support students' learning to meaningfully use the long-division algorithm, as a process in which units of units (etc.) of 10s are distributed into the given number of groups (divisor), while exchanges to smaller units enable such distribution when there are not enough larger units (e.g., to divide 294 into 7 equal groups one exchanges two 100s into twenty 10s, adds the nine 10s, and distribute those twenty-nine 10s by placing four such units in each of the groups, leaving one 10 that needs to be further exchanged, etc.).

Enacting CMRP with Urban ELLs

We have begun to enact a CMRP approach through piloting instruction designed to promote the aforementioned conceptual transitions in multiplicative reasoning in urban ELLs. This pilot work was conducted at an elementary school with over 85% of students whose native language is Spanish. It included work with individual 4th graders who were identified as having disabilities in mathematics, and with K–5 teachers in the school. The latter included workshops with teachers that focused on their own mathematical understandings (e.g., of the place-value, base-ten number system), on children's developmental pathways to mathematical conceptions (from rote-counting, through cardinality and counting-all to counting-on, and to additive and multiplicative reasoning with whole numbers), and on the

CMRP, adaptive teaching approach. Besides the workshops, we have been partnering with two teachers (grade 3 and grade 4) to co-plan and co-teach their students as a medium for enacting and demonstrating the adaptive teaching.

Our pilot work highlights how a CMRP may provide ELLs with instruction that gives them the opportunity to work on cognitively challenging mathematical tasks (e.g., Brown et al., 2011; Campbell, Adams, & Davis, 2007; Moschkovich, 2007). Moschkovich (2002) asserted that equitable mathematics instruction for ELLs must move beyond focusing on acquiring mathematical vocabulary and recognizing multiple meanings for terms. By analyzing students' available conceptions and fostering their problem solving in conceptually tailored tasks, CMRP seems to enable engaging in and linking mathematical processes and expressions. In particular, by fostering students' movement from concrete to figurative to abstract representations, CMRP seemed to promote a focus on *their* mathematical reasoning and justifying. When students justified their solutions, they had an opportunity to do so in multiple ways—in English and in one's native language. Thus, students were highly engaged in discussions with their peers as they shared their strategies. Further, when justifying their results, students have used physical objects and/or diagrams to represent the composite units (e.g., boxes of cookies) involved in the problem. Strategic use of representations supported students' development of more advanced mathematical structures and fostered their productive participation in mathematical practices. Preliminary results of enacting CMRP to promote ELLs' advancement from additive to multiplicative reasoning indicated substantial impact.

Discussion

In this paper we proposed a Culturally-Mathematically Relevant Pedagogy (CMRP) approach, including three key aspects: adaptive teaching, sensible tasks, and meaningful manipulatives. Central to all three aspects is students' engagement in mathematical practices they can make sense of via their prior conceptions. The three aspects of CMRP are interrelated, as indicated by the Teaching Triad. That is, a CMRP approach links instructional goals to funds of knowledge that low SES and ELLs bring to a lesson. If traditional lessons focused on one vertex of the *Teaching Triad* (mathematics), reform lessons on two (tasks/activities and mathematics), CMRP lessons are innovative in that the focus is on all three vertices (student resources/conceptions, tasks/activities, and intended mathematics). Each CMRP lesson begins with tasks that build on student mathematics. This supports gradual use of challenging tasks by making sure the tasks are tailored to students for whom the instruction is being designed.

In essence, CMRP is a mathematically focused implementation of Culturally Relevant Pedagogy (CRP). When implementing challenging tasks, CMRP uses situations that are supposedly meaningful to the students. These tasks build on students' abilities rather than deficiencies and allow for multiple pathways to their success. The "M" part of this approach focuses on strategic tailoring of problem situations and representations that make sense to students given their current understandings. The goal of this strategic tailoring is to promote students' mathematical progression, in our case to promote transition along our six-scheme continuum of multiplicative reasoning. For example, if a student has developed multiplicative double counting (mDC) only for visible items, the student would engage in tasks that would support her development of mDC for *hidden* items. Another student who had developed mDC for abstract items could engage in tasks that foster Same Unit Coordination (SUC) with visible composite units. The articulation of the schemes involved in multiplicative reasoning augments our sensitivity to students' mathematical progression. By linking the operation of division with the operation of multiplication, students can meaningfully solve problems that they otherwise may have been able to solve only with the use of an algorithm. In turn, we foster bringing forth each of the schemes to support its use for more advanced algebraic reasoning.

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TEACHERS' AND STUDENTS' PERCEPTIONS OF CLASSROOM DISCUSSIONS

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We argue that teachers and students view classroom discussions differently. Teachers view the work of facilitating discussions weaving students' contributions together into a coherent discussion of the subject at hand. Students, on the other hand, view the work of participating in discussions as negotiating relationships with their peers. We support these claims with analysis of the appreciation and judgment (Martin & White, 2005) made by high school geometry students and teachers in conversations around an animated representation of instruction.

Keywords: Classroom Discourse; High School Education; Reasoning and Proof

Objectives or Purposes of the Study

In this paper we support the hypothesis that teachers and students view classroom discussions differently from each other, in particular, they view different difficulties involved with engaging in classroom discussions. From the point of view of the teacher the work of facilitating discussions involves weaving students' contributions together into a coherent discussion of the subject at hand, while from the point of view of the students, the work of participating in discussions is much more relational and involves negotiating relationships with their peers.

We have collected teacher and student responses to an animated classroom episode, The Square (which can be viewed at lessonsketch.org), in which a class engages with the question, What can one say about the angle bisectors of a quadrilateral? In the full paper we examine the different ways that teachers and students interpret discursive moves visible in this episode. Here we present examples showcasing the ways that teachers and students interpret a canonical teacher discourse move, revoicing (O'Connor & Michaels, 1993).

Perspective(s) or Theoretical Framework

Within the educational research and reform communities, teachers are encouraged to facilitate discussions in class, building on student contributions (Chapin, O'Connor, & Anderson, 2003; Inagaki, Hatano, & Morita, 1998; Kazemi, 1998; NCTM, 2000; White, 2003; Yackel, Cobb, & Wood, 1999). When researchers examine the difficulties of this work they often focus on the difficulties of listening to and understanding student contributions, responding to unexpected student contributions, choosing the most appropriate discourse move in response to students contributions, and weaving together a coherent argument from student contributions (Even & Gottlib, 2011; Sherin, 2002; Stein et al., 2008). If researchers were aware of students' experience their account of the teacher's challenges would be more complete.

Methods or Modes of Inquiry

To analyze the data in this study we look at the way that the participants used appraisal resources (Martin & Rose, 2003; Martin & White, 2005) in their talk about instructional scenarios. We look at two dimensions of appraisal, appreciation and judgment. Appreciation resources are used to appraise the worth of things in the world and judgment resources are used to appraise the deeds of people. These resources operate at the level of the text rather than the clause because, while they rely on word choice and sentential structure, they cannot be reduced to one or the other. Using these linguistic tools we compare teacher and student talk around an animated instructional episode involving revoicing. This comparison highlights the different ways that teachers and students evaluate the instructional episode, either in terms of the mathematical ideas at play or the people who voice those ideas.

We pull from two data sources; study group sessions with experienced geometry teachers and focus group sessions with classes of high school geometry students. The data collection with each group centered on gathering the group's reaction to an animated episode of geometry instruction. These data sources provide the opportunity to compare the reactions of teachers and students to the same instructional episode.

In the animated episode, The Square, the class is working on the problem, What can one say about the angle bisectors of a quadrilateral. In particular, a student, Alpha, has brought up the case of a square and claimed that the diagonals bisect each other. The teacher picked up this claim and reminds Alpha that the problem is about angle bisectors, not diagonals. In response to this Alpha revises his claim to say that the diagonals cut the square in half. The teacher revoices Alpha's revised claim and asks the class to elaborate on it. Another student, Beta, makes the claim that diagonals are also angle bisectors. The teacher revoices this claim and asks the class if they agree with Beta's statement. Gamma replies that the claim is obviously false since it is not true for the case of rectangles.

Below we describe examples of reactions that the two groups of participants had to this exchange. In the full paper we present analysis of all the conversations that participants had in response to this animated episode. In this proposal we limit our analysis to two examples. In the transcripts below, text in bold shows appreciations and text in italics shows judgments.

The first example comes from a study group session with experienced geometry teachers. The participants were viewing The Square and had just seen the excerpt described above. Tabitha paused the viewing and began a conversation about how the animated teacher handled the students' contributions. In the transcript below, another participant, Tina, questioned how the animated teacher dealt with Beta's contribution by suggesting that the animated teacher should have stressed the condition that the quadrilateral is a square. Tabitha agreed with Tina and Tina made the recommendation that the animated teacher redirect the discussion to focus on squares and the distinction between angle bisectors and diagonals.

Table 1: Transcript from Study Group with Experienced Geometry Teachers

1057	Tina	But see I think *she was* ***wrong*** **in stating it how she said it**. I think the teacher said "listen-" *she* ***should have said*** **something about "listen to what Beta just said. Diagonals are also the angle bisectors of a square."**
1058	Tabitha	**Of a square**.
1059	Tina	***I would've added*** **"of a square."**
		…
1061	Tina	Right. **"So let's think about it as a square. Are the angle bisectors and the diagonals the same?"**

In Table 1 the participants negatively appreciate the teacher's response to Beta's contribution, as well as negatively judge the teacher. By providing alternative responses to Beta's contribution they make tacit, negative appreciations on the response given by the animated teacher. Each alternative provided is in contrast to the animated teacher's comment, "Listen to what Beta has just said. That the diagonals are also the angle bisectors. Do people agree with that?" The contrast between the animated teacher's comment and the participants' rephrasing is set up in turn 1057, when Tina referred the animated teacher's response as "how she said it." Through the use of grammatical resources Tina made a negative appreciation of the animated teacher's statement of the type composition/distortion (Martin & White, 2005, p. 56). Tabitha's comment in turn 1058 strengthens this appreciation, by reiterating the piece of information that was missing from the animated teacher's comment, "of a square." In turn 1061 Tina reiterates the appreciation by providing a new alternative response to Beta's comment that is further from the animated teacher's response. In this turn Tina suggests that the animated teacher abandon the revoicing move and ask students to think about a new question.

The second example comes from a focus group with a class of high school geometry students. The participants had just watched the clip from The Square that is described above. In the transcript in Table 2 participants brought up the interpretation that Alpha might be annoyed by Beta's restatement of his contribution. Tasha saw that Beta could be annoying to Alpha because Alpha knew that he had the right idea, but that Beta was going to be the one to get the credit for the idea since she said it more clearly. Jack expanded on this interpretation by pointing out that other students admire students who share correct answers. The researcher asked if students are concerned about having their ideas heard in class and Nissim pointed out that students see this as a kind of theft when a student rewords another student's answer.

Table 2: Transcript from Focus Group with High School Geometry Students

149	Researcher	So say more about Alpha being annoyed.
		…
155	Tasha	I can kind of agree with that because um like you just know that you *have it* ***right*** too but they're kinda *the one that's going to get the credit for it* but I mean *you had the same idea*, you *just didn't have the words to say it*.
156	Researcher	Yeah it's like someone *stole your idea*.
		…
165	Jack	When you answer a question it makes people be like "*oh they knew that, they're smart*" and then so when someone else answers the question they're like "*oh my I did not get to like explain my [inaudible]*."
166		[students laugh]
167	Researcher	Yeah so that's what you want to do, you want to make sure your ideas are heard that's the only thing you care about?
168	Nissim	I think it's sort of like a [inaudible] when you go up to answer the question yeah [inaudible] answer it, then if you *word it* ***wrong*** then someone else *words it* ***correctly*** the whole class kinda feels like they *stole your answer*.

In Table 2, the participants discussed the increased social status that students associate with contributing correct ideas to the conversation. They did this through lexical judgment resources of the security ("correctly"), capacity ("smart") and inclination ("credit") types to positively judge students who contribute correct ideas and negatively judge students who are unable to communicate their ideas. They mention students who "have it right," "get the credit," "had the same idea," "[are] smart," "did not get to … explain," "word it wrong," and "[word] it correctly." Students who have correct ideas, get credit, and are able to communicate their ideas are judged positively, while students who cannot communicate their ideas are judged negatively.

Results

From the analysis of these two examples, and the other discussions which we do not have room to report on here, it can be seen that teachers are more likely to appraise the ideas that are at play in the classroom discussion while students are more likely to appraise the people involved in the discussion. We contend that these different appraisal patterns reflect the different professional obligations (Herbst, 2010) or commitments that teachers and students are responding to. While teachers seems to be concerned with the mathematical content of the discussion, in response to the disciplinary obligation, students seem to be concerned with the interactions between students around content, related to the interpersonal obligation. This suggests that students' instructional experiences, and the quality of mathematical discussions, would improve if teachers were positioned to attend to the students' interpersonal concerns.

Discussion and/or Conclusions

Mathematical discussions in classrooms are often justified by claims that could be seen to appeal to teachers' individual and disciplinary professional obligations (to attend to students' cognition or motivation, or to attend to the features of the ideas at stake). However, in this study we show that, in the students' experience, this work has a large interpersonal dimension. Within discussions, students not only deepen their own understandings and construct shared mathematical knowledge, but they also relate to each other around the mathematical ideas that are being shared. We argue for a more nuanced way of examining discourse moves that pays attention not only to the impact that they have on individual student learning, or the shaping of the content, but also to how it invites students to interact with each other around the content.

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SECONDARY MATHEMATICS TEACHERS' ENACTMENT OF CONTENT LITERACY STRATEGIES

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Content literacy strategies offer teachers another set of tools for developing student reasoning and sense making, ongoing goals of mathematics education. However, the nature of strategy use determines whether these goals will be met. This paper presents the results of a qualitative inquiry into the content literacy practices of 17 mathematics teachers engaged in an ongoing content literacy professional development. We analyzed classroom observations, lesson plans, and online discussions to determine the nature of mathematics teachers' use of content literacy strategies. We describe contrasting ways teachers enacted content literacy strategies, provide examples of the varying approaches, and describe influences on implementation.

Keywords: Instructional Activities and Practices; Teacher Education–Inservice/Professional Development

Purpose

Content literacy is "the ability to use reading, writing, talking, listening, and viewing to learn subject matter in a given discipline" (Vacca & Vacca, 2005, p. 184). Incorporating content literacy instruction into content courses can improve student learning and support learners in developing understanding of knowledge accepted by the discipline, enabling them to critique and influence that knowledge (Draper, 2002; Moje, 2008). Previous studies have examined teachers' beliefs about use of content literacy strategies in content courses, amount of strategy use, and influences on amount of use, but not the specific nature of teachers' incorporation of content literacy strategies into their instruction (Alvermann, O'Brien, & Dillon, 1990). This study examined the nature of mathematics teachers' implementation of content literacy strategies.

Perspectives

Focus in the use of literacy strategies in the content areas has shifted from the use of strategies for memorization and exposition of knowledge and the development of technical reading and writing skills towards a more constructivist view in which students construct meaning through discussion, reading, and writing activities. Current content literacy approaches incorporate strategies that use a reasoning and sense making approach to actively engage students in learning content (Fisher & Ivey, 2005; Moje, 2008).

Previous research has shown that teachers' decisions to use content literacy strategies are influenced by a variety of factors. Preservice teachers' use was influenced by views of the content, their philosophy of teaching, relevant curriculum materials, administrative policies, and their cooperating teacher's desires and teaching style (Bean, 1997; Sturtevant, 1996). Strategy use was also constrained by limited time, large class sizes, students with academic or personal problems, and instability in teaching assignments (Sturtevant, 1996). Cantrell and Callaway (2008) found that inservice teachers' self-efficacy influenced their use of content literacy strategies.

There is little research that specifically examines the nature of teachers' use of literacy in secondary classrooms. Alvermann et al. (1990) examined the nature of teachers' discussions of content area readings in science, social studies, language arts, and health classes. They found the type of discussion depended upon the teachers' purpose for the lesson and that there was often a discrepancy between teachers' definitions of discussions and their actual practices. This study suggests the importance of examining whether or not teachers incorporate literacy instruction and the need to understand the nature of implementation. Our study extends previous work by examining the nature of content literacy strategy use by secondary mathematics teachers engaged in a long term professional development project specifically focused on content area literacy.

Methodology

This study examined the practice of 17 secondary mathematics teachers in a project on integrating literacy strategies into secondary science and mathematics courses. Project activities consisted of a week-long summer workshop, three follow-on workshops, classroom visits by project staff, and online discussion. Workshops introduced the role of language in content learning and instructional strategies for integrating literacy into instruction. Teachers engaged in activities that modeled specific research-based strategies for vocabulary, writing, and reading in mathematics and science. These strategies were chosen for their ability to support development of student understanding and reasoning in mathematics and science. In order to understand the nature of teachers' literacy strategy use, yearly data were collected from three classroom observations, two lesson plans, reflections, student work, and discussion posts from each teacher. Two researchers independently examined data and coded strategy use. This paper presents findings from the mathematics teachers' lessons from the first two years of the project.

Findings

Although all teachers incorporated new content literacy strategies into their instruction, they did so in varying ways. Upon analysis of teachers' use of content literacy strategies, typical features of strategy use were discerned. Two contrasting approaches to the incorporation of literacy were identified: Rehearsal and Reorganization. In the Rehearsal approach, teachers primarily used literacy strategies to revisit and rehearse content. In the Reorganization approach, teachers used literacy strategies to support students in developing deeper conceptual understanding. We also identified a third Transitional pattern of use that contained components of the other patterns. While teachers seemed to predominately exhibit one pattern, some teachers at various times used a Rehearsal pattern, and at other times a Reorganization pattern. Table 1 presents characteristics of the Rehearsal and Reorganization approaches.

Table1: Rehearsal and Reorganization Approaches to Content Literacy Strategy Use

	Rehearsal Approach Emphasizes:	**Reorganization Approach** Emphasizes:
Vocabulary	• Memorize formal definitions • Definitions taught first, followed by concept development	• Engage in constructing meaning of terms • Integrate vocabulary and conceptual development
Reading	• Decode, vocabulary, and text structure • Acquire information/facts/ vocabulary	• Draw on background knowledge and text structure to support comprehension • Active interaction with text supporting conceptual understanding
Writing	• Write formal reports • Take notes/record information • Support memorization or reinforcement of knowledge • Assess student knowledge	• Process/construct new knowledge • Reflect on prior learning/clarify thinking • Explain and justify one's steps/process to further understanding • Support students in making thinking explicit

Implementation of Content Literacy Strategies

This section presents examples of how teachers used specific vocabulary, reading, and writing strategies with either a Rehearsal or a Reorganization approach.

Vocabulary development. The Frayer Model and Verbal Visual Word Association (VVWA) are word categorization activities designed to develop concept understanding (Billmeyer & Barton, 1998). With the Frayer Model, students write their own definition for the concept, describe its characteristics, and provide examples and non-examples. In the similar VVWA students write their definition, provide a visual representation and a personal association. A teacher who used the Frayer Model with a Rehearsal approach described making sure to "give an answer for each of the four sections of the Frayer Model during my lecture." He then had students copy these during the lecture or transcribe them from their notes. In contrast, a teacher using a Reorganization approach described how she used the VVWA strategy to develop understanding of the concepts, providing "some activity where they 'discover' the concept. … Then we fill out the VVWA, but I ask them to tell me what a good definition would be."

Reading. Anticipation Guides support students in thoughtful reading and interpretation of text (Forget, 2004). When presented with a set of statements that rephrase ideas in the text, students initially read each statement and conclude whether it is true or false. After discussing their choices with a partner, students are motivated to read and interpret the related text, seeking evidence that supports their previous decisions, and then justify their choices with evidence from the text. Anticipation Guide uses identified as Rehearsal focused student attention on finding facts from the reading. Statements required little thought to answer. Anticipation Guide uses that were classified as Reorganization involved students in reading text carefully, making inferences from the text, or engaged students in examining misconceptions. In one such example, the statement "3.14 is an irrational number" engaged a class in a long discussion as students distinguished an irrational number from its common decimal approximation.

Writing. Writing can be used in many ways to support understanding of mathematical ideas. Students might respond to a statement, summarize learning, or describe or justify their problem solving approach or their solution. Such writing leads students to reflect on experiences, organize thoughts, and consolidate understanding (Burns, 2004; Forget, 2004). When writing consisted of recalling information or recording what was presented, it was coded as Rehearsal. In one such example, students were asked to describe steps for solving an inequality and provide an example. In doing so, they needed to recall or look up the rules they had been given, providing an opportunity to reexamine this process. In an example characterized as Reorganization, students conducted a probability experiment. They calculated their own experimental probability and that of the entire class and wrote a paragraph describing results and proposing an explanation for the differences in experimental probability. In this example, writing led students to reconsider the results of the probility activity, make meaning of the situation, and propose an explanation.

Discussion

The data from this study show that teachers tended to use content literacy strategies in ways that aligned with their current practices and instructional goals. As O'Brien, Stewart, and Moje (1995) suggested, this tendency sometimes resulted in a mismatch between content literacy strategies based on socioconstructivist theories and teachers' goals and practices. We found this discrepancy did not necessarily result in a failure to use content literacy strategies, but rather in some teachers modifying the strategies in ways that focused more on transmission of knowledge than on the development of conceptual understanding. All teachers incorporated content literacy strategies into their instruction, but modified them in ways that fit their own teaching approach.

When strategies were implemented according to a Rehearsal approach, the teachers' instructional goals were typically for students to learn procedures or facts or to review specific information. Teachers with such learning goals used strategies that helped students take in and retain information and implemented learning activities in ways to maximize opportunities to do so. When teachers described learning goals of developing conceptual understanding, they used literacy strategies in a Transitional or a Reorganization pattern. In the Reorganization pattern, the strategies were often integrated with instructional activities focused on a common conceptual goal, with time provided for student reflection and discussion.

In strategy use identified as Transitional, the teacher's goals were aligned with conceptual understanding, but aspects of lesson design or implementation resulted in not accomplishing these goals. There are multiple challenges for teachers adopting literacy strategies in content area teaching. For some teachers the incorporation of content literacy strategies required shifts in previous instructional patterns, such as in the amount of time allotted for particular instructional activities, connections between lesson activities, and the role of discussion. Our findings suggest that if related instructional activities are not already in place in a teacher's practice, considerable learning and rethinking of the learning process may be necessary for a teacher to use literacy strategies to support meaningful understanding of mathematics concepts.

Conclusion

Content literacy strategies offer teachers of mathematics another set of tools for use in developing student reasoning and sense making, ongoing goals of mathematics education. However, the ways the strategies are used determines whether these goals will be met. We found teachers' implementation of content literacy strategies to be variable and adaptable. Teachers' use of literacy strategies was influenced by multiple factors, including learning goals, prior teaching practices, and pressures from limited class time. These findings suggest that it may not be enough for teachers to value literacy use in mathematics instruction. They must understand the importance of reasoning and the role of literacy in supporting understanding of mathematics**.**

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TRANSITIONING FROM EXECUTING PROCEDURES TO ROBUST UNDERSTANDING OF ALGEBRA

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Algebra instruction should help students navigate the transition from executing procedures to a more robust understanding of algebra. There is little empirical evidence, however, linking promising instructional practices to such understandings. The Algebra Teaching Study seeks to develop tools for measuring connections between algebra teaching and students' learning outcomes. To demonstrate the use of these tools, we present a comparison of two algebra classrooms, where instructional differences were related to differences in student outcomes.

Keywords: Algebra and Algebraic Thinking; Instructional Activities and Practices; Middle School Education; Problem Solving

Introduction

While growing attention has been given to algebra teaching and learning, there is still a lack of empirical evidence linking teachers' instruction and students' understanding of algebra. The purpose of the Algebra Teaching Study is to develop measurement tools to determine which classroom practices help students navigate the transition from executing procedures to the development of robust understanding of algebra. Our classroom observation scheme, Teaching for Robust Understanding in Mathematics (TRU Math), is being developed using classroom observations in Michigan and California. To measure students' gains in robust understanding, we articulated a set of dimensions defining robust understanding of algebra, which we call Robustness Criteria (RC). We adapted and constructed tasks to measure student understanding along these dimensions. We illustrate these tools by presenting comparative cases in which differences in instructional practices of two classes related to differences in gains in algebra achievement. For this paper, we focused on algebraic representations, due to their central role in algebra. We first present our framework for measuring robust understanding of algebra.

Theoretical Framework

The study focused on student understanding of algebra word problems. This is due to their central role in the algebra curriculum, as well as students' struggles with them. Focusing on word problems enabled us to examine a range of student skills related to robust understanding, including: sense making, modeling, representational and procedural skills. In this section, we will elaborate our definition of robust understanding.

Robustness Criteria Framework

The robustness criteria were developed in consultation with a large body of literature, including: the *Principles and Standards for School Mathematics* (NCTM, 2000), the Common Core State Standards for Mathematics (2010), studies of how students solve problems (e.g., Schoenfeld, 2004), and algebraic habits of mind (Driscoll, 1999). The robustness criteria (RCs) serve two major purposes in our project: guiding task selection and analysis, and focusing our attention during classroom observations, for example, on teacher actions that might promote fluency with algebraic representations. Below, we list our five criteria. We elaborate RC3 because it is the focus of this paper.

RC #1: Navigate language to make sense of the problem situation.

RC #2: Identify relevant quantities and relationships between them.

RC #3: Represent quantitative relationships. Relating and representing multiple varying quantities is a core feature of a functions-based approach to algebra (e.g., Chazan, 2000). Algebraic representations include coordinate graphs, bivariate tables, diagram or pictures, and variable equations (or systems of equations). Further, the Understanding Patterns, Relations, and Functions Standard states that students should be able to "represent, analyze, and generalize a variety of patterns and tables, graphs, words, and when possible, symbolic rules" (NCTM, 2000, p. 222). Schoenfeld (2004) identified building a situation model and building a diagram or other appropriate representations as being important aspects of student knowledge for solving word problems. The use of multiple representations facilitates students' development of mathematical concepts (e.g., Stein et al., 2009) and their efforts to carry out problem solving tasks (e.g., Greeno & Hall, 1997). Tasks with high cognitive demand should have the potential to be represented in multiple ways (Stein et al., 2009). Therefore, students should be given the opportunity to use multiple representations in problem solving activities.

RC #4: Executing algebraic procedures and checking solutions.

RC #5: Explain and justify reasoning.

These criteria represent the proficiencies required to solve rich, contextual algebraic word problems (Wernet et al., 2011), which we henceforth refer to as contextual algebraic tasks.

Method

Over the course of two years, we have collected data from ten 8th-grade algebra classrooms. We administered pre-tests at the beginning of the school year to document students' initial proficiencies with contextual algebraic tasks. During the school year, we observed each classroom eight times, capturing lessons involving contextual algebraic tasks whenever possible. We then administered post-tests at the end of the school year to document students' growth in understanding as a result of a year of classroom instruction.

The pre- and post-test assessments were comprised of three, multi-part tasks drawn largely from the Mathematics Assessment Resource Service (MARS) assessments. Students' work received two types of scores: a holistic score based on rubrics that aligned with the MARS rubrics, and RC-specific scores, with points awarded related to each of the robustness criteria.

For the purposes of scoring, RC3 (interpreting quantitative relationships) was further elaborated to RC3a (generating representations) and RC3b (interpreting and making connections between representations). The RC 3a scores reflect, in part, students' spontaneous generation of a representation to complete a task in addition to the correctness of the representation generated. These scores were then compiled to provide overall class scores, allowing us to capture student growth, at the classroom level, in robust understanding of algebra across the school year.

Across the observed classrooms, teachers used a variety of curricula (four of the classrooms used an NSF-funded curriculum while the others used a traditional, district adopted text). Thus, the videos we have collected provide a range of lessons from those focusing on traditional story problems to those using more open-ended tasks. This collection of videos has been used both to identify teaching moves warranting attention in our observational tool, as well as to test the observational tool during the development process.

For the purposes of this paper we made a holistic characterization of the instructional practices around representations. We focused on five randomly selected students from each of two classrooms, A and B, in a comparative case study. Class A was comprised of low-tracked students in a suburban school in the Midwest and used an NSF-funded curriculum. Class B included de-tracked students and used a traditional text in a large urban setting on the West Coast. The case study addressed the following questions:

1. How does student use of representations in written algebraic tasks differ between two 8th grade algebra classrooms?
2. How do teachers' approaches to supporting students' use of representations differ between these two classrooms?
3. How does variation in student performance on tasks involving representations relate to the differences in instructional practices?

Illustrative Results

In our observations of Class A and Class B, we noticed significant differences in classroom practices. Students in Class A interacted with representations on a regular basis. Using a functions-based approach to learning algebra (cf. Chazan, 2000), tasks in the curriculum regularly asked students to generate equations, tables, and graphs, and to make connections between these representations. Additionally, Teacher A consistently pressed and supported students to make explicit their generation of representations and make connections between them through questioning, re-voicing, drawing attention to context and other visual cues, and building on previous knowledge.

In contrast, students in Class B spent much of their time taking notes on specific procedures, working on exercises and showing steps in their work. During independent or partner work time, students were primarily engaged in practicing procedures demonstrated in class lecture. Students had few opportunities to generate or interpret representations.

There were differences in students' learning growth both in terms of overall scores and in terms of students' use of representations, measured by RCs 3a (generating representations) and 3b (interpreting and making connections between representations). Table 1 shows the average total scores across the two classrooms on the pre- and post-assessments and the performance in understanding the use of the algebraic representations. Recall that students in Class A were placed in a low-tracked classroom and scored lower than Class B on the pre-assessment. By the end of the study, however, the performance gap had closed; Class A showed evidence of greater growth in RC3. Although students in Class B correctly generated and interpreted representations more frequently on the pre-assessment, the students in Class A finished the school year with higher scores on tasks involving representations.

Thus, initial evidence indicates that there are differences in students' growth in RC3 across classrooms, and that our assessments can capture differential growth.

Table 1: Scores across Two Classrooms

	RC3a		RC3b		Total score		Difference
	Pre	Post	Pre	Post	Pretest	Post-test	
Class A	0.6	4	3.6	6	20%	41%	21%
Class B	1.6	3	4.4	5	26%	32%	6%

The results shown illustrate the potential to document the relationship between students' learning with factors related to teachers' specific instructional practices.

Discussion

National discussions increasingly emphasize the importance of all students studying algebra by eighth or ninth grade. If the push for all students to study algebra is to lead to robust understanding, rather than only superficial skill with algorithms, mathematics educators must have a better understanding of which classroom processes will help build robust understanding.

Our Algebra Teaching Study has developed a system for classroom observation and task-based forms for student assessment that can be used for research linking differences among patterns of activity in classrooms to differences in student acquisition of robust understanding. The cross-case comparison presented here shows that our measures document such differences. Although a comparison across two classrooms is only suggestive of connections between teaching and learning, future work with larger numbers of classrooms can reveal whether such connections are typical. Development of these tools is a significant first step in an important line of research that is scant in the literature.

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DO STUDENTS FLIP OVER THE "FLIPPED CLASSROOM" MODEL FOR LEARNING COLLEGE CALCULUS?

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The "flipped" classroom model, wherein students watch online lectures for homework and do traditional homework sets in class, has fueled significant role transitions for both teachers and students. In this ongoing study of one flipped college calculus course, the role of the teacher transitioned from lecturer to problem set designer. The role of the students is still in flux: Initial reports indicate students prefer their role as collaborators but many indicate their uneasiness regarding the first exam. We are currently investigating the stages of several transitions including the students' changing views and the ongoing reflections of the TAs and instructor.

Keywords: Technology; Flipped Classroom; Calculus

Online video sharing websites have fueled major transitions in the ways students learn about the world, the culture, and mathematics. Thanks to innovative mathematics instructors such as Saul Khan, whose Khan Academy currently boasts over 11 million lessons delivered, some teachers have begun implementing a "flipped" (or inverted) model of instruction wherein students are expected to watch online lectures out of class and spend in-class time working in groups to solve problems related to the lecture content. Many students are enthusiastically embracing this chance to shift from non-engaged stenographers in large lecture classes to self-paced collaborative learners. The goal of this ongoing investigation is to examine the stages of this student transition in the context of one college calculus course. The presentation will discuss the results of five surveys that will be given to the students over the course of the semester. The present proposal describes the results from the first of these reports which measures students' initial views of the flipped approach to teaching and their time allocation patterns.

Theoretical Perspective: Finding the "Sweet Spot" Between Teaching for Procedural and Conceptual Understanding

Despite the huge range of topics displayed in online mathematical instruction videos, there is surprising uniformity in presentation: Either a learned instructor or a set of talking hands demonstrates a single, step-by-step procedure for solving a given problem in, on average, 5.5 minutes. Our current hypothesis is that such presentations implicitly promote procedural skills, but are not designed to support the development of overall conceptual understanding. Following current reform initiatives for grades K–12 (cf., NGA, 2011; NRC, 2001; NCTM 2000), we seek to determine if the method devised by this college instructor defines the sweet spot that promotes both procedural fluency and conceptual understanding.

As Star (2005) suggests, one could argue that the procedural/conceptual debate may be seen as promoting a false dichotomy based on the view that there is a good deal of conceptual understanding that underlies *procedural fluency.* For example, when solving novel problems, students with procedural fluency are able to first choose an appropriate algorithm and then persevere in applying it. In this view, procedural fluency can be seen as using various heuristics (Polya, 1957) and being metacognitively aware of how they relate to the problems at hand (Schoenfeld, 1994). Thus, the overall questions for this study focus on determining (a) the degree to which students learned to teach themselves both procedural fluency and the conceptual understanding that underlies algorithm selection, (b), their affective reaction to this class, and (c) the paths they traversed as they transitioned from passive students to active peer collaborators and internet sleuths.

Methods

Subjects and Data Sources

The subjects for this study are the 27 students enrolled in the "flipped" Calculus I course at a large, southwestern university. Of the population, 41% were engineering majors, 25% were computer science majors, and the remaining students spanning a variety of other majors. Sources of data include the five sets of survey results, notes from interviews with the instructor and the three TAs, and students' scores on the daily problem sets and exams.

Setting

At the outset of the course, the instructor had set up a full outline of topics that would be covered each day along with some question and solutions. He also provided a list of online resources including an interactive tutorial, an online textbook, and links to five to seven suggested work in groups to solve a series of roughly ten increasingly complicated problems, one of which is collected and graded at the end of the 50-minute period. Although the instructor is never present during classes, the Teaching Assistants (TAs) are available to answer questions relating to how a particular problem is solved.

Results and Discussion

The two questions posed at the beginning of this semester study have now come into clearer focus. The first question, regarding procedural versus conceptual fluency, can be answered by noting that while the entire exam consisted of difficult procedural items, the average grade for this class was 52%. This indicates that students were most likely not prepared for the rigor of the exam, which involved understanding and applying skills such as application of trig inverses (based on a strong knowledge of trig), inverse functions (which required a strong understanding of inverses) and other "tricky" questions such piecewise function shown in Figure 1.

Consider the function $\mathrm{f(x)} = \begin{cases} 1 & x \text{ is positive and irrational} \\ 0 & x \text{ is positive and rational} \\ x^2 & \text{otherwise.} \end{cases}$

Determine whether $f(x)$ is continuous and/or differentiable at $x = 0$.

Figure 1: "Tricky" exam question

Despite the low exam average, the overall average semester grade assigned was a C. This was affected by the instructor's "C policy" which basically maintained that if students attended each class (as evidenced by their homework quiz), then they would earn at least a C. Therefore, we will use the final exam, rather than the final grade, as a gauge of the educative nature of this particular approach.

In order to determine the degree to which students were developing conceptual knowledge, we interviewed a few students and followed two pairs of students over the course of the entire semester. What we found was not surprising: students were able to perform the procedures of differentiation and integration for the very basic functions, but they had not developed a conceptual understanding of why these procedures mattered. As just one example, a student in one of the focus groups was given a sheet that included several related rates problems. The student's reaction, after discussing how to solve the problems was, "Why are we still on derivatives? We have been on them for the past week!" To us, this indicates that asking students to view short, 10-minute videos and then only do problems during class time negatives the ability for the instructor to help the students make critical connections. For example, how does what we were studying before relate to what we are studying now? Is the concept of rate of change a sub- or super set of questions in calculus 1?

The second question is, how did students adapt their study habits over the course of the semester? As shown in Figure 3, there was a dramatic shift after the first test (survey 2 was given just before test 1). Prior to taking the first test, the students relied most heavily on online texts and other references provided. However, these declined in use and, after exam 1, the students came to believe that the interactive lesson (a narrated video describing calculus ideas with some interactive components) and the instructor's posted sample problems were most educative and useful to study for the exam.

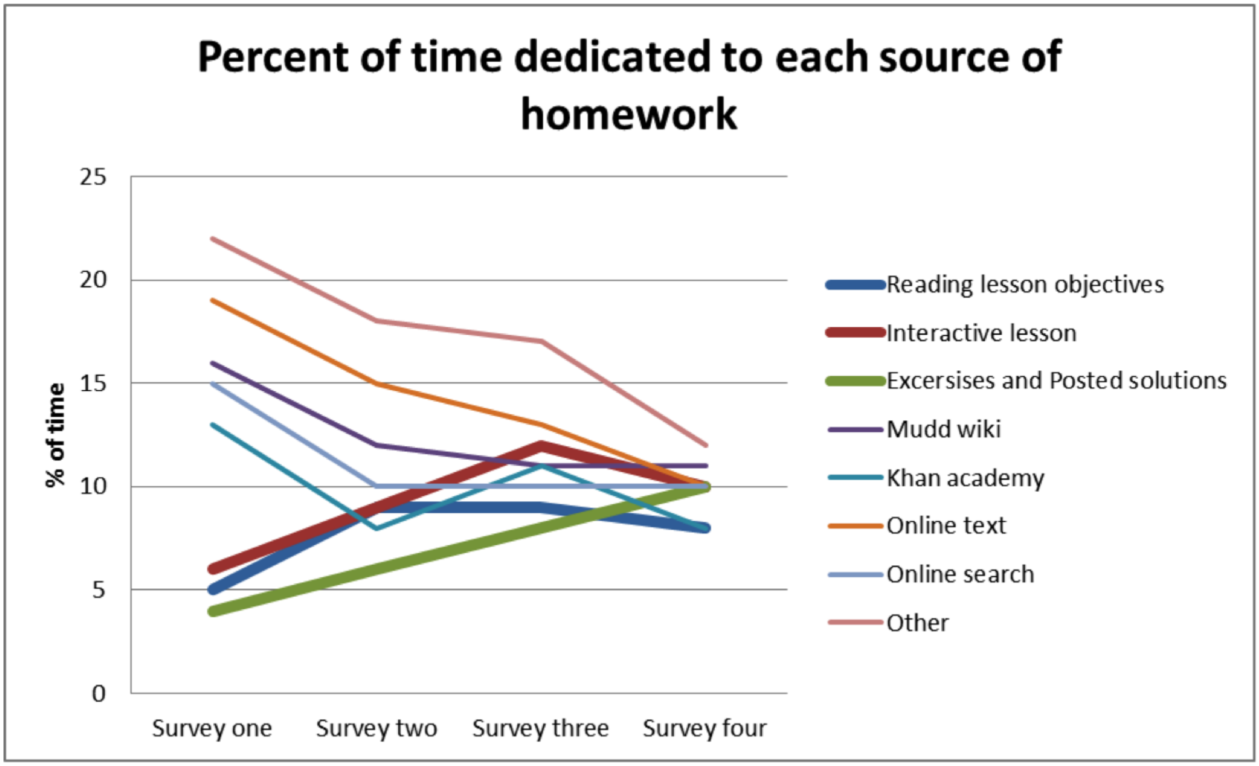

Figure 2: Homework time allotment trends

To answer the question regarding affect in general, we asked the student to rate their enjoinment (enthusiasm) and effort dedicated to this class versus that dedicated to others. To our surprise, the results, shown in Figure 3, indicate that, except for a short dip before the first test, students indicated that they were just a bit more enthusiastic about this set up than a traditional lecture (since the $n = 30$, we cannot claim that these results are significant); however, we can claim that the students' enthusiasm and effort remained very consistent over the course for the entire semester. This continually positive reaction to a class where the professor does not even show up once (and leaves any explaining to the TAs) is alarming. One hypothesis is that the students in our university are very busy; therefore, they appreciate any chance they can get to work on homework. Furthermore, they indicate that even though they are aware that they may not have the most comprehensive grip of the material, they still found the active engagement model to be a more *enjoyable* form of learning.

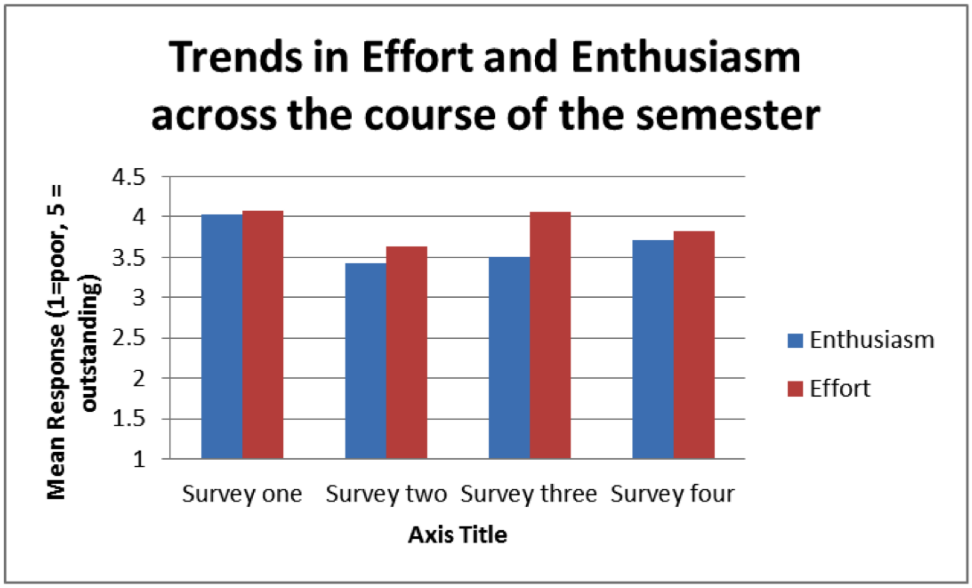

Figure 3: Trends in effort and enthusiasm across the semester

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INTEGRATING INSTRUCTION AND PLAY IN A K–2 LESSON STUDY PROJECT

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Researchers in a lesson study project in Ontario, Canada, facilitated lesson study with nine teacher teams (n = 63) with young children (ages 3–8). This innovative project involves teacher selection of topics of concern, with a focus on content learning (such as spatial reasoning and patterning). Together, teachers and researchers planned and implemented exploratory and public lessons. One unexpected outcome of the project was a heightened attention on the interaction between mathematics instruction and play. Instruction was informed by student play, and was simultaneously being used to inform the design and structures of play-based activities, leading to an integration of instruction and play that questions the assumption that these approaches exist at opposite ends of a continuum of instructional approaches for young children.

Keywords: Elementary School Education; Teacher Education–Inservice/Professional Development; Number Concepts and Operations

Objectives and Background of the Study

Previous research has demonstrated that the goals of kindergarten programs have shifted from a focus on social and psychological development to an academic model with predetermined standards of student performance (Russell, 2011). By engaging in in-depth, sustained professional learning activities known as lesson study, teachers and researchers collaborate to learn more about student capability in mathematics (Fernandez, 2002; Lewis, Perry & Murata, 2006). By engaging with teachers, researchers learn more about what mathematics instruction looks like in the early childhood classroom. The purpose of this study was to: (a) identify the strategies that kindergarten-grade 2 teachers use for exploring mathematics concepts with students, (b) identify how play-based programming supports student understanding, (c) investigate the teacher content knowledge required to support this learning, and (d) develop a greater understanding of student learning trajectories in mathematics for young children. We questioned whether play-based contexts and mathematics instruction need be mutually exclusive and examined how these pedagogies informed one another through an integrated approach with close attention to observing, documenting, and enabling student learning.

Theoretical Framework

The education of young children, especially in the area of mathematics, has never been without contention. In the 180 years since the introduction of publicly funded schooling for young children in North America (the first kindergarten programs), the education community has fluctuated between two opposing views: (1) that children are capable of and enjoy rich mathematical thinking, or (2) that early instruction in mathematics is unnecessary or even harmful to child development (Balfanz, 1999). In parallel, and further dividing the proponents of mathematics education for young children, is the argument over the most suitable and developmentally appropriate approach to mathematics instruction for young children. This argument oscillates between extremes of a teacher-directed or direct-instruction approaches associated with the memorization of rules and algorithms, to student- and/or play-centred approaches fostering exploration with concrete objects and ideas associated with the discovery of patterns and rules along with deep conceptual understanding (Balfanz, 1999).

Why Is Math for Young Children Important?

Research over the past 25 years has shown that "nearly from birth to age 5, young children develop an *everyday or informal mathematics*—including informal ideas of more and less, taking away, shape, size, location, pattern and position—that is surprisingly broad, complex, and sometimes sophisticated" (Ginsburg, Lee & Boyd, 2008, p. 3). Furthermore, young children have a "spontaneous and sometimes explicit interest in mathematical ideas"; Ginsburg, Lee, and Boyd (2008) cite research results from the observation of children at play showing that young children spontaneously count (even up to relatively large numbers) and show interest in quantities ("how many" or "how much") (Saxe, Guberman & Gearhart, 1987; Irwin & Bergham, 1992; Gelman, 1980). They argue that children are capable of handling learning content far more complex than the typically limited curriculum implemented in primary and early elementary school. Baroody, Bajwa, and Eiland (2009) found that the primary cause of problems with basic number combinations among young children "is the lack of opportunities to develop number sense during the preschool and early school years" (p. 69).

Importance of Play

Perry and Dockett (2008) stress the importance of play in the mathematical development of young children. Bergen (2009) describes play as a "medium for learning" that provides opportunities for communicating (even before verbal skills are fully developed), risk taking, confidence building, as well as for developing self-regulation and social skills (p. 416). Play, including imaginative pretense, construction play, and games with rules, promote and enhance logico-mathematical reasoning as well as social understanding and metacognition (Bergen, 2009). As a disposition, play is closely linked to other characteristics valued in mathematics education, including creativity, curiosity, problem posing and problem solving (Ginsburg, 2006; NAEYC/NCTM, 2002; Dockett & Perry, 2007). As Thomas, Warren, & de Vries (2010) write: "play is a pedagogical tool that can enable learning and this learning can be maximized with appropriate, timely and effective adult input" (p. 719). Balfanz (1999) "proposes that intentional teaching of mathematics to young children is both appropriate and desirable" (p. 10). The value of play is not under question. However, play alone does not guarantee mathematical learning will take place. As Seo and Ginsburg (2004) acknowledge: "children do learn from play, but it appears that they can learn much more with artful guidance and challenging activities provided by their teachers" (p. 103). What might this artful guidance look like in classrooms for young children?

Methods of Inquiry

Participants in the lesson study project in Ontario, Canada, included teachers of junior kindergarten through grade 2 (ages 3–8). Nine small groups of teachers (*n* = 63) participated in lesson study, an adaptation of Japanese Lesson Study, which is a professional learning approach where teachers plan lessons and closely observe children's mathematics thinking (Perry, Lewis, & Murata, 2006). As part of this collaborative inquiry approach, we observed key structures in mathematics learning situations, and the intersections of play and instruction. Researchers collected video data of students in three contexts: (a) at play, (b) during small group activities related to mathematics, and (c) during mathematics lessons. Field notes from teacher planning sessions also provided interesting data on teachers' shifting and expanding conceptions of play and instruction.

Based on joint teacher-researcher video viewing and analysis, teams examined the role of play, instruction and the teacher in these contexts, with an action-oriented framework to learn more about the research question: How do play and instruction complement one another to support student learning in mathematics?

Table 1: Data Collection Per Teacher Team (× 9 teams)

	Clinical interviews	Teacher meetings	Observations of free play	Classroom observations
Nature of the Data	Video of clinical interviews: 6 students/ class (pre & post)	Field notes from 8 team meetings of 7 hours each	10 minutes of video per classroom (camera stationed at centre)	Video & field notes from exploratory lessons and public lesson
Quantity	20 hours	50 pgs	50-60 minutes	8 hrs, 30 pgs

We used a design research approach (Brown 1992; Collins, Joseph, & Bielaczyc, 2004; Lamberg & Middleton, 2009) that enabled teachers and researchers to test and refine both theoretical models and products based on observations in classroom contexts. Analysis of data is still underway, and involves coding all data sets using open and axial coding (Charmaz, 2003), and developing matrices to search for patterns in the movements between play and instruction.

Results

Baroody et al. (2006) developed a continuum of four types of teaching: traditional direct-instruction, guided discovery learning via an adult-initiated task, flexible guided discovery learning via a child-initiated task, and unguided discovery learning via a child-initiated task. Guided and flexible guided discovery learning were the most promising according to Baroody et al. In order to test these categories, we analysed teachers and students in math learning situations in their classrooms during lesson study activity. We also asked teachers to describe their practice and their mathematics goals for students. Keeping Baroody's framework in mind, we then focused video and field note data analysis on the two middle categories: guided discovery learning (what we call guided inquiry) and flexible guided discovery learning via a child-initiated task (what we call structured play). Our data revealed that Baroody's continuum was not actually a sequence of locked strategies or even stages, but rather a set of approaches that can be drawn on, and flexibly arranged to maximize student learning in mathematics. Figure 1 represents our analysis of these interconnected approaches to teaching in mathematics classrooms for young children.

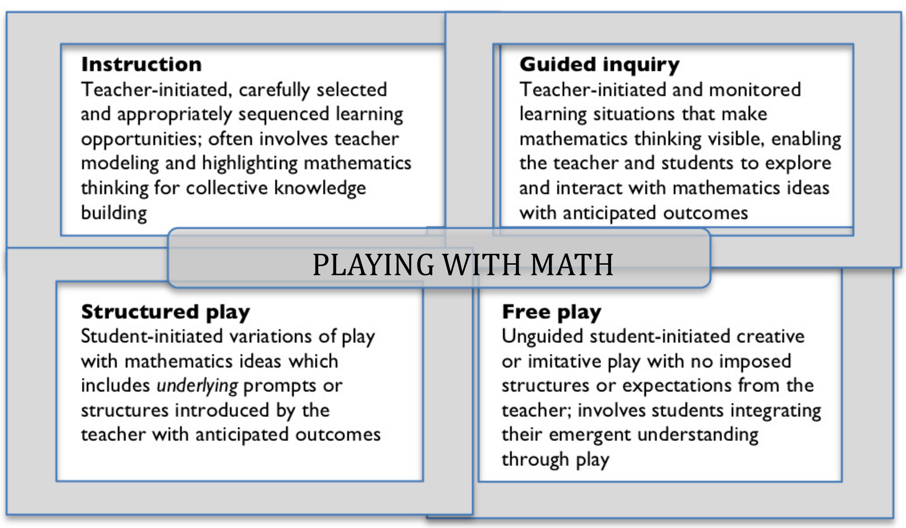

Figure 1: Teaching approaches that supported young children in mathematics

An illustrative example of the intentional yet fluid movement between play and instruction was documented with one team of teacher-researchers inquiring into the composition and decomposition of numbers. To begin, the teachers observed students playing (*free play*) and gathered additional information through clinical interviews which showed that children were able to count (ordinality) but had greater difficulty understanding more and less than five (relative quantity), and how to make five. Subsequently, the team designed some lessons (*instruction with guided inquiry*) to engage children in composing and decomposing five with fingers on two hands as well as five frames with two colours of counters. As an extension, a play station was designed (*guided inquiry*) called "The 5 Bakery," where "customers" ordered a cookie with five toppings. The students coloured a 5 frame with 2 colours to represent toppings and then "bakers" created these cookies using play dough and two colours of glass beads. The five frame was embedded in the activity at the bakery to support both mathematical play. Because of familiarity with this underlying structure, students independently played at the bakery with peers (without adult intervention) to further explore quantities of five (*structured play*). This oscillation between free play, structured play, guided inquiry, and instruction occurred over weeks, and relied on teacher attention to the underlying mathematics content they were encouraging students to play with. These learning experiences also provided a foundation for later variations that capitalized on student interest as well as the now-familiar mathematical structures. For example, on a class walk, the students noticed flowers at bloom. Students improvised to co-create a new play station called “The 10-in-a-pot Flower Shop,” which combined yellow and red tulips to “make 10.” The students recorded their orders on 10 frames, made their flower arrangements and wrote a related number sentence. In focus group interviews, teachers attributed the mathematization of play to their own explicit investigation of mathematics concepts and structures throughout the lesson study professional learning program.

Contributions and Future Research

Next steps include systematic video analysis to count and define *transitions* between the approaches to teaching with the lesson acting as the unit of analysis, connected to longer learning sequences. Cross-case analyses will be generated to further test, refine and amplify a play-instruction framework and to populate the framework with examples from classrooms in the Ontario early childhood context. Practical products of cross-case analyses include a series of short videos on play, instruction and the interaction between approaches to support young children’s mathematics understandings.

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PERSONAL KNOWLEDGE AND ENACTED KNOWLEDGE: EXPLORING THE TRANSITION FROM UNDERSTANDING TO TEACHING PROPORTIONS

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This exploratory study considers how a single teacher's understanding of proportions shapes the experience in his classroom. We built from the knowledge in pieces perspective to consider the teacher's knowledge organization as a way of making sense of his understanding. Then, we considered how his understanding shaped his teaching. We end with a discussion of implications for pursuing this line of research.

Keywords: Teacher Knowledge; Number Concepts; Middle School Education; Mathematical Knowledge for Teaching

Purpose

Teachers' knowledge is instrumental for supporting mathematics learning in classrooms (e.g., Baumert et al., 2010; Hill, Rowan, & Ball, 2005). In standards-based classrooms, teachers are faced with an unlimited range of mathematical ideas as students engage in the valued processes of communicating, arguing, representing, and problem-solving (National Council of Teachers of Mathematics, 2000), thus making mathematics classes more diverse and less predictable as student contributions drive the conversation (Kennedy, 1998). Teachers who lack coherent understandings of the mathematics they teach have only the possibility of teaching disconnected facts (Thompson, Carlson, & Silverman, 2007). Additionally, teachers may find themselves avoiding situations of high intellectual engagement if they cannot accommodate students' unusual or half-formed ideas in the course of a lesson (Kennedy, 2005).

As part of our larger study of teacher knowledge, in this study we are interested in understanding how one teacher made sense of concepts in professional development and, subsequently, how his enactment of his understanding compared in an interview situation and in his 6th grade classroom. Specifically, we wondered whether the teacher's knowledge organization impacted the ways in which he taught the concepts in this own classroom. This exploratory study contributes to the teacher knowledge literature by stepping away from simply quantifying the amount of knowledge the teacher has, to actually seeing how that knowledge is used to drive the implementation of a single lesson.

Theoretical Framework

We rely on the knowledge in pieces epistemology (diSessa, 2006). Knowledge in pieces asserts that each of us has a variety of fine-grained understandings that work in concert with each other to make sense of complex situations. For any given problem situation, we are likely to invoke some number of these pieces to create a more complex, synergistic knowledge. Thus, learning can be seen as (a) developing more fine-grained knowledge pieces, and (b) refining existing pieces so they have more connections between them, allowing a more coherent understanding. Learning occurs when a perturbation causes the learner to reassess an understanding in ways that lead to new understandings or new connections among existing understandings. For example, if a student only understands fractions as *n* pieces of an *m*-sized whole, (e.g., 3 pieces of a 4-piece cake is ¾) that student cannot use that understanding to make sense of 7/4. The student needs to both add a new piece of knowledge about fractions and reassess the existing piece of knowledge to better understand how and when it is appropriate. The development of expertise from this perspective involves building connections and refinements that allow appropriate pieces of knowledge to be invoked in various situations. Knowledge in pieces is important both because it provides a lens for making sense of how a teacher might understand a concept and for understanding that the knowledge invoked in any given situation may or may not reflect the total body of knowledge a person has.

Methods

Walt was a 6th grade teacher with six years of teaching experience after a career in another field. Walt taught at an urban school with a 98% poverty rate. For this study, we considered his implementation of a single lesson on proportional reasoning conducted with his 6th grade students.

Data were collected through two related projects. For one project, we videotaped Walt's 6th grade class as they participated in a one-hour lesson about proportions. The lesson took place in the weeks of the school year. We used two cameras to capture the lesson: one directed at capturing written work on the board or on students' desks and one focused on the teacher or other people talking. As part of a second project, we conducted a 90-minute clinical interview with Walt the following November, immediately after his retirement from the school system, to explore how he understood the concepts of ratio and proportion. The interview prompts consisted of items intended to elicit different aspects of reasoning about ratio and proportion. The items were all situated in the work of teachers, thus asking our participants to respond to sample responses from students or from other teachers.

To understand how content was organized and discussed for Walt and in the PD, we created content maps (Empson, Greenstein, Muldonado, & Roschelle, in preparation) from the videos. To do this, we considered relationships among lexical content to compare what mathematical language is explicitly linked within the two situations. Phrases representing concepts or objects became nodes of our maps. When we noted Walt using two words in similar ways, we recorded that connection as a line connecting the nodes. When Walt described a relationship, we recorded the description of the relationship as an annotation to the line (e.g., a proportion *can be* set-up).

For Walt's clinical interview, this mapping showed Walt's use of language while reasoning about mathematical tasks requiring knowledge of ratio and proportion. The maps of the classroom video show how Walt linked these key concepts in his instructional practice, making those linkages explicit and available for students in his class.

We analyzed each mapping for notable groupings and connections between and among concepts relevant to proportional reasoning. We then compared the mapping of Walt's understanding to the mapping of his teaching to understand how his knowledge was enacted in the classroom.

Results

Due to space constraints, we explore only one key aspect of Walt's proportional reasoning as it related to his classroom. Specifically, we noticed that Walt seemed to have a conceptual separation between the processes of reasoning about relationships and the calculations involved with proportions. This manifested itself in both his interview and his teaching as language about "relationship" and "proportion" formed the center of separate sub-maps within the interview and classroom data. We discuss implications of this for conceptualizing specialized mathematics knowledge for teaching and further research in the Discussion.

In his interview, Walt spoke of contextualized situations involving proportions as being "relationships." Within this realm, he was able to discuss ideas of concrete, real-world quantities being doubled or halved. He related this to street knowledge and used descriptors like "intuitive" and "makes sense" to describe how he was reasoning about the quantities in the tasks. He described proportional relationships in terms of the relationships. For example, he noted that he was looking for how many of one quantity might be in another quantity (multiplicative relationships) and describing the intensive quantity in terms of both its attribute (e.g., flavor) and its equivalence to other ratios in the proportion. However, when he spoke of "proportions" rather than "relationships," Walt's focus shifted to calculating correct answers, creating graphs, and writing equations. This focus elicited more confusing language such as his description that "… what I want them to understand is that we want to maintain an equal amount … we want the ratio always to be equal, that we increase that ratio proportionally." His discussion of proportion also included a relatively large focus on finding a correct amount.

We saw this separation of proportion and relationship in Walt's class as well. In his class, Walt focused extensively on each individual variable in $y = kx$ without tying them together. He expressed to the students that proportions can be graphed, can be gotten done, can be set-up, can be in the form $y = kx$, and can be used to solve problems. There was never any discussion of any definition for proportion nor was there attention paid to the relationships between the *x* and *y*. However, Walt used a number of contextualized situations in which to have students solve proportions. He built much of the lesson around a book called *If You Hopped Like a Frog* (Schwartz, 1999) that featured proportional relationships between frog body length and jump length as well as between chameleon tongue and body length. He also pulled in other real world examples. In the context of discussing these examples, he raised issues of using multiplication to calculate results, using graphs that represented various attributes (e.g., tongue length and body length) and ratios. Unfortunately, with all of the contextualized situations used in class, the focus quickly changed from understanding that there was a relationship between two variables to calculating a single unknown that happened to fit that situation. Thus, there seemed to be a privileging of calculation over reasoning about relationships.

While Walt taught the lesson without explicitly tying the relationships within the contextual situations to the calculations for solving proportions, it was clear, at times, that he was working to convey some kind of connection to his students. Most notably, he repeated multiple times as they worked that the chameleon tongue was ½ the length of its body. This clearly had some kind of proportional meaning for Walt that he was not able to convey in any other way to his students, thus leaving that phrase as the only connection between his contextualized explanations and his calculation-based explanations.

Discussion and Conclusions

We set out to understand how a teacher's organization of knowledge compared to his enactment of a lesson in his classroom. Our hypothesis was that the teacher's knowledge organization would shape the implementation of the lesson. In this case, we noted that Walt had separated his understanding of "relationship" from his understanding of "proportion." In our interview with Walt, he described that relationships were more often contextualized and could be reasoned about using intuitive street knowledge. In contrast, his discussion of proportions was very heavily tied to calculating and graphing to determine correct answers. There was little focus on contextualized reasoning in his discussion of proportions.

In Walt's classroom, we saw a mirror of this reasoning. Walt focused much of the class period on finding a missing value for a given situation. He did raise issues of the relationship between values within the contexts presented, but quickly moved on from those relationships to focus on the calculations. His focus on missing value was consistent with how Walt talked about proportion in the interview. Walt's emphasis on proportions in class was a reflection of his own connection of proportion to the process of calculation rather than the relationship between quantities.

This case study suggests that the organization of teacher knowledge, in fact, may impact the way in which concepts are developed in their classrooms. This is an extension on the research trying to tie teacher knowledge to student learning in that it considers how a teacher understood his content and what that looked like in practice. Clearly a larger body of data as well as empirical results about the implications of the teaching would strengthen this case. But, as an exploratory study, this case raises interesting questions about how teachers might transition between their knowledge of mathematics and their teaching of mathematics.

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INSTRUCTIONAL COHERENCE IN THE MATHEMATICS CLASSROOM: A CROSS-NATIONAL STUDY

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This cross-national study explored instructional coherence by integrating research on discourse and coherence. U.S. teachers referred to instructional coherence as the surface connections between teaching activities, lessons, and topics, and they non-differentially embraced such coherence. In contrast, the Chinese teachers differentiated between surface and real coherence. For real coherence, Chinese teachers emphasized the interconnected mathematical concepts and student thinking. They stressed that only real coherence facilitates learning.

Keywords: Classroom Discourse; Instructional Activities and Practices; Teacher Beliefs; Teacher Knowledge

Purpose of the Study

Although instructional coherence has been recently identified as an important feature of effective classroom instruction (Cai, Kaiser, Perry, & Wong, 2009; Stigler & Hiebert, 1999), thus far, we do not have a clear understanding about the meaning of instructional coherence. In addition, even though prior studies have observed differences in instructional coherence between U.S. and East Asian classrooms (Leung, 2005; Wang & Murphy, 2004), we know little about how teachers themselves view instructional coherence (Cai et al., 2009). The purpose of this cross-cultural study is to examine, from an integrated perspective of discourse and coherence, how U.S. and Chinese teachers view instructional coherence in the classroom.

Theoretical Basis of the Study

Discourse coherence reflects the degree of meaning related to topics that affects readers or listeners' understanding (Dore, 1985). van Dijk (1977) proposed a two-level model for analyzing discourse coherence: micro and macro. Regardless of its level, the eventual goal of discourse is to enable the readers or listeners to process and comprehend it. van Dijk and Kintsch (1983), proposed that for the same *text* (what was read or heard), one might either comprehend its semantic content, resulting in a mental model called a *text base*, or one might actively make inferences and integrate the content of the text into one's existing knowledge system, resulting in a *situation model*. A coherent text contributes to the formation of a text base, which may improve one's ability to remember that text (Kintsch, 1986; Mannes & Kinstch, 1987). Yet, a coherent text with full information may reduce one's active processing of a text. Thus an incoherent text may better facilitate the formation of situation models, which are the keys to learning from a text.

Comparative studies have revealed two main features of instructional coherence that distinguish mathematical teaching in China (and other Asian countries) from teaching in the U.S.: (a) connected topics and (b) using transitional discourses (Chen & Li, 2010; Leung, 2005; Stigler & Hiebert, 1999). However, few studies (e.g., Cai & Wang, 2010) have directly explored how teachers from different cultures view instructional coherence. Explicit discussion is needed about what instructional coherence actually entails and whether teachers hold culturally different views on it. We hope to fill in some of these gaps through this cross-cultural comparative study.

Method

Subjects and Data Collection

A total of 20 Chinese teachers and 16 U.S. teachers participated in our study. All of these teachers were considered excellent mathematics teachers (e.g., all of the teachers won national or regional teaching awards or recognitions). Chinese teachers taught at the elementary level while the U.S. teachers taught at the secondary level. In addition, all of the U.S. teachers had a master's degree while all but one of the Chinese teachers had a bachelor's degree.

Survey Questions, Data Coding, and Data Analysis

Each of the teachers was given a set of questions: (1) *When people say a lesson is very coherent, what does the word "coherent" mean to you? What are the characteristics of a coherent lesson?* (2) *If you were mentoring a new teacher, how would you guide the new teacher to achieve coherence in her or his teaching?* and (3) *Some people say that a coherent lesson can foster students' learning. Do you agree with this statement? Why?* We analyzed teachers' written responses to each question using a constant comparison method (Gay & Airasian, 2000). The detailed codes of each teacher's responses were sorted and combined into broader categories of content, teaching, and learning. We then triangulated teachers' responses across questions and compared Chinese and U.S. teachers' responses. To ensure reliability, two independent coders went through the complete data sets and the general consistency in coding was confirmed.

Results

Static Nature of Instructional Coherence

Both groups of teachers interweaved static and dynamic aspects of instructional coherence in their responses. Regarding the static nature of instructional coherence, both groups discussed it at the micro and macro levels, arguing that content within and between lessons should be connected respectively (See Table 1).

Table 1: Content Coherence in U.S. and Chinese Teachers' Views

	Macro		Micro	
	U.S.	CH	U.S.	CH
Connection of activities/lessons/topics	3, 5, 6, 8, 9, 10, 12, 13	15	1, 3, 4, 5, 6, 7, 8, 11, 12, 13	4, 12, 13, 16
Connection of knowledge pieces	16	2, 5, 10, 15, 19	2, 14, 15, 16	1, 3, 5, 6, 7, 8, 9, 10, 14, 15, 18, 19, 20

Macro level. As indicated by Table 1, eight U.S. teachers (50%) referred to instructional coherence as connected topics that aligned with content standards. In contrast, five Chinese teachers (25%) considered the coherent nature of the mathematical knowledge system.

Micro level. Ten of the U.S. teachers (62.5%) argued that a coherent lesson should flow logically from one component/activity/thing to the next. To elaborate, they described the *beginning, middle, and end* (BME) of a lesson. Chinese teachers also acknowledged the teaching flow. However, 13 Chinese teachers (65%) emphasized the importance of the interconnection of mathematical ideas within a lesson, which was similar to the pattern observed at the macro level. In order to obtain an understanding of interconnected knowledge pieces, Chinese teachers uniquely stressed studying textbooks, along with studying students, before teaching a lesson.

Dynamic Process of Instructional Coherence

Table 2 summarizes teachers' different emphases on key features of a coherent lesson.

Table 2: Teachers' Different Emphases on the Key Features of Instructional Coherence

	U.S. Teachers	*Chinese Teachers*
Setting up teaching objectives	1, 2, 3, 4, 5, 6, 8, 11, 13, 14	1, 2, 3, 4, 7, 9, 10, 14, 18, 19,20
Managing lesson structures	1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15,	1, 3, 4, 5, 6, 7, 8, 9, 10. 11, 12, 14, 15, 16, 17, 18, 19, 20
Designing appropriate teaching sequence	N/A	3, 5, 9, 11, 12 14, 15, 19
Designing teaching language/questions	10	1, 5, 6, 7, 9, 15, 16, 19, 20
Challenging student thinking	3, 6, 10	1, 3, 7, 10, 11, 14, 17, 18, 19
Dealing with emerging events	13	4, 7, 9, 10, 13, 14, 17, 18, 20

Setting up teaching objectives. Ten U.S. teachers (63%) and eleven Chinese teachers (55%) emphasized that a coherent lesson should have a clear goal or teaching focus, which served as a direction for coherence (see Table 2). Some Chinese teachers discussed "essential and difficult teaching points."

Designing teaching with a progressive sequence. Eight Chinese teachers (40%) stressed that the teaching content should be designed and arranged progressively (see Table 2). CH9 suggested arranging tasks from easy to hard so as to align with students' development. None of the U.S. teachers discussed this aspect.

Designing teaching language/questions. Eight Chinese teachers (40%) emphasized the teacher's language coherence including the use of transitional language (see Table 2). In addition, seven Chinese teachers (35%) stressed careful design of teachers' questions. None of the U.S. teachers mentioned transitional language. Only one U.S. teacher mentioned questioning.

Challenging student thinking. Nine Chinese teachers (45%) argued for challenging students' thinking (see Table 2). Chinese teachers consider a smooth teaching flow without challenging student thinking to be surface coherence, but a smooth flow that challenges student thinking to be real coherence. Although three U.S. teachers (20%) stressed challenging student thinking, none of them differentiated between surface and real coherence.

Dealing with emerging events. Nine Chinese teachers (45%) discussed dealing with unexpected student responses, questions, and difficulties. Most Chinese teachers saw them both as a threat and an opportunity, and thus termed them as "emerging resources." Only one U.S. teacher (6%) discussed dealing with emerging events.

The Impact of Instructional Coherence on Student Learning

When explicitly asked whether a coherent lesson fosters student learning, the majority of Chinese teachers (75%) held very conservative views (5 agreed, 2 disagreed, and 13 partially agreed). The Chinese teachers argued that only *real coherence,* not surface coherence, may foster learning. In contrast, the majority of U.S. teachers (87.5%) expressed unreserved agreement that coherent lessons foster student learning (14 agreed, 1 partially agreed, 1 not sure). U.S. and Chinese teachers' different responses seemed to relate to their different views of the purpose of instructional coherence. Many Chinese teachers pointed out that instructional coherence should serve student thinking (e.g., coherence of students' thinking, gradually deepened thinking) while many U.S. teachers viewed it as a way to facilitate students' completion of tasks.

Discussion

What Does Instructional Coherence Entail? Surface versus Real

This study explored teachers' views of instructional coherence from a cross-cultural comparative perspective. U.S. teachers mainly referred to instructional coherence as the connections between teaching activities, lessons, or topics, which appeared to align with some of the "features" of coherent lessons reported by prior studies (e.g., Chen & Li, 2010; Wang & Murphy, 2004). U.S. teachers' emphases on a complete BME lesson structure also aligned with the levels and the structure of classroom discourse (van

Dijk, 1977). Connected activities and a complete lesson structure may help students construct a text base that may be useful for remembering what was taught (Kintsch, 1986; Mannes & Kinstch, 1987). However, such coherence, in Chinese teachers' eyes, was indeed only surface coherence and not real coherence. Chinese teachers emphasized the interconnected nature of the content and they also stressed gradually deepened student thinking. Chinese teachers' differentiation between surface and real coherence is insightful and invites rethinking of what instructional coherence actually entails.

The Tension between Scripted Plan and Generative Actions

To reach real instructional coherence, Chinese teachers advocated careful design of various aspects of a lesson. Although prior studies have observed important features such as transitional language (Wang & Murphy, 2004), our study reveals that these features mainly result from intentional design rather than in-the-moment decisions. In contrast, U.S. teachers rarely discussed designing lessons. Since teaching is complex, it is hard to imagine how a teacher can achieve a coherent lesson, especially *real coherence,* just by thinking on their feet.

Although Chinese teachers emphasized pre-designed lesson plans, they also highly valued capturing generative, emerging events during teaching. These emerging events often bring unpredictability to the teaching flow. Why, then, do Chinese teachers bother to design a lesson plan beforehand if the designed plan will be inevitably disrupted by emerging events? According to discourse theory (van Dijk & Kintsch, 1983), effective discourse should start from "design" even though unpredictable factors exist.

The significance of this study is threefold. Methodologically, it contributes an analytic framework of instructional coherence, integrating two well-established lines of research on discourse and coherence. Theoretically, our findings encourage a rethinking of the existing literature, particularly regarding the essence of instructional coherence. Practically, our findings have direct implications for classroom instruction and teacher professional development. Future studies may include detailed analyses of classroom instruction to show prototypes of lessons with surface and real coherence and to explore strategies to help teachers achieve instructional coherence.

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EXAMINING THE COGNITIVE DEMAND OF TASKS IN THREE TECHNOLOGY INTENSIVE HIGH SCHOOL ALGEBRA 1 CLASSROOMS

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This brief research report uses frameworks and analytic tools found in current research regarding cognitive demand of mathematical tasks to examine the cognitive demand of technology-based tasks in three 1:1 laptop learning environments. Current frameworks and analytic tools were productive in assessing potential and implemented level of cognitive demand and for indicating factors related to level of implementation. Growth in teachers' ability to create and successfully implement high level tasks was demonstrated, and factors related to decreasing or increasing the level of cognitive demand during implementation are reported.

Keywords: Algebra and Algebraic Thinking; Instructional Activities and Practices; Technology; Teacher Education–Inservice/Professional Development

The *Principles and Standards for School Mathematics* (NCTM, 2000) presents a vision of learning where every student has access to technology to facilitate mathematics learning under the instruction of a skilled teacher. Technologies, including calculators and computers, are viewed as integral tools of learning mathematics that provide visual images of mathematical ideas, facilitate the organization and analysis of data, provide efficient and accurate computations, and support student investigation in all areas of mathematics by allowing students to focus on decision making, reflecting, reasoning, and problem solving. "The existence, versatility, and power of technology make it possible and necessary to reexamine what mathematics students should learn as well as how they can best learn it" (NCTM, 2000, p. 24). One learning context that has the potential to match this vision is a 1:1 laptop learning environment. Within this environment, individual teachers and students are provided with access to laptop computers both during and after the school day. Argueta, Huff, Tingen, and Corn (2011) performed a meta-analysis using state executive reports from six major statewide 1:1 laptop initiatives throughout the United States, and their findings support that 1:1 laptop initiatives have the potential to positively influence the instructional practice of teachers and student outcomes. Specifically, when provided with high quality professional development, teachers increased their use of technology to develop learning materials and their use of higher order questioning. Further, their instructional practices began to shift away from more traditional approaches toward teaching with more reform-oriented approaches. This description of learning is consistent with the *Professional Standards for Teaching Mathematics* (NCTM, 1991), the *Professional Standards for School Mathematics* (NCTM, 2000), and matches the context of the teachers participating in this study.

Context

All three teachers participating in this study work within the same school district at two different high schools. The school system has participated in a 1:1 laptop computing initiative for approximately four years. Both schools are located in rural communities and serve high minority populations. The teachers came to be part of this study through their participation in a larger on-going professional development project focused on Algebra instruction within 1:1 laptop learning environments.

As part of the Algebra component of the professional development project, each Algebra teacher participated in a week long institute where the focus of the institute was to develop teachers' knowledge of algebra, Geometer's Sketchpad, and pedagogical practices. The main technological tool incorporated during the summer institute was The Geometer's Sketchpad (Version 5.0). Pedagogical aspects of the institute focused on practices designed to assist teachers in facilitating mathematical discussions in their classrooms (e.g., Smith & Stein, 2011), and Algebra content from the Common Core State Standards for

Mathematics (CCSSI, 2010) served as the mathematical content. On-going support included twelve hours of online professional development throughout the 2011–2012 school year and three observations per semester when teaching Algebra 1.

Methods

The framework and analytic methods that guided this study were derived from research conducted by Boston and Smith (2009), Smith and Stein (1998, 2011), and Stein, Engle, Smith, and Hughes (2008). The Mathematical Task Framework (Smith & Stein, 1998) served as the overall framework for analysis. Similar to Boston and Smith (2009), the Task Analysis Guide (Smith & Stein, 1998), the Instructional Quality Assessment Academic Rigor in Mathematics rubrics for Potential of the Task and Overall Implementation, as well as the Lesson Checklist (Boston & Wolf, 2006; Matsumura, Slater, Garnier, & Boston, 2008) served as the tools for analysis of tasks. However, what distinguishes this study from the previous research is the technology-intensive environment. The purpose of this study was to examine the cognitive demand of technology-based tasks used by teachers' in 1:1 laptop learning environments.

Tasks included dynamic sketches created by the teacher using The Geometer's Sketchpad and any accompanying worksheets or assignments used during each of three teaching sets conducted with the teachers. According to Simon, Tzur, Heinz, Kinzel, and Smith (2000):

> A teaching set consisted of two classroom observations and three interviews: a pre-lesson interview with the teacher about the first lesson to be observed, an observation of the first mathematics lesson, a second interview in which the teacher was asked about the first lesson and about plans for the second lesson, an observation of the second lesson, and an interview about the second lesson. (p. 583)

All tasks were evaluated for Potential of the Task prior to analyzing the Overall Implementation of the task utilizing video from the teaching sets. Then the average Potential score for each teaching set was compared to the Implementation Score. A Lesson Checklist was also completed for each observation. The findings reported include only the first two teaching sets, which focused on introducing linear and quadratic functions. Exponential functions served as the topic for the third teaching set, but that data is still being analyzed at this time.

Results

Table 1 shows the Potential and Implementation scores for each of the three teachers during the first two teaching sets.

Table 1: Potential and Implementation Scores for Participants

	Teaching Set 1		Teaching Set 2	
Teacher	Potential	Implementation	Potential	Implementation
Mrs. Lewis	3.4	2.6	3.5	4
Mrs. Patterson	2.67	2.33	3.5	3.5
Mr. Phelps	3	1.5	4	3.25

Mrs. Lewis

During the first teaching set, the tasks designed by Mrs. Lewis had the potential to engage the students at a high level of cognitive demand during instruction (3.4). However, the actual implementation score of the tasks (2.6) meant that the tasks did not reach their potential. Based on the Lesson Checklist that was filled out, the level of cognitive demand was diminished because the teacher provided a set procedure, the focus shifted to procedural aspects or correct answers, students were not pressed for high level products and processes. Also, the teacher's modeling during the lesson became directive, and students were not allowed enough time to effectively engage in the task.

For the second teaching set, Mrs. Lewis created tasks that demonstrated the potential for a high level of cognitive demand (3.5), and during implementation the level of cognitive demand increased (4). Several factors that influenced the increase in cognitive demand were based on appropriate time for students to engage with the task, the teacher held students accountable to high level products and processes, and the teacher provided consistent presses for explanation and meaning.

Mrs. Patterson

The tasks designed by Mrs. Patterson for the first teaching set fell just below a high level of cognitive demand (2.67), and the level of cognitive demand decreased during implementation (2.33). Based on the Lesson Checklist several factors contributed to the decrease in cognitive demand. The tasks were not complex enough to sustain student engagement in a high level of thinking, the teacher provided a set procedure, and the focus shifted to procedural aspects of the task or obtaining correct answers.

The potential for cognitive demand of the tasks increased for the second teaching set to a high level (3.5) and were maintained during implementation (3.5). During this teaching set, students had the opportunity to serve as the mathematical authority during the lesson, the teacher held them accountable for high level products and processes, and the teacher made consistent presses for explanation and meaning. All of which helped maintain the level of cognitive demand.

Mr. Phelps

Mr. Phelps demonstrated the greatest difference in potential (3) and implemented (1.5) level of cognitive demand during the first teaching set. Despite the potential for a high level of cognitive demand, the teacher provided a set procedure for solving tasks, focused on the procedural aspects of the task and correct answers, and did not hold students accountable for high level products and processes.

The tasks used by Mr. Phelps in the second teaching set scored the highest potential (4) among all the teachers. While the cognitive demand decreased during implementation (3.5), his use of the tasks with students remained at a high level of cognitive demand. During the second teaching set, Mr. Phelps gave students sufficient time to grapple with the high level of demand of the tasks, provided opportunities for the students to share in the mathematical authority of the class, and consistently pressed for high level products and processes from the students.

Conclusions

Two trends were noted regarding teachers' use of technology-based tasks in technology-intensive Algebra 1 classrooms. First, all teachers demonstrated growth in the potential level of cognitive demand from the first teaching set to the second teaching set. Second, all teachers were able to achieve a high level of cognitive demand during implementation of tasks for the second teaching set. This is a stark comparison to low level of cognitive demand achieved during implementation of tasks during the first teaching set. Taken together, these two trends indicate the potential for growth in teachers' design of cognitively demanding tasks, as well as growth in their ability to implement tasks at a high level in technology-intensive learning environments. These findings are consistent with Argueta et al. (2011) regarding teacher changes in 1:1computing environments when provided with quality professional development. Also, the framework and analytic tools from previous research on tasks and cognitive demand (e.g., Boston & Smith, 2009; Boston & Wolf, 2006; Matsumura et al., 2008; Smith & Stein, 1998, 2011; and Stein et al., 2008) proved to be applicable to technology-intensive environments. Future work as part of this study will include analysis and comparison of results from the third teaching set (exponential functions) to see if current trends hold, examining teachers' use of pedagogical practices to facilitate mathematical discussions, and discourse analysis of the resulting discussions. Another area of exploration may be to examine how the presence of technology influenced the resulting mathematical discussions.

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TRANSITIONING FROM PROCEDURAL THINKING TO MAKING MEANING IN THE MATHEMATICS CLASSROOM

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The purpose of this study was to examine how special education teachers make the transition from procedural thinking to making meaning in the mathematics classroom. Teaching and making meaning of difficult mathematics concepts can be influenced by the interaction of technological, pedagogical and content knowledge (TPACK). Special Education credential candidates learn to make meaning through a communal classroom activity system in which the university instructor creates zones of proximal development where candidates begin to make meaning. Dynabook, a Web-based tool to help college faculty, including special education faculty, instruct teacher candidates how to teach proportional reasoning to children in middle school was the foundation for this communal classroom activity system.

Keywords: Proportional Reasoning; TPACK; Special Education

Introduction

The purpose of this study was to examine how special education teachers make the transition from procedural thinking to making meaning in the mathematics classroom. Teaching and making meaning of difficult mathematics concepts can be influenced by the interaction of technological, pedagogical and content knowledge (TPACK; Mishra & Koehler, 2006). Special Education credential candidates learn to make meaning through a communal classroom activity system in which the university instructor creates zones of proximal development where candidates begin to make connections between past knowledge and new information. *Dynabook*, a web-based tool to help college faculty instruct teacher candidates how to teach proportional reasoning to children in middle school, was the foundation for this communal classroom activity system. The *Dynabook* includes dynamic representations, videos of children solving problems, and examples of practice problems. These features are intended to help teacher educators teach middle school teachers about how children think about proportionality, including their misconceptions and the strategies they use when solving problems. In addition, university instructors can create a social learning environment with teacher candidates as they work on math content and share work and ideas through a shared-work space. They model how middle school teachers should build discourse into their classrooms so their students can collaboratively build a deeper understanding of math content.

Theoretical Framework

Boyd and Bargerhuff (2009) completed an extensive literature review exploring research that intersects middle school mathematics with special education. They claim that in special education, college faculty are still instructing pre-service teachers to teach children to solve problems procedurally; in mathematics education, college faculty are working with pre-service teachers in student-centered, constructivist ways to teach children to solve problems more conceptually. These researchers admit that learning explicit procedures in mathematics is important, but that special education teachers tend to focus on procedures too much. Special education methodology is more likely to emphasize task analysis and specific, measurable objectives, often targeting procedural rather than conceptual skills. The propensity of special education to use these approaches is not surprising, given some of the common characteristics of students with disabilities, who often struggle in areas such as short-term memory, visual and auditory processing, and executive functions. Furthermore, past research in the field of special education has demonstrated more effective outcomes for students with disabilities when teacher-directed instruction is used (Kroesbergen & Van Luit, 2003).

Those who write about special education teachers and mathematics instruction come to the conclusions that it is important for teachers to not only know mathematics content, but they also need to know mathematics content pedagogy (Griffen, Jitendra, & League, 2009). Special education teachers need to provide children with opportunities to elaborate their ideas to make their reasoning explicit through the use of why and how questions, to engage more deeply in their ideas through deep discussions, and to use visuals and interactive materials to help make abstract concepts more concrete.

Boyd and Bargerhuff (2009) suggest teacher preparation programs provide a mathematics methods and intervention course that includes both a general and special education focus. As pre-service mathematics and special education teacher candidates develop their understanding of mathematics content, and explore teaching tools and strategies for teaching this content, they should also consider the accommodations and other interventions students with learning differences require to support them in mastering challenging content. Boyd and Bargerhuff insist that candidates enrich their own understanding of proportional reasoning beyond the over-simplified notion of cross-multiplication. While cross-multiplication is important and valuable, this narrow view provides an insufficient basis for later algebra learning.

To learn about proportional reasoning more meaningfully, teacher candidates need to build connections through a coherent learning progression with adequate support for the challenges of maintaining interest and engagement (Stein, Engle, Smith, & Hughes, 2008). Engagement with meaningful mathematical ideas depends on the kinds of tasks candidates are given (Schoenfeld, 1985), the tools and representations they are able to use (Sfard & McClain, 2002), available supports when they get stuck, and of course on the pedagogical talent of their instructor. Additionally, technology can support new more engaging tasks, better tools and representations, and can provide layered supports when students need them. Technology cannot substitute for good pedagogy, but it can encourage and support good pedagogy.

Emerging technological advances combined with Shulman's (1987) work on pedagogical content knowledge (PCK) have lead to the technological pedagogical and content knowledge (TPACK) framework (Mishra & Koehler, 2006). Shulman (1987) defined PCK as the professional understanding of teaching, or "how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (p. 8). TPACK extends this definition to how a teacher utilizes the dynamic interplay of technology, pedagogical skills, and content knowledge to represent concepts in different ways to engage learners. It represents the corpus of knowledge that an expert teacher utilizes to create an effective learning environment.

Method

Participants

A purposive sample of 13 students was recruited from the Mild to Moderate Special Education Program at a California State University in Fall of 2011.

Design

Researchers used a mixed-methods design to explore how special education teachers developed knowledge in the area of mathematics in an advanced curriculum class focused on preparing teacher candidates to teach junior high children with learning differences in math and reading. A chain of evidence is needed to establish that certain kinds of content knowledge support useful practices and that those practices support more positive outcomes for children (Darling-Hammond et al., 2005). For this research project, we planned to implement research methods that provided links for this chain of evidence. First, we recognize that teachers must gain knowledge and develop competencies. Second, they must translate their knowledge and competencies into effective practices. To this end, we collected the following types of data: pre- and post-surveys, pre- and posttest of pedagogical content knowledge, and video observations of class sessions.

Instruction

Thirteen pre-service and intern special education teacher candidates participated in two three-hour classes dedicated to interacting with the ratio section of the *Dynabook*. Participants had varying levels of mathematics proficiency and teaching experience. Over two class periods, teacher candidates utilized the *Dynabook* as part of a neatly choreographed classroom assignment designed by the university instructors and outlined on the assignment page in the *Dynabook*. First, candidates were asked to complete a ten-minute pretest on pedagogy and proportional reasoning. Prior to the first *Dynabook* class session, each candidate had completed an on-line survey about his or her teaching efficacy. Second, candidates logged into *Dynabook* with individualized passcodes that allowed them to save the work they completed and post it onto the shared workspace. On the assignment page, candidates were directed to watch embedded video of how the *Dynabook* is utilized at the university level and how aspects of UDL are embedded and modeled throughout the software. This particular activity helped situate and define the candidates' role within this classroom activity system, as it was important for candidates to take on the role of a teacher who was preparing to teach ratio during the next class session. Third, candidates were asked to solve a ratio word problem with a partner, discuss answers, and post solutions to the shared workspace. The university instructor opened the shared workspace so all anonymously posted solutions could be viewed and discussed by the entire class. Fourth, after the class discussion, candidates watched instructional videos that provided a framework for discussing conceptual shifts in proportional reasoning (Khoury, 2002; Labato, Ellis, Charles, & Zbiek, 2010; Lamon, 1999).

During the second session, candidates began the class by watching a video of a student incorrectly solving the problem that the candidates solved and discussed during the previous class. The student in the video made a typical procedural error by inappropriately applying cross multiplication without checking if her answer made sense. Second, after watching the student attempt to solve the problem, candidates worked in pairs to discuss the student's level of proportional understanding and how they could help the student reach the next level of proportional reasoning. Third, to demonstrate how each pair would teach the student in a way that addresses her misconception, candidates wrote teaching scripts using *Xtranormal,* a Web-based software to create animated teaching videos. Finally, videos were shared with the class and discussed in terms of content and pedagogy.

Research Question

1. How did teacher candidates' content and pedagogical knowledge of ratio change after using *Dynabook*?

Results

Early into the first class session, teacher candidates were reluctant to discuss math reasoning and evasive when asked discussion questions. By the end of the *Dynabook* sessions, they were sharing their mathematical thinking by discussing their own solutions to problems and remarking on the mathematical thinking represented by the videos. Participants were beginning to understand the idea that there are many ways to explain ratio problems and they would need conceptual understanding to reach students who may need multiple means of instruction. For example, one candidate said that she was impressed how the variety of explanations in the shared-work space really demonstrated how many different ways a person could solve one problem:

> I was just thinking it is really cool how everyone is describing it in a different way, and we can point out to students how there is not just one way to explain this problem, it is a really cool example—look at all these teachers in the room, and they all came up with 20 different ways to explain it.

They showed an increased understanding of student thinking by creating and discussing scripts to address a student's misconceptions. Teacher candidates were surveyed about their attitudes toward teaching proportionality and familiarity with terms such as TPACK. On average, they showed increases in self-efficacy for teaching ratio conceptually to struggling learners. They were also more confident with addressing the Common Core standards for teaching ratio, such as generalizing from patterns and making sense of word problems.

Conclusion

In past iterations of the *Dynabook*, credential candidates were only interested in learning the one "best" way to teach ratio and requested a video of a teacher explaining ratio to a student. After working with the *Dynabook* and creating scripts to teach a student with misconceptions, they were all willing and enthusiastic to discuss the mathematical thinking behind the videos.

After utilizing *Dynabook* in a well-designed curricular activity system, these teacher candidates were reintroduced to concepts such as ratio and were better able to recognize and understand the math content. The candidates realized that they needed to go back to this curriculum to review and remember what they learned in those grades. Following their use of the *Dynabook*, they were able to talk more precisely about ratio and how to assess students' understanding of ratio. They were more confident in their ability to teach the subject. After a relatively quick review, candidates were able to discuss their solutions to ratio problems and analyze other perspectives that they may not have considered. Moreover, the candidates were enthusiastic during the discussions, often carrying them over into breaks and after class. They reported increases in understanding TPACK and felt more confident teaching ratio using principles of TPACK. Also, the *Dynabook* activities and discussions led them to think about their ability to solve proportionality problems and recognize misconceptions in children.

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EXPLORING THE IMPACT OF PROFESSIONAL DEVELOPMENT ON K–3 TEACHERS' PRACTICES AND THEIR STUDENTS' UNDERSTANDING

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The purpose of this study is to develop paired case studies of K–3 teachers and subsets of their students to explore how teachers shape their teaching practices to develop students' mathematical habits of mind after participating in Primarily Math, a program for K–3 mathematics specialists. Data were collected from classroom observations and interviews across 2011/12 from three K–2 classrooms in one school, and focused on teachers' uses of representations, questions, and examples, as observable instances of mathematical knowledge for teaching. Teachers credit Primarily Math with helping them become more reflective and with equipping them to more intentionally design lessons that deepen student understanding.

Keywords: Teacher Education–Inservice/Professional Development; Elementary School Education; Mathematical Knowledge for Teaching; Instructional Activities and Practices

While much is known about the effects of professional development on mathematics teachers' knowledge and beliefs about mathematics, less is known about the impact of professional development on teaching practices, and even less about the impact on students. The purpose of this study is to develop paired case studies of K–3 teachers and subsets of their students to investigate how teachers shape their teaching practices to develop students' mathematical habits of mind after participating in a longitudinal professional development program for K–3 mathematics specialists. This study is part of a much larger study, whose central research questions is: How do teachers translate the mathematical attitudes, knowledge, and habits of mind emphasized during Primarily Math into measurable changes in teaching practice?

Primarily Math is part of NebraskaMATH, a $9.2 million NSF targeted partnership, and includes an 18-credit-hour program for K–3 mathematics specialist certification. The coursework includes three courses each focused on mathematics and pedagogy. The overall purpose of Primarily Math is to improve achievement in mathematics for all students and to narrow achievement gaps of at-risk populations.

A main focus of Primarily Math courses is to develop teachers' *habits of mind as mathematical thinkers*, and in turn to help them consider how to develop similar habits in their students. Habits of mind of mathematical thinkers include ways of productive mathematical thinking that are creative and persistent in solving problems; people with effective mathematical habits of mind are good at solving problems and communicating their reasoning to others. Children's mathematical habits of mind include "curiosity, imagination, inventiveness, risk-taking, creativity, and persistence…[viewing] mathematics as sensible, useful and worthwhile and… themselves as capable of thinking mathematically… appreciate the beauty and creativity that is at the heart of mathematics" (Sarama & Clements, 2009, pp. 6–7). The National Council of Teachers of Mathematics (2006) and the Common Core State Standards for Mathematics call for an increase in developing students' reasoning and sense-making skills. Focusing on developing teachers' and students' mathematical habits of mind is our approach to this.

Beyond habits of mind, Primarily Math also focused on teacher noticing (Sherin, Jacobs, & Philipp, 2011), and attending to student understanding. Together, these are a main vehicle for teachers to develop students' mathematical habits of mind. Kilpatrick, Swafford, and Findell (2001) define quality instruction as, "a function of teacher's knowledge and use of mathematical content and a teachers' attention to and handling of students" (p. 315). Teachers can attend to their students in a number of ways, including attempting to understand a student's developing understandings. Attempting to understand means a teacher needs to find a way to see the mathematics through the child's eyes. This is not an easy task for a teacher, as student understanding often occurs in a non-linear manner, is variable, and tumultuous in nature (Edwards, Gandini, & Forman, 2011). The lack of visible student understanding is sometimes attributed to a variety of factors including language and culture (Sarama & Clements, 2009). However, knowing the factors that make students' developing understandings invisible does not help the teacher to deepen their students developing understandings and mathematical habits of mind. Examples of how teachers make developing understandings visible are limited; one contribution of this study to the field is to describe how three K–2 teachers make the developing understandings of their students visible, as teachers work to develop their students' mathematical habits of mind.

Methods

This brief research report focuses on three paired cases. The three teachers were chosen from their cohort of 35 to represent strong and average mathematics achievement; these choices were made following extensive qualitative and quantitative analysis of teachers' trajectories across their Primarily Math coursework (Smith & Shen, under review). The three teachers all teach at the same school (Bluebird Elementary School [all names are pseudonyms]), and took courses June 2009 to June 2010, and participated in ongoing professional learning community meetings through spring 2012. Fifteen observations of each classroom were collected during the 2011/12 school year, with pre- and post-observation teacher interviews conducted around each set of two to three consecutive observations. During observations, researchers regularly interacted with students to ask them about their thinking and understanding, and to capture selected copies of student work. This research project is guided by several questions, including:

- Within each teacher's mathematical teaching practices, what types of questions does she ask, what types of examples (e.g., worked examples) does she use, and what types of representations does she demonstrate or elicit with students during math lessons to develop students' mathematical habits of mind?
- What do children's mathematical habits of mind look like in classrooms of Primarily Math participant teachers? What connections can be observed between teachers' and students' mathematical habits of mind?

Bluebird Elementary School is a Title 1 School. There are 650 multicultural students in prekindergarten through 5th grade. 75.6% of students qualify for free or reduced lunch, 18.4% of the students participate in the English Language Learner program, and the school has a 20% mobility rate. The teachers involved in this study are Ms. Summer, Ms. Spring, and Ms. Autumn (see Table 1); all were Caucasian females. During the year of data collection, the school district was in its first year of implementation of new mathematics textbooks: *Math Expressions*.

Preliminary data analysis conducted includes coding transcripts of interviews and videotaped observations of Ms. Spring; analyses of data from Ms. Summer and Ms. Autumn will during summer 2012. Researchers used MAXQDA10 to code transcripts in vivo (Creswell, 1998), focusing on questions, representations, examples, and observed mathematical habits of mind. Codes were then grouped into broader themes as patterns emerged across observations and teachers.

Table 1: Teacher Demographics

Teacher	Grade	Years of Experience	Class size
Ms. Summer	Kindergarten	9	18-23
Ms. Spring	First grade	33	18-20
Ms. Autumn	Second Grade	10	18-22

Findings

Four themes emerged during preliminary analyses, related to choices teachers make to develop mathematical habits of mind in students and make developing understandings visible: instructional choices, teacher moves, use of representations, and reflection. These themes spanned both interviews and observations, across all three teachers and grade levels.

Teachers making intentional instructional choices to develop students' mathematical habits of mind was marked by instances of teachers making choices about the design of the lesson or task, the representations being used and the design of the classroom. During observations, researchers noted apparent instructional choices, and then asked about these during the post-observation interviews. Teachers then could confirm the intentionality of these decisions.

Teacher moves that develop students' mathematical habits of mind include repeating, revoicing, defining and asking clarifying questions. In one example, the class is trying to come to an understanding of how a total is defined. Ms. Spring calls on several students to ask, "What is a total?" and then repeats what the student says. Through this process the class is beginning to create a definition for total.

A second move that Ms. Spring uses is revoicing, in which she adds mathematical precision or clarity to a student utterance.

Victor: Because 9 is at the bottom and 10 is at the top.
Ms. Spring: And 10 plus 9 makes 19 together.

Victor is referring to a chart with 2 rows of 10 envelopes. Ms. Spring makes a slight change to what Victor said to make a more complete mathematical statement.

Adding mathematical clarity and precision also occurred in the form of clarifying questions. Clarifying questions occurred often and are one of the tools that Ms. Spring uses to make a students' developing understanding visible to other students, and thus develop students' mathematical habits of mind.

Ms. Spring: Would it go at the top of the mountain or at the bottom? How do you know it goes at the bottom?
Kiya (points to the bottom of the mountain): Because it is not the biggest number.
Ms. Spring: And is it always the biggest number that has to go at the top? If it is the biggest than how come 5 didn't go at the top?

In this conversation, Ms. Spring is trying to help Kiya clarify her understanding about why a number is considered either a partner (addend) or total.

Use of representations to develop students' mathematical habits of mind and make developing understandings visible was seen both by the teachers providing students with representations and the students creating representations in different situations. Evidence of this was found throughout all observations. One example is in student solutions to daily math problems. For example, one problem being solved was: I have 1 nickel. I have 7 pennies. I buy a car for 10¢. How much money do I have left? José used drawings of coins to display how he solved the problem, first by drawing all the coins, then by crossing out those needed to purchase the toy car to see how much is left.

Teachers being reflective develop the mathematical habits of mind of their students is a theme that emerged from the interviews. For instance, Ms. Spring spent considerable time during the interviews reflecting on her district's new textbook series (*Math Expressions*). She discussed how before her Primarily Math participation, she would not have known words like "partners" and "totals" and she would not have combined her instruction of addition and subtraction. Many of Ms. Spring's reflective statements intertwined what she was noticing about her students and the mathematics about which they were developing understandings and habits of mind.

Discussion and Conclusion

During interviews, teachers reflected on the impact of their Primarily Math participation on their teaching practices. Since their district was implementing a new, much less traditional, textbook during the year of observations, teachers saw firsthand how their Primarily Math experiences gave them advantages over their peers, especially in understanding the language, representations, and examples provided in the textbook. Teachers saw deeper levels of student understanding and mathematical habits of mind developing as a result of their intentionality in planning representations, examples, and questions. All three teachers reflected about their levels of understanding of student understanding and mathematical habits of mind, and planned instruction to include examples and representations they thought would best support deeper student understanding and development of mathematical habits of mind.

Further analyses will include coding Ms. Summer and Ms. Autumn's videos and interviews, and include cross-case analyses. Since the teachers were chosen to represent strong and average learning trajectories across Primarily Math courses, data will be examined to determine if similar patterns emerge in their teaching practices and attempts to understand student thinking and mathematical habits of mind. Connections among student work, conversations and interviews with students, and teacher data will support conclusions about connections among teachers' and students' observed mathematical habits of mind.

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PREDICTION QUESTIONS IN THE ELEMENTARY CLASSROOM: TRANSITIONING FROM A PROFESSIONAL DEVELOPMENT EXPERIENCE TO CLASSROOM ENACTMENT

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In this study, we use prediction questions as a vehicle for professional development and share factors that hinder and promote experienced elementary school teachers' transfer of these professional development experiences into their classroom practice.

Keywords: Teacher Education–Inservice/Professional Development

Purpose

There is a burgeoning concern among researchers, educators, and policymakers to develop teacher capacities through professional development (PD) as a means for improving instructional practices to increase student learning and achievement. In this paper we share results of a study, investigating how teachers transition their knowledge from a PD experience into classroom practice; in particular, how does use of prediction questions serve as a catalyst for promoting reform-teaching practice?

Perspective

The goal of PD initiatives is to improve instructional practice as research confirms the significance of instructional practice on student learning (McLaughlin & Talbert, 1993). Three characteristics of effective PD are summarized by Edmondson (2009): (a) PD is *content-focused*, (b) teachers are provided with *opportunities for active practice*, and (c) PD is *embedded in teachers' daily work*. But, in order for PD to have a tangible impact on practice, teachers must find ways to transition from methodologies explored within a PD context to meaningful enactment into their classroom. Yet, the ways in which teachers may transfer and implement knowledge varies; Joyce and Showers (2002, p. 102) outline a continuum of "levels of transfer" (see Table 1).

Table 1: Levels of Transfer

Level	Description
1: Imitative	Exact replication of what was demonstrated in a training session
2: Mechanical	Applies an idea or strategy learned to other similar contexts, but in a way that varies little from how it was presented.
3: Routine	Activities become identified with specific models of teaching." At this level, lower-order and concrete curriculum objectives are often noted.
4: Integrated	Begins to integrate methods used in multiple contexts and can understand when and why it is appropriate to use.
5: Executive	Complete understanding of theories underlying various models learned, a comfortable level of appropriate use for varieties of models of teaching

This pilot study is designed around Edmondson's three characteristics and uses prediction questions as the catalyst for monitoring student learning and adjusting teaching to meet students' needs. We define prediction as reasoning about the mathematical ideas using previous knowledge or patterns *prior* to formal instruction (Kasmer, 2009). Prediction does not imply a simple premature guess; it is a sophisticated reasoning process in which students must activate their prior knowledge and connect concepts from previous learning. This study specifically investigated the following research question: In what ways does a teacher's use of prediction questions transfer from a PD experience to classroom practice?

Modes of Inquiry

Professional Development Teacher Intervention

Three teachers are highlighted in this section (Susan, 4th grade; Derrick, 5th grade; and Judy, 6th grade). Their teaching experiences were 15 years, 12 years, and 30 years, respectively. Since none of the teachers previously used prediction questions, the researcher initially met with each teacher to share the purpose and research supporting the effectiveness of this paradigm. Then the researcher taught a model lesson incorporating the use of prediction questions in each of the teacher's classrooms followed by a debriefing session. Each teacher taught 8–10 lessons incorporating prediction questions developed by the researcher based on the mathematical content of the lessons.

The teachers were asked to use the following lesson protocol throughout the study. At the onset of the lesson, the teacher poses one or two prediction questions. Students respond in writing, providing both an answer and explanation. After previewing each student's paper the teacher then elicits a variety of prediction responses with supportive reasoning. At this point, the teacher does not confirm the accuracy of the responses and should assess students' thinking in order to make instructional decisions pertaining to how to address ill-formed ideas and misconceptions and proceed with the lesson. At the end of the lesson, key mathematical ideas are coalesced and misconceptions revealed as the teacher encourages students to resolve discrepancies between initial prediction responses and outcomes.

Data Collection and Analysis

Data were drawn from various sources in order to triangulate the data (Denzin, 2006) and enhance the validity of research findings. These data sources reveal information about how the teachers transferred the practice of using prediction questions into their classroom.

Classroom observations. Classroom observations, collected monthly and augmented by field notes and video episodes, documented the ways in which the teachers transferred the use of prediction questions into their classrooms. In this study, we examined videos from Susan's classroom and observations from Derrick and Judy's classroom.

Teacher journals and debriefing sessions. The teachers submitted a weekly journal, reflecting upon their experiences with prediction questions, responding to questions such as: *What aspects of using prediction in your classroom, if any, have been beneficial to your teaching in terms of gauging students' understanding/misunderstandings? What instructional decisions did you make based on students' prediction responses?*

Our analysis focuses on the qualitative characteristics of each teacher's use and integration of prediction questions with an emphasis on a subset of the RTOP (Reform Teaching Observation Protocol) indicators (Pilburn & Sawada, 2000) that focus on monitoring student learning. Indicators we chose to guide our analysis includes: (a) Instructional strategies and activities respected students' prior knowledge and the pre-conceptions inherent therein; (b) Student questions and comments often determined the focus and direction of classroom discourse.

Results

In this section, we share snapshots of classroom interactions during the sixth week of the study that seem to be typical of each teacher's use of prediction questions.

Susan

Susan did not even demonstrate an imitative use of prediction questions when introducing probability. She asked her students to make predictions and had them look at these questions at the end of the lesson, yet she did not use her students' responses to either inform her own instruction nor have students reflect about their own initial misconceptions, resolve discrepancies or make connections. Susan's lesson was teacher-directed. She encouraged students to share their thinking; however, the focus and direction of the lesson was not determined by ideas originating from the students. Neither Susan nor the students made explicit connections between the prediction questions and newly learned information.

Derrick

In Derrick's fifth grade classroom, students completed a lesson on ordering decimals. Initially, Derrick asked his students to predict whether 3.07 or 3.4 would cover more or less area on a 10×10 grid. A number of students responded that 3.07 would cover more area than 3.4 because there were more digits past the decimal point. One student remarked that two zeros could be added to 3.4 (3.400), and there would now be more digits past the decimal point. Another student responded that two zeros could be added to 3.07 (3.0700) and there would now be more digits in 3.07. He commented during the post-lesson debriefing, "I knew I had to make an instructional decision based on their responses, but I could only show them on a 10×10 grid the difference between 3.07 and 3.4, but I couldn't think of any other way to explain it to them. I really didn't know what else to do." Derrick proceeded with the lesson as he originally planned, unable to make instructional adjustments to directly respond to his students' needs. At the end of the lesson, he only asked "is everyone clear now about the difference in shading between 3.07 and 3.4?"

Derrick's transfer of the use of prediction questions appears mechanical. He respected his students' knowledge and actively encouraged his students to share their strategies and conjectures. Derrick understood the rationale and intent of asking prediction questions both as a means for confronting students' misconceptions and to focus the lesson based on students' current understanding. Though he desired to use his students' thinking to direct the lesson, Derrick was unable to do so; he lacked the pedagogical content knowledge necessary to provide additional ways for his students to visualize, conceptualize, and compare decimals to confront their ill-formed ideas.

Judy

In Judy's sixth grade class, students were exploring strategies for representing fraction amounts larger than one (both mixed numbers and improper fractions). Judy posed this prediction question to begin her lesson: *Can you predict how many ¼ are in 7¼? Explain your reasoning.* A number of students suggested that there would be 7 fourths in 7 since the ¼ was the same in both numbers (¼ and 7¼) and 7 was different. Some students responded 28, reasoning that there are 4 fourths in 1 and since they have 7, multiply 7×4 to get 28. Others thought 11; reasoning that $7 \times 1 = 7$ and $7 + 4 = 11$. A few students suggested 29, stating they remembered the procedure. During this time, Judy wrote the students' responses on the board and summarized their reasoning. Then she told the students, "let's begin a new problem and we will revisit your predictions once you finish." Next, students worked in groups to represent numbers as improper fractions and mixed numbers on a number line. Judy circulated among groups, listening to students' explanations, interacting and questioning students who demonstrated misconceptions revealed through the prediction questions. For example, Judy asked a student whose response to the prediction question was 28, "Can you show me how many ¼s are in 7¼ using a picture?" The student drew seven circles and divided them into quarters. Judy encouraged the student to count the number of fourths; as the student counted, she realized she had forgotten the extra ¼ in 7¼ concluding there were actually 28 + 1 or 29 fourths. At the end of the lesson Judy restated the prediction question, "How many ¼s are in 7 ¼? Do you still want to stick with your original prediction? This is your opportunity to make a change if you want." Judy then led a discussion to help students make connections and resolve the discrepancies in their thinking prior to the lesson.

Judy's transfer of the use of prediction questions is integrated. She used prediction questions to solicit students' thinking and strategies, representing this thinking in an organized way for the entire class. Judy used students' responses to focus the classroom lesson and actively confront student misconceptions. She intentionally asked her students to make connections among concepts and strategies. Judy internalized how prediction questions could be used as a powerful tool for anticipating and building student thinking.

Conclusions

Through this professional development opportunity, teachers did transfer the practice of integrating prediction questions at various levels into their mathematics lessons. It was hoped that the prediction questions would provide an instructional focus and enable teachers to actively assess what the students already knew and misconceptions or ill-formed ideas that would need to be addressed during the lesson, as was illustrated through Judy's enactment of this instructional practice. Susan was unable to incorporate the use of prediction questions in a non-literal way. For Derrick, an incomplete understanding of pedagogical content knowledge interfered with his ability to confront students' misunderstandings.

This study suggests the need for sustained PD that explicitly provides more support for teachers' pedagogical content knowledge. In order for teachers to gather evidence of the student's knowledge and make well-informed instructional decisions based on this knowledge, teachers need additional experiences anticipating students' responses and explicitly connecting student thinking to related mathematical and pedagogical knowledge prior to instruction. In addition, involving teachers in the writing of the prediction questions may help teachers plan for their lessons more effectively as they anticipate students' responses and potential ways to adapt the lesson to address these responses ahead of time. The use of prediction questions has merit and is worthy of inclusion into PD opportunities.

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TEACHER CHANGE AND AGENCY IN A CURRICULUM DEVELOPMENT PROJECT

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We report on research conducted within a large NSF-funded curriculum project which has resulted in a new fourth-year high school mathematics curriculum. Teachers were engaged in the development process in that they provided ideas for content, contributed materials, and provided significant critique of the materials. We find that teachers who have participated in the project indicate they have improved knowledge and skills in mathematics, that their students made changes in their participation in the classroom, and, most significantly, that teachers have made substantial changes to their teaching practice. We claim these changes are evidence of teacher agency, a crucial element of successful professional development.

Keywords: Teacher Education–Inservice/Professional Development; Curriculum

Introduction

Recently, mathematics educators have called for new and focused studies of professional development as a way to improve mathematics teaching at the K–12 level (e.g., Darling-Hammond, Chung Wei, Andree, Richardson, & Orphanos, 2009). Often teacher change is an explicit goal of professional development. In this report, we address teacher change with the intention of connecting and using it to support the notion of teacher agency. This focus on agency has been studied extensively in other educational and psychological venues (e.g., Ray, 2009), but there is very little research on teacher agency in the mathematics education body of research. In this paper, we report on a research study conducted within a large curriculum project [CP] to examine an innovative type of professional development. In particular, we examine teacher-reported change and how this change relates to teacher agency.

Curriculum Development Project

Currently, many states require, or will require, high school students to complete a mathematics course beyond Algebra II in their fourth year (Cavenaugh, 2008). Additionally, there is a growing movement among schools to integrate mathematics with science, engineering, and technology (National Center on Education and the Economy [NCEE], 2007). As a result of these policies, mathematics educators and engineers are collaborating on an NSF-funded project with the objective of creating, implementing, and evaluating a new high school curriculum that integrates mathematics and engineering concepts (Young, Keene, Norwood, Chelst, Edwards, & Pugalee, 2012). The curriculum provides the venue for students to make decisions concerning contextual problems using mathematical methods; hence, the curriculum is called “mathematics for decision-making” [MDM]. The curriculum is intended for high school seniors who have completed two years of algebra and one year of geometry. During the summers of 2008–2011, 232 teachers participated in 11 workshops where they learned the content of the MDM curriculum and ways to teach it.

Research Background

Previously, we showed that the CP is an effective form of professional development (Keene, Dietz, & Holstein, 2011). Here, we extend this research by examining teacher agency because we believe agency is an important element of professional development. In the larger sociocultural structure, human agency is defined as “the process through which people intentionally change themselves or their situations through their own actions” (Ray, 2009, p. 116). When applied to teachers, agency is the belief that teachers have the ability to influence their environment while being shaped by social and individual factors (Lasky, 2005, p. 900). Thus, teacher agency is the desire and ability of teachers to take an active role in making important decisions for their classrooms and their students. Teachers are also shaped by social factors, like their schools, peers, and districts, and by individual factors such as elements of their own teacher identity.

Under this definition, the CP presents a situation where teachers can exercise their own agency. In the CP, teachers have the ability to choose whether or not to teach the curriculum, how much of it to teach, how they teach it, how many workshops to attend, how active they are in the CP, and whether or not they would like to give feedback. The CP teachers assert their own agency repeatedly throughout this process.

Methodology

Participants

The participants in this study came from the group of teachers who participated in a summer workshop (108 teachers at the time of this study). The project team developed and facilitated workshops for the purpose of introducing the volunteering teachers to the MDM curriculum. Participation following summer workshops included: (a) piloting the materials, (b) providing feedback on the materials, and/or (c) contributing to online forums.

Teacher Change Survey and Teacher Artifacts

A survey was utilized as one way to study how teachers reported change. The survey was adapted from a portion of Garet, Porter, Desimone, Birman, and Yoon (2001) Teacher Activity Survey. The CP teachers were given questions from the survey that specifically asked about the effectiveness of their involvement in the CP. The adapted survey consisted of 40 Likert-scale items to collect information about teacher change. In this paper, we report only on the items studying teachers’ perceived change of their practice because we believe that these items most closely relate to teacher agency. Of the 108 teachers who were a part of the CP at the time of this study, 39 (36%) completed the survey. In addition, throughout 2008–2010, we collected a number of teacher artifacts from the summer workshops as well as from throughout the school year. For this paper, we only discuss the teacher change survey results and supporting quotes from teachers’ daily journals from the summer workshops.

Results

We feel attending to teacher agency is an important element of successful professional development, and as Ray (2009) suggests, agency is the process to intentional change. Table 1 details teachers’ responses to the survey items regarding teachers’ perceived change in their practices. Table 1 also shows the results of hypothesis testing on the results. We use an approximate *z*-test to check if the means of the response values are significantly different from 2 (minor change). As the table shows, all but the last question showed a significant difference from 2. Note that the last question’s mean was significantly different from 1 (no change).

Table 1: Summary Data of Teacher Responses to Part 3 of Teacher Survey

To what extent have you made each of the following changes in your teaching practices as a result of being part of the CP?

			No Change (1)		Minor Change (2)		Moderate Change (3)		Significant Change (4)	
	N	Mean	*n*	%	*n*	%	*n*	%	*n*	%
The mathematics curriculum content	34	2.82*	4	11.8%	8	23.5%	12	35.3%	10	29.4%
The cognitive challenge of math classroom activities	35	2.86*	3	8.6%	7	20.0%	17	48.6%	8	22.9%
The instructional methods I employ	34	2.65*	4	11.8%	10	29.4%	14	41.1%	6	17.7%
The types or mix of assessments I use to evaluate students	34	2.56*	3	8.9%	14	41.2%	12	35.3%	5	14.7%
The ways I use technology in instruction (calculator or computer)	34	2.79*	4	11.8%	10	29.4%	9	26.5%	11	32.4%
The approaches I take to student diversity	33	2.09	7	21.2%	17	51.5%	8	24.2%	1	3.0%

*$p < 0.001$

The results from this set of questions show that teachers perceive that they have taken action and changed their teaching practice. For all but the last question, at least 50% of the teachers answered that they made either moderate or significant changes in their practice. We discuss two of the questions whose mean values were significantly higher than 2 in the survey (due to space constraints) and support these results with quotes from the teachers' workshop journals.

The Instructional Methods I Employ

Of the teachers who completed the survey, 58% (20 of 34) said they made at least moderate change in the instructional methods they use. One teacher explained why the MDM curriculum helps foster this change: "It is very tempting to continue business as usual… teach, homework, quiz repeat… Implementing this kind of curriculum at a school will take a teacher who is willing to change their mindset… If the teachers can't change the students never will." Another teacher explained how the MDM curriculum fosters this change by promoting "a problem based learning environment in the classroom. My life does not have to consist of worksheets. Students will ultimately get more out of the problem if we let them do it and not always walk them through step by step." The results from the survey coupled with these teachers' quotes from their journals are interesting particularly because the summer workshops and any further support given to the teachers focus primarily on content and not classroom instruction. The fact that the teachers decided to take action and change their teaching methods is noteworthy.

The Ways I Use Technology in Instruction (Calculator or Computer)

As a result of participating in the CP, teachers reported substantial changes in the ways they used technology in their instruction. This survey prompt received the most "significant change" responses of any of our survey questions (11 of 34). Implementation of the MDM curriculum requires computers on a regular basis and many of the chapters require students to employ Microsoft® Excel. We find teachers' interest in and commitment to incorporating Excel into instruction notable because it is difficult for teachers to introduce this type of technology in traditional mathematics classrooms. This introduction is the biggest indicator of teacher agency, as it takes significant effort for teachers to incorporate this technology. Teachers' journal entries indicate that many were very concerned about the technology required to teach portions of the MDM curriculum as well as the challenge of instructing students on how to use the software. For example, one teacher wrote, "My biggest concern at this point is the availability of computers when I start teaching this." Another teacher shared the same concern, but also offered a way to handle it: "I am very concerned about the amount of technology used because I know I can't get into a computer lab every day. I would definitely need some 'work by hand' material (more so than technology material)." The survey showed that many teachers have made changes in their technology use even though they faced these challenges, indicating that their desire to act on their decisions and effect change.

Conclusions and Implications

We have presented the results of a research study on teacher change which support our hypothesis that teacher-author partnering is an effective form of professional development that supports teacher agency. A relatively large sample of the participating teachers reported at least a minor change, and often a moderate or significant change in their teaching practice in several different areas. By participating in the CP, teachers illuminate their desire to take action in their practice and then actually change their practice. We believe that this is an important professional quality for teachers to have.

However, a large majority of the U.S. is in the process of implementing the Common Core State Standards (CCSSI, 2010). With this new opportunity comes the possibility that teachers will be required to employ a narrowed curriculum, outside-mandated instructional strategies (possibly scripts), and external assessments, all of which may lead to less autonomy and reduced teacher agency. We suggest actively involving teachers in a curriculum project, like the one described here, provides an avenue for teachers to choose action and make choices in their teaching and thus increase their sense of agency and improving instruction in the classroom.

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INVESTIGATING COMMUNITY COLLEGE MATHEMATICS INSTRUCTORS' PRACTICAL RATIONALITY THROUGH PROFESSIONAL DEVELOPMENT

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This study explores community college mathematics instructors' rationales for their teaching decisions. Using Herbst and Chazan's theory of the practical rationality of mathematics teaching, conversations among faculty during professional development sessions for teaching trigonometry are analyzed to better understand the norms of the classroom and influences on instructional decisions. This knowledge can be instrumental in developing sustainable strategies to assist teachers in improving their teaching.

Keywords: Teacher Education–Inservice/Professional Development; Post-Secondary Education; Instructional Activities and Practices

This study explores community college mathematics instructors' rationales for their teaching decisions. Using Herbst and Chazan's (2011) theory of the practical rationality of mathematics teaching, I analyze conversations among faculty during professional development sessions targeting trigonometry or pre-calculus instructors. The sessions also sought to probe the norms of teaching mathematics in community colleges and to establish how community college instructors' professional obligations influence their teaching decisions. This knowledge can be instrumental in developing strategies to assist teachers in improving their teaching.

More concretely I seek answers to the following questions:

1. Why do community college trigonometry instructors teach in the way that they do?
2. What are the norms in these instructional situations?
3. What are the reasons they give for the instructional decisions they talk about in professional development?
4. How do professional obligations (from Herbst and Chazan's theory of the practical rationality of mathematics teaching) influence their instructional decisions? Where in these obligations do their justifications fall?

Rationale

Mathematics courses at community colleges comprise 51% of the total enrolment of undergraduates in mathematics. A large portion of this enrolment is in developmental mathematics (57%) and college level mathematics which is a pre-requisite for calculus (19%, Lutzer, Rodi, Kirkman, & Maxwell, 2007). Thus community colleges play a large role in preparing students who plan to pursue STEM fields, but do not enter college prepared to take calculus. While community colleges emphasize the importance of good teaching and are organized in a way that suggests a commitment to teaching (e.g., faculty are devoted almost entirely to teaching, classes are typically small, Cohen & Brawer, 2008), there is little empirical research on teaching that occurs in community colleges and even less on improving the teaching (Mesa, Celis, & Lande, 2011).

Most community college mathematics faculty hold their highest degree in mathematics (70% of full-time and 49% of part-time faculty), with a smaller portion of faculty holding their highest degree in mathematics education (18% & 27%, Lutzer et al., 2007). Thus unlike K–12 teachers, most community college mathematics instructors have little or no formal education in teaching. As a result much of the instruction that takes place in these classrooms is based on how the instructors themselves learned mathematics—providing information in a teacher-centered classroom (Mesa, Celis, & Lande, 2011; Anderson, 2011)—with an estimated 78% of pre-calculus level mathematics in community colleges taught using standard lecture methods (Lutzer et al., 2007).

While there have been efforts to reform mathematics instruction in community colleges (Blair, 2006), there is no system in place to educate the faculty about reform-oriented teaching practices. This study approaches the improvement of mathematics instruction by looking at a professional development sessions for community college faculty teaching trigonometry or pre-calculus as a means to better understand their reasons for making the teaching decisions.

Conceptual Framework

Drawing on Herbst and Chazan's (2011) theory of the practical rationality of mathematics teaching, I explore the norms of teaching mathematics in community colleges and community college instructor's professional obligations. This theory of practical rationality views teaching as a natural phenomenon; teachers' actions are in response to the conditions and constraints of the environment.

This theory holds that teaching decisions are shaped by the presence of two sets of regulatory elements: norms and professional obligations. The norms of an instructional situation are mutual expectations between the teacher and students of who can do what and when. Norms are in part shaped by the professional obligations that are specific to an individual holding the position of a mathematics teacher. Professional obligations can be divided into four categories—*disciplinary*, *individual*, *interpersonal* (to the class a whole), and *institutional*—which aid in organizing the justifications that instructors make about their teaching decisions.

Context

Throughout a series of five three-hour professional development sessions for trigonometry and pre-calculus instructors from regional community colleges, I explore how teachers talk about their instructional decisions and identify—through their justifications of these choices—the professional obligations that influence these decisions.

Two sections of the professional development were conducted with a total of twenty instructors of trigonometry and pre-calculus from community colleges in the region. The participants were both full-time and part-time instructors with a wide range of backgrounds and years of experience teaching.

Each session included exploration of a trigonometric topic and involved the use of either an animation depicting a community college trigonometry classroom or other video to encourage discussion about instruction. Throughout the sessions the participants were asked questions and encouraged to talk about their reasons for making instructional decisions. Each session was audio and video recorded. The recordings and transcripts of these recordings for each session are the data used for this research.

Analysis

This study uses two methods to analyze the discourse in these professional development sessions. The first analysis aims to better understand the norms, professional obligations, and instructional decisions while the second aims to gain a more nuanced understanding of the tensions instructors face when making instructional decisions.

Analysis 1

This analysis uses open coding with special attention being given to the norms, breaches of the norms, and professional obligations used when justifying instructional decisions. Each justification is also categorized as the result of a breached norm or as one or more of the four professional obligations, with the goal of creating sub-categories within the four professional obligations. The intended result of this analysis is the creation of a *decision space*—the compilation of the influences on instructional decisions.

Analysis 2

To gain a more nuanced understanding of the tensions instructors face when making instructional decisions, the instructors' justifications will be further analyzed using Systemic Functional Linguistics (SFL, Eggins, 2004). SFL was chosen because of its theoretical grounding and comprehensive approach to discourse analysis; SFL focuses on the relationship between language and the context in which the language is being used.

This analysis will make it possible to look at the *position* of an instructor in the decision space when making a specific decision in a given context at a certain point in time. This position would indicate both the professional obligations that are included in their decision making and which professional obligations are driving the decision.

Discussion

Based on the data analyzed so far, I theorize that there exists a *decision space* that the instructor uses (not necessarily consciously) to make their instructional decisions and while this decision space remains constant, there are many variables locating an instructor's current *position* in that space—what obligations are more or less influential for a specific decision at a certain point in time in a given context. Being located in a different position may result in different justifications for their choices or different choices entirely.

For example, instructors talk about using or not using group work in their classroom. Some instructors justify using group work as a way to get students to communicate about and explore the mathematics as a means to better develop their mathematical ideas (obligations to the individual, the class as a whole, and the discipline). Whereas other instructors justify not using group work on the basis that not all students participate equally and therefore not all students get anything out of group work (obligation to the individual and class as a whole); that when the mathematics is not presented correctly it may just confuse the students more or the student may walk away with an incorrect notion of the mathematics (obligation to the individual and the discipline); and that it takes too much class time given the amount of content that needs to be covered in the course (obligation to the institution).

After viewing a clip of a student interview on a topic that had just been covered in class, which showed that the student was still struggling with many of the concepts presented, one faculty member reconsidered his earlier position of not using group work in class based on the realization that students may not be getting out of the traditional lecture what that faculty member would have expected.

This example illustrates the instructors' acknowledgement of their professional obligations as well as the potential for a change of position in the decision space. Because instructional decisions may have competing obligations, an instructor having more awareness or knowledge may influence the weight of a certain obligation in a given situation. In this case the instructor realized that students may leave a traditional lecture without having the expected knowledge, and began to question if lecturing is in fact fulfilling their obligation to the individual student in the way he expected. As a result instructors may begin to consider other methods of teaching that are more student-centered. While student-centered teaching methods were not previously considered an option—in part because of classroom norms and his or her professional obligations—this instructor began to discuss how the individual student may benefit from more student-centered practices, which may either outweigh or better contribute to their other professional obligations.

This example illustrates how knowledge of instructors' justifications for teaching decisions can better inform the design of professional development. Interventions can then be chosen to help instructors realize—rather than be told—ways of improving their teaching. Changes that occur in this manner are much more likely to lead to sustainable change.

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LIMITING STUDENTS' TRANSITIONS IN MATHEMATICAL COMMUNICATION

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This paper describes how facilitators' assumptions during mathematical communication—about the unconventional mathematical discourse practices of some bilingual Latina students—limited attention given to these students' semiotic resources preventing them from transitioning into greater understanding mathematical concepts although students' initial ideas were correct.

Keywords: Equity and Diversity; Instructional Activities and Practices; Reasoning and Proof

Communication in mathematics has become one of the major goals of and learning processes in mathematics education (Chapin, O'Connor, & Anderson, 2009; Sfard, 2008). Learning mathematics is deemed as an active process, closely connected to receptive, expressive, and interpersonal communicative actions (Bereiteer & Scandamalia, 2003). Appropriation of mathematical concepts is facilitated through using one's own terms to express own ideas— languaging (Wittgenstein, 1965) and even tensional, transformative processes that promote understanding (Razfar, Khisty, & Chval, 2010). These communicative perspectives are delineated in the standards. For example, the National Council of Teachers of Mathematics (2000) recommends problem solving and a focus on communication in the teaching mathematics to develop expertise in mathematical practice students need to make sense of problems, construct viable arguments, and look for and express their reasoning. Then teaching and learning of mathematics encompasses the appropriation of mathematical discourse practices, which reach far beyond using operations and terms (Moschkovich, 2004), so students become "mathematically powerful" by blending their understandings with mathematical knowledge and practices (Romberg, 1994). Though, discourse defines what is acceptable and marginalizes viewpoints central to other discourses (Gee, 1990), a situation that challenges an equitable promotion of the communicative process in mathematics.

This study centers on Latina students in mainstream classrooms that are not receiving special services, but still who struggle in mathematics. I believe this population is often overlooked but needs more attention. The U.S. NCES (2011) reports 18% of 4th graders and 27% of 8th graders are performing below the basic level in mathematics; 28% of 4th-grade and 39% of 8th-grade Hispanic or Latina/o students perform at below the basic level in mathematics. Among students reported as ELLs, 41% of 4th graders and 71% of 8th graders perform at this level also. Therefore, the students deemed as struggling in mathematics represent a high percentage of the general student population, the Hispanic/Latina/o, and the ELL student populations.

Focus, Setting, and Methods of the Study

This paper describes how some communication processes limited opportunities for some Latina students to participate and understand mathematics (Gutierrez, 2008). Although they were initially correct, the low expectations, assumptions, and lack of attention they encountered stifled their own understandings and perspectives and forced them to acquiesce and adapt their learning according to the instructor' ways and expectations. The context of the study comprises a math club (Khisty, 2004) where the participating Latina/o students were all bilingual (Spanish and English) and mostly U.S.-born and from a Mexican background. They met twice a week, ninety minutes each time, and the program ran for over three years (2006–2009), having an average of 17 students per semester. The program began with their 3rd and ended with their 6th grade.

Students had a self-selected enrollment. The hosting school has a dual-language program and is located in a low-income neighborhood in a Midwestern urban area where the student population is nearly 100% Latina/o, low income, and over 60% English Language Learners (ELLs). From the pool of participating students, I selected two focal students—Betty and Nora—with low mathematics performance in their regular classroom and examined the quality of their participation in small groups around the mathematics problem solving tasks during the entire program. I focused on the opportunities that each student had to express her ideas and how these were taken up by both the rest of peers in the group and the instructors or facilitators, who were Latina/o bilingual pre-service teachers. The data included videotaped interactions, fieldnotes, and student work. I selected 30 episodes showing focal students' interactions. Through a comparative and contrastive process (Miles & Huberman, 1994), I explored recurrent interaction patterns regarding the level of support that the facilitators and peers provided to different students in the different groups. These patterns were originally sorted in categories either as communicatively supportive interactions—or not. Because of the emphasis of this paper, the examples presented here relate to interactions with limited communicative interactions.

Using a framework based on Wittgenstein's (1965) idea of "languaging" and Vygotsky's (1978) concepts of language interpersonal and intrapersonal functions, I explored the following question: *How do social interactions around students with low mathematics performance mediate their opportunities to understand and participate in mathematics?*

Findings

Results from analysis show that 56% of Betty's and 52% of Nora's communicative interactions were limiting (i.e., students ideas were rarely elaborated on or acknowledged). In contrast, students with high mathematics performance within the same groups had over 75% supportive (i.e., students' ideas and struggles received attention and support) communicative interactions. These results indicate that students with low performance seem to have less support during communicative interactions around mathematics than students with higher mathematics performance. The following selected examples portray limiting communicative interactions.

The first example comes from Betty's group while interacting with Fabiola, Candy, Elsita and a facilitator. They worked on a task naming fractions in order to solve a word puzzle. At first, the facilitator asked students to describe the task and reacted differently to each student.

Betty's explanation was not elaborated by the facilitator. Perhaps it seemed unclear to her. Betty mostly relied on gestures to describe her ideas. Her narrative itself was imprecise compared to Fabiola's. Betty used general terms such as "it," "this," "here," and "there," which made the content of her message ambiguous, as in the following example:

> *Betty*: We need like, this one's already cuz maybe there's three right here [*points to pie charts*]; and then the thing is, there's like three and two [*points to numbers, keeps hand there*]; so, like if there's three and there's two right here [*points to charts with pen in right hand*], then that's a fraction.

Betty used gestures as an important part of her explanation, while Fabiola used gestures to complement her idea. Fabiola used specific terms, which made her message more direct.

> *Fabiola*: Yeah, because the fractions are here (points to circles). So, I think we need to put the letter in here [*taps empty boxes and numbers*], that we see the fraction here [*points to circles and holds pen there with right hand*], and we put the letter up here [*points to boxes with left index*].

Despite these differences, Betty's explanation helped another peer start understanding the task. These alternate ways of providing explanations may depict different levels at which meaning is constructed. For Betty and her peer, having gestures accompanied with a few general terms seemed to help their understanding; for the others, the use of more specific terms accompanied by few gestures worked better. At this point, the reception of Betty's "unconventional" explanation had been ignored and no attempt was made to understand what she meant.

Later, Betty understood the illustrations in the task differently, and the facilitator and peers did not attempt to adapt to this form. This fact forced Betty to change her perspective (although correct) without understanding why. While the group solved the task, Betty focused her attention on the white sectors of the pie chart (M), which led her to name it 1/8 (see Figure 1). This selection differed from everyone else's and what the task asked, which focused on the black sectors instead. Although Betty explained: "*It's one covered. It's one out of eight,*" which was a correct description, her idea was rejected without elaboration. The conflict arose from the difference in how the participants paid attention to the illustrations. The facilitator and the group corrected Betty, claiming that there was only one correct answer (O) to complete the phrase.

Emphasis on the "only" correct answer to overshadowed the possibility of understanding the representations in divergent, but equally correct ways. It is not clear whether Betty understood the differences between the two answers, but she accepted what the facilitator determined as correct. The facilitator and most peers focused on solving the puzzle, and there was no chance of negotiating mathematical meaning. Therefore, this process limited Betty's and the group's transitioning onto deeper understanding of fractions.

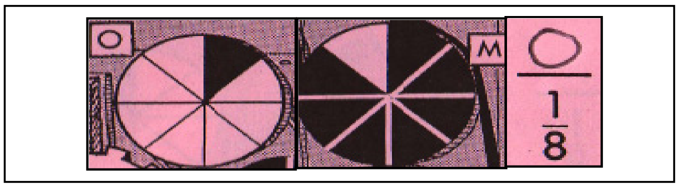

Figure 1: What is 1/8?

In the second example, Nora worked with Yolanda, Pilar, and a facilitator on a probability task called "Bubble Gum." Stressing conceptual understanding, the problem included the following question: "*Why do you think that three cents would be the most that Mrs. Hernández would spend to please her little twins?*" The problem first presented a gum machine with only red and white balls. Since the group was confused, Nora, however, reread and explained:

> Nora: [*extending left arm and holding up three fingers*] Ella tiene tres chanzas de sacar dos bolas con el mismo color./*She has three chanzas to pull out two balls of the same color*.
>
> Nora: Y si echa uno. Entonces tiene, ahí va una chanza. Y si echa el otro penny, ahí tiene otra chanza! /*And if she puts in one, then she has, that was one chanza* [points up as if making a tally mark twice] *And if she puts in another penny, there goes another chanza*!

The facilitator, although impressed by Nora's explanation, moved on to other peers who still seemed confused. Chanza is a term that is neither English nor Spanish, but it corresponds to the English "chance." Nora's explanation was neither used nor brought up again. This suggests that even though the facilitator seemed impressed by Nora's answer, she did not pay close attention to it. After this interchange, the facilitator supported the other students solve the problem.

As Nora's idea was not elaborated on, she was forced to tackle the task in procedural ways following the facilitator's directives. Confused by the process she took longer than her peers. The facilitator explained in a rather condescending way. Nora was the first in the group to have an insight with great potential to promote better understanding of the task. Her Spanglish term *chanza* was a powerful discursive resource that was ignored and perhaps not understood, but the lack of negotiation of this term promoted a limiting communicative mathematical interaction and forced Nora to follow sequential steps to an answer, but not to transition to understanding deeper the mathematical concept embedded in the term *chanza*.

Discussion

Unsuccessful communicative processes became evident when quality attention was not equitably distributed. It seems that students' mathematical status transfers beyond classroom settings. Assumptions about what is "correct" affected how facilitators noticed students. Students with more conventional ways of talking seemed to have greater support. If we believe students should develop understanding from their own perspective and move to more "efficient" ways of doing mathematics, it is crucial that we also make substantial effort to understand students' ideas before discounting or ignoring them. If something we hear or see seems unclear, then that is the point we need to work on, so we can see and listen to what students are saying to help us move with them in the transition onto deeper understandings. This means to build a common platform of understanding to build on collaboratively. Communicative approaches must validate different sources of communication. Unconventional or hybrid terms are not necessarily incorrect; instead, they might already be infused with rich mathematical content that we ought to include in order to strengthen the meaning-making process and equitable communicative opportunities for all, especially for those students with least conventional mathematical discursive practices, which stem from transitions moving along the continuum of understanding mathematics.

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SECONDARY MATHEMATICS TEACHERS' EXPERIENCES WITH HIGH-STAKES EXAMINATIONS IN A TIME OF CURRICULUM CHANGE

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This paper presents preliminary findings on research concerned with understanding secondary mathematics teachers' experiences teaching mathematics courses that have a provincially mandated high-stakes examination at the end of the course. The timing of the research project was such that the secondary mathematics teachers were also experiencing a curriculum change. Three experienced secondary mathematics teachers from three different schools and contexts were interviewed and philosophical hermeneutics was used as a theoretical framework to develop the method and analyze the secondary mathematics teachers' narratives of their experiences. Preliminary findings indicate that the secondary mathematics teachers' emotions, assessment, and teaching practices are influenced by both the high-stakes examination and the curriculum change.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Assessment and Evaluation; High School Education

This paper presents early findings of research inquiring into the ways in which secondary mathematics teachers experience teaching mathematics courses where the final assessment is a provincially mandated examination. The research is being conducted in Alberta where grade 12 students write diploma examinations worth 50% of their final mark in the course. Research (Dager Wilson, 2007; Volante, 2006; Webber, Aitken, Lupart, & Scott, 2009) has indicated that the high weighting of diploma examinations leads them to be considered high-stakes examinations for students. This research asks whether the educators teaching theses courses also see the diploma examinations as being high-stakes for themselves. While the study was not initially focused on examining the impact of curricular change on the teachers' experiences with diploma examinations, new secondary mathematics curricula are being implemented in Alberta in the 2011–2013 school years, thus managing that change is on secondary mathematics teachers' minds.

Purpose of the Inquiry

The purpose of this inquiry is to inquire into and develop an understanding of secondary mathematics educators' experiences teaching a course that involves the students writing a diploma examination. Understanding secondary mathematics teachers' experiences can provide insight into high-stakes examination development and implementation. Their experiences can also highlight policies that schools and school boards have in place that affect teachers both positively and negatively. Media attention to the results of student performance on the diploma examinations also shapes how secondary mathematics teachers experience teaching grade 12 mathematics courses in Alberta.

Having been a teacher and an examination developer, seeing both sides of what can sometimes be a wide gulf, led to me want to delve deeper into secondary mathematics teachers' experiences to understand how they are experiencing teaching courses that involve students writing a provincially mandated high-stakes examination. Clandinin, Murphy, Huber, and Orr (2009) comment that

> stories of school are increasingly driven by standardized achievement plotlines. Present accountability policies in Canadian schools place increased emphasis on achievement testing and mandated assessment practices reflected in provincial policies on yearly testing across the country. (p. 81)

Large-scale testing in Alberta, in Canada, and in the United States has a long and controversial history. Understanding how teachers experience these high-stakes examinations is crucial to administrators and policy makers.

Methodology

This research uses a philosophical hermeneutic framework to develop a personal understanding of others' experiences. Gadamer (1975/2004) said that "[u]nderstanding and interpretation are indissolubly bound together" (p. 400). As I am developing my understanding of others' experiences, I am also interpreting them within the context of my experiences. Like Gadamer, I believe that I cannot separate myself from my experiences, but I can work towards extending my horizon of understanding to that of whom and what I am trying to understand. D. G. Smith (1991) writes

> we can only made sense of the world from within a particular 'horizon' which provides the starting point for our thoughts and actions. Understanding between persons is possible only to the degree that people can initiate a conversation between themselves and bring about a 'fusion' of their different horizons into a new understanding which they can hold in common. (p. 193)

The understanding that I develop is then a shared one between the participants in the inquiry and me. From the fusion of horizons, I do not experience what the participants have experienced, but I understand how they would have experienced an event they way they did based on their history. Philosophical hermeneutics helped me develop an understanding of another's experiences while recognizing my own experiences.

I engage in questioning how I understand something by coming back to the participants to either confirm or challenge my understanding. This process of questioning my developed understanding is engaging in the hermeneutic circle (Ellis, 1998; Gadamer, 1975/2004; D. G. Smith, 1991; J. K. Smith, 1993). The hermeneutic circle involves considering an experience or an understanding within the context and the history of the person who had the experience. As a researcher working within the hermeneutic tradition, I consider the part of the experience within the whole of the participants' lives and within the larger context that they are working and living.

Participants

Participants in this research project are three experienced secondary mathematics educators who are teaching one or more secondary mathematics courses requiring the students to write a diploma examination. Susan has been teaching for approximately ten years and, at the time of the research, teaching at an online school in a large city. Susan has taught in four different schools in the same city and has taught a variety of secondary mathematics courses. Marla has been teaching for approximately eight years with all of her teaching experience at a large urban school. Marla has taught mostly what is considered the non-academic mathematics courses throughout her career. Vanessa has been teaching for approximately nine years and has experience teaching mathematics from grades five through twelve. Vanessa is currently teaching at a secondary school in a smaller, affluent city and teaches mostly the higher academic mathematics courses.

Method

Each of the participants was provided with a general written invitation to participate in the research project and an invitation to complete a pre-interview activity (Ellis, 2006) to bring to our first meeting. The purpose of the pre-interview activity was to provide our conversation a place to start, a way to get to know each other and for me to find out more about what was important to whom I was interviewing. Ellis (2006) writes "a conversational relationship can be established through discussions of the pre-interview activity products, thus building rapport for remaining parts of the interview" (p. 121). Marla was the only participant to complete the pre-interview activity and her response to the prompt "Draw a good day for you as a secondary mathematics teacher, and a bad day for you as a secondary mathematics teacher" provided the basis for much of our first conversation.

The interviews were unstructured and the conversations that we had did not follow a specific pathway but flowed as ideas came to us. As I planned to interview each participant more than once and over a period of eight months, I felt free to let them and I explore ideas and threads as they came to us. Gadamer

(1975/2004) writes, “the more genuine a conversation is, the less its conduct lies within the will of either partner. Thus a genuine conversation is never the one we wanted to conduct…No one knows in advance what will ‘come out’ of a conversation” (p. 385). Each of the conversations was different and focused on different aspects of mathematics education and teaching in general. As Ellis (2006) writes, I was learning “what the topic of the research [was] about for the participant” (p. 113) not pushing a particular agenda. Though the topic of research was identified to the participants, I chose not to direct the conversation to a specific set of experiences and the theme of change appeared in all interviews.

The interviews took place at a location and time of the participant’s choosing. The interviews were conducted both in person and over a distance. The interviews were transcribed and then reviewed for accuracy. The participants had an opportunity to review the transcript and make corrections, deletions, or additions as they saw fit.

At the time of this paper, I have interviewed both Vanessa and Marla on three different occasions, several weeks apart, and interviewed Susan once. Several more interviews with each of the participants are planned through the end of the 2011–2012 school year.

Discussion

During our interviews, each teacher expressed concern, frustration and a desire to not want to be teaching a grade 12 mathematics course in the first year of curriculum implementation. Both Marla and Susan, indicated an uneasiness regarding the diploma examinations adjusting to a change in curricular focus from content to process, though their concern was not stated explicitly. Vanessa, on the other hand, explicitly commented on her concern and what I interpreted as distrust of the government agency responsible for producing the examinations in providing information regarding the examination in a timely fashion so that she could adequately “prepare” her students for the upcoming examination.

Vanessa commented on recent change to the examinations that occurred in the fall of 2009 where the decision was made by the minister of education at the time to remove the written portions of the mathematics and science diploma examinations that left the questions on the examination to be machine scoreable only. She said:

> So I thought, all right I’ll teach them how to write a multiple-choice test. So now I actually spend days teaching them how to write a multiple-choice test it’s kind of my big whatever, you pulled my written, I’ll teach them how to write this. I’ll teach them how to graph it … You’re going to make it 50% of their mark on a multiple choice 40-question test, I’ll teach them how to write multiple-choice test.

Vanessa had earlier in the conversation stated a confidence in her ability to predict what was going to be the focus of the questions on the diploma examination, and this radical change in examination format shook her confidence. She responded to the change in the examination format by changing some of her teaching styles and strategies. Vanessa has not changed her classroom examinations to model the format of the diploma exam as her examinations contain questions where students have to explain their thinking. Vanessa indicated that she believes that communication and explaining mathematical process is important even though the diploma examination does not explicitly assess mathematical communication.

As the school year continued and the end of the term got nearer, each conversation focused more and more on the upcoming curriculum change and the ‘new’ diploma examination. Vanessa expressed a concern regarding the style of the question that was being asked on the examination. Her concern was that if the style of the question, or what was being asked of the students, does not change, her teaching focus would not change. For Vanessa, the diploma examination format and content was influential on her teaching practice. And for all three participants, teaching a course with a diploma examination at the end without knowing what the examination would look like before beginning teaching the course was a source of worry and concern.

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TEACHING AND LEARNING OF MATHEMATICS THROUGH PROBLEM-SOLVING ACTIVITIES IN KENYAN SECONDARY SCHOOLS

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The study sought to determine the effect that teaching through problem-solving activities has on students' performance in mathematics, their attitude, their perception on the teaching approach and the effect the approach has on syllabus coverage. About 420 form two (tenth grade) students from four high schools in Siaya County, Kenya participated in the research in 2011. The study used a quantitative research method that followed a control quasi-experimental design with pre-test and post-test measurements of attitude, confidence and classroom interaction; and a baseline, post-test for students' achievement. Results showed combined control variables of students' attitude, perception on teaching approach, and parents' level of education having effect on students' performance. Also, experimental group had higher performance on post achievement test items that demanded high-order reasoning.

Keywords: Problem Solving; Experimental Design; Attitude towards Mathematics

Introduction

Background of the Study

Past Kenya Certificate of Secondary Education (KCSE) examination results have shown that more than 70% of students attain failing grades of D-E in mathematics each year. For instance, in the years 1979, 1983, 2002, and 2006, students who attained D-E grades were 72.7%, 63%, 71.56%, and 79.2%, respectively (Miheso-O'Connor, 2011). KCSE is graded on a descending scale of A. Kenya National Examinations Council (KNEC) reports for 2005 and 2006 revealed that most mistakes made by students in mathematics are misconceptions and misunderstandings on the application of algorithms in various topics. The Government of Kenya has made attempts to improve the performance; however, it has not significantly improved in mathematics.

Purpose of the Study

The purpose was to examine the effects that problem-solving approach to teaching and learning of mathematics have on students' performance in mathematics in the Kenyan secondary schools, their attitude towards mathematics, confidence in mathematics and classroom interaction; also to asses syllabus coverage within the stipulated time.

Research questions: (a) how does a problem-solving teaching approach influence students' performance in mathematics? (b) What effect does a problem-solving teaching approach have on students' attitude towards mathematics? (c) What are students' perceptions of a problem-solving teaching approach? (d) What effect does this approach have on syllabus coverage?

Framing of Teaching through Problem Solving

Lesh and Zawojewski (2007) defined problem solving as: "A task, or goal-directed activity, becomes a problem when the 'problem solver' needs to develop a more productive way of thinking about the given situation" (p. 782). This study modeled problem-solving teaching approach where students carried out activities as individuals, then discussed their findings in small groups. Teachers, with the help of the students, harmonized the reports from various groups and made conclusions on the concept of the lesson. This approach, involved all students in the lesson as mathematics should be communicated as an active construction rather than inert body of delivered facts and skills (Swan, 2007).

I put students at the center of learning and encouraged planning of the activities that focused students to the concept, in line with the observations by Hiebert, Morris, Berk, and Jansen (2007) who noted that teachers should focus teaching in terms of students' learning.

Methodology

Research Design

I used a quantitative quasi experimental research design (Kazdin, 1998; Trochim, 2006), which followed pre-test and post-test measurements of attitude, confidence and classroom interaction; and a baseline, post-test for students' achievement on control group design approach.

Sampling and Participants

I conducted the study in four public schools (two girls' and two boys') in Siaya County, one of the 47 counties in Kenya in May-July, 2011. The participants comprised form two (10th grade) students in the four high schools and eight trained mathematics teachers of the chosen classes.

I used a total of eight classes (two from each school) with a total of 429 students out of about 12,800 students in the county. Schools were selected through purposive non-proportional quota sampling to get two girls' and two boys' schools. The two classes per school were conveniently sampled as per the sampled teachers whom I selected through purposive expert sampling.

Variables

The independent variables included a problem-solving teaching approach and parental education. Dependent variables were: (a) students' conceptual and cognitive knowledge growth, (b) students' attitude towards teaching of mathematics, (c) students' perception on the use of the approach, and (d) rate of syllabus coverage.

Instrumentation and Measures

For students' conceptual growth, a Centre for Mathematics Science and Technology Education in Africa [CEMASTEA] (2004) 40-item multiple choice mathematics achievement test instrument was adapted and used. The items were tested for both construct and content validities, with a reliability coefficient of 0.81 from statistical internal consistency reliability test.

Perception was operationalized by two indicators of confidence and classroom interaction subscales. A Fennema-Sherman (1986) Mathematics Attitudes Scale (MAS) was adapted for the subscales and attitude towards mathematics. It is a five-point scale Likert-type questionnaire with 30 items. A KIE (2002) guideline was used for the syllabus coverage.

Procedures

Pre-test and baseline items were administered to both groups on the first day of the classroom observation period that lasted for four weeks. Post-test items were administered to both groups on the last day of the intervention period.

Data Analysis

The data were analyzed using Data Analysis and Statistical Software (STATA) version 12 for both descriptive and inferential analyses. A *t*-test and multiple regressions (MR) analysis were used at $P < 0.05$.

Findings

Conceptual and Cognitive Growth

Conceptual and cognitive growth was determined by the achievement test. The intervention topics were applications of gradients in equations of straight lines, areas of quadrilaterals and triangles, and trigonometry I (i.e., sines, cosines and tangents of angles less than 90°). Analyses of post-test results showed consistencies in the higher mean scores for the experimental group on intervention items (i.e. 39 to

42) as shown in Figure 1. These were from the application of gradients in equations of straight lines and trigonometry I.

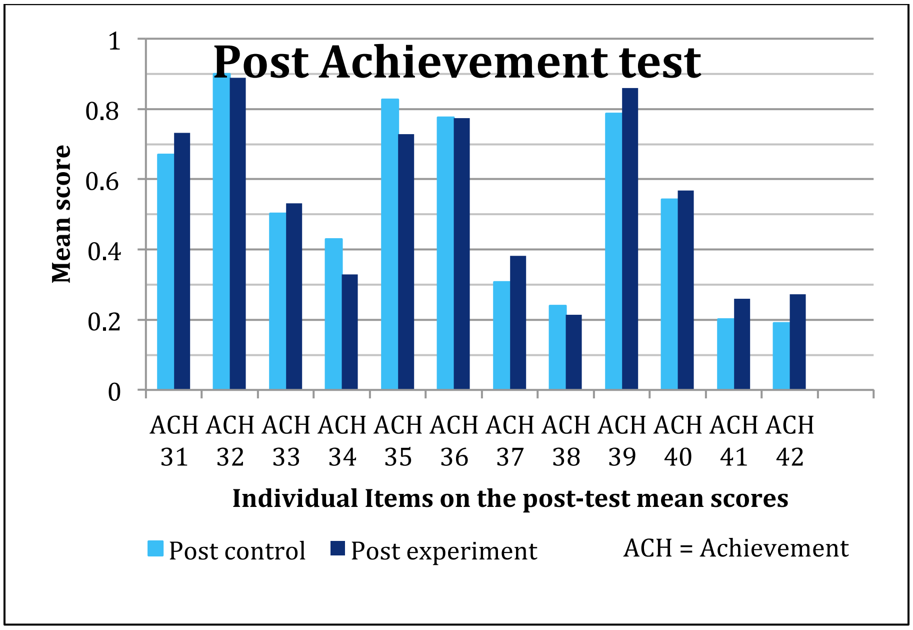

Figure 1: Bar graph on the distribution of the post-test individual items means score on achievement test for the experimental and control groups

These results show that the teaching approach used—that is, teaching through a problem-solving approach—may have contributed to this outcome since the students had no prior experience with these topics unlike the other two topics. Further analysis on these items showed significant differences on items 39 and 42 with $t = 1.96$ and 1.98 respectively at $\alpha = 0.05$. Items 39 and 42 assessed synthesis and evaluation levels respectively, according to Bloom's (1956) taxonomy of cognitive objectives categorization, (i.e. high order thinking skills).

Attitude towards Mathematics, Confidence in Mathematics and Classroom Interactions

Regression analysis showed a significant difference on the combined control variables and confidence over performance as dependent variable as shown in Table 1 below.

Table 1: Regression Analysis Table of the Predictors over the Dependent Variable

Predictors	Coefficient	Standard Error	Beta	t-test	P>|t|
Parents' level of education	-0.24	0.13	-0.12	-1.85	0.07
Attitude	-0.03	0.06	-0.04	-0.45	0.65
Confidence	0.11	0.04	0.29	2.98	0.00
Classroom Interaction	-0.02	0.04	-0.05	-0.57	0.57
Constants	5.50	1.72		3.21	0.00

Dependent Variable = Posttest scores on Achievement

Discussion and Conclusion

Discussion

The experimental group students outperformed the control group on the post-test, and particularly on the items that required creativity and deep understanding of the concept, that is high-order thinking skills (HOTS). It is in support of the research by Lesh and Zawojewski, (2007) that revealed that engaging learners in problem solving with mathematics concepts supports them in developing HOTS and also of research by Cohen, Lotan, Scarloss, and Arellano (1999) that showed that students who were taught through classroom activities gained significantly more than students in comparison classes on questions requiring HOTS.

The attitude change of the students did not show statistical significance and this could be an instrument issue in which a ceiling effect resulted from a pre-test, and it was in line with the findings of the Third International Mathematics and Science Study [TIMSS] (1999) that showed that most students from developing countries, in which Kenya belongs, had higher mean scores on positive attitude towards mathematics than students from developed countries.

The students' confidences in mathematics had low scores in both pre-test and pot-test. The low scores may explain the poor performance in mathematics in KCSE examinations, since findings revealed a significant correlation between students' achievement in mathematics and their confidence in mathematics. Both experimental and control groups taught the topics and completed one week earlier than the suggested time by KIE.

Conclusion

The findings have shown that there was better performance from experimental group on questions that demanded reasoning than the control group. This is an indication that continuous practice of a problem-solving teaching approach could improve performance of mathematics in Kenyan secondary schools.

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A PROCESS OF CHANGE IN TEACHERS: EXPERIENCE IN A PROFESSIONAL DEVELOPMENT PROGRAM

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The paper documents and analyzes the changes in the vision of mathematics that a group of teachers experienced as a result of their participation in a professional development program. The paper further explains the process through which they were able to emigrate from a purely "computational" interpretation of the mathematics contents to a more conceptual type of signification, and evidence is provided to attest that said change had a bearing on some of their didactic practices in the classroom. The change process—basically made up of three moments (learning of mathematics contents by way of problem resolution, where justification of strategies and results and metacognitive practices were emphasized; the study of the children's learning difficulties; and the design, application and analysis of didactic material)—can serve as a model for future professional development.

Keywords: Teacher Education–Inservice/Professional Development; Teacher Beliefs; Rational Numbers; Reasoning and Proof

Background and Interpretative Framework

Recently the attention of research has been focused on the changes produced in teachers as a result of their participation in professional development (Sowder, 2007). In specific terms, the experts—e.g., such as Schifter (1998)—have studied the transformation in the conception of the nature of mathematics that takes place among teachers involved in those programs. In her lessons, the researcher referenced seeks to help the teachers learn how to think in conceptual terms in mathematics, as well as for them to learn that their own mathematics learning experience can become food for thought. According to testimonials given by participants in her program, this has renewed the teacher's comprehension of mathematics as well as their beliefs concerning the discipline. Reid and Zack (2010), also interested in understanding the about turn in the mathematics perspectives of teachers-in-training, albeit by way of exploring their attitudes and beliefs, have revealed that said changes have a bearing on their teaching.

Complementarily, specialists in teacher professional development have suggested models to document their didactic evolution during their professional re-training (e.g., Guskey, 2002). One of such models consists of examining the manners in which the teachers question their students: a mentor with a calculational orientation will focus on procedures to obtain answers; those with a conceptual orientation will take into account a "rich conception of situations, ideas, and relationships among ideas" (Sowder, 2007, p. 194). While still other researchers (e.g., Fennema et al, 1996, and Schifter, 1995, quoted in Sowder, 2007, p. 195) have suggested models based on phases.

In order to evaluate the depth and breadth of mathematics conceptualization attained by the teachers who took part in the study, as well as the evolution of their didactic practices, below is a model—made up of four phases—that will act as the interpretative framework and that was formulated on the basis of the models cited above.

First Phase. The teacher focuses on teaching computational definitions and routines.
Second Phase. The teacher explores new ways of teaching, albeit using fixed routines.
Third Phase. The teacher focuses on the student and delves into his/her learning difficulties.
Fourth Phase. The teacher systematically investigates his students' learning difficulties, as well as the "great mathematics ideas," and takes said information into consideration in making his didactic decisions (in the classroom or in planning his lessons).

Methodology and Data Recovery

The changes reported here arose within the framework of a program, whose objective was to professionalize the mathematics teaching of teachers in practice. The program lasted for four years and sixty teachers participated. The study plans were articulated around two guiding axis, as follows: one of a theoretical nature that was consolidated in the courses; and another of a methodologico-practical nature that was consolidated by way of ten Development Projects (Dp), each of which covered six teachers.

The author was responsible for one of the Dps in which the objectives were: for participating teachers to re-signify mathematics contents; for them to recognize the children's thought processes and reasoning; and for them to experience the design, application and evaluation of didactic materials. The following activities were undertaken in order to achieve the objectives: (a) systematic study of the mathematics contents that was undertaken in a Problem Resolution Workshop; (b) definition of a classroom problem; (c) exploratory study and diagnosis (among students and teachers) of the mathematics contents associated with the classroom problem; (d) design of a didactic material whose purpose was to aid in the solution of the classroom problem; (e) application of the didactic material in the teacher's "experimental classroom"; (f) analysis of the intervention; (g) modification of the didactic material; and (h) teacher self-reflection.

This paper reports partial findings concerning the conceptual and didactic transformation process experienced by the group of six teachers attached to the Dp.

The empirical data for the study were recovered by different means, namely, for the Workshop there are documents written by the teachers—homework, exams and personal testimonials; for the diagnostic work, documentation concerning the instruments applied and analysis of results were maintained; for the classroom intervention, the results of the pre-test and post-test applied to the children, as well as the comparative analysis of those productions are also on file; moreover, video records, teacher field notes and the final assignment report were also kept; there are also three videotaped interviews of each of the teachers who took part in the PD.

Empirical Findings: Inflection Points in the Process of Change

The transformation of the mathematics ideas and of their didactics, which it was possible to see among the teachers attached to the Dp, arose in keeping with a lengthy process (cf Sowder, 2007) of four years in which three Moments stood out. Said Moments are described and analyzed below using the Phases described at the end of the first section as guidance. The evolution process is illustrated through the case of one of the teachers attached to the PD, Miss Luisa Ramos (see Ramos, 2009) (case studies are very often included in the reports that deal with changes in teachers, cf. Sowder, 2007).

When they first began the program, the six teachers in the Dp possessed a purely operational or "computational" interpretation of mathematics (Harel & Behr, 1995, quoted in Sowder et al., 1998, p. 128), pursuant to which the algorithms are their prototypical and unique objects, very much like what is described in the First Phase. The latter was verified in their interventions in the Courses and in the PD, and can be clearly perceived in Luisa's Workshop testimonial (see 2009):

> At the beginning of the Workshop, I resolved the exercise without doing any analysis. I felt that the rule of three applied to all missing value problems. I did not personally have a need to resolve problems using different methods; I was only interested in finding the result. (p. 18)

The first Moment of change arose during the Workshop. Focused on the search for re-signification and reinforcement of teacher mathematics knowledge, like in many other instruction proposals aimed at teachers (Schifter, 1998; Sowder et al., 1998), the Workshop—of one year duration—dealt with resolution of ratio and proportion tasks. It methodically promoted the explanation and (locally) deductive justification of the mathematics definitions and processes involved; metacognitive practices were systematically fostered (e.g., by way of writing a testimonial in which each teacher reflected individually upon his/her difficulties in learning mathematics and the changes of his/her mathematics beliefs and conceptions); and collaborative work was also promoted.

These practices raised a dual challenge for Workshop participants. On the one hand they had to provide support for a great number of algorithms and other mathematics resources that they used to accept as unquestionable truths in their daily teaching activities; while on the other, they had to become aware of the difficulties and changes that this represented for them, in addition to socializing their thoughts. In her testimonial, Miss Luisa shares the manner in which she faced those challenges toward the end of the Workshop, when she speaks of her resolution of the following task: “Laura has some pictures in the computer to which she applied a scalar factor of 6/5. She now wants to return them to their original size. [To do this, she only has to] subtract 1/5 from them. Is that solution correct? Justify the answer.” In her testimonial (Ramos, 2009, p. 18), Luisa offers the following proposed resolution and justification for the task:

> I considered the initial unit to be 5/5. Since the picture was increased by 1/5, it became 6/5, which is the new unit. So our new figure has 6 equal parts and to reduce it we have to take one piece away, which would be 1/6. So we would have: 6/6 – 1/6 = 5/6, which is our operator that will be applied to return the picture to its original size. In this resolution [I was able to] understand the problem and analyze it in depth [and] justify the fraction as an operator. [With respect to the group work, I had difficulty] believing that my method is the correct method and to establish which method has the most meaning for what was being requested; and as a group we had trouble reaching an agreement and common convincement.

One can see that Luisa understood the re-unitization process that was brought into play as a result of the successive application of scalar factor. She also understood the effects of the operativity implied in the problem resolution. In her self-reflection, she attests to her conceptual comprehension of the mathematics notions involved and to the fact that she became aware of the advantages that stem from the culture of justification fostered in the Workshop. She furthermore identified the obstacles that must necessarily be overcome in order to truly work in a team.

Like other professional development (such as that of Schifter, 1998, p. 56), the program presented here was based on the principle that “the teachers have to build their instruction around the children’s thoughts,” a principle that became reality through the diagnostic work described in c). Said work marked another Moment of change among the teachers. Miss Luisa carried out her diagnostic work with two groups of secondary school students and their teachers, and it was on the basis of that work that she typified in detail the following categories: “without proportional reasoning”; “traces of proportional reasoning” and “with proportional reasoning,” for which she based herself on operational considerations albeit especially of a conceptual handling. These categories served as a guide to analyze the results of the written instruments that she applied to the children before and after her intervention in the classroom, and they account for her evolution toward a vision of mathematics that is conceptually broader, richer and more meaningful, and that coincides with what is described in the Third Phase of the interpretative model.

The third turning point Moment that definitively marked the course of change in Miss Luisa arose during the application of her didactic material. It was there that she saw for herself and was able to assess her students’ achievements as of when she began to modify her visions of mathematics and her ways of teaching it (cf. Guskey, 2002). The objective of her didactic sequence was to enable her students to carry out processes of transferring knowledge between science and mathematics, and she based it on the idea that the “cross-cutting” condition of the contents, and of situationality and of collaborative work, can foster said processes of transfer in the classroom setting. Her new conceptualization of mathematics and of their didactics, very much akin to what is described in the Fourth Phase, is clearly revealed in the notes that she wrote after her intervention in her “experimental classroom”:

> In the class we spoke of mathematics while referring to other subjects, like nutrition, health, economics or chemistry. Mathematics topics were raised and discussed on an equal footing with other topics from other fields of knowledge. During the third class session, for instance, we spoke of the health impact of ingesting the sugars contained in soft drinks, and many of the students reflected on what foods are nutritional and which are not and on a balanced diet. As such during the intervention we broke away from the traditional teaching scheme (in which the teacher begins the class by giving

mathematics definitions and algorithms, after which the students do some exercises and possibly some applications), and resorted to another scheme based on a more contextualized and situated teaching. In the latter the idea was to promote different ways of resolving a problem, as well as positive and distant transfer [of proportionality themes] by handling knowledge in a cross-cutting fashion and by working collaboratively. (Ramos, 2009, p. 103)

Final Remarks and Possible Contributions

The paper presents a trajectory of professional development of mathematics teachers. Although that trajectory is partially supported by others, proposed by experts, the conjunction of all of its components and some of its specific practices, such as metacognitive practices, make it singular. To demonstrate its feasibility, a description is given of critical moments that arose during the evolution process of the conceptions and didactic practices of said trajectory's participants. This evidence provides the criteria for broadening the model used to evaluate the changes in teachers, described in the first section, including a new phase in which the teacher is considered in her role as a scholar of mathematics emphasizing justification, metacognitive and group collaboration practices that the teacher can implement during the process of resolving mathematics tasks.

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TRANSITION IN LIMBO: HOW COMPETING PROFESSIONAL OBLIGATIONS CAN LEAD TO PEDAGOGICAL INERTIA

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We describe a high school geometry teacher who perceives a need for change, expresses a desire to change, and yet finds herself struggling to enact change in her practice. Rather than focusing on beliefs or knowledge, we examine the professional obligations underlying her expressed intentions, and suggest that competing obligations made it difficult for her to make progress towards her expressed goals. We share implications for professional development.

Keywords: Teacher Change; Practical Rationality; Teacher Goals; Teacher Beliefs

Purpose

The purpose of this case study is to examine the factors that make transitions in teaching difficult, focusing particularly on the professional obligations inherent in the position of high school mathematics teacher.

Framework

The difficulties surrounding teacher change are often explained as resulting from deficiencies in mathematical knowledge or divergence in beliefs about teaching (e.g., Manouchehri & Goodman, 2000; Wilson & Lloyd, 2000). These explanations have been problematized by the notion that teachers do not operate as free willing individuals; they take *positions* in systems and take on *roles* which come with constraints, often tacit, on possible actions that they are able to take (Herbst, 2010; Herbst & Chazan, 2003). This perspective emphasizes that the decisions and actions of teachers are driven not only by their personal beliefs and knowledge, but also by the rationality demanded by the practice of teaching itself. This *practical rationality* consists of deeply ingrained norms of teaching and obligations to stakeholders such as the student(s) in the classroom, the institution (school), and the discipline of mathematics (Herbst, 2010; Herbst & Chazan, 2003). Obligations in particular present dilemmas for teachers, as teaching actions often favor one obligation at the expense of another, resulting in dilemmas of practice (Ball, 1993).

Methods

Participants and Context

This ongoing study is being conducted through a professional development project involving two high school teachers, a university professor (the lead author), and a graduate student (the second author). The professional development is being provided as a follow-up to a two-week long summer institute which included over 60 K–12 science and mathematics teachers. The follow-up professional development involves a series of meetings in which the two participating teachers are asked to make explicit their goals for teaching and examine their practice in light of those goals through the analysis of video captured in one of their classes (van Es & Sherin, 2010). The role of the professional development leaders is to elicit teachers' beliefs and goals, facilitate discussions, and provide the teachers with exposure to research related to a self-identified "performance gap." During the first meeting, the teachers were asked to create a "goals mapping" where they identified what was "closest to the heart" of their teaching (Herbel-Eisenmann & Cirillo, 2009). Afterwards they shared and described this mapping with the rest of the group. During the second meeting, the teachers had a chance to revise their goals mapping and reflect on their current teaching practices with respect to their self-identified performance gap. Then the teachers were asked to share a lesson plan for an upcoming lesson, which was discussed in the group in an effort to support each teacher in addressing her performance gap.

The teachers both worked a suburban school on the East Coast of the United States. The teacher featured in this article, Becky (a pseudonym), chose to study her practice in a self-contained special education geometry class with twelve students.

Data Analysis

In this paper we examine transcribed audio recordings of the first two meetings with the teachers. We used a grounded theory approach (Corbin & Strauss, 2008) to code and analyze the data. After an initial pass through the data, we separated initial codes into emerging theoretical categories: goals, previous experiences, teaching moves/norms, and obligations. "Goals" consisted of utterances where Becky described what she wanted to achieve with her teaching (but not necessarily what practices would achieve these goals). "Previous experiences" consisted of utterances where Becky described a specific past episode or gave a general description of typical experiences from past teaching. "Teaching moves/norms" consisted of utterances where Becky self-identified with, expressed appreciation for, or implied that she would engage in a particular teaching practice. "Obligations" consisted of utterances where Becky either justified a teaching action or described a constraint or her teaching by explicitly or implicitly appealing to duties derived from her position as a mathematics teacher. In some cases, Becky voiced obligations that we considered to be inherent to the profession of teaching mathematics, such as helping students understand mathematics. In other cases, obligations were inferred from the use of the words "have to" or "need to" with respect to a particular teaching practice.

Subcodes within each of these categories were established through a constant comparison process (Corbin & Strauss, 2008). Multiple codes were allowed for the same utterance; in some cases a single utterance was considered to represent both a goal and an obligation (it is reasonable that teachers, who choose their own profession, will have many goals which coincide with their obligations). Each author separately coded the data. Codes were refined through discussion, a portion of the data was recoded separately (inter-rater reliability was .66), and remaining discrepancies were resolved through discussion.

Results

The goals, experiences, teaching moves/norms, and obligations we inferred from Becky's statements are provided in Table 1. (The limited number of disciplinary obligations voiced by Becky may be attributed to the fact that during the second meeting Becky was discussing a particular geometry lesson that emphasized mathematical vocabulary).

One can see the conflicts among and within the categories. Becky's performance gap, as she described it, was "to do less talking and repeating myself and I want them to do more talking." She expressed the goal of engaging her students in active learning: "I want my classroom to be a place where learning is not just jammed down their throat. I want them to be active in their learning." However, in her previous experiences, she saw students as not wanting to talk about math ("If I am not talking everyone will just sit there and stare").

Becky's described obligations also appeared to make it difficult for her to put her goal of promoting active learning into practice. In the second meeting, Becky brought a plan for an upcoming lesson on parallel lines crossed by a transversal. Prior to describing the lesson, Becky referenced experiences in which students struggled with vocabulary: "words like adjacent, supplementary, complementary, linear pair, vertical, they are just in a cloud in their brain." She also expressed a disciplinary obligation for teaching those words, saying, "They have to be able to use big-people words, grown up words…. They can't just say 'across from,' 'next to.'" She struggled to envision a lesson in which students could engage in active learning and still learn the standard mathematical terminology. When asked why the students could not use "across," Becky hesitated and said, "I don't think I'm allowed," a statement that suggests that teaching the standard terminology was an obligation that did in fact guide her instruction.

Table 1: Subcodes in Each Theoretical Category

Goals	Previous Experiences	Teaching Moves/Norms	Obligations
• Promote active learning • Promote interaction • Reach all students • Motivate students • Make students comfortable • Learn to get better through experience	• Explaining and repeating isn't working • Students **do not** understand math terms or notation • Students **do** understand angle measurement • Students do not like to talk about math • Students are confused with multiple terms for the same idea	• Use nonstandard terms • Give specific instructions on process • Scaffold if necessary • Connect nonstandard and standard terms • Use multiple representations • Keep it simple (only explain one way) • Activate prior knowledge	To school: • Cover material (ensure mastery) To students: • help them make sense of ideas • make them comfortable • do not confuse them • provide a safe/respectful environment • prepare them for citizenship To discipline: • teach standard terms

In response to Becky's expressed desire to turn over more of the math talk to her students, the university professor described a research project where elementary students collectively defined "even numbers" through a nonlinear class discussion (Ball, 1993). This discussion involved the use of nonstandard terms but also actively engaged students in the mathematical act of defining a new set of numbers called "Sean Numbers." Becky did not seem to appreciate the impact of the approach on the active learning of the students and instead appeals to disciplinary obligations: "Well, I would want them to know [the definition of even numbers]…. 'Cause they are important." Becky maintained that multiple terms for the same concept would be confusing, and that allowing the students to come up with nonstandard terms "doesn't seem helpful" and would interfere with learning the standard terms later.

In these statements, we see an obligation that recurred in both meetings: an obligation to not confuse her students. For example, Becky wanted to minimize variation in explanations, saying, "I think about the type of kids these are, if I say something differently it might confuse them." This obligation, coupled with the obligation to teach standard terminology, made it difficult for Becky to enact practices that might have led to attaining her goal of promoting active learning.

Discussion

We believe that the tensions between the different obligations described by Becky present obstacles to substantial changes in her practice. While she wants to promote active learning and provide more opportunities for students to talk, she doubts whether her students will be capable of negotiating the different meanings and terms that might arise if they are allowed more freedom. She is bound both by the obligation not to confuse them and her disciplinary obligations to teach the standard terminology (see Pimm, 1987, for a description of tensions surrounding the development of a mathematics register). These are not unlike the tensions described by Ball (1993) in the Sean Numbers episode, and show that Becky's conflict comes not from a deficiency of knowledge or a lack of appropriate goals for her students, but from the competing obligations she perceives as inherent in her practice.

Becky's case is valuable because it shows that changes to instruction are not likely to result from changes in knowledge and beliefs alone. Becky has goals that she is unable to enact. Understanding the constraints implied by the practical rationality of teaching is important because "durable change in instruction will need not only to provide new and better resources but also to be able to deal with the inertia and possible reactions from established practice" (Herbst, 2010, p. 50). Professional development must anticipate these reactions.

For example, while there is much research which describes the necessity of confusion and struggle in the learning process (Hiebert & Grouws, 2007), Becky seems to operate within the obligation that the teacher should not confuse students. This is an obligation which has value and cannot simply be dismissed, but it can also be understood to operate at different levels and time frames. It may be acceptable for a teacher to introduce confusion if she is reasonably confident that the confusion will be temporary and will ultimately help meet her other obligations. Likewise, allowing students to initially invent their own terms before providing the conventional terminology might help promote active learning, but Becky was unsure whether this was "allowed." She needed reassurance that this strategy would not conflict with obligations to teach the standard terms and to help students make sense of ideas. The ongoing challenge for us as professional development providers is to help teachers like Becky explore paths of transition in her teaching while still honoring her perceived professional obligations.

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STUDENTS' MIMICRY OF INTENTIONAL TEACHER GESTURES

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Poster Summary

The poster includes a synopsis of some relevant results of prior research in this area. These findings prompted the design of this research study, the goal of which is to investigate the impact of teachers' gestures on students' mathematical understanding. The main focus of the poster is on the process and rationale behind the gesture design. The topic of slope was chosen. We wanted to create gestures that demonstrated the kinetic dimension of slope rather than the static graphical representation often used on the blackboard. The preliminary findings support the hypothesis and suggest a need for further research.

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THE INFLUENCE OF A TEACHER'S ORIENTATION ON HER MATHEMATICAL KNOWLEDGE FOR TEACHING AND GOALS

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Keywords: Mathematical Knowledge for Teaching; Teacher Knowledge; Teacher Education–Inservice/Professional Development; Instructional Activities and Practices

Mathematics instruction in the United States has been described as procedural and disconnected, with a primary focus on developing students' calculational abilities rather than their understanding of concepts and how they are connected (Ma, 1999; Stigler & Hiebert, 1999). Researchers have identified mathematical knowledge for teaching (MKT) as a key link between content knowledge and support of student learning (Hill, Ball, & Schilling, 2008; Silverman & Thompson, 2008). Earlier research has also shown that a teacher's image of mathematics, referred to as a mathematical teaching orientation, influences her classroom practice (A. G. Thompson, Philipp, Thompson, & Boyd, 1994). Other research (Webb, 2011) have revealed the interaction of a teacher's goals and pedagogical powerful content knowledge.

In this presentation, we provide findings that illustrate how images of mathematics that result from a teacher's experiences as a mathematics student can be adapted in the context of professional development designed to confront their prior conceptions of instruction and support their ability to implement a conceptually oriented curriculum. We also examine how a teacher's instructional goals might adapt as the teacher advances in her understandings and images of what students are capable of learning. We describe our emergent theoretical framework for examining these shifts, and then illustrate how this framework is used to analyze a teacher's classroom practice.

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SUSTAINING KNOWLEDGE OF AND SELF-EFFICACY IN USING FORMATIVE ASSESSMENT

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This poster examines sustained growth in teachers' knowledge of and self-efficacy in using formative assessment over a three-year period. Teachers were randomly assigned to the FA-then-NAV *group, who received professional development in formative assessment in the first year and in using networked classroom technology to implement formative assessment in the second year, or the* FA-and-NAV *group, who received professional development in using networked technology to implement formative assessment in two consecutive years. Teachers in each group reported gains in knowledge and self-efficacy each year, and these gains were sustained in the year following, when no professional development was received.*

Keywords: Assessment and Evaluation; Technology

The poster report on data collected in a research project, *The Effects of Formative Assessment in a Networked Classroom on Student Learning of Algebraic Concepts (FANC)* funded by the National Science Foundation. The working definition of formative assessment was taken from two sources: "the process used by teachers to recognize, and respond to, student learning in order to enhance that learning, during learning" (Cowie & Bell, 1999, p. 32), and "interactive process between teaching and learning where teachers collect evidence about student achievement in order to adjust instruction to better meet students learning needs, in real time" (Wiliam, 2007). In Project FANC technology was paired with formative assessment because technology may provide a solution to the problems encountered when using formative assessment.

Instruments on seven teacher constructs—knowledge about general assessment, knowledge about formative assessment, self-efficacy in formative assessment, the perceived value of technology, interest in technology, self-efficacy in general technology, and confidence in classroom technology—were used to gather data from teachers in Project FANC over the two years they participated in professional development activities and a year after the conclusion of the training. This poster focuses on the collected information on the constructs of knowledge about formative assessment and self-efficacy in using formative assessment.

Within each model, teachers gained in their knowledge of formative assessment and self-efficacy in using formative assessment each year they participated in the professional development. What is more striking is that they also continued to gain in their knowledge of formative assessment and their self-efficacy in using formative assessment in the year following the completion of the professional development. That is, regardless of how the professional development emphasized formative assessment, after three years both groups had gained, and had gained approximately the same amount, even though the trajectories were different.

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COLLEGE MATHEMATICS STUDENTS' PERCEPTIONS OF JOURNAL WRITING

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Keywords: Instructional Activities and Practices; Post-Secondary Education

Purpose and Methods

Current literature describes many benefits of writing in mathematics, such as more powerful learning (Borasi & Rose, 1989) and deeper procedural and conceptual understandings (Jurdak & Zein, 1998). However, most college mathematics students have not had a previous experience with writing in mathematics (Borasi & Rose, 1989). Therefore, the purpose of this study was to determine if college mathematics students' perceptions of writing in mathematics align with the benefits outlined in the literature.

The participants of this study were the students enrolled in two sections of "Elementary Calculus with Trigonometry." There were 60 students in one section and 59 students in the other section. Data for this study were collected through an open-ended questionnaire and analyzed using code mapping (Anfara, Brown, & Mangione, 2002).

Results and Conclusions

Fifty-two students (43%) believed that writing in mathematics afforded benefits to their learning. The three most common benefits mentioned were helping students engage in critical thinking, building procedural knowledge, and deepening conceptual understandings of the mathematical content, which is consistent with previous literature (Jurdak & Zein, 1998; Borasi & Rose, 1989). From the students' descriptions, engaging in critical thinking encompasses actively and skillfully conceptualizing, analyzing, and synthesizing information. Students also reported that writing in a mathematics course would be beneficial to their learning because it would allow them to understand, remember, or make sense of mathematical procedures as well as to gain a better conceptual understanding of those mathematics procedures. On the other hand, 67 students (57%) reported disadvantages to writing in mathematics. Students reported that writing in mathematics was a waste of time, was not important, and was not an appropriate task in a mathematics course. These were also consistent with previous literature (Williams & Wynne, 2001; Powell, 1997). These findings offer a basis for further investigation to better understand how these perceptions limit or enhance the benefits of writing in mathematics as well as how to aid students in transitioning along a continuum of learning mathematics.

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TWO COLLEGE MATHEMATICS TEACHERS' INTERACTIONS WITH STUDENTS IN DIFFERENTIAL CALCULUS CLASSROOMS AND OFFICE HOURS

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***Rationale*:** Since face-to-face interactions are a very common situation for teachers to help students learn mathematics, it is important for mathematics teachers to facilitate meaningful interactions with their students. In fact, there is an increasing expectation that mathematics teachers engage their students in certain kinds of discourse as a means of increasing student learning (Webb, Nemer, & Ing, 2006). However, few researchers have examined, especially at the college level, how mathematics teachers facilitate classroom interactions when they "make sense of students' reasoning, respond to unexpected questions, analyze students' errors, develop meaningful assessments of student learning, and provide on-the-spot representations, examples, or explanations for ideas that arise spontaneously in the real-time practice of teaching" (Wagner, Speer, & Rossa, 2007, p. 250). This study is designed to contribute in this area of research and extend our understanding of mathematics interactions. I applied conversation analysis (CA) on video recorded data of two college calculus teachers' interactions in their classrooms and during office hours. In particular, I examined what sequential structures around the teachers' questions engage students in a meaningful mathematics discussion.

***Findings*:** I found that most observed lessons were characterized as teacher-centered instruction—the teachers' talk dominated the classroom interactions. CA on the video data revealed that the teachers' domination was attributed to their ways of turn-taking (Ten Have, 2007). The teachers did most of the talking and occasionally provided conversational cues that allowed students to take a turn to speak. For example, when the teachers made a pause after a question, it was students' turn to talk. In contrast, when the teachers did not make a pause after a question, no responses were expected. In fact, students did not respond to such questions and therefore the teachers kept talking. Overall, in the teacher-centered classroom environment, teachers have the power to implicitly decide when the students have opportunities to talk; and the students on the other hand have a good sense of when they are given those opportunities. However, during office hours, the students were more engaged and took an active role in the discussion, especially when they decided how to proceed in their solution. This difference was found in the turn-taking patterns of interactions such as overlaps of talk and pausing.

Significance: Because CA is not widely applied in mathematics education research, the details of mathematics interactions, as described here, have mostly been unexamined. This study plots a new direction in investigating mathematical interactions in classrooms or other instructional situations. For example, if the classroom activities are student-centered or inquiry-oriented, patterns of teacher–student interactions will be illustrated in different ways. This study suggests that we need to further examine conversational structures such as turn-taking and question design across different types of instruction.

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Chapter 8

Teaching and Learning of Specific Content

Early Algebra, Algebra, and Number Concepts

Research Reports

Brief Research Reports

Poster Presentations

Geometry, Measurement, and Spatial Reasoning

Research Reports

Brief Research Reports

Poster Presentations

Data Analysis, Statistics and Probability

Research Reports

Poster Presentations

EXPONENTIATION IS NOT REPEATED MULTIPLICATION: DEVELOPING EXPONENTIATION AS A CONTINUOUS OPERATION

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The concept of exponents has been shown to be problematic for students, especially when expanding it from the domain of positive whole numbers to that of exponents that are negative and later rational. This paper presents a theoretical analysis of the concept of exponentiation as a continuous operation and examines the deficiencies of existing approaches to teaching it. Two complementary theoretical frameworks are used to suggest an alternative definition for exponentiation and guiding principles for the development of a teaching trajectory, and then to analyze an example of the hypothetical learning of a student who goes through the first task in the trajectory. The paper concludes with some possible implications on curriculum and task design, as well as on the development of mathematical operations.

Keywords: Algebra and Algebraic Thinking; Instructional Activities and Practices

The Problem with Exponents

Research has found the concept of exponents problematic for both students and teachers of all levels (Confrey & Smith, 1994; Elstak, 2007; Goldin & Herscovics, 1991; Weber, 2002). The most common definition that students have for exponents is that of repeated multiplication, for example $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ which is two multiplied by itself five times. This limited view of exponentiation, although simple to understand, prevents students from understanding the behavior of exponents in non-natural-number powers. For example, if exponentiation is repeated multiplication then something to the power of zero might be seen as ambiguous—should it be zero or one? Moreover, there is no meaning to multiplying something by itself a negative number of times. An extension to fractional powers that preserves the sense of repeated multiplication is an impossible task, and often fractional exponents are presented as "a different way to write radicals" and connected to repeated multiplication in an artificial way. As they finish their high school unit on exponents, student end up not being able to look at exponentiation as a *continuous* process (in terms of the exponent value) and, in the best case scenario, have a few different models connected to one another through loose logic.

Although it is possible to develop the concept of exponentiation as a case-based operation, this approach may result in various negative implications, such as the perception that exponentiation is always an increasing operation, an inability to work with exponents as continuous functions later in the curriculum and difficulties in understanding the rate of change of an exponential function and its derivative. One other problem that is rarely addressed in the mathematics education literature lies in the fact that students do not have any qualitative sense of the changing growth rate of an exponential function that results in an inability to explain such ideas as compounding or population growth without having to calculate its value numerically.

Existing Approaches

There have been cases of teachers who tried to develop students' understanding of the exponents' rules through the process of proof and mathematical consistency by moving from one rule to another in deductive manner with the goal of allowing the students to see the connection between them. This, however, does not create a single view of exponents that students can work with across domains, and it always remains as a sequence of logical operations that explains the various cases of exponents (positive, negative, zero, and rational). The research community has made several attempts at developing a conceptual understanding of the concept of exponents while at the same time aiming at building a single view of the operation across domains.

The Functional Approach

The teaching of algebra in a functional-based approach was first suggested by Goldin and Herscovics (1991), later to be tested by Elstak (2007) in a teaching experiment. In this approach, the understanding of negative, zero and rational exponents comes from constructing the definition of an exponential function starting from natural exponents, and later investigating this function to expand the notion to other domains. In Elstak's teaching experiment, some of his students were able to logically connect the different cases of exponents, but could not give a single definition for all of them.

Developing Algorithms

Weber (2002) suggested that students be presented with a description of an algorithm to compute exponents, which later they express formally. The students wrote the exponential expressions as products of factors, and then completed activities in which they debated about the nature of rational exponents. Although Weber explained the expected learning path using APOS theory (Dubinsky, 1991), it was not clear how the "debate" stage helped the students develop a conceptual understanding of rational exponents that aligned with their original definition of the operation.

Exponents as Splitting

Confrey and Smith (1994, 1995) suggested that exponents be developed through the idea of splitting. Basing their design on students' familiarity with the idea of fair share and splitting, they gave special attention to the rate of change of the function, which was not given by other researchers. They aimed at developing in students the multiplicative comparison between the sizes of the quantity at different stages, with a focus on a multiplicative rate of change as being fixed throughout the work with exponents, in contrast to the varying additive rate of change. This basis allows students to develop a comprehensive view of exponents of positive base with integer powers (positive, negative, and zero). To extend the domain to rational bases and exponents, they relied on the contrast between the counting and the splitting worlds and offered a logical explanation that accounts for rational exponents. In contrast to their initial work that relied on splitting, the expansion to rational exponents does not offer a cognitively intuitive model to work with, and the case-based view is left unresolved.

What Is Missing?

Although some of those studies contain valuable insights, each has its own drawbacks with regards to the development of a comprehensive conceptualization of exponentiation as a continuous operation relying on a single image. One shared limitation of all of the above approaches is that they begin working in natural numbers and give the students a limited concept of exponents, later to be expanded in one way or another. Expansion to the real number domain is accomplished through formal explanations, definition or logical reasoning, but is not based on the original image the students have. Even in the splitting world, which is an extremely powerful idea based on students' own experience, there is no real meaning to the exponent itself that accounts for both positive and negative values. It ends up being a logical process explaining to the students that in the new "multiplicative" world negative exponents result in division. Rational powers are similarly not well addressed in the splitting world ending up being understood empirically through the identification of patterns. It is extremely hard to think of doing three and a half split operations since the splitting is done on countable sets. In essence, trying to build students' understanding based on a limited domain (natural number) as a starting point requires that students adjust their definition of exponents for every new domain extension, sometimes conflicting with the original one they had, resulting in a disconnected set of definitions.

Another limitation of the approaches is an overuse of calculations as a way to "understand" exponents, indicating a move towards an empirical instead of conceptual understanding. This is evident in the algorithmic and functional approaches, and even in the work based on splitting, there is still some emphasis on generating sample values in order to understand rate of change.

Theoretical Frameworks

One goal when designing the following teaching trajectory is that it will allow students to develop a coherent understanding of the concept of exponentiation as a continuous operation with no contradictions as the domain of the exponent expands from positive whole numbers, to negative whole numbers (and zero), to fractional exponents, and finally to all real numbers. In order to design such a trajectory and to analyze its usefulness, I am using two different theoretical frameworks, which I argue to be complementary (Simon, 2009), and offer principles for the arrangement of the content and provide constructs that explain the learning of the student.

For the overall content organization approach and the analysis of the core understandings that are the foundation of the concept of exponents, I used the design principles as laid out by Davydov (1975). One of Davydov's main points is that the material should be organized, moving from the general to the specific. That is, the students should move from learning about the concept in its most general form and later to work through different cases which are manifestations of it in such a way that they develop a complete understanding of the concept. A principal goal in designing the teaching trajectory was to develop an image of exponents that students can repeatedly use in different cases without ever having to deal with contradictions. This is significantly different than building on the specific case of positive whole numbers and later expanding it to negative and fractional exponents.

A guiding example of the development of a concept from general to specific is the development of multiplication (Davydov, 1992). Davydov asserted that multiplication should not be viewed as repeated addition, raising similar issues to the ones described earlier with exponentiation being viewed as repeated multiplication. In his measurement-based approach, he distinguished between the way one *calculates* the value of a multiplication expression and the *image* of multiplication. Creating an image of multiplication as a change of unit of measurement allows Davydov to develop with his students a more general view of multiplication that also includes multiplication of fractions. My work builds on the previous work not because it is intended to address the issue of teaching multiplication, but because this example is used to demonstrate how it is possible to create a single image of a concept that can be used across cases.

However, Davydov's framework cannot be used to analyze the learning of the students from a cognitive perspective, and to complement it I used Simon et al's (2010) framework for learning through activity, building on Piaget's (2001) concept of reflective abstraction. This framework seeks to explain the transition from the point at which a student did not understand something to the point at which he or she does understand it (p. 77). The following are three principles I used from Simon et al's framework: the importance of a learner's activity, the importance of the learner's reflection, and the distinction between reflective abstractions and empirical learning processes (p. 74). I also note the goal-directed nature of the student's activity, as well as the development of the logical necessity that brings the student to anticipate the relevant results.

Principles in Developing the Concept of Exponents

The development of a single image of exponentiation is the key to developing a *continuous* concept of exponents, including natural and rational bases, as well as real numbers as exponents. The following are guiding principles for designing a trajectory that leads to such concept:

Separating the Image and the Calculation

Students should be able to consider the calculation of any exponential expression separately from the image of exponentiation. Whereas the calculation of positive, negative and rational exponents may involve different operations, creating a single image of exponents will allow them to reason about those as one concept and not as a disconnected set, and carry the same image they expand their domain of operation.

Beginning with a Qualitative Understanding of Exponents

One of the challenges in working with exponents is that the resulting values of the operation quickly become too big (or too small). This does not allow students to develop a real sense of the rate of change since they need to manipulate large numbers at the same time. Beginning with a qualitative (vs. quantitative) experience of exponents that builds on a visual representation will allow the students to build an understanding of the operation's effects and develop an expectation for the result of using different bases, without ever calculating the numerical value of the operation.

Using Physical Quantities of a Continuous Nature

In order to develop in students the understanding of exponents in the integer and rational domains, students should not be limited to working with discrete quantities. Quantities such as length, area, and volume allow them to perform the operation of exponentiation using any value for the exponent and experience the change in a continuous way. Moreover, these quantities are also measureable and visible so they lend themselves to the use of multiple representations.

Using Whole Numbers as a Case within the Continuous Domain

Beginning the calculations of exponents with whole numbers is a way for students to find the result of the operation. The suggested approach introduces whole numbers only once the *image* of exponents is established and without changing it moves to whole numbers as a particular case of *calculating* the result.

Building on Students' Intuitive Models

Confrey and Smith used a concept from the students' world that can be used intuitively as a basis for developing the mathematical concept, and it provides a strong foundation and an important entry point for the teaching trajectory.

Applying several of these principles together poses a great challenge. Continuous models in nature (e.g., temperature change, decay of matter) tend not to be directly measurable which means they cannot be used in a physical activity. The other models are mostly discrete (similar to the ones shown in the splitting approach). An activity that uses a quantity such as length and requires the students to change its size by, for example, stretching it, does not result in an exponential change. In order to create a true exponential change in a quantity, the students have to consistently change the power/speed they use while stretching the quantity, which is not a natural thing to do. In order to overcome such difficulties the suggested teaching trajectory relies on a technological environment that allows the students to change the exponent *continuously* and observe the change in the quantity.

What Is Exponentiation?

The starting point for building a continuous view of exponentiation is accepting that repeated multiplication is only a way to *calculate* exponents in the case of whole numbers, and not the *image* of it. This was understood by previous researchers—Confrey and Smith, for example, developed the image of exponentiation as a splitting operation and repeated multiplication was a way to calculate the number of elements after a sequence of splits. Even though exponentiation is a multiplicative idea at its core, this idea is not enough for building a mental model that students can relate to as they expand the concept and learn to calculate it in different cases.

One of the main differences between addition, multiplication and exponentiation is that the last one is an operation which produces a changing growth/reduction rate. It is also the basis for understanding the extreme changes that happen when exponentiation is repeated.

The basis for understanding the changing rate of exponentiation is that the growth/reduction in exponentiation is always *relative* to the quantity it operates on. Building on that understanding, I define the following *image* of exponentiation as the goal for the trajectory:

Exponentiation is an operation that *changes* a given quantity *multiplicatively* based on the *current size* of the quantity.

A few notes with regards to this image of exponentiation:

1. Through defining exponentiation as a general changing operation, students are not limited to thinking that "exponentiation makes bigger" which is sometimes assumed from the definition of repeated multiplication. The image can be used in the cases of positive as well as negative and fractional exponents.
2. The multiplicative rate as the change factor allows for fractional rate bases, which is important for building the single view across bases that was the goal of the instruction.
3. There is an underlying assumption, although not explicit, that the students can think of a change as related to a given size, meaning a proportional change.
4. The question of calculating the value of the exponential expression is left unanswered in this definition. It is essential that the students have a single model to build on, and that they see the calculations as based on specific cases within that image.

Developing the image of exponentiation is a first step in helping the students to construct the different cases of this image (in the various domains for bases and exponents). The image of exponentiation is reinforced by the use of symbolic language in a more formal mathematical definition and model which includes the following components:

1. Q_0 – the quantity being operated on, at the initial stage (time=0)
2. a – the multiplicative rate of change (natural or fractional) in a given time unit
3. x – the change in time units (as related to the time of the initial stage)
4. Q_x – the resulting quantity at time=x (this can be at any point in time)
5. a^x – the accumulated multiplicative change of the initial quantity (Q_x divided by Q_0).

Representation Model

The use of multiple representations and the utilization of dynamic control and manipulation, made possible using technology, can have a positive impact on students' learning in algebra (Kieran, 2006) and these are incorporated as part of the representational model. In addition, the representation must allow the use of continuous quantities and make it possible for the students to engage with the image without relying on specific calculations.

The Context of the Problem

Based on the desired characteristics of the image of the concept of exponentiation as described earlier, it is essential that the change described in the problem is one that is based on the current size of the quantity being changed. The suggested context problem (other variations exist of course) is the following:

Magic caterpillars need to eat a certain amount of leaves. The length (amount) that each caterpillar needs to eat is proportional to the caterpillar's length. The eating transforms the caterpillar - they end up being as long as the length of the leaves they ate (for simplification they "eat" in straight lines). We will examine the changes in the length of the caterpillars.

The context was chosen to support the development of a complete view of exponentiation:

1. It provides freedom in the selection of the initial quantity to be operated on, as well as the introduction of various bases, in the form of different types of caterpillars that change differently, being more (or less) hungry
2. Its structure allows for the introduction of both growth and decay, since the proportion of the food they need might be smaller or bigger than the caterpillars themselves
3. In the general form of the problem there is no mention of any period of time in which the food needs to be consumed. It will be introduced as a "feeding cycle" later as the students move to the quantitative section in which they calculate the values of the growth. This also allows for the use

of multiple-size feeding cycles, which is the basis for working with any rational exponents, and developing a continuous view of the operation

The Visual Representation

The representation includes two critical components:

(a) Dynamic graphical representation of the caterpillar. Using a narrow rectangle as a representation of the caterpillar allows the students to focus on the length as the relevant quantity. Also, using the length representation answers the need for a continuous quantity, which can grow or shrink dynamically. This eliminates the problem caused by using discrete properties.

(b) Horizontal time axis with a slide bar. The use of a slide bar represents a continuous view of the exponent value allowing the students to work in fractional and integer values. Offering a dynamic manipulation of one element, the use of a slide bar is a known representation for time progression which is familiar to students (consider YouTube for example).

Bringing all the pieces together, and using the elements of the mathematical model with the representation model and the problem context, we have the following:

1. Q_0 is the size of the caterpillar at the given initial state (when the observation begins)
2. "a" is the property of the caterpillar which defines how much food it eat at each time unit
3. "x" is the change in time represented by the slide bar. The location of the marker on the slide bar shows how far they are (and in which direction) from the initial state
4. Q_x is the size of the caterpillar (the rectangular bar) at any point in time (the slide bar)
5. a^x is the result of calculating (or predicting) the change from the initial to the current state

Analysis of the Hypothetical Learning

The affordances of the definition and model described above are best demonstrated through an analysis of a student's *hypothetical* learning process. What follows is a description of the *hypothesized* learning of a student who performs two activities from the first task in a teaching trajectory which is based on the principles above. A full description of the teaching trajectory and analysis of the hypothesized learning as for each step is beyond the scope of this paper.

During the first three activities of the trajectory (not detailed here), the students work with quantities (the length property of the caterpillars, with no particular numerical value but of a comparable magnitude) that change in either exponential or linear form and understand that exponential growth (or decay) is faster than linear one, once the quantity reaches a certain size. They can explain the relationship between the size of the caterpillar and its growth (or decay) and focus on the fact that the bigger the given quantity is, the bigger the change is. Also, students are introduced to the formal mathematical notation. They now move to the fourth activity.

Activity 4: Comparing Different Quantities

In this activity, the student works with two caterpillars with the same base for the exponent and compares their growth, in absolute (additive) and proportional terms. Working in the same base and the same time period with different-sized caterpillars focuses the student on the initial size of the caterpillars. In reflecting on his activity he is expected to understand the logical necessity of the initial quantity explaining the difference in the resulting quantity, building on his prior knowledge of the relationship between the quantity and the change, and knowing that the initial quantity is the only attribute in which the caterpillars vary. I describe now the steps in the activity that lead to this understanding.

In the first step of this process, the student works with one caterpillar and establishes its growth rate, as in previously activities. He does this by using the slide bar marked with units as before to change the time value and compare the resulting size. From this, he can see that the growth is based on a particular base rate. For example, he might observe that the caterpillar grows by a multiple of 3 for every time unit in the case of a base of 3. He does this by comparing the quantity after one time unit, with the quantity at the beginning, or the quantity after x+1 units, with that of after x units.

Once the growth rate is determined, another quantity of a bigger size is introduced and placed next to the original one. The student is told it needs to eat the same proportion as the previous one, and is asked to predict which one will grow more. The student is expected to develop an understanding that the growth of a caterpillar is proportional to its original quantity, and is assumed to know that when multiplying two numbers by the same factor, the larger number will result in a larger product. Building on these two, the student will anticipate that the larger caterpillar will grow more, because "3 times a bigger number is bigger." This understanding is powerful since the student learns about a relationship which is not dependant on whether the given quantity is of a whole or fractional size. At the end of this activity the student concludes that for an equal amount of time units and the same growth rate, the larger quantity will grow by a larger amount. It is important to note that the student generalizes about exponents *without* looking for a numerical pattern, and that the process he goes through is based on reflection about the general process and the understandings developed about the nature of this activity.

Activity 5: Moving Forward and Backward in Time, Only Until the Present

In this activity students can move the scroll bar forward and backward between the present time and a particular point in the future (just to avoid infinite growth/decay in the future). They begin with a quantity of a particular size and are asked to explain whether it is growing (eating more than its size) or shrinking (eating less than its size), based on their observation of the behavior. This should be a simple question for the students, knowing they completed the previous activities in which they learned that when a caterpillar eats more than its size, it grows every day (towards the future) and the opposite in decay. The students are then asked to explain whether the quantity grows bigger or smaller if you move forward in time. The students are then asked to move the bar to a random point on the time line and answer the following question "in order to reach this state and from what you know of the behavior of the caterpillar, would the quantity had to be bigger or smaller before this point in time." Students, building on their "forward" thinking and activity of moving the bar forward from before, see the logical necessity that in order to reach a certain quantity, when a caterpillar is growing, it had to be smaller before (and similarly for decay it had to be bigger before). They are then asked "Knowing that a caterpillar is 'growing' caterpillar, if you look back in time, would that caterpillar be bigger or smaller?" and "how would it look if you move forward in time?" Although this can be checked by the students through the representation, at this point they already see the necessity of the caterpillar being smaller, since it had to grow to reach the given size (in the case of growth). The students learn to anticipate that if the caterpillar is of a growth type ($a>1$), then when moving forward in time (increasing the exponent) the quantity grows, and when moving backward in time (decreasing the exponent) it becomes smaller.

At the end of those five activities, students develop the concept of exponentiation as a change which has a rate proportional to the current size of the quantity, as a factor of time. Also, students can explain the change in a quantity when moving both forward and backward in time, which will serve as a precursor for the development of zero and negative exponents as places on the time line. Moreover, the use of a continuous time line sets the stage for other "time points" which will not be whole numbers. Students never used particular values for quantities so they never calculated the exponential change, and that supports the development of a continuous view of exponentiation and keeping the image of exponentiation separate from the calculation process.

Developing Continuous Concepts in Mathematics

The development of mathematical operations such as addition, multiplication and exponentiation usually begins with the positive whole number case and expands later to negative and rational numbers. I began this paper presenting the implications such an approach might have on the student and suggested principles for designing a teaching sequence for the development of one of those concepts in a continuous manner. Although focused on exponentiation, the importance of the work might be beyond a particular content area, and perhaps also it can serve as an example of how other continuous concepts might be thought of in such fields as calculus and operations in algebra.

Moreover, the approach laid out here might also serve as basis for designing the development of continuous concepts in general. The value of using two different theoretical frameworks is revealed through the examination of the hypothetical learning sequence in which the student on the one hand begins with the general image of exponentiation, as suggested by Davydov, coming from a socio-cultural perspective, but at the same time, develops the understanding which is explained from a constructivist perspective. The use of one framework as the leading one for the overall design of the sequence and another framework to design the activities within the sequence might have applications in other conceptual areas.

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THE IMPACT OF ONLINE ACTIVITIES ON STUDENTS' GENERALIZING STRATEGIES AND JUSTIFICATIONS FOR LINEAR GROWING PATTERNS

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This study explores the impact of working with online learning activities of linear growing patterns (CLIPS) on students transitioning into Grade 9. Fifty students were interviewed about their understanding of linear growing patterns. Twenty-five students had participated in a study involving an experimental instructional approach that emphasized exploration of multiple representations of linear relationships when they were in Grade 8. They were then assessed five months later, and their reasoning compared to twenty-five students who did not take part in the study. Results indicate that students who worked with CLIPS were able to find explicit, generalized rules for patterns and offered higher levels of justifications than their counterparts. These students were also more likely to refine their thinking.

Keywords: Algebra and Algebraic Thinking; Instructional Activities and Practices; Technology

Context

Studies have shown that the transition from primarily arithmetic thinking in elementary school, to algebraic thinking in high school, is difficult for most students (Kieran, 1992). This transition entails moving from a focus on mathematical operations (addition, subtraction, multiplication, division) to thinking about relationships between sets of numbers, and identifying generalized mathematical structures with or without specific numeric values. Traditional algebra is often initially presented in high school as a pre-determined syntax of rules and symbolic language to be memorized by students. Students are expected to master the skills of symbolic manipulation before learning about the purpose and the use of these symbols. In other words, algebra is presented to students with no opportunity for exploration or for meaning making (Kaput, 2000).

In response, a series of online learning objects was designed as an alternative way to introduce the concepts of algebraic relations, specifically linear relations, to Grade 8 students prior to formal algebraic instruction in Grade 9. The activities are based on an approach that emphasizes the observation of relationships among quantities, and among multiple representations, which allows for the construction of understanding rather than rote memorization of procedures. As part of a larger long-term study, I have been investigating the affordances of this instructional approach that prioritizes visual representations of linear relationships – specifically, the building of linear growing patterns and the construction of graphs (e.g., Beatty 2010). Previous research on the lesson sequence has shown that it supports students' progression from working with linear growing patterns as an anchoring representation to considering graphical representations of linear relationships. Students also make connections among different representations – pattern rules, patterns and graphs (Figure 1).

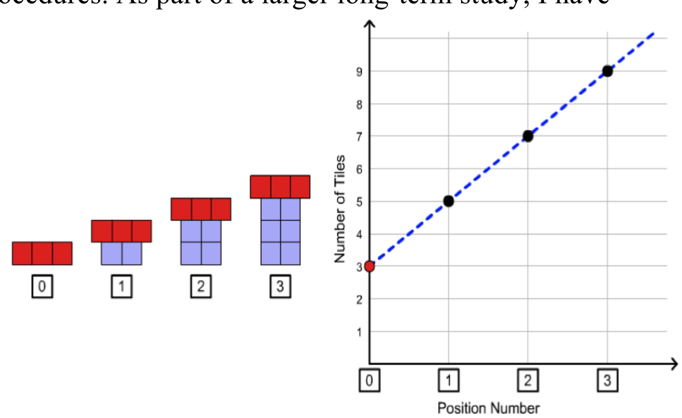

Figure 1: Representations of the rule y=2x+3

The online activities, called CLIPS LGP (Critical Learning Instructional Paths Supports – Linear Growing Patterns) were designed using Flash technology and offered the possibility of combining a proven instructional sequence with unique properties of digital technology. The online activities were integrated into the instruction in five classes of Grade 8 students. The students accessed the online activities for 2 months in order to develop an understanding of linear relationships via linear growing patterns. As part of the instruction, students were supported to develop sophisticated generalizing strategies by considering the explicit relationship between the term number of a pattern and the number of tiles in the pattern, and to express this relationship using pattern rules such as "the number of tiles is equal to the term number x2+3" or "tiles = term number x2+3." Students also engaged in classroom discussions based on the online activities, and developed a disposition for providing justifications for their pattern rules.

In this study we wanted to assess how much content material was retained by these Grade 8 students as they transitioned into Grade 9. We also wanted to compare the problem-solving processes of Grade 9 students who had been part of the CLIPS study in Grade 8 with those who had not in order to determine whether there was a difference in students' generalizing strategies and justifications.

Developing Generalizing Strategies

A main component of algebraic reasoning is the ability to generalize. In the domain of linear relations, particularly when thinking about linear growing patterns, a generalization can be thought of as the articulation of a pattern rule that applies across all cases in the situation (for example figure numbers and number of toothpicks in a linear growing pattern.) Studies have shown that students have difficult moving from particular examples (for instance, focusing on particular iterations of a pattern) towards creating generalizations (a generalized pattern rule that holds for infinite iterations of the pattern). Numerous researchers have reported that the route from working with liner growing patterns to finding generalized rules (and later, algebraic expressions for those rules) is difficult (Kieran, 1992; Orton, Orton & Roper, 1999; Noss et al., 1997). However, we have found in our previous studies that the instructional approach that underpins the CLIPS activities has facilitated students' abilities to find and articulate general rules for linear growing patterns (Beatty & Bruce 2012).

Researchers have identified many generalizing strategies that students adopt when working with problems involving linear relations, including problems based on linear growing patterns (Lannin, 2005; Mason, 1996; Lee, 1996). Below we identify three of these strategies from the least to the most sophisticated. They are presented with reference to a well-known generalizing problem (one that we used in our study), the Toothpick Trees problem (Figure 2). In this problem, students are shown a series of Toothpick Trees and asked to predict how many toothpicks would be needed to build the 10th figure, and how many would be needed to build the 100th figure.

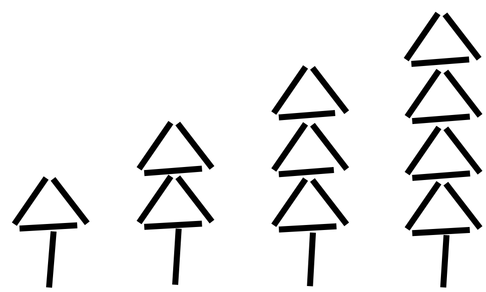

Figure 2: The toothpick trees problem

Counting strategy. Students draw a picture or constructing a model to represent the situation in order to count the desired attributes. For example, students draw the 10th figure and count the number of toothpicks required. The limitation of this strategy is evident to students when they are asked to predict the number of toothpicks for the 100th figure.

Recursive reasoning strategy. Students build on the previous term in the sequence to determine subsequent terms. In our example, students would state that the rule for the pattern is "add 3 each time." To find the 10th figure they add three to the 4th figure, then three more to the 5th figure etc. This strategy generally results in the correct answer to predict the number of toothpicks for "near" terms of a pattern (for example, figure 10) but is problematic for finding the 100th term. It also does not allow for the articulation of the rule, which would allow for the prediction of the number of toothpicks for *any* figure.

Explicit reasoning strategy. Students construct the explicit rule that expresses the co-variation of two sets of data, based on information provided in the situation. An explicit rule can allow for the prediction of any term number in the pattern. An example of an explicit rule would be, "the number of toothpicks is equal to the figure number x3+1."

Research suggests that when working linear relationships and linear growing patterns it is rare for students to go beyond limited kinds of mathematical generalizations – namely counting or recursive reasoning – primarily because these are the strategies that are supported by traditional approaches to teaching patterning and algebra (Noss et al., 1997). However, the instructional approach in CLIPS prioritizes explicit reasoning in order for students to determine and articulate the mathematical structure of linear growing patterns.

Importance of Justification

When students justify their solutions strategies they are able to provide reasoning and evidence to validate their generalization. This has been found to be challenging for most students (Lannin, 2005). However, providing a justification for a generalized rule helps students to see the generalized relationships that exist in the problem context. Just as there is a framework for generalization strategies, there is also a five-level framework for justification strategies (Table 1) (Simon and Blume, 1996). Higher levels of justification have been shown to support higher levels of generalization (Lannin, 2005).

Table 1: Levels of Justification

Level	Descriptions
0	No justification.
1	Appeal to external authority. Reference is made to the fact that a solution is correct because it is stated by some other individual (teacher or a peer who is regarded as more successful) or some other reference material.
2	Empirical evidence. A justification is provided through the correctness of particular example but with no indication of an understanding of <u>why</u> the rule is correct. For instance, "The rule is 'add 3' because for the first figure there are four toothpicks, then you add 3 more for figure 2."
3	Generic example. Deductive justification is expressed for a particular instance, a generic example, which the students uses as a proxy for "any" instance. For example, "I know the rule is "toothpicks = figure number x3+1" because for, say, the fifth figure, there are five triangles, and five times three is fifteen. And then there is one more, so plus one is sixteen."
4	Deductive justification. Validity is given through a deductive argument that is independent of particular instances. For example, "At any figure number, the number of triangles equals the figure number, so that means multiplying the figure number by three, and then there's always an additional one for the trunk."

When working with CLIPS, students engaged in classroom discussions during which they were encouraged to justify their solutions by making connections between their solutions and the context of the problem, with a focus on deductive reasoning.

Methodology

Participants

Fifty Grade 9 students participated. Of these, 25 students had been part of the CLIPS study and 25 had not. The students were drawn from 8 different classrooms in two different school boards with equal number of CLIPS and non-CLIPS students selected from each classroom. Students were interviewed individually for approximately 30 – 35 minutes.

Data Sources and Analysis

In order to track the content knowledge and algebraic reasoning of students, we chose to conduct task-based clinical interviews during which students were asked to describe what they were thinking while solving ten linear relationship problems. This form of interview opens a window into the participants' content knowledge, problem-solving behaviours and reasoning (Koichu & Harel, 2007; Schoenfeld, 2002). In this study, the clinical interviews were semi-structured, which allowed for prompting or questioning students in order to clarify our understanding of the students' reasoning. Validity of the subjects' verbal report corresponds to the extent to which the subjects' talk represents the actual sequence of thoughts mediating solving an interview task (Clement, 2000; Ericsson & Simon, 1993). Therefore, all interviews were digitally video recorded so that verbal report and non-verbal gestures were captured in order to develop a comprehensive analysis of student thinking.

Overall students answered five items that were taken directly from the CLIPS activities, which we termed "near transfer" items because they test the retention of understanding of items that are similar to items students experienced while working with CLIPS. The other five items came from sources such as TIMSS (Third International Math and Science Survey) and NAEP (National Assessment of Educational Progress). We termed these "far transfer" items because they are dissimilar to the CLIPS content, and so assess understanding of underlying conceptual concepts. For this report we will focus on students' responses to one "far transfer" item – the Toothpick Trees problem described above. The students were asked to predict how many toothpicks would be needed for the 10^{th} and 100^{th} figure, and to explain their thinking. However, we did not explicitly ask for a pattern rule in order to determine whether students would use the information presented in the patterns to formulate a general rule that would give the number of toothpicks required for any figure of the pattern.

The scoring guide for these items, based on the generalization framework, is given below.

Table 2: Scoring Guide for Generalization Strategies

Score	Description
0	Incorrect answer
1	Counting strategy. The student drew out the figure(s) and then counted the number of matchsticks/toothpicks (drew out the 10^{th} figure, drew out the 4^{th} to the 10^{th} figure).
2	Recursive strategy. The student has articulated the rule as "add three more each time" or created an ordered table of values that increased by three each time.
3	Explicit strategy. The student has articulated the explicit rule as "matchsticks = figure number x3 + 3" and "toothpicks = figure number x3+1"

Video recordings of task-based interviews were transcribed and coded. Codes were based on the generalization and justification frameworks outlined above.

Results

Generalizing Strategies

Table three shows the level of generalizing strategy demonstrated by students who had experienced CLIPS and those who had not.

Table 3: Generalization Strategies Used by CLIPS and Non-CLIPS Students

	Score 0	Score 1	Score 2	Score 3
CLIPS	0	0	7	18
Non CLIPS	11	8	4	0

Most CLIPS students used an explicit generalizing strategy to find a general rule using the context of the problem in order to find the correct solution. In contrast, many non-CLIPS students did not find a viable solution, and those that did used a counting strategy or recursive reasoning strategy, rather than finding an explicit pattern rule.

Counting strategy. Eight of the 25 non-CLIPS students used a counting strategy for this problem, meaning that they drew the 10th figure and then counted the number of toothpicks. A striking finding was that all eight of the students who used a counting strategy could not make the connection between counting by threes, and multiplying by three. For example, in the transcript below the student is not able to transition to multiplicative thinking in order to predict the 100th term.

Malinda: Well I drew it out and counted and found out that it just kept adding three. So I drew it to the tenth and then counted them to find the right number of toothpicks.
Interviewer: How are you counting? Can you count out loud?
Malinda: Three, six, nine, twelve, like that?
Interviewer: Yeah.
Malinda: (pointing to each triangle as she counts) Three, six, nine, twelve, fifteen, eighteen, twenty-one, twenty-four, twenty-seven, thirty plus one is thirty-one. I was counting by three's.
Interviewer: Could that help you think about how many toothpicks you'd need for any figure number? Like the 100^{th} figure?
Malinda: Um…well…I know that I would draw out three more for each figure. So I would just keep drawing three for each next figure, and then count by threes.

It should be noted that Malinda, like most of the students we interviewed, was considered to be, and considered herself to be, capable of engaging in mathematical operations like multiplication. However, the majority of students who were not part of the CLIPS study could not make the connection between "counting by threes" and multiplying the term number by three.

Differences in recursive thinking. The seven CLIPS students who used recursive thinking created a generalized rule "add three each time plus the one for the trunk" that took into account both the multiplier and the constant of the rule with reference to the figural context of the problem.

> I added three toothpicks every time, like one triangle every time up to the tenth one and got thirty toothpicks and then you have to add the stump part to it. So it's always going up by three each time, but with the little stump so you add one for that. So for 100 you'd add 3 100 times and then add 1.

The two non-CLIPS students who used recursive thinking articulated their pattern rule as "start with four and add three." This is a common way that students are taught to articulate linear growing rules. Students were able to find the number of toothpicks for the 10th tree, but then simply guessed for the 100th term.

> Ok so I started at the first tree with four toothpicks. Then as it added we still had the four and I added it to each new tree every time. So then every time I'd get an answer with three more. I'd added 4 and three and three and three and keep count of where I was until I hit the tenth figure. Then that was my answer. For 100 it would be…maybe 101? I don't know.

Explicit thinking. Eighteen CLIPS students found a generalized rule for this pattern. Most of the students used language and concepts that are part of the CLIPS LGP instruction.

> I looked for the rule. So I could see the triangles were growing, so that meant the multiplier would be times 3. And then the one that stays the same, that's the constant. So for 10 it would be 10x3+1, which is 31. And for 100 it would be 100 x 3 is 300 plus 1 is 301.

Justification Strategies

Transcripts were coded for the level of justification offered by students as they articulated their thinking during the task-based clinical interview. Justifications were scored from Level 0 to Level 4, based on the framework outlined above. Table 4 below shows the level of justification provided by students who had experienced CLIPS and those who had not.

Table 4: Levels of Justification Given by CLIPS and Non-CLIPS Students

	Level 0	Level 1	Level 2	Level 3	Level 4
CLIPS	0	0	2	7	16
Non CLIPS	11	2	8	0	0

Seven CLIPS students offered a Level 3 justification for their rules. An example of a Level 3 justification is as follows: "The tenth tree would have three triangles, which is 3 times 10 or 30. And then you add the one, so it's 31."In this example the student refers to a particular figure number to describe how each component of her rule relates to how her rule determines the number of toothpicks. She notes the ten groups of 3 for the ten triangles in the tenth figure and explains the need to add the one extra for the trunk. In this case a particular example is used to communicate generality across all cases.

Sixteen CLIPS students offered a level 4 justification: "I know my rule is correct because you just multiply the figure number by the group of three for the triangles because the figure number tells how many triangles there are, and triangles are always going to be 3 toothpicks. And then the little trunk means you always add one more." Students clearly explain why the rule applies to all cases of the situation by relating it to the context of the pattern. The students describe the "groups of three" they see in each of the patterns, so the multiplier x3 represents the number of toothpicks in these groups. The extra 1 toothpick (for the trunk) is added to the rule to express the total number of toothpicks needed for any figure number. Unlike a generic example, this justification does not describe a particular instance. Instead, it describes a general relationship that would apply to any case.

In contrast, the majority of non-CLIPS students did not offer a justification, and when asked why their rule worked replied, "I'm not sure" or "because it just does." Two students offered justification that were scored as Level 2, empirical reasoning based on the correctness of one specific example, but with no demonstrated understanding of why this was correct, or how their solution was related to the context of the problem. "From the first to the second tree you add three more so the rule is plus three."

Students' Refining Their Own Thinking

One of the most striking results revealed by an analysis of the interview transcripts is the extent to which students who had been part of the CLIPS project refined their thinking during the course of the interview as a result of explaining their solution process. Overall, there were 28 such episodes coded in the interview transcripts for the 25 CLIPS students. However, there were no such episodes coded for the transcripts of the non-CLIPS students. Non-CLIPS students did not revise their thinking, and when they discovered an error between their rule and the problem context (their rule would lead to an incorrect number of toothpicks) they either gave up or were not aware of the disconnect between their rule and the values given in the problem.

The CLIPS students were more likely to try to find an alternative solution strategy, or to refine their answer based on new evidence. In this example, Deepak had briefly looked at the first figure of the pattern and written "x4." He was then asked to explain his rule. During his explanation, Deepak realized that he had misperceived the structure of the pattern, and that his perception did not coincide with the numerical value of the pattern for each figure number. Rather than dismiss this discrepancy, Deepak went to work to try to discern a rule that would work with all iterations of the pattern.

Deepak: Well I just looked at figure 1, and found that the tree is made up of 4 toothpicks, so the rule is figure number times 4 So, if you look at figure 2, it would be 2 times 4 which is 8, and there are…wait…oh that's not right.

[Deepak spent 54 seconds working on a new rule.]

Deepak: Ok I see what I did wrong. I didn't see it right, I thought the whole tree was made of 4 not 3. But if you check the numbers, it's growing by 3, so the three that are growing are these three that make up the triangles so it's times 3. And then the trunk is made of one, so it's plus 1. And that works with all the figures. So figure 3 is 3 times 3, 9, and then plus 1, 10. So the 10^{th} figure would have 31.

Interviewer: So how many toothpicks would be needed for the 100th figure?
Deepak: Easy! I can just use the rule! 301!

Conclusions

One focus of this study was to assess the enduring understandings students developed while working with the CLIPS activities, and how much of this understanding was retained during the five months from the end of the instructional intervention (June, 2010) to the time of the interviews. The interviews were held near the end of the first semester of school, during November and December 2010. The students had not yet had any formal instruction in linear relations. Given that our intervention was relatively short, these results indicate that students retained a great deal of understanding both of content material, and of the importance of providing justifications for their answers.

Another focus of the study was to compare the thinking of Grade 9 students who had been part of the CLIPS study with those who had not. There were two main areas of algebraic thinking that we assessed – the level of generalizing strategy used by students when solving linear problems, and the level of justification offered for their solutions. We found differences in the kinds of generalization strategies used by CLIPS and non-CLIPS students. In this study, students who had not been part of CLIPS, but who had experienced traditional approaches of instruction, had great difficulty in finding generalized rules for patterns. For those who did find a correct solution for the tenth figure of a linear growing pattern, their solutions were based on counting or, less frequently, recursive reasoning. These limited solutions strategies allowed students to find the number of toothpicks for the tenth figure (a near generalization), but did not aid them in finding the number for the one hundredth figure (a far generalization).

In addition, few students who had not been part of the CLIPS study offered any kind of justification for their solutions. In contrast, students who had been part of CLIPS tended to offer level 3 and 4 justifications. They explained their solutions using the context of the problem, and could articulate why their general rule would work for any case.

Finally, there was the unexpected, yet striking result, of the extent to which CLIPS students revised their thinking. This happened numerous times with the CLIPS students, who, during the course of explaining their solutions, caught and corrected their own mistakes. Past research suggests that students typically do not attempt to revise their rules (Bednarz, Kieran, and Lee, 1996; Mason, 1996; Stacey, 1989). In fact, Cooper and Sakane (1986) suggest that once students select a rule for a pattern, they tend to persist in their claims even when finding a counter example to their hypothesis. Students would rather refute the data presented than modify their original rule. This was the behaviour we observed with the non-CLIPS students, who had no interest in revising their rules.

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TRANSITIONING FROM WHOLE NUMBERS TO INTEGERS

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This paper presents the results of a pre-test, instruction, post-test study that investigated students' integer mental models and how their models changed based on instruction. Sixty-one first graders' responses to questions about the values and order of negative numbers were categorized according to a series of mental models. The models reveal that initially, students over-rely on the values and/or order of whole numbers to varying degrees when working with negative numbers. Focused exploration on the properties of negative numbers helped students transition to more sophisticated mental models compared to only learning about moving in positive and negative directions on number lines.

Keywords: Number Concepts and Operations; Elementary School Education

Purposes of the Study

During students' progression through early elementary school, one of the foundational mathematical understandings they must develop is that of numbers (National Research Council, 2001; National Council of Teachers of Mathematics, 2000). During this time, students learn the counting sequence, numeral names, and numerical values. In particular, students must coordinate their knowledge of number values and order—understanding that moving one number forward in the counting sequence corresponds to a one-unit increase in value—so that they can use this information to solve addition and subtraction problems (Griffin, Case, & Capodilupo, 1995; Fuson, 2004). As students move along the continuum of numerical understanding, difficulty arises when they must expand their number knowledge to include negative numbers and rethink how numerals relate to numerical values: 10 is big in terms of positive numbers but small when made negative; the inclusion of negative numbers also requires students to reevaluate the meanings of addition and subtraction (e.g., adding no longer always results in a larger number) (Bruno & Martinon, 1999; Küchemann, 1980; Murray, 1985). Further, students must navigate the changing meanings of the minus sign (it can mean an operation, a negative sign, or an indication to take the opposite), which later interferes with students' ability to reduce polynomials and manipulate algebra problems (Gallardo & Rojano, 1994; Vlassis, 2004, 2008).

Past research on negative number instruction and student learning explored whether students could learn how to solve integer addition and subtraction problems from a particular method of instruction (Linchevski & Williams, 1999; Schwarz, Kohn, & Resnick, 1993; Thompson & Dreyfus, 1988) or whether one method of instruction is more effective than another (Liebeck, 1990; Janvier, 1985). Results from these and other studies highlight strategies students use to solve integer problems. For example, to solve $-4 + -3$, a student might add $4 + 3 = 7$ and then add a negative sign to get -7 (Bofferding, 2010). Another student might treat negatives as worth zero and get an answer of -4 because adding zero will not change the answer (Bofferding, 2011).

What might account for these differences? One influence on students' solutions to integer addition and subtraction problems is how they think about the values and order of negative numbers in comparison to positive numbers. However, research does not provide a clear picture of what the transition from whole number to integer understanding looks like. This study contributes to this area by addressing the following research questions: (1) What are first grade students' mental models of negative integers in relation to order and value (the elements underlying the central conceptual structure of numbers)? (2) How do students' mental models change based on integer instruction?

Theoretical Framework

The *central conceptual structure of number* (CCSN) or *mental number line* is a mental model or internal structure hypothesized to support numerical thinking (Case, 1996; Griffin, Case, & Capodilupo, 1995). This structure involves four components: number word order (i.e., the counting sequence), a tagging routine for counting objects, numerical values, and written symbols. Although young children may learn how to say the number names in order, they do not initially use this process to quantify sets; similarly, they might compare two sets of objects visually instead of counting and comparing them numerically (Sophian, 1987). Eventually children coordinate the four components, creating the fully integrated CCSN. By referring to this mental model and counting up and down their mental number line, students can add and subtract single-digit numbers (Case, 1996; Griffin, Case, & Capodilupo, 1995).

As students learn about negative numbers, they need to modify their CCSN. Because there are no true physical manifestations of negative numbers, students need to reason that numbers further to the left on the mental number line are smaller than numbers to their right, even if before zero and even if they contain the same numerals. Therefore, while -2 and 2 look similar and are equally far away from zero, the number further to right on the number line is greater. Students must also wrestle with the changing meaning of the minus sign; for example, –4 – –3 does not mean subtract 3 twice (Murray, 1985) but take away –3 from –4.

Situations, such as this one, where students must reorganize their knowledge structures, involve conceptual change. In the context of numbers, students have *initial* mental models that numbers are discrete, which arise from their experiences with objects (Gallistel & Gelman, 1992; Vamvakoussi & Vosniadou, 2004). Students who have an initial mental model for negative numbers might ignore the negative signs and treat the numbers as if they are positive; for example, they might place negative numbers next to their positive counterparts when ordering them (Peled, Mukhopadhyay, & Resnick, 1989). As children have more experiences with a concept, they begin to restructure their initial mental models in order to deal with new, conflicting information; this process results in one of many *synthetic* or *intermediary* mental models. For example, when students learn that negatives exist, they might think that –7 is greater than –3 because 7 is greater than 3 (Stavy, Tsamir, & Tirosh, 2002). With time and experience students will eventually develop the *formal* mental model, where negatives are ordered opposite their positive counterparts on the number line—with zero separating positive and negative values—and numbers decreasing in value to the left and increasing in value to the right. The goal of this study is to address the research questions by using the lens of conceptual change to investigate how students' mental models for the CCSN change during the transition from whole number to integer understanding.

Methods

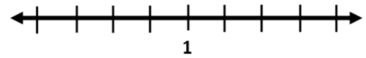
Participants and Site

This study took place at the end of the school year in a diverse elementary school in California. It was important to recruit students who had initial mental models for negative numbers so that I could explore the transitions they make as they learn about the topic. Based on pilot work, I selected and recruited students at the end of their first grade year. Overall, 61 first graders participated.

Materials and Data Collection

Interviews. The study consisted of a pre-test, instructional intervention, and a post-test. Both pre- and post-tests were conducted as individual interviews by a trained graduate student and me. Additionally, the two tests had the same questions, although with different numbers. The goal of the questions was to determine students' understanding of negative number values, order, and symbols (see Table 1). Although we also asked students arithmetic questions, results for those data are not discussed here. After students solved a question, we asked them to explain how they solved it, and students did not receive feedback on whether they answered correctly.

Table 1: All Pre-Test and Post-Test Questions for Each Category

Category	Number	Pre-Test Questions	Post-Test Questions
Counting	1	*Start at five and count backwards as far as you can. Can you count back any further? Is there anything less than <last number said>?*	
Number Line	1	*Fill in the missing numbers on the number line.* 	
Ordering	2	*Put these number cards in order from the least to the greatest. Which is the least? Which is the greatest?*	
		a. $\boxed{2}$, $\boxed{-3}$, $\boxed{0}$, -$\boxed{9}$, $\boxed{3}$, $\boxed{8}$, $\boxed{-5}$	a. $\boxed{5}$, $\boxed{-2}$, $\boxed{-8}$, $\boxed{1}$, $\boxed{7}$, $\boxed{-4}$
		b. $\boxed{6}$, $\boxed{-6}$, $\boxed{4}$, $\boxed{-7}$, $\boxed{3}$, $\boxed{-1}$	b. $\boxed{3}$, $\boxed{8}$, $\boxed{-9}$, $\boxed{-7}$, $\boxed{0}$, -$\boxed{5}$
Greater	7	*What are these two numbers? Circle the one that is greater.*	
		a. 8 vs. 6	a. 6 vs. 4
		b. 3 vs. –9	b. 5 vs. –7
		c. –2 vs. –7	c. –3 vs. –1
		d. –5 vs. 3	d. –8 vs. 4
		e. –8 vs. –2	e. –6 vs. –2
		Two children are playing a game and trying to get the highest score. Circle who is winning.	
		f. Abigail: 4 vs. Joseph: –7	f. Amy: 5 vs. Ken: –9
		g. Crystal: –7 vs. Leon: –3	g. Dan: –8 vs. Will: –6

Instructional intervention. Based on their performance on the pre-test, students were stratified and randomly assigned to one of three instructional groups. The groups were designed to provide students with differing levels of exposure to negative numbers so that I could explore the ways in which their mental models change. I taught each instructional group for eight, 45-minute sessions and followed strict lesson plans to maintain proper instructional treatments.

The *Integer Operations* group only had exposure to negatives. They learned how to use a number line to move *more positive, more negative, less positive, and less negative,* but they did not learn specifics about negative numbers. The *Integer Properties* group learned about the values and order of negative numbers as well as how to tell the difference between positive and negative numbers and minus and negative signs. Finally, the *Combined Integer Instruction* group had three lessons in common with the *Integer Properties* group and five in common with the *Integer Operations* group. Therefore, they learned the order and value of negatives as well how to use movements of more and less on a number line to add and subtract integers.

Data Analysis

To analyze the data, I first transcribed students' responses to the pre- and post-test questions and classified them as correct or incorrect. The counting and number line tasks needed to include negative numbers to be considered correct. Based on the methods employed by Vosniadou and Brewer (1992) to identify mental models of the Earth, I first formulated possible integer mental models using previous results from the literature (Peled et al., 1989) and the CCSN as a guide. For each test, I then coded students according to the mental model that would account for the pattern of their responses to the interview questions. As I coded, I created additional mental model categories when necessary to capture differences in students' responses that were not depicted by the initial mental models hypothesized from the literature. As found by Vosniadou and Brewer, some students fell into categories that were mixed versions of two other categories, which suggests they have inconsistent mental models. Once all students were categorized according to an integer mental model, I compared the percentage of students with each mental model in each instructional group before and after instruction.

Results

Overall, across the pre-test and post-test, students demonstrated a variety of mental models for integers, reflecting initial, intermediary, or formal understanding of the concept.

Initial Mental Models (in ascending order)

Whole number mental model. Students with a whole number mental model treated all negatives as if they were positive. When counting backward they stopped at zero or one, their number lines only included whole numbers, they ordered negative numbers next to positive numbers (e.g., 0, 3, –5, –7, 8, –9) and choose greater integers based on absolute value (e.g., –9 > 3).

Continuous zero mental model. Although students with a continuous zero mental model also treated negatives as if they were positive, these students used repeating zeroes on their number lines and/or when counting backward as demonstrated by Student 403: "Five, four, three, two, one, zero, zero, zero." Similarly, Bishop, Lamb, Philipp, Schappelle, and Whitacre (2011) had three first graders label multiple zeroes when playing a number line game. Their treatment of zero suggests that these students have some idea that the mental number line continues indefinitely less than one.

Absolute value mental model. The absolute value mental model is the first which involves students noticing a difference between positive and negative numbers. Students with this mental model separated the negative numbers from the positive numbers when ordering them (sometimes correctly) but continued to claim that negative numbers have the same values as their positive counterparts. Therefore, while they might have correctly ordered negatives before zero, they still claimed that –9 is greater than 3 and –7 is greater than –2.

Symbolic mental model. Students with the symbolic mental model correctly counted into the negatives. Additionally, when filling in the number line, they often did so correctly but used their own notation to indicate negative numbers. For example, one student wrote "N3" for negative three, while another student used an "X" instead of "N." Consequently, when ordering and determining which integer was greater, they treated all negative numbers as if they were whole numbers because the problem notation did not match their invented notation. This result highlights the importance of the role of symbols in the central conceptual structure of integers. It is possible they could have ordered the numbers using their own notation, but, unfortunately, this was not tested. Therefore, the students' understanding in this category may be understated.

Ordered nothings mental model. A few students not only separated negative numbers from positive numbers when ordering them but also treated negatives as worth zero. One student justified her order of the numbers (–3, –5, –9, 0, 2, 3, 8) explaining that "nine minus nine is zero," as is five minus five and three minus three. Later, when comparing –2 and –7 she explained that although 7 is greater than 2, both of these numbers were zero.

Intermediary Mental Models (in ascending order)

Separated value mental model. The separated value model is the first example of a mental model where students start to add to their CCSN instead of trying to fit negative numbers into positive number rules. Students with this mental model could correctly order integers and could determine the larger of two negative integers *or* the larger of two positive integers. However, when given a group of positive and negative integers, the students ignored the negative numbers when determining which number in the group was greatest or least. Therefore, a student in this category might say that –3 is greater than –5 and 2 is greater than –3 but when given –5, –3, 2, and 8 would say that 2 is the least. This behavior may arise from students thinking that anything less than zero is not a real quantity.

Equal/Unequal magnitude (mixed) mental model. Students with this mixed model gave responses consistent with having a whole number mental model for some order and value questions and a magnitude mental model for others.

Magnitude mental model. Students with a magnitude mental model ordered negatives before zero but either reversed their order (e.g., –1, –6, –7, 3, 4, 6) and claimed that –7 is greater than –2 or ordered the negatives correctly but thought that numbers further away from zero were larger, again claiming that –7 is greater than –2.

Equal/Unequal integer (mixed) mental model. Several students treated negatives as positive while ordering them but then correctly determined the greater of two integers for all combinations of integer pairs. These students had both whole number and integer mental models.

Dual value (mixed) mental model. Students with the dual value mixed mental model always identified negatives as smaller than positive numbers but sometimes treated negatives further away from zero (e.g., –9) as larger and sometimes correctly identified them as smaller than negative numbers closer to zero (e.g., –1). It is possible that students with these mixed mental models were in the process of transitioning from relying on one mental model to the other. On the other hand, their responses could have been influenced by the context of the question.

Formal Integer Mental Model

Students with a formal integer mental model correctly ordered negative numbers in relation to positive numbers and consistently identified the greater integer regardless of whether two or several integers were presented.

Changes in Integer Mental Models

While students across instructional groups had a similar spread of mental models on the pre-test (15–17 students in each group started with initial mental models), there were several changes by the post-test. Table 2 shows how students' mental models shifted in each instructional group. The cells show the percentages of students in each instructional group who started with a particular integer mental model (initial, intermediary, or formal) on the pre-test and ended with a particular mental model on the post-test. Students who started with initial mental models and shifted to formal mental models made the greatest transition.

As shown in Table 2, most students in the Integer Operations group (who did not learn about the properties of negatives) started with initial mental models on the pre-test and still had initial mental models on the post-test. On the contrary, most students in the other two groups transitioned from having initial mental models on the pre-test to having intermediary or formal mental models of integers on the post-test.

Table 2: Percentage of Students with each Mental Model on Pretest and Post-test by Instructional Group

		Mental Models on (Pretest, Post-Test)					
Group	***n***	**(I, I)**	**(M, M)**	**(F, F)**	**(I, M)**	**(M, F)**	**(I, F)**
Combined Instruction	20	20%	0%	10%	**35%**	5%	**30%**
Integer Properties	20	5%	5%	10%	**35%**	10%	**35%**
Integer Operations	21	**62%**	5%	14%	10%	0%	10%

Note. I = Initial Mental Model; M = Intermediary Mental Model; F = Formal Mental Model. A student who falls in the (I, I) category demonstrated an initial mental model on both tests.

To test whether the average differences in mental model advancement were due to the instructional treatments versus chance, I used the Kruskal-Wallis 1-way ANOVA by ranks (Shavelson, 1996). Students who had reached ceiling (i.e., had formal mental models) on the pre-test were eliminated from the analysis, which resulted in 18 people per instructional group. Results indicate a significant effect for instructional group ($H_{observed}$=9.8064, $H_{critical}$=5.99, α<.05). Pairwise comparisons reveal that the Combined Instruction (mean rank = 31) and Integer Properties (mean rank = 34) groups improved significantly in terms of developing more formal mental models for negative numbers than the Integer Operations group (mean rank = 17) (HSD = 7.76, α < .05). There was no significant difference between the Combined Instruction and Integer Properties groups.

Students in the Combined Instruction and Integer Properties groups both had instruction on the properties of negative numbers; however, only one student from the Integer Properties group continued to have an initial mental model for integers at the post-test. Furthermore, this student progressed from treating all negatives as positive (Whole Number Mental Model) to interpreting the value of negative numbers as different from positive numbers (Ordered Nothing Mental Model). On the other hand, four students in the Combined Instruction group continued to have initial mental models for integers at the post-test, and all four treated negative numbers as if they were positive (Absolute Value and Continuous Zero Mental Models). This difference is suggestive that spending focused time (more than 3 days) on integer properties helped students in the Integer Properties group develop more advanced mental models. This result would need to be studied further, though, to determine how generalizable it is and whether it can be replicated.

Discussion and Implications

The results of this study highlight mental models that students develop for integers and provide insight into the process through which students, who have developed the central conceptual structure of number, expand or restructure their conceptual structure to include integers. As found in other research on conceptual change (Vosniadou & Brewer, 1992), the students' initial mental models for integers were constrained based on their current knowledge (in this case, their knowledge of whole numbers). Their understanding of how order, values, and numerical symbols relate led students to interpret negative numbers as a different type of positive number. Some students with initial mental models ordered negatives apart from positive numbers but did not use this information to reason that the values would also be different; meanwhile, others sorted them separately and considered negatives as amounts taken away, equivalent to zero. Further, students with intermediary mental models knew that negatives were less than zero but had not associated them with the order of numbers definitively.

Identifying students' integer mental models can help teachers better understand students' incorrect solutions to integer arithmetic problems. Further, the results of this study have implications for how curricula and instruction could support students' learning of abstract concepts and help move them along the continuum of numerical understanding. Based on the integer mental models identified, several concepts to emphasize in integer instruction over several lessons include distinguishing between the negative sign and the minus sign, understanding the symmetrical nature of the number line, distinguishing between greater distances from zero versus greater numerical values, and identifying the difference between negative numbers and zero. As with other notation and representations, students need formal explorations with negatives because exposure to them does not make their structure obvious. Students in the Integer Operations group did not learn the structure and values of negative numbers through hearing the words "positive" and "negative" without associating the words with their symbols and meaning. Instruction on negatives (and other abstract ideas) needs to help students see the structure of the concept: there is symmetry in the number sequence when zero separates the positive from the negative numbers, numbers increase as we count up through the sequence, and the minus sign takes on new roles (Vlassis, 2008). Furthermore, teachers can use familiar representations to extend students' thinking; classroom number lines or number paths should continue beyond zero into negatives (even in the younger grades), so that as students learn about negative numbers, they receive constant visual feedback of their existence.

Although curriculum developers, researchers, and policymakers continue to place negative numbers late in the curriculum—6th and 7th grades (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010)—this study demonstrates that students are quite capable of learning about integers much earlier than fifth grade. In fact, students in this study were able to learn about negatives as early as the end of first grade, although whether this would be beneficial in the long term remains to be investigated.

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PRE-FRACTIONAL MIDDLE SCHOOL STUDENTS' ALGEBRAIC REASONING

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To understand relationships between students' quantitative reasoning with fractions and their algebraic reasoning, a clinical interview study was conducted with 18 middle and high school students. Six students with each of 3 different multiplicative concepts participated. This paper reports on the 6 students with the most basic multiplicative concept, who were also pre-fractional in that they had yet to construct the first genuine fraction scheme. These students' emerging iterating operations facilitated their algebraic activity, but the lack of a disembedding operation was a significant constraint in developing algebraic equations and expressions.

Keywords: Algebra and Algebraic Thinking, Cognition, Rational Numbers

Based in part on recommendations that improved fractional knowledge is critical for success in learning algebra (National Mathematics Advisory Panel [NMAP], 2008), researchers are starting to investigate how students' fractional knowledge is related to their algebraic reasoning (e.g., Hackenberg & Lee, 2011; Empson, Levi, & Carpenter, 2011). Since this research is in its infancy, there are numerous unexplored issues. One issue is how students who conceive of fractions primarily as parts *within* wholes may be challenged when working on algebra problems. These students' challenges may extend beyond the limitations of their fractional knowledge. However, little is known about how these students' fractional knowledge may assist or limit them in building basic algebraic ideas, such as making generalizations from quantitative relationships (Ellis, 2007; Kieran, 2007) and operating on unknowns (Hackenberg, 2010).

To understand relationships between students' fractional knowledge and their algebraic reasoning in the area of equation writing, a clinical interview study was conducted with 18 middle and high school students. Six students with each of three multiplicative concepts (Steffe, 1994) were invited to participate. These concepts have been found to significantly influence students' fractional knowledge (Hackenberg, 2010; Steffe & Olive, 2010), and they are based on how students produce and coordinate composite units (units of units).

The six students with the most basic multiplicative concept also conceived of fractions primarily as parts within wholes and had not yet constructed the first "genuine" fraction scheme, a *partitive fraction scheme* (Steffe, 2002, p. 305). So, these six students could be considered pre-fractional. That meant that the students did *not* conceive of a fraction like three-fifths as three one-fifths, related to but distinct from the whole. Instead, they thought of three-fifths as embedded within the whole—as five parts with three shaded. This view of fractions relies on being able to separate a quantity represented by a segment or rectangle into parts, a mental action we refer to as *partitioning*. However, it also relies on *not* being able to disembed a part from the whole while keeping the whole mentally intact, a mental action we refer to as *disembedding*. In general, pre-fractional students can learn to partition, but they do not yet disembed.

The purpose of this paper is to investigate relationships between the fractional knowledge and equation writing of the six pre-fractional students in the study. The research questions are:

1. How do pre-fractional students solve algebra problems that involve writing equations to represent relationships among unknowns?
2. How do pre-fractional students solve algebra problems that involve generalizing activity?
3. How are students' pre-fractional ways of operating related to their equation writing and generalizing activity?

A Quantitative and Operational Approach

Quantitative Reasoning

We conceive of students' quantitative reasoning as a basis for building fractional knowledge and algebraic reasoning (Smith & Thompson, 2008; Steffe & Olive, 2012). Approaching fractions as quantities means that we pose problems to students in which fractions are measureable extents, or lengths; these lengths may represent other quantities as well (e.g., weight). Approaching algebraic reasoning from a quantitative perspective means that unknowns are quantities for which a value is not known, but for which a value could be determined. So unknowns are potential values of quantities. In working with students we routinely ask them to make drawings of quantitative relationships, and we aim for students' fraction and algebraic notation to trace the quantitative reasoning in which students engage.

Operations, Schemes, and Concepts

Our work is also based on conceiving of mathematical thinking in terms of people's mental actions, or *operations* (Piaget, 1970; von Glasersfeld, 1995). Operations critical for fractional knowledge include *partitioning* and *disembedding* as mentioned above, as well as *iterating*, which is repeatedly instantiating a fractional part to make a larger fraction. Operations such as these are *interiorized* physical actions—that is, they arise from re-processing physical actions in such a way that they can be performed mentally, without having to be carried out materially.

Operations are the components of *schemes*, goal-directed ways of operating that consist of three parts: an assimilated situation, activity, and a result (von Glasersfeld, 1995). For example, if a student has constructed a partitive fraction scheme, then a situation of the scheme is a request to make a new length that is 3/5 of a foot. The activity of the scheme involves partitioning the foot into five equal parts, disembedding one of those parts, and iterating the part to make three such parts. The student then assesses the result of her activity in relation to her expectations.

For us, a *concept* is the result of a scheme that people have interiorized. For example, a student who has interiorized the result of her partitive fraction scheme can take that result, three-fifths consisting of three one-fifths, as a basis for carrying out more activity. This student could engage in problems such as determining what the result of partitioning each of the fifths into two equal parts would be, or how to re-make the whole if the given length is three-fifths of the whole.

Characteristics of Pre-Fractional Students

Students who are pre-fractional struggle in a variety of ways. For example, Olive and Vomvoridi (2006) have analyzed the case of Tim, who had not constructed a partitive fraction scheme by his sixth grade year. At that time, one feature of Tim's fraction scheme was that both a unit fraction and the whole referred to the same partitioned image: One-sixth meant a whole partitioned into six equal parts, and six-sixths meant the same partitioned whole. This idea about fractions led Tim to add up parts regardless of size. For example, in adding ½ and ¼, Tim said the answer would be 1/5 because ½ was one part and ¼ was four parts.

In short, pre-fractional students can engage in equal-partitioning of lengths (Biddlecomb, 2002; Steffe & Olive, 2010), but they cannot take a partitioned length as given prior to engaging in activity. For example, to share a 1-foot length of licorice fairly among five people, these students have to actually partition—they cannot imagine the partitioned length prior to making it. Research also shows that constructing a disembedding operation requires a significant reorganization of these students' ways of operating that can take as long as two years (Steffe & Cobb, 1988; Steffe & Olive, 2010).

Methods

Seven seventh grade students, 10 eighth grade students, and one tenth grade student participated in this clinical interview study. Participant selection occurred via classroom observations, consultation with students' teachers, and one-on-one, task-based selection interviews to assess students' multiplicative concepts. Six students with each multiplicative concept were invited to participate; this paper focuses on the six students with the most basic multiplicative concept. Three of these students were enrolled in a seventh grade mathematics class for struggling students; the other three students were taking an eighth grade pre-algebra class. The three seventh grade students and one of the eighth grade students received special education support for one period per day. All pre-fractional students had received some instruction in their mathematics classes on unknowns and equation solving.

Students participated in two 45-minute, semi-structured interviews, a fractions interview and an algebra interview. All students completed the fractions interview prior to the algebra interview, but the time between interviews varied from 3 weeks to 4 months. The interview protocols were refined in a prior pilot study (Hackenberg, 2009) and were designed so that the reasoning involved in the fractions interview was a foundation for solving problems in the algebra interview. For example, one fractions interview task was the following: “A 65-cm stack of CDs is 5 times the height of another stack. Can you make a drawing of the situation and determine the height of the other stack?” In the algebra interview, students were posed a similar situation but both heights were unknown. Students were asked to make a drawing and write equations to represent the situation. In addition, students completed a written fractions assessment (Norton & Wilkins, 2009) to triangulate claims about their fractional knowledge. This assessment confirmed that the students identified as pre-fractional were pre-fractional.

Each interview was video-recorded with two cameras, one focused on the interaction between the researcher and student, and one focused on the student’s written work. The videos were mixed into one file for analysis, which occurred in three overlapping phases. The first phase of the analysis was to formulate a model (Steffe & Thompson, 2000) of each student’s fraction operations, schemes, and concepts; equation writing and solving; and generalizing activity, to the extent possible over two interactions. Toward this end, the researchers viewed videofiles and took detailed analytic notes (Cobb & Gravemeijer, 2008), which included transcriptions, data summaries, memos, and conjectures. The resultant models provided the basis for responding to the first two research questions for this paper.

In the second phase of the analysis, the researchers looked across the students to articulate differences in how students with different multiplicative concepts solved the problems in each interview. Products of this phase included written syntheses of the ways of operating of students with a particular multiplicative concept, which provided an important backdrop for responding to the three research questions in this paper. Finally, in the third phase of analysis researchers examined how the operations, schemes, and concepts that constituted students’ fractional knowledge were involved in students’ equation writing and generalizing activity. This phase was the basis for responding to the third research question for this paper.

Analysis and Findings

Equation Writing and Multiplicative Relationships

Two of the six pre-fractional students, with significant coaching, wrote equations to represent multiplicative relationships between unknowns that were correct from the researchers’ perspectives. Our analysis suggests that the students’ emerging iterating operations were one reason these students were successful, but that the lack of a disembedding operation was a major source of the difficulties that even these two students experienced in conceiving of unknowns in multiplicative contexts. In this section we present one student’s work on the first problem in the algebra interview to substantiate these claims.

The first problem in the algebra interview was the following:

> A1. *Cord Problem.* Stephen has a cord for his iPod that is some number of feet long. His cord is five times the length of Rebecca’s cord. Could you draw a picture of this situation? Can you write an equation for this situation? Can you write another equation?

Initially all pre-fractional students made a drawing for A1 in which one of the lengths (represented by a segment or rectangle) was a little more than half of the other. Only two students refined their pictures to make a more accurate representation by iterating a shorter segment five times to make a longer segment. Only one of these two students, 7th grader Henry, wrote a multiplicative equation for A1 that was correct from the researchers’ perspectives.

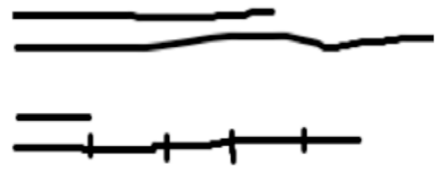

Figure 1: Henry's work on A1

In Henry's initial picture for A1, the segment representing Rebecca's cord length was longer than Stephen's, and Stephen's segment was a little more than half of Rebecca's (Figure 1, top two segments). Without any intervention from the interviewer, Henry reinitiated his activity and drew a small segment. Then he drew a copy of that segment below, and he proceeded to draw four more copies, pausing after each copy (but not lifting his pen). So he repeated one cord length five times, and this new segment represented the other cord length (Figure 1, lower two segments; hash marks have been added for clarity). Henry called the long segment Rebecca's and the short segment Stephen's. However, Henry switched these meanings when the interviewer restated the problem. Spontaneously initiating the repeating of a segment was novel, and it suggested that Henry had constructed an iterating operation that he would need for constructing more advanced fractional knowledge.

Henry's initial equation for A1 was "$S \cdot$ O = Rcord," which he said meant "Stephen's cord times what Rebecca's cord is, equals Rebecca's cord." He said that he wrote an "O" to "leave it open," since he did not know the length of Rebecca's cord. Then, in discussion with the interviewer about how many of Rebecca's segments would fit into Stephen's in Henry's picture, Henry generated a correct equation, "$R \cdot 5$ = Stephens cord." In explanation, he changed the 5 to a 4, saying, "No, Rebecca's times four equals Stephen's cord, 'cause she already has one [of the segments]." This conflation suggests the lack of a disembedding operation: Rather than consider Rebecca's cord as one part of Stephen's, Henry appeared to think of Stephen's as five parts, one of which had to be Rebecca's, leaving Stephen with only four parts.

The researcher then posed a numerical example in order to test the equation: "Let's say Stephen's cord length is 15 feet; how long is Rebecca's cord?" Henry spent nearly 6 minutes determining Rebecca's cord length. He initially thought it would be 10 feet. Then he tried 5 feet and arrived at 15 feet. He appeared to be iterating an amount three times, because then he said "three, 9 feet." In this process, Henry extended the segment for Stephen's cord length by another segment the size of Rebecca's cord length, so Stephen's length then consisted of six segments (later Henry crossed off this part following questioning from the interviewer). We note that confusing "five times" and "five more than" is a sign of not having constructed iteration (Steffe & Olive, 2010, p. 182). So, despite Henry's later correction of his drawing, this work throws some doubt on whether Henry had indeed constructed an iterating operation for segments.

To explain his answer of 9 feet for Rebecca's length, Henry counted by threes along the first four parts of Stephen's segment, and then counted by ones ("13, 14, 15") along the fifth part. When the interviewer repeated back to Henry how he had counted along Stephen's segment, Henry changed his mind. "Yeah, hers is like three feet," he said.

Finally, the interviewer asked Henry about his equation—whether he wanted to use 4 or 5. Henry said four, although under questioning he agreed that 3 x 4 was not 15. Following that exchange, Henry changed his equation back to "$R \cdot 5$ = Stephens cord." When asked for any other equations he could write for the situation, he wrote "$5 \cdot 3 = 15$, $3 \cdot 5 = 15$, and $3 \div 15 = 5$."

Our current interpretation of Henry's work relies on the fractional operations we could attribute to him. Although the evidence is not incontrovertible regarding Henry's operation of iteration, Henry appeared to have something like iteration available—or becoming available—based on how he made his drawing for A1. The spontaneous change that he independently made in his drawing allowed him to create a quantitative foundation for his algebraic work that was a key reference during the rest of his activity. Since only one other pre-fractional student in the study made a similar drawing, we infer that creating this kind of drawing to show one segment and another that is five times longer is not a trivial achievement for a pre-fractional student.

In addition, although Henry received significant support from the interviewer in order to write a correct equation, we suggest that his emerging operation of iteration allowed him to make sense of the support that the interviewer offered in terms of questions about his picture. In contrast, none of the other pre-fractional students wrote a similar equation with similar questions—not even the pre-fractional student who generated a drawing similar to Henry's.

However, we suggest that Henry's lack of a disembedding operation, as shown most clearly in him changing the 5 to a 4 in his equation, was a constraint for him. That is, although with the support of the interviewer's questions Henry did return to a correct equation, it's not clear whether using 5 was a logical necessity for him. Indeed, without a disembedding operation it would be unlikely for Henry to make sense of a segment that is five times another, because that relationship appears to require thinking about the other segment as both embedded in and disembedded from the longer segment. So, without that operation, it would be more natural for Henry to think of the longer segment as "four more" than the original segment. This analysis indicates that writing equations representing multiplicative relationships between quantities would be quite challenging for students without disembedding operations.

Making Generalizations: Solving The Border Problem

In contrast to their work on A1, five pre-fractional students solved parts (a), (c), and (d) of the Border Problem, which has been used to introduce ideas of unknowns and variables to middle school students (Boaler & Humphreys, 2005):

A7. *Border Problem.* Below is a 10 by 10 grid with the squares on the border shaded.

(a) Without counting one-by-one, and without writing anything down, can you find a way to determine how many squares are on the border?
(b) Can you find another method?
(c) Can you apply your first method to a 6 by 6 grid?
(d) How would you describe in words how to use your first method on any grid?
(e) How would you use algebra to write an expression to communicate your first method to someone?

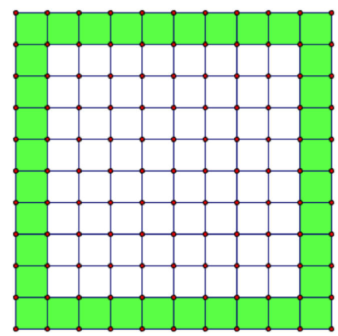

Figure 2: Border problem

All six pre-fractional students initially thought that there were 40 squares. Upon counting to check, five students adjusted their initial idea based on observations about counting the corner squares of the grid twice. Two students adjusted by subtracting 4 from 40. Three students, including Henry, adjusted by adding 10 and 10 for the top and bottom sides, and then adding 8 and 8 for the left and right sides (eliminating both corner squares from these sides). All five students applied their method to a new grid, a 6 by 6 grid, and verbally described their method to some degree. The two most detailed verbalizations were from Henry and another 7th grade student, Courtney, which we state below.

Due to time constraints, only these two students were asked to use algebra to communicate their methods (part (e)). Courtney said she did not know how to do that, even after discussion with the interviewer about using a letter to represent the number of unit squares in one row of the grid. However, Courtney did then apply her method correctly to a 15 by 15 grid without drawing that grid. Henry also had a discussion about part (e) with the interviewer, who suggested that x could represent the number of unit squares in one row. After asking Henry what x was in each of the first two grids (the 10 by 10 and 6 by 6), the interviewer asked if Henry could use x to write down an expression for the number of squares on the border. Henry wrote "x = top row 10" and then underneath "x = top row 6." Then he added the 10 and the 6 to get 16. So, no student made a correct solution to part (e) from the perspectives of the researchers.

Yet the five students who solved parts (a), (c), and (d) did generate a method for determining the number of squares on the border, used it on a grid of different size, and verbalized the patterns they observed. We assess that in doing so, they engaged in two forms of generalizing activity (Ellis, 2007): They extended their reasoning beyond the range in which it originated, and they began to identify commonalities across cases. However, we propose that the students' lack of a disembedding operation constrained the nature of their generalizations and prevented them from writing an algebraic expression. Although these conclusions were made from analysis of all data, we use Henry and Courtney as examples for explaining them, in part because these two students demonstrated some of the more advanced thinking of the pre-fractional students.

Henry's generalizing activity. In describing in words how to use his method on any size grid (part (d)), Henry said, "I'd tell them to do the top first, see how much in a row it would be [pointing at a row]. And then do the bottom, which is the same. And then after that, like, whatever number's at the end [corner], go to the next box [down] on the other side and put, like, put how much it is. Don't use the same number two times." When asked if by his last statement he meant "don't count a corner square again if you've already counted it," Henry agreed. The interviewer then asked how Henry knew, in the 6 by 6 grid, that the other side had to be four, and whether the four had any relationship to the six. Henry's response was inaudible. When the interviewer asked the same question about 8 and 10 in the 10 by 10 grid, Henry said "Huh?" and proceeded to label his drawing with numerals.

From this data excerpt, we conclude that Henry did not articulate the relationship between the 4 and the 6 and the 8 and the 10 structurally. In other words, he did not appear to see 4 as embedded in 6 and also separate from the 6 in terms of the side lengths of the grid (and similarly for 8 and 10). This means that in thinking about the grid he did not disembed 4 from 6 (or 2 from 6) while leaving the 6 intact—and we infer he did not do so because he had not constructed a disembedding operation. His comments do provide evidence that he knew two different numbers should be involved—that a person can't just add the same number four times as he initially did. But the lack of a disembedding operation contributed to Henry's generalization about adding the number of unit squares in the top and bottom rows, and then adding a different pair of numbers for the other sides of the grid. This generalization might lead to writing something like $x + x + y + y$ as an algebraic expression, but it would not lead to something like $x + x + x - 2 + x - 2$.

Courtney's generalizing activity. In contrast with Henry, Courtney subtracted 4 from 40 in solving the Border Problem. In writing down her method, she first wrote multiplication signs in between each of the four tens, changing them to addition signs under questioning from the interviewer. In verbally describing her method, she said, "Since a square has ten [sic] sides, on each one, I'd add ten plus ten four times and then I subtracted four 'cause I counted all four ends [corners], and I counted them twice. So I subtracted four since there are four sides. For the 10 by 10 I got 40 and then I subtracted 4 and I got 36." To clarify, the interviewer asked Courtney why she added 10 four times, and Courtney said it was because the square had four sides. At this point the interviewer did not probe for clarity about reasons for subtracting four. However, prior to this data excerpt and within it, Courtney said she subtracted four in order to not count corner unit squares twice. In fact, except for stating that she subtracted four due to there being four sides, her generalization does not seem problematic in any obvious way.

Yet based on our model of Courtney, we claim that she did not write an algebraic expression for her generalization in part because of how she thought about the "ten plus ten four times" and the 40. Since we knew Courtney had not constructed a disembedding operation, we knew that taking a number (such as 10) some number of times was a significant cognitive load for her (Steffe, 1994). When students like Courtney take a number (10) and repeat it, they do not consider these numbers (four 10s) as both embedded in and disembedded from the result (40). Instead, it's like the tens disappear after they have been used. In short, Courtney's method really was not 10 + 10 + 10 + 10 – 4, structurally; we infer she did not generate awareness of the 10s as segments of the 40 in the process of and after computation. This conclusion is supported by Courtney's multiple ways of describing and notating the use of 10s to make 40. This analysis indicates that it was quite reasonable for Courtney *not* to know how to use the interviewer's suggestion to let x represent the number of squares in one row in writing an expression for her method, since the relationship between the number of unit squares on each side and the total number of border squares was rather ephemeral for her.

Discussions and Conclusions

This study contributes to understanding why pre-fractional students struggle with algebra. In particular, it suggests that these students' iterating operations may facilitate their representation of multiplicative relationships between quantities, and that these students' lack of a disembedding operation is a significant constraint in developing algebraic equations and expressions. It also suggests that students' fractional operations shape the generalizing activity in which they engage. For example, without a disembedding operation, pre-fractional students will be unlikely to distinguish amounts that are both contained within and separate from other amounts in a quantitative situation—and doing so is critical for creating a structural view of many situations that can be represented with algebraic notation.

Implications for algebra instruction for pre-fractional middle school students include the pressing need to develop curricular materials that provide support for helping these students advance their fractional and algebraic knowledge simultaneously. These materials need to be based on the ways and means of operating of the students so that these students will not be left out of making mathematical progress, and so that their mathematical thinking will not remain invisible or under-valued in an environment where extant curricular materials assume operations that these students are yet to construct.

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FRACTION OPERATION ALGORITHMIC THINKING: LEVERAGING STUDENTS' USE OF EQUIVALENCE AS A TOOL

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The data used for the qualitative analysis reported here were generated as part of a larger study to understand and characterize teacher practice related to engaging students in algorithmic thinking associated with the fraction operations of addition, subtraction, multiplication and division. This paper presents ways in which teachers used students' emergent ideas to leverage the use of equivalence as a tool, rather than a procedure, to support students as they work to develop algorithms for operating with fractions.

Keywords: Rational Numbers; Instructional Activities and Practices; Middle School Education

Purpose

Prior work on teacher practice acknowledges the complexity of instruction when teachers aim to engage students in authentic mathematical activity where the instructional path is not specified and teachers themselves engage in sense-making as they make instructional decisions (Ball & Bass, 2003; Kazemi & Stipek, 2001; Stein, Smith, Henningsen, & Silver, 2000). In their review of the collective literature on teaching and classroom practice, Franke, Kazemi and Battey (2007) offer that effective teaching involves more than having a rich task or eliciting students' thinking. They argue that the field would benefit if the complexity of teacher practice were examined using a domain-specific approach leading to the identification of routines of practice, or core activities, that should occur regularly within particular mathematical domains.

From an instructional perspective, fraction operations are especially complex (Lamon, 2005; Ma, 1999; Borko et al., 1992). The literature (e.g., Kamii & Warrington, 1999) has documented that students can invent, or reinvent, procedures for operating with fractions. However, there has been little consideration of the role that a teacher might play in supporting students to construct such strategies and procedures. In this paper we draw from our work with four experienced and "skillful" teachers whose approach to teaching fraction operations involves positioning student to invent, or reinvent, their own procedures for operating with fractions. It is argued that the ways in which the teachers leveraged student reasoning to draw out perspectives on equivalence is an important aspect of teacher practice associated with instruction that emphasizes a *guided-reinvention* approach to fraction-based algorithm development (Gravemeijer & van Galen, 2003).

Theoretical Framework

In their discussion of a *guided-reinvention* approach to algorithm development, Gravemeijer and van Galen (2003) emphasize that instead of concretizing mathematical algorithms for students, teachers can use an instructional approach where students develop or reinvent algorithms for themselves. Given the opportunity to reinvent mathematics in somewhat the manner that it played out historically, students can experience mathematical knowledge as a product of their own activity. "The core idea is that students develop mathematical concepts, notations, and procedures as organizing tools when solving problems" (Gravemeijer & van Galen, 2003, p. 117). Related to guided-reinvention is the notion of emergent-modeling (Gravemeijer, 2004). When instruction is designed to support emergent-modeling, instead of trying to concretize mathematical knowledge, the objective is to help students model their own informal mathematical ideas. From this informal modeling, more formal ways of reasoning can emerge. The teacher plays a role in supporting this development. This work characterizes practice where teachers supported students' mathematical activity related to fraction operations, and the role of equivalence, without taking over the guided-reinvention process or reducing the cognitive demands of the work.

Equivalence concepts are fundamental if students are going to be able to operate meaningfully with fractional quantities. The flexibility to understand and view fractional quantities as having many names all representing the same number, the ability to generate equivalent fractions meaningfully, and the ability to perceive the relationship between equivalent fraction representations, are important features in algorithm development (Lamon, 2005). The students in this study explored equivalence as a conceptual idea and as a skill in an instructional unit that preceded the unit where data was collected for the study reported here. In the data we focus on ways in which teachers drew from students' informal reasoning in order to support the notion of equivalence as a tool when operating with fractions. It was not suggested to students in advance that they needed to have or use equivalent fractions. It emerged from their mathematical activity. It was present in their informal work when making sense of and solving problems that would lead to adding, subtracting, multiplying and dividing fractions.

Methodology

The settings for this study were the classrooms of four sixth-grade teachers and their students. Each of the teachers used the Connected Mathematics Project (CMP) II instructional unit *Bits and Pieces II: Using Fraction Operations* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006a) as their primary curriculum source. This unit uses a guided-reinvention approach to developing meaning for fraction operations. It allows algorithms to arise through student engagement with both contextual and number-based situations. In this setting, assumptions can be made about the tasks used and about the fraction-related concepts that were developed prior to, and during the unit on fraction operations. In the timeline for the sixth graders who are part of this study, students came to the fraction operation unit with previous experiences that supported their understanding and ability to use equivalent fractions. Prior to implementing the *Bits and Pieces II* unit, the *Bits and Pieces I: Understanding Fractions, Decimals and Percents* (Lappan, Fey, Fitzgerald, Friel & Phillips, 2006b) unit was also implemented.

This study used a qualitative design. During the teaching of the *Bits and Pieces II* unit, classroom lessons were videotaped each day during the 5–6 weeks it took to cover the unit. In addition, the teachers wore an audio recorder during each lesson. The audio recorder was used to record the small group conversations teachers had with students. When a teacher completed a lesson, they also audio recorded a short 5-minute reflection on the lesson. When visiting, the researchers engaged in participant observation. This included observing, taking field notes, interacting with students during small group work time, and meeting with the teacher after the lesson to seek their perspectives on the lesson. During the summer the researchers and teachers came together for three days to discuss their teaching. The three days of summer work were also videotaped for data analysis. Ways teachers purposefully leveraged the use of equivalence as a mathematical reasoning tool was one of the topics discussed.

Data analysis was guided by Erickson's (1986) interpretive methods and participant observational fieldwork, which addresses the need to understand the social actions that take place in a setting. The multiple data sources allowed for triangulation. The school-year data was transcribed and analyzed for emerging themes. The analysis led to characterizations for leveraging equivalence as a tool. It focused on themes related to what teacher elicited from students during whole class discussions when they were sharing strategies for solving problems, and how these elicitations positioned students to move from informal to formal mathematical reasoning.

Findings

In order to capture how a teacher might leverage students' informal reasoning with equivalence in support of helping them articulate strategies for operating with fractions, characterizations of practice are provided for each addition/subtraction, multiplication, and division. Specific project teachers are not identified in the dialogue. These findings are presented as a collective view of what was observed in the classroom data and what emerged in the collaborative work that took place during the summer workshop.

Addition/Subtraction

The work on addition and subtraction began with a task where students drew on past work that involved partitioning and naming fractional quantities. The problem, referred to here as The Land Problem (see Lappan et al., 2006a, pp. 17–19 for full problem), used an area model where square sections of land where divided into smaller sections for farming. Initially, the task asked students to determine what fraction of a section of land each farmer owned. Depending upon how students partitioned the land, various equivalent fractional names emerge for different farmers. As part of their arguments, students visually partitioned their map into equal-size parts and showed, for example, that Bouck owned 1/16 of a section or Foley owned 5/16 of a section. Some students would cut out a farmer's section, for example a Kreb's piece, and then show 32 Krebs-size pieces filled Section 18 and that Bouck's land could also be called two 32nds of the section. Figure 1a shows a map where a teacher recorded the fractional values that emerged from students' work. An important idea that emerged from this part of the problem was that collectively students offered more than one possible fractional name for each section.

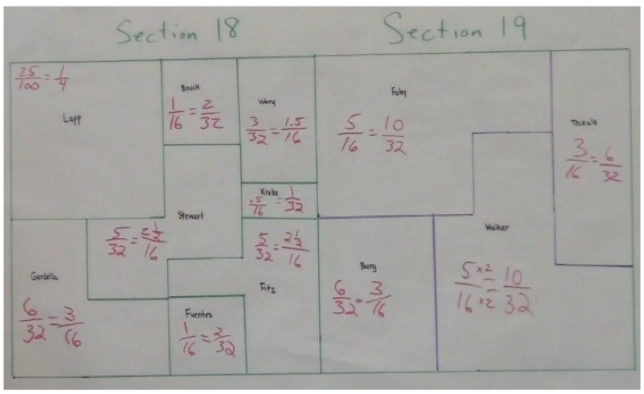

Figure 1a. Land map solutions

(1) 5/16 + 3/16 = 8/16 = ½

(2) 10/32 + 6/32 = 16/32

(3) 10/32 + 6/32 = 16/32 = ½

(4) 5/16 + 6/32 = 16/32

(5) 5/16 + 3/16 = 8/16

Figure 1b. Number sentences

The next part of the Land Problem asked students to combine various sections of land and write a number sentence for their solution. One problem posed was: *Lapp and Bouck combine their land. What fractions of a section do they now own together?* These number sentences were offered by students during discussion: 4/16 + 1/16 = 5/16 and 8/32 + 2/32 = 10/32. Here, as is typical of students who solve this problem, they used fractions with common denominators to write their number sentences. This emerges intuitively. Students did not do this because they were prompted to. When presenting their number sentences, students were asked to show on the map, how they knew their number sentences were true. The teacher then extended students' ideas to draw upon equivalence as a tool by asking them to consider ideas like the following:

- When we put Lapp plus Bouck together some of you said the answer was 10/32 and some of you said the answer is 5/16. Are those amounts the same? Or are they different?
- I am going to throw up another example. I have some kids who look at Lapp and say that Lapp is ¼ of Section 18. Is Lapp ¼ of that section? [class says "yes"] And we are supposed to add Bouck to it. So for example, I could say that Bouck is 1/16. Is Bouck 1/16? [class says "yes"] So I am going to write the number sentence ¼ + 1/16 = 5/16. Is that a true statement?

In response to the later scenario, some said "yes" and others said "no." The teacher asked students to talk to their groups and prove if it was true. The class conversation then went:

T: Tara I heard you say something. Would you share it?
Tara: It is right but if you wanted to make it an easier addition problem to do, you could change the ¼ into 4/16. Then you would have the same denominators.
T: Would that make it easier?
Class: Yes.
T: How many of you agree with that?
Class: [Most students raise hand.]
T: Is this sentence right here [1/4 + 1/16 = 5/16] a true sentence?
Class: Yes.
T: Can someone say what it is about this sentence [1/4 + 1/16 = 5/16] that makes it hard to say if that is right or wrong?
Sam: Because the denominators are different?
T: What does that tell us about the size of the pieces.
Liam: They are different.
T: We are talking about a unit here [points to Lapp on map] that is fourths and then a unit here [points to Bouck on map] that is sixteenths. And it is kind of hard to put that together and say what it is.

There was a similar discussion when the teacher posed the following: *Foley and Burg combine their land. What fraction of a section will they now own together?* Figure 1b contains a string of number sentences that emerged during this discussion. Again, students were asked to use the Land Problem map to argue that their solutions were sensible. A student offered number sentence 1 in Figure 1b. Another student then offered number sentence 2 in Figure 1b.

T: You didn't get the same fraction that the other group had…Can someone talk to us about that? One is 8/16 and one is 16/32. Who is right?
Drew: They are both equal.
T: How do you know they are both equal?
Kayla: Because 8 times 2 is 16 and 16 times 2 is 32.
T: So if you have 16/32 of the whole section, how much to you have?
Kayla: ½. [Writes number sentence 3 in Figure 1b.]
T: So, Isabel, what does that say about both answers?
Isabel: They are both equal.
T: Let me ask another question. I had a kid last year that did it a different way. He said Foley was 5/16. Then he looked at Burg [on map] and wrote 6/32. Then for the answer, he wrote 16/32. [See number sentence 4 in Figure 1b.] Would that number sentence work?
Class: [There were both yes and no responses.]
T: Talk with your table. [Students talk.] Daniel.
Daniel: The answer works but not the sentence. But the answer is the right answer.
T: Oh. So the answer works but not the sentence.
Leah: If it gives the right answer then it is true.
Others: No.
Tara: Really, you can change 6/32 to 3/16 and 16/32 is 8/16…[this gets recapped and number sentence 5 in Figure 1b is recorded.]
T: Let me ask again, is this sentence [5/16 + 6/32 = 16/32] a true sentence?
Class: [some said yes and some said no]
T: Yes. This is a true sentence. Because we know that Foley really is 5/16. We know Burg really is 5/32. But, what helps us think about it? Daniel, I heard you say you don't like this sentence. What is it about that sentence that made it hard for you?
Daniel: There are different denominators

T: Yeah. The pieces are not the same size. So what were you guys doing to make these easier for you? So they weren't confusing.
Lacey: Changing the denominators.
T: Yeah. You were changing the denominators and then adding the amounts.

Using equivalence as a tool, the teacher leveraged students' reasoning to draw out several ideas. One idea was to make explicit why students were choosing fractions with common denominators. They were doing this intuitively. By leveraging equivalence as a tool, students were able to explain why this was helpful. This is a key component of the algorithm they are working toward. A second idea involved using equivalence to compare three different solutions (i.e., 8/16, 16/32, and ½) in order to verify they were all correct. This supported students' ability to read and work with mathematical symbolism. When students move on to problems that do not have a context, they will need to use equivalence as a tool to show others how they are working with and manipulating quantities.

Multiplication

There were opportunities to leverage equivalence as a tool when working with fraction multiplication. In one conversation students were finding fractional sections of fractional parts (parts of parts) in a scenario that involved brownie pans (see Lappan et al., 2006a, pp. 32–33 for full problem). For example, *What fraction of a pan will I have if I buy ¾ of a pan that is ½ full?* In these scenarios, the problems were presented as "part of part" problems. At this point in the work, an algorithm was not established nor pushed for explicitly. When modeling ¼ of 2/3, two different diagrammatic approaches were used leading to two different number sentences: ¼ of 2/3 = 2/12 and ¼ of 2/3 = 1/6. Students were asked to consider whether these were both true and how they knew. Students used their diagrams and equivalence as a tool to argue that both 2/12 of a pan and 1/6 of a pan were the same amount.

After working through numerous brownie pan problems, a student offered that when she wrote her number sentences she noticed that it looked like you could just multiply the numerators across and the denominators across and it would work too. Many students were still trying to understand what ¼ of 2/3 meant and so the teacher suggested that this student continue to draw her brownie pan models and test her idea to see if worked across numerous problems.

On the second day of the unit, students were introduced to multiplication symbolism where ¾ of ½ is formally written as ¾ × ½. They were also asked to use estimation and number sense to consider whether the following problems would lead to products greater than or less than one whole: 5/6 × ½, 5/6 × 1, 5/6 × 2, and 3/7 × 2. The student who had been contemplating how to operate symbolically started the following discussion.

Libby: When there is a whole number, not so much when estimating, but remember how I told you before [referring to her idea to multiply numerators and multiply denominators]. For 5/6 × 2, couldn't you turn the two into 12/6 and do it my way and I could figure it out?"
T: [Rewrites 5/6 × 2 as 5/6 × 12/6 on board.] Change this into 12/6?
Libby: Yeah.
T: I don't know why not. It is another name for 2. Right?
Libby: Then I could do 5 times 12 and 6 times 6.
T: If that way works. It seems like every time you tried it, it has matched your model. I don't know if I would want to draw all those things but you could. For 2 you would have to draw two whole pans.
Ginny: I agree with Libby on her way but I think you could do it in a simpler way. You could turn two into two halves.
T: Would that be one? [writes 2/2 = 1].
Ginny: No.
T: So that is not equal. That seems like I would be finding 3/7 of 1 instead of 3/7 of 2 if I made it 2 halves. This is an excellent discussion and it is exactly what I want everyone to be doing…You are thinking and I love it. Keep thinking.

Here the teacher prompted students to draw upon equivalence as a tool. We also saw the student, Libby, inquiring about the use of equivalence as a tool. In her understanding of fractions, she recognized that a fraction can have many names. She was also starting to realize that by choosing a specific equivalent name, she could use her theory about how to multiply fractions symbolically. This is an idea the entire class would eventually explore together.

Division

With division the long-term goal was to support development of the common denominator algorithm for fraction division. Most of the work used quotative division problem contexts. For example: *You have 7/8 of a pound of hamburger. If you make patties that are each 3/8 of a pound, how many patties can you make?* The initial problems involved simple fractions with common denominators, then simple fractions with unlike denominators, and finally mixed numbers with both common and unlike denominators. In the initial days of the work on division, the focus was on creating a picture or visual representation for the problem and writing a corresponding number sentence. Students were using drawings, rate tables, and number sentences to talk about why 7/8 ÷ 3/8 was like finding how many 3 eighths are in 7 eighths or how many groups of 3 are in 7.

In this scenario, students had moved to working with mixed numbers. The problem being worked on was *You have 2 2/3 pounds of hamburger and you are making 2/3 pound patties. How many patties can you make?* A student presented a picture where three wholes were partitioned into thirds. Two and two-thirds was marked. The student then marked and counted out how many two-thirds were in 2 2/3. The number sentence they wrote was 2 2/3 ÷ 2/3 = 4.

T: Did anyone have a different number sentence then what she had written there? She had 2 2/3 ÷ 2/3 = 4 which is correct. But I think there is another number sentence that could help make the answer stand out even better.
Cody: 8/3 divided by 2/3 equals 4.
T: Can you write that number sentence up there? [pause] Look at that number sentence. We are purposefully putting these up there so you can look at those and start seeing if there is a faster way to do this then drawing a picture or making these [rate] tables that we are making. 8/3 divided by 2/3 is 4. I can see that really easily, but I had a hard time seeing it with 2 2/3 ÷ 2/3. So keep thinking about that.

Next, students worked work on the problem *You have 2 1/4 pounds of hamburger and you are making 3/8 pound patties. How many patties can you make?* In his diagram, a student partitioned each pound of hamburger into eighths and marked groups of 3/8. Along with a drawing to support an answer of 6, the student presenting his work wrote the number sentence 9/4 ÷ 3/8 = 6.

T: I am looking at his number sentence. I am having a hard time seeing that the answer is 6. Does any one have a way that we could write that number sentence that could help us see the answer better. I saw some other number sentences on peoples' work.
Chris: You could write 18/8 ÷ 3/8 equals 6. [This is recorded on the work being displayed.]
T: Why is the first sentence [9/4 ÷ 3/8 = 6] so hard to deal with?
Ali: We don't have common denominators.

The class continued to discuss why having common denominators were helpful. In these examples the teacher was drawing out the basis behind using the common denominator algorithm. The students' number sentences did not capture how they were using common-size parts in their drawings. Leveraging equivalence as a tool was one way to draw out a connection between students' diagrams, their symbolism and a potential algorithmic approach.

Discussion and Significance

The contribution of this work is an articulation of specific ways that teachers might leverage equivalence for a particular fraction-based operation without reducing the cognitive complexity of the students' work. In the bigger picture of supporting algorithm development, there were connections made between symbolism, visual models, and equivalence. The leveraging of equivalence as a tool supported students to make their implicit or informal ideas, found in their various representations, explicit for public discussion. While the data presented did not share the actual emergence and articulation of specific algorithms for each operation, it highlighted ways a teacher might use equivalence as a tool to support students to invent (or reinvent) for themselves algorithmic procedures for operating with fractions based on their informal work. This focus on leveraging equivalence as a tool is in contrast to presenting equivalence as a rote procedural step as is common when instruction presents algorithmic procedures as ready-made.

While it was not the direct focus of this paper, an important part of the work students were doing involved developing visual representations or models for scenarios that enacted the four fraction operations. Students then attached symbolism in the form of number sentences to their visual representations. The data shared revealed ways in which the visual representations and the symbolic representations were important in the algorithm-development process. While there was not enough space here to display full development from informal to formal, equivalence is presented as one important tool that teachers might leverage to help students engage in mathematical reasoning that supported the emergence of meaningful procedures and algorithms for fraction operations.

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WHAT SENSE DO CHILDREN MAKE OF NEGATIVE DOLLARS?

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We interviewed 40 students in Grade 7 to investigate their integer reasoning. In one task, children were asked to write and interpret equations related to a story problem about borrowing money from a friend. Their responses reflect different perspectives concerning the relationship between this real-world situation and various numerical representations. We identify distinct ways in which integers were used and interpreted. All of the students solved the story problem correctly. Few thought about the story as involving negative numbers. When asked to interpret an equation involving negative numbers in relation to the story, about half related it to the story in an unconventional fashion, which contrasts with typical textbook approaches. These findings raise questions about the role of money and other contexts in integer instruction.

Keywords: Middle School Education

Introduction

The use of contexts is common in integer instruction. In a review of fifth and sixth grade textbooks adopted by the state of California, we found that 94% of these used the context of money in instruction concerning integers. Elevation (89%) and temperature (89%) were also popular contexts (Whitacre et al., 2011). There are conventional ways in which textbook authors relate these contexts to the integers and to integer arithmetic. These conventions may or may not jibe with children's intuitions. In interviews with 40 seventh graders, we investigated how children made sense of a story problem concerning borrowing money from a friend, and how the children saw this context as related (or not) to various equations. We report on children's ways of reasoning about the relationship between the context and the equations. These new findings concerning children's reasoning about integers suggest implications for integer instruction.

Theoretical Framework: A Children's Mathematical Thinking Perspective

We approach this study from a children's mathematical thinking perspective. In the tradition of Cognitively Guided Instruction (CGI), we value children's mathematics. We take seriously the nature of that mathematics, even if it is incorrect from an expert perspective. We endeavor to see the mathematics through children's eyes in order to better understand the sense that they make (Bishop et al., 2011; Lamb et al., in press). We do this because the ultimate goal of our research is to find ways to better support children's learning of mathematics (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Carpenter, Franke, & Levi, 2003; Empson & Levi, 2011).

A Distinction Informed by the History of Mathematics

There is a somewhat subtle distinction that is relevant to the analysis presented in this paper. As a matter of background, the notion of a negative number was controversial historically. Many mathematicians resisted the idea of negative numbers because numbers were associated with magnitudes, and magnitudes less than zero seemed nonsensical. During the 13th through the 18th centuries, many Western mathematicians used negative numbers in algebra, although these remained hotly debated and were not generally regarded as legitimate numbers (Gallardo, 2002; Hefendehl-Hebeker, 1991; Henley, 1999).

Taking account of the history of negative numbers, it would be tempting to say the following: "Positive numbers existed before negatives. Furthermore, even after negatives came into use, it took a long time for them to become understood and accepted as legitimate numbers." We would take exception with one aspect of this account—the use of the word *positive*. Prior to the advent of negative numbers, there was no such thing as a positive number. Positive and negative are opposites. Their meaning derives from the contrast between the two. Before the notion of a negative arose, there were simply numbers. In keeping with the language that many children have used in our interviews, we refer to these as *regular numbers.*

Consider the integers. It is commonplace today to refer to the positive integers and the natural numbers as one in the same—at least among mathematicians or in mathematics classes at the middle school level and above. Historically, however, the natural numbers predate the integers. More importantly, there is a conceptual distinction between these sets. The positive numbers are signed numbers. The natural numbers are not. Put another way, signed numbers may be thought of as *directed magnitudes*. That is, they convey two distinct pieces of information, direction (sign) and magnitude (absolute value). The positive numbers have this property. The natural numbers, by contrast, have only magnitude.

This distinction is relevant to the experience of children learning mathematics today. For the first so many years of a child's life, numbers are not signed. Children learn to count with natural numbers. At some point, they learn about zero. Later, they encounter nonnegative rational numbers, which they come to know as fractions and decimals. Typically, a child's introduction to negatives (and positives) comes after all this. In fact, we have found that many elementary children who have some familiarity with negative numbers have never heard of positive numbers. When they learn about integer arithmetic, which typically occurs in middle school, children are told that the regular numbers are actually positive. However, there is a sense in which this distinction remains salient. The distinction between positive numbers and regular numbers plays a part in the analysis that we present.

Methods

We interviewed 40 children in Grade 7 during the spring of 2011. The interviews were conducted at public schools in an urban area in California. All Grades 5, 6, and 7 mathematics textbooks adopted by the California Department of Education contain material concerning integers and integer arithmetic. Thus, we expected that the students we interviewed all would have received substantial instruction on integers. Indeed, we know from the interviews that all 40 of the children were familiar with negative numbers and were able to solve (at least some) problems involving integer arithmetic.

The interviews consisted of a range of tasks, including open number sentences, number comparisons, and story problems. Interviews were conducted at the children's school sites, during the school day. Interviews were videotaped and typically lasted between 60 and 90 minutes.

Task: The Money Problem

In this report, we focus on the following story problem:

> *Yesterday, you borrowed $8 from a friend to buy a school t-shirt. Today, you borrowed another $5 from the same friend to buy lunch. What's the situation now?*

The interviewer would often clarify the question by asking, "Do you owe your friend money? Does your friend owe you money? How much money?" Students were asked to solve the problem and to explain their thinking. They were also asked to write a number sentence that would represent the problem, including its solution, and to explain how this equation related to the story. Students were then asked if they could write additional equations that would also represent the story. Next, they were presented with three equations, one by one. They were told that these had been written by other children to represent the same story. Students were asked to tell whether or not they thought each equation matched the story and to explain why. The equations shown to students were the following:

i. -8 + -5 = -13
ii. -8 – 5 = 13
iii. 8 + 5 = 13

Order of presentation varied. Typically, if one of equations i–iii matched an equation that the child had written, this one was shown first.

Analysis of Children's Responses

We first analyzed a subset of the data qualitatively, focusing on children's solution strategies and underlying ways of reasoning. We used principles of grounded theory (primarily the constant comparative method) to identify emergent, distinguishing themes in students' reasoning (Strauss & Corbin, 1998). Once a set of codes was generated that fit this subset of the data, these were used to code the remainder of the data. In particular, our analysis of children's responses to the story problem revealed an interesting theme, that of *perspective*. This is a category of codes. Within that category, we identify three ways of reasoning with regard to perspective: *conventional*, *unconventional*, and *perspectiveless*. These three ways of reasoning were used to code the responses of all 40 children. These apply to children's explanations of their own equations, as well as to their interpretations of equations i, ii, and iii.

As a reliability check, 25% of the data was double-coded. For 10 of the 40 children's responses, two researchers independently coded those responses using the perspective codes. Coders agreed on the ways of reasoning of 9 of the 10 children (90%). The one instance of a disagreement was a case of coder error, and it was corrected. We also coded children's responses to the story problem as correct or incorrect, and we recorded the equations that they wrote. We then tabulated the results.

We report here on the percentage of children who solved the story problem correctly, the percentages of children who wrote certain equations to represent the story problem, and the percentages of children who interpreted equations involving negatives from a conventional or an unconventional perspective.

Results

First, we describe the three different ways of reasoning. Then we provide specific examples of each way of reasoning. Finally, we report the results of our analysis of the responses of all 40 children who were interviewed.

Ways of Reasoning

We describe the ways of reasoning that were identified.

Perspectiveless. The child sees regular numbers as an appropriate representation of amounts of money. Thus, 8 + 5 = 13 would appropriately represent the situation because 8 represents $8, 5 represents $5, and 13 represents $13. The fact that this was money borrowed from a friend is relevant only in that the answer of 13 refers to a total amount of money borrowed/owed. However, the same equation could just as well represent the situation from the lender's side. In either case, $8 plus $5 is $13. The numbers are being used here only to communicate magnitude, not direction (of owing).

Unconventional perspective. The child views positive numbers as representing money that the *borrower* gained, even though the money was borrowed. From this perspective, 8 + 5 = 13 matches the story because 8 represents $8 given to me by a friend, 5 represents an additional $5 given to me by that friend, and 13 represents the total of $13 that I acquired. For children reasoning from this perspective, -8+ -5 = -13 does not match the story from the borrower's side, but it could match the story from the lender's side because the lender lost money by lending it.

Children whose responses were coded as reflecting an unconventional perspective were taking perspective into account. They reasoned about the appropriateness of using positive or negative numbers to represent the situation. Negative numbers had meaning for these students, albeit an unconventional meaning. We use the term *unconventional* because, in our review of textbook approaches to integer instruction, we did not find negatives used in this way. Furthermore, the unconventional perspective contrasts directly with the conventional perspective, which is typical of textbook treatments.

Conventional perspective. The child views negative numbers and subtraction as representing debt or net loss. For example, -8 + -5 = -13 is seen as matching the story about borrowing money from a friend because -8 represents a debt of $8, -5 represents an additional debt of $5, and -13 represents the total debt of $13. From this perspective, positive numbers would be used to represent the lender's situation. That is, 8 + 5 = 13 would not describe the situation from the borrower's side, but it would describe the situation from the lender's side.

Children's Responses

Below, we present examples of particular children's responses that illustrate these distinct ways of reasoning.

Perspectiveless. Elisa wrote 8 + 5 = 13 as her equation. Like the other children who wrote this equation, her explanation was perspectiveless:

Interviewer: Okay. And can you explain to me how this equation matches the story?
Elisa: Okay. So, yesterday I borrowed eight dollars from my friend to buy a school t-shirt. So, I have eight dollars from my friend [points at "8" in equation]. And then today I borrowed five from the same friend, and I bought a lunch. And then plus five that I borrowed from her [circles "+ 5" in equation] equals thirteen dollars that I borrowed from her [circles "13" in equation]. And what's the situation? I owe her thirteen dollars.

Elisa made sense of the story and was well aware of who owed money to whom. However, this directional information was not conveyed in her equation. It belonged to what she knew the equation represented—how much money she owed her friend. The equation itself simply conveyed magnitude information: the sum of $8 and $5 is $13.

Unconventional Perspective. Tommy also wrote 8 + 5 = 13 to represent the story. When he was shown -8 + –5 = 13, he said that it would not work to describe the borrower's situation. Tommy's explanation is an example of unconventional perspective:

Interviewer: Can you read that one for me? [Interviewer reveals -8 + -5 = -13 on paper]
Tommy: Negative eight plus negative five equals negative thirteen.
Interviewer: Okay, and what do you think about that? Do you think that that describes the situation, or the story?
Tommy: Um, no. Because it's like, it's basically like saying that they owe you because it's like, it's like you're not taking any money. It's like they're taking your money. Because it's negative, which means it's like, it's kind of lower; it's lower than zero. Like, so then it's like they're owing you money, instead of you owing them.

Tommy interpreted negative numbers as indicating someone "taking" money. He viewed the situation from his side (as the borrower), and so he said that the equation would not fit because the negatives would mean that money had been taken from him, rather than given to him.

Conventional Perspective. Jen initially wrote -8 – 5 = x as her equation. At the interviewer's request, she later rewrote her equation, replacing x with a known number (-13). Her explanation conveyed a conventional perspective:

Interviewer: Okay, so tell me about what you wrote.
Jen: I wrote negative eight minus five equals x [pause] because, um, because if I borrow money, it's like I'm, I'm like losing money. No, it's not I'm losing money. It's like I'm, I'm borrowing someone else's money. So, on my side, it would be a negative. And then on another day, I'm also borrowing money, so that's also like a negative.
Interviewer: Okay. And what would the x be here?

Jen: It'd be like how much money I borrowed in total.
Interviewer: Mm-kay. And could you rewrite this and replace x with the answer?
[Jen writes -8 – 5 = -13]
Interviewer: Okay. So, why did you write negative thirteen for the answer there?
Jen: Because negative eight minus five is negative thirteen.
Interviewer: And how does that relate to the story?
Jen: It relates to the story because, if I borrow eight and five, I owe them thirteen. So, it's like a negative thirteen dollars on my side.

Jen articulated a view of negatives as representing debt. For her, -8 represented $8 that she had borrowed, – 5 represented $5 that she had borrowed, and -13 represented the total amount that she owed to her friend.

Overall Results

All 40 of the 7th graders (100%) solved the story problem correctly. That is, they said something like, "I owe my friend $13." They were then asked to write one or more equations to represent the story. Of the 40 students, 33 (82.5%) wrote 8 + 5 = 13 as one of their equations. Usually, this was the first equation that they wrote. Often it was the only equation that they wrote. Rarely did students write equations involving negative numbers. Those who wrote multiple equations typically created variations that also involved natural numbers. The equation 5 + 8 = 13 was the most common second choice. In each case that a child wrote 8 + 5 = 13, the child's explanation for that equation was perspectiveless. The child talked about the numbers as representing amounts of money, and there was no evidence that he or she intended to convey information about the direction of borrowing/owing with the signs of the numbers. These children were using regular numbers.

Only 8 of the 40 students (20%) wrote an equation involving negative numbers. Six children wrote -8 + -5 = -13, and two children wrote -8 – 5 = -13. Children's explanations for these equations naturally addressed the issue of perspective since the children had made a choice to use negative numbers. Thus, only 20% of the children took perspective into account in writing an equation to represent the situation.

The children were then shown equations i, ii, and iii. We report here on the perspective reflected in their interpretations of equations i and ii (-8 + -5 = -13 and -8 – 5 = -13). Often children's responses explicitly addressed perspective for either i or ii but not both. In cases where perspective was explicitly addressed for both, the perspective was always consistent. For these reasons, we group children's responses to i and ii. Of the 40 children who were shown equations i and ii, 19 of them (47.5%) articulated a conventional perspective. They interpreted the negative numbers in the given equation as representing debt. For them, the given equation was appropriate for describing the borrower's situation. An additional 19 children (47.5%) expressed an unconventional perspective in interpreting one or both of equations i and ii. These children viewed negative numbers as representing money lost or taken away. For these children, the given equation described the lender's situation. Two of the children (5%) were not given any perspective code for i or ii. It was not clear that they had any way of interpreting negative numbers with respect to the context.

Discussion

We have identified three distinct ways in which children reason about equations like 8 + 5 = 13 and -8 + -5 = -13 in relation to a story about borrowing money. Our analysis has involved two key ideas that arose in children's mathematical thinking. The first is the distinction between regular numbers and positive numbers. When children wrote 8 + 5 = 13 to represent the situation, they were using regular numbers. They viewed this equation as related to the context in terms of the numbers and the operation involved. The children were aware that the number 13 represented $13 borrowed from a friend; however, information about the direction of borrowing/owing was not contained in the equation.

The second key idea involves another distinction, and it applies only to those children whose mathematical worlds include positive and negative numbers *and* who see these as related in some way to the context of borrowing/owing money. Our analysis of the responses of children like these revealed two distinct perspectives, conventional and unconventional. Of the 38 children who were able to interpret negative numbers in relation to the story, exactly half of them expressed a conventional perspective. They interpreted the negative numbers in the given equations as representing debt, or money owed by the borrower to the lender. The other half of the children expressed an unconventional perspective. They interpreted negative numbers in the opposite fashion, as expressing money lost by, or taken away from, the lender. Both groups of children consistently viewed the equations as making sense for describing the situation of one person in the story (either the borrower or the lender) and not the other. In this respect, the two perspectives are incompatible. Their interpretations disagree with one another.

Implications

Story problems in real-world contexts are widely used in integer instruction, and stories involving money are among the most common of these. The reason we have used the terms *conventional* and *unconventional* to describe students' reasoning in relation to these contexts is that the conventional perspective is typical of the way that positive and negative numbers are used to represent debt in mathematics textbooks. That is, the conventional perspective is the accepted perspective of the mathematical community. It is also reflected in the standard notation used in bank statements, utility bills, and so on. Negative numbers are used to denote a debt or debit; positive numbers are used to denote a deposit or credit. Thus, it is noteworthy that this convention is not consistent with the reasoning of many children. In particular, our review of textbook approaches to integer instruction suggests that the vast majority of 7th graders have encountered money contexts in their instruction concerning integers. Yet approximately half of the 7th graders that we interviewed interpreted the relationship between integers and money from the unconventional perspective, which contrasts with the interpretation reflected in the textbooks.

These findings raise questions concerning the roles of contexts such as money in integer instruction. Our group has begun to question what it means to make sense of integers and integer arithmetic. Often making sense in K–8 mathematics seems to be defined in terms or relating numbers and operations to quantities in the world. At least in the case of the integers, we believe that this notion may be in need of revision. Certainly, a mature understanding of integers includes the ability to relate them to quantities in the world. However, this does not entail that reasoning about real-world quantities like money should serve as a *source* for children's mathematical intuitions. On the contrary, we can imagine children developing a deep, purely mathematical understanding of the integers. This understanding could then be superimposed upon real-world situations, in the way that we do as mathematically literate adults.

We hope that the distinctions that we have discussed may be useful to both researchers and practitioners who are interested in the teaching and learning of integers and of integer arithmetic. In particular, sensitivity to the distinction between regular numbers and positive numbers can inform the language that we use with students and the care that we take in introducing them to the notion of signed numbers. Likewise, sensitivity to the issue of perspective can inform instructional decisions. It reminds us of the importance of eliciting the details of students' thinking and of having explicit discussions of different ways of reasoning.

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INTERPRETING AVERAGE RATE OF CHANGE IN CONTEXT

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This paper draws on a models and modeling perspective to investigate students' abilities to describe and interpret non-constant average rates of change in the context of exponential decay. The results show that about half of the students could describe the behavior of the function and its rate of change, with more than half of those also referring to the problem context. Two themes in the students' responses are discussed: describing the context as a way of describing the function, and difficulties related to the use of language in describing changing phenomena.

Keywords: Advanced Mathematical Thinking; Modeling

Students' difficulties when creating and interpreting functions as models of changing phenomena are well documented (Carlson et al., 2002; Thompson, 1994). Students must be able to simultaneously attend to both the changing values of the output of a function and the rate of that change as the input values vary over intervals in the domain (Oerhtman, Carlson & Thompson, 2008). The complexity of such reasoning has proven difficult even for high achieving undergraduate students (Carlson, 1998). Other researchers have emphasized the role of context in the development of students' reasoning about changing phenomena (Confrey & Smith, 1994; Michelsen, 2006; Shternberg & Yerushalmy, 2003). Michelsen (2006) argues that one source of difficulty in applying functions in context is that students fail to treat variables as related quantities that change and hence have difficulties in recognizing that functions are tools for describing, explaining and predicting the relationships among changing quantities.

In this paper, we examine how students interpreted and attended to function values, average rates of change and changing average rates of change in the context of a discharging capacitor.

Theoretical Background, Design, Methodology and Methods

Our research used a contextual modeling perspective on the teaching and learning of modeling (Kaiser & Sriraman, 2006) to design a model development sequence (Lesh et al., 2003) to motivate students to develop the mathematics needed to make sense of meaningful situations. Each task in such a sequence engages students in multiple cycles of descriptions, interpretations, conjectures and explanations that are iteratively refined while interacting with other students. The sequence formed the basis for a six-week summer course for students preparing to enter their university studies in 2010 and 2011. This paper analyzes the responses of 51 students (15 females and 36 males) on a task given to the students at the end of the sequence. All but one participant had completed four years of study of high school mathematics; 29 students had studied calculus in high school.

In the task, the students were given a data set of the voltage drop across a discharging capacitor for 50 seconds and were asked to: (a) find an equation of the form $y = a \cdot b^x$ that could be used to describe the data; (b) give an interpretation of the constants a and b in this equation; (c) find the point in time when the voltage across the capacitor was 0.05 V; (d) compute the average rate of change over three subintervals of time; and (e) write two or three sentences interpreting the average rate of change data in (d). In this paper, we report on the analysis of the students' responses to the last part of this question. Figure 1 shows a graph of the voltage data, modeled by an exponential decay function, decreasing with an increasing average rate of change across the sub-intervals of time. Since the function is decreasing the average rate of change is negative. However, the average rate of change is getting successively less negative and closer to zero. Hence the average rate of change is increasing. In terms of the context, this means that the voltage across the capacitor is decreasing at an increasing rate.

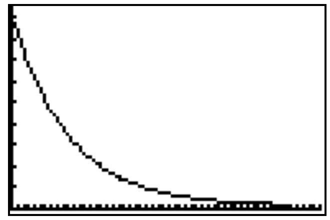

Figure 1: A plot of the voltage data provided to the students

The analysis of the students' written work was done in three phases. First, the written work of the 49 students who answered the question were read focusing on the extent to which the students made references to and distinctions among the behavior of the function, the average rate of change, and the context. This resulted in the identification of a preliminary set of categories and themes capturing the variations of the students' answers. In the second phase, each student's answer was re-read and coded with respect to which aspects of the changing phenomena the student described in terms of (1) the behavior of the function, (2) the average rate of change, and (3) the context. The student responses in each category were also coded correct, incorrect, or not addressed. Further, the answers were classified and grouped in the three categories listed in Table 1. This coding was carried out independently by two of the researchers and discrepancies were discussed and resolved. In the third phase of the analysis, the students' answers were revisited in the search of patterns, commonalities and differences.

Results

Table 1 summarizes how the students simultaneously attended to function values, average rates of change and changing average rates of change in the context of a discharging capacitor. Approximately 49% of the students could correctly describe both the behavior of the function and the average rate of change with more than half of those students (58% or 14 out of 24) also including a reference to the context of the voltage drop across the discharging capacitor.

Table 1: Students' Interpretations of Average Rate of Change

Category	# total students	# correct answers	# incorrect answers
(R+C+F) Students interpreting the average rate of change with reference to both the behavior of the function and the context.	19 (39%)	14 (29%)	5 (10%)
(R+F) Students interpreting the average rate of change with reference to the behavior of the function but without any reference to the context	15 (31%)	10 (20%)	5 (10%)
(R) Students interpreting the average rate of change without any references to context or to the behavior of the function	15 (31%)	5 (10%)	10 (20%)
Total	49 (100%)	29 (59%)	20 (40)

Note: All percentages are calculated relative the whole population and are rounded to the nearest whole percent.

An example of a correct answer in the **(R+C+F)** category is: "The function is decreasing at an increasing rate. You can come to this conclusion because the voltage (or *y* values on the table) is decreasing and the average rate of change values are becoming less negative, which means that it is increasing." This student provides a correct description of both the behavior of the function and the average rate of change, while making a reference to the context. A typical a correct answer in the **(R+F)** category is: "The average rate of change from the intervals above [referring to his calculations] show that as the time increases the function decreases at an increasing average rate of change." This student correctly distinguishes between the function values, which are decreasing, and the average rate of change, which is increasing. This is done with reference to prior numerical calculations, but without reference to the voltage change across the capacitor. An example in the **(R)** category is: "The average rate of change is increasing because it is becoming less negative." This student answers the question as it is posed, but does not go beyond the question to describe the function nor the context.

Describing the Context as a Way to Describe the Function

Using the context as a way to describe the function was a frequent characteristic found in the students' answers. One student used "the capacitor is discharging" to mean that the function is decreasing: "As time passes by, the capacitor is discharging at an increasing rate." Another student wrote: "Between the intervals 5 to 10, the charge of the capacitor dropped at a rate of .43 v/s. For every second, the charge went down by –.43 volts. Between 20 to 25, for every second, the charge of the capacitor went down by –.05 volts. Between 40 to 45 seconds, for every second, the charge of the capacitor went down by –.0004 volts. I can also say that the charge was decreasing at a decreasing rate because the numbers were getting more negative." Although this student exhibits conflated ideas about the behavior of the function and the average rate of change, the description of the behavior of the function is closely tied to the context: "the charge of the capacitor dropped" and "the charge went down." This example also points to students' difficulties related to the language of change when rates are negative. The student correctly states that "the charge of the capacitor dropped at a rate of .43 v/s." But then the student states that "the charge went down by –.43 volts," thus mixing everyday language (went down) and the rate of the change as a signed quantity. In the second statement, the use of the magnitude of the rate would have been more appropriate.

Difficulties Related to the Use of Language in Describing Changing Phenomena

The role of language became increasingly apparent in our analysis of students' answers. The ability to use notions and concepts in a mathematically meaningful way, while keeping these meanings distinguished from similar everyday language is crucial for being able to formulate precise descriptive statements. For example, some students described the magnitude at which the rate of change changed rather than describing this change as a signed quantity. This may have blurred the interpretation of the average rate of change since the magnitude (the absolute value) of the average rate of change is decreasing whereas the signed average rate of change is increasing. The following example shows how easy it is to shift between conflicting formulations: "In (d) the average rate of change is negative and thus we know that voltage is decreasing. However, the rate at which voltage is decreasing is less and less each seconds. This statement can be concluded because comparing the first, middle, and ending intervals shows that the average rate of change is becoming less negative. Thus we see that the average rate of change is decreasing at an increasing rate." In the first and third sentence of this student's answer, the average rate of change is seen and used as a signed quantity, enabling the student to correctly describe both the behavior of the function ("voltage is decreasing") and the average rate of change ("is becoming less negative"). In the second sentence, the average rate of change is seen and used in a magnitude sense, leading the student to incorrectly conclude that the rate of change of the voltage is decreasing. In the fourth sentence, the student's statement ("the average rate of change is decreasing") is true about the magnitude about the average rate of change, but not about the signed quantity.

Everyday language mixed with mathematical terminology sometimes obscured the clarity and precision of what the students were trying to convey. For example, one student wrote: "The average rate of change is constantly increasing. This means that voltage across the capacitor is still decreasing, but at a increasing rate." Here, the ambiguity is with the use of "constantly." From a mathematical point of view, the first statement about the average rate of change is incorrect; it is not increasing constantly (at a constant rate). However, in an everyday use of the word "constantly," it makes perfect sense, meaning that the average rate of change is increasing "all the time" over the intervals under consideration.

Discussion and Conclusions

Describing the behavior of the voltage drop of a discharging capacitor, modeled by an exponential function, revealed a number of conceptual and contextual challenges for the students. The conceptual challenges resides, in part, in attending simultaneously to the global features the behavior of the function and the average rate of change, and in coordinating one's understanding of the change in function values with the rate of that change over various subintervals, especially when the rates are negative but increasing. The contextual challenges arise from the difficulties in everyday language in describing magnitude of the voltage drop, since this magnitude decreases, while the signed average rate increases as it becomes less negative. Everyday language for describing the change in rate appears in conflict with formal mathematical language for describing that change.

Overall, half of the students were able to give meaningful interpretations of the data and descriptions of change in context. Students' difficulties in distinguishing the function values and changes in the average rate of change when the average rates of change were negative suggest the need for closer attention to developing the concept of a negative rate of change in context and using language to express that concept.

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USING DISTANCE AS A DEFINITION FOR ABSOLUTE VALUE

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Students tend to struggle with both the algebra and the logical reasoning required to solve absolute value equations (and inequalities) through the algebraic "reasoning by cases" technique. Recent research of the epistemology of absolute value has focused on how to make this technique more accessible. But what if there was a way to simply avoid the algebraic difficulties while simultaneously revealing the logic of having multiple solutions (or ranges of solutions)? Following the format of a design-based case study, we used a definition of absolute value based on distance to create and implement an instructional sequence in an eighth grade classroom. While having more conceptual parts to keep track of, students using this distance-based definition were more likely to solve absolute value equations successfully by avoiding the consistent algebraic errors found in the control group.

Keywords: Algebra and Algebraic Thinking; Design Experiments; Middle School Education

Theoretical Framework

The first attempts to study student conceptions of absolute value focused on simply illuminating the subject, either pedagogically or mathematically. For example, Brumfiel (1980) laid out a mathematical framework for how absolute value can be defined in equivalent ways. Two of the five definitions he listed are geometric, based on distances for $|x|$ and $|x-y|$, while the other three are arithmetic: $|x| = x$ if $x \geq 0$ or $|x| = -x$ if $x < 0$, $|x| = +\sqrt{x^2}$, or $|x| = \max\{x;-x\}$. Chiarugi, Fracassina, and Furinghetti (1990) found that instruction needs to include situations to motivate the algebraic approach most students had simply proceduralized.

Recent research on student conceptions of absolute value has been influenced by two pairs of significant journal articles. The first pair focus on the didactic effectiveness of instruction of arithmetic definitions. Wilhelmi, Godino, and Lacasta (2007) explored an epistemic network created by the three arithmetic definitions presented by Brumfiel (1980) plus a fourth piecewise functional definition related to solving absolute value equalities by splitting the domain into two pieces. Sierpinska, Bobos, and Pruncut (2011) created different approaches to teaching absolute value using the piecewise functional definition in a college setting. They found that a visual approach, using graphs to demonstrate the generation of cases, showed the greatest effectiveness. They concluded that the dual nature of the instruction, both graphical and algebraic, helped students to reason through problems instead of blindly following a procedural technique.

The second pair of articles present lessons based on Brumfiel's (1980) geometric definitions of absolute value that were mostly ignored in the previous pair of studies. Ponce (2008) recognized the difficulties students have with the arithmetic definitions and noticed a theme in previous research (Arcidiacono, 1983; Wallace, 1988; Horak, 1994; Wei, 2005) that students are more successful when using a number line than those who stick with algebraic strategies. His students solve absolute value equalities and inequalities centered at zero by translating them to story problems and number lines. Ellis and Bryson (2011) proposed an improvement to this distance-based definition by relocating the center to match the problem, instead of always being at the origin. Their method asks the students "to connect the symbolic expression $|x-b| = c$ to the verbal phrase '*x* is *c* units from *b* in either direction'" (p. 593). Like the Sierpinska et al. (2011) study, this process is meant to help students reason through problems instead of simply following an algebraic procedure. However, by grounding the problem through story problems and number lines, the algebra can be avoided completely.

Given the reported success of using a distance-based definition of absolute value in the second pair of studies, the lack of its inclusion in the didactic analysis in the first pair of studies opens an opportunity for further research, especially since studies up to this point have not formally investigated using a distance-based definition in a classroom. We wish to explore the epistemic, cognitive, and instructional implications of a choice of absolute value definition that avoids the algebra heretofore deemed necessary to solve such problems. We believe that this definition will have the benefits of the visual approach studied by Sierpinska et al. (2011) while moving beyond its success, because students will not need to rely on the reasoning by cases procedure either to generate or verify their answers.

Method

We used a design-based case study (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) in two eighth grade teaching practicum math classes located at a mid-size, Midwestern public school to explore the implications of solving absolute value equality problems using either the $|x-y|$ distance definition (seven students in one class) versus the piecewise functional definition (14 students in the other class). Both groups of students had been introduced to absolute value of numbers three months before our instruction, using the $|x|$ distance definition to describe how to find the absolute value of positive and negative values. Students had also covered one-step and two-step algebraic solutions to linear equations, the role of the equal sign in equations, and identifying points on a number line.

Piecewise functional definition instruction began by motivating the need for two solutions to algebraic absolute value equality problems like $|x-A|=B$ with an arithmetic example: $|x|=3$. Note that we specifically chose to use the two solutions of $\pm(x-A)=B$, and not $(x-A)=\pm B$, because the former matches the operation of applying the absolute value when $x-A<0$ in the piecewise functional definition. Discussion of several examples of problems of this form were led by one of the researchers and the students before completing a homework assignment covering $|x-A|$ and $|x+A|$ problems with positive and negative solutions.

Distance definition instruction began for this class by motivating the need for two solutions to absolute value equality problems with a story problem: Jake and Sarah are in a tall building. Jake is waiting on the 10th floor for Sarah. If Sarah is 6 floors away, what floor is Sarah on? After it was established that there are two possible solutions to this problem through the use of a vertical number line, several more problem scenarios relating to distance were solved and discussed as a class using number lines. The following day, students continued the topic with a new story problem using temperature, a context chosen to enable a sensible negative solution. Then the task was reversed: starting with a number line picture, showing a center and a distance from the center, can you create a story problem that matches the number line? Finally, the symbolic notation of absolute value equalities, $|x-A|=B$, was connected to the idea of center and distance. After examples of translating number lines and story problems into the symbolic notation were discussed, students were given a fifteen question worksheet that asked them to translate from each of the three representations used (symbolic notation, number line, story problem) to the other two, then find the solutions.

Results

Both groups of students initially had difficulty understanding why there would be two solutions. One student in each class suggested that there could be two solutions, and this idea quickly caught on. From here, however, each class went on to have different sets of difficulties. In the piecewise functional definition instruction, using the dual solutions of $|x|=3$ to explain why we get two solutions to the problem $|x-A|=B$ confused the students. There were several questions concerning where the second (negative) case came from, and it took many examples completed as a class for some of the students to begin to catch on to the procedure. On the homework, two students completely forgot to include a negative equations. But the other 12 students who set up the cases correctly repeatedly made multiple types of algebraic and arithmetic errors. Thus, while most of the students eventually figured out the need for two solutions, and could set up the algebraic equations to obtain both solutions, only three students consistently (>95% of the time) found both correct solutions.

On the first day of distance definition instruction, the students rapidly became proficient in finding the solutions to story problems through the use of number lines. By the end of the second day, most students were independently translating the symbolic notation into story problems or number lines. From the problems assigned the second day, six of the seven students consistently solved the problems correctly, regardless of which representation was initially given. However, even though students were successful in finding the solutions to problems overall, generating different representations showed weaknesses in their comprehension of each representation.

When translating from story problems to symbolic equations, errors included one student putting a negative sign in front of the absolute value, one student writing an incomplete equation, and two students giving no equation at all. Students made fewer errors when translating from number lines to equations, with one student forgetting the absolute value symbols, and another writing equations of the form $|A+B| = x$ and $|A-B| = x$.

When translating from number lines or equations to story problems, students were roughly split between reusing cover stories that had already been established in class and creating new cover stories. Regardless, errors included inappropriate context for negative solutions, implied change in only one direction (instead of both), inconsistent starting points for change in both directions, situations that only promoted adding or subtracting values instead of using absolute value, simply repeating the absolute value notation using words, or no story problem given. Thus, students had difficulties in writing appropriate story problems, even when reusing contexts presented in class.

It was clear that students preferred to use number lines over the other two representations to solve absolute value equalities. During the second day, when students had a choice of translating symbolic notation into number lines or story problems, the researcher observed students consistently choosing to use number lines to solve the problems. In addition, while some students did not produce other representations when translating, every student always generated a number line when given a story problem or a symbolic representation. Amazingly, even though number lines were their favored representation, students had more difficulties with number lines than either of the other two representations. In translating from story problems and equations to number lines, the most frequent errors were labeling equally spaced distances on their number line as representing different values and creating inappropriate tick marks (such as using increments of five to count up to 23). Other errors included having one too many/few tick marks between the center and the edge and using the same number of unlabeled tick marks regardless of the numbers involved in the problem. Six of the seven students made at least one of these errors, and most students made multiple errors. However, errors in creating each representation did not affect students' ability to obtain correct solutions to the problems.

Conclusion

Students who learned the piecewise functional definition eventually were able to accept the idea that two solutions were necessary and set up the correct pair of equations, but algebraic difficulties consistently got in their way to finding the correct solutions. Students who learned the distance definition also had difficulties in generating story problems, symbolic notation, and number lines. However, these difficulties did not prevent them from successfully finding solutions. The researcher observed several of these students lifting the values out of the given representation and then solving the problems using arithmetic. In each representation, the students successfully established which number was the center and which number was the distance from the center, and then found center + distance and center – distance. Even though students had several misconceptions about number lines, the concept of a center and distance was clear enough. With the solution in hand, other representations were then created.

The value we discovered in presenting the distance definition to these students was thus twofold. It completely avoided the algebraic difficulties that prevent students from successfully finding solutions via the reasoning by cases procedure. These difficulties, seen in previously cited studies, were also seen in the piecewise functional definition group. More importantly, it appears that the distance definition promotes an understanding of the values involved in an absolute value equality that is basic enough to avoid common misconceptions surrounding number lines. By connecting symbolic notation to number lines and story problems, students were able to find and utilize the ideas of center and distance from the center in each representation.

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STUDENTS' TRANSITION FROM PROTO-RATIO TO RATIO AND PROPORTIONAL REASONING THROUGH CO-SPLITTING AND EQUIPARTITIONING

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This paper presents preliminary findings on the strategies used by third-grade students while working with the topic of co-splitting, a form of covariation, and how students develop an understanding of the concept based on prior learning experiences within equipartitioning. The range of these strategies are discussed as to how they pertain to a movement from proto-ratio reasoning to more fluent and sophisticated ratio and proportional reasoning, eliciting implications of equipartitioning as a strong foundation for a transformative move in thinking.

Keywords: Elementary School Education, Learning Trajectories, Rational Numbers

The purpose of this work is to address the transitional levels third grade students pass through in moving from early proto-ratio reasoning to more formal ratio reasoning. The following research question was formulated: *What are the different strategies of third-grade students working with co-splitting tasks based on their knowledge of equipartitioning?* A study of this nature is important because it recognizes equipartitioning and splitting as a foundation for multiplicative proto-ratio reasoning, which holds the potential to lead to ratio and proportional reasoning earlier and more directly than through additive compositions.

Literature and Framework

Ratio and proportion have been well-documented as difficult concepts for children to understand (e.g., see Behr, Harel, Post, & Lesh, 1992). Descriptions of protoquantitative ratio reasoning (Resnick & Greeno, 1990) and proto-ratio reasoning (Singer, Kohn, & Resnick, 1997) have been noted to only account for the development of additive properties of measure numbers (Singer & Resnick, 1992). Lamon (1993) found that some sixth grade students were capable of formal ratio reasoning, treating ratio as an invariable composite unit in solving proportion tasks. However, Streefland (1984, 1985) pushed for an earlier introduction of ratio reasoning, and Confrey (1994) too has argued that the concept of ratio can be accessed even at the early elementary grades with its cognitive underpinnings lying within an understanding of splitting and equipartitioning. She demonstrated that 3rd–5th graders could develop *ratio constructs* simultaneously as they worked with multiplication, division, slope and similarity. Nonetheless, there is a gap in the literature as to what the intermediary understandings are between these later and earlier treatments of the concept, what types of strategies are seen to coincide with those understandings, and what types of tasks may aid in eliciting those strategies.

Our research team has developed a learning trajectory for equipartitioning (Confrey, Rupp, Maloney, & Nguyen, in review), built on the context of fair sharing (Squire & Bryant, 2002; Confrey, Maloney, Nguyen, Mojica, & Myers, 2009) which relies on sharing collections (Pepper & Hunting, 1998) and partitioning a whole (Pothier & Swada, 1983). Co-splitting has been identified as a key construct at its upper levels where fair sharing is extended to multiple wholes among multiple people. Co-splitting is a specific, "primitive" form a covariation which establishes a ratio relationship between two quantities, such that any multiplicative change in one quantity is coordinated with the same multiplicative change in the other quantity. Moreover, these changes always occur in the same direction: increasing or decreasing. The term co-splitting was conceived to make explicit the nature of relationships in the tasks and problem-types utilized in equipartitioning that differ from many ways covariation is used in other places in the literature, which include coordination of changes that are additive or other (e.g., see Rizutti, 1991; Thompson, 1994; Confrey & Smith, 1995; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002).

Methods

The site of the study was a small charter school in an urban setting in the southeastern United States. A sample of eight students was purposefully chosen based on those students presence during the equipartitioning portion of an earlier teaching experiment. For a sample with maximum variation, prior student work and observations by the classroom teacher and the researchers were considered, and for gender balance, four females and four males were selected.

Two, semi-structured clinical interviews were conducted by the same interviewer with each of the eight students. These were limited to 45 minutes each so as to keep the student's interest, and therefore two sessions were necessary to complete the tasks, but both interviews were video-recorded and occurred within the same week for every student. The collected data were first open coded, considering single statements, responses, and mathematical moves (physical actions relevant to solving a problem or explaining how a problem was solved) to be the units of data. In a second round of coding, specific codes were developed based on common themes across the students that were relevant to the research question.

For the clinical interviews, we adopted the notion of what Streefland (1991) called "distribution" problems, asking students to determine how a number of people and a number of pizzas could be fairly distributed across multiple tables so that all receive the same size share. His representation for the problem displayed a circle and number of pizzas over a number of people sharing (hereafter a Streefland diagram, Figure 1). Students were presented with three tasks: (1) 24 pizzas and 18 people, (2) 12 pizzas and 8 people, and (3) 25 pizzas and 15 people.

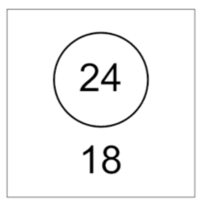
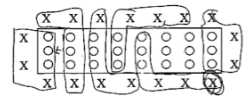
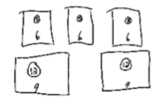
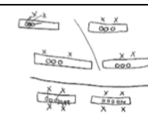
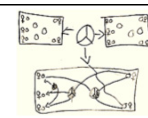
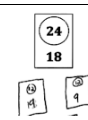
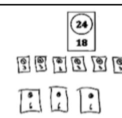


Figure 1: Streefland diagram for 24 pizzas and 18 people

Results

There were four types of strategies utilized by the third-graders working with co-splitting tasks. More than one level was observed for some strategy types, all eight of which have been ordered from lowest to highest sophistication and are shown in Table 1 below. There was also one dominant facilitating strategy that showed up across these main strategies, which was the use of manipulatives or drawing physical representations. In doing so, the students relied on experiences from equipartitioning collections of objects and equipartitioning single wholes, particularly dealing in rounds and establishing one-to-one correspondences. This often led them to the determination of the ratio unit (smallest whole number ratio within a set of proportions) and/or the unit ratio (quantity per 1)—which represented the fair share in these problems. Most all of the students used more than one of the strategies during the course of the interviews. No students initially attempted to create an arrangement that did not involve a splitting action on the people and pizzas. However, all but one student were able to justify presented arrangements that involved different numbers of people and pizzas at one or more tables, and more than half went on to later create at least one arrangement involving unequal quantities of people and pizzas across the tables. Therefore, almost every student reached a combining ratios level at some point.

Table 1: Observed and Hypothesized Strategies

Type	Strategy	**Brief Description:** Students determine the number of tables by …	Exemplar of Student Work
Guess	Guess	… random choice – do not specify mathematical reasoning (often an initial approach)	
Adjustment	Incremental Adjustment	… adding/subtracting tables from previously attempted numbers	
	Factor-based Adjustment	… multiplying/dividing tables from previously attempted numbers	
Splitting and Co-Splitting	Split and Check	… splitting (partitive) one quantity, and then checking the same split on the other quantity	
	Partitive Co-split	… splitting (partitive) both quantities	
	Inverse Co-split	... inverse splitting (doubling or *n* times) both quantities	
Combining Ratios	Combining Like Ratio Units	… combining like unit ratios, ratio units, or compound ratio units, to form equivalent fair shares	
	Hypothesized: Combining Unlike Ratio Units	… combining unlike unit ratios, ratio units, or compound ratio units, to form equivalent fair shares	*From above – create two tables: one with 3 pizzas and 2 people and the other with 9 pizzas and 6 people*

Discussion and Conclusions

An understanding of what it means to preserve the fair share (unit ratio) was vital to students being able to properly justify their responses and work flexibly with different splits and strategies. However, finding the unit ratio corresponding to a fair share did not imply that students would use a more sophisticated strategy, such as combining ratio units. The students in this study were able to find a unit ratio readily, which appeared to be based on equipartitioning knowledge and skills, but only one student used the combining ratio units strategy fluently.

Proficiency in this upper-level equipartitioning construct of co-splitting clearly required knowledge of fairly sharing collections and single wholes, and an understanding of the principle of continuity, which states that fair sharing of a single whole is possible for any number of people. However, the construct is also further enhanced by understandings of compensation and reallocation. Therefore, mastery of *all* the lower levels in the equipartitioning learning trajectory provides a solid path along which the construct of co-splitting is understood and the strategy of co-splitting can be used to solve these types of sharing problems. Experiences with co-splitting tasks such as these will allow students reaching the combining ratios strategy to establish a multiplicative proto-ratio understanding, which can be later built upon and used in conjunction with ratio comparison tasks to create a more formal understanding of ratio.

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CONCEPTUAL ALGEBRA READINESS: MEASURING AND DEVELOPING TRANSITIONS IN THINKING

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A key transition for students is from arithmetical thinking to algebraic reasoning. Measuring the impact of materials and approaches intended to promote this transition requires assessment tools that address more than symbol manipulation skills and computational fluency. This paper provides a brief overview of our perspective of conceptual algebra readiness, our initial attempt to measure this construct, and a brief glimpse of how we are using the test results to develop activities to help students develop conceptual algebra readiness.

Keywords: Algebra and Algebraic Thinking; Learning Trajectories (or Progressions)

Introduction

This study is part of a larger project to prepare students to be conceptually ready for algebra, (conceptual algebra readiness). To accomplish this goal, we first attempted to gain a better understanding of conceptual algebra readiness. Second, we attempted to measure this construct. Finally, we are using these results to develop and refine activities that promote this construct. This paper will address our attempts to determine test items and constructs that measure conceptual algebra readiness and also its utility as a potential predictor of success in algebra.

Through a state Mathematics Science Partnership grant with a local school system, we are developing weekly problem solving activities for students in grades 4–7 to improve their conceptual algebra readiness. In the year prior to the implementation of the activities in classrooms, we conducted a study of eighth grade students in a different, but neighboring, school system to try to determine what problems and constructs were indicators of conceptual algebra readiness.

Our reasoning is that helping students understand certain underlying concepts will better prepare them for success in algebra. And if we can measure students' understanding of these underlying concepts or constructs we can (1) develop and refine activities which will help students gain the appropriate conceptual understandings, and (2) predict success in algebra. If we can measure algebra readiness and predict success in algebra then we will have a way to measure the success of our project.

Measuring the impact of materials and approaches intended to promote conceptual understanding of algebra requires an assessment tool that addresses more than symbol manipulation skills and computational fluency. There are a multitude of tests that purport to measure algebra readiness. A brief examination of some of these tests reveals that most are focused on the skills necessary to succeed in algebra (e.g., computational fluency, fractions, decimals, percent, inequalities, etc.). While these tests attempt to measure algebra readiness, they typically measure knowledge of students' arithmetic skills or pre-existing algebraic understandings and do not get at the conceptual nature of algebra—the real difference between algebra and arithmetic—the ability to generalize and work with generalizations. Other, existing tests are measures of symbolic manipulation. However, these tests fail to address the need to assess students' conceptual readiness for algebra.

Theories and Perspectives of Early Algebraic Learning

The field of research on early algebra—developing algebraic reasoning prior to formal algebra—is quite diverse with several different approaches and theories guiding these approaches (Blanton & Kaput, 2003; Carraher, Schliemann, & Brizuela, 2000; NCTM 2000; Schifter, 1999). Our project has attempted to translate (Lagemann, 2009) these theories and approaches into usable knowledge in order to determine "what works." Under further examination, we found that no one theory could effectively explain how children learn algebra and there was no consensus. Consequently we used multiple theories and approaches as a theoretical basis for the test we developed.

Methodology

Using the previously described theoretical perspectives and others which are too extensive to include in this paper, we developed a test for eighth grade students designed to measure conceptual algebra readiness. The test consisted of 15 items with sub-parts for a total of 23 responses. Items were either scored correct, "1" or incorrect, "0."

The study took place in two middle schools, grades 6–8, from a Midwestern city school system, population 6,344 with 49% of the students receiving free or reduced lunches. The state average is 48%. At the beginning of the school year, the test was given to all eighth grade students taking algebra and to eighth graders taking pre-algebra whose parents had completed a student permission form. At the end of the school year the students' grades were collected in algebra and pre-algebra and the algebra students' End of Course Assessment (ECA) on the state mandated test in algebra. Table 1 shows the number of students in each group who took the test at the beginning of the year. The number in parentheses is the number of students who we received their final grade and /or ECA score.

Table 1: Samples Sizes

School	#Pre-algebra	#Algebra
B	80 (64)	47 (45)
K	41 (36)	34 (29)
Total	121 (100)	81 (74)

Results and Discussion

As part of the analyses we computed the mean number of correct responses and standard deviation of each group on the test. In each school the algebra students scored higher than the pre-algebra students, as expected.

Table 2: Mean and Standard Deviation

School	Pre-algebra Mean (STD)	Algebra Mean (STD)
B	11.4125 (3.22)	17.213 (2.36)
K	10.8049 (3.06)	15.4706 (3.50)
Combined	11.16 (3.16) n = 100	16.58 (2.68) n = 74

We combined the pre-algebra students from both scores and the algebra students and conducted a two-tailed *t*-test of the means. The test was statistically significant at the $p = .01$ level.

On all items, the algebra students scored better than the pre-algebra students. How might this chart inform our test construction and design of student activities? Item #3 asks students to generalize:

> *A machine that puts kites together costs $12.00. The cost of materials to make one kite is $3.00 per kite.*
> *1c. What if you do not know the number of kites you need to make.*
> *How much will it cost to make 'n' kites?*

Here we expected differences between the groups; 46% and 74% respectively. We expected the Algebra students' ECA score would correlate with this item. We ran a basic Pearson correlation between the test item #3 and students ECA scores where we found a low correlation of 0.16. This item represents a key focus of our project and we will need to consider why there was not higher correlation.

Items #18–21 ask students to graph points on a coordinate plane.

Plot the following points on the graph provided: (-3,4), (0,-2), (2,3),(4, -1)

Both groups did well with small differences. This suggests that graphing ordered pairs on a coordinate plane need not be a major focus of our student activities. Our next test iteration may focus on the concept of slope instead.

A more modest correlation, 0.48, occurred between item #5 and students' ECA scores which dealt with general problem. In the algebra group, 64% of the students had this item correct and 36% of the pre-algebra students had the item correct.

A protractor and a compass cost $3.00. If the protractor costs $.80 more than the compass, how much does each cost?

Here there is a major difference between the two groups. We will keep this question on our next test iteration and develop more student activities similar to this problem. Consequently, to improve the conceptual algebra readiness of students in grades 4–7, we will focus more on problem solving. This reinforces one of our underlying premises of the project; a problem solving approach is a viable way of helping children develop algebra readiness.

Another modest correlation; 0.47 occurred between item #22 (a generalization problem where students were asked to choose the best deal) and students' ECA scores.

Raymond has some money His grandmother offers him two deals:
Deal 1: She will double his money
Deal 2: She will triple his money and then take away 7
Raymond wants to choose the best deal. What should he do?

We expected a difference between the pre-algebra and the algebra groups: 20% and 47 %. This suggests that these types of problems might be a focus of student activities we design for the project.

Conclusions

We are using these analyses to make revisions to a conceptual algebra readiness test we developed and to the classroom activities we are developing. Conceptual algebra readiness is a difficult concept to measure. However, our initial attempts to measure it have proven fruitful in helping us refine student activities and in developing the second iteration of a test.

Our study was limited in that we do not yet have the ECA scores from the students who were in pre-algebra. We also expected higher correlations on the item analysis.

Our next phase is to refine our activities for students based on our findings from this study. We would like to give the algebra readiness test to students who have used our activities, to assess how well our project is preparing students conceptually for algebra. Ideally, we would like to test students with multiple years of experience using our activities in grades 4–7. Our aim is to both measure conceptual algebra readiness and to better prepare students for algebra.

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EXPLORING EQUIVALENCE OF EXPRESSIONS THROUGH SPREADSHEETS

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In this study, I investigated how two first year university students in an introductory algebra class developed ideas about equivalence by using spreadsheet-based tasks. Using a teaching experiment method I proposed a small-scale hypothetical learning trajectory focused on three topics; variable, explicit form, and methods for determining equivalence. Participants showed progress along these three landmarks as indicated by a pre-survey and post-interview data as well as discourse data from the three teaching sessions. Evidence from this study suggests that spreadsheet-based tasks may support some students with investigating, conjecturing, and reasoning about the equivalence of algebraic expressions.

Keywords: Algebra and Algebraic Thinking; Design Experiments; Technology

Purpose

Algebra has been called a *gatekeeper* for future opportunities in education and employment (e.g., Knuth, Alibali, Weinberg, & Stephens, 2005), and research indicates that students who successfully pass algebra requirements have more opportunities than students that do not (National Mathematics Advisory Panel, 2008). Equivalence of expressions sits at an important crossroads in the learning of algebra because it relies on an understanding of *variable* and a relational view of the equal sign, something with documented challenges for younger students (e.g., Knuth, Stephens, McNeil, & Alibali, 2006).

Learning Equivalence through Technology

Although research on equivalence of algebraic expressions is somewhat limited there are some studies that indicate technology could be a potentially useful instructional tool (e.g., Kieran & Saldanha, 2005). Interested in the affordances of technology in learning equivalence, I chose to focus on spreadsheets for three primary reasons; they are economically viable and widely available, there are similarities between algebra-syntax and cell-formula programming, and a main feature of spreadsheets is the capability to generate large tables of values to inspect and reason about equivalence. The research question for this study was, *What is the nature of learning of algebra students on equivalence of expressions by interacting with tasks in a spreadsheet environment?*

Theoretical Framework

To address the research question I conducted a teaching experiment. A core purpose of this method is to better understand student thinking by generating models of learning (Lesh & Kelly, 2000). A main feature of teaching experiments is the close interaction of the researcher with the instructor whether it is as a consultant or a dual role as in this study. The main rationale for conducting a teaching experiment in this study was two-fold. First, I was interested in studying a change in knowledge (i.e. learning) that took place over time. Second, although this was a small-scale introductory study, any notions of generalizability would hopefully be useful to practicing teachers. Thus, it was important to strike a productive balance between creating a research-friendly environment while maximizing the authenticity of classroom interactions.

Hypothetical Learning Trajectory on Equivalence

Underlying the teaching experiment method is an articulation of how participants' understanding of equivalence changed over the sessions called a *hypothetical learning trajectory* [HLT] (Simon, 1995). This trajectory is the researcher's hypothesis of how participants will progress through the teaching experiment, which includes certain landmarks of understanding. The trajectory and landmarks should be informed by research whenever possible. Because the research literature on equivalence is less developed compared to other mathematical topics, there is minimal elaboration of what an HLT on equivalence of expressions might look like. In the next few paragraphs I outline three important landmarks as one possible trajectory.

Landmark 1 – Variable. Researchers (e.g., Usiskin, 1988) have indicated many different ways that the term *variable* can be interpreted, ranging from a specific unknown value, to representing a set of values, to a "*temporally indeterminate number* whose fate is to become determinate at a certain point" (Bardini, Radford, & Sabena, 2005, p. 129). Others (e.g., Wagner, 1981) have pointed out how students misinterpret variables in particular situations. The general nature of variables is essential in seeing the corresponding generality in two equivalent expressions and so it was included as the initial landmark of the HLT.

Landmark 2 – Explicit form. The second landmark is writing a rule in explicit form. This landmark was chosen specifically because of the spreadsheet intervention. The ability to generate large tables of values using a spreadsheet might lend participants to reason recursively. Recursive equivalence is in some ways more complicated, and explicit form is in many cases simpler when talking about relationships of certain uncountable sets. Researchers (e.g., Lannin, 2004) have claimed that it is more natural for students to reason recursively and others such as Swafford and Langrall (2000) report findings indicating that some students' ability to represent situations symbolically is related to the likelihood they would symbolize a relationship explicitly.

Landmark 3 – Equivalence methods. The final landmark is creating a method to reason about equivalence. Kieran and Drijvers (2006) refer to one approach for determining equivalence as *numerical*. Using the method a student reasons two expressions are equivalent if the same input values correspond to the same output values for both expressions. In contrast, Kieran (2007) provides an alternative approach referred to as the *transformational* approach that uses algebraic properties (such as the distributive property) to reason if two expressions are equivalent. This final landmark is the end goal of the HLT, for a student to develop a rationale for determining whether two expressions are equivalent or not.

Method

Two college algebra students met with the teacher-researcher over three 60–90 minute teaching sessions in a three-week time period with an individual post-interview a few days after the final teaching session. The sessions consisted of two to three tasks per session designed to create opportunities to encounter equivalence in different ways such as geometric patterns and relationships between numbers. In between sessions a colleague played the role of consultant helping with planning and making any modifications to the planned tasks for the sessions.

Data Collection

Throughout the sessions participants' paper-and-pencil and electronic work was collected. Video was recorded and later transcribed both in large group and on the participants' computers by use of screen capture software. Participants' responses on a pre-survey and post-interview were also collected. The first part of the survey focused on comfort and frequency of spreadsheet use. The second part focused on equivalence of algebraic expressions. In the post-interview, participants were asked open-ended questions about their experiences, as well as five identical questions from the pre-survey on identifying equivalence. Analysis of these items was conducted comparing the answers provided on the pre-survey with post-interview responses.

Analysis of sessions. To analyze how participants progressed along the HLT a discourse analysis was conducted focusing on two types of talk; how they talked about the spreadsheet, and how they talked about equivalence of expressions. Along with coding on expressions and equivalence, Gibbons (2002) *mode continuum* was used as a framework to analyze participants' talk about spreadsheets. The mode continuum describes how explicitly or implicitly participants talk refers to objects with shared meaning. Table 1 below describes the three subcategories along the continuum that was coded.

Table 1: Example Coding of Participant Discourse

Modal Code	Description	Example
Situation embedded	Shared meaning of language, often using pronouns such as *this*, *that*, or *it*.	"It will give you what that equals"
Specific Language	Objects in the statement are referred to explicitly, the speaker reconstructs experience through their description	"In C2 I guess you'd do A1 plus B2"
Object-Generic	Speaker talks non-temporally, the statement is about an object not a specific event.	"Spreadsheets are amazing"

Results

In this section I provide results in the form of a case study. Due to space limitations, I only provide Xandra's results from the discourse analysis and movement along the HLT. Xandra's discourse analysis shows a gradual shift from situation embedded language (66% in session 1) to more specific language (64% in session 3). Xandra seemed to have a solid working definition of variable as evidenced by her ability to relate cell-formula names to their algebraic analogues. She was able to justify why the expression 3*(G1 + 2) was equivalent to 3*G1 + 6 via the distributive property. It was unclear however, when Xandra used the drag-fill feature whether she interpreted the copying and pasting to be a move from a single situation (i.e., B1 = 3*A1) to a more global situation (i.e., column B = 3 times column A) analogous to a students' movement from interpreting a literal symbol as an unknown to a variable.

For the final two landmarks on the HLT, Xandra demonstrated the ability to look for patterns and describe relationships both recursively and explicitly. She often articulated expressions multiplicatively instead of additively consistent with an explicit form approach for the tasks provided. From the beginning of the first session, Xandra used the spreadsheet to investigate equivalence. Across the sessions, Xandra was more likely to justify equivalence through the use of properties (34%) such as the distributive property, and she made over twice as many claims about why expressions were equivalent than Melissa. Overall, Xandra showed strengths in her understanding of variable, her ability to generate expressions in explicit form, and fluency in reasoning about equivalence. Her post-interview told a somewhat different story.

Xandra had only one correct answer on the pre-survey's five questions focused on determining whether two expressions were equivalent. Identical questions were given during the post-interview. Xandra confidently answered all five incorrectly. After providing her answers she was prompted, "is there a way you could use the spreadsheet to convince someone that…[these expressions are] equivalent or they're not equivalent?" One by one, she worked through the problems using the spreadsheet and convinced herself that her original conclusions were all incorrect.

Conclusions

The results of this study indicate that Xandra used spreadsheet based language with increasing specificity as the sessions progressed. This lends evidence to the claim that her ability to communicate in the technological discourse increased indicative of a change in knowledge. In Xandra's case she was able to re-convince herself of the equivalence or non-equivalence of two algebraic expressions by using the spreadsheet. Admittedly, it is unclear whether Xandra's move to use the spreadsheet was the reason for the change in her solutions, or whether a similar occurrence would have resulted by similar presses for justification by the instructor/researcher.

While generalizable claims are difficult to make given the small convenient sample size and participants' general interest in technology, both participants did show movement along the HLT. Variables were seen as generalized quantities, participants articulated the advantages of writing expressions in explicit form, and they applied different methods for determining equivalence. These landmarks provide a productive starting point for future work on unpacking students' notions of equivalence of algebraic expressions.

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PERFORMANCE TRENDS IN ALGEBRAIC REASONING: 1996 TO 2011

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This ongoing study focuses on the algebra knowledge of 4th and 8th grade students as measured by the 1996 through 2011 NAEP mathematics assessments. Over that period, 149 items have been used allowing analysis of performance on topics such as understanding and use of algebraic expressions, understanding of numeric patterns, and ability to use equations and inequalities to solve problems. For example, performance on a grade 8 item assessing ability to write the equation of a line passing through a point increased significantly from 2007 to 2011, suggesting that there is improved coverage of this topic at the middle school level. In contrast, there has been little change on three items requiring equations to solve word problems. Knowledge of such patterns provides information about what areas of the mathematics curriculum are effective and what areas need to be strengthened.

Keywords: Algebra and Algebraic Thinking; Assessment and Evaluation

The National Assessment of Educational Progress (NAEP) has collected data on the performance of elementary, middle, and high school students in the United States since the late 1960s. Of the content areas that NAEP has assessed, there has been more growth in mathematics performance at the elementary and middle school levels than any other subject area (Kloosterman & Walcott, 2007). Growth in mathematics performance at the high school level has been modest, although major changes in the grade 12 assessment between the 2000 and 2005 administrations make patterns of growth after 2000 difficult to quantify.

In addition to reporting overall results, NAEP has reported results by content strand (number and operation; geometry; measurement; algebra; and data analysis, probability, and statistics) since 1990. Like the overall trends in mathematics performance, the trends in performance in the algebra strand for students in grade 8 showed substantial improvement between 1990 and 2011. In contrast, there was substantial improvement between 1990 and 2005 at grade 4 but there was only a 3-point gain between the 2005 and 2011 administrations and the algebra score did not change at all between 2007 and 2011 (NAEP Data Explorer, 2012). Given the importance of algebra and algebraic thinking (National Mathematics Advisory Panel, 2008), this study analyzes grade 4 and grade 8 NAEP algebra data from 1996 through 2011 with the intent of explaining what NAEP assesses with respect to algebra and the extent to which gains have varied across topics within algebra (patterns, graphing, linear equations, etc.).

Background

The National Center for Education Statistics (NCES) provides reports on overall findings of each NAEP assessment. The Main NAEP assessments, which are the basis for this study and a bit different from the Long-Term Trend Assessments (Kloosterman & Walcott, 2007), provide overall results for the nation as a whole, by state, and by demographic subgroup. The reports also provide technical information such as sampling procedures, item development and scoring procedures, and details of statistical analyses performed on the data. Because NAEP is based on a representative national sample of students at the grades where it is administered (4, 8, 12), results based on NAEP data are valid for the United States as a whole. Although any one student completes at most 25 NAEP items, different items are completed by different students so when results are pooled across students, there is information on a wide variety of skills. The 2003 Main NAEP assessment, for example, used 182 items at grade 4 and 189 items at grade 8. Of these, 26 of the grade 4 and 48 of the grade 8 items were in the algebra strand. After each administration, roughly one-fourth to one-third of the items are replaced so that there are enough items to track tends over time while allowing for updates to keep the assessment consistent with changes in curriculum. Most items that have been retired are released to the public (see http://nces.ed.gov/nationsreportcard/itmrlsx/). One of the reasons that items are retired is that they no longer represent what is being taught in schools and thus released items are not necessarily representative of the NAEP assessment as a whole.

Since the early years of NAEP, there have been interpretive reports based on the specific items used for the mathematics assessment. Some of these reports have focused specifically on NAEP algebra items. For example, Chazen et al. (2007) reported that gains in the algebra strand were greater than gains in any other content strand for grades 4 and 8 in recent years but that gaps in performance based on race/ethnicity persist. These researchers also found that performance on items used at both grades 4 and 8 was always higher at grade 8 although the amount of difference between grades varied substantially by item. Looking at performance of grade 8 students on NAEP, Sowder, Wearne, Martin, and Strutchens (2004) reported on 10 algebra items that were used in 1990, 1992, 1996, and 2000. Of those items, performance increased significantly from 1990 to 2000 on both pattern items, three of six items involving algebraic expressions or equations, and both items involving graphing. In contrast, performance increased significantly on only 1 of 8 algebra items administered from 1996 to 2000. In a similar analysis, Kloosterman et al. (2004) found significant gains on all 5 pattern and informal grade 4 algebra items used from 1990 to 2000 but just 2 of 5 items used only in 1996 and 2000. These types of analyses show that progress varies depending on time frame and also on the specific task assigned.

NAEP Conceptual Framework

The NAEP assessment has always been based on a conceptual framework outlining content and grade or age level assessed, sampling characteristics, item format, and additional issues such as use of calculators and complexity or difficulty of items (e.g., National Assessment Governing Board, 2010). This framework, which is updated periodically, affects both what and who is assessed. For example, the current framework includes provisions for accommodations for students with disabilities in the sample whereas the framework used 20 years ago included students with disabilities only when they could complete the assessment without accommodations. For the purposes of this study, it is assumed that the NAEP frameworks are adequate for item development along with collection and reporting of performance data. It is also assumed that NAEP items used in the study are appropriate measures of algebraic reasoning. In other words, this study defines student understanding of algebra as the responses given on the NAEP assessment.

Method

Because the sampling procedure for Main NAEP was the same from 1996 to 2011, this study focuses on all items that included algebraic concepts used during that period. Given that previous research, including the research described earlier in this report, has reported on some of the algebra items used between 1996 and 2003, there will be a special focus on performance between 2003 and 2011. Note that only items that have been released can be reported verbatim, but the content of non-released items can be described in general terms and thus non-released items are also included in the analyses.

All items were coded by two members of the research team into categories including (a) algebraic expressions; (b) patterns, relations, and functions; and (c) mathematical reasoning in algebra. Based on those codes and the need to have enough items in any one sub-category to make generalizations, the research team is currently building tables of results for the items on topics where there are four or more items. Tables showing items and results for all categories for grades 4 and 8 will be available on the project website (ceep.indiana.edu/ImplicationsFromNAEP/) in time for discussion of those tables at PME.

Results

Table 1, provided as an example of the tables that are being developed, shows the percentage of students correctly answering items related to understanding and use of algebraic expressions at grade 8. The table includes 5 released items, which are reported verbatim, and one non-released item, which is only described. As was the case in previous reports of item-level performance on NAEP, results reported in Table 1 show that performance on some of the individual items has been relatively stable but performance on others has changed. Item 2, in which students were told that N equals a specific perfect square and then asked to identify the square root of N, had the greatest gain with a 6% increase in performance over a 4-year period. Item 6, where students had to identify the equation of a line passing through a given point, had a 5% increase in performance over 4 years. These results suggest that seeing square roots in symbolic format and seeing the connection between equations and graphs are topics that are getting more attention at the middle school level. The relatively low performance on item 1 shows that while students have experience at using common formulas in middle school, they have difficulty when formulas are described rather than explicitly stated. Item 3 shows that almost ¾ of students can write a simple algebraic expression while items 4 and 5 show that when expressions get a bit more complex, performance drops significantly.

Discussion

Although NAEP provides overall results by content strand, those results are too general to give a good sense of exactly how performance changes over time on specific types of mathematics tasks. Item-level analyses help to alleviate this problem. Between 1996 and 2011, there were 55 grade 4 and 94 grade 8 items that involved some sort of algebraic reasoning and most were used for more than one year. With an item pool this large, there are enough items on relatively specific algebra-related topics to get a sense of the algebra knowledge of 4th and 8th graders as well as a sense of how knowledge in each of those areas is changing over time.

Table 1: Performance on Grade 8 Items Involving Understanding and Use of Algebraic Expressions

	Percent Correct				
Item Description	2011	2009	2007	2005	2003
1. The temperature in degrees Celsius can be found by subtracting 32 from the temperature in degrees Fahrenheit and multiplying the result by 5/9. If the temperature of a furnace is 393 degrees in Fahrenheit, what is it in degrees in Celsius, to the nearest degree? (calculator available)			37	35	35
2. Find a number given its square root. (secure item)			64	61	58
3. If *m* represents the total number of months that Jill worked and *p* represents Jill's average monthly pay, which of the following expressions represents Jill's total pay for the months she worked?			73	72	73
4. At the school carnival, Carmen sold 3 times as many hot dogs as Shawn. The two of them sold 152 hot dogs altogether. How many hot dogs did Carmen sell?			47	47	47
5. Robert has x books. Marie has twice as many books as Robert has. Together they have 18 books. Which of the following equations can be used to find the number of books that Robert has?	53	52	52		
6. Which of the following is an equation of a line that passes through the point (0, 5) and has a negative slope?	31	29	26		

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LEARNING TRAJECTORIES: THE DEVELOPMENT OF EARLY FUNCTIONAL THINKING WITH GEOMETRIC GROWING PATTERNS

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Learning trajectories have gained recent attention and prominence in mathematics education. This research investigated the development of a tentative learning trajectory in relation to students' algebraic reasoning with geometric growing patterns. Design research was used to develop a local instruction theory, to which figural reasoning was central. A hypothetical learning trajectory was developed, including a learning goal, instructional activities, and a learning progression. Four teaching experiments were conducted in sixth grade classrooms. Based on the analysis of several sources of data, changes were warranted for the hypothetical learning trajectory, and a revised learning progression is outlined and discussed.

Keywords: Algebra and Algebraic Thinking; Design Experiments; Learning Trajectories (or Progressions)

Learning trajectories have gained recent attention and prominence in mathematics education. They played a major role in the development of the Common Core State Standards in Mathematics (Common Core State Standards Initiative, 2010); when possible, the research around students' learning of mathematical concepts was used to design appropriate mathematical standards for the grade levels (Daro, Mosher, & Corcoran, 2011). Empirically-based learning trajectories, however, are not available in all domains of mathematics. In a 2011 report (Daro et al., 2011), the status of learning trajectory research in mathematics was summarized, citing rigorous research in several domains. Although quality research exists and is ongoing, algebra was identified as an area lacking research on learning trajectories.

In this paper, I summarize one aspect of my dissertation research in the development of a learning trajectory in relation to students' algebraic reasoning with geometric growing patterns. Growing patterns have characteristics which make them unique and ideal for supporting students' development of functional thinking. Shapes comprise geometric growing patterns; these shapes and their configurations can range from very simple to exceedingly complex.

This study used design research to develop a local instruction theory: "a theory about the process by which students learn a given topic in mathematics and theories about the means of support for that learning process" (Gravemeijer & van Eerde, 2009, p. 510). One aspect of a local instruction theory is a hypothetical learning trajectory. As defined by Simon (1995), a hypothetical learning trajectory consists of three components: (1) a learning goal, (2) learning activities, and (3) the hypothetical learning progression by which students' thinking might evolve.

Hypothetical Learning Trajectory

The learning goal for this learning trajectory was the development of students' functional thinking. Smith (2008) defines functional thinking as "representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances" (p. 143). The functional relationship is the relationship that is identified between the stage number and some aspect of the geometric growing pattern. Figural reasoning is central to this conjectured local instruction theory and "relies on relationships that could be drawn visually from a given set of particular instances" (Rivera & Becker, 2005, p. 199).

Four mathematical practices along a learning progression were proposed: (1) identifying and articulating the growth in a geometric growing pattern using figural reasoning, (2) translating figural reasoning to numerical reasoning, (3) identifying and articulating a relationship between the stage number and a quantifiable aspect of the geometric growing pattern, and (4) using variables as varying quantities for generalization of the linear function. Figural reasoning is foundational to the first three mathematical practices; the relationship that is articulated should be based on figural reasoning. It was anticipated that students would first articulate a relationship using words. Variables would be used as a more succinct expression of the functional relationship in the fourth mathematical practice. Six lessons were designed for use in sixth grade classrooms to support students' progression through these four mathematical practices.

Methods

This study consisted of four teaching experiments, with two teaching experiments conducted simultaneously in two macrocycles. Each macrocycle consisted of the three phases of the design research process: design of instructional materials, classroom-based teaching experiments, and retrospective analyses (Gravemeijer & Cobb, 2006). A revised version of the instructional materials was implemented in the second macrocycle.

Participants

Teaching experiments were carried out with classes of sixth grade students. Four teachers (two pairs of teachers from two schools) participated in the study. One class was selected from each teacher's schedule. All students in the classes selected received instruction by the researcher and were recruited for participation in data collection procedures (Table 1).

Table 1: Student Participants in Teaching Experiments

Teaching Experiment	Total Students in Class	Number of Student Participants	Number of Participants Allowing:		
			Video-Recording	Collection of Student Work	Interviewing
TE1	23	12 (52.2%)	10	12	8
TE2	23	16 (69.6%)	16	16	13
TE3	28	13 (46.4%)	12	13	9
TE4	24	13 (54.2%)	12	13	9

Data Collection and Analysis

Several sources of data were used in this study: a pretest and posttest, co-researcher and witness classroom observations, whole-class and small group video-recording, daily reflection audio-recording with research team, student interviews, and artifact collection of student classwork and SMART board files. At the conclusion of the first macrocycle, all data from the teaching episodes were compiled for retrospective analysis. Retrospective analysis at the conclusion of the study included a thorough organization of the data, rescoring of all pretests and posttests, field notes on several forms of data collection, some statistical analysis, and extensive coding of both *a priori* and emergent themes.

Findings

Overall, figural reasoning strategies increased and numeric reasoning strategies decreased throughout the course of the instructional sequence. Whole-class discussions established the expectation that students base calculations on a way of seeing identified through engagement in the first mathematical practice. Although the result of the second mathematical practice was a numerical expression, it was consistently conveyed that this expression must be grounded in the pattern's physical structure. This allowed students to identify a generalizable calculation method.

Although other aspects of the instructional design brought out and supported students' functional thinking, the three-column table was a particularly effective tool. Typically, the first three stages of the growing pattern were articulated in the three-column table. The numerical calculations (based on figural reasoning) were then extended to other stage numbers. Students successfully used the thinking from the middle column of the three-column table to extend and generalize a calculation involving the stage number that generated the total number of pattern blocks, chairs, etc. Thus, the three-column table supported students' functional thinking.

Students' representations of the functional relationships were supported by the extended use of the three-column table. In the first macrocycle, students' representations progressed from rules with words to semi-symbolic rules, to full symbolic rules incorporating variables. Very few rules progressed to this full symbolic representation, in part because of the extensive time needed to guide students through this process. In the second macrocycle, this process was facilitated by the use of the three-column table; the numerical reasoning was extended and generalized in the three-column table by substituting a variable for the stage number.

Revised Hypothetical Learning Trajectory

The findings of this research warranted changes to the hypothetical learning trajectory. Functional thinking remains the goal, but increasingly sophisticated representations of this thinking can be considered an extension. The results of this study suggest that the learning progression of the hypothetical learning trajectory can be refined (Figure 1). The first three practices illustrate students' learning progression with figural reasoning, translating figural reasoning to numerical reasoning, and identifying the functional relationship. The bottom section of the diagram illustrates potential progressions in students' representation of the functional relationships.

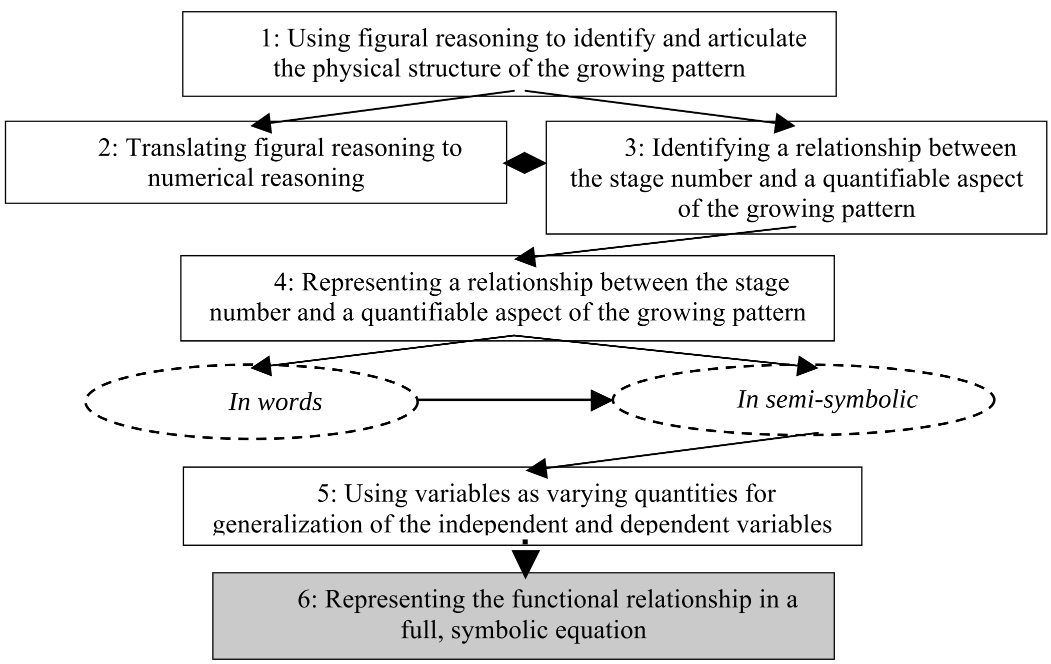

Figure 1: Revised learning progression

Students' responses in this study indicated that a focus on growth elicits recursive descriptions of the patterns. Rephrasing the first mathematical practice focuses on the physical structure of the pattern, rather than the growth that occurs as a change from stage to stage. The second mathematical practice remains the same, but the third mathematical practice has been broken into two different mathematical practices, first focusing on identifying a functional relationship and second focusing on representing the functional relationship. Two avenues of representation are suggested. The first path (left first) is suggested when the ways of seeing the growing patterns are not too complex to be represented in words. The second path (right first) is suggested with more complex patterns, for which the representations in words can be unwieldy.

The introduction to variables as varying quantities can effectively build upon the representations that students generate using familiar mathematical symbols, i.e., numerals and symbols for operations (the fifth mathematical practice). Representation of the independent variable was successful in this study, but further work is necessary around using two variables to represent functional relationships. A sixth mathematical practice has been added to the learning progression. This practice was achieved on two occasions in this study, and these occurred in whole-class discussions in the first macrocycle. More effective progression to this mathematical practice likely requires more time and exploration with geometric growing patterns. This worthy extension of the goal should be considered with further research.

Conclusion

This instructional sequence was intended to be an introduction to functions and the use of variables in representing functional relationships. Neither the mathematical context of geometric growing patterns nor the duration of the experience is sufficient for bringing students to a thorough understanding of functional relationships or how to express these relationships in full, symbolic form. However, this research provides an empirically-based, tentative learning trajectory for students' development of functional thinking in algebra. It is a place to begin with additional research to determine if this sequence of mathematical practices is valid, and how tasks might be improved to support students' development of functional thinking at other ages.

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RESEARCH ON STUDENTS' ALGEBRAIC THINKING: WHAT IS USEFUL FOR TEACHERS?

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This systematic review of mathematics education literature identified over 800 manuscripts that addressed students' algebraic thinking. Initial analysis of articles led to a framework examining the math involved in a misconception or way of thinking, assessment problems, interviews with students describing their thinking, and suggested instructional strategies to develop understanding. The articles were divided into five domains of algebraic thinking based on the Common Core State Standards: Variables & Expressions, Algebraic Relations, Analysis of Change, Patterns & Functions, and Modeling & Word Problems. This report summarizes what we know from research in each domain that may be useful for teachers.

Keywords: Algebra and Algebraic Thinking; Mathematical Knowledge for Teaching; Teacher Knowledge

Algebra is the "gatekeeper" to higher education and employment because, rather than helping students develop mathematical competence and gain access to higher education, it screens out many students. For instance, in the Los Angeles School District, "48,000 ninth-graders took beginning Algebra; 44% flunked, nearly twice the failure rate as in English… It triggers dropouts more than any single subject" (Helfand, 2006, p. 1). The literature indicates students' comprehension of algebraic ideas is often not what it seems. Errors can result from simple carelessness or forgotten rules, but studies indicate common struggles with more complex sources. Some "errors are widespread among students of different ages, independent of the course of their previous learning of algebra" (Demby, 1997, p. 48). Students' have significant mental hurdles in making the transition from arithmetic to algebra (Booth, 1984; MacGregor & Stacey, 1997; Moseley & Brenner, 2009). This project captured research on students' algebraic thinking that might be useful for preservice and early career teachers, created resources based on findings, and is now helping teachers make use of that resource. We are currently in the second year of a three-year Fund for the Improvement of Secondary Education (FIPSE) grant, developing the *Center for Algebraic Thinking.*

Theoretical Framework and Background

This systematic review of the mathematics education literature identified 858 manuscripts that addressed students' algebraic thinking. Articles typically included a discussion of the math involved in a misconception or way of thinking about algebra, assessment problems, transcripts of interviews with students describing their thinking with problems, and suggested instructional strategies to address misconceptions or ways to develop understanding. Accordingly, five questions served as the framework for the synthesis of research:

1. What Common Core State Standard(s) does this research address?
2. What is the symbolic representation of thinking with the idea? (What does it look like?)
3. How do students think about the algebraic idea? (What does it sound like?)
4. What are the underlying mathematical issues involved?
5. What research-based strategies/tools could a teacher use to help students understand?

Findings from the readings contributed to three resources: a wiki-based Encyclopedia of Algebraic Thinking (www.algebraicthinking.org), a Formative Assessment Database, and Instructional Modules for Mathematics Methods courses. The above questions defined the framework for the encyclopedia entries. In addition, most manuscripts included a problem or problems that were designed to elicit a range of students' thinking around a particular algebraic concept that we collected into the Formative Assessment Database. Besides project resources, videotape of students' describing their thinking as they solve algebra problems are incorporated into the Instructional Modules.

Method

We divided articles into five domains based on the Common Core State Standards (CCSS): Variables & Expressions, Algebraic Relations, Analyzing Change, Patterns & Functions, and Modeling & Word Problems. Articles were read by a Domain Team, consisting of three to four readers who were mathematics faculty, teacher education faculty, or public school mathematics teachers from institutions in Oregon and Colorado. The director of the project identified potential worthwhile articles, categorized them by domain, and sent them to the appropriate team. Each team member read approximately 35–50 pages of articles each month for eight months. Each reading did not necessarily provide clear responses to each question, so, in some cases, the distillation of the reading into the framework was incomplete. In cases in which an article applied to more than one domain, each relevant Domain Team would read the article. Each Domain Team took responsibility for reading and discussing the articles and populating a single document with responses to each of the framework questions, with an emphasis on what would be useful to algebra teachers in middle and high school.

Each Domain Team coded the findings from the readings into themes, determining the nature of each theme as they progressed with the reading of the manuscripts. Themes were fluid throughout the process, breaking into smaller themes or combining into one theme as insights were gained from the readings. Each theme was translated into a single document with references to one or more readings. Each document addressed each of the five framework questions, where possible. Of the original 858 articles read, 680 articles were ultimately distilled into the resources. After one month of reading, teams convened to discuss their findings and suitability of the framework and made adjustments to the process. At the end of eight months, teams convened to compare findings, discover connections across domains, and refine entries. Finally, theme documents were loaded as entries into the Encyclopedia of Algebraic Thinking. The following are highlights of the research across the five domains.

Results

Two hundred twenty-eight (228) articles were identified that discussed **Variables and Expressions**. Of those, 137 were potentially relevant for teachers The majority of research fit into themes identified by Kuchemann (1981): *Letter evaluated, Letter not used, Letter used as an object, Letter as a specific unknown, Letter used a generalized number,* and *Letter used as a variable*. Four other themes also arose from examination of the literature: *Representation, Acceptance of Lack of Closure, the Process-Product Dilemma,* and *Conservation of Variable*. An example of the research findings is that students struggle to understand the stability and *Conservation of Variables*: when a letter can represent a specific unknown *or* a range of values as well as if the value of letters can change within or across problems. For instance, students were asked when the statement L+M+N=L+P+N was true (always, never or sometimes). A high proportion of students, in multiple studies, chose *never* (Steinle et al., 2009), believing M and P would never be the same amount because they are different letters. In an equation such as $x + x + x = 12$, some students believe that the same letter in an expression does not necessarily stand for the same number. A student who believes that 10 + 1 + 1 is an acceptable answer, is unlikely to make sense of the explanation that $x + x + x = 12$ is equivalent to $3x = 12$ and then $x = 4$ (Fujii, 2003). In another study, a student given the equation $x + 5 = x + x$ responded that the second x on the right side had to be 5, but the other x's could be anything (Wagner & Parker, 1993). Hearing that "x can be any number" repeatedly in class, students may logically take that principle to the extreme.

One hundred seventy-nine (179) articles were identified that discussed **Algebraic Relations**, including equations and inequalities. Of those, 135 were potentially relevant for teachers. Some related to pre-requisite knowledge: Negative Numbers and Rational Numbers. Other themes related to the structure of problems: One and Two Step Equations (x on One Side), Solving Equations with Variables on Both Sides, Solutions Involving Zero, and Inequalities. Finally, themes related to students' conceptual understanding: Flexible Use of Solution Strategies, Student Intuition and Informal Procedures, and Translating Word Problems into Equations. An example of the findings is that mathematical structure of a problem can present difficulties for students. When students approach a problem, they rely on procedures demonstrated in that day's lesson. Successful students rely on relational thinking—rewriting the problem into a form more easily understood (Demby, 1997). Once developed, students can be more successful when approaching problems that involve special characteristics such as trivial equations (x=2), equations with no solution (0x=7), equations with a solution of zero (3x=0), and with infinite solutions (2x+4=2x+4).

One hundred sixty-three (163) articles were identified that related to students' understanding of **Analysis of Change** (Graphing). Of those, 120 were potentially relevant for teachers. Three themes emerged: Creating Graphs (connecting graphs to real world data and problems with scaling), Interpreting Graphs (connecting graphs to algebraic relationships and viewing graphical representations as literal pictures), and Describing Analysis of Change (understanding slope and rate of change).

For instance, in regard to Creating Graphs, students often learn mechanical steps of graphing a set of points without understanding the real world situation they represent or understanding the relationship between the variables they are plotting. Widespread use of graphing calculators to teach graphing can complicate matters, as unequal scaling of the axes can distort the shape of the graph, further impairing student understanding of the situation being graphed (Mitchelmore & Cavanagh, 2000). Students should be involved in gathering data, creating a graph, and analyzing variables, starting as early as the elementary grades, before students study negative numbers. There is some research (i.e., Mevarech & Kramarski, 1997) that claims traditional approaches to teaching graphing that begin by first breaking the process of producing a graph into step-by-step items in a procedure may in fact perpetuate the perceived problem that graphing is a difficult topic. A preferred approach is to present students with a purposeful task in a familiar context, and students will be able to act intuitively to use line graphs.

One hundred forty-six (146) articles were identified that related to students' understanding of **Patterns and Functions**. Of those, 104 were potentially relevant for teachers. The domain team found the following themes: Reasoning and Interpreting; Translating Across Representations; Developing the Concept of Function; and Building, Transforming, and Generalizing. An example of findings from the research is that students need to see a wide variety of functions (Even, 1998). Students who only see continuous functions come to believe that functions must be continuous; students who see only smooth functions (e.g., no cusps) believe that functions must be smooth. Students should work with discontinuous functions, piecewise-defined functions, functions with holes in the domain, functions with discrete domains, with non-numeric domain and/or range, etc.

One hundred forty-two (142) articles were identified that related to students' understanding of **Modeling and Word Problems**. Of those, 108 were potentially relevant for teachers. The modeling domain team found that articles fell into several broad categories: Syntactic Issues in Problem Solving, Choosing Strategies that Result in Meaningful Solutions, Drawing Models as an Intermediate Solution Step, and Issues with English Language Learners (ELL). An example of findings from the research include that students may be overly focused on the structure of the narrative, losing the connection between the context and mathematical expression. The problem, "There are six times as many students as professors" is frequently translated as 6S = P, where S=students and P=professors (Fisher, 1988). In order to help children understand word problems, teachers often focus on key words such as "more" and "times." This strategy is useful but limited because key words don't help students understand the problem situation. Key words can also be misleading because the same word may mean different things in different situations.

Conclusion

The project has transformed the research on students' algebraic thinking into entries of an *Encyclopedia of Algebraic Thinking*, a *Formative Assessment Database*, and Instructional Modules that will are currently being incorporated into Math Methods courses at our four institutions. These resources are the heart of our current preparation of preservice teachers.

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INDIVIDUAL DIFFERENCES IN STUDENTS' UNDERSTANDING OF MATHEMATICAL EQUIVALENCE

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Many children have a limited understanding of the equal sign ("="), interpreting it as an *operator* (i.e., as a symbol to "do" something; Carpenter et al., 2003) without understanding that the left and right sides of an equation need to represent the same amount (i.e., a *relational* view). This is worth exploring because conceptual understanding of the equal sign is at the basis of algebraic thinking (Franke et al., 2008). Instructional interventions on the equal sign are not uniformly successful and investigating the sources of students' individual differences solving non-canonical equivalence problems (e.g., 4 + 3 + 7 = 6 + __) may shed light on this phenomenon. In the present study, we investigated the domain-general and domain-specific factors that account for differences in children's performance on non-canonical equivalence problems.

Fifty-six third- and fourth-grade students from a suburban school district in Canada were asked to solve a series of non-canonical equivalence problems before and after receiving explicit instruction on the equal sign. As a group, the students showed significantly higher scores after instruction, but a full 30% were unsuccessful on more than half of the items on the posttest. We conducted individual interviews to determine the source of the variability, and measured (a) general cognitive ability, (b) working memory, (c) mathematical fluency, (d) conceptual understanding of equivalence, and (e) ability to rate and generate definitions of the equal sign.

Regression analyses indicated that of all the variables, only general ability and mathematical fluency were significant predictors of performance on the posttest. Furthermore, qualitative analyses of the interview data revealed that 39% of the students generated an operational definition of the equal sign (Operational group), 38% of them generated a definition that included both operational and relational features (Combined group), and 23% generated a relational definition (Relational group). We found a significant effect of definition type on posttest performance, $F(2, 53) = 6.93$, $p < .01$. Post-hoc tests revealed no significant difference between the Combined and Relational groups, but each outperformed the Operational group, ($p < .0167$ in both cases).

These findings suggest that performance on problems that assess the understanding and use of the equivalence concept is related to content-specific skills, namely the ability to generate an accurate definition of the equal sign. More specifically, holding an operational definition does not hinder successful problem solving as long as it is accompanied by some element of relational understanding. This finding suggests that learning the meaning of the equal sign is not straightforward; rather, it appears to be initially context-bound and fragmented (Lawler, 1981).

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CHALLENGES IN ASSESSING COLLEGE STUDENTS' CONCEPTION OF DUALITY: CASE OF INFINITY

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The paradoxical nature of infinity makes it challenging for students to conceptualize. Understanding the process-object duality of infinity is essential for having a formed conception of infinity, which is crucial for students to be successful in advanced mathematics courses. The purpose of this study was to examine college students' conception of duality and determine whether or not they possessed a dual process-object view of infinity. More importantly, (1) how is the duality conception externalized and expressed by college students, and (2) how could students' conception of duality be assessed?

Theoretical Framework

The study is supported by APOS theory to model the development of the duality conception (Dubinsky, Weller, McDonald, & Brown, 2005) via *actions*, *processes*, *objects*, and *schema*. The schema represents the "process-object duality" (Monaghan, 2001), and this dual nature of mathematical constructs can be observed through students' representations (Sfard, 1991, Gray & Tall, 1994). At the same time, scholars warn about the danger of assumption that comes with process-object duality and emphasize dynamic nature of the duality conception (Falk, 2010; Bingolbali & Monaghan, 2008) arguing that care needs to be taken in interpreting students' representation of infinity. The proposed study examines challenges in coding and assessing students' conception of duality as well as addresses diversity and variations among students' conception of duality (e.g., cases where the student's process view is dominant and the object view is recessive, cases where students' object view is dominant and process view is recessive).

Method of Inquiry and Preliminary Results

The research sample included N = 192 college students taking Calculus sequence courses at one of the U.S. southwestern universities. Multiple measures to assess students' externalization of their conception of infinity were used in the study. The first measure was a survey addressing students' concept-definition of infinity with the following statement: "When you think of infinity what comes to your mind?" The second construct was a survey assessing students' concept image of infinity in the form of "draw infinity" task with a direct statement: "Draw infinity in the space provided." The third measure was a task that includes a contextualized problem. Rating scale "duality-abstractness" was used to determine students' positioning toward duality of infinity concept. The study found that the coding and assessing college students' conception of duality is a challenging and complex process due to the dynamic nature of the conception that is (1) task-dependent and (2) context-dependent. The results of this study could be used as a springboard to further analyze cognitive obstacles in college students' understanding of infinity concept.

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RELATIONSHIPS BETWEEN REPRESENTATIONAL FLEXIBILITY AND NUMBER CONCEPTS: DRAWING ARRAYS AND (MIS)UNDERSTANDING SQUARE NUMBER PROPERTIES

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A useful factor in navigating the transition from misunderstanding to understanding in math may be *representational flexibility* (RF: Nistal et al., 2009), a complex of representation abilities. One such ability is the capacity to see numbers as objects with spatial and numerical properties. Considering that number concepts (e.g., squares, primes, and composites) are central to math and understanding them may be required to use higher-order operations and reasoning (e.g., functions and integration), more work is needed to explain how they are acquired through representational experiences. A positive relationship was predicted such that students able to represent square number spatially and numerically should demonstrate greater conceptual knowledge of square number whereas those not able should reflect greater misunderstanding.

Method

Thirty fifth-graders were interviewed on two items to examine the relationship between RF (i.e., constructing spatial and numerical representations of a square number) and conceptual knowledge (i.e., explain square number in terms of its properties). Square arrays were first defined; *A **square array** is a special rectangular array that has the same number of rows as it has columns. A square array represents a whole number, called a **square number***. A spatial-numeric pattern of the first four square numbers (see Figure 1) was described; *The first four square numbers and their arrays are shown below*. The Item 1 instruction was, *Draw a square array for the next square number after 16*; a dot matrix on which to draw the array and a place to write the the square number were presented. Item 2 asked, *Can a square number be a prime number? Why or why not?* Item 1 was analysed for patterns of accuracy in students' drawn arrays and written square numbers. Explanations observed in students' responses to Item 2 were analyzed for features of an underlying conceptual model of square number.

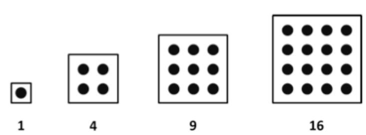

Figure 1: Visual pattern of first four arrays

Results and Discussion

RF and conceptual understanding of square number were found to be positively related and several interesting misunderstandings emerged from students' explanations. Only students who constructed equivalent array and numerical representations ever demonstrated understandings of square number properties whereas those who constructed nonequivalent representations always reflected misunderstandings about those properties. The results suggest that number concepts should be promoted by helping students represent key properties of concepts. The implications for navigating transitions from misunderstanding to understanding will be discussed.

NOTICING RELEVANT PROBLEM FEATURES: EXPERIENCE AFFECTS STUDENTS' RECONSTRUCTION AND SOLVING OF EQUIVALENCE PROBLEMS

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Elementary and middle school students often fail to correctly solve mathematical equivalence problems, which are problems that have operations on both sides of the equal sign (e.g., 3 + 4 + 5 = 3 + __) (e.g., Perry, Church, & Goldin-Meadow, 1988). These difficulties with mathematical equivalence have far-reaching effects, as understanding of the equal sign is associated with students' abilities to solve algebraic equations—a skill considered essential for long-term success in mathematics (Knuth, Stephens, McNeil, & Alibali, 2006).

This study investigated a possible causal mechanism for these difficulties, namely that the way we teach math during elementary school creates fundamental misconceptions about equations. Specifically, students' experience appears to follow three operational patterns that hinder performance on equivalence problems (McNeil & Alibali, 2005). First, students learn the perceptual pattern that equations always follow an "operations = answer" format. Second, students learn the conceptual pattern that the equal sign means to "calculate the total" or "put the answer." Finally, they learn the procedural pattern that the correct way to solve an equation is to "perform all the given operations on all the given numbers." These operational patterns may hinder students' performance is by affecting what they *notice*, or encode, about problems.

It would be inappropriate to investigate the causal basis of young students' performance by strengthening their misconceptions. However, some information about the mechanisms can be obtained through research with adults. Previous research has shown that activating adults' knowledge of the operational patterns with a priming task disrupts their ability to solve equivalence problems (McNeil & Alibali, 2005). One remaining question is how exactly these experiences affect students—does activating knowledge of the three patterns affect not only the way they solve the problems, but also what they notice about the equations? In the current experiment, adults (N = 138) were primed with either the operational patterns or with neutral stimuli. Knowledge activation condition affected the problem features that participants noticed. Specifically, participants who were primed with operational patterns both encoded and solved the equations as if they were traditional arithmetic problems, leading to less accurate performance than participants primed with neutral stimuli $t(134) = -3.09$, $p = .002$ for encoding, $t(120) = -2.56$, $p = .012$ for solving). These findings suggest that one way in which traditional mathematics education might hinder children's ability to solve equivalence problems is by leading them to misencode relevant problem features.

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LEARNING TRAJECTORIES FROM THE ARITHMETIC/ALGEBRA DIVIDE

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The concept of a *Learning Trajectory* has recently acquired new importance as the organizing principle for the new Common Core State Standards in Mathematics (CPRE, 2011). There are several definitions of a learning trajectory within the research profession; for the purpose of this work we take the definition of Clements and Sarama (2009): A ***Learning Trajectory*** (LT) of a particular mathematical concept consists of three components:

- a specific mathematical goal,
- a developmental path along which students' thinking and comprehension develops and
- a set of instructional activities that help students move along that path.

The effectiveness (efficacy) of the LT/CCSS in Mathematics approach depends on the classroom on the teachers' ability to incorporate the developmental information contained therein into the classroom instruction.

> For that to happen, teachers are going to have to find ways to attend more closely and regularly to each of their students during instruction to determine where they are in their progress toward meeting the standards, and the kinds of problems they might be having along the way. Then teachers must use that information to decide what to do to help each student continue to progress, to provide students with feedback, and help them overcome their particular problems to get back on a path toward success. This is what is known as *adaptive instruction* and it is what practice must look like in a standards-based system. (Consortium for Policy Research in Education Report [CPRE], 2011)

How to assure the required level of teachers' involvement, how to prepare them professionally for the new challenges of the job—are questions which are preoccupying American educators at present (*Education Week*, February 1 through April 3, 2012).

Two Learning Trajectories from Arithmetic/Algebra divide, "Number Sense as the Gateway to Algebra" and "Linear Equations," will be presented on the background of the Arithmetic/Algebra concept map. Both trajectories had been obtained with the help of several cycles of *Teaching Research* (TR) NYCity Model, each developing a new iteration. The TR NYCity underlying their development involves the teacher in every component of adaptive instruction specified above. The choice of LT's developed by the TR methodology responds to the local educational needs; the most critical area along the mathematics education spectrum (middle school, high school, remedial college) is the transition from arithmetic to algebra. The aim of both trajectories is to guide the teacher in the design of adaptive instruction along that transition.

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"SOME STUDENTS ARE ADVANCED AND CAN JUST FACTOR": ALGEBRAIC PROCEDURES AND STUDENTS' INSTRUCTIONAL IDENTITIES

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In mathematics learning, students' identities impact whether and how students choose to engage in mathematical activities, as well as their attitudes and dispositions towards mathematics (Bishop, 2012; Boaler, 1999; Cobb, Gresalfi, & Hodge, 2009). With this poster we consider the instructional identities (Aaron, 2011; Aaron & Herbst, 2010) that students may enact in an Algebra I class. The construct of instructional identities is helpful to highlight the interaction between students' dispositions towards certain actions and the instructional context in which they are acting. In this study we focused on three questions: *What value did students ascribe to each of three strategies for finding the zeroes of a polynomial—factoring, using the quadratic formula, and using a calculator? What were the justifications for each of these evaluations? What instructional identities could be distilled from the students' values of each of the strategies?* Taken together, these questions inquire into students' instructional identities as they are played out in the everyday work that students do in an Algebra I class.

To explore these questions, we interviewed 11 students from two high schools about their perspectives about the three strategies for finding the zeroes of a polynomial. We used Toulmin's (1958) model of argumentation to decompose students' arguments into six components: *data*, *claim, warrant*, *backing*, *qualifier*, and *rebuttal*. We have found that students often base their evaluations of different algebraic procedures on either the abilities of a student (e.g., factoring is good for students who are advanced enough to use it) or on the appropriateness of the procedure for a particular task (e.g., factoring is more efficient for simple polynomials). Moreover, students are slightly more likely to determine the appropriateness of a particular procedure based on their perceived ability of the student using that procedure. This indicates that students' may place greater emphasis on how they perceive themselves as learners than what a particular task calls for when approaching problems in Algebra. Understanding how these factors influence students' instructional identities can help to explain whether and how students choose to engage in particular algebraic activities.

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FROM CONFUSION TO CLARITY: HOW COMPLEX INSTRUCTION SUPPORTS UNDERSTANDING OF LINEAR FUNCTIONS THROUGH NON-LINEAR PATTERNS

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The standards published by the National Council of Teachers of Mathematics emphasized that all students are capable of learning mathematics and should be given access to developing high-level mathematical proficiency (Tate, 1997). However, the perception of mathematical understanding as only accessible to a select few still lingers (Boaler, 2008). Gaps in levels of proficiency persist because students view themselves as either capable or incapable of success in mathematics (Cobb et al., 2009).

Research suggests that Complex Instruction (CI) provides teachers with a powerful tool for promoting rigorous and equitable learning (Cohen et al., 1999). However, even among teachers with the same training, implementation of CI varies significantly (Lotan, 1989). This study focuses on the critical case in which all the conditions of CI are met, as closely as possible (Yin, 2009). In one specific task within a broader curricular effort, students explore patterns of non-linear functions, as an introduction to the upcoming unit on linear functions. The teacher admits that many of his colleagues feel this design is backwards, and they worry that such a challenging task may block access to learning. He responded, “Watching kids wrestle with this task affirms for me that it is the right sequence. Asking them to do something, much, much harder makes the subsequent stuff more obvious to them” (personal communication, March 9, 2012). This case provides an example of how introducing student to complex mathematical ideas first can encourage deeper understanding and proficiency, with proper instructional support provided by CI. Students gain access to this proficiency through peer collaboration, which is supported by multiple ability tasks, appropriate classroom norms and the teacher’s efforts to assign competence to low-achieving students. The teacher’s effective use of these aspects of CI facilitates each student’s transition from exploring challenging concepts with non-linear functions to developing more precise proficiency with linear functions.

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ROBUSTNESS CRITERIA AS A FRAMEWORK TO CAPTURE STUDENTS' ALGEBRAIC UNDERSTANDING THROUGH CONTEXTUAL ALGEBRAIC TASKS

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Our project aims to develop research tools to provide empirical evidence for links between teaching practices and students' understanding of algebra. Each year, students were pre- and post-tested using an assortment of Contextual Algebraic Tasks (CATs)—rich word problems adapted from tasks developed by the Mathematics Assessment Resource Service ([MARS], available at http://www.noycefdn.org/resources.php). By focusing on these tasks, we are able to examine a range of student skills using a set of criteria we developed, Robustness Criteria (RCs), which includes sense making, modeling, representational, and procedural skills.

We use the RCs in part as a framework to capture students' understanding. The five RCs are: (1) interpreting text and context in problem statements; (2) identifying and relating salient quantities; (3) generating, interpreting, and making connections between representations to solve problems; (4) executing algebraic procedures and calculations and checking plausibility of results; and (5) providing explanations that give insight into the depth of students' algebraic thinking. The criteria were developed based on literature on problem solving and algebra, particularly functions (e.g., Chazan, 2000; Schoenfeld, 2004). With this poster, we address how our scoring rubrics have evolved to provide information on students' understanding according to the RCs.

Our early rubrics were based on existing, well-tested MARS rubrics, which provide an overall score for each task based on correctness of student responses. We decided, however, that it was important to capture student understanding at the fine-grained level of individual RCs to answer our overall research questions. As a result, in Year 2 we developed new rubrics that not only provide an overall correctness score, but also scores for each RC. RCs can be assessed by looking at both students' answers to specific parts of the task and their use of particular strategies. The overall scores using these new rubrics were highly correlated with the MARS scores. Moreover, among the eight scorers across our research sites, the correlation of the scores for each RC and the overall score was above 0.700.

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MULTIPLICATIVE THINKING AND STRATEGY USE OF A STUDENT WITH MATHEMATICS LEARNING DISABILITIES AFTER STANDARDIZED INTERVENTION

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Fraction equivalency is one of the most relentless areas of difficulty in mathematics performance and understanding for all students (National Center for Educational Statistics, 2009; Siegler, Carpenter, Fennell, Geary, Lewis, Okamoto, et al., 2010). Unfortunately, students with mathematics learning disabilities (MLD) perform far worse on tests of fraction concepts compared to typical students and even students who struggle with mathematics but do not have disabilities (Hecht, Vagi, & Torgesen, 2007; Mazzocco & Devlin, 2008). The researcher designed an intervention for equivalency concepts based in the ratio interpretation (Lamon, 1993; Streefland, 1993). The researcher delivered the intervention to several students with MLD through research-based, systematic instructional methods led by teaching scripts (Fuchs & Fuchs, 2006).

Methods

One student in the third grade ("Bill") is the focus of this report. The researcher chose third graders to receive supplemental intervention in fraction equivalency due to grade level content specifications. The focal student had a label of "learning disability" from the school district and failed a curriculum-based pretest of fraction equivalency. Data sources were curriculum-based measures (CBM) and two clinical interviews given before and after 20 one-half hour supplemental teaching sessions. The intervention provided instruction on relationships found within a fractional unit (Lamon, 1993) and how to use representations and mathematical operations to generate equivalent fractional units (Streefland, 1993). Analyses of interview data involved several stages of identifying, sorting, and analyzing involved in thematic analyses as described by Grbich (2007).

Findings

Bill's performance on the CBMs and post interviews did not improve significantly after intervention. His inability to understand or iterate the fraction unit continued. Despite the ratio-based instruction, Bill viewed each equivalency problem as a sharing or partitioning situation. Although Bill's notions of fractions through partitioning were at a level of pre-coordination with potential for further development (Empson, Junk, Dominquez, & Turner, 2005), the scripted intervention was too static and did not allow for development of Bill's informal notions of fractions and fraction equivalency. Instructional experiences for students with MLD must change from static teacher driven approaches to iterative processes of student activity and teacher support based in what a student currently conceives of a mathematics concept.

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TEACHING FOR ROBUST UNDERSTANDING IN MATHEMATICS

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Our study focuses on developing a classroom observation scheme for capturing and analyzing teaching practices hypothesized to foster students' development of robust understanding of algebra. We aim to provide a tool to address questions such as: "What are the critical aspects of Algebra classrooms?" by capturing critical aspects of classroom interaction through real-time classroom observations. Our coding scheme, Dimensions of Teaching for Robust Understanding in Mathematics (TRU MATH), builds on the work of existing classroom observation tools, such as IQA (Junker et al., 2005), but includes an algebra-specific focus. In this poster, we will share the rationale for TRU MATH, and report on the results of the use of this scheme in twelve Algebra 1 classrooms.

TRU MATH focuses on 6 dimensions of classroom practice: (1) Important Mathematics; (2) Cognitive Demand; (3) Access; (4) Agency, Authority, and Accountability; (5) Uses of Assessment; and (6) Algebra Content-Specifics addressing the following "essential" questions about mathematics classrooms: (1) Did the lesson engage the students and teacher in working on mathematics consistent with the Common Core Standards? (Common Core State Standard Initiative, 2010); (2) Did students engage in "productive struggle" with the mathematics? (Henningsen & Stein, 1997); (3) Did all students have the opportunity to engage with the learning? (Cohen & Lotan, 1997); (4) Who had a voice in the classroom discussion and ownership over the mathematical ideas? (Engle & Conent, 2002); (5) Did instruction seek to reveal what students know and build on it? (Black, Harrison, Lee, Marshall, & Wiliam, 2003); (6) Did students engage in practices that support solving algebra word problems? These dimensions are scored on rubrics specific to particular facets of classroom interaction, such as: the launch of a task, whole class discussion of mathematical ideas, or the connecting of ideas to prior knowledge. Using these scores, we can create profiles of Algebra teaching across these dimensions which can be correlated with student performance on contextual algebraic tasks (described in another proposal) to provide insight that supports and improves the teaching of Algebra.

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BEYOND LOW ACHIEVEMENT: IDENTIFYING MATHEMATICAL LEARNING DISABILITIES THROUGH ATYPICAL UNDERSTANDINGS

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It is estimated that approximately 6% of school-aged children have mathematical learning disabilities (MLDs; Shalev, 2007), which are cognitively-based differences in numerical processing that result in persistent and significant problems learning even the most basic mathematics. Subject identification remains *the* central issue in studies of MLDs, because there is no assessment to accurately identify students with MLDs (Mazzocco, 2007). I present two related studies that attempt to gain traction on differentiating general low math achievement from MLDs, by identifying the atypical understandings displayed by students with MLDs.

Study 1 involves a detailed analysis of two college students with MLDs engaged in weekly videotaped tutoring sessions on the topic of basic fraction concepts (Lewis, 2011). In addition to having low math achievement, these students met rigorous response-to-intervention learning disabilities criteria (Fletcher, Lyons, Fuchs, & Barnes, 2007). Microgenetic analysis revealed that the students with MLDs relied upon a small number of atypical understandings. For example, these students tended to focus on the fractional complement (rather than the fractional quantity), and represented 1/2 as the act of halving rather than the quantity 1/2. These atypical understandings accounted for nearly all the student's incorrect answers, resisted standard instruction, and were detrimental to the student's ability to learn more complex fraction concepts.

In Study 2, to determine the prevalence of these atypical understandings, a 16 question paper-and-pencil assessment was designed and was administered to 384 seventh and eighth grade students. The assessment was scored for evidence of these atypical understandings (1 point for each answer that was consistent with an identified atypical understanding). Most students (61%) demonstrated no atypical understandings, and only 11% of the students were classified as having high levels of atypical understandings (3 or more atypical understanding points).

Study 1 identified that students with MLDs demonstrated atypical understandings of fractional quantity and Study 2 indicated that the prevalence of these atypical understandings were not common in a larger sample of younger students. This suggests that identifying atypical understandings may be a way to begin to differentiate low achievement from MLDs. Accurately identifying students with MLDs is a necessary first step toward better understanding the very nature of MLDs. Only then can we understand the unique difficulties students with MLDs face and provide tailored instruction to help these students navigate the mathematical transitions.

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VIEWS OF LESS PREPARED STUDENTS ON INSTRUCTION INCORPORATING THE COMPARISON OF MULTIPLE STRATEGIES IN ALGEBRA I

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Increasing evidence suggests that the practice of comparing and discussing multiple solution strategies to mathematics problems is beneficial for students' learning (e.g., Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Star & Rittle-Johnson, 2008). Yet considerable doubt remains among practitioners as to whether multiple strategies instruction is beneficial to *all* students or only to high-achievers.

In the current study, we ask: How did nine less prepared students view the practice of learning multiple strategies for solving math problems, after participating in full, year-long algebra courses "infused" with multiple strategies that were taught by regular classroom teachers? The nine students interviewed attended five different middle and high schools in four different parts of a single large metropolitan area, yet were similar in that they were all under-prepared for algebra as evidenced by their September algebra readiness exam scores.

When students were asked whether they had ever had teachers emphasize more than one way to solve a math problem *prior* to the intervention year, seven of the nine students responded that they had never or rarely had teachers emphasize multiple strategies.

When asked about advantages of the approach, students stated that learning more than one way to solve a problem helped them to *find a method that "worked for them"* (6); improved their *understanding of methods and concepts* (6); improved their *understanding of specific mathematical methods* (5); improved their *efficiency and speed* (4); improved their *accuracy in finding correct answers* (2); and *reduced their anxiety about mathematics* (2).

Students also described several curriculum features as advantageous, particularly the guiding *structured comparison questions* and the *clarity* of the step-by-step examples presentation.

Responding to the question, "What are the disadvantages of learning more than one way to solve a problem?" five students stated that there were *no disadvantages*. Four cited possible *confusion*, such as learning too many methods and forgetting one, as a disadvantage. All nine students stated that there were more advantages than disadvantages to multiple strategies.

Concerns about overwhelming low-achieving students were largely not substantiated among the students that we interviewed. In fact, students cited the advantage of *improved* understanding repeatedly throughout the interviews, and felt that the risk of confusion was minor. When developing instructional plans for struggling learners, we suggest that teachers and researchers should take struggling students' own views about multiple strategies into account.

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RELATING CHILDREN'S TOPOLOGICAL UNDERSTANDINGS WITH NUMBER UNDERSTANDING THROUGH SUBITIZING

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Keywords: Cognition; Number Concepts and Operations; Geometry and Geometrical and Spatial Thinking

Theoretical Framework

Piaget (1941/1964) explains that for number to be understood and expressed by an individual, a simultaneous coordination of the serial (sequential) and algebraic (classification) thinking structures must occur. This study focuses on how three developmental characteristics related to topology (separation, proximity, and enclosure) relate to children's coordination of serial and algebraic thinking structures. Both separation and proximity are defined by Piaget and Inhelder (1948/1956) as the perceptual joining or separating of objects due to a particular orientation of a set of objects. Enclosure is defined as a type of "between separate elements" (Piaget & Inhelder, 1948/1956, p. 80) and is dependent upon the dimension an individual perceives an object as being enclosed within. The purpose of this study was to investigate the relationship between children's topological and number understandings as it relates to conceptual subitizing.

Methods

Two 4-year-old children, Mary and Neil, from a rural area in the southeastern United States were chosen as participants due to their inability to conserve number consistently, yet ability to exhibit 1:1 correspondence and keep track of items perceptually. A teaching experiment was used to better witness the "essential mistakes" (Steffe & Thompson, 1991, p. 20) children make, informing researchers' of children's models of learning. The six teaching experiment sessions included tasks to assess children's notion of number and topological thinking structures, while focusing on children's ability to conceptually subitize and draw what they "saw."

Results and Conclusions

The children each drew upon both topological perceptions as well as their understanding of number when expressing and drawing what they "saw" when subitizing numbers four or fewer. However, children did not consistently coordinate these thinking structures. As Neil and Mary looked at dots, they seemed to attend to location of objects and then quantity of objects; quantity of objects and then location of objects; or both location and quantity simultaneously. When quantity was attended to before space, the dots drawn by the children were represented as one line or two equal lines. When space was attended to before quantity "action" was described, preventing a re-visitation of the space. The one time space and quantity were attended to simultaneously a group of four objects were decomposed into two groups of two objects, respectively. Separation and proximity seemed to relate to participants' ability to coordinate both serial and algebraic thinking structures.

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A COMPARISON OF COLLEGE INSTRUCTORS' AND STUDENTS' CONCEPTUALIZATIONS OF SLOPE

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This study investigates university calculus students' and instructors' understandings of slope using eleven conceptualizations proposed by Moore-Russo and colleagues (Moore-Russo, Conner, & Rugg, 2011; Stanton & Moore-Russo, 2012) and based on the earlier work of Stump (1999, 2001a, 2001b). Data were collected from 65 university students enrolled in two sections of an introductory calculus course and 26 mathematics professors attending a regional conference of postsecondary mathematics instructors in the same geographic area as the university. The students and instructors responded to the same five items related to slope, and each response was coded to indicate which conceptualizations were demonstrated.

Based on all responses to the five items, students most commonly conceptualized slope as a *parametric coefficient* (m), a *behavior indicator* (increasing or decreasing line), and as an *algebraic ratio* (y_2-y_1/x_2-x_1). The conceptualizations most frequently used by instructors were *functional property* (rate of change), *calculus conception* (derivative or slope of tangent line), and *geometric ratio* (rise over run). When considering whether the participants used one of the conceptualizations on any of the five items, *behavior indicator*, *parametric coefficient*, and *algebraic ratio* were still the most common among students, but *real world situation* moved ahead of *calculus conception* for instructors.

Students' responses demonstrated a limited diversity in conceptualizations of slope, most often interpreting slope as a coefficient or ratio that describes a line's behavior. Students demonstrated a procedural emphasis with little indication of the meaning for the covarying quantities involved or the physical and real world applications. Instructors' responses provided evidence of relatively diverse conceptualizations of slope, viewing slope as a ratio of covarying quantities with utility in a variety of applied contexts. Despite this diversity, instructors rarely used the *behavior indicator* conceptualization of slope prominent in students' responses. While these results may suggest that college instructors do not utilize students' prior knowledge as a foundation for more advanced notions of slope, further investigation of instructional emphasis and the possibility of a hierarchical relationship among the various slope conceptualizations is needed to better understand this discrepancy.

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LEARNING TO REPRESENT, REPRESENTING TO LEARN

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Keywords: Problem Solving; Algebra and Algebraic Thinking; Reasoning and Proof

Representation is central to learning and doing mathematics. As a mathematical practice, it involves more than simply learning to use and interpret canonical representations (e.g., Cartesian graphs). Representational practice also involves learning to invent, communicate, and reason with representations as tools for problem-solving (Greeno & Hall, 1997). Engaging in representational practice in these ways can support the development of complex reasoning and justification skills (Maher, Powell, & Uptegrove, 2009). But students may need explicit opportunities to learn how to engage in representation practice in these ways. This study explored how students' engagement in representational practice changed through participating in a five-week summer school exploratory algebra class for middle school students. The teaching intervention was explicitly designed to foster mathematical practices, including representation (Boaler et al., in preparation). The study focused on a heterogeneous group of boys, who had no prior access to this type of math instruction. Analyses of the boys' math journals showed that their written representations became more sophisticated over the five weeks as evidenced by (1) greater variety of representational forms, (2) use of multiple representations for a single problem, and (3) making connections between representations. Analysis of their small group interactions around a quadratic growth pattern task provided evidence of how they learned to create, invent, communicate, and reason with representations while problem-solving. Representation emerged as a tool for the boys to act upon the mathematics, supporting both their engagement with content (quadratics, first and second differences, and Gauss's pairing method) and other mathematical practices (justification, generalization, and making use of structure). The boys' interactions also suggested that representation supported their persistence and collaboration. Changes in their discourse revealed how negotiating multiple representations positioned them to act with agency. These findings imply that representation may be generative for students. Through learning to represent, students gained tools to support further learning of mathematics. While this study focused on a small group of students in a university-led intervention, it provides a compelling "image of the possible" (Shulman, 1983). When given access to experiences designed to explicitly foster representational practice, the boys showed substantial changes in both their engagement in representation and mathematics more broadly. With the transition to the Common Core Standards fast approaching, this study suggests that students may need access to explicit opportunities to learn mathematical practices such as representation.

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SUPPORTING CONCEPTUAL REPRESENTATIONS OF FRACTION DIVISION BY ACTIVATING PRIOR KNOWLEDGE DOMAINS

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Keywords: Number Concepts and Operations; Rational Numbers; Problem Solving

Fluency with fraction operations is foundational for algebra, but often a source of difficulty (NMAP, 2008). Both young students and adults struggle to understand fraction division, and many of their errors suggest that they spontaneously attempt to transfer other operations with fractions concepts to make sense of fraction division (Ma, 1999; Siegler, Thompson, & Schneider, 2011). A structure-mapping framework (Gentner, 1983) predicts this negative transfer, since fraction division and other operations on fractions share surface similarities that encourage students to draw on this prior knowledge. This framework further predicts that students should benefit more from a structurally similar analogue, such as whole number division, which shares the abstract division structure with fraction division. Therefore, we hypothesized that students would have stronger conceptual representations of fraction division after studying whole number division than after studying fraction subtraction. To test this hypothesis, 47 undergraduates were randomly assigned to compare fraction division problems to either whole number division or fraction subtraction problems. Each participant completed a fraction division pretest, 3 comparison-to-an-analogue problems, a picture generation task, and a fraction division posttest.

The data indicated that participants transferred and adapted conceptual representations of their given analogue to fraction division. Overall, participants who compared fraction division to whole number division were more likely to demonstrate conceptual representations that included the grouping structure common to fraction division (68.2%) than those who compared fraction division to fraction subtraction (40.0%; Fisher's exact test, $p = .05$). In contrast, students who compared to fraction subtraction were more likely to draw pictures that included features of fraction subtraction. Furthermore, students who drew correct pictures gained more from pretest to posttest than did students who drew incorrect pictures on an item that required them to generate a story to correspond with a given fraction division equation, $t(45) = 2.07$, $p = .04$.

Though this study was carried out with adults, for whom fraction division is not a novel concept, we still found that activating different types of related knowledge shaped their immediate conceptualization of the problems. This work has implications for curricular sequencing, as students often learn fraction division along with other fraction operations. This proximity may contribute to negative transfer from domains with similar surface features. Instead, students may be better served by instruction that highlights whole number division as a structural analogue.

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EXPLORING THE IMPACT OF KNOWLEDGE OF MULTIPLE STRATEGIES ON STUDENTS' LEARNING ABOUT PROPORTIONS

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Keywords: Problem Solving; Middle School Education; Learning Progressions; Rational Numbers

Proportional reasoning is widely considered to be a major goal of mathematics education in the middle grades. The literature identifies three strategies that are commonly used by students in solving simple proportion problems: cross multiplication, equivalent fractions, and unit rate. In past research, scholars have expressed concern that students rely too heavily on cross multiplication when solving these types of problems and have advocated delaying instruction on cross multiplication in favor of both unit rate and equivalent fractions (Cramer & Post, 1993; Stanley, McGowan, et al., 2003). In fact, there is evidence that, when instruction is delayed on cross multiplication, students tend to use the unit rate strategy most frequently (Cramer & Post, 1993; Post, Behr, & Lesh, 1988). As part of a larger project investigating the effectiveness of an intervention on ratio and proportion problem solving, we assessed students' strategy repertoire for solving proportion problems and to what extent students' prior knowledge of one or more strategies impacted their learning from the curricular intervention. A sample of 430 seventh grade students completed a pretest (that took 45 minutes) and then participated in a six-week scripted intervention that focused on ratio and proportion concepts (including percent) and solving ratio and proportion word problem solving.

Contrary to our expectations, the participants in the present study relied most heavily on equivalent fractions when solving proportions at pretest, rather than either cross multiplication or unit rate. This suggests that perhaps some of the past claims about student reliance on cross multiplication may now be outdated. Consistent with the literature on cross multiplication, we found that exclusive reliance of cross multiplication at pretest had a negative impact on learning from our instructional intervention. Prior knowledge of equivalent fractions, unit rate, or multiple strategies, however, had a positive impact on students' abilities to learn from our intervention. The students who were able to benefit the most from the intervention were students who demonstrated procedural flexibility, defined as the ability to select the most appropriate strategy for a given problem (Star & Rittle-Johnson, 2009), at pretest. This finding provides further support for the growing body of literature that suggests that students' learning is enhanced when they have the opportunity to learn multiple methods and compare and contrast them.

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STUDENTS WITH MATHEMATICAL LEARNING DISABILITIES RESPONSES TO WORD PROBLEMS

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This poster presents findings from a project that combines two disparate perspectives that generally do not intersect in academe: special education and mathematics education. Although the two perspectives are usually separate at the research level, they do interact in the field. Shalev (2007) estimates that approximately 6% of the population has a mathematical learning disability (MLD), a disorder that leads to problems with many aspects of number sense, including counting, magnitude estimations, word problems, and retrieval of basic facts (Jordan, 2007).

Mathematics education research has long focused on how children understand mathematics. In particular, there has been considerable research using the framework of Cognitively Guided Instruction (CGI) to explicate how typically developing children understand addition and subtraction (Carpenter et al., 1999). This framework has helped mathematics educators understand the typical developmental trajectory of these skills. However, there has been limited research using the CGI framework with students who have MLD. Most of the research on mathematics for students with MLD has focused on the students' mathematical deficits, but there has been limited research into what they do understand about mathematics. This means that most intervention studies for this population have been designed from this deficit perspective and may be built on faulty assumptions because they do not take into account what the student does understand. This current study examines what students with MLD already understand about addition and subtraction, and how this compares with students without MLD. This is important preliminary information in order to be able to later design effective interventions.

This poster presents findings from a comparative study of students with and without MLD. This study examines the following research questions: (1) How do the types of word problems solved accurately by students with and without MLD compare? (2) How do the patterns of strategy use among students with and without MLD compare? The study involves individual clinical interviews of ten third grade students with MLD and ten without MLD, using the CGI framework on a variety of tasks involving counting tasks, and addition and subtraction word problems. This poster presentation will present the preliminary analysis of the study. Implications for designing future interventions will also be shared.

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STUDENT REASONING ABOUT THE INVERTIBLE MATRIX THEOREM IN LINEAR ALGEBRA

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I report on how a community of learners in a linear algebra classroom reasoned about the Invertible Matrix Theorem (IMT) over time. The IMT is a core theorem in linear algebra in that it connects many fundamental concepts together through the notion of equivalency. As the semester progressed, the class developed the IMT in an emergent fashion, adding equivalent statements to the theorem as the students explored related concepts. As such, the various equivalencies took form and developed meaning as students came to reason about the ways in which key ideas involved were connected.

Data for this study came from the third iteration of a semester-long classroom teaching experiment (Cobb, 2000) in an inquiry-oriented introductory linear algebra course in the southwestern United States. There were 30 students in the course, and the text *Linear Algebra and its Applications* (Lay, 2003) was used. Data sources included video and transcript of whole class discussion relevant to the development of and reasoning about the IMT, and these occurred on 10 of the 31 days of the semester. From this, 109 relevant arguments were identified, where *argument* is defined as "an act of communication intended to lend support to a claim" (Aberdein, 2009, p. 1). Through grounded analysis, these arguments were broken down into clauses of separate ideas and coded as either concept statements from the IMT or interpretations of those concept statements. Adjacency matrices were then used to organize and analyze this information.

An adjacency matrix depicts how the vertices of a particular directed graph are connected. For a given directed graph, an adjacency matrix is an $n \times n$ matrix $[a_{ij}]$ with one row and one column for each of the n vertices in the digraph, and entry $a_{ij} = k$ indicated k edges from the i^{th} vertex to the j^{th} vertex (Chartrand & Lesniak, 2005). The adjacency matrices analyzed in this study correspond to directed graphs in which the vertices are the aforementioned coded clauses from arguments regarding the IMT, and the edges are directed in such a way as to match the implication offered by the speaker(s). Adjacency matrix analysis revealed not only what mathematics developed in the class, but also the various structures of argumentation and what concepts were included within particular arguments. Results will include centrality of concept statements, density and continuity of ideas (Tiberghien & Malkoun, 2009), and an analysis of argumentation patterns over the semester that provides a summary of the emergence and shift of the collective's mathematical ways of reasoning about the IMT.

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A RETROSPECTIVE ANALYSIS OF STUDENTS' THINKING ABOUT VOLUME MEASUREMENT ACROSS GRADES 2–5

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This paper describes a retrospective analysis of data collected during a 4-year longitudinal study on children's thinking about measurement through a teaching experiment methodology. It focuses on results from individual interviews with two students on volume measurement. Data analysis was guided by Sarama and Clements' (2009) learning trajectory on volume measurement. Results indicate that (1) both students progressed through levels of the learning trajectory during the study, (2) different representations of 3-D objects (e.g., physical objects, cubes, pictorial representations) influenced their strategies, and (3) their individually constructed definitions for the term "volume" affected their decisions in volume comparisons.

Keywords: Measurement; Learning Trajectories; Geometry and Geometrical and Spatial Thinking; Elementary School Education

Background and Rationale

Measurement can bridge two critical domains of mathematics, geometry and number, as well as provide conceptual support to those domains (Clements & Sarama, 2007). Research has shown that children have difficulty in fully grasping the concept of volume measurement (Battista & Clements, 1996; Ben-Haim, Lappan, & Houang, 1985; Enochs & Gabel, 1984). In measurement contexts, including volume, many children apply formulas to get the answers without understanding the meaning of these formulas (Clements & Battista, 1992). Nation-wide assessments also revealed student difficulties in solving volume measurement problems. The 1977–1978 results from the National Assessment of Educational Progress (NAEP) showed that less than 50 percent of the students at grade 5 through 8 were able to answer questions correctly about the volume measurement of three dimensional (cube) arrays (Ben-Haim et al., 1985). Students' errors stemmed mostly from counting the number of visible faces of cubes shown, counting the number of visible cubes in the diagram, estimating the number of faces of cubes shown in a given diagram, and double counting cubes (Battista & Clements, 1996; Ben-Haim et al., 1985). The researchers stressed that many students were unable to enumerate the cubes correctly in such an array. Overall, these studies showed that students could not correctly solve volume measurement tasks because they were (a) not correctly applying the volume formula, (b) not correctly counting the number of cubes in 3-D arrays, or (c) confounding volume and surface area measurement. Although the current research tells us much about children's thinking in volume measurement, missing is longitudinal work showing how students' thinking about volume measurement grows throughout their development and with the instruction they receive.

Theoretical Framework

The theoretical perspective guiding this study is described by the framework of *hierarchic interactionalism,* which indicates "the influence and interaction of global and local [domain specific] cognitive levels and the interactions of innate competencies, internal resources and experience" (Clements & Sarama, 2007, p. 464). Of the 12 tenets of *hierarchic interactionalism*, the learning trajectories tenet (Sarama & Clements, 2009) is the most germane to this report. It guided the design of the longitudinal study and informed the development of instructional tasks. A hypothetical learning trajectory consists of three components: a learning goal, a likely path for learning as students progress through levels of thinking, and the instruction that guides students along the path (Sarama & Clements, 2009). Furthermore, the learning trajectory for volume measurement was utilized as the data analysis tool for this study.

Sarama and Clements (2009) asserted that students' understanding of volume measurement gradually improves with the instruction they receive in addition to their natural development. They defined the volume measurement trajectory through eight levels. According to the trajectory, children initially focus on external aspects of arrays as sets of faces. Later, students develop an appreciation of the internal structure of 3-D arrays. They gradually become capable of counting the number of cubes contained in objects, one by one or in a pattern of rows and columns and layers. This level is called Primitive 3-D Array Counter (PAC). The next level, Capacity Relater and Repeater (CRR), focuses more on volume as capacity. At the CRR level, a student "fills a container repeatedly with a unit and counting how many. With teaching, [a student] understands that fewer larger than smaller objects or units will be needed to fill a given container" (Sarama & Clements, p. 307). The next level, Partial 3-D Structurer (PS), describes student counting in terms of rows or columns (or units of units) of a solid built with unit cubes. The next more complex level of volume involves thinking in terms of layers of unit cubes and is called 3-D Row and Column Structurer (VRCS). The highest level described in the volume measurement learning trajectory is 3-D Array Structurer (AR). At this level, students can mentally de/compose 3-D arrays into layers. These levels are used to describe student's thinking for a particular task or teaching episode rather than to define the student's overall thinking about volume measurement.

Purpose

The aim of this study was to investigate students' thinking in volume measurement over a four-year period within the context of a teaching experiment.

Research Question 1: How do students develop coherent knowledge and integrated strategies for volume measurement across Grade 2 through Grade 5?

Research Question 2: How are students' abilities for spatial thinking, algebraic reasoning, or proportional reasoning related to their measurement knowledge and strategies?

Method

The sample for this report consisted of two children (Ryan and Owen) from a Midwestern public school. Each student represents just one of seven case studies within a four-year longitudinal study investigating children's thinking and learning across length, area, and volume for Grades 2–5. Instructional tasks were developed within the context of a teaching experiment (Steffe & Thompson, 2000). The teaching experiment consisted of a series of teaching episodes for which the research team generated a set of tasks and predictions for student responses and then later checked student responses against these predictions. Each teaching episode was an individual, semi-structured interview, which lasted 15 to 40 minutes. The interview tasks were informed by the learning trajectory for volume measurement (Sarama & Clements, 2009). Before the first teaching episode, an initial assessment was administered in the form of a clinical interview in which the interview tasks were posed without feedback or instruction. All interviews, including the initial assessments and teaching episodes, were videotaped, transcribed, and analyzed by a group of researchers, the authors.

During the four-year teaching experiment, Owen encountered 30 volume measurement tasks within 12 interviews, and Ryan encountered 30 volume measurement tasks within 11 interviews when they were in third, fourth and fifth grades in addition to their initial assessment interview during second grade. The teaching episode tasks represented volume with a variety of objects (e.g., physical objects, cubes, pictorial representations). Additionally, the tasks required a variety of actions: filling a container with water, packing a box with the unit cubes, building a prism with unit cubes, and finding the volume given only linear measurements. Some of the tasks required students to draw 3-D objects.

Results and Discussion

Five themes emerged from the analysis of the data for both students: (a) finding volume of rectangular prisms, (b) relating the size and the number of units, (c) visualizing representations of the 3-D objects, (d) flexible unit sense, and (e) fractions and volume measurement. Due to space limitations, only the first three themes will be presented in this paper.

Owen's Thinking on Volume Measurement

a) Finding volume of rectangular prisms. In the second grade spring semester—during his initial assessment, Owen was asked to determine the number of cubes needed to make a 3 in × 2 in × 2 in prism presented as a physical object with the individual cubes displayed. While holding the prism and showing the interviewer what he counted, Owen said, "12…because there is 3 here [showing one whole surface of one face] …and 3 here [showing another whole surface of one face]." Later, he changed his answer to 6 (correct). While determining the number of cubes, Owen mentally constructed and counted layers (composite units) of 6 in the figure. However, when asked how many cubes altogether would be needed to fill the partially filled box of 3 units × 3 units × 4 units (Figure 1), he looked at the figure and said "about 35" without any apparent strategy.

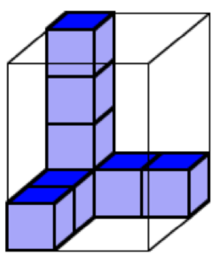

Figure 1

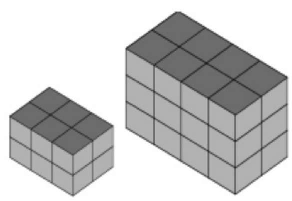

Figure 2

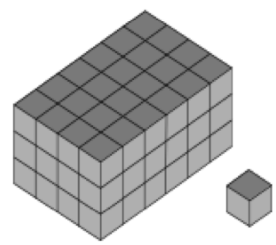

Figure 3

Approximately 12 months later, in the third grade spring semester, Owen was asked to compare the volume of the two prisms in Figure 2. Although he reported a correct additive comparison of 12 more cubes would be needed to build the larger prism, to determine how many blocks would be needed to make the smaller prism (3 units × 2 units × 2 units), he incorrectly answered 16. When asked to build the first figure with the actual cubes, Owen built the figure by using an appropriate row structuring strategy, with 12 cubes. He made two separate rows of 3, placed them next to each other, then explained that there were two layers of 6 in the figure, and changed his answer from 16 to 12. In response to an extension of the same task, Owen said that he would need 36 cubes to build the second figure (4 units × 2 units × 3 units) shown on the paper. Pointing to one of the lateral faces of the figure, he explained 6 plus 6 is 12 and then pointed to the front face and lateral face respectively and said "another 12." Owen appeared to have attended to the surface area of the vertical faces only. When asked to build the figure with cubes to check his answer, he constructed rows and layers correctly and changed his answer to 24.

Throughout the interview, Owen did not initially determine the number of cubes correctly. Instead he first attended to the surface area or counted the squares on the lateral faces. These actions are consistent with the PAC level. On the other hand, when asked to build, Owen correctly resolved the tasks at the PS level by skip-counting and thinking about volume in an organized way; he was adding the number of units in rows.

Approximately eleven months later, in the spring semester of fourth grade, Owen was asked to compare the volume of a rectangular prism to a unit cube (Figure 3). Owen stated that one cube on the side was one of the cubes in the rectangular prism and counted the number of cubes on the bottom. He said that 5 times 4 was 20 and 20 times 3 was 60. In this task, he saw that there were 3 horizontal layers in the solid. However, he calculated the number of cubes in each layer incorrectly; he counted the length as 5 instead of 6, likely in an attempt to avoid double counting a row. Nevertheless, this showed that he could think at the

VRCS level by counting layers of units multiplicatively. Later, in the same interview, Owen was given the same task again (see Fig. 1) and asked how many cubes would be needed to fill the outlined box. By attempting to count the cubes individually Owen gave an incorrect answer of 27. He did not use layers or multiplicative thinking although he used the strategy of structuring in terms of layers for the previous tasks of the same interview. Owen may have failed to resolve this task successfully because of the high visualization demand of the task.

Approximately eight months later, in the first interview of fifth grade, Owen was given a solid box (4 in × 3 in × 3 in), which did not have any unit indication on it and a collection of 1-inch cubes and was asked how many small boxes it would take to fill the big box (Figure 4). Owen struggled in the beginning and gave an incorrect answer of 12. After building one vertical layer of 4 in × 3 in × 1 in, which aligned to one face of the box, he changed his answer. He found the number of cubes in one layer, 12, and thought there were "3 rows [layers]" so 36. In this instance, Owen showed the VRCS level of thinking as evidenced by his tendency to think about prisms in terms of layers built from cubes.

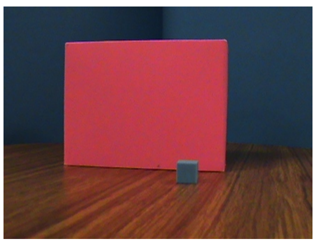

Figure 4

Figure 5

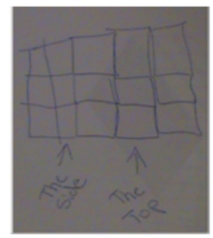

Figure 6

One of the tasks for the second interview of the semester was about finding the volume of the room compared to a cubic meter and a cubic decimeter (Figure 5). After iterating a meter stick across the floor of the room, 6 times for the length and 3 times for the width of the base of the room, Owen reasoned correctly that 18 cubic meters would fit in the bottom layer of the room. He said that there would be "3 of those [layers] going high…so 18 times 3." He concluded that therefore, 54 cubic meters would fit into the room. He also measured all three dimensions of the cubic meter and one dimension of the cubic decimeter block with a meter stick and found that 10 cubic decimeters would fit along each edge of the cubic meter. Thus, he multiplied 10 times 10 times 10 and found a product of 1000 to represent the number of cubic decimeters in one cubic meter. In order to determine the number of cubic decimeters that would fit into the room, he multiplied 54 by 1000. Owen found approximate values for the measurement of the dimensions of the room (actual room size: 7 m × 4 m × 3 m). The student applied multiplication and applied AR level strategies to resolve this task by using only the linear measurement of the 3-D prisms, by building and manipulating composite units of cubic decimeters as well as cubic meters, and by mentally decomposing arrays into layers, rows, and columns.

b) Relating size and number of units. Owen encountered a number of tasks requiring him to relate the size and number of units. Starting in third grade, as suggested in the volume measurement trajectory CRR level, he was aware that different units would give a different volume measurement and he recognized an inverse relationship between the size and number of units.

c) Visualizing representations of the 3-D objects. In the spring semester of fourth grade, Owen was asked to draw something that was 3 times as big as a 1-unit cube. Owen drew three squares in a 3×1 form. Next, he was asked to draw a picture of a solid three times the volume of a 3×2×1 solid. He created the drawing shown in Figure 6. With an aerial perspective, he could visualize the new solid from a top view. In the follow up interview, while looking at his old drawing, Owen explained that there would be 2 layers of 9 so that there would be 18 cubes. The figures Owen drew did not have 3-D perspective; however, he could use his own representations to determine the number of cubes in the figures.

Ryan's Thinking on Volume Measurement

a) Finding volume of rectangular prisms. In the second grade spring semester initial assessment, Ryan was asked how many cubes altogether would be needed to fill the partially filled box of dimensions 3 units × 3 units × 4 units (Figure 1). Ryan drew line segments extending the edge lines of the cubes and gave an incorrect answer of "18."

In the third grade spring semester, when asked how many of these cubes were necessary to make the smaller shape in Figure 2, Ryan gave the answer 25 (incorrect) and explained that he counted 4 for each lateral side, 6 for the front side, and 6 for the ceiling. He did not count the cubes on the base and also added the numbers incorrectly. Ryan was asked to build the physical representation of the shape with actual unit cubes. After building he noticed his mistake and said, "I was counting them wrong…I was counting the sides…I should have been counting cubes." He also took the cube in the top corner and said that he was counting that cube as two although he should have counted it as one. His final answer of 12 was correct after being allowed to build the structure with unit cubes. Ryan's realization of his incorrect strategy might have influenced his volume definition throughout the task, which changed to "how many cubes there are." In the next task of the same interview, while finding the number of cubes in the second figure of dimensions 4 units × 3 units × 2 units (Figure 2), Ryan curtailed his previous strategy of counting the faces of cubes. However, he still answered incorrectly (28 cubes) because he only counted the visible faces of some cubes. When he was asked to build the physical representation of the figure, he built the actual figure row by row and reached the correct answer. Ryan's responses and strategies, while resolving these tasks, were consistent with the PAC level. Specifically, he initially counted the outer faces of cubes. On the other hand, while building the shape with the actual cubes, Ryan could think more systematically in terms of columns as exemplified in the PS level. Ryan used a higher level of strategy while using 3-D physical objects than when resolving the task with the figural representations on the paper.

Seven months later, in the fall semester of fourth grade, Ryan was shown the actual cubes and container in Figure 7 and asked how many blocks would be needed to fill the container. Ryan counted by fives up to 20 by showing the bottom rows of 5 and said that there would be about 20 blocks. Then he counted by 20s up to 100 reporting a correct answer. When the same type of question was asked for a larger container (8 in × 10 in × 5 in) and with missing cubes in the first layer (Figure 8), Ryan failed to give a correct answer for the number of blocks needed to fill the whole container. In order to prompt the student, the interviewer took the extended rows and columns so the shape was changed back to its original version, and reminded Ryan that he said 100 blocks would fill the previous container. Next, the interviewer added 4 more cubes to the side of 4 and repeated the question. Ryan explained that the complete part had 4, the missing part also had 4 and explained that if they filled the missing part, 40 would go in the first layer, 80 in two layers, and so forth; he added composite units of 40s until he reached a correct response of 200. In the first task of the interview, Ryan counted first row by row and after finding the number of cubes for one layer, iterated the numbers by adding for each layer. However, he could not implement the same strategy properly when both rows and columns were missing on the bottom layer or when the numbers got bigger than 5 or 6 in a row.

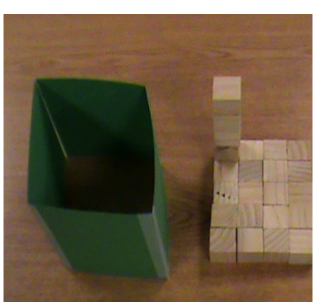

Figure 7

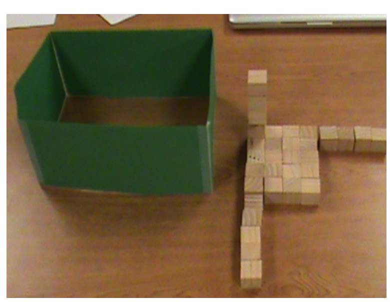

Figure 8

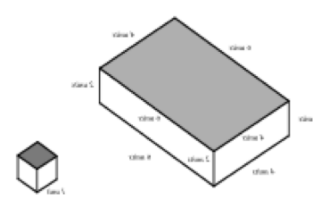

Figure 9

Although Ryan showed VRCS level strategy of counting composite units and then counting layers by skip counting in the first task, he did not use this strategy when the mental structuring demand increased as in the later task. Once he knew how many cubes were in one layer, he used repeated addition to add up the blocks in all the layers. This strategy is a typical VRCS strategy in the learning trajectory.

Eight months later, in the spring semester of fourth grade, Ryan was shown the task represented in Figure 9 and was asked, "If this cube has a volume of one, what is the volume of this solid? How many cubes would it take to fill the box?" Ryan found the volume by adding all of the number labels given for the edges. This task representation did not elicit his thinking clearly whereas the previous task was useful to probe his thinking about structuring in volume measurement.

Approximately five months later, in the first interview of fifth grade, Ryan was given a rectangular prism (4 in × 3 in × 3 in), which was wrapped with paper, and a 1-inch cube (Figure 4). He was asked how many of the cubes it would take to fill the box. By iterating the cube along one side of the prism, Ryan said "it is about 36….1,2,3,4." He pointed to a row of 4 on the bottom of the lateral face by iterating the cube and continued to iterate vertically by counting "4, 8, 12." Then he continued to count by pointing with his finger on the upper edge of the box vocalizing 12, 24, 36. Later, when given a centimeter cube and asked how many of those cubes would fit into prism, Ryan made a composite unit of 8-centimeter cubes to represent each 1-inch cube. Next, he reasoned correctly and multiplied 36 by 8 and gave an answer of 288. Ryan showed VRCS level of thinking by mentally decomposing a 3-D array into layers. Additionally, his strategy of building a composite unit of 8-centimeter cubes was consistent with AR level.

b) Relating size and number of units. In the spring semester of third grade, Ryan's volume definition was, "the number of cubes in a shape." In order to determine whether he was able to relate the unit size and volume, he was posed a task requiring him to compare the volume of a 4 in × 2 in × 2 in rectangular wooden prism with a 4 cm × 3 cm × 2 cm rectangular yellow plastic prism. He was told, "Another person compared the figures and thought that since the yellow prism has 24 cubes and the wooden one has 16, the yellow figure has a larger volume. Do you agree?" Ryan thought, by his definition of volume, that the volume of the block made of centimeter cubes was greater even though he articulated that two wooden cubes are just like 24-centimeter cubes. Similarly, in the spring semester of the fifth grade, he was given a task relating different unit sizes and the number of units in volume measurement. When asked which objects would melt into more water, Ryan thought that the collection of smaller cubes would melt into more water because there were more of them even though he noticed the difference in the unit size. He stated that they would need four of the larger cubes to "make" the six of the smaller cubes. We suggest that Ryan's volume definition had not substantially changed through third and fourth grade, nor had he conceptualized volume as the amount of space an object would take up.

c) Visualizing representations of the 3-D objects. In the spring semester of fifth grade, Ryan was asked to make a drawing on paper of a 1-inch cube that he was handed, and second to draw a picture of 3 in × 2 in × 2 in figure shown in a two-point perspective drawing with shading. He drew a square and called it a cube. He struggled to copy the 2-D drawing of the 3-D image.

Conclusions and Implications

According to the results, when finding the volume of rectangular prisms, both Owen and Ryan demonstrated abilities in each of the levels of PAC, PS, VRCS, and finally AR. However, both students also employed lower level strategies for some tasks.

Consistent with prior research, (e.g., Battista & Clements, 1996; Ben-Haim et al., 1985), initially, both students had the tendency to count the outer faces of the cubes instead of the number of cubes, demonstrating PAC level strategies. After one or two semesters, they could see the row, column, and layer structures in the rectangular prisms and count the number of cubes by creating composite units (row, column, layers). In later semesters, both students used multiplication as repeated addition while thinking in terms of rows, columns, and layers made of unit cubes.

Ryan mostly used multiplication when the figures were represented physically but not pictorially. These findings suggest that it is important to provide a variety of representational forms of 3-D objects in volume measurement tasks in order to help students solidify the meaning of multiplication in volume measurement. In addition, the formula for the volume of a rectangular prism may not make sense to students without understanding why they need to multiply the linear dimensions (Clements & Battista 1992).

In spring semester of fourth grade, although Ryan and Owen struggled to determine the volume of rectangular prisms without any grid on them, they were able to resolve some of the tasks when prompted to use a 3-D unit. For example, both students could resolve the task represented in Figure 4 by using the unit cube given. Ryan iterated one unit cube along the edges of the cube to find the number of unit cubes in each row, column, and layer. Owen was prompted with multiple cubes aligned as a vertical layer of the prism. Therefore, giving a 3-D solid unit was helpful for both students.

Students' thinking differed according to the physical versus pictorial representations provided in the tasks. When both students were posed tasks requiring them to imagine rows, columns, and layers, and some aspects of the figure were obscured, the students sometimes gave incorrect answers and used incorrect strategies. For the task represented in Figure 1, Owen struggled to visualize the complete rows, columns and layers if the box was full of cubes. Similarly, Ryan could not resolve the task (Figure 8) when there were hidden rows and columns on the bottom layer, as the visualization demand was higher.

Based on students' responses, it can be claimed that students' strategies about relating the unit size and the total number of units in the objects were influenced by their volume definition. For Ryan, who defined volume as, "the number of cubes in a shape," the volume of the block made of more (centimeter) cubes was greater than the one made of fewer inch cubes even though he noticed the difference in the unit size. Therefore, while preparing the instructional materials for students, using different size units and comparing the volume of objects with those units should be considered, especially for the students who have a misconception that the volume is the number of units instead of the amount of 3-D space an object takes up.

Moreover, both students had difficulties in drawing or copying 3-D figures made of cubes on paper. Although they could calculate the volume of those objects, their drawings did not correctly represent the actual figures. We claim that Owen could interpret his own drawing and that he could hold a correct mental representation in his mind for the object. This showed that the students' representations might be identified as incorrect, even though they have correct mental representations for the figures in their minds. Students should be given opportunities to draw representations and at the same time given opportunities to articulate what they are seeing from the pictures. This might let teachers understand how students think and visualize the 3-D objects.

Lastly, on most of the tasks requiring students to interpret 2-D drawings or pictures of 3-D figures, students resolved the task if they were allowed to build the figures with the actual unit cubes. Building the figures with cubes apparently helped students identify their mistakes and change their strategy to count in terms of rows, columns, and layers. Therefore, students should be given opportunities to build the shapes, which are shown on paper.

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ADDRESSING TRANSITIONAL CHALLENGES TO TEACHING WITH DYNAMIC GEOMETRY IN A COLLABORATIVE ONLINE ENVIRONMENT

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This report describes navigating transitions along continuums of professional learning. We discuss current approaches and introduce a discourse-centered educational model to introduce secondary mathematics teachers to dynamic geometry environments (DGEs). We present how our course will engage teachers in a collaborative online environment, known as Virtual Math Teams with GeoGebra (VMTwG), focusing on significant mathematical discourse.

Keywords: Classroom Discourse; Geometry and Geometrical and Spatial Thinking; Teacher Education–Inservice/Professional Development; Technology

To effectively improve mathematics education to be aligned with Common Core State Standards for Mathematics (CCSSM), teachers need to understand and learn to use innovative technologies and pedagogies. For instance, the CCSSM recommends, "students consider the available tools [such as] dynamic geometry software…to explore and deepen their understanding of concepts." To accomplish this recommendation, middle and high school teachers first need to be equipped to engage students meaningfully in lessons that incorporate dynamic geometry software and other technologies. However, the research literature in mathematics education contains few investigations on teacher professional learning in the use of dynamic geometry software. To address this scarcity, this report has two aims. First, it discusses what the research literature in secondary mathematics education offers about how to support practicing teachers' use of dynamic geometry. Second, it presents a model for professional learning that seeks to empower and support mathematics teachers to appreciate and engage in new approaches using dynamic geometry and, particularly, within a cyber-learning environment.

This work is important because it is a key component of our collaborative research project between investigators at Drexel and Rutgers Universities,[1] that integrates a dynamic geometry environment (DGE) with collaborative tools for mathematics exploration to provide a range of opportunities for students to engage in significant mathematical discourse, and develops supporting professional learning opportunities for practicing teachers. This project incorporates the use of GeoGebra, which currently only exists in single user mode, and an application of computer supported collaborative learning, known as Virtual Math Teams (VMT), which the project is extending to include the first multi-user, dynamic DGE, known as Virtual Math Teams with GeoGebra (VMTwG).

Professional learning of DGEs and use of online collaborative environments each includes navigating transitions through several stages. We offer theoretical considerations and examine challenges that occur when teachers move along a novice-expert continuum, using similar environments, as reported in the literature. We are in the first year of a four-year project; therefore, we present preliminary results from our design-based research (Brown, 1992; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). We offer approaches to the challenges through our design for professional learning courses that intend to support teachers through their transitions. Finally, we identify issues we anticipate emerging from this work that will be important to examine carefully, and we make suggestions for future research.

Theoretical Framework

The theoretical framework for the discourse focus in VMTwG is situated in theories of knowledge and meaning making in mathematics. Using Vygotsky's theory of social mediation, we believe that higher functions of human thought are first learned socially, as part of interactions among people, and that knowledge is an evolving achievement of interpersonal meaning making. We view mathematics as discourse, using a participation metaphor (Sfard, 2001, 2008) and believe that norms can be established to engender productive mathematical discourse (Michaels, O'Connor, & Resnick, 2008). In line with this and adding to the Vygotskian notion of the zone of proximal development, a group of peers has the ability, though its discourse within a computer supported collaborative learning environment, to develop new knowledge that exceeds the capabilities of any one member of the group (Stahl, 2009).

Collaborative problem solving among learners working as small teams is an interactive, layered building of meaning. Through their discursive interaction teams create objects and, in turn, these objects shape and advance the discourse. Further, the team's discursive interactions occasion their reflections on relations among objects and dynamics among relations, as well as reasoning and problem-solving heuristics. The interactive work leaves the team with tools for future collaboration (Powell & Lai, 2009). The public and permanent nature of online mathematical collaboration allows classmates to follow their own and other groups' mathematical accomplishments, observing and reflecting on their colleagues' developing knowledge, successful mathematical collaboration, and shifts in discourse (Silverman, 2011).

The research literature describes gradual transitions learning DGEs: as teachers are introduced to DGEs, they use the environment first to produce static handouts, and then for presentation of dynamic visualizations. Once teachers feel comfortable and self-confident, they allow their students to actively discover mathematical concepts through discovery learning and lesson enrichment. Some teachers do not move past the presentation stage, and many take several years to transition to discovery learning (Laborde, 2007; Lu, 2008a; Preiner, 2008).

The mathematics education literature reports teachers rarely find time to engage in learning processes capable of transforming their teaching practice or significantly modifying their highly constrained curricula (Assude, 2005; Cuban, Kirkpatrick, & Peck, 2001; Laborde, 2007).

Methods

Using time they have scheduled for masters-level courses, we invite practicing teachers to identify their least favorite topic to teach, or a topic their students have difficulty learning, as one VMTwG lesson to incorporate into their practice. To help in-service teachers navigate the transitions reported in the literature, we are developing an online course in VMTwG that will catapult teachers' learning beyond the initial phases directly to preparing teachers to engage their students in learning mathematics through significant mathematical discourse. We propose to accomplish this through a series of synchronous and asynchronous activities in which teachers function in collaborative teams. Customized activities are being adapted from existing curricula, aligned with the Common Core State Standards, and through reflections on course readings, which include learning VMT and GeoGebra. Teachers will also learn to help students develop discourse strategies that lead to productive and accountable collaboration.

Within their team's "chat room," teachers first become familiar with the VMTwG environment, which includes a chat panel, a whiteboard tab, a wiki page tab for summaries and reflections, and a multi-user, dynamic version of GeoGebra, where users can define dynamic objects and where diagrams can be dragged around the screen. Each team has a common GeoGebra tab, and each team member has a GeoGebra tab for individual work.

For each course module, team members are prompted do some mathematics asynchronously in their GeoGebra tab, and post their noticings and wonderings to a group wiki page. The team then meets synchronously at its scheduled time, and chats about interesting noticings and wonderings. In the same synchronous session, the team collaborates to solve open-ended mathematics problems on the common GeoGebra tab, guided by prompts to engage in mathematical discourse surrounding discovery learning, with team members taking turns to accomplish the activity and to explain reasons for their actions.

Constructions are validated by showing that the properties and relations of their diagram remain intact when they drag any basic object that they used to construct it (Powell & Dicker, 2011). Each member of the team will be accountable to the whole team, ensuring that every member is capable of accomplishing each task. To help teachers become better prepared to orchestrate significant mathematics discourse with their students, they reflect on their experience and how they will structure a similar activity for their students, empowering them as they develop their discourse pedagogy.

Finally, team members will reflect on the logs of their discourse and the discourse of the other teams, which will be captured in VMTwG, to identify successful discourse moves and discourse moves that may have hindered progress, moving them along the continuum of first learning and then learning how to support significant mathematics discourse among their students, posting their reflections on the team's wiki page. There will be a final synchronous session in which the team discusses the most interesting discourse reflections.

Throughout this course, cycles of problem solving followed by analysis, discussion, and reflection on the solutions in small groups move teachers towards the goal of facilitating the transition from doing math and supporting each other's mathematical development to synthesis and reflection on the significant mathematical ideas that transcend particular solutions or solution method (Silverman, 2011). This is the first of two professional learning courses. In the second course, teachers will implement what they have learned in their classrooms, within the context of their current curriculum; with mentoring and resources to support this effort.

Preliminary Results

Engaged in design-based research, we are seeing what an emerging educational model using GeoGebra might look like in a collaborative synchronous environment. We have just begun to collect data from our research team and from groups of teachers participating in our formative evaluation, in the form of whiteboard summaries, logs of chat and of GeoGebra moves. From the data collected, we are developing codes to analyze the mathematics discourse. This data has led to the following two preliminary results:

First, we see that systematic turn taking in GeoGebra, is necessary for both technical and collaborative reasons, with clarification of who is doing what.

Second, in an online environment, we have found it necessary to simulate classroom procedures of periodically calling the class together to make meaning of an activity. To do this, we have refined the model to have multiple stages: asynchronous (noticing and wondering, getting everyone on the same page), synchronous (talk about the activity, and then doing it), asynchronous (reflections on the math, the discourse, and the VMTwG system), and for some modules, synchronous (reflection on discourse moves).

Discussion and Implications

Some of the issues we anticipate emerging that will be useful to investigate are as follows: How do we assess teachers' transition along the novice-expert continuum? How can we provide optimal scaffolding so that the teachers emerge from a one semester course with sufficient skill and confidence to implement what they have learned with their students, and mentor other teachers, the following semester? As the original cohort of teachers transition along the novice-expert continuum with VMTwG, how will their transition correlate with students' performance?

Endnote

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DOES BLOCK PLAY SUPPORT CHILDREN'S NUMERACY DEVELOPMENT?

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Block play engaged by young children has been found to predict their subsequent mathematical and literacy competence in formal schooling (Clements & Sarama, 2007; Wolfgang et al., 2001). Moreover, studies have shown that adult number talk has been found to relate to children's emergent numeracy skills (e.g., Levine et al., 2010). However, little is known whether parents' engagement in block play also supports preschoolers' early acquisition of numeracy skills. Thus, the current research investigated whether the complexity of block play engaged by the parents is related to their children's numeracy abilities a year later. Twenty-seven children (14 girls, 13 boys) between the ages of 31–41 months old and their parents participated in a 30-minute play session at their home. Our findings reveal that the preschoolers' numeracy competence is positively related to the complexity of block play engaged by their parents.

Keywords: Early Childhood Education; Geometry and Geometrical and Spatial Thinking; Number Concepts and Operation

Introduction

The acquisition of mathematical knowledge at a young age is important as research has shown that children already exhibit individual mathematical differences before the age of four (Klibanoff, Levine, Huttenlocher, Vasilyeva, & Hedges, 2006); and these differences are predictive of children's later mathematical knowledge throughout elementary school (Duncan et al., 2007). For example, the amount of parental number talk children received between 14 and 30 months old is predictive of their acquisition of cardinality of number words when they are 46 months old (Levine et al., 2010). These findings reveal that there are already children who are at an academic disadvantage before entering formal school. Furthermore, they underscore the fact that it is important to support early mathematical learning and development via a mathematically-enriched home and/or preschool environment to facilitate young children's subsequent mathematical competence.

One type of mathematically-enriched activity that has been found to benefit children's subsequent mathematical competence is block play (e.g., Clements & Sarama, 2007; Wolfgang, Stannard, & Jones, 2001). For example, block play has been found to predict five- to seven- year-old children's spatial skills and geometric knowledge (Casey et al., 2008). Furthermore, engaging in block play has been reported to facilitate the development of a number of mathematical concepts and skills (MacDonald, 2001). Specifically, higher levels of block building by preschoolers are predictive of: subsequent mathematics scores on standardized tests in the 7th grade, the number of mathematics classes and honours classes taken in high school, average mathematics grades in secondary school, greater reading ability in elementary school, and a faster rate of growth in reading skills (Hanline, Milton, & Phelps, 2010; Wolfgang et al., 2001).

Despite these findings on the benefits of block play in supporting academic skills, the National Research Council committee on Early Childhood Mathematics (Cross, Woods, & Schweingruber, 2009) has strongly recommended that children between three and six years old learn geometry (along with numeracy) as they are not currently provided with adequate opportunities to engage in developmentally appropriate early childhood math activities. This sentiment is supported by a survey with preschool teachers revealing that engaging in geometry is their least favorite topic compared to counting (Sarama, 2002). Additionally, there is a dearth of research on the role of adult geometric talk and play on children's mathematics competence. Hence, there is a pressing need to (1) examine the relationship between engaging in block play and numeracy development and (2) understand how adults—parents and early childhood educators—scaffold the development of numeracy through their geometric input using blocks prior to kindergarten and formal schooling.

Thus, the present research investigated the relationship between the level of complexity of block play engaged by parents with their children during play and children's subsequent numeracy competence a year later. Competencies in geometry and numeracy have been reported to being closely related to each other. For example, young children with visual-spatial deficits also experience difficulties performing numerical tasks (e.g., Semrud-Clikeman & Hynd, 1990). Previous studies have determined a child's level of block play by taking a snapshot of each child's final construction and then determining which level this final creation most resembled (Hanline et al., 2010; Wolfgang et al., 2001). A limitation of this approach is that the final construction may not accurately reflect a child's grasp of spatial and geometric knowledge, especially for young children with incomplete mastery of motor and coordination skills. An example is a child building a castle/tower who was unable to put a roof properly on the castle/tower due to limited motor skills. Based on the final creation, one would determine the child's block play to be at the level of block stacking, instead of at the level of block bridging which is considered more advanced according to Johnson's (1966) developmental stages of block play. The current study sought to capture more accurately the level of block play in preschool children (and their parents) by observing the entire block manipulation process.

Method

Participants, Materials and Procedure

Twenty-seven native English-speaking children (14 girls; 13 boys) aged 31 to 41 months old (*M age* = 35.55, *SD* = 2.65) and their primary caregiver (the mother was the primary caregiver in 24 observations) participated in a 30-minute play session at home. The participants were part of a longitudinal study that examined young children's mathematics development. The family socio-economic status (SES) was measured by the mother's education level, a reliable proxy for SES (Catts et al., 2001). The highest education level attained by mothers was as follows: 8% of mothers with high school or lower, 30% with college/trade, and 62% with university, graduate or professional education. All families were two-parent households.

A standard set of toys such as puppets, shapes, and foam blocks were provided in order to minimize differences in the immediate environment. The parent-child dyad play sessions were recorded, transcribed and the parent's levels of block play were coded using the Observer XT 8.0 system and software program (Noldus Information Technology, 2008). A weighted average was used to compute the average level of block play complexity for each parent. Twenty-two percent of the visits were secondary coded by a trained research assistant. The levels of block play were coded based on Johnson's (1966) developmental stages of block play (see Table 1).

One year after the home-visit, the children were individually administered the Test of Early Mathematical Abilities version 3 (TEMA-3) (Ginsburg & Baroody, 2003) to assess their numeracy competence. The TEMA-3 is a standardized early mathematics test that is administered orally and assesses both informal (e.g., numbering) and formal (e.g., written representation of numbers) mathematical concepts. The test has a total of 72 questions; however, the testing stopped once the children answered five consecutive questions incorrectly. The raw TEMA-3 score for each child was computed and used in the analysis.

Results

Our results indicate that caregivers produced 348 instances of block play during the 30-minute play session. About 20.4% of the play session was spent in block play. Most children had some exposure to mathematical-oriented games in their homes; about 62% of the children played frequently with blocks. As shown in Table 1, out of the eight levels of block play, the most frequent levels modeled by parents were in the lower levels, specifically, levels 1 (40%), and levels 2 (40%), followed by levels 4 (7%), and levels 8 (5%). Examples of Level 7 and 8 block play engaged by two parents are presented in Table 2. The inter-coder reliability based on 22% of the play sessions was as follows: Cohen's Kappa at 92% and the population coefficient (Rho) at 97%.

Furthermore, our findings reveal that the average level of block play engaged by parents (M = 2.15, SD = .77) were positively and significantly correlated with the children's TEMA-3 raw score (M = 12.44, SD = 7.16), [$r(25) = .40$, $p = .037$], indicating that parents who engaged in higher levels of block play had children who received higher scores on the numeracy task one year later.

Table 1: Frequency/Instances of Block Play Levels

Block Play Level	Total instances across families Parent(s)	Proportion of total types Parent(s)	Number of families using the type Parent(s)
1: Carrying involves carrying, grouping, and exploring the properties of the blocks but not using them to build	140	0.40	25
2: Simple Building involves horizontal and vertical rows being stacked	142	0.40	27
3: Bidimensional Building involves superimposing two or more horizontal or vertical rows to make a longer or higher row	6	0.02	5
4: Bridging involves connecting two blocks by a third block to form a roof/bridge over the space between them	23	0.07	12
5: Enclosing involves placing blocks in such a way that they enclose a space	9	0.03	10
6: Tridimensional Enclosing involves superimposing two or more enclosures to make additional layers	3	0.01	3
7: Making Decorative Patterns involves any pattern that follows an AB, ABA, or AABB pattern with the shape or colour of the blocks	6	0.02	2
8: Representational Play involves using the block constructions to represent things	19	0.05	8

Table 2: Examples of Levels of Block Play

Level 7:

Mother: Make this side of the castle the same as that side! See how we used the red rectangle on that side with the blue cylinder on top and then another red rectangle on top of that. Why don't you try to make this side the exact same as that?

Level 8:

There were many parents who would direct their children to try and do representational play and pretend that their block construction was something else.

Mother: Let's pretend that this is a fire station. This block could be the door, this space here could be where all of the fire trucks are kept, and on this block is where the firefighters wait to get called.

Discussion

The results reveal that children's emergent numeracy skills are positively related to the levels of complexity in block play by their parents. However, our findings also indicate that the parents spent more time engaging at the lower levels of block play with their children. This set of findings is consistent with the findings that most parents are unsure of how to engage in mathematics activities with their children (Cannon & Ginsburg, 2008).

Ideally, children between three and six years old should be engaged in developmentally appropriate early childhood math activities such as blocks to learn geometry and numeracy (Cross, Woods, & Schweingruber, 2009). Levels of block play are indicative of having advanced understanding of geometry and spatial skills, therefore, parents should be encouraged to move beyond carrying blocks around and simple building to enrich children's understanding of early mathematical development.

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EVOLUTION OF NUMBER AND PERMUTATION ACTIVITIES WITHIN AN ELEMENTARY 3D VISUALIZATION TEACHING-LEARNING TRAJECTORY

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This research team is developing a teaching-learning trajectory on 3D visualization for elementary children. This paper focuses on how specific teaching activities that connect to number and permutation concepts have evolved since the project began four years ago. Implications for mathematics learning at the elementary level are discussed.

Keywords: Geometry and Geometrical and Spatial Thinking; Learning Trajectories (or Progressions); Elementary School Education; Instructional Activities and Practices

Theoretical Frameworks, Context and Methods

We define the 3D visualization construct as the capacity to move among various representations of solid geometric figures. This design research project responds to the need to develop coherent classroom activities around 3D visualization that will impact learners' capacities to engage in higher-level mathematics and science at a later stage.

The *spatial operation capacity* (SOC) framework (van Niekerk, 1997; Sack & van Niekerk, 2009) that guides this study exposes children to activities that require them to act on a variety of figures made from wooden cubes or their 2D images including physical and mental use of transformations, to develop skills necessary for solving spatial problems.

The SOC framework (see Figure 1) uses: full-scale figures, that, in this study, are created from loose cubes or Soma figures, made from glued unit cubes (see Figure 2); conventional 2D pictures; abstract representations such as front, top and side views or numeric top-views that do not obviously resemble the 3D figures; and, verbal descriptions using appropriate mathematical language that may be accompanied by gestures (Sack & Vazquez, 2008). When designing tasks the research team selects one of the SOC representations to be the stimulus object that learners must act upon in specific ways. The product may be among any of the SOC representations.

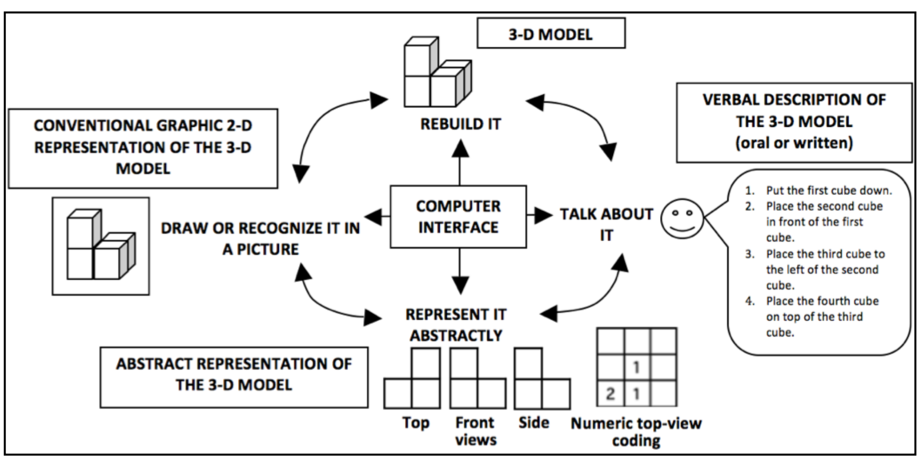


Figure 1: Multiple representations within 3-D visualization

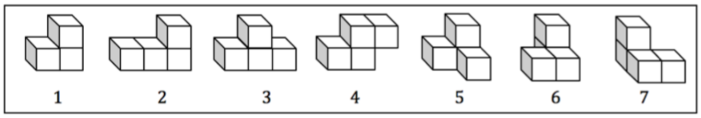


Figure 2: The seven Soma figures

In addition, the project utilizes a dynamic computer interface, Geocadabra (Lecluse, 2005). Through its Construction Box module, complex, multi-cube structures can be viewed as 2-D conventional representations or as abstract top, side and front views or numeric top-view codings. These options can be (de)selected according to instructional goals. The interface integrates the SOC representations in the form of a dynamic image that can be moved with the mouse to provide the same views as if moving about the 3-D object. These images replicate the attributes of the physical 3D objects by behaving as such in learners' minds. This becomes the basis upon which later mathematical thinking occurs. The model extends as the child develops his or her own problems based upon the objects that were recently defined (Connell, 2001).

The study is conducted in an urban elementary school within a large southwestern US public school district. More than 50% of its students are designated "At Risk" and as English Language Learners. The study occurs weekly for one hour in teacher-researcher, Vazquez' 3rd-grade classroom within the school's existing after-school program. The research team uses socially mediated instructional approaches to support a problem-solving environment that fosters children's creativity according to readiness and interest.

Based on design-research principles (e.g., Cobb, et al, 2003) this research team has developed a teaching and learning trajectory to develop children's 3D visualization capacity. In this particular context, the teaching and learning trajectory refers to several knowledge-development pathways, including: sequencing of learning tasks and/or their delivery; interpretation of children's learning and development of 3D constructs as tasks are enacted; and trials of modified or new tasks to further enhance learning about 3D constructs.

Each experimental lesson is followed by a retrospective analysis in which the research team determines the actual outcomes and then plans the next lesson. This may be an iteration of the last lesson to improve the outcomes, a rejection of the last lesson if it failed to produce adequate progress toward the desired outcomes, or a change in direction if unexpected, but interesting, outcomes arose that are worthy of more attention. Data corpus consists of formal and informal interviews, video-recordings and transcriptions, field notes, student products and lesson notes.

Results and Discussion

Initial Trajectory Summary

At first, children solve problems with 3D models and 2D task cards using loose cubes and then the Soma figures. The task cards show assemblies of two Soma combinations in different orientations requiring figure identification and classification. Thus, learners become familiar with the SOC 3D and 2D conventional graphic representations. Then, children begin to digitally reproduce figures printed in a customized manual (e.g., see Figure 3). Through these tasks the children begin to coordinate numeric top-view codings with 2D pictures. They assemble various combinations of two Soma figures and produce their own printable task cards using the Construction Box. They draw the top-view numeric codings for others to identify which two Soma figures correspond to their coding diagrams. This sequence within the project's original trajectory has been retained with few changes for subsequent cohorts.

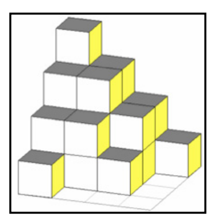


Figure 3: Task 1*f*

Permutations: Fixed Volume Rectangular Prisms

Through pre-program interviews, the research team has confirmed that many children find rectangular prisms made of unit cubes more difficult to visualize and enumerate than irregular structures such as those in Figure 3 (Sack & Vazquez, 2010). The research team developed a problem: *Ms. Moral must ship 24* *cube-shaped shoeboxes to a nearby city. How can the boxes be arranged if they must be shipped in a rectangular prism container?* Twenty-four has a relatively large number of factors leading to a variety of permutations across all factor sets. The research team hypothesized that the children would use the $l \times w \times h$ volume formula from their regular classroom. However, they surprisingly recorded each prism using the top-view numeric notation learned in our program.

The children's development of multiplication constructs during their academic day centers on visual representations, such as, area models, number-line jumps and various pictorial grouping models. Rectangular volume follows much later in the curriculum. Children begin to recognize the area model within the top-view numeric coding of rectangular prisms. Several different 24-cube prisms are constructed. The discussion about the numbers of rows, number in each row, and number of stacks is enacted many times and allows the children to conceptualize volume deeply through the top-view numeric representation. They discover that by rotating one prism onto a different face a permutation of its representation can be created and recorded (e.g., Figure 4.)

Figure 4: Children's top-view numeric grids and volume calculations

At the end of the year, the children are challenged to create a rectangular cake pattern with all seven Soma figures laying flat except for #5, #6 and #7, which have one cube sticking up from the base rectangle, forming three candles. They draw the top-view coding for each design. One will be selected as a template for the end-of-year party cake. See Figure 5 for an example.

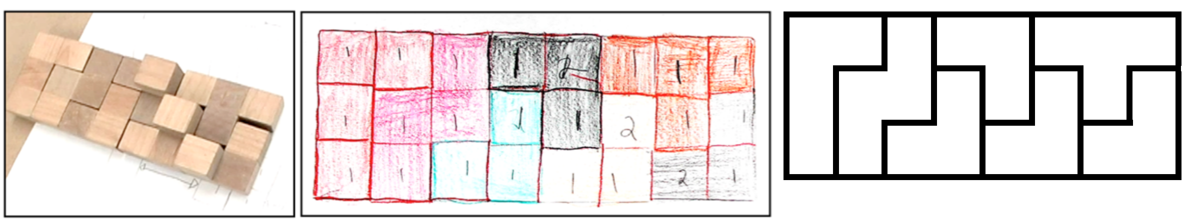
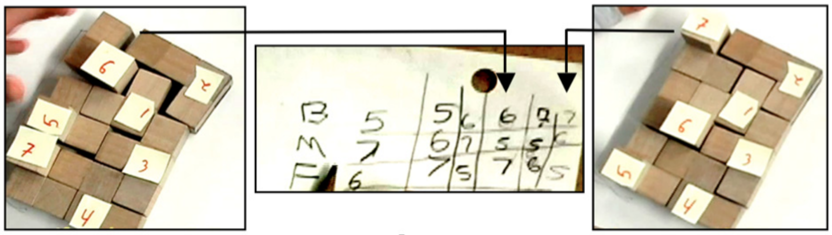


Figure 5: A cake pattern made from the 7 Soma figures and its top-view numeric coding

During the first cohort, Sara (pseudonym), noticed that the bases of Somas #5, #6 and #7 all formed the same L-shape and these could be interchanged providing a hands-on permutation experience. During the academic day, Sara struggled with analytical skills in which she was deficient relative to the 3rd grade norms. However, this program provided an opportunity for her to engage her relatively strong visual skills to solve quite difficult spatial problems. Figure 6 shows a tabulation that Sara created spontaneously during her explanation to the whole group. In Sara's chart, F represents the L-shape in the front of the figure, closest to her; B represents the L-shape at the back; M represents the L-shape in the middle of the figure, where Soma figures #5, #6 and #7 are positioned in Figure 6a to the left and Figure 6c to the right. Teacher Vazquez had asked Sara to label her Soma figures to keep the focus on their positions rather than on their identification each time they were moved about within the base pattern.



a. b. c.

Figure 6: Sara's cake permutations

Connections were made to academic day class work in which children determined how many different words, real or not, could be made from 3 random letters, such as T, E, and A; or, how many different 3-digit numbers could be made from 3 random digits. Sara compared the Soma figures, #5, #6, and #7, with her letters for the cake positions, F, M, and B. Sara was now able to explain how many words or numbers can be made much in the way different candle positions could be determined on her cake pattern. This was a significant connection for this child.

The ability to create multiple patterns by interchanging these figures (including Soma #1) has emerged with each cohort and is a key group discussion to connect permutations within the SOC framework.

Conclusions

This program focuses on 3D visualization in the geometry strand of the elementary mathematics curriculum. The specific teaching activities that connect to number and permutation concepts integrate all of the SOC representations. These exemplars show how children make meaningful connections through worthwhile problems across the SOC framework in ways that help them build the capacity to abstract and generalize. This is largely due to this research team's deliberate efforts to honor and encourage children to share their thinking publicly and safely in the classroom. The research team believes that these experiences will provide learners a solid foundation for extending their spatial processing knowledge to other STEM domains.

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USING THE GSP AS A PSYCHOLOGICAL "TOOL" TO RECONCEPTUALIZE SOME PROPERTIES OF ISOSCELES TRAPEZOIDS

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A semester-long pilot teaching-experiment on the teaching-learning of geometry was conducted with four sophomore pre-service teachers. They had taken no geometry courses in college and their knowledge was rooted only in their high-school experiences. The main goal was to use the Geometer's Sketchpad (GSP) as a mediating "tool" to develop their understanding of triangles and quadrilaterals. Here, we analyze a task about trapezoids posed to one of the pre-service teachers, with the pseudonym of Michael, who advanced his understanding of congruence and similarity of triangles at a faster rate than the other teachers.

Keywords: Geometry; Technology; Teacher Education–Preservice

Rationale

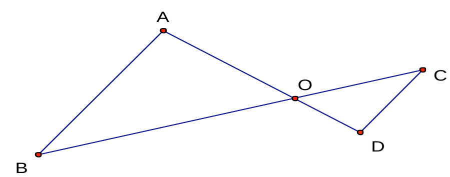
The study was framed within the Vygotskian theory of conceptual development. This theory considers that *material tools*, *psychological tools*, and *social interaction* mediate cognitive development. Prominent examples of psychological tools in mathematics are place-value systems, number line, *x-y* coordinate plane, equations, and the language of mathematics. These devices are, more often than not, perceived as pieces of information rather than "tools" to be used to organize and construct mathematical knowledge and understanding (Kinard & Kozulin, 2008). Likewise, the GSP can be used as a *technical tool* to make constructions rather than as a *psychological tool* to explore and better understand definitions and properties of two-dimensional geometric figures. We concur with Laborde (1993) and Hoyles (1996) who argue that dynamic environments, like GSP, lead us to change the way we think about solutions of geometric tasks. Mariotti (2001), Jiang (2002), Christou, Mousoulides, Pittalis, and Pitta-Pantazi (2004), De Villiers (2004), Mariotti (2006), and Laborde, Kynigos, Hollebrands, and Strässer (2006) also argue about the appropriateness of geometric tasks for dynamic environments as well as the guiding role of the instructor to facilitate argumentations, explanations, justifications, and proofs.

Methodology

A semester-long, pilot teaching-experiment on the teaching-learning of triangles and quadrilaterals was conducted with four sophomore pre-service teachers who had taken no geometry courses in college. The pilot study had two goals. The first was to use the GSP in such a way that it would be not only perceived as a *technical tool* but also used as a *psychological tool*. The second was to pilot the tasks designed for a research study to be conducted with pre-service high-school teachers. The purpose of each task was to allow freedom to explore, to make conjectures, and to prove it. Each pre-service teacher participated in nineteen, 90-minute, one-to-one weekly interviews. The guiding principle of the teaching-experiment was to use semi-structured tasks to mediate the three-way interaction between the pre-service teachers, the GSP, and the interviewer. Two researchers participated in the study. The interviewer was one of the researchers and the other was the participant observer. The tasks for the first thirteen interviews were designed to re-conceptualize the notions of congruence and similarity of triangles as well as some properties of parallelograms, rectangles, and squares. In the fourteenth interview, Michael was given the task to *transform* a drawing of a trapezoid into a drawing of an isosceles one. Here we analyze Michael's case for two reasons: (1) he was able to re-conceptualize congruence and similarity of triangles faster than the other participants, and (2) we hypothesized that congruence and similarity of triangles would help him to re-conceptualize quadrilaterals in a systematic way. Given the space constraints, we only present short excerpts of this interview and mainly describe it in order to analyze (a) Michael's strategies, and (b) the efforts of the interviewer to indirectly guide his geometric activity.

The Semi-structured Task Posed to Michael

A GSP *drawing* of a geometric figure is called *robust* when the dragging-function preserves the Euclidean definitions and properties of that figure. In contrast, a GSP *drawing* is called *non-robust* when the dragging-function continually gives us a drawing of a different geometric figure. We view a geometric task as *structured* when the questions are related to a *robust drawing* and as *semi-structured* when the questions are related to a *non-robust drawing*. The task given to Michael, in this interview, is an illustration of a semi-structured task. In Drawing 1 the side AB is parallel to the side CD. Michael was asked first to *drag* points and segments and to observe *what* kinds of quadrilaterals would result. Second, he was asked to transform the trapezoid ACDB into an isosceles one.



Drawing 1

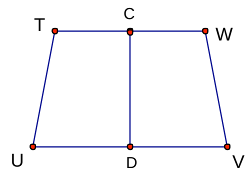
With respect to the first question, Michael dragged points and segments and formed parallelograms, rectangles, squares, and trapezoids using the property of opposite parallel sides and the lengths of the diagonals. In prior interviews, he had investigated and proved some properties of parallelograms, rectangles, and squares using congruent triangles. With respect to the second question, Michael initially explored the drawing, at random, without taking into consideration any specific properties of isosceles trapezoids.

Unsuccessful Transformation of Drawing 1 into an Isosceles Trapezoid

The researcher asked Michael to observe and explore Drawing 1. He observed that triangles AOB and DCO are similar and tried to transform Drawing 1 into an isosceles trapezoid by dragging sides and vertices, but he was not sure what to do. Then, the researcher intervened asking him to explain *how* he was trying to do this transformation. Michael said he was trying to make *the base angles congruent,* but he continued dragging the vertices at random. Therefore, the researcher decided to facilitate Michael's activity and advised him to construct, on a new screen, an isosceles trapezoid. He constructed an angle and his goal was to mirror the constructed angle and get an equal angle at the other endpoint of the horizontal line segment using the GSP as a *technical* tool. However, Michael did not know *what* line should be the mirror. Then, he constructed an angle at the other endpoint of the horizontal line segment and measured both angles. He moved the non-common side of one of the angles until the two measurements were roughly the same. Finally, he constructed a line segment parallel to the initial horizontal line segment forming a *non-robust* isosceles trapezoid. Then, by indirect questions, the researcher helped Michael to establish an association between *isosceles triangles* and *isosceles trapezoids*. Although Michael made the connection that the non-parallel sides are congruent in isosceles trapezoids, he was unable to use this fact to construct a *robust* isosceles trapezoid. The construction of *robust drawings* requires a purposeful use of the GSP. This is to say that Michael also needed to use the GSP as a tool to organize and guide his thinking rather than just as a measuring tool. This is what we have called the *transformation* of the GSP into a psychological tool (Sáenz-Ludlow & Athanasopoulou, 2008).

Inferring and Proving a Property of Isosceles Trapezoids

The researcher continued asking Michael if he knew other properties of isosceles trapezoids. He gave no answer, so the researcher suggested to Michael to join the midpoints of the parallel sides of the isosceles trapezoid he had constructed (Drawing 2).



Drawing 2

Michael measured the angles formed between CD and the bases UV and TW and then he said, “The line CD is perpendicular to the bases.” The researcher asked him if he could prove this statement. He constructed line segments CU and CV. He proved the congruence of triangles ΔCTU and ΔCVW by SAS as well as the congruence of triangles ΔCUD and ΔCVD by SSS. Using the implication of the later congruence ∠UDC=∠CDV and the property that these angles are supplementary, he concluded that CD is perpendicular to the base UV. He also added that by a similar argument CD is perpendicular to the base TW. Then, the researcher asked Michael if CD is a line of reflection. He said,

> 265 I think it is. Let’s see….[He uses the GSP as a *technical tool* to mark CD, to highlight points T and U, and to reflect them with respect to CD. Points T and U fall over the points W and V]. Yes, CD is a line of symmetry; it is a *mirror*. It is symmetric.

Michael used the reflecting function of the GSP to verify the symmetry of the drawing with respect to the line CD. That is, he used the GSP to verify his implicit idea of some type of symmetry in isosceles trapezoids. He was looking for the mirror! Now the GSP was not only a technical tool, it was also a tool to guide his thinking—a *psychological tool*. With some hints on the part of the researcher, he was able to set in motion the proof of his statement using his consolidated knowledge of congruence of triangles.

Inferring and Proving one More Property of Isosceles Trapezoids

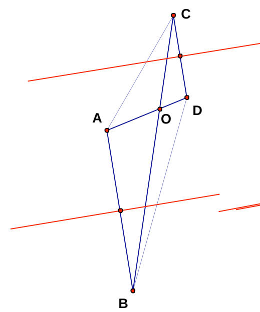
The researcher guided Michael to infer another property of isosceles trapezoids. The researcher asked Michael to copy Drawing 2, to hide the line segment CD, and to construct the diagonals TV and WU. Michael explored the drawing by *measuring* the diagonals and the sub-segments created by the intersection of the diagonals and tabulated these measurements. He observed that the diagonals did not bisect each other and he added that “if they bisected each other, the quadrilateral could be a parallelogram, or a rectangle, or a rhombus, or a square.” Then, he conjectured that the diagonals were congruent. The researcher asked Michael if he could prove it. He proved that ΔTUV and ΔWVU were congruent by SAS and inferred the congruence of the diagonals.

The above episode indicates that Michael was learning to guide his own geometric activity. He used the GSP as a *technical* and *psychological tool* when he measured the diagonals and the sub-segments determined by the intersection, tabulated all the measurements, and conjectured that in isosceles trapezoids the diagonals were congruent. The fact that he was able to conjecture and prove a new statement about isosceles trapezoids indicates a progress in his geometric activity. This episode also indicates that Michael is starting to gain control on his geometric thinking aided by the GSP and his interactions with the researcher.

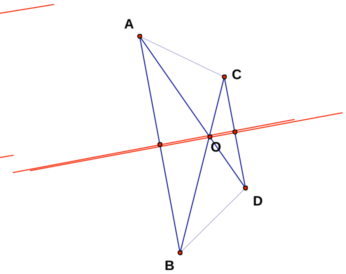
Back to the Initial Question

The researcher suggested to Michael to go back to Drawing 1 and to try to transform the trapezoid ABDC into an isosceles one. He measured the diagonals and then dragged a vertex in an effort to make them congruent. This strategy created a non-robust geometric drawing, so the researcher asked Michael to do the transformation without measuring but by using either of the two properties of isosceles trapezoids he had proved.

After several failed attempts to construct equal diagonals without measuring, Michael came up with the idea to construct perpendicular lines through the midpoints of the bases AB and CD. Then, by dragging different vertices, he saw that when the two perpendicular bisectors of the bases became one and the same, the intersection of the diagonals fell over the perpendicular bisector (Drawings 3 and 4) making the trapezoid ABDC, at that moment, an isosceles trapezoid. In other words, Michael had used the GSP as a *technical* and *psychological tool* to generate a strategy that involved one of the properties he had proved before in isosceles trapezoids.



Drawing 3



Drawing 4

Concluding Remarks

The analysis indicates that Michael, with the indirect guidance of the researcher and with the mediation of the GSP, was able to conjecture and prove two properties of isosceles trapezoids. One was that the segment joining the midpoints of the bases is also perpendicular to them. The other was that the diagonals are congruent. He used his consolidated knowledge about congruence of triangles to prove the statements. Further, Michael was able to use the GSP as a *psychological tool* to *transform* a drawing of a trapezoid into a drawing of an isosceles one by applying the first property.

The analysis also indicates that Michael transcended the *technical* aspects of the GSP and started to use it as a *psychological tool* not only to conjecture and prove properties of isosceles trapezoids but also to *transform* a trapezoid into an isosceles one. In sum, the semi-structured task for the GSP mediated the three-way interaction between the Michael, the GSP, and the researcher. In addition, the semi-structured task, the GSP, and the indirect guidance of the researcher contributed to improve the development of Michael's geometric activity and initiated *self-regulation* of his geometric activity.

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GENDER DIFFERENCES OF HIGH AND LOW PERFORMING STUDENTS' SPATIAL REASONING

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Keywords: Equity and Diversity, Gender, Geometry and Geometrical and Spatial Thinking

Mental rotation is the ability to look at an object and visualize its rotation in three-dimensional space. Males, in general, have stronger spatial reasoning skills, and typically outperform their female counterparts on mental rotation tasks (Linn & Petersen, 1985). The purpose of this study was to examine differences between two groups of sixth grade students' spatial reasoning skills. Specifically, How do high and low performing students spatially reason, and what gender differences exist between high and low performing students as they spatially reason? Using a mixed methods approach, we investigated students' spatial development by gender within and between control and experimental groups. Both student groups (two experimental and one control) studied Earth/Space concepts related to the Solar System, which integrated various mathematical concepts. Students' understanding was documented before, during, and after project implementation. The quantitative data source used to assess students' pre and post understandings was the Lunar Phases Concept Inventory (LPCI, Lindell & Olsen, 2002), which is a 20-item multiple-choice instrument that assesses eight science domains and four mathematics spatial domains. To examine students' spatial reasoning skills, twenty-four students participated in video-recorded clinical interviews. Students were selected based on their highest and lowest change scores on the LPCI from pre to post. Eight students (2 low female, 2 high female, 2 low male, 2 high male) per teacher were interviewed. Overall, the high performing students (males and females) made positive gains from pre to post on every LPCI mathematics domain. The low experimental females made slight gains on both spatial projection and periodic patterns, whereas the low control males only made slight gains on periodic patterns. The low control females made slight gains on geometric spatial visualization, and both the low control and low experimental males did not have any positive gains on cardinal directions. All the male subgroups, except the low experimental males, had higher post scores on spatial projection. Students were then asked to sketch the top and bottom view of Figure 1. Students who sketched the top and bottom view holistically (viewed the object as a whole and first sketched the outline of the shape) were more successful mentally manipulating figures. However, students (mostly low males and low females) who sketched the top and bottom view discretely (sketched each individual block separately) were not as successful mentally manipulating objects. Additionally, students who approached the task holistically, scored higher on the post assessment for the mathematics domains of the LPCI. Understanding how males and females (low and high) reason spatially may lead to more focused interventions that better promote spatial skills for *all* students.

Figure 1

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A COMPARISON OF THREE STUDENTS' RESPONSES TO AREA INVARIANCE TASKS ACROSS GRADES 2–5

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The theme of area invariance is one of the most difficult concepts related to area measurement. Students struggle to conceptually understand how two regions may look different but have equivalent areas (Lehrer, 2003). The authors of this report define area conservation to be a subset of area invariance. With area conservation, the child observes a region being decomposed into parts, as well as how those parts are recomposed to form a new region that has the same area as the initial region; whereas, with area invariance, the child does not observe the transformation directly. Historically, research has been conducted related to area conservation, but more research needs to be conducted on students reasoning about area invariance.

The data discussed in this report come from a longitudinal teaching experiment (Steffe & Thompson, 2000) with 16 students from a Midwestern public school. The study was designed to investigate children's thinking and learning about length, area, and volume measurement across grades 2-5. In this report, we consider a subset of that data. Although area invariance was not a focus of the longitudinal study, area invariance is related to area measurement. Hence, results were re-analyzed retrospectively with respect to the development of area invariance.

Our results suggest that some students can reason qualitatively about area invariance, some can reason quantitatively about area invariance, and others can reason flexibly about area invariance, integrating qualitative and quantitative arguments. Abby, Anselm, and Owen followed consistent and steady paths for developing area conservation competencies but in different ways. Throughout seven semesters, Anselm never made an error related to area invariance. Anselm demonstrated an ability to reason qualitatively and quantitatively about two-dimensional space throughout the entire study, from grade 2 through grade 5. Owen illustrated steady progress in his reasoning about area invariance. Initially, he conserved shape and area qualitatively by mentally decomposing and recomposing, but he could not reason quantitatively. However, he demonstrated an ability to reason qualitatively and quantitatively about two-dimensional space by the end of fifth grade. In contrast to Anselm and Owen, Abby was inconsistent in her responses to area invariance tasks. She initially recognized that different shapes could have the same amount of area. However, in later grades, she did not reason about area invariance of two shapes either qualitatively or quantitatively.

These results indicate that young children may not be able to attend to two dimensions simultaneously when attempting to reason about area invariance. Furthermore, reasoning about shape, reasoning about area, and reasoning about space seem to be intertwined concepts. These results also suggest that reasoning about area invariance qualitatively is disconnected from reasoning about area invariance quantitatively. Thus, reasoning flexibly both qualitatively and quantitatively about area invariance is non-trivial.

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BODIES IN MOTION: LEARNING MATHEMATICS AND SHI-DO-KAN KARATE

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> There is no such thing as a computational person, whose mind is like computer software… Real people have embodied minds whose conceptual systems arise from, are shaped by, and are given meaning through *living human bodies*. The neural structures of our brains produce conceptual systems and linguistic structures that cannot be adequately accounted for by formal systems that only manipulate symbols (Lakoff & Johnson, 1999, 6, emphasis mine).

Mathematics educators rarely acknowledge the role of the human body in producing mathematical knowledge. Gesture research describes the communicative function of spontaneous movements (Sfard, 2008) but less on deliberate movements and their meaning-making function. Work on embodied cognition reports the interaction between physical objects and mathematical knowing (Lakoff & Núñez, 2000; McGarvey & Thom, 2010), but focuses more on the objects as representations of mathematical concepts and less on students' bodies. Human bodies have implications for mathematical knowing—they balance, move through space, and reach out to grasp objects, all of which require spatial reasoning, transformations, etc. Understanding the tacit knowledge required for these deliberate movements pushes the boundaries of mathematical knowing, broadening conceptions of how students make meaning of three-dimensional space.

On this poster, I theorize data from an ethnography of a karate dojo. I mathematize the bodily movements of karate students to understand their opportunities to develop tacit and explicit knowledge of mathematics as they learn *kata*. Kata are choreographed series of movements that are scaffolded over time to increase in complexity of directions, angles, and distances of movement. Students are expected to verbally describe the movements, developing both informal and formal mathematical language for them (e.g., using terms like higher, forward, 65%, 45 degrees). Using pictures and other representations of the kata alongside transcripts of student and teacher discourse, I depict students' opportunities to embody (and in some cases, talk about), spatial relationships, transformations, and spatial reasoning (NCTM, 1999). I discuss the human body in motion as a source of funds of mathematical knowledge (González, Andrade, Civil, & Moll, 2001) and a resource for modeling and model-eliciting activities (Lesh & Doerr, 2000).

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CONCEPTUAL UNDERSTANDING OF DIMENSIONAL CHANGE

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Keywords: Spatial Thinking; Teacher Knowledge; Mathematical Knowledge for Teaching; Teacher Education

Dimensional reasoning includes the ability to predict changes in measures across dimensions, such as predicting changes in area or volume given variations in edge length. This study examined methods of determining dimensional change for pre and in-service teachers (n = 90). Data were collected from problem situations where information for a computational solution was readily available and in problem situations where it was necessary to generalize the effects of dimensional change to obtain a solution. Results suggest that teachers in this sample do not exhibit flexible and generalized knowledge of geometry and dimensional change. This is a concern as teachers who emphasize computational solutions more heavily than conceptual understanding in their own mathematical solutions may neglect to emphasize conceptual understanding in geometry instruction. The results from this study suggest a more balanced geometry instructional focus that includes formulas, computation and memorized characteristics as well as exploration, generalization, and flexibility concerning relationships between shapes, measures and dimensions.

The teachers in this study relied heavily on computation strategies using familiar formulas to determine resultant change from changes in dimensional measures. One example of using formulas and computation is for the question: If a box has a volume of 10, what would its volume be if all its sides were doubled? Participants were able to identify the length, width, and height for a box with a volume of 10, double all of the sides, compute the new volume, and then compare old and new volumes. When figures became too complex for computation, or formulas could not be recalled, teachers were less successful in obtaining correct answers. This difficulty may indicate a lack of a generalized concept of dimensional change.

Several possible rationales for the exhibited lack of dimensional reasoning exist. Participants may lack conceptual knowledge about what is meant by the word “dimension.” Arcavi (2003) recognized that visualization is a “key component of reasoning” (p. 235). Participants may not be able to visualize the dimensional change occurring in each problem. Because many participants did not utilize dimensional change from one to two dimensions to predict change from two to three dimensions, nor did they utilize changes in simple figures to predict changes in more difficult figures, it appears that participants may not recognize that a pattern or generalization exists for dimensional change. More research is needed to identify why specific dimensional reasoning deficits exist and determine ways to mediate them.

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WHY ORDER DOES NOT MATTER: AN APPEAL TO IGNORANCE

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Within the limited field of research on teachers' probabilistic knowledge, incorrect, inconsistent and even inexplicable responses to probabilistic tasks are most often accounted for by utilizing theories, frameworks and models, which are based upon heuristic and informal reasoning. More recently, the emergence of new research based upon informal logical fallacies has been proving effective in accounting for certain normatively incorrect responses to probabilistic tasks. This article contributes to this emerging area of research by demonstrating how a particular informal logical fallacy, known as "an appeal to ignorance," can be used to account for a specific set of normatively incorrect responses to a novel probabilistic task, which were provided by prospective elementary and secondary mathematics teachers.

Keywords: Cognition; Probability; Teacher Education–Preservice; Teacher Knowledge

The objective of this article, in general, is to contribute to the limited amount of research on (prospective) "teachers' probabilistic knowledge" (Jones, Langrall, & Mooney, 2007, p. 933; Stohl, 2005). In specific, the objective of this article is to contribute to the well-established domain of research, which accounts for (through various theories, models and frameworks) incorrect, inconsistent, sometimes inexplicable responses to a range of probabilistic tasks (Abrahamson, 2009; Chernoff, 2009; Kahneman & Tversky, 1972; Konold, 1989; Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993; LeCoutre, 1992; Tversky & Kahneman, 1974).

To meet the general and specific objectives stated, prospective teachers were asked to determine and justify which of two student responses provided the correct answer and explanation to a question, which involved determining the probability that a three-child family has two daughters and one son. In addition to contributing a twist to a task recently introduced to the research literature, we also utilize a novel lens to account for certain responses to that task. In our analysis, we demonstrate that logically fallacious reasoning, more specifically, in this instance, an appeal to ignorance (i.e., there is no evidence for *p*; therefore, *not-p*) accounts for certain prospective teachers' normatively incorrect responses to the task. By meeting the general and specific objectives presented, this article will correspond and contribute to a continuation or, stated in terms associated with the theme of PMENA 2012, a *transition*—from heuristic reasoning (e.g., Kahneman & Tversky, 1972) and informal reasoning (e.g., Konold, 1989) to logically fallacious reasoning (e.g., Chernoff & Russell, 2011a, 2011b, in press)—in how researchers account for particular prospective teachers' responses to probabilistic tasks.

A Brief Summary of Prior Research

Research into probabilistic thinking and the teaching and learning of probability has, in the past, seen a focus on normatively incorrect responses. Worthy of note, the focus on normatively incorrect responses does not, in any way, suggest a negative view of the mind (Kahneman, 2011). The theories, models and frameworks associated with heuristic and informal reasoning—rooted in the notions of conceptual analysis (Thompson, 2008, Von Glaserveld, 1995), grounded theory (Strauss & Corbin, 1998) and abduction (Peirce, 1931)—have, traditionally, accounted for normatively incorrect responses to probabilistic tasks within the field of mathematics education. Chernoff (2009a, 2009b, 2009c, 2011, in press) and Chernoff and Russell (2011a, 2011b, in press) have provided detailed accounts of the theories, models and frameworks associated with heuristic and informal reasoning in the field of mathematics education and are recommended—given the 8-page limitation associated with the present venue—to the reader.

More recently, a burgeoning area of research suggests that fallacious reasoning, more specifically, the use of informal logical fallacies, can account for certain normatively incorrect responses to probabilistic tasks. For example, Chernoff and Russell (2011a) demonstrated that certain prospective mathematics teachers—when asked to identify which event (i.e., outcome or subset of the sample space) from five flips of a fair coin was least likely to occur—did not use the representativeness heuristic (Kahneman & Tversky, 1972), the outcome approach (Konold, 1989) or the equiprobability bias (Lecoutre, 1992), but, instead, utilized a particular informal logical fallacy, the fallacy of composition: when an individual infers something to be true about the *whole* based upon truths associated with *parts* of the *whole* (e.g., coins (the parts) are equiprobable; events (the whole) are comprised of coins; therefore, events are equiprobable, which is not necessarily true). Worthy of note, the fallacy of composition accounted for both normatively correct and incorrect responses to the new relative likelihood comparison task.

In subsequent research, Chernoff and Russell (2011b, in press) applied the fallacy of composition to a more traditional relative likelihood comparison. Prospective mathematics teachers were asked to determine which of five possible coin flip sequences—not events—were least likely to occur. As was the case in their prior research (e.g., Chernoff & Russell, 2011a), the fallacy of composition accounted for normatively incorrect responses to the task. More specifically, the researchers demonstrated that participants reference the equiprobability of the coin, note that the sequence is comprised of flips of a fair coin and, as such, fallaciously determine that the sequence of coin flips should also have a heads to tails ratio of one to one. In other words, the properties associated with the fair coin (the parts), which make up the sequence (the whole), are expected in the sequence. Once again, the fallacy of composition, not the traditional theories, models and frameworks associated with heuristic and informal reasoning, accounted for certain normatively incorrect responses to a probabilistic task.

Chernoff and Russell (2011a, 2011b, in press) contend, based on their research utilizing the fallacy of composition, that they have (re)opened a new area of investigation for those researching probabilistic thinking and the teaching and learning of probability. However, they also contend that more research will allow individuals to determine to what extent informal logical fallacies and fallacious reasoning can account for normatively incorrect responses to a variety of probabilistic tasks. The former and latter contentions have provided the motivation for us to determine whether or not another informal logical fallacy, an appeal to ignorance, can account for normatively incorrect responses to a probabilistic task that has recently been introduced to the research literature.

Theoretical Framework

Of the numerous informal fallacies that could, potentially, be utilized as a theoretical framework (e.g., equivocation, begging the question, the fallacy of composition, the fallacy of division and others), our analysis of results will rely, specifically, on one particular informal logical fallacy: *an appeal to ignorance*, which, essentially, "is an argument for or against a proposition [*p*] on the basis of a lack of evidence against or for it" (Curtis, 2011, para. 3). However, and worthy of note, an appeal to ignorance can come in one of two forms: (1) there is no evidence against *p*. therefore, *p* and (2) there is no evidence for *p*. therefore, *not-p*. To be more specific, our analysis of results will rely on the second form of an appeal to ignorance: there is no evidence for *p*. therefore *not-p*. Stated in more colloquial terms, the reader may be familiar with the following phrase: "the absence of evidence is not evidence of absence." For example, consider the following question: Is there a lawn mower in my garage? If one does not look inside my garage, the absence of evidence does not amount to evidence of an absence of a lawn mower because there may, in fact, be a lawn mover in my garage. Thus, and stated once again in colloquial terms, an individual who declares the absence of evidence as evidence of absence is employing (the second form of) an appeal to ignorance. Our attention, in the analysis of results, will focus on a set of individuals utilizing the absence of evidence as evidence of absence. Stated, in the terms of our example, our analysis of the results will focus on those individuals who do not look inside my garage and use their lack of evidence of a lawn mower in my garage to declare that there is no lawn mower in my garage. However, our research does not focus on garages and lawn mowers; instead, our research focuses on whether order matters or order does not matter for a particular probabilistic task.

The Jane or Dianne Task

The Jane or Dianne task, presented below in Figure 1, represents an alteration to the original "two boys and a girl task" (Chernoff & Zazkis, 2011, p. 21), which was utilized in previous research (Chernoff & Zazkis, 2011) and introduced by Chernoff and Zazkis (2010).

> What is the probability that a three-child family has two daughters and one son?
>
> **Jane's explanation:** Out of the four possible outcomes (3 daughters, 0 sons; 2 daughters, 1 son; 1 daughter, 2 sons; and 0 daughters, 3 sons) only one outcome (2 daughters, 1 son) is favourable, so the probability is one-fourth.
>
> **Dianne's explanation:** Out of the eight possible outcomes (daughter, daughter, daughter; daughter, daughter, son; daughter, son, daughter; son, daughter, daughter; daughter, son, son; son, daughter, son; son, son, daughter; son, son, son) only three outcomes (daughter, daughter, son; daughter, son, daughter; son, daughter, daughter) are favourable, so the probability is three-eighths
>
> ________________________ 's explanation is correct because.....

Figure 1: The Jane or Dianne task

Fundamentally, the two boys and a girl task is the same as the Jane or Dianne task. In other words, the core of the task, the probability question, that is, what is the probability that a three-child family has two daughters and one son, is the same in both tasks. Previously, the task was utilized in order to elicit insight into prospective secondary school mathematics teachers' pedagogical approaches. In this new version of the task, however, the focus is not pedagogical, but, rather, on which response prospective mathematics teachers deem mathematically correct and, relatedly, which explanation is deemed appropriate. Stated in more general terms, the task has been altered in order to contribute to the limited amount of research on what Jones, Langrall, and Mooney (2007) called "teachers' probabilistic knowledge" (p. 933).

Participants

The (*n* =) 130 participants in our research were comprised of 52 (40%) prospective elementary school teachers (PESTs) and 78 (60%) prospective secondary school teachers (PSSTs). The PESTs were enrolled in a methodology course designed for teaching elementary school mathematics and the PSSTs were enrolled in a methodology course designed for teaching secondary school mathematics. The 52 PESTs were from two different classes (each containing approximately 25 students) and, similarly, the 78 PSSTs were from three different classes (each containing approximately 25 students). For both the PEST's and the PSSTs, the topic of probability had not yet been addressed in their methodology courses. Instead, content, strategies and approaches garnered from research and practice related to the teaching and learning of probability were addressed after the data for this research was collected. To collect the data, participants were asked and given as much time as required to determine, via written response, which of the two explanations, Jane's or Dianne's, was correct and, further, to justify their choice also via written response.

Results

As seen in Table 1 below, there was, roughly, an even split between those individuals who declared and explained why Jane and Dianne's response was correct. Roughly half (51%) of the participants declared that Dianne's explanation was correct. More specifically, 40 of the 78 PSSTs (51%) and 26 of the 52 (50%) of the PESTS chose Dianne and her explanation. Thus, there was little difference between the percentage of PESTs and PSSTs that chose Dianne and her explanation. Worthy of note, not all 66 of the 130 participants provided an appropriate justification for why Dianne's explanation was correct. Interestingly, 9 of the participants (7%) chose an option that was not presented to them. These individuals may have indicated that neither choice and explanation was correct or that both of the choices and explanations were correct. These individuals have been placed in the ‘Other’ column in Table 1. As mentioned, there are certain pagination limitations associated with the current venue and, as such, the responses from those individuals who chose Dianne or who fell into the ‘Other’ category will not be part of this analysis of the results.

Table 1: Numerical Results

Participants	Jane	Dianne	Other
PESTs (52)	23	26	3
PSSTs (78)	32	40	6
Total (130)	55 (42%)	66 (51%)	9 (7%)

Instead, our analysis of the results will focus on the 55 participants (42%) that chose Jane and her explanation. More specifically, 23 of the 52 PESTs (44%) and 32 of the 78 (41%) of the PSSTs chose Jane and her explanation, which, as was the case with Dianne, shows little difference between the percentages associated with the PESTs and PSSTs. The 55 participants who chose Jane and her explanation did not have similar justifications for why Jane’s response and explanation was considered correct. As such, the 55 responses from those individuals who chose Jane have been further categorized in Table 2 below.

Table 2: Numerical Results within Jane Responses

Reference	order	no order	order & question
PESTs (23)	15	8	8
PSSTs (32)	30	2	13
Total (55)	45	10	21

Jane responses were organized into two distinct categories: those responses that referenced order and those responses that did not reference order. Of the 55 participants that chose Jane’s response, 45 (82%) referenced order in their justification and 10 (18%) did not reference order in their response justifications. More specifically, of the 23 PESTs that chose Jane’s response, 15 (65%) referenced order and 8 (35%) did not make reference to order and, of the 32 PSSTs that chose Jane’s response, 30 (94%) referenced order in the justifications and 2 (6%) did not reference order in their justifications, which represents a departure from the previous even split seen between PESTs and PSSTs.

Refining “Jane” responses a step further, of the 45 people that referenced order in their response justifications, 21 individuals (47%) also made reference to the question, that is, what is the probability that a three-child family has two daughters and one son, in their response justifications. More specifically, 8 of the 15 PESTs (53%) and 13 of the 30 PSSTs (43%) specifically referenced order and the question. The 21 responses referencing both order and the question, which represent, concurrently, 16% (21/130) of all participants, 38% (21/55) of those who chose Jane’s response and 47% (21/45) of those who chose Jane’s response and referenced order in their justifications, are featured in the analysis of results.

Analysis of Results

Given the consistency associated with all 21 of the responses that referenced both order and the question, 5 of the 8 responses from the PESTs and 5 of the 13 responses from the PSSTs, that is, 10 in total, are presented for analysis.

PESTs Response Justifications

In what follows, we analyse five exemplary responses, from Sam, Rebecca, Carla, Ernie and Woody, which all evidence an appeal to ignorance. We begin by considering the responses of Carla and Woody.

Carla: Dianne's explanation gives possibilities of 2 girls and one boy *plus birth order possibilities – this is not what the question asked.*

Woody: Dianne reuses some of the possibilities multiple times. *GGB is the same as GBG*. *The question is not asking anything about the order in which they were born*.

As italicized in the responses from Carla and Woody above, both individuals reference that birth order should not be taken into consideration. On the one hand, Woody, declaring that "GGB is the same as GBG," is implicitly declaring that order does not matter. Carla, on the other hand, is more explicit in declaring that Dianne's explanation calculates the possibilities "plus" the birth order, which can be interpreted to mean that the order is in addition to what the question is asking. Both Woody and Carla, however, are quite clear in declaring why they have concluded that birth order does not matter. Essentially, both individuals make it clear that the question does not "ask" about the order. As seen in the responses from Sam, Rebecca and Ernie, which are presented below and similarly italicized as above, they, too, reference that the question does not "say anything" or "mention" or "never asked" (respectively) about birth order.

Sam: This explanation is better because *the question doesn't say anything about birth order*. It just wants to know the probability of 2 daughters and 1 son; *whether or not this occurs as DDS, DSD, or SDD does not matter*. I think they have a 1/4 chance of the 2 daughters and 1 son outcome.

Rebecca: There are no repetitions of the number of each sex of child in Jane's. *The question doesn't mention birth order and therefore there is no need to consider that GGB and BGG is the same thing* in answering this particular question.

Ernie: *The question never asked about order* so the only total is assumed and therefore needed ~ *disregard order patterns* so 4 possible outcomes.

Further, the responses from Sam, Rebecca and Ernie indicate that there is no need to consider birth order or one can, as stated by Ernie, "disregard the order patterns." Alternatively stated, for all three responses, the question does not make reference to order mattering and, as such, order does not matter, which, ultimately, leads to choosing Dianne and her explanation as correct.

Considered from within an appeal to ignorance framework, the responses from Sam, Rebecca and Ernie and, further, from Carla and Woody (and the other 3 PESTs who referenced both order and the question in their responses), all note that the question does not provide evidence that order matters (i.e., there is no evidence for *p*) and, as a result, order does not matter (i.e., therefore *not-p*), which, ultimately, predicates a justification for why Dianne's response and her explanation are correct. Similar results are found within the responses from the PSSTs.

PSSTs Response Justifications

In what follows, we analyse five exemplary responses, from Frasier, Eddie, Robin, Paul and Glen, which also all evidence an appeal to ignorance. We first consider the responses of Paul and Eddie.

Paul: There are four different combinations that are possible *because it didn't specifically say that order mattered*. So generally, you can have 2 sons, 1 daughter; 2 daughters, 1 son; 3 sons; 3 daughters *in any order*.

Eddie: *There is no specification as to what order the daughters & sons have to be born in. Therefore, there is only four possible outcomes* causing a one in four chance.

As seen in the responses from Paul and Eddie, they make reference to the question not specifying that order mattered. Worthy of note, we are inferring in these particular responses that "it" for Paul and "there is no specifications" for Eddie are implicit references to the question. Working from this inference, for both Paul and Eddie, the reason that there are only four possible outcomes or that the different events can happen in any order are predicated on the question not providing evidence that order mattered. The responses from Frasier, Robin and Glen, presented below, are more explicit in their reference to the question.

Frasier: *The question does not state the order of the siblings matters* → they simply want a 2 girl + 1 boy family. *Dianne has multiples of the same outcome* such as DDS, and DSD.
Robin: *The question did not ask* what the probability is that the family has 2 daughters and one son *in that order, so there are only 4 possible outcomes*.
Glen: *The question does not specify that the order of the children matters*, they just want 2 daughters and a son. It shouldn't matter what order they come out in. *As far as the question is concerned, the DDS, DSD, SDD are all the same outcome*.

In fact, the above three responses are quite explicit in declaring that the questions does not "state", "ask" or "specify" that the order of the children matters. Further, and working from the notion that the question does not specify that the order matters, all three participants conclude that the order does not matter, albeit in different ways (e.g., "multiples of the same outcome"; "there are only four possible outcomes"; "DDS, DSD, SDD are all the same outcome"). Alternatively stated, for the responses of Frasier, Robin and Glen, the question does not specify that order of the children matters and, as such, order does not matter, which, ultimately, leads to choosing Dianne and her explanation as correct.

Considered from within an appeal to ignorance framework, the responses from Paul, Eddie, Frasier, Robin and Glen (and the other 8 PSSTs who referenced both order and the question in their responses) all note that the question does not provide evidence that order matters (i.e., there is no evidence for *p*) and, as a result, order does not matter (i.e., therefore *not*-*p*), which, ultimately, acts as a justification for why Dianne's response and her explanation are correct.

Concluding Remarks

As demonstrated in the analysis of results, all 10 responses that were analyzed can be framed within the informal logical fallacy know as an appeal to ignorance. More specifically, all 10 responses made reference, whether implicit or explicit, to the question not "stating", "asking", "indicating" or "declaring" that order mattered (i.e., there is no evidence for *p*) and, as such, the responses further concluded that the order (of the outcomes) does not matter (i.e., therefore *not*-*p*), which was represented differently by different individuals and which led, ultimately, to their decision to choose Dianne's response and explanation. Although only 10 responses were presented in the analysis of results, we further note, based upon the striking similarities between the 10 responses presented and the 11 responses not presented, that the informal logical fallacy, known as an appeal to ignorance, accounts for 100% (21/21) of the participants whose responses referenced both order and the question, which also represents, concurrently, 47% (21/45) of the responses who chose Jane's response and referenced order in their justifications, 38% (21/55) of those responses who chose Jane's response and 16% (21/130) of all the participants involved in the current research.

Discussion

Research into the teaching and learning of probability and probabilistic thinking has focused on accounting for normatively incorrect, sometimes inexplicable responses to a variety of probabilistic tasks. Stemming from these investigations, a number of theories, models and frameworks have been developed to account for and to make sense of particular responses. Traditionally, this particular domain of research has been focused heuristic and informal reasoning. More recently, a emerging thread of research has (re)opened informal logical fallacies as a fresh perspective to account for certain response justifications. While it has been established that the fallacy of composition is able to account for incorrect responses to comparisons of relative likelihood (Chernoff & Russell, 2011a, 2011b, in press), it had not been determined to what extent informal logical fallacies can describe response justifications to other probabilistic tasks other than relative likelihood comparisons and which other fallacies could be utilized. Building upon this emerging thread of research, this article has demonstrated that another informal logical fallacy, other than the fallacy of composition, is able to account for particular responses to probabilistic tasks. More specifically, in this case, an appeal to ignorance can be added to the fallacy of composition as another particular informal logical fallacy that is able to account for certain responses to probabilistic tasks. In other words, it can be argued that this article further strengthens the use of logical fallacies as a new area of investigation for future research on probabilistic thinking, the teaching and learning of probability and teachers' probabilistic knowledge. Despite what can be considered as early "success" with the use of particular informal logical fallacies, for example, the fallacy of composition and an appeal to ignorance, more research will determine to what extent logical fallacies play a part of teachers' and students' knowledge of probability. Speaking inductively for a moment, it appears that we are off to a good start.

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STRATEGIES AND DIFFICULTIES THAT UNIVERSITY STUDENTS DEVELOP THROUGH THE MODELING OF RANDOM PHENOMENA BY SIMULATION

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This paper reports the results of a research about the strategies and difficulties developed by university students in the process of modeling and simulating of random phenomena in an environment of a spreadsheet. The results indicate that students had difficulties to identify key components of the problems, which are crucial to formulate a simulation model. We have identified three different schemes to generate the results of the key components, which only one of them is correct; this scheme is based in the generation of random numbers. In consequence during this investigation it was observed that the process of the instrumentation of the spreadsheet to simulate random phenomena it is complex.

Keywords: Spreadsheet; Modeling; Probability

Purposes of the Study

The computer simulation has been suggested for many researchers and organizations as a pedagogical tool for the probability and statistics teaching, but above all, the statistics software, spreadsheets and computer technology have been penetrating even more inside laboratories and schools (Biehler, 1991; NCTM, 2000). In this way, the modeling of random phenomena by simulation has been constituted as an important part of research in statistics education during the last years. As a consequence of this, some studies were undertaken in order to know their effectiveness in probability and statistics teaching and learning (Maxara & Biehler, 2006; Lee & Mojica, 2008; Chaput, Girard & Henry, 2011).

The computer simulation integrates different aspects that are important in mathematics teaching, and particularly, in probability teaching:

1. It requires an activity of mathematical modeling in which students develop some skills, such as making assumptions to simplify the problem, identify and symbolize variables and parameters, as well as formulate the model by taking into account the assumptions and conditions, to finally, solve them and interpret the results.
2. When it is possible an analytic solution of the problem, the experimental results that are generated by the simulation can be contrasted with theoretical results. In some cases, in which the analytic solution is not possible or complex, the simulation is an important and fundamental tool.
3. It allows to work with abstract issues in concrete words, and above all, when the simulation is performed in computer environments that are equipped with representations (graphics, symbols, numbers) bound together, which make possible a visualization and feedback of the different parts of the model.

Among the advantages of using simulation as a method to solve problems of probability and statistics, Biehler (1991) mentions:

1. The possibility of formulating models in concrete terms instead of expressing the ideas by means of symbolic models (representational aspect).
2. Students can process the data generated more easily than data generated using analytical and combinatory methods (computational aspect).
3. It is possible to begin with the design of the experimental environment instead of starting with the calculations (concept-model aspect).

There are different proposals to formulate a simulation model in a computer environment. Some examples are Gnanadesikan, Scheaffer, and Swift (1987) that propose a complete and detailed a process:

1. State the problem clearly.
2. Define the key components.
3. State the underlying assumptions.
4. Select a model to generate the outcomes for a key component.
5. Define and conduct a trial.
6. Record of observation of interest.
7. Repeat steps 5 and 6 a large number of times.
8. Summarize the information and draw conclusions.

On the other side, Albright (2010) proposes a three step process:

1. Construct a model that uses random numbers.
2. Evaluate the model many times using different random numbers each time.
3. Analyze the results statistically.

The implementation of the simulation in probability and statistics teaching could be given in different ways. NCTM (2000) recommends that the probability's problems could be first investigated through simulations in order to get an estimated result, and after that, to use a theoretical model in order to find the exact result.

One of the most important aspects in the implementation of simulation as a pedagogical tool is the computer tool utilized, because its design determinate some potentialities and constraints to the mathematical activity that students develop through the interaction with it. A special kind of tool in which it's easy to simulate different random phenomena is the spreadsheet. However, its use in mathematics education remains still relegated, despite its potential to handled quantitative information (Haspekian, 2005).

Although, the advantages that simulation offers, it is necessary to do a deep analysis about its didactic potentially. Our interest in this work has been the research of the potential that a spreadsheet can have in order to modeling random phenomena in a basic probability university course, and the difficulties that the students present in the different stages of the process of modeling. Particularly, we ask the following questions: (a) what are the potentialities and constraints that the spreadsheets have to the simulation of random phenomena? (b) what are the strategies that students can develop? and (c) what are the difficulties that students can find in the process of modeling?

Theoretical Framework

In the development of this work, we have adopted an instrumental approach of the mathematical cognition (Artigue, 2002). The most basic notions of this approach consist in the meanings of *instrument* and *artifact*. The artifact is a material or abstract object that is available for certain activities. Examples of this are the language, the calculator and the spreadsheet. On the other hand, the instrument is a personal construct that can be developed by handling an artifact in a progressive way.

An artifact becomes a instrument when the subject achieves to appropriate of the artifact and establishes meaningful relationships for doing a specific kind of work (a mathematical one, in this case), this means that the subject can use and control this artifact to achieve their goals, and to integrate it to their activities (Verillón & Rabardel, 2005). The process of the transformation of the artifact into an instrument is called *instrumental genesis.*

The process of instrumental genesis evolves in two interrelated directions. The first one is directed to the artifact, loading it progressively with potentialities, and eventually transforming it for specific uses; this is called the *instrumentalization* of the artifact. Secondly, instrumental genesis is directed towards the subject, leading to the development or appropriation of schemes of instrumented action, which progressively take shape as techniques that permit an effective response to given tasks. The latter direction is properly called *instrumentation*. In order to understand and promote instrumental genesis for learners, it is necessary to identify the constraints induced by the instrument (Artigue, 2002). The restrictions are the result of the tool's design. In this way, the use of a tool is not a unidirectional process, but a dialectic process between the subject that acts over the instrument, and the instrument, which acts over the thinking of the subject.

In the process of instrumental genesis, the user develops mental schemes for specific tasks. In these schemes, the technical knowledge or in other words, the skills to use the artifact and the knowledge of the specific domain of the mathematical content, are intertwined or complimented (Drijvers & Trouche, 2008). An instrument is a mixed entity constituted by one piece of artifact and other part with the personal schemes that the users develop through doing specific tasks. In the case of a mathematical task, a mental scheme involves the global strategy of solution, as well as the technical resources that the artifact offers, and the mathematical concepts in which underpin the strategy.

Potentialities and Constraints of the Spreadsheet to Random Phenomena Simulation

Excel spreadsheet possesses diverse potentialities and a framework of representational aspects of calculation and communication. In the communication aspect, a spreadsheet requires that the students work with an interactive algebra-like language, which focuses their attention on a rigorous syntax. This is why it is said that spreadsheets help to translate a problem by means and algebraic code (Haspekian, 2005). Into the representational aspect, the spreadsheet has multiple representations that allows several semiotic registers that can be presented in simultaneous ways on screen, such as the case of the formulas register to express relations between cells, numerical register to represent data or results of calculations, graphics register that allows user several types of graphical representations dynamically linked to the numerical data. And finally, for the aspect of calculation, Excel has an extended range of formulas that make possible formulating models, generating data and making calculations. Other important element is the numerical feedback obtained when working with a formula, which allows students to experiment, speculate, and help them to find mistakes. On the specific case of simulation of random phenomena, Excel spreadsheet has several commands to generate pseudorandom numbers. Under the case of discrete random phenomena which are simulated through models of urns (as it is in our case), Excel has two commands that generates numbers provided from a uniform distribution: *rand() and randbetween(bottom,top).* The function *rand()* returns a random number between 0 and 1, meanwhile the function *randbetween(bottom,top)* returns an entire random number between the limits specified. The figure 1 and 2, show those formulas and the resulted generates are shown:

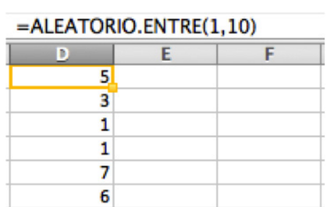


Figure 1: randbetween(1,10)

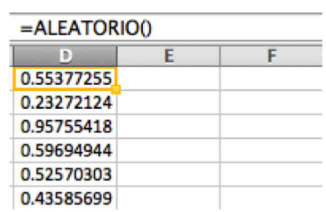


Figure 2: rand()

The pseudorandom numbers generated by *rand()* or *randbetween(inf,sup)* have two properties that make them comparable to in fact random numbers:

1. Any number between 0 and 1 has the same probability to be generated.
2. The numbers generated are independent between each other.

Excel spreadsheet, also as we know, works with arithmetic formulas, conditionals and statistics that make possible the conditions of a problem and realize the analysis of the results. The copy/paste option and F9 permits to evaluate the model as many times doable and therefore a quick feedback.

Methodology

This study was done with 22 students between 19–20 years old of the undergraduate program of Information Systems. The students were enrolled in a basic course of probability and statistics in the first semester of the school year 2011–2012. This course emphasized the simulation as complement of the probability theoretical approach in some themes. Students weren't trained in a previous course about use of Excel to simulation and the study started with the their basic knowledge of Excel spreadsheet. Simple examples of simulation were raised in the class and some assignments to be complement it took place, therefore contrast results of the simulation with the theory achievement. The present work shows the results from the following three activities:

Activity 1: Simulate the rolling of two dices:

a) *Add of points resulting of faces up and determine the probability that the result is 7.*
b) *Subtract points resulting of faces up and determine the probability that the difference of points is 3.*
c) *Multiplicate the points of the faces up and calculate the probability that the product of points is bigger than 10.*

The purpose of this activity was to introduce students to simulation environment in the Excel´s spreadsheet, to observe with what formulas they will be working for and what difficulties they will be confronted with to finally formulate the model. According with the instrumental theory, this activity represents the starting point for the analysis of the instrumentation developed by the students in the spreadsheet.

Activity 2: If a friend of yours thinks in a number between 1 and 100 ¿what is the probability of making it divisible by 6 or 10?

This activity require the application of the rule of the addition of the probabilities; this is it, $\mathrm{P(AUB)} = \mathrm{P(A)} + \mathrm{P(B)} - \mathrm{P(A \cap B)}$, where A is an event that represents divisible numbers by 6, B is an event that represents divisible numbers by 10 and $\mathrm{P(A \cap B)}$ is event that represents divisible numbers by 6 and by 10.

Activity 3: There are 3 urns with black and white balls (Urn 1: 3 white and 2 black. Urn 2: 1 white and 3 black. Urn 3: 6 white and 2 black). There is selected one ball from one urn. What is the probability that a ball color white shows up? There is an assumption of equal probabilities to select any of the urns.

The purpose of this activity was to formulate a model of simulation to the calculus of the total probability of a event; this is it, $P(B) = P(U1)P(B/U1) + P(U2)P(B/U2) + P(U3)/P(BU3)$, where U1, U2 and U3 are events Urn 1, Urn 2, Urn 3 respectively, and B is the event that represents to which white ball is selected. Formulate the model requires identify 2 stages; the first one consist in decide the urn in a random way, and second stage; select the ball from that urn.

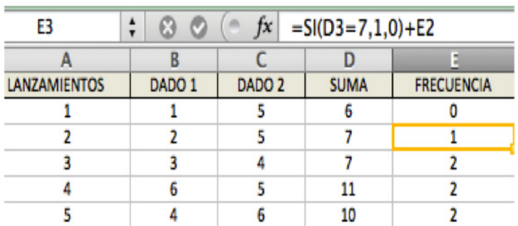
In the analysis of the information we have present the strategies and the elements of the spreadsheet that were used by students in each phase of the process of the modeling described by Gnanadesikan, Scheaffer, and Swift (1987) and Albright (2010).

Results

Formulation of the Models through Random Nnumbers

The formulation of a simulation's model of random phenomena contemplates the comprehension of the problem, identify the key components, make assumptions and build a symbolic expression through one or more commands to obtaining results.

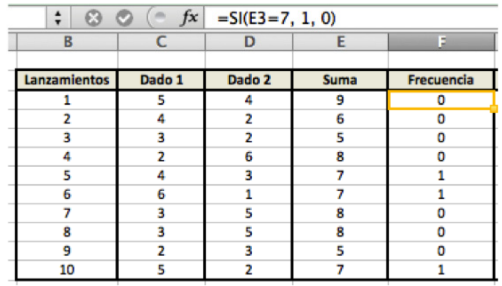
In the context of the activity 1, the key components are the results of the dices. The assumption is that the dices are symmetrical, because each face of them is equally possible. One result is obtained adding, subtracting or multiplying the points of two dices -referring that case-. In this activity the students didn't show difficulties in identify the key components, which are essential part to make the simulation correctly. Figures 3 and 4 show the model made it by two students on the case by adding points:



E3 =SI(D3=7,1,0)+E2

LANZAMIENTOS	DADO 1	DADO 2	SUMA	FRECUENCIA
1	1	5	6	0
2	2	5	7	1
3	3	4	7	2
4	6	5	11	2
5	4	6	10	2

Figure 3: Model constructed by Luis



=SI(E3=7, 1, 0)

Lanzamientos	Dado 1	Dado 2	Suma	Frecuencia
1	5	4	9	0
2	4	2	6	0
3	3	2	5	0
4	2	6	8	0
5	4	3	7	1
6	6	1	7	1
7	3	5	8	0
8	3	5	8	0
9	2	3	5	0
10	5	2	7	1

Figure 4: Model constructed by Silvia

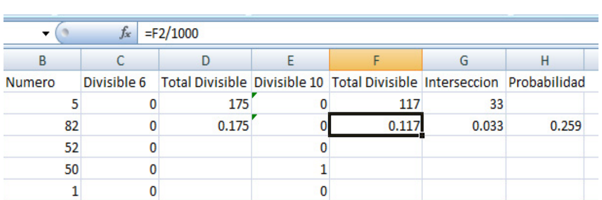
In the case of the activity 2, the key component is the number that is thought. There is the assumption of any number between 1 and 100 are equally probable to happen, therefore the result from the key component is obtained generating random numbers between 1 and 100. In this case, students had difficulties to identify the key component and most of them started with favorable results (divisible by 6 or by 10 or both), when the correct one was to generate firstly the key component and lately to identify favorable results.

In other hand, in the activity 3, the key components are the selection of the urn (urn1, urn2 or urn3) and the selection of one of the balls (black or white). The assumptions are that each urn has the same probability to be selected and for the balls option is that each ball has the same assumption of equal probability. As the same way that it was in the other activities, here was observed that students do showed difficulties to identify the key components, however some of them identified the balls selection (second part) but omitted the urn selection (first part).

In conclusion, we have identified three schemes for the formulation of the model:

1. Students, who use the space of the spreadsheet like a notebook, make calculus in manual manner and write it down the results. They don't use the potential of the spreadsheet.
2. Students, who use formulas to calculate probabilities, but do not request to generate random number how it is supposed to be in a model of simulation.
3. Students, who use formulas with random numbers to simulate the key components.

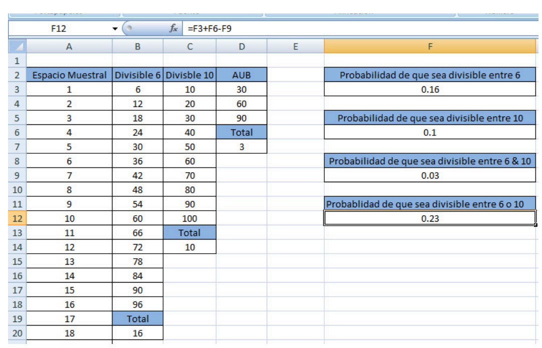
In the following figures of one the students who participated in the activity 2 which results generated from a random command (Figure 5) and another one where no random formula results to accomplish the divisibility condition (Figure 6).



=F2/1000

Numero	Divisible 6	Total Divisible	Divisible 10	Total Divisible	Interseccion	Probabilidad
5	0	175	0	117	33	
82	0	0.175	0	0.117	0.033	0.259
52	0		0			
50	0		1			
1	0		0			

Figure 5: Model constructed by José



F12 =F3+F6-F9

Espacio Muestral	Divisible 6	Divisble 10	AUB		Probabilidad de que sea divisible entre 6
1	6	10	30		0.16
2	12	20	60		
3	18	30	90		Probabilidad de que sea divisible entre 10
4	24	40	Total		0.1
5	30	50	3		
6	36	60			Probabilidad de que sea divisible entre 6 & 10
7	42	70			0.03
8	48	80			
9	54	90			Probablidad de que sea divisible entre 6 o 10
10	60	100			0.23
11	66	Total			
12	72	10			
13	78				
14	84				
15	90				
16	96				
17	Total				
18	16				

Figure 6: Model constructed by Ana

For his part in Figure 7, the student Angel shows his work in activity 3. Where, first at all, he made calculations in manual method using formula of total probability, then simulates extraction of balls and calculates frequencies to 10,000 cases. Results of both approaches match up given certainty to precision of his problem.

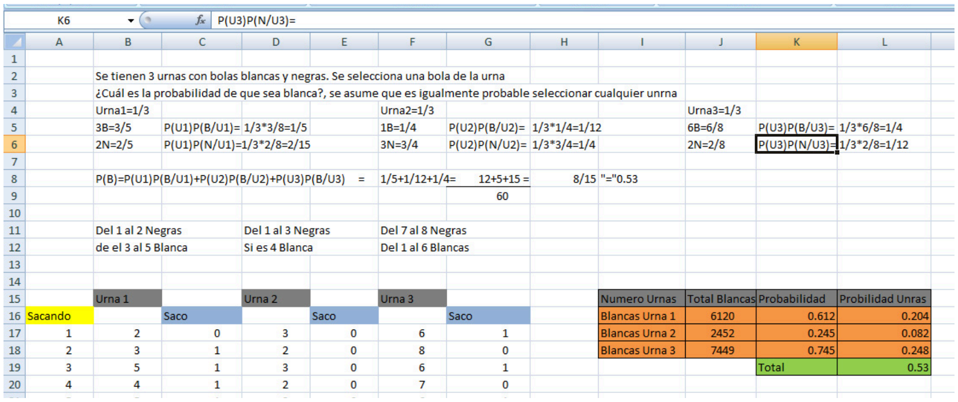


Figure 7: Work developed by Angel in activity 3

The Table 1, show the results of the different schemes developed by the students and the commands utilized to formulate the model of simulation.

Table 1: Classification of Students by Schemas and Commands Utilized

Activity	*Frequency of students by schema*			*Commands used to generate the model*
	1	*2*	*3*	
1	8	0	14	Randbetween(1,6) A1+A2, Sum(A1,A2), A1-A2, A1*A2
2	7	9	6	Randbetween (1,100)
3	3	14	5	To the selection of the urn: Randbetween(1,3) To the selection of the balls: Randbetween (1,5), Randbetween (1,4), Randbetween (1,8).

Evaluation of the Model

The evaluation of the model consist in generate a case and record the outcome of interest, and then repeat it many times and see if the results satisfy the conditions of the problem. In the first activity, the analysis reveals that the students who generated the results of the key components though a random function (scheme 3), use the function “copy of formulas” to generate many cases (hundreds or thousands) as it is shown in figures 1 and 2. Other students used F9 in order to repeat the case in the same cell, and afterwards, they copied again. The students who used scheme 1 (data direct introduction) had the difficulty to generate many cases. It is also important to highlight that some students found out some mistakes in their model when the result obtained was not what they were waiting for. This important function of the spreadsheet allows students to monitor a problem solution process.

In activity 2 the students who did not use a random formula to generate the results of the key components (thought number), could not evaluate their model as a case of simulation, however, they got correct results through probability's classic formula. In the case of the students, who generated randomly the results (e.g., José, Figure 5), identified the favorable cases through the formula *residue(number,number_divisor),* combined with the conditional formula *if(logical test,value_if_true_value_if_false).* Afterwards, when students observed that some correct results were produced, they copied the formula to solve many cases (see Figure 5).

On the other side, in activity 3, only 5 students used random numbers in order to build a model and as mentioned before, the simulation was partial (the extraction of the ball in the urn selected), and this is the reason why the evaluation of the model consisted in identifying if the random generated number corresponded to the white or black ball, for which they used the conditional formula *if(logical test, value_if_true_value_if_false).*

Table 2: Commands Utilized to Obtain Conditions and to Evaluate the Model

Activity	*Commands used to identify favorable results*
1	To additions=7: IF(D6=7,1,0 To substractions=3: IF(G6=3,1,0) To products > 10: IF(J6>10,1,0)
2	To divisors of 6: IF(RESIDUE(B1,6)=0,1,0) To divisors of 10: IF(RESIDUE(B1,10)=0,1,0) To divisors 6 or 10: =COUNTIF(C2:C1000,1,E2:E1000,1),

3	IF(B17=1,0,IF(B17=2,0,1))
	IF(D17=4,1,0)
	IF(F17=7,0,IF(F17=8,0,1))

Statistic Analysis of the Results

In this phase, we focus on summarize the information obtained by the model and draw conclusions. In specific, it means register observations of interest (favorable events), and calculation of the relative frequencies.

In activity 1, most of the students who generated the model through of a random command followed a schema of identify and accumulate the favorable cases in the same column, what means introduce a recursive formula starting at the second line, how is showed in case Luis (Figure 1). This part was cause of difficulties for some students, who only can identify the favorable cases but no one accumulated the results as is showed in Silvia's work. A schema more simple was identified the favorable cases and make additions in determined cells, nevertheless, any student didn't show that kind of schema.

In the activity 2, the register of the favorable results and calculation of the frequencies did it by the students using a random phenomena through formulas like *countif(range,criteria)* and *countif(range,criteria1,range,criteria2,).* In other hand, activity 3, using 0 and 1 as variable indicator to identify favorable results, it helped to calculate the frequency through the formula *countif(range,criteria)* or *sum(number1,number2)*

Table 3: Commands Used for the Register Favorable Results and Calculation of Frequencies

Activity	*Commands used to count favorable results*
1	=IF(D5=7,1,0)+E4
2	=COUNTIF(E2:E1000,1)
3	=COUNTIF(C17:C10016,1)

Conclusions

The results obtained in this research have showed that the process of the instrumentation of the spreadsheet in the modeling of random phenomena it is not an easy process, even for the students with a previous programming knowledge, as it was the case of the subjects involved in our study. The process of simulation works with a methodology that demands from students the use of intertwined strategies between the knowledge they already have of the topic and the technical knowledge of the resources that the spreadsheet has.

As it was observed the empty knowledge of the functions of the spreadsheet (potentialities) and constraints, from where depends the level of instrumentation, by taking in consideration spreadsheet wasn´t conceived as an educational tool. In other hand, the theoretical approach that the students are used in probability courses since secondary school is part of the influence that relegate the use of the Excel potentialities to the typical calculus functions, because for them results more simple and easy use it in that way.

The incorrect schemas developed for the formulation of the model show the difficulties that the students had in identify the key components of the problems, therefore they opted to use other strategies as to introduce in a direct way the data or generating through no random formulas to accomplish the problem. According with this our conclusion states that the competences to develop models of simulation in a spreadsheet, requires a planned process to show to the students the methodology of simulation, as well it is needed a better knowledge of the potentialities and constraints of the tool.

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NAVIGATING THE TRANSITION FROM DESCRIPTIVE TO INFERENTIAL STATISTICS THROUGH INFORMAL STATISTICAL INFERENCE

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Keywords: Data Analysis and Statistics; Probability; High School Education

Students' difficulties in understanding formal statistical inference in introductory statistics courses are well known in the field of statistics education. Recent research efforts have focused on "informal" statistical inference to understand how students begin to reason about data as they transition from descriptive statistics to formal statistical inference (Pfannkuch, 2011). In "informal" statistical inference, students generalize about populations based on sample data without the formalities of constructing a confidence interval or conducting a hypothesis test. This qualitative study investigates: (a) How does students' informal inferential reasoning develop over a series of three informal statistical inference tasks? and (b) What are the characteristics of students' informal inferential reasoning as they make informal generalizations?

Influenced by the task design framework proposed by Zieffler et al. (2008) and research on students' difficulties and misconceptions in introductory statistics, this study used a framework that focused on three tasks explicitly linked to the concepts of descriptive statistics, probability, and the sampling distribution. Task-based interviews (Goldin, 2000) were conducted with three pairs of high school students taking statistics for college credit. We used Makar and Rubin's (2009) principles for informal statistical inference to document the development of students' informal inferential reasoning as evidenced by the extent to which they: (a) made generalizations that extended beyond the data at hand, (b) used the data as evidence for their generalizations, and (c) indicated a level of certainty in their generalizations.

As the student pairs worked through the three tasks, we found evidence of how their informal inferential reasoning developed. As the complexity in comparing the distributions increased from a focus on mean and skewness to include variation and proportional reasoning, students focused on individual elements rather than the overall distribution in making generalizations. In the second task, students' generalizations progressed to extending beyond the data as they estimated population probabilities and proportions based on the data they had collected. Another characteristic of students' inferential reasoning occurred during the third task when the normality of the sampling distribution was not accessed by students as they decided on the validity of a claim about a population.

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UNDERSTANDING RISK THROUGH THE COORDINATION BETWEEN LIKELIHOOD AND IMPACT

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Keywords: Data Analysis and Statistics; Probability; Design Experiments; High School Education

The purpose of the study is to investigate secondary students' understanding of risk with respect to the coordination between likelihood and impact. There are two objectives: (1) to investigate students' informal knowledge of risk and their informal coordination between likelihood and impact; (2) to explore ways in which the formal instruction can foster the coordination between likelihood and impact. In this study, risk of an event is defined as the measure of both likelihood and impact (Pratt et al., 2011). The study uses the design experiment approach (Cobb & Gravemeijer, 2008) to explore risk literacy education within the context of secondary school mathematics. In the research, there were two cycles of the design experiment. The first cycle took place in a grade eleven classroom at a secondary school in Ontario during the probability and statistics unit (23 participants, all boys). The second design experiment was conducted in a different educational setting, i.e., a different independent school in Ontario in a grade eleven enriched classroom during the probability and statistics unit (23 participants, 19 girls and 4 boys).

During the initial written assessment, students were given a question concerning safety of nuclear power plants. Out of 19 students who completed the initial assessment, eight out of 19 students made arguments based on impact. Based on the initial assessment, students were divided in five groups and given an activity in which they had to assess the safety of nuclear power plants using data including likelihood and impact. In all five groups students estimated that the probability was small. In the second part of the activity, students were asked to estimate an impact of an accident based on empirical data provided. The students were then asked to assess the risk of both nuclear and coal accident by combining the data on probability and impact. Most of the students, decided to multiply the two values which is a sound mathematical technique. However, when asked to provide reasons for the operation, the students were not able to provide adequate explanations. This suggests that the coordination between likelihood and impact needs to be presented in a more meaningful way.

For the second cycle, the results of the initial assessment were very similar to the first group. However, for the group activity, instead of the introduction of the product, the students were provided the set of coordinates with the probability on the vertical axis and the impact on the horizontal axis. The students were then asked to plot the values for the probability and impact for both nuclear power plants and coal power plants. Based on the results, there is evidence that students already possess the idea that the assessment of risk requires combination of likelihood and impact. Out of the two mathematical ways of coordinating the two, the graphical representation yielded a more of conceptual understanding of the nature of risk.

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Chapter 9

Technology

Research Reports

Brief Research Reports

Poster Presentations

TEACHING PRACTICES, TECHNOLOGY AND STUDENT LEARNING

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While computing technologies are widely available in secondary schools, these technologies have had only limited impact on changing classroom practices. Partly, this can be attributed to an underdeveloped understanding of the role of the teacher in engaging in classroom practices that can support student learning with technology. In this study, we analyzed the teaching practices that supported students' learning of a conceptually rich and deep topic (the average rate of change) when using an exploratory computer simulation environment. The results illustrate the demands placed on teachers when faced with the multiplicity of student ideas generated by their interactions with the simulation and three aspects of a teaching practice in response to those demands. These findings contribute to evolving frameworks for understanding meaningful and productive technology use in teaching secondary mathematics.

Keywords: Teacher Knowledge; Technology; Modeling; Advanced Mathematical Thinking

Over the past three decades, much research has focused on the potential for computing technology to impact K–16 mathematics education. Graphing calculators, internet access, and (most recently) interactive whiteboards are now widely available in secondary schools and colleges. But the widespread availability of computing technology has had only limited impact in making the kinds of changes to classroom practices envisioned by research. While many factors contribute to the successful adoption of any technology, one crucial factor in any kind of change to classroom practices is the teacher (Godwin & Sutherland, 2004; Ruthven, Deaney, & Hennessy, 2009). An underlying assumption of this study is that our understanding of the role of the teacher in supporting learning with computing technologies is underdeveloped.

The need to understand the relationship between pedagogy and student learning with technology was identified in the early 1990s by Hoyles and Noss (1992) as they observed "the inescapable and perhaps unpalatable fact that simply by interacting with an environment, children are unlikely to come to appreciate the mathematics which lies behind its pedagogical intent" (p. 31); they also noted the sparseness of research that addresses the nature of pedagogies that can support student learning with computer environments. More recently, Ruthven and colleagues have noted that the teaching practices associated with the widespread use of graphing technology have received relatively little attention from researchers (Ruthven et al., 2009). Ruthven et al. argue for the development of teachers' craft knowledge to support their classroom use of technology. This perspective is in contrast to a less situated approach to teachers' knowledge that is characterized by the TPACK (technological pedagogical content knowledge) construct (Mishra & Koehler, 2006; Neiss, 2005).

The larger goal of this study is to contribute to the development of a model of teaching practices that support student learning with exploratory computer simulations. To that end, we investigated the teaching in a pre-college classroom setting where the students used a computer simulation to study of the average rate of change, a traditionally difficult, yet conceptually rich and foundational topic in mathematics. Our study was guided by the following question: what was the nature of the teaching practices that supported students' learning of average rate of change when using an exploratory computer simulation?

Theoretical Background

Much recent work on the relationship between teaching practices and technology has drawn on the TPACK model, often examining the preparation of teachers or the professional development of in-service teachers (e.g., Bowers & Stephen, 2011; Neiss, 2005). However, Graham (2011) and others have criticized the TPACK model for lacking clear theoretical distinctions between the elements of the model, a lack of precision in definitions, and difficulties in discriminating between the proposed constructs of "technological content knowledge" and "technological knowledge." The fuzziness at the boundaries of the TPACK model may call into question the existence of the proposed constructs or it may simply point to the need for empirical work on teaching practices that can inform revisions and clarity within the model. Our purpose in this paper is not to critique the TPACK model, but rather to study teaching practices to better understand the role of the teacher when using computer technology, in particular an exploratory computer simulation. As Hoyles and Noss suggested in 1992, such a pedagogy would include introducing a mathematical agenda, a progressive sequence of computer tasks, related paper-and-pencil work and class discussions of computer-based work, and small group activities to bring together computer and non-computer work. Ruthven and colleagues (2009) argue that, when using graphing software, the teacher plays a fundamental role in making the mathematical relationships meaningful for students by supporting the mathematical interpretation of the technology-based representations. Our goal in this study is to contribute to a clearer understanding of the nature of teaching practices with computer technology, particularly as students come to understand the concept of average rate of change.

Over the last twenty years, researchers have documented the difficulties that students encounter in learning to interpret models of changing phenomena (Carlson et al., 2002; Thompson, 1994). In this paper, we draw on a modeling approach to student learning that Kaiser and Sriraman (2006) identify as a "contextual modelling" perspective. This perspective emphasizes the design of activities that motivate students to develop the mathematics needed to make sense of meaningful situations. Much work done within this perspective draws on model eliciting activities developed by Lesh and colleagues (e.g., Lesh & Zawojewski, 2007). Such activities confront the student with the need to develop a model that can be used to describe, explain or predict the behavior of familiar or meaningful situations. Considerably less research has focused on model exploration activities, where students explore the mathematical characteristics of the model. In this paper, we focus on a set of model exploration activities using a computer simulation environment, accompanied by student presentations and teacher-led discussions that focused on the underlying structure of the model, on the strengths of various representations, and on ways of using representations productively. Thus, for this study, we designed an instructional sequence that began with a modeling activity to elicit the construct of average rate of change, followed by model exploration tasks that examined the underlying mathematical structure and its representations. The focus of this study is on the role of the teacher in facilitating student presentations and leading class discussions that support students' understandings of how to represent the average rate of change.

Research Design and Methodology

This study used design-based research as an approach to studying teaching and learning as it occurs within the complexity of a naturalistic classroom setting (Cobb et al., 2003). This approach is intended to generate principles of practice, in this case related to teaching with computer simulations. We draw on the multi-tiered design experiment (Lesh & Kelly, 2000), which provides a framework for collecting and interpreting data at the researcher level, the teacher level and the student level. Central to our analytic approach is the notion that, as researchers, we examine the teacher's actions in the classroom and her interpretations of those actions, which are in turn influenced by the students' interactions with the tasks in the simulation environment. The researchers and the teacher (the third author) collaboratively developed the tasks that were designed to support students in understanding the concept of average rate of change.

Simulation Environment and Task Design

We began the instructional sequence with a model-eliciting activity, using the physical situation of motion along a straight line. Students created graphs using their own bodily motion and a motion detector and wrote verbal descriptions of that motion. This included comparative situations of faster and slower constant speed, changing speed and changing direction. Following the model-eliciting activity, the students engaged in a sequence of model exploration tasks. These tasks were designed to help students to think about the underlying structure of the model of constant and non-constant motion. An important goal of these tasks was to engage students in using informal and formal language to describe the average rate of change and to develop their understanding of the representational systems for describing change. As argued earlier, this brings with it a concomitant role for the teacher in using instructional strategies that will support students in interpreting the mathematical relationships intended in the tasks and instantiated in the computer environment.

The model exploration tasks used SimCalc Mathworlds (Kaput & Roschelle, 1996). This computer simulation environment was designed around the context of one-dimensional motion to explore the relationship among position, velocity and acceleration, the connections between variable rates and accumulation, and an understanding of mean values. The drag-and-drop environment makes use of piecewise linear functions to create position or velocity graphs; these graphs drive the one-dimensional motion of cartoon-like characters in the linked WalkingWorld. The MathWorlds environment reversed and extended the representational space of the model-eliciting activity with the motion detector where bodily motion created a position graph; in the simulation environment, the students created velocity graphs that generated the cybernetic motion of a character. From the simulated motion, the students created position graphs, thus developing an understanding of how the position graph could be constructed by calculating the area between the velocity graph and the *x*-axis. In exploring this linked relationship among the characters' motion, the velocity graph and the position graph, students began to reason about the position of characters solely from information about the velocity of the characters. This model exploration task provided an opportunity for students to develop their abilities to interpret position information from a velocity graph and velocity information from a position graph. Subsequent model exploration tasks introduced the concepts of average velocity, negative velocities, linearly increasing and decreasing velocities and their associated position graphs.

Context and Participants

The sequence of model exploration tasks was part of a larger set of modeling tasks that formed the basis for a six-week course for students who were preparing to enter their university studies. The teacher and the first author collaborated in the development of the entire set of tasks for the course. The teacher had three years of experience teaching secondary and college students; this was her second year teaching the summer course. There were 17 students in the course all of whom volunteered to participate in the study. Three of the participants were female and 14 were male. All participants had completed four years of study of high school mathematics; 11 students had studied calculus in high school and six had not studied any calculus. The model exploration tasks were done individually at a computer; however, the participants were encouraged to discuss their work with each other. Following each task in the sequence, there was a whole-class discussion that usually involved students in presenting the results of the work produced during the model exploration tasks. The class discussion following these tasks focused on the mathematical structure of the model and on the relationships among different representational systems.

Data Collection and Analysis

Consistent with the methodology of multi-tiered design experiments, data for this study were collected at two levels: the level of the teacher and the level of the students. The data sources at the teacher level included videotapes of all class sessions, written field notes and memos, class materials such as worksheets and a record of board work, the teacher's lesson plans and annotations made by the teacher during the lesson. Following each lesson, there was a debriefing session with the teacher, which captured the teacher's reflections on the lesson and any changes to the plans for subsequent lessons. These debriefing sessions were audio-taped and transcribed. The model exploration activities with the simulation world took place over a total of six lessons; each lesson lasted one hour and 50 minutes. Central to our analytic approach is the notion that as researchers we examine the teacher's descriptions, interpretations, and analyses of artifacts of practice that were developed, examined and refined during our collaborative work on the design and teaching of these six lessons. In this paper, we only report on the analysis of the teacher level data, although we acknowledge that this analysis was influenced by the data at the student level.

The analysis of the data took place in three phases. Consistent with the iterative approach of design-based research, the first phase of analysis took place during the six weeks of teaching. In this phase, the research team met with the teacher and regularly engaged in discussion about the model exploration tasks, the progress of the class as a whole, and our observations about students' thinking about average rate of change and their language for expressing their ideas. Analytic memos were written by members of the research team to document their emerging understandings of the teaching practices and observations about student learning.

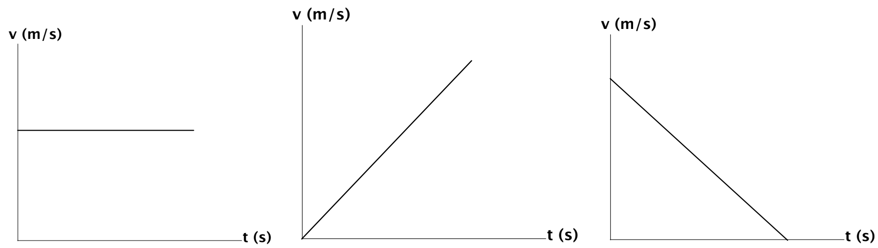
In the second phase of the analysis, the research team viewed the videotapes and wrote a detailed script of each lesson, identifying the nature of the teacher's activity in each segment of the lesson and its time-stamp and duration. Following the principles of grounded theory (Strauss & Corbin, 1998), preliminary codes were developed to categorize what the teacher did in the classroom. Drawing on this analysis, the research team identified a set of approximately six to eight video segments within each lesson that captured recurrent themes and for which we wanted the teacher's retrospective perspectives and interpretations. These video segments were the basis for video stimulated recall with the teacher and gave further insights into the teaching practices from the perspective of the teacher. This in turn led to further refinement of the coding scheme. In the third phase of the analysis, we coded the videotapes of the six lessons using the revised coding scheme. As we analyzed the teaching practices, we sought confirming and disconfirming evidence in the teacher's lesson plans and annotations during the lesson, and with the teacher's perspective on the lesson from the de-briefing interviews and the post lesson video stimulated recall. This led to the formulation of the results in three broad categories: (1) pressing students for representations; (2) harvesting student ideas; and (3) sorting out and refining student ideas. In this paper, we report on the results in the first category: pressing students for representations.

Results

A representational press occurs when the teacher applies pressure on students for the purpose of furthering the students' emerging understandings of the representations of average rate of change, which in this case occurred within the computer simulation environment and in students' related work. This related work could be any one of the forms of the following representations: language (both written and spoken); table; symbolic (such as function notation and algebraic expressions); iconic or graphical; and enactments (either cybernetically in the simulation world or bodily in the real physical world). We found three categories of representational presses that the teacher engaged in: (1) explicitly inserting a representation into the discussion to support connections to other representations; (2) pressing the students to give interpretations of their representations in terms of the context of the task, while articulating arguments that justify their interpretations; and (3) pressing students to use representations to clarify a situation or question. Due to space limitations, we report here only on the second and third categories.

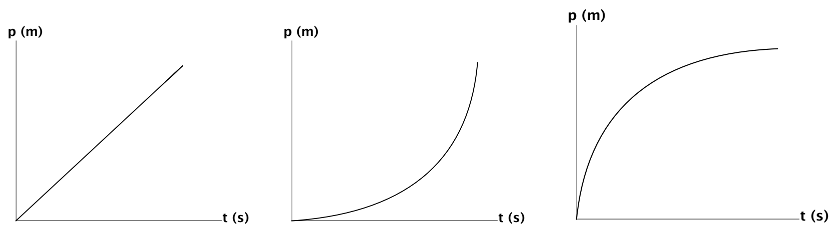
Interpreting Representations

In this episode, we illustrate how the teacher pressed the students to give interpretations of their graphs in terms of the context of the task while articulating arguments that would justify their interpretations. This episode occurred in the fourth day in the sequence of the six lessons. The teacher led a whole class discussion about the characteristics of three different linear velocity graphs and their corresponding position graphs, which had been the focus of the tasks with the simulation environment. The three velocity graphs are shown in Figure 1 and their corresponding position graphs are shown in Figure 2.



(a) (b) (c)

Figure 1: Comparing three velocity graphs from the simulation environment



(a) (b) (c)

Figure 2: Comparing the corresponding position graphs

During the whole class discussion, the teacher labeled the graphs on the blackboard with the students' verbal interpretations of the graphs. Graph (a) was described as constant velocity and constant speed; graph (b) was described as increasing velocity, increasing speed, and acceleration; and graph (c) was described as decreasing velocity, decreasing speed, and acceleration. The position graphs shown in Figure 2 were interpreted as: (a) linear, increasing position; (b) curved, accelerating, increasing position, "walk slow then fast"; and (c) accelerating, increasing position, "walk fast then slow." In the following excerpt from the class discussion, the teacher focused students' attention on the velocity graph (c) in Figure 1. In this exchange, we see the teacher pressing students (1) for the use of appropriate language to describe the graph, (2) for making connections between cybernetic and physical enactments, and (3) for understanding the meaning of the relationship between a constant or linearly changing velocity graph and its associated position graph.

1 *Tchr:* How would you describe this motion here [graph (c) in Figure 1]?
2 *Chris:* Uhmm, it's deceleration [inaudible]
3 *Tchr:* Okay, so we also have acceleration here, okay, uhmm, because why?
4 [Several students talking]
5 *Tchr:* Because why?
6 *Chris*: Umm, as the…because the velocity is changing
7 *Tchr:* Um, how would you have to walk? If you were trying to match that graph from the third day we did *Hiker* [an earlier activity]? You're holding the motion detector. How would you tell the person to walk?
8 *Quent:* For which one? [Teacher points to graph (c) in Figure 1]
9 *Vic:* You tell him to walk away from the censor;
10 *Quent:* Real fast
11 *Tchr:* Real fast
12 *Vic:* And then slowing down
13 *Tchr:* And then slow… Okay.

This episode began with the teacher pressing the students to interpret the decreasing velocity graph from the simulation environment and to verbalize deceleration as changing velocity. In turn 7, the teacher pressed for a description of this changing velocity in terms of enacted physical motion. She invited the students to describe an enactment of the motion in terms of a device (the motion detector) that could measure and record the physical motion of a person walking. In this way, the teacher engaged the students in generating verbal descriptions of simulated motion that were explicitly connected to physical motions that the students had experienced earlier.

Using Representations to Clarify Situations

In this episode, we illustrate how the teacher pressed a student to insert a representation into an argument so as to support and clarify his reasoning about a specific situation. The teacher had posed the following question to the class for homework: "If two people take a walk and end together, have the same velocity throughout the walk, then both must have walked for the same amount of time. True or false?" This task was designed with some intentional ambiguity around what it means to "have the same velocity" that the students would need to resolve in answering the question. In the class discussion the next day, the teacher polled the students and made public the result of the poll: all of the students, except one, thought that the claim in the posed question was true. The teacher decided to hear about the false argument:

1 *Vic:* It says take a walk. It doesn't say that they started the same time, so one [person] can have already been going at… that for a while so… they could have…. at the same time so… Let's say [inaudible] one's going somewhat faster and the other one could be going somewhat slower, but the slower one started earlier… so they end together, at the same place at the same time… but… this does not seem, I mean… they had their own velocities, uh for the walk… that is to say that, they both had the exact same velocities.
2 *Tchr:* Is this bouncing off of Vic or new idea? [to Jorge who is holding up his hand]
3 *Jorge:* I have a new idea. Uh, it says that they "have the same velocity". If they didn't have the same velocity and one person was already ahead of the other then they would never end up at the same time.
4 *Tchr:* Uh huh
5 *Jorge:* Like if two people are walking at 4 meters per second – how are they gonna end up at the same place in the same amount of time if one already started walking.
6 *Tchr:* So what do you take "same" to mean?
7 *Jorge:* That… basically two people are walking at the same time, and one walks for a longer dist[ance], for a longer amount of time, then he'll walk more distance.
8 *Tchr:* Okay.
9 *Vic:* Um
10 *Tchr:* [To *Vic*] Do you have a rebuttle to that?
11 *Vic:* Uh huh
12 *Tchr:* You want to argue with that?
13 *Vic:* Yes, um, that's still not taking into account that someone could have already been ahead of the other [person]. But going into, the velocity, um, but it's still, making the velocity constant. It isn't saying that it has, that is, that it has to have the exact same velocity. It says "have the same velocity throughout the walk." That could mean anything. That could even just mean constant velocity.

In the first turn, Vic offers the argument that the "same" velocity means that the walkers each had their own "same" constant velocity throughout the walk. But, in turn 5, Jorge makes clear that he has interpreted "same" velocity to mean the same as each other: both are walking at a constant velocity of "4 meters per second." In turn 6, the teacher acknowledges the ambiguity of the meaning of the "same" and in turn 10 invites Vic to further his argument.

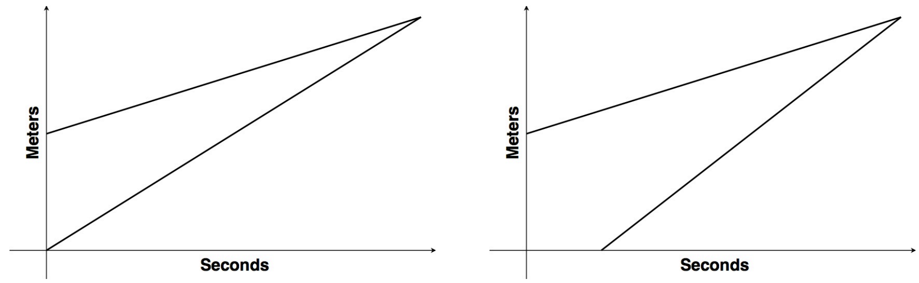


Figure 3: Representing the "same velocity" with different times

After checking with the students in the class for their understanding of Vic's argument the teacher asked Vic: "Do you think that you can demonstrate what you are talking about?," a suggestion Vic quickly takes up; he goes to the blackboard and draws the graph shown on the left in Figure 3. This graph shows "the slower one" (as Vic expressed in turn 1) starting behind the other walker in terms of position (as expressed in turn 13), but both walkers walk the same amount of time and hence this is not a counterargument to the original claim. As Vic elaborates his thinking, he correctly revises his graph to the one shown on the right in Figure 3, which shows the slow walker being ahead of the fast walker, but the walkers walk for different amounts of time, an argument that convinces many students that the original claim is false. The teacher had not (and could not) fully anticipate all of the students' arguments and pressing for representations was helpful to her in understanding the complexity of the students' arguments.

Discussion and Conclusions

Students' difficulties in learning to interpret rates of change, particularly in the context of one-dimensional motion, are well known in the research literature. Computing technology would seem to hold great potential for helping students to understand this rich and yet challenging concept. However, the relationship between pedagogy and student learning with technology is still an area in need of research (Hoyles & Noss, 1992; Ruthven et al., 2009). The computer technology provided a flexible way for students to represent their ideas and to manipulate them. As students engaged with the tasks in the environment, and the related non-computer tasks where they had to interpret the meaning of graphs and give verbal descriptions or arguments justifying their representation, more student ideas were generated and conflicts among interpretations arose that needed to be resolved by mathematical reasoning. The technology also provided a common frame of reference for small group conversations and whole class discussions. However, as Hoyles and Noss (1992) warned, one cannot assume that the students fully understand the representations in the computing environment. The generation of student ideas and the need for students to interpret and give meaning to the representations in the computer environment place new demands on the craft knowledge of the teacher. In this study, we found the emergence of a teaching practice that responded to these new demands, namely pressing for representations. Through this practice, the teacher pressed the students to articulate the connections among representations, to make interpretations of their representations while giving arguments to justify their interpretations, and to use representations to clarify situations and resolve questions.

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DEVELOPING TPACK ALONGSIDE PROFESSIONAL VISION OF TEACHING MATHEMATICS WITH TECHNOLOGY

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This study questions the extent to which a course in Mathematical Problem Solving with Technology was developing TPACK in mathematics preservice teachers. In order to measure the development of TPACK, both quantitative and qualitative data were collected. Preliminary findings are promising. Preservice teachers developed a vision of technology use in the classroom that better aligned with the vision outlined in the NCTM Technology Principle. Students reported they had sufficient opportunities to work with different technologies throughout the course. Moreover, students reported they could choose technologies that enhance the mathematics for a lesson.

Keywords: Teacher Education–Preservice; Technology; Problem Solving

Introduction

In seminal works (Niess, 2005, 2008; Mishra & Koehler, 2006) that have culminated in the description of a framework by which to study the development of Technology, Pedagogy And Content Knowledge (TPACK), mathematics teacher educators have envisioned teacher education programs that integrate technology instruction with content and pedagogy. These programs would provide preservice teachers with learning opportunities that might help them amend personal philosophies of teaching to reflect a deep understanding of teaching with technology. A picture of how to accomplish this integration is emerging in the field, including Zbiek and Hollebrands' (2008) position that the ways in which technology is integrated into teachers' classrooms is influenced by their conceptions of technology, mathematics, learning and teaching. Furthermore, Zbiek and Hollebrands (2008) recommend that preservice teachers be given opportunities to use technology as a mathematics learner and then reflect on those experiences from a pedagogical perspective.

We have begun to develop our own vision of what it means to enact these principles in the development of preservice secondary mathematics teachers (PSMTs) and to honor the rich connections between technology, mathematics, and teaching. This paper reports on a study of 39 PSMTs enrolled in two sections of a course, *Mathematical Problem Solving with Technology*. In this course, PSMTs are expected to revisit their own learning of secondary mathematics and investigate mathematical concepts by way of problem solving with various technological tools. Taught with an eye toward immersion learning, the PMSTs in our course are engaged almost entirely in lab-based activities and discussion of the mathematical, pedagogical and technological principles they encounter along the way.

Methodology

As a means to inform our own practice, we engaged in research to uncover the extent to which our course was supporting the development of TPACK in our preservice teachers. We set about to explore two research questions: (1) What is the vision of teaching mathematics with technology held by PSMTs prior to and at the conclusion of a semester of concentrated experiences utilizing technology for mathematical problem solving? (2) To what extent does our course influence TPACK of our PSMTs?

Participants and Setting

The participants of this study were enrolled in a semester-long course, *Mathematical Problem Solving with Technology*, during the 2010-11 academic year. PSMTs pursuing licensure to teach secondary mathematics typically take this course during their sophomore year. As our PSMTs enroll in the majority of their education courses in the junior year, those enrolled in this course have not yet taken any methods courses or completed any field placements in a K–12 setting. Of the 39 participants, three had graduate student standing but had not taught secondary mathematics at the beginning this course. During the fall semester, 18 participants were recruited and during the spring semester, 21 students were recruited.

Data Collection

Data were collected with the purpose of measuring the development of TPACK as well as linking that development to practice. In order to accomplish this goal, two distinct perspectives were taken. First, we wanted a quantitative tool by which to capture growth in TPACK over time. We selected a survey intended to measure TPACK (Zelkowski et al, under review) and administered it as a pre/post measure during the first and last weeks of the course. The survey is divided into multiple sections with items assessing each of the domains within the TPACK framework, but we chose to focus our analysis on six items shown to be reliable and valid in measuring perceptions of TPACK (Zelkowski et al, under review).

Table 1: Summary of Selected TPACK Survey Items

Item	Prompt
Item 1	I can use strategies that combine mathematics, technologies, and teaching approaches that I learned about in my coursework in my classroom.
Item 2	I can choose technologies that enhance the mathematics for a lesson.
Item 3	I can select technologies to use in my classroom that enhance what I teach, how I teach, and what students learn.
Item 4	I can teach lessons that appropriately combine mathematics, technologies, and teaching approaches.
Item 5	I can teach lessons that appropriately combine algebra, technologies, and teaching approaches.
Item 6	I can teach lessons that appropriately combine geometry, technologies, and teaching approaches.

It was also important for us to take a deeper look at individual development throughout the course in order to index any global shifts in TPACK to specific facets of the course. In order to examine the opportunities we were providing our students to develop TPACK and to more closely examine any shifts in TPACK that were captured by the survey, samples of student reflective writing were collected of which two samples were used in the current analysis.

Principles of Mathematical Problem Solving with Technology. Within the first month of the course, students were asked to read three items and write a reflection paper. The three items were selected to convey to our students the underlying principles of the course. First, we selected the NCTM Technology Principle (NCTM, 2000). Second, we selected two chapters from *Teaching Mathematics through Problem Solving K–12* (Schoen & Charles, 2003). The chapter written by Hiebert and Wearne (2003) was selected for its overview of problem solving and the vision it provided of teaching and learning. The chapter written by Zbiek (2003) was selected for its attention to the role of technology in a classroom where problem solving is valued. Students were asked to respond to three prompts:

1. *How is the perspective taken in the readings similar or different from your own experiences learning mathematics?*
2. *How does it compare to your own beliefs about teaching?*
3. *What ideas did you find yourself (dis)agreeing with?*

Final Examination. A final examination prompt was provided asking students to describe their vision of responsible use of technology in the classroom.

> *Many of you have reflected on the use of technology in mathematics education and used a statement similar to, "technology is a benefit to the mathematics classroom as long as it is used responsibly." Reflect on this statement and explain to me what "responsible use of technology" looks like in the mathematics classroom. Do not define it in terms of what it is NOT—I am not interested in hearing about examples of irresponsible uses of technology and these will detract from your answer. Instead, use your experiences and any readings you have completed for this class to craft a reasonable*

definition or standard by which I could determine if technology were being used to support teaching and learning of mathematics in your classroom.

This prompt was devised in response to themes identified in classroom discussions throughout the semester. It is not uncommon for PSMTs to begin to categorize their experiences, both past and current, as appropriate/inappropriate. In both semesters, PSMTs invoked the phrase “responsible use” to differentiate between technology practices that they endorsed (“responsible”) and those they did not (“irresponsible”). This prompt was aimed at assessing PSMTs’ outgoing vision of “responsible use.”

Data Analysis

Analysis of written reflection. The two reflections described in the previous section were chosen because of their timing, one in the fifth week of the semester and the other at the completion of the course. Furthermore, the nature of the assignments has the potential to illustrate changes in our PSMTs’ visions of teaching mathematics with technology.

Both top-down methods (Miles & Huberman, 1994) and grounded theory (Strauss & Corbin, 1990) were used to develop and apply a coding instrument. Within NCTM’s Technology Principle (2000) and also Zbiek (2003), there are many smaller statements about the envisioned role of technology in the mathematics classroom. From these sources, we created a framework by which to analyze the vision our PSMTs held of teaching mathematics with technology. From each statement indicating a potential role for technology, a code was developed. For example, the statement “With calculators and computers students can examine more examples or representational forms” (NCTM, 2000, p. 23) yielded two initial codes, “Examples” and “Representational Forms.” While Doerr and Zangor (2000) classified calculator use into five broad categories: Computational, Transformational (changing the nature of the task), Data Collection and Analysis, Visualizing, and Checking (confirming conjectures, understanding multiple symbolic forms), we initially identified 25 specific roles technology plays in the classroom. We later refined this list to 17 as coding progressed, including combining “Representational Forms” with “Visualization,” as it was difficult to parse out differences such as this in our PSMTs’ writing. The final list of codes is listed in alphabetical order in Table 2.

Table 2: Framework for Examining *Vision of Technology for Teaching*

Perceived Roles of Technology in the Mathematics Classroom		
Assessment	Exploring Conjectures	Problem posing
Communication	Extends Range of Problems	Problem Solving/Reasoning
Differentiated Learning	Mathematical Change	Reflection
Efficiency/Accuracy	Mathematical Connections	Supplementation
Engagement	Organize/Analyze Data	Visualization/Representational Forms
Examples	Present/document	

TPACK surveys. Pre- and post-survey data were collected and analyzed to determine overall shifts in perceived TPACK amongst PSMTs taking a semester-long course in problem solving with technology. In order to compare the results of the administrations of the TPACK survey, individual scores were tabulated by summing the responses given by an individual to each of the six items in terms of the 5-point Likert scale values (1 = Strongly Disagree, 5 = Strongly Agree). Quartile scores for each administration were calculated and compared using a box-and-whisker plot. In order to delve deeper and assess growth with individual items, a novel data visualization was constructed utilizing color as a means to assess the extremity of individual responses as well as the overall change in PSMT responses between administrations. A detailed description of this data visualization is provided along with the results of the analysis.

Results

In response to the research questions outlined above, the results will be organized in three distinct sections. First, we will present the results of the qualitative analysis of the two collected samples of reflective writing. Then, we will present the results of a quantitative analysis of the TPACK survey data.

Incoming Vision of Teaching with Technology

Utilizing the framework in Table 2, we coded each of PSMT's reflections on the role of technology in the teaching and learning of mathematics. While we could recognize at least one of the roles in most of the reflection papers, there were two reflection papers that received zero codes. These two PSMTs did not share a vision with regard to the inclusion of technology in the classroom. One stayed close to a positive, yet non-specific message, that technology should be used to enrich the students' experience, while the other challenged the essentiality of technology in the classroom altogether, stating:

> *Why is it so essential, as described in the Technology Principle, that technology be used in the application and problem solving of math (NCTM, 2000)? My thought is that shouldn't anything in math class be able to be solved without the aid of technology?*

In the remaining 37 reflections, we were able to recognize anywhere between one and nine of the roles (and on average 3.35), whether the PSMT agreed, disagreed or simply summarized the author's position. The most-agreed-with roles include Supplementation (49%), Efficiency/Accuracy (36%), Problem Solving/Reasoning (28%), and Visualization and Representational Forms (41%) (see Figure 1 [red columns]). Of the 17 roles, some seemed to garner more argument than others. Engagement (33%), Visualization/Representations Forms (41%), and Mathematical Change (13%) were mentioned more than most of the other roles both positively and negatively. While 33% of our students agreed that technology engages students, 10% disagreed, claiming technology was a distraction. Walter calls some of the statements made in the Technology Principle "too sweeping" and explains,

> *I submit that if a student is not motivated by the task at hand, then a computer provides many more distractions for them. This is not to suggest that one should not use a computer, only that I disagree with the implication that utilizing a computer task is a cure for those that are easily distracted.*

While this provides a snapshot of the specified vision our PSMTs had at the start of the semester, there was also an undercurrent of concern about the use of technology in the classroom in general. The most common concern was that technology would prevent students from achieving mastery of mathematics. More than 25% of our students specifically mentioned their concern that students will either miss out on learning basic skills (generally arithmetic) or that this knowledge initially gained will atrophy once technology is in hand. Even more expressed concern that technology would replace a student's basic understandings of mathematics, with 49% choosing to quote, paraphrase or reinterpret the statement, "technology should not be used as a replacement for basic understandings and intuitions; rather it should be used to foster the understanding and intuitions," (NCTM, 2000, p. 24). This was, by far, the most cited passage in all three readings. In addition to these concerns, some PSMTs expressed a sense of nostalgia and favored tradition over technology. Mary says, *"I got through all of those high school courses with just a graphing calculator and passed with flying colors, so why do we need all this technology in our classrooms now?"* This narrow view of education based on personal success or failure permeates the initial reflection papers and causes many to question NCTM's assessment of technology as *essential* in the classroom. A student who feels that they have achieved success in school mathematics potentially sees himself or herself as a counterexample to the essentiality of technology. There is a general reluctance to consider what experiences they may have missed and to judge their own stories as complete and representative of mainstream.

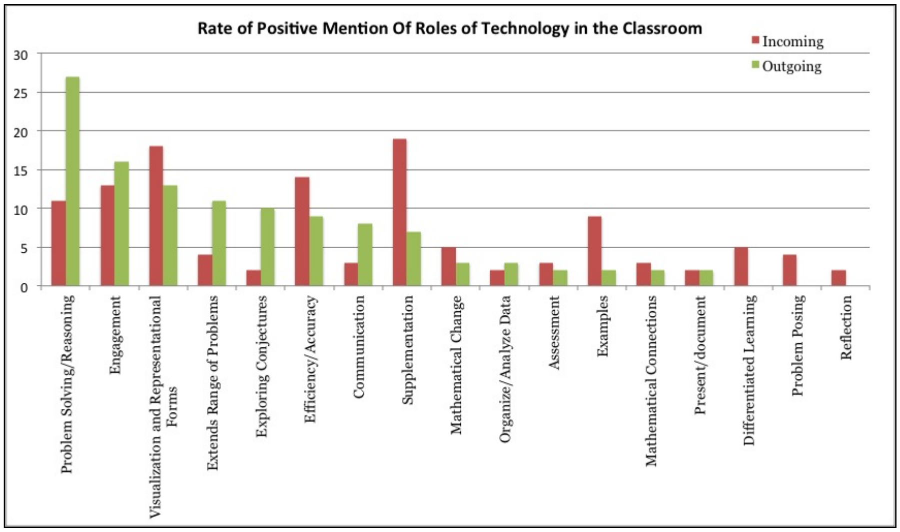

Figure 1: Incoming vision (red) and outgoing vision (green) of teaching with technology

Outgoing Vision of Teaching with Technology

In order to gauge the vision of teaching with technology that our PSMTs had when they exited our course, we conducted a similar analysis of their responses to the Final Examination Prompt. In this assignment, our PSMTs were explicitly asked to share their vision of "responsible use of technology in the classroom." In this sense, the assignment was different from the first reflection in that the PSMTs were not asked for their reactions to an expressed vision, but rather to create their own.

Unlike their initial reflections, every PSMT identified at least one specific role that technology plays in the mathematics classroom. On average, students identified three roles with the maximum being eight. Figure 1 (green columns) is a depiction of the overall vision. Again, if we take a look at the most identified roles, we get an indication of the vision of the group. Almost 70% of our PSMTs made mention of the role technology plays in problem solving and reasoning in the classroom. Engagement, Visualization and Representational Forms, Efficiency/ Accuracy and Supplementation are still among the most cited. However almost three times as many PSMTs as did initially indicated technology extends the range of problems that can be used in the classroom and aided in Communication, and five times as many noted technology could be used to Explore Conjectures. The number of students who indicated technology should be used to supplement pencil-and-paper instruction dropped by a factor of three.

TPACK Survey Data

In order to compare the results of the administrations of the TPACK survey, individual scores were tabulated by summing the responses given by an individual to each of the six items in terms of the 5-point Likert scale values (1 = Strongly Disagree, 5 = Strongly Agree).

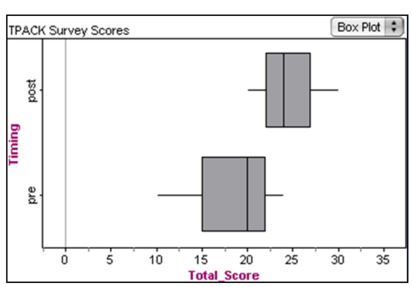

Figure 2: Box plot with whiskers comparing TPACK scores for two administrations

These ordinal data are displayed in a box and whiskers plot in Figure 2. The median score increased from 20 on the pre-survey to a 24 on the post-survey. Furthermore, the first quartile of the post-survey and third quartile of the pre-survey are equal, implying that 75% of the post-survey scores were higher than 75% of the pre-survey scores.

To get a deeper sense of these shifts, we composed a color-coordinated image of the data set. The individual responses to each item given on the pre- and post-survey are shown in Table 3. Each row pertains to an individual PSMT and the rows have been sorted in decreasing value according to the sum of the Pre-test scores. The left table contains data collected by the pre-survey. The middle table contains data collected by the post-survey, and the right table contains calculated differences indicating shifts in responses. These were calculated by subtracting pre-survey responses from post-survey responses. Due to page limitations, we have provided only a portion of the table to give the reader a sense of the trends in the data.

Table 3: Color-Coded TPACK Survey Where Individual Responses Have Been Highlighted

Student	Item 1	Item 2	Item 3	Item 4	Item 5	Item 6	Total (Pre)
1	5	5	4	4	3	3	24
3	4	4	4	4	4	4	24
Stephanie	5	4	4	4	4	2	23
Walter	3	3	2	2	3	3	16
21	2	2	2	3	3	3	15
22	3	2	2	3	2	2	14
24	2	2	2	3	2	2	13
Hannah	2	2	2	2	2	2	12
Wilma	2	2	2	2	2	2	12
26	2	2	2	2	1	1	10
Overall Means	3.4	3.2	3.1	3.3	3.0	2.7	18.7

Item 1	Item 2	Item 3	Item 4	Item 5	Item 6	Total (Post)
5	5	5	4	5	5	29
4	4	4	3	3	4	22
5	5	5	5	5	5	30
5	5	5	5	4	5	29
4	5	5	5	5	5	29
4	4	4	4	4	4	24
4	5	5	4	4	4	26
4	4	3	4	4	4	23
3	4	4	3	4	4	22
4	4	4	3	4	4	23
4.1	4.3	4.2	3.9	4.0	4.0	24.5

Item 1	Item 2	Item 3	Item 4	Item 5	Item 6	Difference
0	0	1	0	2	2	5
0	0	0	-1	-1	0	-2
0	1	1	1	1	3	7
2	2	3	3	1	2	13
2	3	3	2	2	2	14
1	2	2	1	2	2	10
2	3	3	1	2	2	13
2	2	1	2	2	2	11
1	2	2	1	2	2	10
2	2	2	1	3	3	13
0.7	1.1	1.1	0.7	0.9	1.3	5.8

In each table, color has been used to visually differentiate the responses with the more saturated colors indicating more extreme views. In the first two tables, green (responses 4 & 5) indicates that the student has agreed with the prompt and red (responses 1 & 2) indicates that the student has disagreed with the prompt. White (response 3) would indicate a Neutral response. In the final table, green (positive integers) indicates a positive shift while red (negative integers) indicates a negative shift.

Interpreting the tables means looking at how the color patterns change. If we compare the pre-survey and post-survey scores from all 29 participants, we find dramatic color shifts. The cloud of red at the bottom of the pre-survey data disappears in the post-survey data and is replaced by light green and even some dark green cells. At the top of the tables, we see a dark green cloud emerge in the post-survey data replacing the light green and white that was present in the pre-survey data. Red and pink have virtually disappeared from the post-survey data, demonstrating that PSMTs disagreed with very few prompts after the course had completed.

This is also reflected in the overall color tone of the difference table, which is almost entirely composed of white and shades of green. Very few items showed a negative shift from pre-survey to post-survey, and these are indicated by the pink cells.

Findings and Implications

This study seeks to further the research in the field of TPACK by testing the hypothesis set forward by Zbiek and Hollebrands (2008) that a key experience for PSMT should be to use technology as a mathematics learner and then reflect on those experiences from a pedagogical perspective. Our results show that a course in Problem Solving with Technology that provides opportunities for PSMTs to reengage with school mathematics using a problem-based curriculum in a technology-rich environment has a positive impact on PSMT TPACK development as well as on PSMTs vision of teaching with technology. This vision was communicated in part by examining specific experiences and readings encountered during the course and presenting them as illustrating examples of what the 'responsible use' of technology looks like in the classroom. We find that we can draw two conclusions.

Finding 1: Our students developed a vision of technology use in the classroom that better aligned with the vision outlined in the NCTM Technology Principle. If we take a closer look at the nature of the roles that we were able to identify in the incoming visions of PSMTs, Supplementation (49%), Efficiency/Accuracy (36%), and Examples (23%) are roles that suggest our PSMTs are envisioning technology as "computational" tools rather than "transformational" or "visualizing" tools (Doerr & Zangor, 2000). In contrast, we find the roles of Visualization and Representational Forms (41%) as well as Problem Solving/Reasoning (28%). This may be due to the brief exposure our PSMTs had to problem solving using dynamic geometry software in the five weeks prior to the submission of this reflection. It was clear that many were enamored with the ability to generate dynamic geometric objects for study and had begun to envision their personal independence in mathematical problem solving. It is likely that even this brief exposure had an impact on the PSMTs' vision. It is unclear whether that vision was truly aligned with that of NCTM, but many referenced these roles positively. Whereas their initial vision favored using technology to "do mathematics", the outgoing vision seems to favor technology for learning mathematics. PSMTs more readily identified roles that were "transformational" or "visualizing" (Doerr & Zangor, 2000). Furthermore, PSMTs readily accepted the role technology plays in generating and sustatining classroom discussion of mathematics and collaborative work habits, something that was missing from their initial vision.

Finding 2: A course in problem solving with technology can have an impact on the TPACK development of PSMTs. The results of our analysis of pre- and post-survey data show a clear increase in the TPACK of our PSMTs. The items that saw the greatest gains were, "*I can choose technologies that enhance the mathematics for a lesson," "I can select technologies to use in my classroom that enhance what I teach, how I teach, and what students learn,"* and, *"I can teach lessons that appropriately combine geometry, technologies, and teaching approaches."* Comments made in their final exam reflections would support this finding as well. It is clear that PSMTs are thinking more about what it would take to enact their vision of teaching with technology and the complexity of that practice.

Conclusion

Many factors affect the development of PSMTs' use of appropriate technology tools. Olive and Leatham (2000) have documented that using technology as a tool for learning mathematics is not enough to ensure PSMTs will use technology as a teaching and learning tool in their own classrooms. Many PSMTs need sustained interactions with technology throughout their teacher education programs, especially in the context of content and pedagogy courses, combined with positive experiences that would challenge their deeply rooted beliefs. However, unless students are given opportunities to reflect on their beliefs and come face-to-face with them, it will be difficult for them to relinquish their fears and mistrust (Fleener, 1995).

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YOUNG CHILDREN'S UNDERSTANDING OF ANGLE IN A DYNAMIC GEOMETRY ENVIRONMENT

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This paper examines the effect of the use of dynamic geometry environments on children's thinking about angle. Using a driving angle model in Sketchpad, kindergarten children were able to develop an understanding of angle as "turn," that is, of angle as describing an amount of turn. Gestures and motion played an important role in their developing conceptions.

Keywords: Geometry and Geometrical and Spatial Thinking; Elementary School Education; Angle; Gestures

Introduction

The concept of angle is multifaceted and can pose challenges to learners, even into secondary school (Close, 1982, Mitchelmore & White, 1995). Despite these difficulties, children show sensitivity to the concept of angle from very early years (Spelke, Gilmore, & McCarthy, 2011). Angles are normally introduced to children quite late in formal school settings. For example, in British Columbia, they are introduced in grade 6 (12 years old), even though students are expected to describe, compare, and construct 2-D shapes, including triangles, squares, rectangles and circles in grade 2. The strong capacity of young children to attend to and identify angles in various physical contexts motivated us to see whether a more dynamic conception of angle—namely, angle-as-turn—might support their developing understanding at an earlier age.

We have been investigating other geometry-related concepts at this age too, using DGEs, including shape identification, symmetry and parallel lines (Sinclair, Moss, & Jones, 2010; Sinclair & Kaur, 2011). Previous research reports on the effectiveness of Turtle Geometry (Logo) for teaching the concept of angle (Clements, Battista, Sarama, & Swaminathan, 1996; Simmons & Cope, 1990). However, we believe that DGE might be helpful in thinking of angles as turns and rotation more effectively. In this paper, we report on an exploratory study conducted with a split class of kindergarten/grade 1 children (ages 5—6) working with angle using *The Geometer's Sketchpad*. We focus on the emergence of the concept of angle-as-turn and discuss the specific mediating role of the use of the software on this thinking.

Children's Understanding of Angle

In the research literature, the concept of angle is shown to have different perspectives, namely: angle as a geometric shape, union of two rays with a common end point (static); angle as movement; angle as rotation (dynamic); angle as measure; and, amount of turning (Close, 1982; Henderson & Taimina, 2005). Due to different prevalent definitions of the term angle, teachers often face difficulty in knowing what definition of angle to use (Close, 1982). Mitchelmore (and colleagues) and Clements (and colleagues) have done abundant research in the area of angle concept over the past twenty years. Much research has been conducted on the development of the concept of angles, focusing at the grades 3, 4 and higher levels. Mitchelmore and White (1995) suggest that angles occur in a wide variety of physical situations that are not easily correlated. Despite the excellent knowledge of all situations, specific features of each situation strongly hinder recognition of the common features required for defining the angle concept (Mitchelmore, 1998). Mitchelmore and White (1995) proposed that children initially acquire a body of disconnected angle knowledge situated in a large number of everyday experiences; they then group situations to form angle contexts such as turns and corners; and finally they form an abstract angle concept by recognizing similarities across several angle contexts.

Later works of Mitchelmore involved teaching experiments (White & Mitchelmore, 2003) in which they divided angle situations into three clusters—2 line angles (corners of room, intersecting roads, pairs of scissors), 1-line angles (doors, windshield wipers), and 0-line angles (the turning of a doorknob or a wheel). The situation is more problematic for students where the two arms (of angle) are not clearly visible. Research using *Logo* shows that students tend to visualize the turn of turtle as turn of their body but making these turns involves writing numerical commands (Clements, Battista, Sarama, & Swaminathan, 1996). The DGE does not involve the writing of the commands and can thus be used at an earlier age to develop more qualitative understanding of angle. We believed that the DGE approach would be helpful in developing the dynamic as well as static concept of angle.

Theoretical Perspective

In previous research, we have found Sfard's (2008) "commognition" approach is suitable for analysing the geometric learning of students interacting with DGEs (see Sinclair, Moss, & Jones, 2010; Sinclair & Kaur, 2011). For Sfard, thinking is a type of discursive activity. Sfard's approach is based on a participationist vision of learning, in which learning mathematics involves initiation into the well-defined discourse of the mathematical community. The mathematical discourse has four characteristic features: word use (vocabulary), visual mediators (the visual means with which the communication is mediated), routines (the *meta-discursive rules* that navigate the flow of communication) and narratives (any text that can be accepted as true such as axioms, definitions and theorems in mathematics). Learning geometry can thus be defined as the process through which a learner changes her ways of communicating through these four characteristic features. We have previously presented a developmental trajectory related to identifying shapes in terms of different levels of discourse and now we are trying to do the same thing with angles, but we will look first at how the different components of the discourse change as the students work within the DGE. We are particularly interested in investigating how the students might move between different word uses and to examine the informal language they use to talk about angles.

Similarly, given the importance of gestures in communication of abstract ideas (Cook & Goldin-Meadow, 2006), and their potential to communicate temporal conceptions of mathematics (Núñez, 2003; Sinclair & Gol Tabaghi, 2010), we chose to extend Sfard's approach to incorporate gestural forms of visual mediators. Given the fact that we are working with very young children who have had little exposure to a mathematical discourse around angle, we will be interested in seeing whether they make use of kind of "mismatch" gestures that Goldin-Meadow (2004) describes as indicating a readiness-to-learn. Kita (2000) focuses on the cognitive functions of gestures, which play an important role in communication. He points out that the production of a gesture helps speakers organize rich spatio-motoric information, where spatio-motoric thinking organizes information differently than analytic thinking (which is used for speech). We thus expect that children will use gestures to convey spatio-motoric information, even though they might not be able to convey the analytic thinking used in speech. Moreover, children's gesture might be non-redundant with their speech. Our goal in looking at the gestures will be to see how they communicate different ideas about angles; particularly the mobile ones associate with angle-as-turn. We are less interested in classifying the students' gestures in terms of McNeill's (1992) categories than in understanding the embodied, conceptual basis of the gestures.

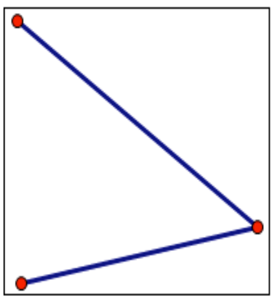
Exploring the Concept of Angle

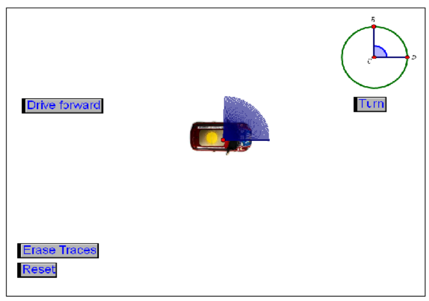
Participants and Tasks

We worked with kindergarten/grade1 children (aged 5—6) from a school in a rural low SES town in the northern part of British Columbia. There are 20 children with diverse ethnic backgrounds and with a wide range of academic abilities. We designed lessons related to angle along with the classroom teacher, who has a Master's degree in mathematics education and has been developing her practice of using DGEs for a couple of years. The teacher and students worked with angles in different ways using Sketchpad for six lessons in a whole class setting with an IWB (Interactive Whiteboard). Each lesson lasted approximately 30 minutes and was conducted in a group with the children seated on a carpet in front of a screen. Lessons were videotaped and transcribed.

Dynamic Angle Sketches

In this study, we used two different sketches to explore the concept of angle with the children. We began with a simple angle diagram (Fig.1). In the sketch, dragging the vertex of an arm of the angle changes the angle. The purpose of using this sketch was to enable children to focus on the standard form of angle as a geometric shape and to build an understanding of angle through its behavioural properties. The research suggests that children have difficulty seeing a static angle as a turn. The situation is more problematic when the two arms of the angle are not clearly visible. The second sketch used is a "driving angle model," which shows both a static as well as dynamic sense of angle (Fig.2a, 2b). It includes a car that can move forward as well as turn around a point. The turning is controlled by a little dial (which has two arms and a centre). No numbers are used. There are four action buttons (Turn, Drive Forward, Erase Traces and Reset) that control the movement of the car. Students can regulate motion and turns to create different shapes like random paths, squares, rectangles, and so on.





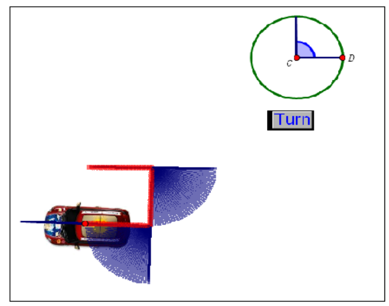


Figure 1: Angle as a Shape **Figures 2a and 2b: Driving Angle Model**

The traces offer a visible, geometric record of the amount of turn. The purpose of this sketch was to move to a more dynamic presentation of angle related to a real life context, where the focus of the children would be more on the continuous behaviour of the turning wheels and would enable them to see the process of turning along with the final product (position after a specific angle turn).

Classroom Discussion

The classroom teacher tried to support the discourse of angle-as-turn as she worked with the sketches. In what follows, we report on the children's work with the first sketch and then their developing sense of angle as they worked with the second one.

Introducing the angle as a geometric shape. The teacher began by showing the children the sketch in Figure 1 and asking them what they saw. Initially, the students attended to the features of the figure like points, circle etc. One student uttered the word "angle," although, when prompted, didn't elaborate. Two students, Colin and Jasmine, compared the sketch with triangle. Kristian was the first to impose motion on the diagram:

Teacher: Kristian, what do you see? (*Pointing towards figure1 on the screen*)
Kristian: I see the point is going back and that point is going up…*Inaudible*…

Kristian focussed here on the two arms of the angle, not mentioning explicitly the point at which they meet. The teacher asked about other similar examples and the children described what they saw in terms of concrete objects like a swing, slide, the letter "w," a house, the bottom of a hill, and a nose in their responses. Thus, initially their discourse was dominated by everyday language and they made little or no use of mathematical words—and their comparisons involved very loose visual resemblance. Following initial discussion, the teacher dragged the vertex of one arm of the angle.

Teacher: What happens if I do this? (*dragging one vertex of one arm of the angle*)
Will: It turns wider

Van Zoest, L. R., Lo, J.-J., & Kratky, J. L. (Eds.). (2012). *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Kalamazoo, MI: Western Michigan University.

Teacher: It does turn wider. What about if I do this? (*dragging the vertex further*)
Morris: It widens even more now.
Teacher: It widens even more. What am I doing if I am moving this line? What kind of movement is that?
Jasmine, Will, Kristian: Square, it's a clock
Teacher: It's a clock movement?
Students: Yeah
Teacher: What am I doing to the line?
Students: Moving, clock.

When the teacher drags the vertex of one arm of the angle to make it wider, Will and Morris describe the change in terms of widening; Will also uses the word "turn" to describe the motion. Several students describe the movement of the arm of the angle as the movement of the clock. This is the first instantiation of angle-as-turn, which the students associated with a clock. Note that they do not explicitly use the word "angle," but they talk about the same diagram they saw before, now using an entirely new comparison, which becomes shared widely. The teacher finishes the lesson by talking about the space between the two arms and showing them how to mark an angle between two arms in *Sketchpad*, which can be done using the Marker tool.

Introducing driving angle model. In the subsequent lesson, the teacher presented the driving angle model to the students and asked them what they saw. Initially children focused more on features like color of lines, circle and points. The teacher prompted them to find the similarity between the two sketches (Figures 1 and 2a). Most of the students focused on describing the absence of the car in first sketch (Fig.1) and the presence of the circle in the second sketch (Fig. 2a). During comparison, only the second sketch was displayed on the IWB.

Will: And there is a blue one on this one
Teacher: There is a blue one on this one. That's the same?
Will: Yeah, Surrounded by a circle (*draws circle in the air to show the surroundedness. Repeats gestures a few times*)

Will recalled the angle diagram of first sketch and noticed its presence in both the sketches, but his description "blue one on this one" showed a dominance of everyday language. Despite the use of word "angle" by the teacher (while marking the angles) at the end of previous lesson, none of the students used the word in their comparison of the two sketches. During the classroom discussion, the children asserted that in order to drive the car, the wheels should be turned.

Teacher: Okay, I have another question for you guys. What does the car do?
Will: It drives on the ground. (*gesturing with both arms, turning both his arms together from left to right*)
Teacher: It does drive on the ground. So we're going to pretend our screen is the ground. How is it going to drive on the ground? What ways can it drive? Jasmine?
Jasmine: You have to turn the wheel. (*With both arms turning as if turning a steering wheel*)

Will and Jasmine both used gestures to describe the turning of the car. Many children in the class imitated their gestures. Jasmine used the gesture of turning the steering wheel as a visual mediator while associating the motion of car with the turning of wheels. The teacher then pressed the Turn button and the car turned by the same amount indicated by the angle dial, also leaving a trace of the turn behind.

Teacher: Okay…I am going to move something for you, here. Okay… If I tell you that this is kind of like the car's steering wheel, what do you think that car is going to turn like now? What do you think is going to happen? Think about it. (*Making the turning angle smaller in the size*), What is it going to do now?
Students: Turn, turn
Teacher: Turn…is it going to turn the same?

Students: Yeah.

The teacher told the students that the angle dial was like the car's steering wheel and she changed the size of the angle in the dial. Then she prompted the students to predict the outcome on pressing the Turn button; several students predicted that car would turn although they could not predict the amount of turn. This shows that students did not associate the turn of the car with the angle in the dial: they had no routine for assessing the magnitude of the turn. The teacher invited the children to compare the size of the angular turns. She changed the size of the angle and asked the children to decide whether this turn was smaller or larger than the previous amount of turn.

Teacher: The car is stopped…but if you look, that's like right here. The car is turned … turnn ... and stops here. And then I moved this steering wheel up here with the angle (Fig. 3a) and we press turn again and it only went from here to here (Fig.3b). Are these the same?

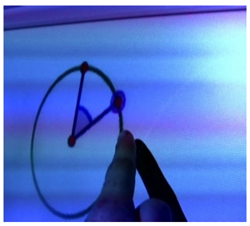


Figure 3a

Figure 3b

Figure 3c: Morris's gestures

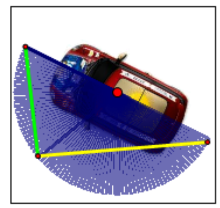


Figure 3d

Students: No, no, no…that one is too bigger (*pointing towards the screen*)
Teacher: Which one is bigger?
Morris: They don't match. There is no match. One is bigger (*Lying on the carpet, gesturing with his fingers as if he is comparing the sizes (Fig. 3c). And then shaking his head no*).

Morris compared the two different amounts of angle turns by comparing the size of the traces. Morris linked the difference in the size of two fingers (Fig. 3c and 3d) with the difference in two amounts of turn (sizes of traces). He used the gestures with two index fingers to explain his reasoning. These are new visual mediators that are used to express the idea of angle. They also point to a new routine for comparing "turn." His gesture here is non-redundant (see Kita, 2000) with his speech since he expresses meanings for angle/turn that are not evident in the speech. Indeed, the children use very few verbal expressions in these lessons. We note that the initial use of concrete visual mediators in first lesson was replaced by the use of embodied mediators in the form of gestures.

Understanding of benchmark angles. The teacher introduced some benchmark angles in the next lesson by using the first sketch with words like "right turn." She showed the 90° angle in the sketch (Fig.4a). The children compared the right angle with the side of a house and corner of a box. The teacher showed a whole movement of the one arms of angle from 0° to 180° by dragging the vertex of one arm (Fig. 4b).

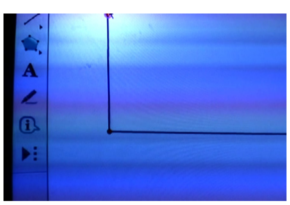


Figure 4a: Sketch showing 90^0

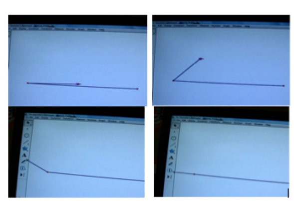


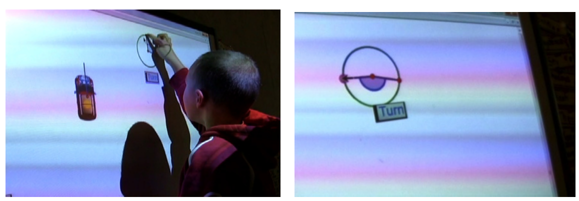
Figure 4b: Movement of one arm from 0^0 to 180^0

The children compared 180° to a line and a road. Morris made a gesture with his hand moving along a straight path, while describing a car moving along a straight road (Figure 5a)—thus picking up the context of the previous day's work. The teacher asked the students how they could make the car turn by 180^0.

Teacher: How can you make that car go that far…go in a straight line…Remember we did the 180 that was a straight line on the board. You said that it looked like a road. What can you do to the steering wheel?
Students: Press it
Teacher: No
Will: Move it
Teacher: How can you move it, Will? Show us. Could you show us that up there, Will? Give it a try. Can you show how to make it go 180? Show us how to go halfway around? (*Will comes on the board and try to move the vertex (point) on angle dial to adjust angle, but fails to move the point* (Fig.5b)). You have to hold it. Tell me when to stop? (*Will hold the vertex while dragging and adjusts angle to approx. 180^0* (Fig.5c))

None of the children, other than Will, were able to recognize that they needed to change the angle in the dial in order to make the car to turn by 180°. Will showed an understanding of the association of the turn of car with the angle in the dial. This shift in the Will's understanding might be the result of seeing the teacher repeatedly changing the angle in the dial during the lesson and of seeing her systematic dragging shown in Fig. 4b.

At the teachers' request, Jasmine made the car turn halfway by pressing the Turn button (Fig. 5d). When the teacher asked about the direction of the car after pressing the Turn button again, both Chloe and Jasmine predicted that the car would complete the full turn.



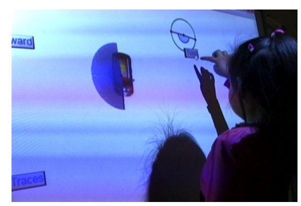


Figure 5a **Figure 5b & 5c: Will adjusting dial to 180^0** **Figure 5d**

Teacher: Why do you think it is going to go all the way around?
Chloe: Because when you press it, it will go…and stop
Teacher: Okay, can you think why it is going to do that?
Chloe: Because it goes that way and stops right there. (*Pointing towards the screen and reflecting, as if imagining the turn* (Fig. 6a))

Figure 6a: Chloe while responding **Figures 6b & 6c: Jasmine gesturing half and full turn**

From Chloe's explanation (including her gestures), it seems that she imagines the car turning and explains her reasoning in terms of the stopping point of the wheel.

Teacher: It is going to close. Why do you think it is going to close?
Jasmine: Because it is like this [*gestures shown in Fig. (6b, 6c)]*
Teacher: Yeah, because it is half way now…oh…that's interesting

While Chloe and Jasmine both visualised the final position of the car successfully, Chloe used words to describe the turn of the car, whereas Jasmine used the gestures to explain the reasoning in terms of half turning and full turning position. Both seemed to understand that two half turns result in a full turn. Once again, their understandings were communicated by the non-redundant gestures, in which arms are used as sides of angle, which provided new visual mediators for their reasoning.

Discussion and Conclusion

At the beginning of the episodes, the children had virtually no mathematical discourse around angle. By working on situations involving turn, which we thought would provide a strong embodied connection for the children, the goal of these lessons was to see whether students could develop a more sophisticated discourse around angle-as-turn—and not necessarily involving any numerical quantification of angle.

The dragging of the one arm of the angle focused the children's attention on the turning behavior of the segment in the diagram. This seemed to act as a visual mediator to which the children associated the movement of a clock. The clock movement metaphor enabled the children to see the angle diagram as turning of arms, and hence, initiated the discourse angle-as-turn. When the children were asked to work with the Drive model, they initially used informal ways to describe the turning of the car by gesturing the turning of a steering wheel. This steering wheel metaphor differs in important ways from the clock metaphor. While the latter focuses on the changing position of the arms (and thus on the changing "size" of the angle between the arms), the former focuses on the angle as a movement from one arm to another (and thus, on the "size" of a given angle).

Only one student was able to associate the turn of car with the angle dial explicitly. Will's recognition of the need to adjust of angle dial to 180^0, in order to move the car halfway, shows that the difficulty in visualizing the 0-arm angle—as reported by White and Mitchelmore (2003)—could be eased by the trace feature of *Sketchpad*, enabling students to see the process of turning as well as the product. Some children were successful in comparing two different amounts of turns. Also, some children were able to reason about half and full turns—in fact, they could explain that a full turn would require the repetition of two half turns. They could not talk about angle and turns using mathematical vocabulary.

At the beginning, the students described the angle diagram in terms of its parts, using concrete, everyday language; they eventually came to associate the notion of "angle" and "turn" to both a clock and a steering wheel—both of which capture something about the idea of angle-as-turn. Several students also used gestures to express the notion of angle, as can be seen in Figures 3 and 6. Finally, the students could also talk about angle in terms of various sizes (especially, the half angle). We argue that the gestures used by the students became part of their visual mediators for the concept of "angle." These gestures were at least in part evoked by the diagrams in *Sketchpad*. For example, the two index fingers shown in Figure 3 arise from the angle dial used by the teacher. The children's use of "turning" gestures in their responses to the car sketch shows that they created gestures as mediators for the purpose of communication. Also, Jasmine's gesture (Figure 6) relating half turn to full turn is a clear example of the effect to dynamic visualization offered by Sketchpad. The traces of half turn in *Sketchpad* acted as a visual mediator and then Jasmine used her gestures of half turn (straight opened arms) to make a full turn (covering the whole movement of arms to close hands) as a means of communication. This interplay between gesture and diagram resonates with Châtelet's (1993) theory of mathematical inventiveness, which de Freitas and Sinclair (2012) discuss in the context of mathematics education; in this case, the dynamism of the diagram seems to evoke quite directly the moving of the children's fingers and hands.

At this early stage, we have focused on word use and visual mediators. There was little sense of any routines used for identifying angles or evaluating their size. Further, given the emergent sense of angle and turn, there weren't any endorsed narratives. However, the frequent use of non-redundant gestures by Morris, Will, Chloe and Jasmine during the lessons indicates, as well as the relatively infrequent use of verbal expression, suggests that the children had a certain "readiness to learn" (Goldin-Meadow, 2004) about angle-as-turn. The next phase of our research, for which we are now in the process of gathering data, involves exploring the way in which the children's emergent sense of angle can be developed into routines for identifying angles and talking about their size. We have some initial evidence that the children are able to identify two angles as being the same even when their arms have different lengths because they use a routine of comparing turn. This would be a very important result given the existing research that shows that children often confuse the size of an angle with the length of its arms (Stavy & Tirosh, 2000).

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THE USE OF CAS IN THE SIMPLIFICATION OF RATIONAL EXPRESSIONS AND EMERGING PAPER-AND-PENCIL TECHNIQUES

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In this paper we analyze and discuss students' performance in a CAS environment related to the simplification of rational expressions. Results indicate that if students have more initial paper-and-pencil techniques, the CAS environment spurs them to deeper theoretical reflections than for students who have fewer techniques.

Keywords: Task-Technique-Theory; CAS; Rational Expression

Background

In the last few years, an area of research interest in mathematics education has developed that deals with the influence of CAS technology in students' algebraic thinking. Thomas, Monaghan, and Pierce (2004), for example, have identified some crucial questions when considering the use of CAS in the learning of algebra: "How does the use of CAS influence student conceptualization? How does the way students work on tasks by hand inform their work in a CAS environment and vice versa?" (p. 166). These paramount questions and those arising from other recent studies (e.g., Kieran & Drijvers, 2006; Hitt & Kieran, 2009; Guzmán, Kieran, & Martínez, 2010, 2011) have driven our interest in this area. In particular, these studies and others have suggested the importance of the technical aspect in algebra learning in CAS environments.

Researchers such as Kieran and Drijvers (2006) have indicated that the use of CAS promotes conceptual understanding if the technical aspect of algebra is taken into account; these researchers have shown specifically that technical and theoretical aspects of algebra co-emerge in students' thinking. In this sense, and related with the simplification of rational expressions, Guzmán, Kieran, and Martínez (2010, 2011) have shown the epistemic role of the use of CAS when students confront their CAS work with their paper-and-pencil work. These studies are related to the transformational activity of algebra (Kieran, 2004)—a characterization of algebra in which the importance of technique acquires relevance in the sense that, within transformational activity, conceptual understanding can come with technique.

Guzmán, Kieran, and Martínez (2010, 2011) have shown that the use of CAS provoked spontaneous theoretical reflections in students, which allowed them to think of new techniques to simplify rational expressions. The use of CAS promoted a change in the students' technique for simplifying rational expressions whose denominator is a binomial (from canceling "literal components" that were repeated in both numerator and denominator to using the polynomial division algorithm). This epistemic role played by the CAS occurred in students whose initial technique was "cancelling literal components," but for whom the notion of cancelling "common factors" and dividing polynomials was absent. Based on our previous studies (Guzmán, Kieran, & Martínez, 2010, 2011), one can therefore ask the following question: What is the role of CAS in students' algebraic thinking if they already have as initial techniques "canceling literal components" and the "long division of polynomials" for simplifying rational algebraic expressions? Does CAS promote other techniques and theories? This paper will deal with this issue.

Theoretical Framework

The Task-Technique-Theory perspective, which is part of the instrumental approach to tool use, has been proposed as a framework for analyzing the processes of teaching and learning in a CAS context (e.g., Artigue, 2002; Lagrange, 2003). This approach encompasses elements from both cognitive ergonomics (Vérillon & Rabardel, 1995) and the anthropological theory of didactics (Chevallard, 1999). There are two directions within the instrumental approach: one in line with the cognitive ergonomics framework, and the other in line with the anthropological theory of didactics. In the former, the focus, according to Drijvers and Trouche (2008), is the development of mental schemes within the process of instrumental genesis. Within this direction, an essential point is the distinction between artifact and instrument.

In line with the anthropological direction, researchers such as Artigue (2002) and Lagrange (2003, 2005) focus on the techniques that students develop while using technology. According to Chevallard (1999), mathematical objects emerge in a system of practices (*praxeologies*) that are characterized by four components: *task*, in which the object is embedded (and expressed in terms of verbs); *technique*, used to solve the task; *technology*, the discourse that explains and justifies the technique; and *theory*, the discourse that provides the structural basis for the technology.

Artigue (2002) and her colleagues have reduced Chevallard's four components to three: *Task, Technique,* and *Theory*, where the term *Theory* combines Chevallard's *technology* and *theory* components. Within this (Task-Technique-Theory) theoretical framework a *technique* is a complex assembly of reasoning and routine work and has both pragmatic and epistemic values (Artigue, 2002). According to Lagrange (2003), technique is a way of doing a task and it plays a pragmatic role (in the sense of accomplishing the task) and an epistemic role. With regard to the epistemic value of technique, Lagrange (2003) has argued that technique plays an epistemic role in that it contributes to an understanding of the mathematical object [in this case the rational expression and its simplified form] that it handles, during its elaboration. Technique also promotes conceptual reflection when the technique is compared with other techniques and when discussed with regard to consistency (p. 271).

According to Lagrange (2005), the consistency and effectiveness of the technique are discussed in the theoretical level; mathematical concepts and properties and a specific language appear. This epistemic value of technique is crucial in studying students' conceptual reflections within a CAS environment. We took into account this Task-Technique-Theory (T-T-T) framework in the designing of the Activity related to the task "simplifying rational expressions," in the conducting of the interview interventions, and in the analysis of the data that were collected.

Unfolding of the Study

In this paper we report and discuss the data of the first two of four Activities designed for a wider research study on a Technical-Theoretical approach in the construction of algebraic knowledge in a CAS environment.

The Design of the Activity

Hitt and Kieran (2009) have pointed out that when taking into account the transformational activity of algebra it is important that the design of the Activity promote the articulation between techniques and theory construction. Since we adopted the T-T-T framework for carrying out the study, the Activities were designed so that technical and theoretical questions were central. We wanted students to have the opportunity to reflect on both technical and theoretical aspects throughout the Activity that was embedded in a CAS environment. It is important to mention here that both paper-and-pencil work and CAS work were intertwined within the Activity. In addition, in this study we use the term task as is defined in the T-T-T framework. As Kieran and Saldanha (2008) state, the Activity is a set of questions related to a central task, in this case the "simplification of rational expressions." In the study, we developed four Activities, each one related to different aspects of the simplification of rational expressions. In this paper we report only the results of the first two Activities, which both involved paper-and-pencil work and CAS work, both with technical and theoretical questions.

Population

This report focuses in the work of one team (two students); the full study included seven teams (two students each team). The participants were 10th grade students (15 years old) in a Mexican public school. The selection of the students was made by their mathematics teacher. None of the students were accustomed to using CAS calculators; consequently, at the outset of the study, all the students received some basic training from the interviewer-researcher on how to use the TI-Voyage 200 calculator for basic symbol manipulation (how to introduce algebraic expressions, the use of the *Solve*, *Expand* and *Factor* commands, the use of the *Enter key* and the use of the "*equal sign*").

Implementation of the Study

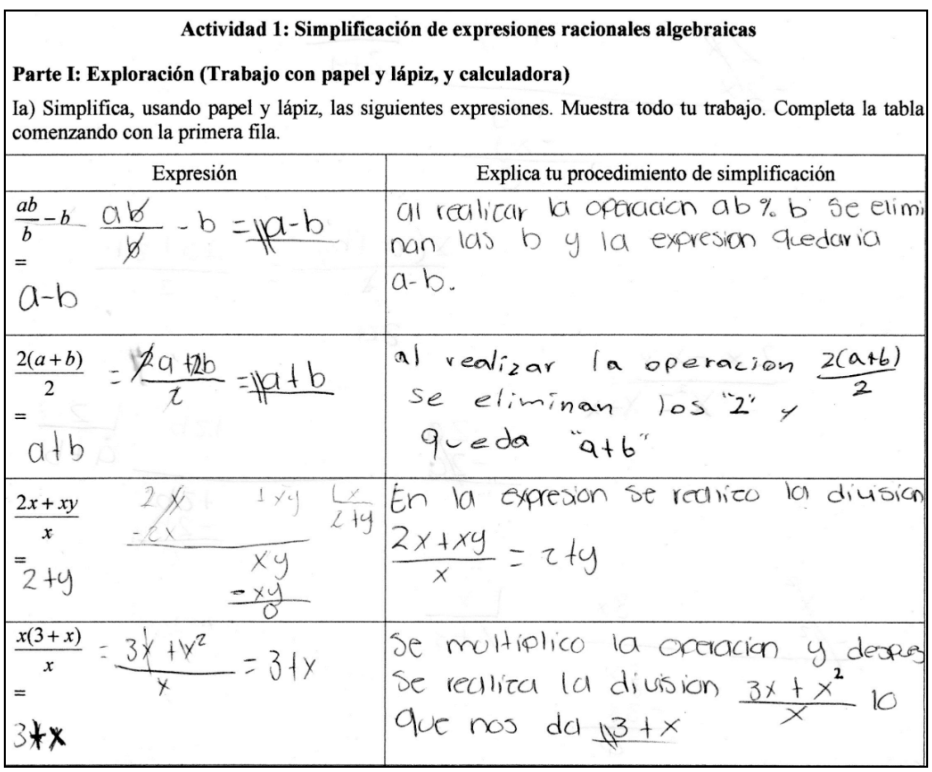
The data collection was carried out by means of interviews conducted by the researcher. Students worked in pairs; each work session lasted between two and three hours (for each Activity). Each team of two students had a set of printed Activity sheets as well as a TI-Voyage 200 calculator. Every interview was audio and video-recorded so as to register the students' performance during the sessions. So, our data sources included the audio and video recordings, the written Activity sheets, and the researcher's field notes.

Analysis and Discussion of Data

In this paper we analyse and discuss the work of one team. The team was chosen for this report because these students (we will call each of them Student A and Student B) used two techniques to carry out the task in the first Activity: Cancelling numbers or literal symbols that are repeated in the numerator and denominator of the rational expression, and at other times applying the long division technique. So the performance of these students fits the question that we try to respond to in this paper. The following analysis and discussion is restricted only to the first two of the designed Activities.

The Paper-and-Pencil Technique and Theory

As was mentioned before, in Activity 1, for those expressions that involved a monomial in the denominator, these students "simplified" the given rational expressions by using two techniques. One technique was cancelling the numbers or literals symbols that were repeated or common to the numerator and denominator. The following Figure 1 illustrates their paper-and-pencil work.



Actividad 1: Simplificación de expresiones racionales algebraicas

Parte I: Exploración (Trabajo con papel y lápiz, y calculadora)

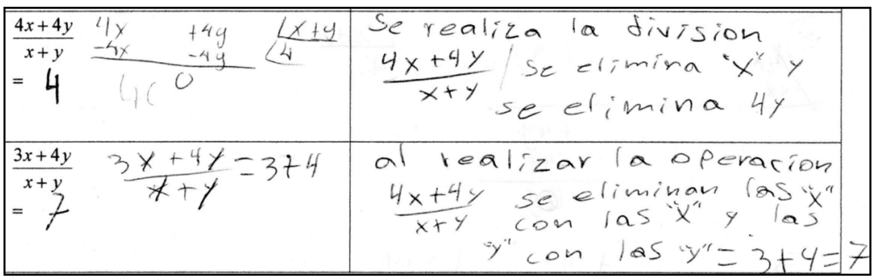
Ia) Simplifica, usando papel y lápiz, las siguientes expresiones. Muestra todo tu trabajo. Completa la tabla comenzando con la primera fila.

Expresión	Explica tu procedimiento de simplificación
$\frac{ab}{b}-b$ $\frac{a\cancel{b}}{\cancel{b}}-b = a-b$ = a-b	al realizar la operacion ab % b se eliminan las b y la expresion quedaria a-b.
$\frac{2(a+b)}{2}$ $=\frac{\cancel{2}a+2b}{2} = a+b$ = a+b	al realizar la operacion $\frac{2(a+b)}{2}$ se eliminan los "2" y queda "a+b"
$\frac{2x+xy}{x}$ = 2+y	En la expresion se realizo la division $\frac{2x+xy}{x} = 2+y$
$\frac{x(3+x)}{x}$ $=\frac{3x+x^2}{x} = 3+x$ = 3+x	Se multiplico la operacion y despues se realiza la division $\frac{3x+x^2}{x}$ lo que nos da 3+x

Figure 1: Students' paper-and-pencil work

In a first moment, the performance of these students was similar to that of others reported in an earlier pilot study in Guzmán, Kieran, and Martínez (2011). Students first expanded the expressions, and after that, they cancelled out the repeated elements in both the numerator and denominator. This technique works if the numerator is a binomial and the denominator is a monomial that is common to both terms of the binomial. The other paper-and-pencil technique that one can see in Figure 1 is the long division algorithm for polynomials. The explanations given by the students of these two techniques were more a description of what they did rather than a theoretical discourse. For instance, for the second expression (see Figure 1) they wrote: "When carrying out the operation… the 2's are cancelled and you are only left with *a*+*b*." For the third expression in Figure 1, their explanation included the terminology of *dividing*.

When the students were faced with expressions whose numerators and denominators were both binomials, they again used the techniques described above. Sometimes they used the long division technique and other times the "cancelling technique." As a result of using this latter technique applied to these kinds of expressions, they made well-known errors (Matz, 1980), that is, they applied the "cancelling technique" no matter whether the number or literal symbol they cancelled out was a common factor of both numerator and denominator or not (see Figure 2).

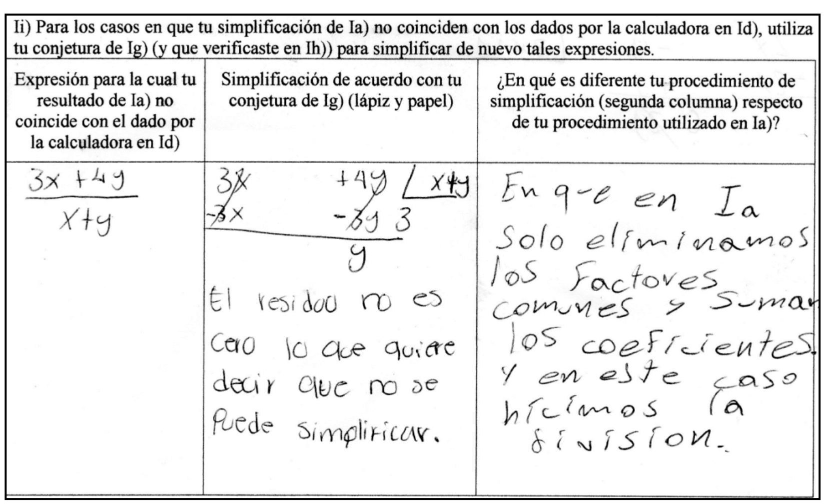


$\frac{4x+4y}{x+y}$ = 4	4x +4y ⌊x+y -4x -4y 4 0 0	Se realiza la division $\frac{4x+4y}{x+y}$ Se elimina "x" y se elimina 4y
$\frac{3x+4y}{x+y}$ = 7	$\frac{3\cancel{x}+4\cancel{y}}{\cancel{x}+\cancel{y}} = 3+4$	al realizar la operacion $\frac{4x+4y}{x+y}$ se eliminan las "x" con las "x" y las "y" con las "y" = 3+4=7

Figure 2: Students' paper-and-pencil work on binomial over binomial expressions

The CAS Work (a First Theoretical Reflection)

Once students confronted their paper-and-pencil results with the CAS results, a theoretical reflection based on their long polynomial division technique emerged. At this point we can see that using a technique is not just a routine work, just as Artigue (2002) has mentioned. The performance of these students fits the results obtained in a previous phase (the pilot study) of the research (see Guzmán, Kieran, & Martínez, 2011). In this main study, the same kind of theoretical reflection was provoked by the use of CAS (see Figure 3).



Ii) Para los casos en que tu simplificación de Ia) no coinciden con los dados por la calculadora en Id), utiliza tu conjetura de Ig) (y que verificaste en Ih)) para simplificar de nuevo tales expresiones.

Expresión para la cual tu resultado de Ia) no coincide con el dado por la calculadora en Id)	Simplificación de acuerdo con tu conjetura de Ig) (lápiz y papel)	¿En qué es diferente tu procedimiento de simplificación (segunda columna) respecto de tu procedimiento utilizado en Ia)?
$\frac{3x+4y}{x+y}$	3x +4y ⌊x+y -3x -3y 3 y El residuo no es cero lo que quiere decir que no se puede simplificar.	En que en Ia solo eliminamos los factores comunes y sumar los coeficientes y en este caso hicimos la division.

Figure 3: Students' reflection based on their CAS work

In this part of the Activity they wrote (see the second column of Figure 3): “the remainder is not zero; that means that the expression cannot be simplified.” As reported in Guzmán, Kieran, and Martínez (2011), we consider this kind of discourse to be a spontaneous theoretical reflection. In the third column of Figure 3, they included terminology of *common factors*. However, because of their previous work, we can say that they did not really understand this aspect (common factors); for them, all numbers or literals repeated in the numerator and denominator are *common factors*. In Activity 2, when these students had the opportunity to explore other cases, the use of CAS played an important role regarding the idea of *common factors* and making this idea more mathematically clear.

Second Theoretical Reflection Based on the CAS Technique

After the first theoretical reflection emerged, the students used their long division technique in order to explain the CAS results each time they found discrepancies between their paper-and-pencil work and their CAS work. Figure 4 illustrates this.

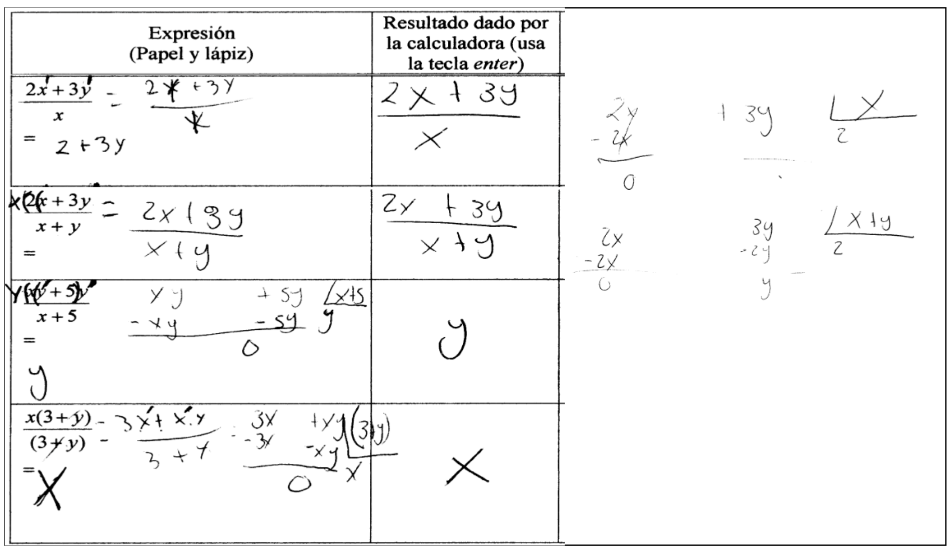


Figure 4: Use of long division technique in order to explain some CAS results

After the students had used CAS, their explanations (based on their theory of the remainder of the long division of polynomials algorithm) for simplifying expressions whose denominator is a monomial went a little bit further; in their discourse they included the words numerator and denominator. In the third column of Figure 5, they wrote: “before, we just eliminated the like terms from N/D [numerator over denominator] and now we know that if the numerator doesn’t have like terms then the expression cannot be simplified.” Compared to their written discourse shown in Figure 3, they had now begun to talk explicitly about the numerator and denominator and to speak about “like terms” instead of their very loose, and poorly understood, formulation involving “common factors.”

Expresión para la cual tu resultado con papel y lápiz no coincide con el dado por la calculadora en IIa)	Explica si es posible o no simplificar tales expresiones. Rehaz los cálculos, si es el caso.	En qué forma es diferente tu simplificación (segunda columna, si es el caso) respecto del procedimiento utilizado en IIa)
$\frac{2x+3y}{x}$ =	no se puede simplifica $2x \quad + 3y \quad \lvert \underline{x}$ $\underline{-2x} \qquad\qquad 2$ $0 \qquad 5y$ que antes solo eliminabamos los	rque antes solo eliminabamos los terminos comunes del $\frac{N}{D}$ y ahora sabemos que si el numerador no tiene terminos comunes no se puede simplificar la expresion.

Figure 5: Explanation as to why the given expression cannot be simplified

However, for the expressions of the form “binomial over binomial” (see the last three expressions of Figure 4), their explanations were (at this moment of the activity) still evolving. The next verbatim extract illustrates this.

Researcher: I heard that you said that in this case it is possible to cancel out elements of the expression [*Referring to the last expression of Figure 4; immediately after they finished the long polynomial division*].

Student A: Yes.

Student B: Because there is a monomial in the bottom …

Student A: It is a binomial, isn’t it?...

Researcher: So, why in the previous one [*Third expression of Figure 4*] is it that, that technique doesn’t work?

Student A: Because there are not the same terms above and below [*Referring to the numerator and denominator*]

Researcher: And in the last [*Expression*] they are?

Student A: [*Nods his head in agreement*]

Researcher: Which ones are those terms you are referring to?

Student A: *3* plus *y* divided by *3* plus *y*.

Researcher: So, there [*Referring to the last expression for the Figure 4*] you identify that both techniques work, dividing or cancelling?

Student A: Yes, but here as well [*Signalling the second expression of Figure 4, and he tries to factor the expression*]… For which one you asked?…

Researcher: For the third one [*Referring to the third expression of Figure 4*]

Student A: Let’s see… [*And he factors the expression, see Figure 4*]… Yes, you need to change the form [*of the expression*]

Student B: You factored the expression

After this, for expressions of the form “binomial over binomial” they explained their techniques in terms of factoring the expressions, even if for some cases there were still some inconsistencies in their explanations—that is, until they used the CAS for another case (see Figure 6).

Introduce en la calculadora	Resultado dado por la calculadora	Describe cómo harías para obtener el mismo resultado que el dado por la calculadora
$\frac{a+a\,x}{x}$	$\frac{a(x+1)}{x}$	Factorizando la expresion

Figure 6: CAS work

Once they used the CAS for simplifying the expression shown in Figure 6 and the CAS gave the result in factored form, this decisively changed their point of view regarding the technique for simplifying rational expressions. From then on, their explanations included the idea of factoring (as seen in the third column of Figure 6).

Conclusions

In this paper we have shown that the CAS environment led students to think in terms of factoring when simplifying rational expressions—something that they had not previously considered in their initial techniques of “cancelling” or using the “long division algorithm for polynomials.” This is in contrast to the findings from our earlier pilot study (Guzmán, Kieran, & Martínez, 2011) where students did not possess both initial simplifying techniques and where their CAS work did not lead to the emergence of the idea of factoring and its role in simplifying rational expressions. While both studies provided evidence for the power of CAS to stimulate theoretical reflection, the findings of this study suggest that if students have more initial paper-and-pencil techniques (even if not completely understood), the CAS work can spur them to deeper theoretical reflections than for students who have fewer techniques.

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DIGITAL TECHNOLOGIES IN MEXICAN HIGH-SCHOOLS

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In this paper we present some of the issues in Mexican public high-schools related to the incorporation of digital technologies in mathematics classrooms; noting that the inclusion of technologies is very isolated, and that schools still lack proper facilities. We also present some of the didactical approaches of teachers in using digital technologies during their lessons; we observe lack of preparation of these lessons and conflictive situations for the learning processes arising from difficulties in the implementation of technologies and generated by deficient technical content and pedagogical knowledge.

Keywords: Technology; Didactical Approaches; High School Education; Educational Changes

Introduction

The use of digital technologies (DT) in schools has become increasingly important in today's societies, due to its inherence in all areas of daily life. But in education, changes have been slow. In fact, research on the impact of computers in classrooms on students' academic performance has shown that the effect has been moderate if any at all (e.g., Papert, 1993; Kilpatrick & Cuban, 1998; Chadwick, 2001; Battista, 2007). The incorporation of DT in classrooms, is a particular challenge for teachers. The importance of teachers' professional development, that strengthen their competencies and knowledge for helping students address the needs of the 21st century, has been a theme of several international educational conferences (e.g., at the International Conference on Education—ICE, and the International Congress on Mathematical Education—ICME), in particular with regard to the need of incorporation of digital technologies. Over a decade ago, the National Council of Teachers of Mathematics (NCTM, 2000) stated that technologies had to be used widely and responsibly in order to achieve a complete mathematical training, and to facilitate visualization of mathematical ideas. However, as Chadwick (2001) cautions, when teachers use technology they should ensure that the means do not cause straying from the educational aims. Technology can be a didactical tool only insofar as it helps in the construction of meanings of the objects of study; we also believe that a responsible use of DT should be supported in results from educational research in order to be successfully implemented. Sacristán, Sandoval and Gil (2009) concluded, from a research conducted on Mexican primary and middle-school teachers, that if teachers are to successfully incorporate DT in their practice, they need to understand how to use these tools in order to create meaningful learning in students.

Research Objectives and Theoretical Framework

Our general research aims to analyze elements of teachers' didactical practices in Mexico, and their relationship with curricular contents and recommendations; one particular aspect that is the focus of this paper, is teachers' didactical practices related to the use of digital technologies in mathematics classrooms.

In our study, we use as framework the categories proposed by Shulman (2001) related to the knowledge base that teachers should have in their professional development; thus we consider pedagogical content knowledge (PCK), curricular knowledge, technological pedagogical and content knowledge (TPACK, that provides understanding on the technological tools that the teacher uses in her/his practice) (Mishra & Koehler, 2006), among others. Llinares (2000) recognizes that a fundamental part of a teacher's practice lies in the choice of the instruments s/he uses (spoken language, modes of symbolic representation, didactical materials, use of technologies in daily practice, etc.) and he emphasizes the importance of her/his understanding of which and how they will be used, and for which aims. On their part, Ponte and Chapman (2006) state that a teacher's knowledge and her/his didactical approach, are mutually dependent in the teacher's activity; this relationship is representative of the organization of the elements for teaching. We have used the above ideas to design our survey questions and to observe characteristics in teachers' development of knowledge, not only taking into account what they know (content knowledge), but what they *do* with what they know (knowledge use), in a twofold way: their mathematical content knowledge (Ball, 1988) and the technology use during their classroom practice. In this way we attempt to assess how meaningful is a teacher's use of DT in her/his classroom.

Methodology and Data Collection

In studying teachers' didactical approaches, we also consider their knowledge in terms of the educational changes and curricular reforms that have occurred in the last decades both at international and at national levels; in particular we consider those related to teaching methodologies (recommended classrooms strategies and dynamics, didactic materials, etc., often based on specific epistemological theories—e.g., constructivism) and the use of DT in the classroom (computers, videos, Internet, specialized software for mathematics, etc.).

In a first research phase, we carried out a documental type of research where we reviewed the diverse programs of study and curricular recommendations for high-school mathematics in Mexico, in order to establish what is considered essential for the teachers' practice. In a second research phase, we carried out a survey of 159 high-school mathematics teachers in different regions of Mexico; through this survey we had some panoramic insights of the ways in which high-school teachers have perceived the educational changes and needs in the 21st century world. In a third research phase, we carried in-classroom observations, as well as pre- and post- interviews, of a subset of the surveyed teachers: 13 teachers in Mexico City, who claimed in the survey to have changed, in the past decade, the way they teach and incorporated DT to their practice, and 3 other teachers, not originally surveyed, who where reputed in their schools to use DT in their lessons. Thus, we observed (and interviewed) a total of 16 teachers, in 5 different public high-schools in Mexico City, for up to two classroom sessions of 60 to 120 minutes, in which they were meant to use DT. It is worth noting that the 5 schools we visited are considered amongst the best public high-schools in the country.

For the analysis of the results, we carried out a correlation of the results, through a methodological triangulation to structurally relate the cualitative, quantitative and documental data (Denzin, 1990; Bryman, 2007). In this way we could analyze the relationships between a teacher's didactical beliefs (including her/his beliefs on educational needs and changes, and on DT tools as didactical instruments; teaching methodologies used in her/his practice; her/his changes in the last decades), what s/he claims to have changed due to educational reforms in Mexico, and her/his actual didactical approaches in the classroom.

Some Results and Sample Data

In this section we present some results derived mainly from the second and third phase, related to the use that the teachers in our study make of DT in their practice. Other results are beyond the scope of this paper.

Some General Results on the Use of DT by the Study's Teachers

The great majority of the 159 teachers surveyed, 91%, agreed that there have been significant educational changes in the last two decades in the world; however only 38% mentioned the use of technology as one of the most significant changes. Nevertheless, the surveyed teachers coincided in that the use of DT has become an essential part of students' development and that it is important to include them in teaching for didactic support. A large majority of them, 73.8%, claimed to use DT to support their mathematics teaching practice. And 65% said they used Internet to search for theoretical information related to the topics studied in their classes, to search for formulae, or to send homework to their students.

In terms of the observed and interviewed teachers, all 16 of them mentioned that they had taken professional development workshops on the use of digital tools, mainly on the use of graphing tools, such as Winplot—which was also the most common use mentioned; and on the use of information and communication tools (i.e., Internet) or office suites; and ten of them had had a short course on some dynamic geometry software (i.e., Cabri, SketchPad or Geogebra).

However, from the interviews, we realized that the way in which these third-phase teachers used DT, was mostly as checking and comparison tools, or to save time by using tools (e.g., Winplot) that would facilitate the construction of graphics. A surprising response was that of a teacher who said he only used DT because it was a requirement of his school.

Furthermore, although, when we interviewed them, all 16 teachers claimed to use DT for their lessons, at the time we visited them only four of them actually used them with their students in our observation sessions. Some of explanations that were given for the lack of use of DT when we observed, were the following:

- they only used DT in class once per school term (e.g., to show students how to use a graphing software) and afterwards students are supposed to use it for homework;
- they use Internet (e.g., email) for sending and receiving students' homework;
- they ask students to research how to use a graphing software at home and turn in computer-plotted graphs as homework;
- the school doesn't have the necessary equipment;
- they don't have access to the school's computer room;
- they only use DT for class preparation.

Therefore, though all these teachers claimed to use DT in their practice, it was clearly a very limited use, if any at all.

Sample Data from the Four Teachers Observed Using Technology

Here we summarize the way in which the four teachers used DT when we first visited them:

Three teachers, whom we will name teachers A, B and C took their own laptops and beamers to their respective classrooms. However, teacher A could only connect his equipment to the ceiling lamp, due to the lack of electrical outlets in his classroom.

Teacher A showed his students a video downloaded from the Internet on the theme of geometrical congruence and similitude that was the theme under study; however the video had no sound, had Portuguese subtitles and was blurry, so that it was difficult to follow and see.

Teacher B had no problems in connecting his equipment. He used a plotting software (Graphmatica) to show his students the domain and range of polynomial functions. Though he did allow a couple of students to play with the software, the DT tool was used only for visualization and the main activity was carried out in paper-and-pencil.

Teacher C also used Graphmatica, but she faced many problems in projecting the images (taking over 20 minutes of a two-hour session). She was teaching the theme of irrational functions and asked the students to type a function in Graphmatica. She began with $y=\sqrt{x}$ typing it on the whiteboard (Figure 1) and asking a student to do it with Graphmatica (Figure 2).

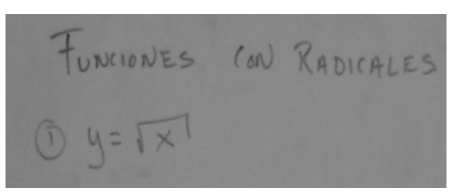

Figure 1: Teacher C writes function on board

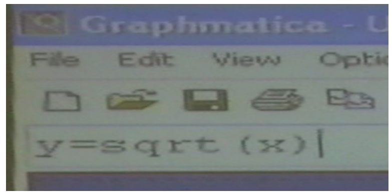

Figure 2: Graphmatica input

She wrote other functions on the board, that students took turns graphing in Graphmatica: $y=-\sqrt{x}$, $y=\sqrt{-x}$, $y=\sqrt{x}+3$, $y=\sqrt{x}-3$. But when she moved on to more complicated functions: $a)\ y=\sqrt{x^2+9}$, $b)\ y=\sqrt{x^2-9}$, $c)\ y=\sqrt{-x^2+9}$, $d)\ y=\sqrt{-x^2-9}$, the way of inputting these functions became more complicated: whereas before they had been using the SQRT command, they now had problems and the teacher changed to using the ½ power instead. But this created further problems; such was the case of function c) which was incorrectly inputted (Figure 3) and produced an incorrect graph (Figure 4).

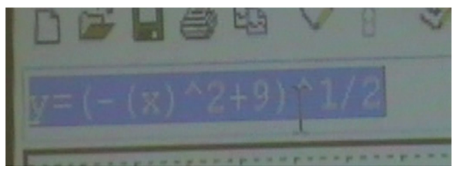

Figure 3: Input in Graphmatica for function c)

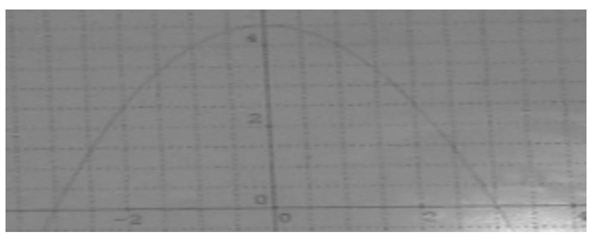

Figure 4: Graph produced by Graphmatica to an incorrect input

The problem with the input of this function was that the ½ power was not placed between parentheses. Therefore the plotted function was actually $y=\frac{(-x^2+9)^1}{2}$, which for x=0, gives y=9/2=4.5. However, the teacher did not notice this; even when a student pointed out that the graph was wrong, that the curve should have cut the y-axis in 3, she replied by saying that the root of the function, where the graph cut the x-axis, was 3, so the student's comment was not correctly taken into account. All the other functions were also incorrectly inputted and thus incorrectly graphed; the teacher, however, did not acknowledge that there was any problem. Another four students also doubted the accuracy of the graphs, but the teacher just said that those were the behaviors of the functions, never correcting the situation.

Teacher D took her students to the school's co mputer room, where each student could use Graphmatica. The topic was the same as Teacher C's, but in this case there were no problems.

Discussion and Concluding Remarks

Though curricular reforms and society's changes are pushing for the inclusion of digital technologies in schools, our results show that is not straightforward. Though the majority of the surveyed teachers are conscious of the changes brought about by digital technologies in the world and how these have permeated daily life, and 73.8% claimed to use computers as teaching aids, when we went to visit the schools we observed a different reality. In our study we observed that the incorporation of DT in Mexican public high-schools is extremely limited, if not nil.

One of the categories that were established in our study, derived from the work of Shulman (2001) and others (as discussed in the theoretical framework section above), is that related to curriculum knowledge: in our study, we wanted to see if and how teachers took into account and used the methodologies, strategies and other recommendations stated in the official programs and curricula, in their practice. But in our study, most teachers observed and interviewed were unable to explain what those recommendations from the curriculum were (let alone put them into practice), even though they had previously claimed explicitly to be using them, including the use of technology.

Another observation is the lack of equipment and facilities for using DT that is seen even at some of the best public high-schools in the country (which were the ones we visited). Most public high-schools are not equipped with computer rooms, and in those that do have them, teachers tend to not use them (in fact, we were able to observe only one teacher using it), arguing problems in accessing those rooms, or lack of training in their use.

When teachers do use DT, the use that is done, tends to be limited to presentation (as in the case of Teacher A), visualization or computing uses (e.g., using graphing software), for checking results produced in paper-and-pencil, or simply for communication (e.g. using email or Internet for sending homework's). It is thus more of a mechanical use (or, simply, for accuracy and saving time, as in the case of the use of plotters) rather than having educational aims, and much less a meaningful harnessing of the potential of DT for enhancing learning. Furthermore, we observed that teachers did not design any activities using DT (other than plotting a graph, or checking a result with the use of DT)

Those few teachers who mentioned to be up-to-date in the use of DT, underestimate their potential as educational aids and lack technical pedagogical and content knowledge (TPACK). They are not conscious of the difficulties that may arise during the implementation of DT in the classroom (such as in the cases of Teachers A and C), and lack the technical knowledge and mathematical content knowledge to deal responsibility with situations such as the one observed with Teacher C. Another deficiency noted, is that two of them had not prepared their lessons, which is an important aspect mentioned by Llinares (2000) that should be part of the professional teaching practice.

It is important to note that although some teachers have tried to adapt to the changes in education, attempting to change their teaching methodologies and attempting to incorporate DT into their practice, the lack of training and support can lead to confusions, misinformation, or even loss of interest or commitment. In fact, during the interviews, some of the high-school teachers complained that the only type of training they had received were on the basic use of office software suite packages, and not in more specific tools for mathematics education (such as dynamic geometry or CAS).

The above results coincide with those reported by Julie et al. (2010) from a survey conducted in Latin America in 2006. We would have expected changes since 2006, but as was noted ten years ago by Cuban, Kilpatrick, and Peck (2001) the incorporation of technology into classrooms has been a very slow process, and this seems to be particularly true in developing countries like Mexico, where the incorporation of technologies in teaching practices is limited. In Sacristán, Parada, and Miranda (2011) we discussed this problem, observing two types of limitations and obstacles: one related to digital divides (illustrated here by the lack of equipment and facilities); the second related to professional development of teachers and the educational system itself.

Hennessy, Ruthven, and Brindley (2005) point to the importance of teacher involvement (rather than a technologically-driven model of technology integration) in effecting classroom change; but they also point that this involvement is undoubtedly influenced by the teachers' working contexts. The little use of technologies we observed in our study, is partly due to lack of proper conditions, but also because teachers have not experienced other uses. Hitt (1998) pointed out that teachers will only feel the need to incorporate technologies to their practice when they experience the effectiveness of a tool or resource in dealing with a problem. Thus, rather than focus on the delivery of technical skills (which are the type of courses the teachers in our study had received), it might be helpful if teachers can participate in professional development models that immerse them—and support them—in the experience of dealing with mathematical situations through technology. However, taking into account the reality of countries such as ours, this may not be so easy.

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THE ROLE OF TECHNOLOGICAL TOOLS IN RELATION TO STUDENTS' MATHEMATICAL THINKING DURING CLASSROOM TASKS

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This study uses the Mathematical Tasks Framework (Stein & Smith, 1998) to assess the cognitive demand of mathematical tasks implemented in four mathematics classrooms, and to investigate the role of technology in both low- and high-level cognitive demand tasks. The metaphor of using technology as an amplifier or reorganizer (Pea, 1987) is used to characterize technology use. Results indicate that when technology is used as an amplifier, it has no influence on the thinking demands of a mathematical task, but when used as a reorganizer it is intimately related to the supporting students' high-level mathematical thinking. Furthermore, this distinction can be an important part of mathematics teachers' technological pedagogical content knowledge (TPACK) (Mishra & Koehler, 2006; Niess et al., 2009) by providing ways to distinguish uses of technology along dimensions that matter for students' mathematical thinking and learning.

Research on the use of instructional technology in secondary mathematics education has proliferated over the last 20 years (e.g., Heid & Blume, 2008; Zbiek, Heid, Blume, & Dick, 2007). There has also been an increased awareness of and interest in students' mathematical thinking and reasoning (Common Core State Standards Initiative, 2010; National Council of Teachers of Mathematics, 2009), However, little research has focused on how the use of technology can support students' mathematical thinking and reasoning more generally.

Theoretical Framework

As the purpose of the this paper is to characterize the use of common classroom technologies in relation to students' mathematical thinking, an array of classroom technologies are considered. The interest in the present study is with digital technologies used specifically as cognitive technologies (Pea, 1987). Pea defines cognitive technologies as those that "help transcend the limitations of the mind (e.g., attention to goals, short-term memory span) in thinking, learning, and problem-solving activities" (Pea, 1987, p. 91). By mediating human thought, cognitive technologies both assist and influence thought and learning. The focus of the present study is on *digital* cognitive technologies that may support students' mathematical activity. A distinction that Pea makes within cognitive technologies is between its use as an *amplifier* or a *reorganizer* of mental activity (1987). That is the focus of the next section.

Amplifier and Reorganizer Metaphors

When technology is used as an amplifier, it performs more accurately or efficiently tedious or time consuming processes that might be done by hand, like arithmetic computations or the generation of standard mathematical representations. In this use of technology, what students do or think about is not changed, but can be done with significantly less time and effort, and more accurately. The use of a scientific calculator for computations while students set up and solve proportions can make their work more efficient and help to avoid basic arithmetic errors in their solutions. However, what students are doing is not changed by the use of the calculator; their cognitive focus is still on setting up and solving proportions whether the calculator is used or not.

As a reorganizer, technology has the potential to support a shift in the focus of students' mathematical thinking and behavior, by producing novel representations which make salient some aspect of a concept which is difficult to make explicit without it, or by providing feedback to students that they would otherwise not have access to. For example, students might use dynamic geometry software (DGS) to construct a triangle and manipulate it in order to look for and make conjectures about the relationship between the lengths of the sides, with the goal of discovering the Triangle Inequality Theorem. By using

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technological tools to generate dynamic and interactive representation, students are able to focus on looking for patterns and making and testing conjectures, rather than on drawing and measuring triangles. This use of technology supports a shift in the focus of students' mathematical activity and thinking.

An important aspect of the type of thinking afforded by the use of technology is the kind of problem or task that calls for its use. Whether technology is used or not, one way that teachers shape students' learning and view of the discipline of mathematics is by the choice of mathematical tasks for instruction (National Council of Teachers of Mathematics, 1991). However, with the introduction of technology comes the need to understand what kinds of tasks utilize the resources provided by the technology to support students' high-level thinking (Hollebrands, Laborde, & StraBer, 2008). A framework for understanding the influence of tasks on students' mathematical thinking is described in the next section.

The Mathematical Tasks Framework

The *Mathematical Tasks Framework* (Stein & Smith, 1998) has been used to describe and differentiate the type of thinking that is called for by a given mathematical task, defined as "a classroom activity, the purpose of which is to focus students' attention on a particular mathematical idea" (Stein, Grover, & Henningsen, 1996, p. 460). This framework distinguishes between low-level cognitive demand, including *memorization* and the use of *procedures without connections* to meaning or concepts, and high-level cognitive demand, including the use *of procedures with connections* to meaning or concepts, and *doing mathematics*, of which non-algorithmic thinking is characteristic. An important characteristic of this taxonomy is that it is not related to specific mathematical content, but rather characterizes different types of thinking that students may engage in while working on a mathematical task.

An important contribution of the *Mathematical Tasks Framework* is the recognition that the thinking requirements of a task may change during its enactment. The task as it appears in curricular materials does not directly influence students' learning by the type of thinking it requires, as those demands may be altered by the teacher when announcing the task to students during instruction, known as the *set up* phase, and again while students are working on the task, referred to as the *implementation* phase. This element of the *Mathematical Tasks Framework* makes it especially suitable for describing the impact of using technology on students' thinking in a classroom context. The research question investigated by this study is: what is the role of technology in relation to the cognitive demand of mathematical tasks?

Research Methods

This study uses a qualitative, observational research design with the goal of understanding the role of technology in supporting the mathematical thinking of students. Four teachers were recruited primarily based on their use of technology for mathematics instruction. One or two units of instruction, as designated by the teacher, were observed in each classroom. Each of the teachers had three years of teaching experience, and had taught the observed unit at least once previously. An overview of the data collection classrooms is given in Table 1.

Data collected at each site included lesson observation field notes, task artifacts and student work on the task, and audio recorded post-lesson interviews with the teacher. Hand-written jottings taken during the observation were developed into a detailed narrative of the lesson immediately following the observation (Emerson, Fretz, & Shaw, 1995), and all post-lesson interviews were transcribed. Using the *Task Analysis Guide* (Stein & Smith, 1998), each task was coded with respect to the cognitive demand of the task as stated in the curriculum, as introduced to the class (set-up), and implementation. In addition, for those tasks which utilized technology, the use of technology was coded as amplifier, reorganizer, both, or neither during the set up and implementation phases. Approximately one-fourth (24%) of the observed tasks were double coded for reliability, with 89% agreement on the cognitive demand, and 86% with regard to the use of technology. All discrepancies were resolved and the consensus code was assigned to the task.

Table 1: Summary of Data Collection Classrooms

	Tasks Observed	Grade/Class Level	Topics	Technologies
Ms. Jones[1]	12	9th grade Integrated Math	• Angle relations • Triangle Inequality • Similarity	• DGS • Scientific Calculators
Ms. Young	17	11th grade Inclusion	• Angle relations • Triangle Inequality	• DGS • Interactive Whiteboard • Scientific Calculators
Mr. Mack	17	6th grade Regular	• Order of operations • Fractions	• Interactive Whiteboard • Scientific Calculators
Ms. Lowe	17	10th grade Advanced	• Points of concurrency in a triangle	• DGS • Interactive Whiteboard • Graphing Calculators

Following data collection and coding, the coding results were summarized in order to observe patterns in the data that could guide qualitative analysis. For example, it had been hypothesized that the use of technology as an amplifier would be associated with low-level cognitive demand tasks. However, the summary of the coding results revealed that technology was used as an amplifier in both low- and high-level cognitive demand tasks across sites. These tasks were analyzed qualitatively in order to understand the role that technology played in these tasks, and how it was related to the thinking demands of the task. The constant comparative method (Glaser, 1965) was used in analyzing different tasks in the same classroom, as well as across classrooms, in order to make generalizations about the relationship between the role of technology and students' mathematical thinking.

Results

A primary concern in this study is how the use of technology might be correlated with the cognitive demand of the mathematical tasks within which that use is situated, and the meaning of those correlations. The hypotheses for this study were that technology is used as an amplifier in low-level tasks, and substantial evidence for the hypothesis that technology is used as a reorganizer (or both) in high-level tasks. The reasoning behind these hypotheses was that teachers would use technology to support a change in students' focus to high-level thinking by offloading computations or the generation of representations to the technological tools, or by providing novel representations capable of supporting conceptual connections that would be difficult or impossible to produce by hand. According to this logic, a teacher that did not utilize technology as a reorganizer would not be attempting to support such a shift, and thus the task would remain at a low-level. Results of the study provide partial evidence for the hypothesis that technology is used as an amplifier in low-level tasks, and substantial evidence for the hypothesis that technology is used as a reorganizer (or both) in high-level tasks.

The results of the coding of tasks in terms of the cognitive demand and the use of technology are reported in Tables 2 and 3. Table 2 shows the distribution of tasks using technology as an amplifier during set up at a low- or high-level, and during the implementation at a low- or high-level, while Table 3 depicts the same for the use of technology as both an amplifier and reorganizer.[2] Technology use during the set up phase refers to the way in which technology was designed to be used in the task as set up by the teacher, but prior to students actually engaging with the task. For example, if the teacher introduced as task in which students were to use DGS to investigate the properties of medians of a triangle, the use of technology was coded as an amplifier and reorganizer during set up.

Table 2: Amplifier Technology Use in Relation to Cognitive Demand[3]

Amplifier Use of Technology	Low-level	High-level
Set up	24	7
Implementation	46	2

Table 3. Reorganizer Technology Use in Relation to Cognitive Demand

Technology Use as Reorganizer	Low-level	High-level
Set up	0	16
Implementation	2	6

Amplifier Use of Technology

As Table 2 indicates, within the sample of tasks set up at a low-level, the use of technology was always intended as an amplifier. Although there exists an association of low-level tasks with amplifier use during set up and implementation, qualitative analysis of these tasks revealed that the way the technology was used was not directly related to the low-level demands of the task. Indeed, technology was also used as an amplifier in high-level tasks, and likewise qualitative analysis revealed no relationship with the cognitive demand of these tasks; it was merely used for displaying the statement or description of a task that would have been high-level without it.

A primary way in which technology was used as an amplifier was in tasks in which the interactive whiteboard (IWB) was used to display lecture notes or practice problems, to project a worksheet while discussing problems or solutions, and in a few cases, it was used in conjunction with DGS in order to provide a dynamic demonstration or example. Another common amplifier use of technology included the use of a calculator for computations while practicing a procedure. For example, students used scientific calculators for arithmetic computations while solving for missing angles in a diagram of parallel lines cut by a transversal.

What all of these tasks had in common is that the cognitive demand of the tasks in which they appeared would not have changed if technology had been used in the way that it was, i.e., as an amplifier. Given the way that the amplifier use of technology is defined, i.e., making some process more accurate or efficient that could be accomplished without it, it makes sense that such a use of technology is not directly related to the cognitive demand. Rather, the association revealed in these data seems to be mediated through the teachers, and the affordances they perceive of the technology available to them in relation to low-level tasks. Thus, the selection of the task may be the primary factor in the cognitive demand when technology is used as an amplifier.

Reorganizer Use of Technology

The use of technology as a reorganizer was strongly associated with the set up and implementation of high-level tasks. As hypothesized, its use as a reorganizer was in all cases related to its use as an amplifier, in the sense that by offloading the construction, labeling, and measuring of mathematical objects to the technological tools there existed the potential for students to shift the focus of their mental activity to such behaviors such as dragging, observing, generalizing, and making and testing conjectures. In general, teachers used a dynamic geometry software package such as GeoGebra or Geometer's Sketchpad to have students investigate and explore the properties of geometric objects such as triangles.

Three of the four teachers this study used technology as both an amplifier and reorganizer to set up tasks at a high-level using DGS within a student-centered exploration. In general, the purpose of using technology in these tasks was to support students in constructing meaning for a mathematical concept or procedure, or to engage in mathematical behavior, such as observing, reasoning, generalizing, and conjecturing.

An example of a task that was set up and implemented at a high level using technology as a reorganizer is taken from Ms. Lowe's classroom. Ms. Lowe created a worksheet to guide students in using

GeoGebra individually at their own computer for most of the period in order to investigate the properties of the centroid of a triangle, i.e., the intersection of the medians[4] of a triangle. She guided students to construct a triangle and the medians of the triangle, to construct the centroid, to measure the segments from the vertex to the centroid, and from the centroid to the midpoint of the opposite side, and then to record these measurements in a table in order to look for a relationship[5] and make and test conjectures. In this case, the opportunity to drag and explore the properties of the medians individually was directly connected to the cognitive demand of the task.

As an example of the type of thinking that students engaged in while working on the task, the following conversation between two students was observed while working on the task:

> *Nick and Brian are dragging their figures and discussing what it is that they're supposed to be noticing.*
> **Nick:** I'm going to make it a right triangle. What would that do? It would stay at the center of the triangle, right?
> **Brian:** look at this.
> *Brian shows Nick his table, pointing out the 6.17 and the 3.08.*
> **Brian:** this one is almost exactly double that one.
> **Nick:** you can't make assumptions from one triangle
> *Both start dragging their triangles.*
> **Nick:** I see something like that, but if you stretch it far enough…
> *They continue dragging their triangles and looking at the measurements.*
> **Nick:** one is always half of the other
> **Brian:** the distance from the vertex is always double the distance to the midpoint.
> **Ms. Lowe:** change it, see if you can disprove it.
> *Starting over with a new triangle, Brian begins to measure the distances from the centroid to the vertex and from the centroid to the midpoint for each median.*
> **Brian:** (as he measures each segment) that is double that, and that is double that, and that is double that.
> *Nick drags his figure.*
> **Nick:** yes, it does stand true. (Field note, 2/7/11)

This excerpt demonstrates how technology can be used as both an amplifier and reorganizer. As an amplifier, students constructed a triangle, the medians of the triangle, and the centroid quickly and precisely, and measured and labeled the angles, the lengths of the medians, and the lengths of the segments. Most students had completed this part of the task within 10 minutes. While students might be able to construct the centroid of a triangle and use a protractor and ruler to make the same measurements, this could be difficult for most students to do accurately in 10 minutes. Furthermore, by dragging the triangle, students are essentially creating many triangles, medians, and centroids. As a reorganizer, dragging does more than just create multiple examples quickly and accurately. One can observe, for example, how the centroid moves in response to a vertex being dragged, or how the location of the centroid is changed as the triangle is changed from an acute triangle, to a right triangle, to an obtuse triangle, and back again. This sort of "real-time" motion of one object in relation to another is simply not possible in a pencil-and-paper environment.

Further evidence of the reorganizer use of technology is that students are not focused on making the measurements, but on using them to discern regularities in the behavior of the segments and on understanding what they mean. Nick's statement, "I'm going to make it a right triangle. What would that do? It would stay at the center of the triangle, right?" indicates the open-ended nature of having students directly manipulating the object created within a DGS, that there are many possibilities to choose from in terms of how to drag the object. It also reveals the making and testing of conjectures that is inherent in the development of a more strategic investigation of an object using dragging. Students must consider the purpose of dragging in terms of an overarching goal, what information would be helpful in achieving that goal, and what sort of dragging might provide that information. Once that move is made, students must assess if the object behaved in the anticipated manner, and if not, why, and what the next move should be in light of this information. The technology acts as a reorganizer by supporting these students' focus on looking for relationships, and making and testing conjectures, which constitute the high-level thinking demands of the task.

The Role of Technology in the Decline of Cognitive Demand

One explanation for the correlation of amplifier use of technology with low-level tasks during implementation is that many tasks that intended to use technology as both an amplifier and reorganizer in a high-level task during set up were implemented at a low-level when students used technology as an amplifier only. In these tasks it was the use of technology as a reorganizer that was intimately connected with the high cognitive demand of these tasks as set up. Thus, when technology failed to act as a reorganizer of students' thinking, the cognitive demand declined during implementation.

In these tasks, students constructed, measured, and manipulated figures, but did not engage in making mathematically meaningful observations, generalizations, or conjectures. For example, in Ms. Jones class, students created triangles and measured side lengths in order to explore the Triangle Inequality Theorem.[6] However, when asked if it were possible to create a triangle in which the sum of two side lengths could be less than the third, some students replied "yes," and very few students wrote a conjecture about the relationship between the lengths of the sides.

In general, these teachers seemed to underestimate the support that students would need in connecting their work with DGS to the mathematical thinking and behavior required by the task. While the affordances of DGS can support high-level thinking, there is nothing about the use of a DGS for an exploratory task that *causes* students to engage in high-level thinking. For example, if students have never been asked to make a conjecture before, providing them with technological tools will not necessarily result in their ability to do so. DGS can support students' ability to make conjectures by providing the opportunity to examine numerous examples to analyze as the basis for a conjecture, and strategically manipulate objects in order to test a conjecture. However, it does not inherently support students' understanding of the importance of examining a variety of examples, what is mathematically meaningful to look for across those examples, how to make a mathematically precise statement as a conjecture, the importance of testing a conjecture or looking for counterexamples, or the difference between a conjecture and a proof. Ultimately, when technology is used as both an amplifier and a reorganizer, teachers must support the shift entailed by its use as a reorganizer. What that support may consist of has been discussed elsewhere (Sherman, in press).

Discussion

The present analysis builds on previous work that makes use of the amplifier and reorganizer distinction (Ben-Zvi, 2000; Laborde, 2002), but extends the distinction by considering how technology might act as an amplifier or reorganizer during the implementation of classroom tasks. The use of technology as an amplifier was generally associated with the interactive whiteboard and calculator, while its use as a reorganizer was almost always in the context of using DGS. It is tempting to explain the difference in technology to the differences in the affordances of these classroom technologies. Research points to the potential of calculators to be used in ways that can support and influence students' thinking (Burrill et al., 2002). Thus, the real issue may be how the affordances of these technologies are perceived by teachers.

A way in which the results of the present study may contribute to research in mathematics education is by characterizing the use of technology in relation to students' thinking in a way that can differentiate superficial from meaningful use of technology for mathematical instruction and learning. These results provide empirical evidence that the mere inclusion of technology does not have any inherent implications for students' opportunity for high-level thinking, but how it is used does.

An understanding of this distinction may be an important element of mathematics teachers' TPACK (Mishra & Koehler, 2006; Niess et al., 2009), by providing a way to critically examine the role of technology in the tasks they enact with their students. Anecdotal evidence indicates that preservice teachers in a secondary methods course were able to learn and use this distinction in evaluating and selecting tasks. Research is needed to examine this claim more carefully, and to determine how it may influence in-service mathematics teachers' selection and design of classroom tasks.

Endnotes

[1] Pseudonyms

[2] For the sake of simplicity, "both amplifier and reorganizer" is used interchangeably with "reorganizer" for the remainder, since no cases of using technology were coded as reorganizer only.

[3] The discrepancy in the number of tasks set up and implemented using technology as an amplifier has two sources. Some tasks were set up by the teacher without any explicit mention of technology as part of the set up of the task, but students initiated its use during while working on the task. In other cases, the task was set up to use technology as both an amplifier and a reorganizer, but utilized technology as only an amplifier during implementation. This also explains why more tasks were set up than implemented using technology as a reorganizer, as shown in Table 3.

[4] A segment connecting the midpoint of a side of a triangle to the opposite vertex.

[5] The relationship that students were intended to discover is that the segment from the midpoint to the centroid is 1/3 the length of the median, and the segment from the centroid to the opposite vertex is 2/3 the length of the median.

[6] The sum of the lengths of any two sides of a triangle is always greater than the length of the third side.

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EXAMINING THE TRANSITION FROM A PART-WHOLE TO PARTITIVE UNDERSTANDING OF FRACTIONS

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Sixth grade students' transitions from part-whole understandings of fractions to more powerful ways of operating were explored as part of a pilot study of student interaction with Candy Factory, an iPad app designed to support the construction of partitive fraction schemes. Students demonstrated a novel strategy for creating fractions of the form (n–1)/n. We will discuss this strategy and consider its potential for supporting students' development of an understanding of composite fractions as measures.

Keywords: Number Concepts and Operations; Learning Trajectories; Middle School Education

As part of a pilot study for the NSF-funded GAMES grant (award #1118571), we observed student engagement with the *Candy Factory* fractions iPad app (see http://ltrg.centers.vt.edu/). The study included observation of two sixth-grade classes using the app in the classroom, during which we noticed several students' use of a "complement" strategy for constructing proper fractions. The purpose of this paper is to consider how and why one student's use of the strategy might have supported the integration of his part-whole and measure conceptions of fractions.

Theoretical Framework

Researchers commonly consider knowledge about fractions in terms of the model proposed by Kieren (1980), in which facets of fractions understanding are described via relationships amongst five identified sub-constructs: part-whole, measure, ratio, quotient, and operator. A part-whole sub-construct includes understanding a fraction as a relation between a whole and a subset of that whole, whereas in the measure sub-construct, fractions are treated as measureable units, such as length. Via a longitudinal classroom teaching experiment, Lamon (2007) found that teaching with intent to encourage students' development of fractions as measures is the most powerful for the emergence of the other sub-constructs. The aforementioned model categorizes fractions knowledge of *adults*, and thus the relationships between the sub-constructs might not be part of *children's* understanding as they construct their knowledge (Thompson & Saldanha, 2003). Steffe and colleagues have identified and continue to refine a hierarchy of students' *fractional schemes* and associated operations such as *partitioning, splitting,* and levels of *unit-coordination* that are necessary and/or sufficient for the schemes' construction (see Steffe & Olive, 2010). The focus of this paper is on the development of a *partitive fraction scheme* (PFS), which is closely related to the measure sub-construct (Norton & Wilkins, 2010).

Methods

We observed two sixth-grade classes, once per week, for about thirty minutes, over a period of seven weeks, as students worked in pairs using the *Candy Factory* app. Our focus was to document students' verbalizations and observable actions within the game that were indicative of their ways of operating with fractions. For more fine-grained analysis of students' ways of operating, we also conducted a paired-student teaching experiment (cf. Steffe & Thompson, 2000) with nine weekly after-school sessions, each lasting approximately thirty minutes, using the app and other manipulatives such as Cuisenaire rods. We chose a participant from each of the two sixth-grade classes, Austin and Jane, because clinical interviews from another study revealed that they each had constructed only a part-whole scheme but not a partitive unit fraction scheme (see Wilkins, Norton, & Boyce, in review). Our focus in this paper is Austin, because he was the more demonstrative and verbal of the pair.

Results

Austin's Initial Ways of Operating with Fractions

While we were able to attribute only a *part-whole scheme* (PWS) to Austin based on our initial interview, our observations from the first few weeks of the teaching experiment led us to believe that Austin constructed a *partitive unit fraction scheme* (PUFS). He consistently arrived at the name for a unit fraction bar by iterating its length from left to right across a non-partitioned whole, both within and outside of the context of the app. When Austin was given the task of finding the length of a "whole" given a fraction rod that he was told was "three-sevenths," he iterated the given piece three times to form the whole and said that he "counted up three-sevenths." Thus, Austin did not appear to have constructed a PFS, for he did not conceive of a proper composite fraction such as three-sevenths as a unit consisting of three one-sevenths units, each of which could be iterated seven times to make the whole unit (Steffe & Olive, 2010).

Austin's Use of a Complement Strategy

In *Candy Factory*, a user "manufactures" a customer order by choosing the number of partitions of a whole "candy bar" and subsequently choosing a number of iterations of a resulting part. In the second week of classroom observations, we noted a pair of students using a "complement" strategy for customer orders that were proper fractions of the whole (see Table 1). We noticed several students using various forms of the strategy in subsequent weeks, including students with less-developed fractions schemes such as Austin. We retrospectively analyzed the teaching experiment data in order to incorporate his use of the strategy, both within and outside the context of the app, in our model of his understanding of fractions.

Table 1: A Complement Strategy

If customer order is perceived as:	"Close" to 1/2	Less than 1/2	Greater than 1/2
Student chooses partitioning to make:	1/2	1/n, for n > 2	1/n, for n > 2
Student chooses iteration to make:	1/2	1/n	(n–1)/n

In the segment below, Jane and Austin were playing a game with fraction rods, which were ostensibly distinguished in size only by color, although we also refer to them as multiples of the unit rod in this article. Concealing his or her actions from his or her partner, one student would choose a whole and create a proper fraction of that whole. The task for the other student was to make the whole, given the fractional part and its name. In the following segment, Jane has just chosen the brown 8-rod as the whole and named the purple 4-rod as two-fourths. [A: Austin; J: Jane; R: Teacher-researcher.]

Protocol 1: October 19, 2011

J: [Handing Austin the purple rod] Two fourths.
A: [Immediately picks up the black 7-rod and lines two purple 4-rods beneath it. He then tosses the black back into the box and pulls out the brown 8-rod. He lines two purples beneath the brown, and then puts it back as well. He pauses for two seconds, clenching his teeth and looking up, as if confused] Two-fourths?
R: Yeah, she said that the purple one was two-fourths.
A: [Gasps. Picks out the orange 10-rod and lines up two purple 4-rods beneath it from left to right, leaving a small gap on the right hand side] Nope. [He then lines up two purples beneath the blue 9-rod] Uh, no, wait. [He then gets the brown 8-rod back out it in his left hand and the blue in his right hand and moves them up in down as if deciding between the two. He then indicates that the brown 8-rod is his answer.]
J: [Three second pause] Are you sure?
A: [Slowly grabs the piece back] I'm not sure [Everyone laughs].

R: Why are you confused, can you explain why you're confused between those [the blue 9-rod and the brown 8-rod held in Austin's hands.]
A: She said two-fourths, and that's equal [lines up the two purples beneath the brown] and that's two and two.

The brown 8-rod was a whole for which the purple 4-rod was a unit fraction, but the purple was not a fourth of that whole. Thus for Austin, choosing the brown 8-rod as the whole required overlooking that there had been only two iterations despite the purple rod's name as two-*fourths*. It is unclear why Austin thought the blue 9-rod might be the whole, but one possibility could involve his use of a PUFS within a complement strategy: the blue is the length of two purple rods, with one-fourth of a purple rod remaining. The following week, there was additional evidence of his use of a complement strategy while solving a similar task. The researcher proposed the task to both students, who worked independently before sharing their results.

Protocol 2: October 26, 2011

R: This [teal 6-rod] is three-fourths of a whole I have in my pocket. Show me what the whole stick looks like.
A: You say three-fourths?
R: Yeah, show me what the whole stick looks like. [Puts a teal piece in front of Jane as well, so that she can concurrently but independently perform the task.]
A: She has more teal than me [in her box of fraction rods].
R: I'll even it out [puts two more teal 6-rods in Austin's box of fraction rods].
A: [Takes out all three teal 6-rods and lines them up end to end. Simultaneously puts his hands at either end of the long result as if to measure the total length. Puts two teal 6-rods back. Gets out a black 7-rod and lines it up beneath the teal rod so that the left sides are aligned. He holds them up together in a single hand, aligning first the left side, and then the right side of the two bars together. He then puts them down and gets out a brown 8-rod and lines it up above the teal 6-rod, so that the left sides are aligned. After looking at the rods for one second, he indicates the brown rod is his response.]

Austin's initial reaction to the task was to treat the composite fraction as if it were a unit fraction; he abandoned this after the result was much longer than any of the available fraction rods. The researcher asks Austin to prepare to present his solution while Jane continues to find a solution. Austin pulls out a blue 9-rod and a black 7-rod again and compares them with the teal 6-rod. He lines up a single tan 1-rod next to the teal rod as if to see what is missing to make the black rod from the teal rod. He then lines up two tan rods next to the teal as if to see what is missing to make the brown 8-rod from the teal. The fact that he immediately found the amount to append onto the teal rod to make the black rod and the amount to append onto the teal rod to make the brown rod indicate that these may have been the actions he was doing mentally beforehand. Thus, Austin chose the brown rod as his response because the complement of the brown rod and the teal rod was one-fourth of the brown rod.

Austin then watches as Jane lines up three red 2-rods beneath the teal and four reds beneath the brown. He appears to be concerned that he might have made a mistake because he hadn't used the red 2-rod—he gets out three reds and compares them to the size of the teal 6-rod before the exchange below.

A: My presentation is, this [the brown 8-rod] is not the right piece.
R: Ok, remind me what the question is.
A: What is three-fourths of this [teal 6-rod]?
R: I said that's three-fourths of the whole.
A: Oh, the whole thing [emphasis on the word whole]. [He then gets out a blue 9-rod, appends three tan 1-rods to the teal 6-rod to make the blue, which he holds up as the answer.] I changed it.

When Austin changed his answer from the brown rod to the blue rod, it was immediately after he noticed that the red rod fit into the teal rod three times. It appears that although he had created the correct whole with a complement strategy, he had not previously conceived of the teal rod as three iterations of the complement of the brown rod and the teal rod.

Discussion

We have described how the cognitive conflict Austin faced when constructing composite fractions reflects the incompatibility of his PWS with his newly formed PUFS. Using a PUFS, students can conceive of the whole as a unit and of a unit fraction such as $1/n$ as a unit measure. Thus, the complement strategy allows such students to conceive of a fraction such as $(n–1)/n$ as the result of action between two units. However, as seen with Austin, the structure of the result of the strategy does not necessarily retain the multiplicative relationship with the unit fraction. That is, although 1 and $1/n$ might be understood via the relation that n iterations of $1/n$ make the whole, and $(n–1)/n$ might be understood as the amount resulting from taking away $1/n$ from the whole, $(n–1)/n$ might not retain the structure of $n–1$ iterations of $1/n$. From our field notes, it appeared that, for students who had already developed a PFS, the result from the use of a complement strategy *did* retain this structure and it was simply a shortcut. However, is unclear whether the use of the complement strategy might interfere with the construction of more advanced schemes, such as an iterative fraction scheme, which is necessary for an understanding of improper fractions (Steffe & Olive, 2010). Future teaching experiments might address limitations of our study, which include the short time frame, close examination of only a single student who utilized the strategy, and exclusive use of a linear representation of fractions.

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THE AFFORDANCES AND CONSTRAINTS IN IMPLEMENTING TECHNOLOGY-ENHANCED LESSONS IN ELEMENTARY MATHEMATICS CLASSES

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This study evaluates the affordances and constraints that elementary teachers in grades 3-5 experience when attempting to implement technology-enhanced activities designed to help students learn elementary number and early algebra concepts. We make use of a new theoretical model of didactic interactions in the presence of technology: the Didactic Tetrahedron, and the notion of teachers' Zone of Free Movement (ZFM) with respect to the implementation of technology. Video records of sequences of three lessons in each of three classrooms implementing Dynamic Number (DN) activities using DN fraction tools, interviews with the teachers and interviews with pairs of students from each classroom working with the DN tools provide the data for establishing the teachers' ZFM and the nature of the interactions among teacher, student, task and technology (the vertices of the didactic tetrahedron).

Keywords: Elementary School Education; Instructional Activities and Practices; Number Concepts and Operations; Technology

Objectives and Purposes of the Study

Key Technologies (developers of *Dynamic Geometry* and *Dynamic Statistics* software) have been awarded a research and development grant from the National Science Foundation to develop *Dynamic Number* (DN) tools for students and teachers in elementary and middle school (Scher & Rasmussen, 2009). Our role in the project is to evaluate the use of the DN tools and curricula activities by both teachers and students, and assess their potential for helping students learn elementary number and early algebra concepts. In working with the Dynamic Number project, the main focus, so far, has been on the mathematical knowledge of the students involved in the project and how Dynamic Number has promoted their mathematical thinking with number concepts (Olive, 2011). While the project is in its third year, it has come to our attention that another component, relating to the teachers, is contributing to both the successes and difficulties with the software. This realization has led us to focus on the teachers and to collect data that will allow us to understand the teachers' point of view in implementing DN lessons in the classroom.

We are focusing on both the affordances and constraints the teachers' experience when using the technology-based activities in their elementary classrooms. The site for this study is a small rural elementary school (grades 3–5) in Southeast United States. It has a student body that is about 80% white with over 50% on free or reduced lunch. The teachers have little access to technology with only one computer lab for the entire school. Although each classroom has some technology, they vary with both the amount and the availability of technology such as interactive white boards and tablets, which creates a unique situation for each teacher as far as planning and implementing technology-enhanced lessons. This diversity has allowed the researchers to evaluate the software-based activities from multiple levels and gives a more comprehensive overview of how both the teachers and the students interact with the software tools in their classrooms and how the tools impact the mathematical understanding of the students as well as alter the instructional methods of the teachers as they use Dynamic Number activities in their classrooms.

Theoretical Framework

The theoretical framework for this study needs to take into account the role of the teacher (or more experienced other) in the didactical situations made possible by the integration of technology. Olive and Makar (2009) put forward a new tetrahedral model derived from Steinbring's (2005) didactic triangle (see

Figure 1) that integrates aspects of instrumentation theory (Verillon & Rabardel, 1995) and the notion of semiotic mediation (Vygotsky, 1978; Saenz-Ludlow & Presmeg, 2006). Olive's and Makar's new tetrahedral model illustrates how interactions among the didactical variables: student, teacher, task and technology (that form the vertices of the tetrahedron) create a space within which new mathematical knowledge and practices may emerge. Olive and Makar state "we place the student at the top of this tetrahedron as, from a constructivist point of view, the student is the one who has to construct the new knowledge and develop the new practices, supported by teacher, task and technology" (p. 168). In this study, however, our main focus will be on the vertex representing the *teacher* and the interactions represented by the edges connecting the teacher to *student, task* and *technology*.

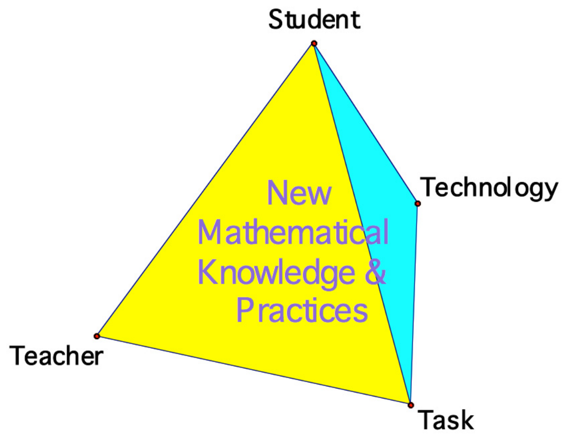

Figure 1: The didactical tetrahedron (from Olive & Makar, 2009, p. 169)

In analyzing these interactions represented by the three edges connected to the *Teacher* vertex, we make use of Valsiner's (1987) Zone of Free Movement (ZFM) which includes access to hardware, software and laboratories, access to teaching materials, support from colleagues, curriculum and assessment requirements, and students' attitudes and abilities.

Modes of Inquiry

The participants in the study are four teachers at a rural elementary school in the southeast, representing grades three through five. Three of the teachers are classroom teachers while one is a support teacher who both works with students needing extra help within the math classroom, as well as pulling out students for small group mathematics instruction. The majority of our data for this paper was collected during the implementation of the lessons involving the Dynamic Number fraction activities, along with semi-structured (Roulston, 2010) interviews of the teachers and interviews with pairs of students interacting with the software. We have been working with the teachers to choose a sequence of lessons to support their instruction on fractions. Working as a group, we chose specific Dynamic Number curriculum activities and then tailored those activities to the teacher's specified grade level to employ in the classroom. The lessons were taught in consecutive weeks so as to be able to capture the arc of the students' understanding of fractions and how the sequence of activities might promote the learning of fractions as well as being able to follow the teacher in their planning and implementation of the lessons with the use of Dynamic Number.

All lessons involving this sequence of activities were videotaped. Evaluation of the implementation of the tasks is ongoing, to ensure that the critical issues from the research perspective are being addressed. This evaluation component utilizes artifacts from the implementation including, but not limited to, videotaped segments and student work, as well as on-site observations and interviews with students and teachers. Audio-recorded interviews have been completed with each of the four teachers using a semi-structured interview guide in order to assess the Dynamic Number project from their perspective. We have also interviewed pairs of students from each class in order to gain the students' perspective. The student interviews were videotaped using the *Screenflow*® application to capture all of their work with the DN tools as the pair worked on related fractions tasks using a laptop computer. The students' interactions with one another and with the interviewer/researcher were captured using a video camera that feeds into the *Screenflow* recording using a picture-in-picture format.

Results

Currently the project is in its final stages with both the videotaping of the third and fourth grade teachers lessons as well as the interviews being completed. The initial findings from the interviews indicate both positive and negative aspects of the use of DN tools in the classroom. The constraints of the software include some activities being above the grade level of the students, as well as not being user friendly and time consuming. The affordances of the software include being a great dynamic visual representation of fractions for the students and the use of Dynamic Number tasks as a teaching tool to deepen the students understanding of fractions. The interviews with the teachers revealed that the DN tools for representing fractions were the only programs that they have worked with that allowed their students to manipulate and create their own fractions, and the teachers believe use of these tools aided students' learning of equivalent fractions, being able to compare fractions to one and adding fractions with a common denominator.

The third-grade teacher used a *Bluetooth* slate in order to give students access to the DN tools while working with the whole class on a projected screen, using one computer. The first class lesson used an activity focused on creating equivalent fractions, given a specific fraction using the DN *Fraction-Rectangle* tool. The second lesson focused on creating combinations of fractions with a given denominator that would sum to one whole using the DN *Fraction Number-line* tool. While the third grade teacher had positive experiences with the equivalent fractions lesson, stating "I think it went great, the kids took it far beyond what I thought they would, and I think they grasped the concept of equivalent fractions a lot faster with the program," she had thought some of the activities were not suitable for students at a third grade level saying that "it would have to be dead on with my standards and with my curriculum and have some training provided for my students." She felt that major modifications of the activities had to be made in order for her third graders to be able to complete them successfully on their own. In relation to this, she often pointed out that she had to spend too much time training her students on being able to manipulate the program for the minimal amount of time that the activities were used in class. The fourth grade teachers had similar sentiments and while they were able to successfully implement the two DN fraction activities in a small group setting, they also had concerns with a majority of the activities being above the grade level of their students.

Discussion

It is the hope that through a more detailed analysis of the lessons and interviews with the teachers, we will be able to delve more deeply into the affordances and constraints the teachers experience when implementing technology-based lessons at the elementary level. Since most of the focus has been on the students' interactions with the software, it is our plan to use the teachers' observations to help further evaluate the effectiveness of the DN software as a technology tool to aid instruction in the elementary classroom. The ability to work with teachers' at multiple grade levels with different students and activities provides a range of data on which to evaluate the potential of the DN tools and activities for enhancing the teaching and learning of fraction concepts and operations.

Our preliminary results do suggest that the potential of the DN tools is directly affected by the teacher's ZFM. Being restricted to using the tools with one computer and projector with the whole class limits what even an expert teacher can accomplish with these tools. The potential could be more likely realized if students could work in smaller groups to be able to better explore with the tools at their own pace; also, if the program allowed for more adaptation by the teachers to create activities more targeted to the grade level of their students. On a more positive note, all the teachers had positive comments on how the interactive visual models of Dynamic Number helped their students be able to connect to the mathematics. They were pleased with how the students were able to represent fractions, and compare side by side the fractions that they were working with. This allowed the students to have direct visuals of the fractions, and be able to compare if they were equivalent, greater than one, less than one, etc., which were all components of the lessons the teachers implemented. In its current form, however, there is a need for ongoing support for the teachers in order to address the limitations of using the DN software in their instruction. A more robust, child-friendly user-interface, with easy-to-use design features for the teacher, could enlarge teachers' *Zone of Free Movement* with respect to implementation.

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MATHEMATICAL REASONING AND TECHNOLOGY INTEGRATION: A STUDY OF PRE-SERVICE SECONDARY MATHEMATICS TEACHERS

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It is important to explore pre-service secondary mathematics teachers' (PSMT) mathematical reasoning as it can influence their beliefs and future instructional practices. A cross-case analysis was conducted with PSMT's to better understand their ability to reason about mathematics and integrate relevant technology. By using the Mathematical Task Framework developed by Stein and Smith (1998), preliminary data analysis indicated that the mathematical task reasoning is lower than desired for preparation to teach certain concepts.

Keywords: Reasoning and Proof; Teacher Education–Preservice; Technology

Purposes

The National Council of Teachers of Mathematics (NCTM, 2000) identifies technology as an important principle for high quality mathematics education because it has the potential to enhance mathematics learning, support effective mathematics teaching, and influence what mathematics is taught. According to their view, technology can promote student understanding through use of multiple representations and experimentation (NCTM, 2000). When technology is used in this way, it has the potential to promote students' mathematical reasoning. There have been some studies to analyze pre-service mathematics teachers' mathematical reasoning with and without technology enhanced environments (Zembat, 2006; Christou et al., 2004). These studies were mostly conducted with elementary pre-service mathematics teachers. Studies investigating the quality of pre-service secondary mathematics teacher's (PSMT) mathematical reasoning with and without technology integration for pre-calculus or calculus subjects are limited in number (Ball, 1990; Simon & Blume, 1994; Zazkis & Campbell, 1996; Pandiscio, 2002).

To contribute to our currently limited understanding of the connections between technology and PSMTs' mathematical reasoning, this study addresses the research question "What is the level of reasoning that each PSMT exhibited in each mathematical task, with and without the use of technology?"

Mathematical Reasoning and Conceptual Understanding of PSMTs

Many pre-service mathematics teachers do not know why they use specific operations or follow a procedure (Ball, 1990; Simon & Blume, 1994; Lloyd, 2006). For example, Ball (1990) found that both secondary and elementary teachers demonstrated a narrow understanding of division that mostly emphasized remembering a particular rule without a conceptual frame attached. NCTM (2000) identifies combination of factual and procedural knowledge with conceptual understanding and mathematical reasoning as the key principle for students' learning of mathematics. PSMTs are expected to teach reasoning skills, but this lack of conceptual understanding often inhibits teachers' efforts to promote reasoning in their own classrooms (Ball, 1990; Simon & Blume, 1994; Lloyd, 2006).

Beliefs about Technology and Mathematical Reasoning

Another NCTM (2000) expectation from mathematics teachers is the integration of technology into the classroom to favor students' reasoning and sense making. The effectiveness of technology integration for students' learning depends on teachers' beliefs and views about technology. Chen (2011) found that pre-service math teachers viewed technology as an instrument to increase the efficiency of the instruction and educational outcomes rather than as a mediating component of students' learning. Chen claims that the different views of technology, either instrumental or substantive view, would determine the quality of reasoning and sense making activities during instruction.

Zembat (2006) found that dynamic geometry environment facilitated PSMTs' reasoning to deal with new situations and applications, whereas paper and pencil environment only allowed a type of reasoning on formulas and procedures to abstractly solve mathematics problems. Contrary to Zembat's findings, Pandiscio's study (2002) showed that technology inclusion did not amplify quality of teachers' reasoning on proofs in geometry.

Cognitive Load Theory (CLT) is an instructional theory that is grounded in the idea that working memory is limited with respect to the amount of information it can hold and the number of operations it can perform on that information (van Gerven et. al., 2003). Cognitive load is comprised of intrinsic load (the complexity of the information or task given), extraneous load (the techniques in which the information is presented), and germane load (the complex schema helping the learner move from novice to expert). One implication of CLT is that minimizing extraneous cognitive load will maximize the germane load (Chipperfield, 2006). This study is based on the hypothesis that the use of technological tools will reduce extraneous load, increasing germane load, and thus increasing the level of reasoning used by the participant.

Theoretical Framework

Stein and Smith's (1998) Mathematical Task Framework identifies three phases through which mathematical tasks pass, with four possible levels of reasoning in each phase. The three phases are (1) tasks as they appear in instructional materials, (2) tasks as set up and presented by teachers, and (3) tasks as implemented by students. All three phases are influential on what students learn. Due to the nature of this study, which did not involve direct instruction, only the first and third phases were addressed. Within each phase, the four possible levels of reasoning from lowest to highest are (i) memorization (reproduction of facts, formulas, or definitions with no connection to concepts or meanings); (ii) procedures without connections (algorithmic thinking without developing mathematical understanding); (iii) procedures with connections (algorithmic thinking with an emphasis on developing mathematical understanding); and (iv) doing mathematics (complex non-algorithmic thinking that demonstrates understanding of mathematical concepts, processes and connections) (Stein & Smith, 1998). To assess the potential of PSMT's ability to teach with mathematical reasoning, we seek to determine which level of reasoning they have exhibited and whether they can make strong arguments for how and why they solved various mathematical tasks of varying difficulty. Additionally, we will utilize the same framework to determine if their level of reasoning is altered by the use of a technological tool in completing similar tasks.

Methods

The participants for this study were secondary mathematics preservice teachers enrolled in a senior level methods course during the fall of 2011 at a large research institution in the southeastern United States. Data were collected from September to December of 2011. Data consisted of one semi-structured, one-on-one interview (approximately 20 minutes in length), one task-based interview (involving students solving mathematics with transcriptions of their verbal reasoning), and course artifacts (e.g. student reflections and lesson plans). The primary source of data used in this analysis was the task protocol with other data sources used to support findings. In this mathematical task interview, each participant was presented with four mathematical tasks, two tasks to be completed with only pencil and paper and two tasks to be completed with technology tools. Technology tools were chosen at the discretion of the participants from those provided by the interviewer, including a Casio® graphing calculator, Microsoft Mathematics 4.0®, SmartBoard Math Tools®, and Geogebra®. The set of tasks to be completed with technology tools were the same as the tasks in the set to be completed without technology tools, with the use of decimals in the tasks to be completed with technology tools. Therefore, Task 1 was similar to Task 3, and Task 2 was similar to Task 4. The level of mathematical reasoning was analyzed by all three researchers, using the four-level scale adapted from the Mathematical Task Framework of Stein and Smith

(1998). A cross-case analysis was conducted where each case was first considered individually (Yin, 2009). Reliability was ensured by triangulation of the multiple sources of data.

Results

Preliminary analysis indicate differences in how PSMT's reasoned through mathematical tasks with and without technology. Task 1 and Task 3 called for solving a 2-variable system of equations. When using only pencil and paper, All participants solved Task 1 with a level 2 reasoning involving procedures without connections, but when using a technology tool to solve the same problem in Task 3, four of the seven participants increased to a reasoning level 3 (procedures with connections) by graphing the equations and determining where they intersected for the solution. Task 2 and Task 4 involved determining the length of an h" square cut from each corner of a 4" × 6" rectangle to maximize the volume of a box that would be formed if the sides were folded up to form an open-top box. Completion of this problem in Task 2 without technology involved four of the participants determining the equation, finding the first derivative, and solving for the roots, while three participants used a guess-and-check method. The same problem was presented in Task 4 with more difficult dimensions and allowed the use of an applet in Geogebra® in addition to the other technology tools. Four of the participants maintained the same level of reasoning as in Task 2, while one participant decreased the level of reasoning and two participants increased their level of reasoning (see Table 1).

Table 1: Preliminary Reasoning Analysis

	Without Technology Tools		With Technology Tools			
	Task 1	Task 2	Task 3	Tool(s)	Task 4	Tool(s)
Participant 1	2	3	3	Casio® GC	3	Geogebra® applet
Participant 2	2	2	2	Casio® GC	3	Geogebra® applet
Participant 3	2	3	3	Casio® GC	2	Geogebra® applet & Casio® GC
Participant 4	2	3	2	Casio® GC	3	Casio® GC
Participant 5	2	2	3	Microsoft Math 4®	3	Geogebra® applet
Participant 6	2	3	3	Casio® GC	3	Geogebra® applet
Participant 7	2	2	2	Casio® GC	2	Geogebra® applet

Transcriptions of the task interviews revealed some reasons why PSMTs may not use a higher level of reasoning when utilizing technology to complete mathematical tasks. Participant 7 enthusiastically claims that technology should be used to "…enhance learning, not as a crutch." However, when asked about why the technology was an advantage for completing Tasks 3 and 4, this participant admitted "…the advantage was due to lack of my math skills." This was evidenced in how this participant completed Tasks 1 and 3, both problems asked to solve a 2-variable system of equations, in the same way with the calculator as with pencil and paper.

Preliminary results also indicated that PSMTs do not feel comfortable integrating a variety of technology tools with instruction. Participant 3 indicated a comfort with graphing calculators, but had only experienced the other tools during that semester. Participant 7 affirmed the same limited focus on technology integration due to "…lack of experience and competency…" with technology tools other than graphing calculators.

Discussion

Visions for mathematics instruction, such as encouraging students' reasoning, their construction of knowledge, and utilizing their understanding with technology integration, are expected from teachers with the emergence of curriculum standards (Frykholm, 1996). Initial findings from this study indicate that the level of reasoning PSTM's are using with and without technology is not ideal in actualizing these reform practices. This has implications for teacher education programs and how these programs can help provide opportunities to use higher level reasoning skills both with and without technology.

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PROSPECTIVE HIGH SCHOOL MATHEMATICS TEACHERS' DESIGN AND IMPLEMENTATION OF DYNAMIC GEOMETRY TASKS

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The purpose of this study was to examine the ways in which prospective teachers designed and implemented technology-based tasks with middle school students enrolled in a high school geometry course. Teachers designed geometry tasks that varied in scope and posed questions that focused students on features of the technology and mathematics in different ways. Analysis of six prospective teachers' implementation of their tasks is provided.

Keywords: Technology; Geometry and Geometrical and Spatial Thinking; Teacher Education–Preservice

Introduction

When preparing teachers to use technology to teach, it is imperative that they understand the specifics of the technology tools they are using. Even more important, teachers need to know how to design and implement mathematical tasks that support students' learning while using these tools. Dick (2011) states, "the value of technology to the teacher lies not so much in the answers it provides, but rather to the questions it affords" (p. 2). The importance for teachers to select and implement worthwhile mathematical tasks to engage all students in learning mathematics has also been emphasized as a significant pedagogical activity by the National Council of Teachers of Mathematics (NCTM, 1991) and shown by researchers to be a critical component of effective mathematics classrooms (Stein, Smith, Henningsen, & Silver, 2000). Some studies have examined how teachers design tasks for students that utilize technology (e.g., Laborde, 2002) or implement technology-based tasks with students (Hollebrands & Heid, 2005; Lee, 2005). And while there is research that has focused on the implementation of non-technology based tasks in mathematics classrooms (e.g., Stein, Grover, & Henningsen, 1996), few studies have considered how teachers design and implement mathematical tasks with students while using technology. The purpose of this study is to examine how prospective mathematics teachers (PSTs) design and implement geometry tasks using *The Geometer's Sketchpad* (GSP) with middle school students enrolled in a high school geometry course.

Background

When dynamic geometry programs were developed and introduced more than two decades ago there was great excitement about and enthusiasm for their uses with students. In particular, educators were encouraged about how students' uses of these tools could support students' movement from a focus on empirical drawings to theoretical generalizations (Cuoco & Goldenberg, 1997; Laborde & Laborde, 1995). However, teachers' adoption and use of this technology in high school classrooms has been slow (e.g., Becker, 2000, Kasten & Sinclair, 2009). This is perhaps related to the lack of time teachers perceive that they have to teach both content and technology skills. It may also be related to the need for teachers to redesign mathematical tasks and lessons to incorporate the use of this tool.

Designing tasks that effectively use technology tools requires teachers to carefully consider mathematics, learning, and teaching. In a three-year study of teachers learning to design tasks to incorporate dynamic geometry technology, Laborde (2002) found that several teachers initially designed tasks for which the technology served as a visual aid or used the tool to do similar non-technology based activities in a more efficient manner. However, over time teachers began to develop tasks that could only exist in the technology-based environment. Transformations of teachers' views about technology-based tasks were mediated by their knowledge of the tool, their knowledge of mathematics, and their apprehensions about what students were learning.

While the design of the task is a critical component of the learning process, its implementation by teachers is also important. How teachers introduce the task and technology, pose questions, and offer assistance to students as they are working can impact what is learned. In particular, how teachers and technology tools focus students' attention on particular aspects of mathematics can influence what students notice and learn from these interactions (Lobato & Ellis, 2002; Lobato, Ellis, & Munoz, 2003). Lobato and Ellis (2002) define the focusing effect of technology as "the regular direction of users attention toward certain subject matter properties over others" (p. 299) and expands this to consider the focusing phenomena, which relates to how other aspects of a classroom environment (e.g., teachers, curricula, tools) "direct attention toward certain mathematical properties over others" (Lobato, Ellis, & Munoz, 2003, p. 1).

In an instructional setting, questions can play an important role in focusing students' attention. Mason (1998) identified three types of questions: questions that test information, questions that seek information, and questions that suggest a particular focus. Questions that test information are particular to classrooms. The teacher poses a question to test what students know (What is the definition of a square?). Questions that seek information are common in our every day experiences (Where is the nearest gas station?). When teachers are implementing technology-based tasks they ask all three types of questions to which Mason refers. And as Lobato and her colleagues suggest, how these questions focus students influences what they learn. The research question that guided this study was: In what ways do prospective teachers focus students on technological features and geometric objects and relationships as they implement the task with students? This paper will focus on the second question.

Methods

Participants were paired and assigned to design, share, and implement a thirty-minute, exploratory task using *The Geometer's Sketchpad* (GSP). PSTs implemented their tasks individually with a middle school student. Following a case study design (Patton, 2002), six PSTs, representing different subgroups in the mathematics education program, were selected for closer analysis (Jessica and Matthew, Audrey and Victoria, Delta and Jonah). For these three pairs, the teaching session was videotaped. There were three components to the task: (1) a pre-constructed GSP file; (2) a Word document posing the task for the middle school student; and (3) a Word document describing anticipated responses, including a list of questions to guide students, and a description of how the task could be differentiated.

To analyze the ways in which teachers were designing and implementing their tasks, we considered how their questions (written and verbal) focused students. Codes were generated to analyze the focus of the questions, prompts, and statements exchanged between the PST and student while working on the task. The direction of the conversation (student to teacher or teacher to student) was also considered.

The data were analyzed using the qualitative data analysis tool, *Transana* (The Board of Regents of the University of Wisconsin System, 2010). The researchers coded a portion of a single transcript linked to the video recording and discussed differences in coding until a consensus was reached. Examples and definitions of codes were developed and refined during this process. This was repeated until there was a high level of agreement among the coders. Then each researcher analyzed one to two video recordings and met with another researcher to discuss commonalities and differences in the ways the PSTs implemented the same task. Codes were analyzed to identify trends and themes in questions and statements made by PSTs and students.

Conclusions and Summary

This study was designed to examine the ways in which prospective teachers designed and implemented geometry tasks using *The Geometer's Sketchpad* (GSP) with middle school students enrolled in a high school geometry course. For three pairs of PSTs, we identified similarities and differences in how their tasks were designed and implemented with students.

The mathematical goals for each of the three tasks were similar: discover a theorem, develop definitions, and describe geometrical relationships. Two of the pairs wanted the student to discover a theorem. In one case (Jessica and Matthew), the theorem was fairly obvious and did not require much investigation on the part of the student. In the other case, the pair of PSTs (Delta and Jonah) wanted students to observe that the hypotenuse of an inscribed right triangle is also the diameter of a circle. Unless the students had previously studied theorems related to circles, it was probably novel and perhaps also interesting. Audrey and Victoria chose to provide students with examples of different triangle centers and through students' interactions with these sketches develop definitions of the centers, identify relationships, and consider similarities and differences among them. In general, the PSTs did not push students to explain why a relationship might be true. Students were simply asked to record what they observed. All three of these tasks seemed to go beyond simply "amplifying" what they might also do on paper (Laborde, 2002). The tasks required students to construct or drag and observe measurements or other invariants; however, how these interactions were encouraged during task implementation varied.

The participants were paired when they designed their tasks and then they implemented individually with different students. For two of the pairs (Jessica and Matthew and Audrey and Victoria) there were similarities in how their questions focused students on mathematics and technology. However, there were striking differences in the way Delta and Jonah implemented their tasks. Because this task required students to construct, and their plan involved different accommodations for supporting students, their decisions about how to guide students in their technological work had profound effects on their interactions and they ways in which they focused students on mathematics and/or technology. The differences in their approaches are interesting and the consequences of the two approaches in terms of what each student learned are unclear. This could be a question that is considered in future research.

There also seemed to be a difference in the type of task and sketch that was designed and how consistently the task designers were able to implement the task with their student. Jessica and Matthew and Audrey and Victoria were working with closed and open-middled tasks, respectively, that were well coordinated with the pre-constructed sketches. These pairs of PSTs were very similar in their implementation of the task and use of GSP with their students. However, Delta and Jonah, worked with an open-ended task that required much construction within GSP and task questions that did not explicitly prompt for measurements or dragging actions, differed dramatically in their implementation.

Although these cases are not enough to make a strong statement, we wonder if it is more straightforward for new teachers, or teachers new to using technology, to implement a task using a pre-constructed sketch that aligns more carefully with Sinclair's (2003) design principles. In particular, it may require more effort in task and sketch design to have a strong connection between task statements that focus a student's attention or prompt for actions in GSP with elements made available in the pre-constructed sketch to draw students' attention and enable such actions. However, such effort might make implementations of the task more consistent across teachers.

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PRIMARY SCHOOL TEACHERS' DISPOSITIONS TOWARD WEB-BASED INSTRUCTION IN MATHEMATICS

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This study investigated 12 primary school teachers' dispositions toward web-based instruction. In terms of cognitive dispositions, more conceptual than procedural or strategic lessons were planned to be used. Still, compared to traditional classroom instruction, less conceptual and strategic activities were designed for web-based instruction. In terms of affective dispositions, the teachers believed that students can benefit from web-based instruction because of its instant feedback and motivating environment. Most of them felt confident and comfortable to conduct web-based instruction. However, most of them believed that web-based instruction can only be used to reinforce conceptual knowledge taught in a traditional classroom setting.

Keywords: Technology; Cognition; Beliefs and Attitudes; Number Concepts and Operations

Introduction

The applications of technology hold profound potential in mathematics teaching and learning (NCTM, 2000). Although the use of web-based instruction (WBI) is a relatively new phenomenon of technology applications, a number of studies have been carried out to determine the potential of WBI for enhancing mathematics teaching and learning in a range of contexts. While teachers have increasing opportunities to teach by utilizing web-based resources, their dispositions regarding the use of WBI in the mathematics classroom should become an important research topic for researchers and educators.

The purpose of this study is to investigate 12 U.S. primary schools teachers' dispositions toward WBI in a mathematics classroom. Teachers with productive dispositions show "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" (National Research Council, 2001, p. 116). Teacher dispositions continue to be essential elements of teacher preparation and teacher quality. In this study, WBI is a teaching setting in which teachers use web-based resources to teach mathematics in a classroom. Traditional classroom instruction (TCI) is contrasted as a teaching setting in which no websites or related resources are applied in a classroom. The significance of this study is to provide useful information that enables administrators and teacher educators to better understand teachers' dispositions toward WBI in mathematics. Consequently, a better focused professional development course for mathematics teaching and learning could be designed, developed, and implemented.

Theoretical Framework and Literature Background

The overall theoretical framework guiding our attempt to explore teachers' dispositions toward using web-based resources and tools to teach students is based on Thornton's perspectives on dispositions (2006). Dispositions are viewed by Thornton as "habits of mind including both cognitive and affective attributes that filter one's knowledge, skills, and beliefs and impact the action one takes in a classroom or professional settings" (p. 62). Drawing from this perspective, we propose to capture not only the affective patterns of teacher beliefs and attitudes toward web-based instruction (WBI) in mathematics, but also the cognitive patterns of knowledge and skills that teachers expect their students to construct through WBI. Based on such a multidimensional perspective on dispositions, this study identified two major themes to facilitate the investigation of teachers' dispositions toward WBI.

The first theme focuses on teachers' cognitive dispositions, addressing the types of mathematics knowledge and skills they expect their students to construct. To advance the measurement of teachers' cognitive dispositions toward WBI, it is important to have an appropriate classification scheme to categorize the types of mathematics knowledge and skills. Based on the National Research Council's (2001) strands of mathematical proficiency and National Center for Education Statistics' mathematics achievement-level descriptions (NCES, 2011), we propose to categorize the mathematics knowledge and skills that expect their students to construct through WBI into three common types: (1) conceptual understanding (CU)—comprehension of mathematical concepts, operations, and relations, (2) procedural fluency (PF)—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately, and (3) strategic competence (SC)—ability to formulate, represent, and solve mathematical problems by integrating conceptual and procedural knowledge.

The second theme focuses on teachers' affective dispositions, addressing their beliefs regarding WBI. Among relevant studies (e.g., Kao & Tsai, 2009; Lin, 2008), teachers' beliefs regarding the benefits of utilizing WBI, the importance of WBI, and their self-efficacy for using WBI are often explored. Lin, for example, investigated whether pre-service teachers think Web-based resources are enjoyable and stimulating, whether they believe it is imperative to utilize WBI, and whether they feel comfortable and confident regarding use of web-based resources in a mathematics classroom. Overall, research has supported positive beliefs on the use of WBI. In addition to the research interests described above, this study is interested in teachers' beliefs about how WBI should be implemented in a mathematics classroom, and in the potential impact of their beliefs on their actions.

Research Questions

The two themes identified above were used to generate the following two clusters of research questions. They are:

1. What are the types of web-based mathematics knowledge and skills that teachers expect their students to construct? What are the differences or similarities between web-based and traditional instruction in terms of the types of mathematics knowledge and skills that teachers expect their students to construct?
2. What are teachers' beliefs regarding the benefits of utilizing web-based instruction (WBI)? What are teachers' perceptions of the importance of WBI? What are teachers' self-efficacy beliefs regarding WBI?

Methodology

Participants

Participants included 12 in-service primary school teachers in southeastern United States. Since insightful perspectives are more likely to be identified from the teachers who have sufficient understanding of the nature and use of web-based instruction (WBI), the teachers were provided four professional development hours regarding how to use web-based resources and tools to deliver instruction and assessment before collecting data. In addition, the teachers were asked to review research and teaching literature to gain sufficient understanding about WBI.

Data Collection and Analysis

To add depth to our descriptions about primary mathematics teachers' dispositions toward WBI, this study collected and analyzed qualitative data in relation to two themes: (1) teachers' cognitive dispositions addressing the types of mathematics knowledge and skills, and (2) teachers' affective dispositions addressing their beliefs regarding WBI.

To enhance the validity, this study adopted the process of triangulation by which multiple methods were used to collect data (Gall et al., 1996). Data were collected through the following methods: First, data were collected from the participating teachers' web-based and traditional instructional plans. The data drawn from teachers' traditional instructional plans were used as a comparative referent to better unpack teachers' tendencies in selecting knowledge and skill types for a WBI setting. Second, data were collected from a survey with several open-end questions regarding the teachers' beliefs about the benefits of utilizing WBI, their perceptions of the importance of WBI, their self-efficacy beliefs regarding WBI, and their beliefs about how WBI should be implemented. Lastly, data were collected from the teachers' discourse which reflected their own overall dispositions toward both WBI and TCI.

Results

As discussed previously, we collected data related to two themes: (1) types of mathematics knowledge and skills in a web-based instruction (WBI) setting and traditional classroom setting (TCI), and (2) teacher beliefs toward WBI. The results below reflect each of the themes.

Types of Mathematics Knowledge and Skills in a Web-Based Instruction Setting

The participating primary teachers selected either addition or multiplication as their lesson topics. We found half of the participating teachers planned similar types of WBI and traditional classroom instruction (TCI) lessons while half of them did not. We also found that the teachers expected their students to construct more conceptual understanding types of activities (CU) than procedural fluency (PF) or strategic competence (SC) lessons in both the WBI and TCI settings. However, it was found that the teachers planned a more procedural fluency type of lessons in a WBI setting than in a TCI setting, and fewer strategic competence types of lessons in a WBI setting than that in a TCI setting. Further, only one of the 36 web-based lessons developed by teachers can be categorized as a strategic lesson. Generally speaking, compared to TCI, lessons for WBI placed more emphasis on conceptual understanding and procedural fluency than on strategic competence.

Teacher Beliefs toward Web-Based Instruction

The findings for teachers' beliefs regarding WBI are grouped according to predetermined questions:

Benefits from utilizing web-based instruction. Several teachers believed that students would benefit from the motivating learning environment provided by WBI. Others believed that WBI is beneficial because it effectively provides instant feedback. Different from the peers, one of teachers believed the benefits of WBI depend on the student's age. She felt web-based activities are more beneficial to older students.

Role of web-based instruction. As we live in a technological society today, there is no surprise that all of the participating teachers agreed that the use of WBI is essential. However, no matter how impressive WBI is, most of them only considered WBI as a supplemental method for enhancing concepts and strategies developed in a TCI setting. Generally speaking, in their perspective, WBI is less important than TCI.

Self-efficacy beliefs. Most of the teachers felt confident about doing WBI. This study found personal factors such as teaching and workshop experience often contributed to teachers' efficacy beliefs.

Discussion

The findings of this study were limited by a small sample of primary school teachers, and the limited number of tasks used in the investigation. Thus, the research results should be considered as insights rather than generalizations. The results of this study raise several questions. Let us discuss the results from the first theme—teachers' cognitive dispositions toward WBI. It is interesting to investigate why the teachers hesitated to plan strategic lessons in a web-based setting. Is it possible that it is less convenient to engage students in strategic lessons than conceptual or procedural lessons in a web-based setting? Are strategic lessons less available than conceptual and procedural lessons on websites? It is also interesting to know why the teachers have planned less conceptual and strategic lessons, and more procedural lessons in web-based settings. Is it possible that the teachers' cognitive dispositions depend on the type of instructional setting or their affective dispositions toward WBI such as the beliefs that conceptual understanding has to be taught in a traditional classroom setting first? Is it possible that the knowledge or skill types of available web-based teaching resources are different from those in traditional classrooms, or the others? It is interesting to examine these questions in the future.

Next, the results show most of the teachers hold similar affective dispositions toward WBI. Most of them believed that WBI is beneficial not because the nature of knowledge and skills provided by WBI but because of pedagogical features in a web-based setting such as providing instant feedback, entertaining and motivating environment, individual advancement and interactive opportunity. What changes should WBI to be made to make teachers feel beneficial due to cognitive growth as well? Most of them also believed that WBI should play a supplemental role for reinforcing concepts taught in a traditional classroom setting. Will this belief change with the progress of technology used to support WBI? Third, most of them believed they are effective and confident web-based instructor. However, although a lot of descriptive reflections regarding each affective question were provided, it remains unclear that how their affective dispositions can be connected as a triad. It is interesting for future study to explore how teachers' affective dispositions are networked.

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STUDENT PERCEPTIONS ABOUT THE USE OF EDMODO

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A case study methodology was used over an academic quarter to analyze students' perceptions regarding the incorporation of the educational blog Edmodo in their algebra class. Students that did not regularly participate in class were willing to be part of the class community on Edmodo. Edmodo contributed to the students' learning experience in their algebra class.

Keywords: Technology; Middle School Education; Beliefs, Attitudes

Introduction and Background of the Study

A significant amount of research into the use of technology in education exists (Conole, 2010) and much has been written about the potential of blogs to support learning in higher education (MacBride & Luehmann, 2008). However, little of this research is based empirically and even fewer of these studies have been peer-reviewed (MacBride & Luehmann, 2008). Moreover, limited research has occurred investigating middle school students' perceptions about the use of blogs in their learning. Therefore, a next logical step would be to consider the impact of a class blog on a middle school mathematics class. The purpose of this study was to explore the students' perceptions of the contribution of a social learning network, Edmodo, to their learning process.

Prior research has been conducted to illuminate the perspectives of teachers regarding educational blogs. Glass and Spiegelman (2007) discovered that blogs provide a platform from which students learned from each other as they shared their ideas. It has also been concluded that the incorporation of a class blog allows teachers to focus on creating additional forms of participation and that the educational value of blogs depends largely on how teachers choose to structure and use them (MacBride & Luehmann, 2008). Finally, according to Smith, Ferguson, and Caris (2003), communication in web courses is profoundly different than classroom courses, and this phenomenon results in greater student-instructor equality.

As noted above, student-teacher discourse on blogs is different from student-teacher interaction in classroom discussions. Blogs facilitate this beyond the traditional classroom in that they engage students in learning until they have achieved success (Spiegelman & Glass, 2008). Mathews's (2009) findings suggest that students increase their use and understanding of mathematical vocabulary by creating a community of discourse based on mathematics. Online discussions also allow students to get updated information easily in that they can access course materials at their own pace and at their own convenience (Chan & Waugh, 2007).

However, limited research pertaining to the perceptions of students and online learning has been conducted. Makri and Kynigos (2007) advocate that online learning provides a rationale for the change in roles and practices of the participants. These findings are based on the integration and use of a blog, both as a communication medium and as a mechanism for provoking reflection. My study focuses on the student perceptions of using the online social networking and blog site Edmodo.

Theoretical Framework: Social Constructivism

Social constructivism is primarily a theory of learning, in which the teacher's role is mainly to structure or guide the student discussion, as opposed to a theory of teaching (Brophy, 2002;Vygotsky, 1978). Additionally, scholars contend that constructivist learning must incorporate authentic learning tasks (Applefield, Huber, & Moallem, 2001). This study draws on a social constructivist perspective (e.g., Vygotsky, 1978) to frame and interpret this study because this theoretical perspective takes into account the critical role of social interaction and dialogue in student learning. Social constructivism provides a powerful lens for studying the use of blogs in education because a central tenet of this theory is that learners construct their own understanding by participating in meaningful shared discourse (Vygotsky, 1978).

A secondary theory I drew upon maintains that classroom experiences must mirror the complexity of society (Bereiter, 2002; Dede, 2000). In 2007, Pew Internet and American Life Project found that 55% of online teenagers, ages 12-13, are avid users of social networking sites. Because blogging has become such a common means of communication among this age group, the consideration of the educational potential of using blogs, such as Edmodo, in middle school instruction is worthwhile (as cited in Lenhart, 2007).

Context and Methods of the Study

A case study methodology was used over an academic quarter to provide an in-depth analysis of students' perceptions regarding the incorporation of the educational blog Edmodo in their algebra class. This study is an intrinsic case study because it is an in-depth exploration of a bounded system and the case itself is of interest (Creswell, 2002; Stake, 1995). The following question guided this study: what are the perceptions of eighth grade students about the use of Edmodo in their algebra class?

Researcher Role

I began teaching the Advanced Algebra class at this school in the fall semester of 2010. Prior to this, I taught math continually for four years at the local community college, as well as at the university level. I have been active as a private math tutor for over 10 years. Additionally, I taught mathematics to seventh and eighth grade students for one semester while completing my student internship. I have my master's degree in Applied Mathematics and a license to teach secondary mathematics. Currently, I am the advanced algebra teacher at a small private school and a mathematics instructor at the local university. My interest in this research topic stemmed from my personal experience with Edmodo while teaching my eighth grade algebra students.

I was the only math instructor for these students and was the principal and only investigator during this study. As a researcher-participant (Creswell, 2002; Stake, 1995), I instructed the students in their daily assignments and guided them in the use of Edmodo. In my role as investigator, I authored the questions and collected and analyzed the data.

Data Sources and Data Collection Procedures

Data was collected during the fourth quarter of the 2010-2011 school year (March 28, 2011 to May 2, 2011). Primary data sources included online journals, weekly questionnaires, online interviews, and one-on-one interviews (Creswell, 2002). Secondary data sources were lesson plans with memos and student work samples.

Data Analysis

The objective of this investigation was to gain insight from the students directly involved in the case study. The main categorizing strategy that I used was coding the data by organizing it into broader themes (Maxwell, 2005). Every weekend during the ongoing study, all text was systematically searched and organized by emergent themes in an iterative pattern over the research period, following the approach of qualitative data analysis outlined by Bogdan and Biklen (2003).

Discourse analysis was used to study the online blog interviews and discussions. Gee (2011) suggested that discourse analysis could be revealing about the identities, relationships, and actions behind language. The remaining posts were analyzed individually and categorized into emergent themes. Also, online journal entries were analyzed individually and separated into themes.

Furthermore, student work samples and lesson plans were used to help analyze the student perceptions of the blog. The collection of data using a variety of methods and sources is one aspect of triangulation (Maxwell, 2005). I triangulated the data to reduce the risk that the conclusion of the study reflects systemic biases. It also allowed me to gain a broader and more secure understanding of student perceptions about the use of Edmodo.

Results

Immediate Feedback and Multiple Perspectives

Edmodo allowed the participants, my students and me, to communicate online. The blog was an extension of the class in that students could go there to work together and get help from each other. This classroom community created the social setting for learning to occur outside of school. Students remained part of their learning environment, even when they were absent from school. Additionally, every member on Edmodo could share resources, such as homework due dates and test information. These findings coincide with those of Chan and Waugh (2007) in that the blog allowed students to get updated information easily because they could access course materials at their own pace and at their own convenience.

Pose Questions, Answer Questions, and Join Other Conversations

The blog is an open forum in which every member can view the discussion threads. This platform encouraged students to work together on their math homework and, through this communication, students engaged in question-answer discussions. For example, a student said, "On homework, when you forget the first step of the problem you can ask people on Edmodo." Moreover, student discussions informed my instruction on the following day. Thus, Edmodo created a student centered learning environment in my class.

Enrichment Opportunity

Students viewed the blog as an enrichment opportunity. For instance, one student commented, "My teacher uses Edmodo to post practice problems." Another student responded, "It's really easy to post stuff and send you messages and also find assignments and polls and all that stuff on the side." Others suggested that Edmodo be used for extra credit opportunities such as having them solve challenging math problems on the discussion board or by creating math-centered online games. Additionally, many students requested a chat application so that they could chat privately, as opposed to using the open discussion forum. Some students suggested that Edmodo allow them to see those blog members that are on the site and those that are not.

Conclusion

Multi-level Communication and Extended Learning Environment Beyond the Classroom

Edmodo not only allowed students to communicate with each other and me outside of class, but also gave them access to course materials and information. Moreover, I was able to assess the students' understanding of the algebra concepts by reading and analyzing their discussions on Edmodo. Additionally, some students that were not comfortable speaking or asking questions in class would pose their questions to me, or to other students in the class, using the blog. These students did not regularly participate in class, but were willing to be involved and to be part of the class community on Edmodo. This may imply that this social network site helped equalize the balance of power among students and their teacher similar to the findings of Smith et al. (2003).

This study was based on the theoretical framework of social constructivism (Vygotsky, 1978). Further research on Edmodo could include the theory of connectivism (Siemens, 2005) to show how learning is a network-forming process. In addition, further research needs to be conducted on how teachers implement the use of blogs to teach mathematics. Furthermore, the apparent differences between teacher implications and actual student uses of educational blogging sites might be of interest, especially with the emergent trend of the integration of technology in all levels of education.

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SUPPORTING STUDENTS' CONSTRUCTIONS OF FRACTIONS SCHEMES: IS THERE AN APP FOR THAT, TOO?

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We share results from a pilot study that investigated the effectiveness of the Candy Factory app in supporting the construction of new fractions schemes. Pre- and post-tests among three sixth-grade classes (two experimental and one control) indicate that substantial growth can be supported through game play, but that app design must account for students' existing ways of operating, as well as additional factors affecting engagement.

Keywords: Learning Trajectories; Middle School Education; Rational Numbers; Technology

Candy Factory is a freely available app for iOS devices, designed by the Learning Transformation Research Group at Virginia Tech (LTRG; see http://ltrg.centers.vt.edu/). The app is designed to elicit the coordination of partitioning and iterating operations, as students produce candy bars of specified sizes, relative to a given whole. Given a whole "candy bar" and a specified "customer order," students select an appropriate number of parts to project into the whole and then decide how many iterations of one part would be needed to produce the customer order. By challenging students in this goal-directed activity of coordinating partitioning and iterating operations, the app should support students' constructions of schemes and operations identified in the theoretical framework section. The purpose of this paper is to report on investigations of how game play supports students' development of new ways of operating.

Theoretical Framework

The partitive unit fraction scheme (PUFS) is the simplest scheme that coordinates partitioning and iterating operations within fractions contexts. Students use the PUFS to partition a continuous whole into equally sized pieces and then iterate any one of those pieces to check whether the pieces are appropriate in size and number. This sequential coordination of partitioning and iterating allows the student to make "an explicit numerical one-to-many comparison" and permits the student's "explicit use of fraction language to refer to that relation" (Steffe, 2002, p. 292). The two tasks illustrated in Figure 1 were designed to assess this way of operating.

Whereas the PUFS involves the sequential application of partitioning and iterating, splitting is the simultaneous composition of these two operations. The sequential operations of partitioning and iterating must be interiorized so that they can be coordinated as part of a single operation; at that point, partitioning and iterating become inverse operations. Students who can operate in this way can solve tasks that require them to partition the given whole in service of a goal that is iterative in nature: "This stick is three times as big as another stick; draw the other stick." The power of the splitting operation has been documented in reports from several teaching experiments (e.g., Norton, 2008; Steffe, 2002) and has been affirmed through quantitative analyses (e.g., Norton & Wilkins, 2009).

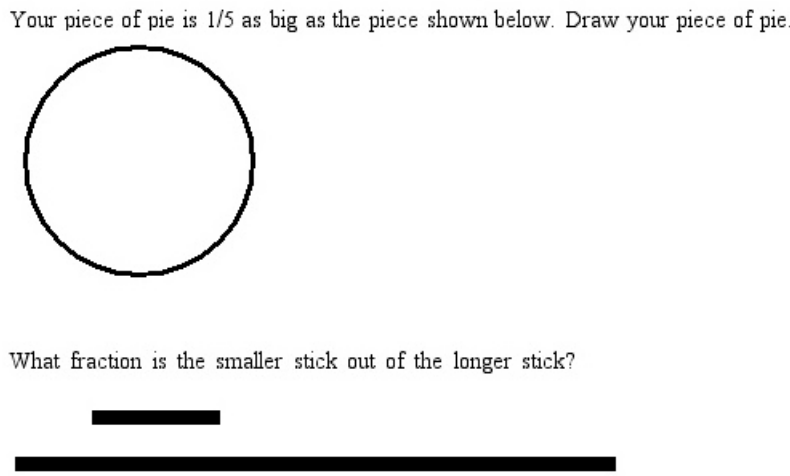

Figure 1: Tasks for assessing the partitive unit fraction scheme

Beyond its role in the construction of the splitting operation, the PUFS can be generalized to support conceptualizations of non-unit proper fractions as iterations of a unit fraction (i.e., *m/n* as *m* iterations of *1/n*). This generalization is referred to as the partitive fraction scheme (PFS). We hypothesize that students' engagement with the Candy Factory app will support their development from part-whole conceptions of fractions to partitive conceptions of fractions, in general. Due to the relationship between the PUFS and splitting, this hypothesis stipulates that the app should also, indirectly, support students' constructions of the splitting operation. Here, we operationalize the hypothesis in three testable parts:

- Hypothesis 1a (H1a): The app will support movement from PWS to PUFS.
- Hypothesis 1b (H1b): The app will support development from PUFS to the more general PFS.
- Hypothesis 2 (H2): Students who construct the PUFS go on to construct splitting operations with little or no instructional support.

Methods

Our study involved 72 students from three sixth-grade classrooms, all taught by the same teacher. One of the classes served as a control group (n = 23); one served as a low performing experimental group (n = 23, including 9 students with special needs); and one served as a high performing experimental group (n = 26, including 14 students on the honors track). We administered a pre-test on August 31, 2011 and as a post-test on November 9, 2011. Not all students took the test at both sittings; as a result, we were left with a working sample of 63 students with data from both the pre- and post-test. The test items on the test included four items each for the PUFS, the splitting operation, and the PFS. The first two authors independently rated student responses for each item. The raters assessed student responses based on all of the written work associated with the item to infer whether there was sufficient indication that the student had operated in a way that was consistent with a particular scheme or operation.

To measure the overall agreement (inter-rater reliability) for the PUFS, the splitting operation, and the PFS, we computed the kappa statistics, K, for the 66 students who took the pre-test, and the 69 students who took the post-test. The kappa statistics for the splitting operation (K_{pre} = .88, K_{post} = .85) and the PFS (K_{pre} = .82, K_{post} = .83) represent "almost perfect" agreement; the kappa statistics for the PUFS (K_{pre} = .76, K_{post} = .82) represent "substantial" and "almost perfect" agreement for the pre- and post-test, respectively (Landis & Koch, 1977, p. 165). These statistics provide strong evidence for the reliability of the ratings for the schemes and operations. For cases in which there was disagreement, the two raters re-examined the cases together to come to a consensus and create one rating for each student.

The two experimental groups played the Candy Factory app for the first 30 minutes of class each Wednesday, for seven consecutive weeks, beginning two weeks after the pre-test and ending two weeks prior to the post-test. On those days, the control group spent the first 30 minutes of class participating in "Jump Starter" review activities across various mathematical topics. No other instruction of fractions, decimals, or percents occurred in any class during this time period.

Descriptive Statistics from Pre- and Post-test

The number of students within each class limited possibilities for statistical analyses, so we use descriptive statistics to objectively characterize growth within and across classes. These results provide early tests of Hypotheses 1a and 1b, and provide further affirmation of Hypothesis 2 (beyond data used in Norton & Wilkins, 2012). We begin with Table 1, which summarizes student growth across all three groups of students.

Table 1: Frequency of Student Construction of PUFS, Splitting, and PFS for the Pre-test and Post-test

	Post								
Pre	000	100	010	001	110	101	011	111	Total
000	**14**	6	1	0	1	0	0	1	23
100	1	**1**	1	0	4	1	0	3	11
010	1	0	**0**	0	2	0	0	1	4
001	0	0	0	**0**	0	0	0	0	0
110	0	0	2	0	**9**	0	0	4	15
101	0	0	0	0	0	**1**	0	0	1
011	0	1	0	0	0	0	**0**	0	1
111	0	0	0	0	4	0	0	**4**	8
Total	16	8	4	0	20	2	0	13	63

Note: Codes represent whether students have (coded 1) or have not constructed (coded 0) PUFS, Splitting, and PFS; e.g., 110 stands for students who have constructed PUFS and Splitting, but not PFS.

Note that the most substantial movement occurred among students who had begun with the PUFS alone (see second row in Table 1). Consistent with Hypothesis 2, eight of these eleven students constructed the splitting operation by the post-test. Other notable changes between pre- and post-test occur in the construction of the PUFS and the generalization of the PUFS to the PFS. Unlike the first change, which should occur with little or no instructional support, these latter two changes were instructional goals built into the design of the Candy Factory app.

Table 2 presents the ratios (*a/b*) for numbers of students by class who had constructed no fraction schemes beyond the part-whole fraction scheme (*b*) and subsequently constructed a PUFS (*a*, H1a); and those who had constructed a PUFS and not a PFS (*b*) and subsequently constructed a PFS (*a*, H1b). Considering the data in Table 2 we find similar growth patterns among the students in the control group and the lower performing experimental group. However, growth among the higher performing experimental group is substantially higher. Because we measured growth from one stage to the next, this difference cannot be explained in terms of existing fractions knowledge. Rather, it seems that the higher performing experimental group engaged with the app more effectively, which might be a consequence of cognitive or social factors not measured in our study. Likewise, growth among the lower-performing experimental group might have been muted due to such factors.

Table 2: Development of Fractions Schemes by Class

	Control	Experimental (special needs inclusion)	Experimental (honors inclusion)
H1a: PWS → PUFS	4/10	4/12	3/5
H1b: PUFS → PFS	1/6	1/8	6/12

Conclusions

Results from this pilot study affirm Hypothesis 2, which stipulates that students who have constructed the PUFS go on to construct the splitting operation with little or no instructional support (Norton & Wilkins, 2012). Given the role splitting plays in the construction of advanced fractions schemes (Steffe & Olive, 2010) and multiplicative reasoning in general (Hackenberg, 2010), this finding further emphasizes the importance of supporting students' construction of the PUFS. Results also indicate that apps like the Candy Factory—which take seriously Olive's (2000) recommendation that we "think carefully about the contributions that the children need to make to the situation in order to build their own mathematical structures" (p. 260)—can provide an effective means to support new ways of operating. In particular, Candy Factory purposefully elicits students' partitioning and iterating operations to provoke the construction of partitive fractions schemes: PUFS and PFS. Descriptive statistics affirm Hypotheses 1a and 1b among the higher performing group. However, given the differences in growth between the two experimental groups, we need to further scrutinize cognitive and social factors that might limit meaningful engagement and, therefore, hinder growth among some students. Future app design, which is ongoing, will need to account for this disparity.

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HIGH SCHOOL TEACHERS' USE OF DYNAMIC SOFTWARE TO EXTEND AND ENHANCE ANALITIC REASONING

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We analyze and document the extent to which the use of a dynamic software helps high school teachers develop ways of representing, exploring, and solving analytic geometric tasks that complement and extend approaches based on the use of analytical methods. As a result teachers exhibited a way of reasoning about the tasks that differs when contrasted with traditional analytic approaches.

Keywords: Technology; Mathematics Problem Solving; Teacher Knowledge for Teaching

Introduction

In this report, we outline a teachers' professional development program that aims to review and enhance their mathematical and didactical knowledge through the use of computational technology. Thus, the research questions that guided the development of the study were: To what extent does the use of a dynamic software provide teachers with an opportunity to transform the nature of routine tasks, found in textbooks, into a series of learning activities involving empirical and formal approaches? What ways of reasoning do teachers construct and exhibit as a result of using a dynamic software during the tasks' solution process? Addressing these questions will provide us important information regarding the type of reasoning that teachers develop as a result of using computational tools in problem solving approaches.

Conceptual Framework

Delving into the teachers' preparation implies to recognize that there are multiple paths or programs and traditions to prepare teachers around the world. Some teaching models recognize the need and importance for teachers to develop the mathematical and didactic knowledge that can help them organize and implement proper conditions for students to learn the subject. This recognition does not imply an agreement on routes for teachers to study and develop the mathematical knowledge for teaching. For example, Even (2011) states that the assumption that "advanced mathematics studies would enhance teachers' knowledge of mathematics, which in turn will contribute to the quality of classroom instruction" (p. 941) needs to be reexamined in terms of what means for teachers to have adequate subject-matter knowledge to become an expert teacher and how a teacher can develop and use that knowledge in her teaching.

> Many practicing teachers, for different reasons, have not learned some of the content they are now required to teach, or they have not learned it in ways that enable them to teach what is now required. … Teachers need support if the goal of mathematical proficiency for all is to be reached. The demands this makes on teacher educators and the enterprise of teacher education are substantial, and often under-appreciated. (Adler et al., 2005, p. 361)

To shed light on the role of advanced mathematical knowledge in teachers' classroom decisions, Zazkis and Mamolo (2011) provide examples where teachers' awareness of that knowledge becomes useful to orient the development of a lesson. They use the construct horizon knowledge to refer to "teachers' advanced mathematical knowledge which allow them a "higher" stance and broader view of the horizon with respect to specific features of the subject itself (inner horizon) and with respect to the major disciplinary ideas and structures…occupying the word in which the object exist (outer horizon)" (p. 10).

We argue that the ways in which teachers study mathematics courses play a crucial role in developing resources and contents to be used in their teaching. That is, it is not sufficient for prospective teachers to take advanced courses; but they also need identify and explore diverse ways of reasoning and reflect on forms to connect mathematical results with other problems or situations. In this context, the systematic use of diverse digital tools can provide teachers with the opportunity to enhance and extend problem-solving approaches that involve the use of paper and pencil.

The Context, Participants and Methods

This report is part of an ongoing research project that aims to guide and orient high school teachers and students to use of several digital tools in problem solving activities. Six high school teachers participated in two hour weekly sessions during one semester. Teachers worked in pairs and they had the opportunity to work on mathematics problems that appear in textbooks with the use of a dynamic software. During the development of the sessions, two researchers coordinated the activities that involved initial pair work followed by pairs' presentation and whole group discussions. The research team formed by mathematicians, mathematics educators and teachers discussed the participants' contributions in terms of features of mathematical reasoning that characterize their approaches and were consistent throughout the sessions (Santos-Trigo & Camacho-Machin, 2009).

The Task

Points A(0, 0) and B(6, 0) determine a side (base) of a triangle. Find the locus of the third vertex C which is moved in such a way that the product of the tangents of angles formed with side AB (base) is always equal to 4 (Lehmann, 1980, p. 186)

To present how teachers approached the task, we organize the pairs' work in episodes that involves comprehension of the task statement; the construction of a dynamic model; the exploration of the model, and extension of the task. In addition, we contrast the software approach with the use of analytic methods to deal with this type of task.

Comprehension of the Task

Participants who used the software to represent the information of the task, began by drawing segment AB and a line AD to locate the third vertex. At this stage, the questions that helped them to represent the task involved: How can we find the third vertex? What about if we draw a particular angle? Is it possible to find the other asked angle? How can we determine the value of angle B by introducing the tangent product condition? The discussion of these types of questions led them to find a particular case where the conditions of the task statement were held.

The Dynamic Model and Model Exploration

During the comprehension phase of the task, the participants not only spent time making sense of the conditions embedded in the problem statement; but also they started thinking of ways to move objects within the problem representation or configuration. For example, drawing a circle with center at A and radius AD became important to control the movement of line AD. The use of the software allows moving the position of line AD maintaining the condition of the problem. That is, the measurement of angle B was determined by solving $\arctan\left(\frac{4}{\tan(45.24)}\right)$ and then side AB was rotated this angle (–75.85) around point B. On the rotated segment, a line was drawn and this line intersects line OD at E. Hence, the participants used the software to find the locus of point E (third vertex) when point D is moved around the circle. Figure 1 shows the resulting locus.

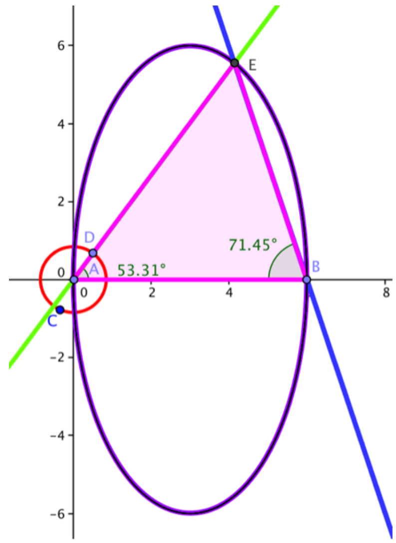

Figure 1: What is the locus of point E when point D is moved around the circle with center at A?

An Analytic Approach

The participants also thought and solved the task analytically by using the coordinate system and representing and operating the task conditions algebraically. They chose point E(x,y) (third vertex) on the locus and related the tangent conditions to the slopes of the lines that pass by the sides of the triangle. That is, they calculated the slopes of lines and expressed its product as:

$$\left(\frac{y}{x}\right)\left(\frac{y}{-(x-6)}\right) = 4 \text{ which led to the equation } \frac{(x-3)^2}{9} + \frac{y^2}{36} = 1$$

An Extension Episode

A question that the participants posed was: What happens to the ellipse when product of the tangents of the base angles of the triangle is another constant? For example, when the constant is –4. Again, the use of the software became important in exploring other cases. Can this approach be applied to others families of problems? What kind of learning opportunities can this dynamic approach offer to teachers and students? It is evident that the use of the tool offers opportunities for high school teachers to transform some routine problems into a set of activities that fosters mathematical reflection and connections between concepts.

Discussion and Remarks

We argue that mathematical tasks are the vehicle for teachers to both delve into mathematics concepts and to promote their students' mathematics knowledge and reflection. Hence, there is evidence that the use of a dynamic software provides teachers with an opportunity to transform routine problems (found in textbooks) into a set of activities where they exhibited not only different ways of reasoning to approach the tasks; but also ways of connecting analytic methods with their geometric meaning.

Dynamic models of the tasks became a platform to explore not only different forms to represent emerging relations; but also ways to extend and connect the initial statement of the task with a set of mathematical relations. In this process, it was possible to generate graphic behaviors of particular relations (locus of particular objects) without defining the algebraic model. In general terms, the use of the tool offered the participants the opportunity to examine graphically relations that later can be explored and contrasted algebraically. In this context, the use of the tool complements or extends mathematical reflection that learners engage in algebraic approaches. We contend that the use of computational tools plays a crucial role in extending high school teachers' mathematical knowledge.

Teachers were aware that the use of the tools opened up a window to explore tasks in a way that values visual and empirical arguments. In addition, the loci of points that emerged as a result of moving particular elements within the dynamic configuration become a source to launch a set of mathematical conjectures that needed to be supported. The use of the software made easy to explore cases in which the initial conditions of the task were changed. For example, what is locus generated by the third vertex when the product of tangents associated with the angles is –4? Here again, the use of the software showed that the locus became a hyperbola. While exploring different constants for the tangents product, they directed their attention to find intervals associated with the constant values to generate conic sections that involve circles, ellipses and hyperbolas. They also recognized that analytic approaches are useful to verify results obtained through the use of the tool. In this context, both approaches (the software and the analytic) are complementary and helpful for teachers and students to solve mathematical tasks. Throughout the development of the sessions, teachers recognized that routine problems could be conceptualized as departure point to engage in mathematical reflection.

Heid and Blume (2008) mentioned that "…as a reorganizer, [the use of] technology extends the existing mathematics curriculum by increasing the number and nature of examples that students encounter; as a reorganizer, technology changes the nature and arrangement of the curriculum" (p. 422) and it is clear that the study of analytic geometry with the use of computational tools need to be reorganized not only as a curriculum content; but also as a way to study the conic sections in intertwined manner favoring visual, empirical, and formal approaches.

Finally, we content that high school teachers should discuss mathematical tasks and the use of computational tools within a community that includes mathematicians, educators and practicing teachers. This type of interaction allows the community to discuss not only mathematical contents, problem solving strategies, and ways to support conjectures; but also possible didactic routes to implement problem solving approaches that enhance the use of computational tools.

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TECHNOLOGY INTEGRATION IN PRE-SERVICE SECONDARY MATHEMATICS TEACHER EDUCATION: PROSPECTS, PRIORITIES, AND PROBLEMS

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This poster presentation aims to examine prospects, priorities, and problems of technology integration in preservice secondary mathematics teacher education. We employed autoethnographic reflective and reflexive accounts of experiences (Ellis, Adams, & Bochner, 2011) through a teacher education program as participant observer and instructors. The first co-author brought his etic perspective as a participant observer, and the second and third coauthors portrayed their emic perspective as instructors in mathematics methods course and student supervisors during residency. We developed narrative accounts of lived/living experiences of observing, teaching, and supervising students in relation to technology integration in mathematics methods course and subsequent practices during residency.

We reviewed related literatures to backup revelations and portrayals. Nordin et al. (2010) investigated the pedagogical usability of a digital module prototype that integrated dynamic geometry software—Geometer's Sketchpad (GSP)—in mathematics teaching. They used pedagogical usability criteria that included student control, student activities, objective-oriented, application, value added, motivation, knowledge value, flexibility, and response. Their prototype digital modules met the pedagogical usability criteria that facilitated integration of GSP in mathematics teaching. Hixon and So (2009) discussed five specific benefits of technology integration—exposure to various pedagogical environments, shared experiences, reflectivity, cognitive development of students, and knowledge of technology integration. We agree with Stigler and Hiebert (1999) that "one other approach to understanding the difficulties of integrating IT in the classroom stems from seeing teaching as a complex cultural activity" (p. 97) and this complexity of culture sometimes becomes a barrier for change in teaching and learning.

We identified key aspects of prospects, priorities, and challenges of technology integration in the preservice mathematics teacher education in terms of relevancy, applicability, sufficiency, and extensions (RASE) as a model within the institutional cultures of teaching and learning mathematics with technology from university to schools. Preliminary findings indicated that technology integration in preservice secondary mathematics teacher education program had a high prospect for reform based mathematics teaching as a priority, but still there were latent challenges of transferring these prospects from university to schools. Success of technology integration in mathematics education depends upon its use either as a means of presentation and instruction or it is used as a means of learning and development.

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WHO'S AT BAT AND DOES IT MATTER? USING AGENT-BASED BASEBALL MODELS TO PROMOTE MATHEMATICAL REASONING

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Keywords: Technology; Reasoning and Proof; Middle School Education

This research examines the possibility of using computer models in mathematics classrooms to promote reasoning among students. There are several motivating factors for the design of these models. First, students often rely heavily on the memorization of equations and procedures when problem solving. David Hammer' suggests that students have a "range of epistemological resources, the activation of which depends on context" (Hammer & Elby, 2002, p. 24). Thus, students likely possess the resources necessary to reason about mathematics problems; however, the context of the mathematics classroom instead causes many students to activate resources that lead them to search for and use equations. One goal of this software is to create an environment that backgrounds equations and instead encourages students to utilize their own reasoning skills in solving problems. Second, Liping Ma's (1999) work with elementary mathematics teachers illustrates U.S. teachers' lack of understanding of the concepts behind equations. Ma suggests that teachers' difficulty might be due to missing connections and links in their knowledge. These missing links are likely to cause students to develop incomplete understandings, as well. Thus, a second goal of this software is to provide students with a space in which to explore mathematics concepts before attempting to connect those concepts to equations.

Researchers have shown that students can develop important knowledge through their own experimentation and construction (Papert & Harel, 1991), and agent-based modeling environments such as NetLogo (Wilensky, 1999) can be useful in creating spaces for such experimentation. This research centers on the design of two NetLogo models that allow students to experiment with the impact of a baseball team's line-up on its overall performance. Using this software, users can input baseball/softball players' statistics, choose a batting line-up comprised of any nine players, and simulate a series of games to see how many runs the team is able to score each game. Students are provided with a variety of displays to use when reasoning about a team's performance (including graphs indicating changes in individual and team statistics, as well as a visual of players' movement around the field). Additionally, users can make changes to a team's line-up and then simulate a second series of games in order to compare differences in individual and team performance given different batting line-ups. If used in a classroom, this software could encourage students to access epistemological resources that they might not typically access in a mathematics classroom through the foregrounding of reasoning and backgrounding of formulas.

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TRACING STUDENTS' ACCOUNTABILITY AND EMPOWERMENT IN AN ONLINE SYNCHRONOUS ENVIRONMENT

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Keywords: Classroom Discourse; Technology; High School Education; Affect, Emotions, Beliefs, and Attitudes

Tracing students' chat and whiteboard interactions in an online, synchronous environment as they collaborated to solve cognitively demanding, open-ended mathematical problems, our poster will document the emergence of discourse that was accountable to community and to standards of reasoning. Moreover, our analysis indicates that the students also transitioned into empowering stances about their ability to think mathematically.

Conditions in the online learning environment enable students to engage in the mathematics on their own terms, however, issues and challenges arise in how students' collaborate and form a learning community. With the increased demand for distant learning, it is important that we understand how students' engage with one another online without contemporaneous teacher involvement (Howell, Williams, & Linday, 2003). Our analysis revealed that the students displayed accountability to the learning community, accountability to the accepted forms of reasoning, as well as epistemological and mathematical empowerment as they practiced mathematics in an online environment. As they worked in small groups to solve open-ended mathematics problems they shifted away from individual, competitive work to more group cooperation and collaboration. They developed their own unique practice, expectations of one another, and most importantly, how to work together as a unique learning community. Our data also show that this accountability was an empowering experience, one that may affect their relationship with the subject of mathematics.

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A THEORETICAL FRAMEWORK FOR IMPROVING MATHEMATICS TEACHERS' MANIPULATION OF CLASSROOM TECHNOLOGY

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Keywords: Learning Outcomes; Pedagogy; Reform; Technology

Thirty years ago, the available mathematical tools for teachers and students consisted mainly of chalkboards, slide-rulers, paper-and-pencil, and geometry sets. This situation has radically changed with technologies like calculators, graphing calculators, and computer software permeating modern mathematics classrooms; this rapid growth opens up the potential for change in the didactical field, as it affects decisions on *how* mathematics should be taught in the classroom with such aids. This poster presentation explores, through examples of technological implementation in mathematics classrooms, a theoretical lens for differentiating between effective and ineffective uses of technology. In particular, we use the metaphor of a double-edged sword to frame our discussions with illustrative episodes chosen from actual teaching practices.

Research has strongly advocated for the use of technology (Dede, 2008; Keengwe & Onchwari, 2008; Leung, 2006; NCTM, 2000). With all the availability and advocacy, technology appears to have the potential to address several desirable goals in mathematics education, like engaging students in meaningful tasks, allowing multiple representations, and creating critical thinkers. However, this is all conditioned on the appropriate use of technology to target these problems: the benefit education will reap from the technology "depends on what models of teaching and learning we use. If technology is simply used to automate traditional models of teaching and learning, then it'll have very little impact on schools" (O'Neil, 1995, p. 6). The power of technology is directly linked to its ability to open new pedagogical pathways for student learning. "Smart boards" are only as "smart" as the individuals using them.

Technology is a double-edged sword because it cuts in two directions: It can cut new pedagogical pathways, or if mishandled, severely "cut" the educational experience. Like a sharp two-edged sword, technology must be handled properly. While acknowledging the positive impact that technology can have, this poster presentation recasts technological merits in terms of the pedagogical objectives to argue that technology is beneficial to instructional activities only insomuch as it expands the pedagogical possibilities of a lesson. Such a view is a goal-based filter for judging technological merit, and is not emphasized in the extant literature.

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INFLUENCE OF TEACHER BELIEF ON TECHNOLOGY INTEGRATION IN TECHNOLOGY ENHANCED MATHEMATICS CURRICULUM

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Keywords: Curriculum; Middle School Education; Teacher Beliefs; Technology

The extent of technology use in mathematics instruction is generally a pedagogical choice of the teacher, potentially moderated by the availability of technology resources in the school. Over the past two decades, curriculum developers have integrated technology into curriculum materials on a regular basis. At the middle grades, scientific calculators are often assumed, but graphing calculator technology is typically less common. As this technology becomes more widely used across levels, mathematics teachers need to be ready to use such technology in teaching to enhance mathematics learning.

Research Questions

1. What are teachers' beliefs about the use of technology in teaching middle grades mathematics, particularly when using a curriculum (University of Chicago School Mathematics [UCSMP] *Transition Mathematics* [Third Edition]) in which technology (e.g., graphing calculators, geometric drawing tools) is integrated into the curriculum materials? Specifically, is technology considered a tool that hinders or enhances mathematics learning?
2. What are factors that influence changes in teachers' beliefs towards the use of technology?

Data Collection

Results are based on secondary analysis of data collected from 7 teachers who participated for an entire school year in an evaluation study of the UCSMP *Transition Mathematics* (Third Edition) curriculum, a middle grades curriculum which integrates technology on a regular basis. Teachers completed initial and final questionnaires related to their use of technology. Interviews conducted as part of classroom observations provided a means to validate questionnaire responses.

Data Analysis

ATLAS TI 6.2 was used to code teacher interview data to find common themes and issues regarding teachers' beliefs towards the use of technology, including beliefs about student performance and the implementation of technology into the curriculum. SPSS 20 was used to analyze changes over time in questionnaire responses.

Results

Analyses indicate that teachers generally began the teaching of the curriculum with little experience using graphing technology and tended to resist its use due to lack of familiarity with it, doubts about its effectiveness, or concerns about how it was embedded into the curriculum. Positive changes in teacher beliefs towards technology integration occurred after seeing improvement in student performance.

Conclusion

The findings of this study indicate there is potential for technology integrated curriculum to change teachers' beliefs about its use for mathematics instruction at the middle grades. However, technology support for teachers is necessary when implementing such curriculum so teachers feel confident in making pedagogical choices about the use of technology in their instructional practices.

Chapter 10

Theory and Research Methods

Research Reports

Brief Research Reports

Poster Presentations

ENACTING RESPONSIVE RESEARCH STRATEGIES IN MATHEMATICS EDUCATION: ANSWERABILITY AND ACTION IN CLOSE-TO-THE-CLASSROOM ETHNOGRAPHIC WORK

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Researchers in education broadly, and in mathematics education in particular, have made progress in defining culturally responsive or relevant pedagogies and have documented a variety of contexts where these pedagogies have supported the mathematical learning of various groups of non-majority children. However, little attention has been paid to examining research practices in light of the demand to become culturally relevant. The purpose of this paper is to examine the impact of our conscious choice to make our methodological work responsive to the children involved.

In the last two decades, researchers in education broadly, and in mathematics education in particular, have made progress in defining *culturally relevant pedagogies* (Erickson & Mohatt, 1982; Ladson-Billings, 1995; 1997) and have documented a variety of contexts where these pedagogies have supported the mathematical learning of various groups of non-majority children (e.g., Brenner, 1998; Gutstein, Lipman, Hernandez, & de los Reyes, 1997; Ladson-Billings, 1994). As researchers concerned with equity issues within mathematics education, we share the goal of "helping students to accept and affirm their cultural identity while developing critical perspectives that challenge inequities that schools (and other institutions) perpetuate" (Ladson-Billings, 1995, p. 467). However, in addition to asking what pedagogies might meet this goal for children, we want our own research projects to work toward these goals. In addition, we would like to move beyond producing findings that are aimed at helping children to affirm their identities – in relation to both culture and mathematics—toward ensuring that our moment-to-moment engagements with our student participants also work toward these equity commitments. In response to this desire, we have begun to ask ourselves the following researchable questions:

- In what ways are our own research strategies responsive to the participants in our study?
- What tensions arise when we, as mathematics education researchers, make data collection decisions based on our desire to be responsive?

Purpose

The purpose of this paper is to explore the questions above within the context of a three-year ethnographic research project. The project's goal is to document the mathematical strengths of young children attending a rural school that serves a predominantly African American community. To do this work, we are following a cohort of children from preschool to first grade and collecting data about their mathematical learning in a variety of contexts (e.g., formal lessons, play, parent involvement events, assessment interviews) using videotape and fieldnotes as our primary data collection tools. As we approach the end of our second year of data collection, we find ourselves thinking a great deal about the responsibility we have as researchers, mathematics educators, and adults toward the children we visit each week and find that more and more of our actions during data collection are guided by our perceived responsibilities toward these particular children rather than toward the research project more broadly. For this presentation, we would like to closely examine three critical sites during our weekly data collection efforts where we have chosen to alter our actions within the school context out of a desire to be responsive our student participants.

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Literature Review

In defining culturally relevant pedagogy, Ladson-Billings (1995) differentiated her work from previous scholarship concerned with the cultural differences by arguing that previous work treated these differences as neutral. She posited that it was not enough to teach children the dominant communicative practices used in schools, but said that students must also be given opportunities to critically examine schooling practices. More specifically, through ethnographic analysis of successful teachers of African American students, she argued that culturally relevant pedagogies include a commitment to students' academic success, cultural competence, and critical sensibilities (Ladson-Billings, 1994, 1995).

Broadly, both within and outside of mathematics education, work focused on culturally relevant or responsive teaching has focused on ethnically homogenous classrooms, primarily in African American or Latino/a contexts (Morrison, Robbins, & Rose, 2008). As Morrison, Robbins, and Rose (2008) note in their meta-analysis of 45 research studies about culturally relevant pedagogy, this focus on ethnically monolithic classrooms is problematic because the knowledge base may not help teachers who want to teach in culturally responsive ways in diverse classrooms. But it is also problematic because it can contribute to an assumption that all children who claim similar ethnic or racial identities find the same schooling practices to be productive. Following Schmeichel (2012), we were wary of essentializing the students we work with by suggesting that particular pedagogies and practices were directly related to students' cultures and ethnicities, rather than, for example, individual temperaments, schooling histories, community norms, gender, class, etc. Scholars have dealt with this challenge in a variety of ways. For example, Ladson-Billings (1997) used footnotes to temper claims about the universality of certain features of African American culture. Civil and Khan (2001) dealt with this challenge by grounding claims about parent knowledge in a Latino/a community by emphasizing the hyper-local nature of that knowledge and the ethnographic investigation that had led to the understandings.

Theoretical Framework

In thinking about how to apply the body of work on culturally relevant teaching practices to research methods, we faced a number of challenges. Most importantly, teaching and research are quite different endeavors so it was difficult to think about how recommendations for teaching practice might translate for us as researchers. We, for example, had little control over what children experienced in the classroom moment-to-moment and thus had few opportunities to either work to align classroom experiences with at-home communication practices or to introduce opportunities for children to critically examine their school and classroom.

Second, little explicit direction on what constituted culturally relevant research practices existed and what we did find felt like an uncomfortable fit given our own subject positioning. For example, in discussing her own research methods, Ladson-Billings (1995) drew on the work of Collins (1990), a black feminist theorist, to articulate a research stance based on four big ideas: concrete experiences as central to meaning, dialogue as central to assessing truth claims, caring as central to the research endeavor, and personal responsibility as critical. While we felt drawn to these precepts, we felt we could not unproblematically take them up. A central idea in Collins' (1990) work is that it presents a "Black women's standpoint," informed by living as an "outsider-within" (p. 16). As white women working with primarily African American students we did not want to assume that we could fully comprehend this research stance only from reading published work. This is not to say that communication across lines of difference is impossible, only that it is important to be careful in these moments. Finally, we wanted to be sensitive to the two broad issues raised by our review of literature on culturally relevant or responsive pedagogies: the occasional sidling of critical perspectives and the danger of essentializing groups of children, again especially considering our own outsider status in the community.

To meet all of the above challenges, we decided to draw on the ideas of a theorist commonly used by our colleagues in literacy, Mikhail Bakhtin. In particular, we took up Bakhtin's notion of *answerability* as a guiding principle for our own in-the-moment decisions in the field as well as a framework for analyzing the extent to which our project was meeting Ladson-Billing's goals of helping children to both affirm their identities and develop a critical stance toward schooling. In a small collection of work (Bakhtin, 1990, 1993), Bakhtin articulated the ethical stance of answerability wherein "I myself – as the one who is actually thinking and who is answerable for his act of thinking – I am not present in the theoretically valid judgment" (Bakhtin, 1990, p. 4). Here Bakhtin rejects a priori ethical standards and argues that it is only in the moment with other human beings that we can determine what it means to be ethical or responsive to them. We cannot, Bakhtin argues, draw on alibis from other places to justify our behavior, although we may be informed by them, whether those alibi's come from state curriculum standards, NCTM guiding principles, culturally relevant practices, or commonly accepted research norms. Rather, we must be responsive to the demands of the people before us. Hicks (1996) writes about this stance as "more similar to faithfulness, even love, than adherence to a set of norms" (p. 107). We felt this stance captured the heart of culturally relevant pedagogy while addressing the concerns raised above. Following Bakhtin, we were not constrained by a set a principles and practices that better described the work of classroom teachers than our own work, and we could view our responsiveness to the children in front of us in light of their many legitimate demands for consideration including, but not limited to, culture, developmental stage, economic status, gender, and temperament.

Modes of Inquiry

As mentioned above, this project is situated within a larger three-year ethnography following a cohort of students from preschool to first grade. The data collection is primarily based on weekly visits that include video taping and writing fieldnotes about informal mathematical play, formal mathematics lessons, parent activities, and assessment interviews with the researcher. Our student participants include 16 children attending a rural, low-SES school. Thirteen of the 16 students are African American; one is European American; one is a recent Indian immigrant; and one is Hispanic. The children are currently in kindergarten. The research team includes two European American women (one the PI), one Asian American woman, and one Korean woman. Each week, three members of the team visited the school, with two researchers collecting data in the classroom.

During the first year of data collection, we took up relatively traditional participant observer roles (Erickson, 1986) video taping the classroom during math lessons while frequently speaking to students. The Pre-K teacher included little formal mathematics in the day, which meant that a great deal of our data collection occurred during play. Students handled manipulatives and engaged in mathematical thinking during unstructured activities. We loaded fieldnotes and video clips into a qualitative data analysis program, which we used to code the data for both mathematical content (such as problem solving and cardinality) and social features of the classroom (such as peer play and teacher interaction).

Moving into the second year of the study, we did not plan to change data collection methods. We began by recording and taking notes about the kindergarten math lessons, which were whole group. Many of the students who had expressed excitement and accomplishment the previous year were visibly upset during the math lessons. For example, over a period of three weeks, we observed five children crying during mathematics. In response to this situation, we made two significant changes to the project. First, we made an offer to the teacher, which was accepted, to have one member of the research team take a small number of students out of the classroom to work each week with the goal of both addressing mathematical needs and reducing unhappiness and anxiety for these students and their classmates. Second, we began to note, collect data about, and specifically code for our interactions in the classroom that were designed to be responsive.

For the first time in the 18 months of data collection, we began to intentionally video record interactions between the children and the other researcher in the room. Initially, we coded these moments as *researcher interaction*, but as our coding and theory became more sophisticated we also coded using the word *answerability*. For this presentation, we more closely analyzed data collected in relation to the small group of students we removed from the classroom and from video and fieldnote episodes marked with the codes *researcher interaction* and *answerability*. Following ethnographic (Emerson, Fretz, & Shaw, 1995; Erickson, 1986) analysis strategies, we searched these episodes for common themes, significant disparities, and social meaning. The following section describes three key sites we identified where our answerability as researchers was most apparent.

Critical Moments for Answerability

Removing Students from the Room

Our choice to tutor three students each week is perhaps the clearest, most systematic example of our enactment of answerable, or responsive, research methods. This practice was not a part of the original research plan and initially we did not know how data would be collected because the researcher working with the children could not teach and operate a video camera at the same time and a stand-alone camera proved too distracting. In addition, data collection was complicated by the inclusion of one child whose parents had not agreed to video taping; however, we included this child in the group because we believed it would be a positive experience for him. Ultimately, we relied on researcher journals and audio taping.

At the request of the classroom teacher, the focus of the small group was counting. In making recommendations for the small group, the classroom teacher expressed anxiety about the selected students' scores on benchmark tests and a desire to see these scores go up. Although the researcher working with the children was not unconcerned with their performance on assessments, she made a conscious choice to emphasize positive interactions with mathematics in the small group rather than tasks strictly related to the benchmark assessments on counting.

For example, during one session the researcher disregarded her plan to work on counting skills in favor of measuring objects because students said they had been studying measurement in class, but when asked what objects they had measured, responded "nothing." The students had listened to their teacher talk about measuring, had watched her measure, but they had not yet had the opportunity to measure themselves. In the small group that day, students chose objects to measure with various non-standard units. Although students practiced some counting during the measurement activity, the researcher leading the session felt some discomfort in abandoning the goals set by the teacher. However, in the moment, providing the children with an engaging experienced connected to their immediate learning seemed more responsive. Although the small group did give selected students opportunities to engage in more hands-on experiences in mathematics, it presented a few problems as well. First, students not chosen for pull out regularly begged to be included. Second, previous data collection plans for the researcher who worked with the small group had to be abandoned.

Introducing Mathematics to the End of the Day

The classroom moments most frequently coded for researcher interaction and answerability occurred during the last twenty minutes of the school day. Routinely, the mathematics lesson ended well before students needed to line up for the bus. The teacher and paraprofessional's typical practice was to pass out backpacks and folders throughout the last twenty minutes of the day while the children sat quietly at their desks. Typically, the researchers would sit near children during this time and chat quietly.

However, on one occasion after a geometry lesson in which students identified solid figures on a worksheet but did not handle any figures themselves, the PI got a box of solid figures down off the shelf and passed it around to the students at the table where she was sitting. Students immediately grabbed for the shapes, some stacking up multiple figures, some experimenting to see which figures would roll. During this interaction, the children and the PI both used quite a bit of geometric vocabulary from the lesson, including "cylinder," "cone," "cube," "circle," and "face." After a few moments, the children started to become loud and the PI shushed them. She also intervened on several occasions to ensure that all children at the table had access to at least one figure when one little girl tried to collect them all.

The decision to pass out these materials, even in the moment, felt uncomfortable because this action violated both the norms of the classroom—materials are not taken out during the last twenty minutes of the day – and the norms of ethnographic research—the participants define the social rules and ethnographers try to adopt them in the least intrusive way possible. However, informed by the theoretical language of culturally relevant pedagogy, which called for adopting a critical stance toward dominant schooling practices, and answerability, which called for a responsiveness to the children in the moment, the PI made the decision to do something uncomfortable. The result was an opportunity as a researcher to see what sense students were able to make of these figures, which features they noted and talked about, and what they found interesting. It was also an opportunity for an experienced classroom teacher to model what it might look like to engage students in geometric thinking in a more hands-on way and to give students the opportunity to experience mathematics in ways that felt engaging and fun. On the video, students' faces are far more animated during these moments than while completing the worksheet.

However, this moment also created complications. While passing out shapes to the table the PI was sitting with seemed possible, passing out shapes to the entire classroom felt like too much of an intrusion. As a result, a little more than half of the children did not get to participate. In addition, because she initiated the activity, the PI became responsible for the behavior of the children in her group, which shifted her role in relation to them not just in the moment but in future interactions.

Putting Down the Camera

The PK teacher who we began the project with was a 20-year veteran of the classroom. As a result, few lessons spiraled out of control and those that did were quickly adjusted. Although we had our own opinions about the teaching, we never felt that the PK teacher was in need of our help. In contrast, as a third-year teacher, the kindergarten teacher occasionally found herself in the midst of lessons that were not going the way she intended. During these lessons, as part of our orientation toward responsiveness, we began to move around the room as classroom helpers, sitting with small tables of children and directing their progress.

For example, in one activity, students were asked to roll a number on a die, write the numeral, write the number word, and color the correct number of spaces on a tens frame. Most students were able to do each of these tasks, but had a great deal of difficulty interpreting where on the sheet they were supposed to write each component. Both researchers in the classroom began to help groups of children. Some of the video clips show wavering footage as the researcher tried to continue taping while pointing and explaining. In other cases, the video simply shuts off as the researcher attended to the children in front of her. Over the course of the semester, this switch from researcher to teacher occurred during three lessons in significant ways. Again, this move demonstrated a responsiveness to the children in the room that helped them to feel successful and academically accomplished in mathematics in ways that probably would have been unlikely with out the researchers' intervention. Additionally, although we don't yet have evidence, these sorts of interactions may help to build relationships that will make parents more comfortable with us and our questions during parent events.

However, these moves were not without consequences for us as researchers. For example, in a lesson during the week following the one described above, the PI is repeatedly interrupted by a little girl saying "Can you help me now? Can you help me now?" while videotaping a boy who is completing a task independently. Similarly, although the total amount of time when we chose to stop taping was small, there were some moments we lost that later we wished we had on tape.

Discussion

Asking ourselves whether our research strategies were culturally responsive led us to a point where we felt obliged to continually ask ourselves whether our practice as researchers was answerable to the children in the room and as a result toward stances in the classroom that we would not have adopted if we had only been considering our roles as researchers. In many ways, the dilemmas described in this report are related to long-standing conversations in the field of qualitative research, where a number of scholars have argued that researchers, who are privileged in many ways, have ethical obligations to positively impact the people with whom they work (e.g., Duneier, 1999; Weis & Fine, 2000). However, as others point out (Bogdan & Biklen, 2003), decisions to involve oneself change what is possible in the research relationship. We believe that the historical failure of schools, in mathematics and beyond, to include and to educate all children places the same ethical burden on researchers as on classrooms teachers—to provide opportunities for children to experience academic success, cultural competence, and critical engagement (Ladson-Billings, 1994, 1995). We also believe Bakhtin's notion of answerability provided a way of framing research decisions with an appropriate emphasis on the children in the room. At the same time, there are possibly unresolvable tensions involved in making an ethical stance such a large part of one's work. Through inviting children (even implicitly) to critique classroom practices, we risked our unproblematic relationship with the classroom teacher, which is essential to gaining the access necessary to doing this work. Similarly, by engaging with children during lessons we lost our status as objective observers.

These tensions need to be explored in both philosophical and empirical ways. For example, we continue to question each other's decisions in the classroom and to ask each other to articulate the ethical principles by which we are making these decisions. Empirically, we are seeking to document our own roles in the classroom (a practice supported by the presence of multiple researchers) and to code, analyze and theorize these interactions as we would any other classroom episode. In putting this forward, we hope to launch a conversation with other researchers about the ways we can use our mathematical and pedagogical knowledge to support children while also carrying out research on current schooling practices.

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DEMONSTRATING THE USEFULNESS OF THE PARTICIPATORY-ANTICIPATORY DISTINCTION

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This paper demonstrates the explanatory power of the participatory-anticipatory distinction postulated by Tzur and Simon (2004). Using data from our current project, we show how this lens allows us to make sense of seeming inconsistencies in a student's development of a mathematical conception. More broadly, the distinction allows us to observe, analyze and conceptualize stages of learning a particular concept and, in turn, generate more thorough accounts of student learning. Furthermore, we draw attention to its usefulness in informing the design of subsequent tasks.

Keywords: Rational Numbers; Learning Trajectories; Design Experiments

Tzur and Simon (2004) postulated different levels of abstraction in the learning of a concept (see also Tzur, 2007; Tzur & Lambert, 2011). Specifically, they characterized two levels representing different learned anticipations (reviewed below). In our current project, this distinction provided a critical lens for our analyses. In this paper, we demonstrate the explanatory power of these fine-grained distinctions in accounting for student learning and, its usefulness in informing the generation of tasks.

The Problem

As we worked with Kylie, a 4th-grade student, we were struck by the following sequence. We presented her with a series of tasks designed to develop an understanding of the relationship between mixed numbers and improper fractions. In order to identify the equivalent improper fraction for a mixed number, she was shown a bar[1] made up of whole and fractional units (representing a mixed number). She partitioned the whole units into fractional parts. Having subdivided the bar, she was able to verbalize or write the equivalent improper fraction. As she worked through the sequence, she began to anticipate what the equivalent improper fraction would be *without* partitioning the bars. For example, when she was shown a quantity that was 6 and 1/4 units long and asked to express it as an improper fraction, she quickly answered, "Twenty-five fourths." Furthermore, she was able to explain her answer, "If that was all broken up into fours ..., four times six is twenty-four. Plus one is ... twenty-five fourths." She was able to respond similarly to several other tasks of this type.

We then presented the task without setting up the figure on the screen. We asked her for an improper fraction equivalent to 2 5/7. She was unable to provide a correct response. She answered, "Um …(pause) twenty-seven fifths? No — twenty-five sevenths?" How could we explain the discrepancy between what Kylie had done on the prior tasks and her inability to complete this task? She had previously shown that she could create a mixed number from the numeral representation. She could start with a mixed number represented on the screen and anticipate the equivalent improper fraction. Why was she not able to do this most recent task?

Background and Methodology

This report is based on research conducted during the second year of the five-year Measuring Approach to Rational Number (MARN) project.[2] The project is focused on two goals: (1) increasing understanding of how students learn through their activity, and (2) understanding how students can effectively learn fraction and ratio concepts based on activities grounded in measurement. The design builds upon aspects of the Elkonin–Davydov elementary curriculum (Davydov, Gorbov, Mikulina, & Savel'eva, 1995; Davydov et al., 1999) as well as research on rational number learning. In our second year, we are conducting one-on-one teaching experiments based on task sequences we have developed.

The data in this paper comes from a one-on-one teaching experiment with Kylie, a fourth grader in New York City. We began working with her in the fall of 2011 twice a week for hour-long sessions. Our team conducts an on-going analysis of the data after each session. We make inferences about her understanding and modify our trajectory for the upcoming session.

Conceptual Framework

We now review the conceptual framework we use in this research project, including the participatory/anticipatory distinction.

Abstraction from One's Activity

An emerging body of work (Tzur & Simon, 2004; Simon, Tzur, Heinz, & Kinzel, 2004; Simon et al, 2010) builds upon Piaget (2001) and Von Glaserfield's (1995) constructs of *goal-directed activity, reflection,* and *abstraction* in order to understand the process(es) of conceptual learning. *Goal-directed activity* is used to describe both the mental and physical activity of a learner. Conceiving of the activity as goal-directed supports the researcher in understanding the learner's choice of activity and attending to what the learner is focused on; the learner's goal determines her focus. *Reflection* refers to the ability of a learner to notice (consciously or not) commonalities in his or her experience. A*bstraction* is the process by which conceptual learning occurs. Abstraction involves reflection on one's goal directed activity. This reflection or noticing of commonalities in one's activity results in an anticipation, the ability to know the effect of that activity without actually engaging in it. We elaborate further on the idea of learned anticipation in the next section.

Participatory and Anticipatory Stages

Building upon the learning through activity framework,[3] Tzur and Simon (2004) postulated two stages of abstraction as a student is developing a new mathematical conception. The first stage, termed *participatory,* refers to the idea that a learner can anticipate the result of an activity. However this anticipation is limited.

> At the participatory (first) stage, the learner has learned to anticipate the effects of an activity and may also be able to explain why the effects derive from the activity. However, this knowledge is only available to the learners *in the context of the activity* through which it was developed. "In the context of the activity" means either that the learner is engaged in the activity or is somehow (e.g., chance, social interaction) prompted to use or think about the activity. (Tzur & Simon, 2004, pp. 12–13)

The second stage, which is labeled *anticipatory,* occurs when the student can anticipate the need to call upon the activity in response to a particular type of task. At this point they are not calling on the activity alone, but the activity *and* the learned anticipation of the effect of that activity.

Analysis

The sessions analyzed here occurred after our initial work with Kylie on fractions. Kylie completed a variety of tasks designed to foster the development of an understanding of a fraction as a partial measurement unit. Many of these tasks were done using the computer program JavaBars (Biddlecomb & Olive, 2000). After this preliminary work, she was able to create and identify fractional quantities, including mixed numbers. In the sessions described, Kylie worked on a series of tasks designed to foster an understanding of how to express equivalent mixed numbers and improper fractions.

Converting Improper Fractions to Mixed Numbers

The tasks created for this session used the context of measuring a beam and communicating to the owner of a hardware store the length of the beam. Using JavaBars, the student was presented with a bar (the beam) that was unmarked, a bar that represented the unit the store owner had, and a long "measuring strip" that had been marked with fractional parts (see Figure 1). The student was asked to measure the beam with the measuring strip.

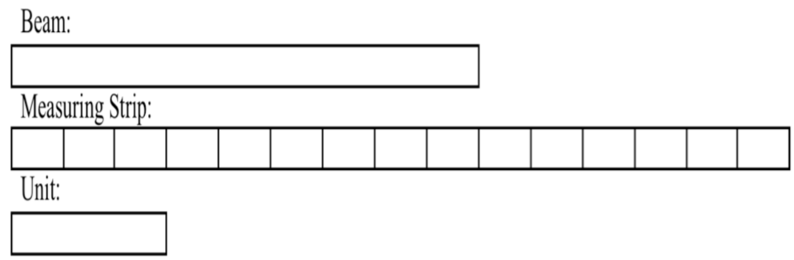

Figure 1: Context for the task

The activity she used to solve these tasks was the following. First, she measured the quantity with the long strip to determine the number of pieces in the quantity. She then measured the unit with the long measuring strip to determining the size of pieces. Because the earlier work had begun with mixed numbers, Kylie tended to give the answer as a mixed number either using division or more informal strategies. However, she was also able to give the answer as an improper fraction.

Kylie: It is two and one fifth.
Researcher: Or?
Kylie: Eh or eleven fifths.

In this example, we see she can anticipate what the quantity will be when it is grouped into units. She does not have to measure with multiple units. She can anticipate what the result of her activity of measuring with the unit would be given that she knows the number of partial units in the quantity and the number of partial units in the unit

After completing several tasks and demonstrating an anticipation of the results, she was asked a similar question but without the context.

Researcher: Okay. How about if I told you I had something that was ... thirteen thirds. Could you tell me another way to say it?
Kylie: (pause) Three ... no. Ten and three thirds? …
Researcher: Okay. Why do you say ten and three thirds?
Kylie: … cause there's ten … well, (pause) well, ten units and three thirds.
Researcher: (pause) you're just looking at these numbers separately?
Kylie: Yeah. Mhmm.
Researcher: Okay.
Kylie: Thirty-one thirds?

Although she had just demonstrated that she could anticipate what would happen if she measured an improper fraction like thirteen-thirds with a unit, she could not answer this question. In order to reconcile this disparity, we used the participatory-anticipatory distinction. The distinction suggests that two seemingly identical tasks could in fact demand different levels of abstraction, that is, different anticipations. We examined the distinction between the latest task and the previous ones in order to begin to articulate the differences in understanding. In the previous tasks, her attention was directed to the bar on the screen. Although she did not need to complete the activity in order to anticipate the result, she was cued to think about the activity. She saw the onscreen situation as being a question of measuring the unit and fractional parts of the unit. In the most recent question, she was not prompted to think about these particular measurement activities. In fact there was no measurement expressly asked for. We will present a detailed analysis of the tasks Kylie could not do after we present some additional data

Converting Mixed Numbers to Improper Fractions

This same phenomenon happened when Kylie was engaged in finding equivalent improper fractions from mixed numbers. In these tasks, was given a unit and asked to make a bar of a given length, such as three and two-fifths. She would create this quantity by iterating the unit three times, partitioning another unit into five pieces, pulling out two of the pieces and joining them to the three unit bar. After the creation of the quantity, the researcher asked how long the bar would be if it were cut into fifths. Her activity in these tasks involved partitioning each of the units into five pieces and then counting all of the pieces. When the numbers became cumbersome to count, she used multiplication to help her determine the number of the pieces. In addition, after she completed the initial task, the language used by the researcher changed so that she was asked what improper fraction the quantity was equal to, instead of how long it would be if it were cut into pieces. The use of the term improper fraction did not seem to confuse her or change her activity.

After completing several of these tasks, she began to anticipate the answer without partitioning the units. In the excerpt below, the researcher had made a bar that was five and five sixths units long.

Researcher: Do you know what improper fraction it's equal to?
Kylie: Uh ... (pause) If it was all cut up into...sixths?
Researcher: If this were all sixths, how many would there be?
Kylie: (pause) Oh! Thirty-five ...
Researcher: Why thirty-five?
Kylie: Cause there's five sixths over here, and if it was all cut up, this part would be thirty and then that would be five.
Researcher: Okay. How do you know this part would be thirty?
Kylie: Cause each one of these is six.
Researcher: Uh huh.
Kylie: Six, twelve, eighteen, twenty-four….
Researcher: How do you know each one of those is six?
Kylie: Cause this is sixths.

In this excerpt, she can anticipate the result of cutting the bars into sixths without having to perform the activity.

After successfully demonstrating she could anticipate the results of partitioning a mixed number on several tasks, she was asked the following.

Researcher: I have a candy bar that's ten and a third units long. Can you tell me what the improper fraction would be?
Kylie: (pause) ten thirds? Ten... (pause) ten ...

Similar to what happened earlier, she suddenly is unable to answer the question.

Enlisting the Participatory-Anticipatory Distinction

An implication of the participatory-anticipatory distinction is that if a difference in performance can be attributed to this distinction, the tasks must have demanded a different level of abstraction from the learner. By *task*, we must consider not just the written or oral articulation of the task, but its position in a sequence of tasks and the tools available to solve the task. Let us first look closely at the tasks in this section that Kylie was able to do correctly. Kylie was given (or asked to draw) a bar representation of the mixed number. She was then asked to identify the improper fraction equivalent, which she understood as the number of fractional parts if the whole units were also broken up so the whole bar was partitioned into equal parts. Through her activity of partitioning the whole units and determining the total number of fractional parts, she came to be able to anticipate the number without actually partitioning the unit. Thus, the tasks prompted her to look at the fractional part, consider the number of parts that the wholes would be broken into and then total the parts—often through multiplication followed by addition. To summarize, Kylie was carrying out the task of finding the number of partial units that would measure a bar that was initially measured in both whole units and partial units.

Did the last task (candy bar of length 10 1/3 units) require the same level of abstraction? Or did this task require an anticipatory level of knowing, whereas the prior tasks required only a participatory level? We argue for the usefulness of the latter. In this last task Kylie was asked to convert from a mixed number to an improper fraction. How was this different? Wasn't that what she was doing before? No. Before, the mixed number specified a bar to draw (sometimes drawn by the researcher), and then determine how many partial units were in that bar. Her focus was not on the equivalence of two representations. In the last task, she was asked to change a number written one way into a number written another way. She needed to know (to anticipate) to call on her prior activity, that is anticipate that if she thought about drawing a bar and partitioning it, that she would know the number of partial units and therefore the improper fraction. However, this was an anticipation that she had not yet developed. The following arrow diagram represents this claim.

Initially, Kylie developed an anticipation of the effect of her activity sequence: $A \rightarrow E$ (A is the activity, E is the effect). This anticipation was learned in a particular type of task for which the learner had a particular goal (e.g., determine the number of partial units in the bar on the screen). However the concept that was being developed (the researcher's instructional goal), represented by the last (anticipatory) task, required that she anticipate the need to call on that activity in response to a task that differed from the task through which the original anticipation ($A \rightarrow E$) was developed. This new anticipation that was needed can be represented as the relationship between a new goal G_1 and the activity A which is already linked to effect E by the original anticipation. Thus, the anticipatory stage requires the anticipation represented as $G_1 \rightarrow (A \rightarrow E)$.

Kylie did not have the anticipation between G_1 and the activity. She did not know (had not developed the anticipation) to call on her partitioning-and-totaling activity sequence in response to this equivalence question. When Kylie responded 10 1/3 = 10/3, the researcher created a bar that was 10 and 1/3 units long and prompted her to use the bars to see if she was correct.

> *Researcher:* Okay. So you said, you said it's ten thirds, right?
> *Kylie:* Mmhmm
> *Researcher:* Figure it out, is that ten thirds?
> *Kylie:* No.
> *Researcher:* How much is it?
> *Kylie:* It's ... (pause) one two three. Uh, three, six, nine, twelve, fifteen, eighteen … twenty- one, twenty-four ... (pause) twenty-seven? (pause) thirty. Thirty-one. It's thirty-one thirds.

The data excerpt above is consistent with the analysis. When prompted for the activity, she can make use of her anticipation regarding the number of partial units. We see she can successfully anticipate the results when prompted to think about the activity. It is clear that the anticipation required for the two tasks is different. When prompted to think about the activity, she needs to anticipate the results of the activity. When given a task that does not explicitly refer to the activity, she needs to anticipate the activity she needs to call upon. These two stages require distinct levels of understanding. It is clear she has the first anticipation, while the second type of abstraction remains outside of her current understanding. We leave it to the reader to make a similar argument relative to the first data segment (improper fraction to mixed number).

The diagram below (see Figure 2) is meant to represent Kylie's knowledge. The vertical arrows represent the anticipation that Kylie developed about the relationship between the diagram showing a mixed number and its equivalent improper fraction. The right arrow shows the reverse. The question Kylie is focused on for each vertical arrow has to do with the size of the bar. The dotted arrow reflects the lack of anticipation of the relationship between improper fractions and mixed numbers in response to the create-the-equivalent question.

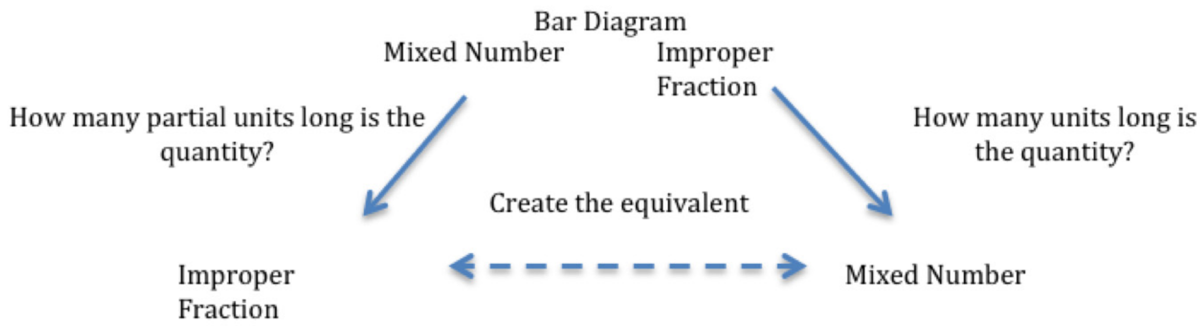

Figure 2: Diagram of Kylie's knowledge

The following data segment further strengthens the analysis. Kylie watched the researcher make a candy bar on the screen that was 2 and 5/7 units long. He then directed her the paper in front of her.

Researcher: Okay. How many... can you tell me two and five sevenths as ... (writing) as a, as an improper fraction?
Kylie: (pause) um. (pause) … twenty-five sevenths?
Researcher: Twenty five sevenths, look up there, does that look like twenty-five sevenths?
Kylie: No.
Researcher: What does it look like?
Kylie: … seven, fourteen … fifteen, sixteen, seventeen, eighteen, nineteen, nineteen sevenths.
Researcher: Nineteen sevenths.

Although the bar was on the screen in front of her, Kylie did not use it. One cannot be sure that she was aware that it was the length indicated on the paper. However, her failure to consider it when answering the question can be seen as her inability to connect the task and the activity she had used.

Conclusion

The data analysis provided demonstrates the explanatory power of differentiating between the anticipatory and participatory stages of conceptual learning. In Tzur and Simon (2004), the distinction was demonstrated with "the next day phenomenon," a common experience of educators. These data are even more compelling as the contrasting problems come one after the other in the same session (and with multiple examples). Without this construct, we would have struggled with understanding what seemed like inconsistent knowing on Kylie's part.

Researchers often expect that newly learned concepts might be inconsistently called on (Siegler, 1995). However, what percentage of those situations could be explained by this distinction? The distinction allows us to observe, analyze and conceptualize stages of learning a particular concept. It permits us to generate more thorough accounts of student learning. In some cases, we can use the distinction to anticipate aspects of a hypothetical learning trajectory (Simon, 1995) and in other cases, it allows us to notice in the data when we have failed to anticipate the challenge of moving from a participatory level to an anticipatory level. Explaining a data sequence as a move from participatory to anticipatory has a significant effect on task design in our teaching experiments. We can design tasks that aim directly at the new anticipation needed, the anticipation of the activity (linked to the effect) in response to the new goal. Continuing to provide experience at the participatory level would not benefit the student.

We close with a note about the use of the participatory-anticipatory distinction. The claim that an anticipation is at a participatory or anticipatory stage is relative to the particular concept in question and the related learner goal and activity. Thus, Kylie's ability to look at a bar measured in whole and partial units and anticipate the measurement in partial units only is neither participatory nor anticipatory. Rather it is useful to think of it as participatory relative to understanding conversion of mixed numbers to improper fractions, the goal of making the conversion, and the activity of partitioning whole units into partial units.

Endnotes

[1] This was done on a computer using JavaBars (Biddlecomb & Olive, 2000). The focus in these activities was only on length (horizontal dimension).

[2] This paper is based upon work supported by the National Science Foundation under Grant No. DRL-1020154. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation.

[3] Originally, this work was based on a specific elaboration referred to as "reflection on activity-effect relationships" (Simon et al., 2004). Whereas Tzur has continued to work with that elaboration, Simon has chosen to embark on a program of research, using the underlying concepts of reflection, activity, and abstraction, to conduct particularly rigorous teaching experiments (see Simon et al., 2010) to build a strong empirical base for elaborating a mechanism or mechanisms for conceptual learning.

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TRANSITIONING INTO CONTEMPORARY THEORY: CRITICAL POSTMODERN THEORY IN MATHEMATICS EDUCATION RESEARCH

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In this theoretical paper, the authors provide an overview of mathematics education as a research domain, identifying and briefly discussing four transitions or historical moments in mathematics education research. Using the Instructional Triangle as a point of reference for the dynamics of mathematics instruction, they illustrate how mathematics education researchers working in different moments explore different questions and use different theoretical perspectives. The authors then provide brief summaries of critical theory and postmodern theory, and suggest critical postmodern theory (CPT) as a hybrid theory that offers new possibilities for conceptualizing and conducting mathematics education research.

Keywords: Research Methods

Introduction

In this theoretical paper, to critically examine and deconstruct the persistent inequities of mathematics education or, more specifically, to open up the "fictions, fantasies and plays of power inherent in mathematics education" (Walkerdine, 2004, p. viii), we make a case for considering critical postmodern theory (CPT) (Kincheloe & McLaren, 1994; Stinson & Bullock, 2012a) in mathematics education research. We believe that CPT provides a means to make visible the Trojan Horse of the *mathematics for all* rhetoric (Martin, 2003). We structure the paper into two sections. In the first section, we provide an overview of mathematics education as a research domain, identifying and briefly discussing four transitions or moments in mathematics education research. We use the familiar *Instructional Triangle* (see National Research Council, 2001, p. 314) as a point of reference for the dynamics of mathematics teaching and learning to illustrate how mathematics education researchers working in different moments explore different questions and use different theoretical perspectives. In the second section, we provide brief summaries of critical theory and postmodern theory, and suggest CPT as a hybrid theory that offers new possibilities for conceptualizing and conducting mathematics education research. (For a significantly revised and expanded version of this argument see Stinson & Bullock, 2012a, 2012b.)

Theoretical Transitions in Mathematics Education Research

Our intent here is not to offer a comprehensive history of mathematics education as a research domain, that has been done elsewhere (Kilpatrick, 1992). But rather to briefly outline four transitions or historical moments of mathematics education research: the process–product moment (1970s–), the interpretivist–constructivist moment (1980s–), the social-turn moment (mid 1980s–), and the sociopolitical-turn moment (2000s–). We do not see these moments as linear phases of progress but rather as distinct yet overlapping and simultaneously operating theoretical perspectives or paradigms. Therefore, we do not identify end dates. Furthermore, we understand that our attempt to mark the beginning of a moment within a specific decade is somewhat misleading, given that there have been education scholars and researchers (mavericks) who began developing different possibilities for mathematics education research long before the decades that we identify (e.g., Marilyn Frankenstein [1983/1987] began exploring the sociopolitical implications of critical mathematics education several years before the sociopolitical-turn moment of the 2000s).

Because mathematics education draws from a number of disciplines, it is surprisingly difficult to characterize, and research in mathematics education is perhaps even more difficult to define (Silver & Kilpatrick, 1994). Nonetheless, as we acknowledge the difficulty in "defining" mathematics education research, we start our discussion with the 1970s and identify this decade as the beginning of the process–product moment. Most of the research in this moment attempts to quantify effective mathematics teaching;

quantitative statistical inference is the primary methodology. Here, mathematics teachers' classroom practices are described (process) and linked to student outcomes (product); limited effort is made to describe the decision-making processes of teachers or students (e.g., Good & Grouws, 1979). Securely embedded in the Enlightenment (i.e., the Age of Reason), this moment is theoretical grounded in positivism. Its aim is to predict social phenomena by "objectively" observing and measuring a "reasonable" universe. In the late 1970s and early 1980s, however, mathematics education researchers began transitioning away from a reliance on statistical inference. An analysis of manuscripts submitted to and published by the *Journal for Research in Mathematics Education* between the early 1970s to the mid 1990s showed that by the end of the 1990s "mathematics education had outgrown its dependence on statistical techniques in favor of qualitative methodologies adapted from such disparate research disciplines as anthropology, psychology, history, philosophy, and sociology" (Lester & Lambdin, 2003, p. 1676).

And because research methodologies are inextricably linked to theoretical perspectives (LeCompte, Preissle, & Tesch, 1993), this favoring of qualitative methodologies transitioned some mathematics education researchers into new theoretical perspectives such as interpretivism and constructivism. Although embedded in the Enlightenment, within the interpretivist–constructivist moment (1980s–), the aim of research is not to predict social phenomena, but rather to understand it (e.g., Steffe & Tzur, 1994; Thompson, 1984). Here, mathematics teaching and learning is examined within the dynamic interactions between teachers-and-students and students-and-students as they engage with mathematics in the classroom; often illustrated in the familiar *Instructional Triangle* (see National Research Council, 2001, p. 314).

But as mathematics education researchers continue to explore the complexities of mathematics teaching and learning, adapting theoretical perspectives and methodologies from other disciplines, some begin to understand the indispensable requirement of exploring not only the complexities of the Instructional Triangle but also the complexities of contextualizing students, teachers, and mathematics (Stinson, 2006). In so doing, they make the *social turn* in mathematics education research (Lerman, 2000). The social turn signals something different in mathematics education research, namely, the emergence of theoretical perspectives that "see meaning, thinking, and reasoning as products of social activity" (Lerman, 2000, p. 23) (e.g., Boaler, 1998; Carraher, Carraher, & Schliemann, 1987; Cobb, Perlwitz, & Underwood, 1996). Lerman cautioned, however, that the greatest challenge for mathematics education researchers who work within the social turn "is to develop accounts that bring together agency, individual trajectories…and the cultural, historical, and social origins of the ways people think, behave, reason, and understand the world" (p. 36). Researchers in this moment in general do not abandon psychology altogether—a discipline that has had a seminal influence (Kilpatrick, 1992)—but rather call for a sociocultural, discursive psychology in which mathematics teaching and learning might be understood as a particular moment in the zoom of a lens (Lerman, 2001).

By zooming out, researchers explore not only the complexities of the concentric contexts in which the Instructional Triangle is embedded (e.g., classroom, school, district, community, society) but also the multiplicities of the sociocultural and sociohistorical discourses that construct and continuously shape those contexts (Weissglass, 2002). By zooming in, researchers explore the dynamic complexities of how sociocultural and sociohistorical discourses have constructed and continuously shape students, teachers, and mathematics—thus, the possibility of the very existence of the triangle. This back-and-forth zooming of the lens motivates different questions to explore regarding the contextualization of the triangle as well as students, teachers, and mathematics. This back-and-forth zooming has also resulted in a small (but growing) number of mathematics education researchers abandoning theoretical perspectives that investigate understanding social phenomena such as interpretivism or constructivism to embracing theoretical perspectives that investigate emancipation from or deconstruction of social phenomena such as critical theory, critical race theory, feminist theory, and postmodern theory. In so doing, these researchers have adopted "a degree of social consciousness and responsibility in seeing the wider social and political picture" of mathematics education research (Gates & Vistro-Yu, 2003, p. 63).

Seeing the wider social and political picture characterizes the sociopolitical-turn moment (2000s–) in mathematics education research. Gutiérrez (2010) marked the *sociopolitical turn* as signaling "the shift in theoretical perspectives that see knowledge, power, and identity as interwoven and arising from (and constituted within) social discourses" (p. 4). Researchers who position their work within the sociopolitical-turn moment use familiar theoretical perspectives in novel and unexpected ways and/or embrace contemporary theoretical perspectives to formulate different questions and possibilities for mathematics education (e.g., Berry, 2008; Gutstein, 2003; Martin, 2010; Walshaw, 2001). The sociopolitical-turn moment, as we envision it, permits mathematics education researchers to problematize the Instructional Triangle—its existence, its assumptions, and its implications—by maintaining the exhausting process of concurrently zooming out and zooming in on the triangle only to zoom out and in yet again. This simultaneous zooming out/in steals the innocence of the Instructional Triangle, deconstructing it, as the discursive binaries used to name the vertices, and thus the triangle, are put under erasure (cf. Derrida, 1974/1997).

Here, students, teachers, and mathematics are understood as discursive formations (cf. Foucault, 1969/1972), named and re-named (but not determined) within hegemonic sociocultural, sociohistorical, and sociopolitical assumptions, conditions, and power relations. With this simultaneously zooming out/in, the vertices are no longer brought into focus, but become monsters, no longer intelligible, as they resist the surveilling and disciplining gazes of normalization (cf. Foucault, 1977/1995). As the vertices become unintelligible, it provides different possibilities for the vertices; thus, different possibilities for the Instructional Triangle and mathematics teaching and learning in general. The sociopolitical-turn moment has the potential to move mathematics education researchers away from the research agenda that explores "primarily questions of *how* to improve possibilities for teaching and learning of mathematics, toward a research agenda strongly concerned with the question of *why* mathematics education" (Pais, Stentoft, & Valero, 2010, p. 369, emphasis in original). In exploring this—in many ways, forbidden—why question, mathematics education as a research domain is cracked wide open, revealing its inclusions and exclusions (Skovsmose, 2005). Within the sociopolitical-turn moment, we believe that CPT provides a means to not only ask this forbidden why question but also other why and how questions, opening up different possibilities for mathematics education research.

Working Against Theoretical Fundamentalism

In this section, we briefly summarize critical theory and postmodern theory from our current understandings of these complex and far-reaching theories, and suggest that concepts from both theoretical perspectives might be used side by side—like tools pulled from a tool box—to short-circuit systems of power (Foucault, 1975/1996b). Although some researchers might view conflicting theoretical perspectives as incompatible, they also can be viewed as *complementary* (i.e., exploring different aspects of the same phenomena) or *incommensurable* (i.e., using different languages rather than really being incompatible) (Sfard, 2003). We believe that to capture the complexities and multiplicities of contexts when making sense of social phenomena, it often requires sifting data through one theoretical sieve, analyzing what is captured, and then catching that which remains with the next sieve of theory. Effective use of theory, therefore, requires that the researcher assume the responsibility of scholarly work; that is, the difficult intellectual work of studying the strengths and weaknesses and the convergences and divergences of different theoretical concepts pulled from (at times) conflicting theoretical perspectives (Paul & Marfo, 2001).

Critical Theory

Critical theory emerges from a Marxist tradition within the Frankfurt School (*circa* 1920) of challenging asymmetrical power relationships (Bottomore, 1991). As an activist and emancipatory project, critical theory calls its claimant to question the structures that are developed and maintained by "constructors" (Skovsmose, 2005, p. 140) and manifested as false consciousness for those who are constructed within hegemonic power. Hegemony constructs people as *objects*—those who are acted upon, rather than *Subjects*, those who act—who become so entrenched in their own oppressive condition that they do not realize their own subjugation or their complicity in the perpetuation of unjust social and economic systems (Freire, 1970/2000). Employing critical theory, therefore, requires the researcher to use her or his scholarship to dismantle the constructors' hegemonic power and the reproduction and execution of that power through institutions such as media and schools (Slott, 2002). She or he must consider how her or his scholarship—and even her or his language—supports or subverts hegemonic assumptions (Agger, 1991). In so doing, the critical theorist questions the production, validation, dissemination, and reproduction of knowledge through these institutions. Critical theorists, therefore, call for all efforts to disseminate knowledge to be accompanied by an investigation of not only its relation to ideology and power but also the subjectivities of the researcher (Leistyna & Woodrum, 1996). Through this investigation, critical theorists aim to transform existing power relations in a redemptive struggle for the humanization of people (Freire, 1970/2000). As a modernist project, embedded in the Enlightenment, critical theorists believe that as marginalized groups become critically aware of their "true" situation, intervene in its reality, and take charge of their destiny, they will exercise their right to engage in the sociohistorical transformation of their society (Crotty, 1998).

Postmodern Theory

Postmodern theory is a critique of the Enlightenment that rejects any static foundational systems of logic, resulting in truth—and thus, knowledge—becoming fluid and avoiding absolution (Seidman, 1994). Postmodern thought, however, is not a denial of the existence of truth but rather an acceptance of multiple forms of truth, made and remade within sociocultural, sociohistorical, and sociopolitical discourses (Foucault, 1984/1996a). But here discourses are no longer the mere intersections of things and words that might be spoken, heard, or read but rather "practices that systematically form the objects of which they speak" (Foucault, 1969/1972, p. 49). Knowledge then, for the postmodern theorist, is a discursive formation (cf. Foucault, 1969/1972); it no longer maintains its privileged status as an objective order of things but rather is subjected to and limited by the very sociocultural, sociohistorical, and sociopolitical assumptions, conditions, and power relations against which "true" knowledge within the Enlightenment claimed immunity (cf. Foucault, 1970/1994). Working in postmodern theory, therefore, is "a movement of 'unmaking'" (R. Wolin, cited in Crotty, 1998, p. 192). This unmaking pulls apart or deconstructs (cf. Derrida, 1974/1997) reductionist discursive binaries—truth/untruth, rational/irrational, objective/subjective, man/woman, white/black, teacher/student—as a means to unsettle and displace binary hierarchies, to uncover their historically contingent origin and politically charged roles, there inclusions and exclusions. The aim of deconstruction, however, is not to provide a "better" or "truer" foundation for knowledge and society but rather to dislodge the dominance (i.e., power) of discursive binary hierarchies, creating a social space that is tolerant of difference, ambiguity, and playful innovations which support autonomy and democracy (Seidman, 1994). In embracing difference and ambiguity, the postmodern theorist rejects the single story or grand meta-narrative (Lyotard, 1979/1984) that attempts to sanitize knowledge of difference and ambiguity. Here, the single story or grand meta-narrative of "science" is merely an illusion because it is not possible to control historical events that escape the clutches of reason and rationality (Usher & Edwards, 1994); objectivity is a mere fiction.

Critical Postmodern Theory

Employing concepts from critical theory and postmodern theory—or any other theoretical combination—side by side is messy work that is "necessary and fruitful in 'the search for meaning'" (Cook, as cited in Lather, 2010, p. 9). Working against theoretical fundamentalism (Lather, 2006), CPT operates as a differential consciousness, which Sandoval (2004) described as representative of the variance that emerges out of correlations, intensities, junctures, and crises. As we consider critical theory and postmodern theory independently, we encounter such variance from which CPT—the synergy of the two—emerges (Kincheloe & McLaren, 1994). To illustrate this synergy, we provide an example of how oppression (or marginalization) and resistance might be reconceptualized when considering the both-and theoretical perspective of critical postmodern theory rather than the either-or perspective.

While both critical theorists and postmodern theorists are concerned with oppression and resistance, their approaches are indeed significantly different. Critical theory addresses oppression by focusing, often to the point of tunnel vision, on the oppressed. Critical theorists see liberation or emancipation for the oppressed as a worthy and attainable goal achieved through praxis—a recursive process of critical reflection followed by action—what Lather (1991) defined as "philosophy becoming practical" (p. 11). Through praxis, the critical theorist works on behalf of the oppressed frequently without regard for ethical relations with the oppressor. The goal for the critical theorists becomes for the oppressed to reverse the oppressor/oppressed binary, for the oppressed to assume the position of power held by the oppressor. Once this reversal or power shift occurs, too often there is no further action (World history repeatedly validates this claim). This reversal leads us to see critical theory as a contradiction upon itself as an emancipatory project. By restricting itself to the oppressor/oppressed binary, the oppressed can assume no position beyond that of oppressor. This limiting of possibilities is still oppressive and yields no real sense of liberation. To speak more broadly, in the surge for liberation, the critical theorist is often seduced into overturning one régime of truth with yet another régime (cf. Foucault, 1977/1980).

Postmodern theory, on the other hand, provides a way out of this contradiction; it advocates for the erasure of all boundaries through decentralization, thus eliminating the need for emancipation, as it is not necessary to free one who is not bound. By deconstructing the binary between the oppressed and the oppressor and placing both binaries (i.e., oppressor/oppressed and oppressed/oppressor) under erasure, postmodern theory addresses the contradiction within critical theory by leaving the subject (i.e., the individual) open to infinite possibilities. Through deconstructing reductionist binaries and troubling emancipatory régimes of truth, the subject lives in a perpetual state of becoming her or his best self, while working within/against sociocultural, sociohistorical, and sociopolitical discourses. The irruption of the oppressed/oppressor binary eliminates the need for the us-them or self-other argument, allowing researchers to work the hyphen that separates the two (Fine, 1994). It is within this hyphenated space that ethics gains prominence. To exist with others within the hyphen, the subject must constantly be aware of the incompleteness of her or his ethical dealings with her or his self and with others. The emancipation of critical theory is too often not without casualties; postmodern theory requires a continuous ethical awareness of and responsibility for these casualties.

Closing Thoughts

Postmodern theorists in general advise caution with the emancipatory nature of critical theory because "any emancipatory perspective presupposes values which cannot be agreed upon universally or permanently" (Brown & Jones, 2001, p. 4). This cautious stance, however, causes critics of postmodern theory to claim that it "is an obstacle to the formation of open and radical perspectives that challenge inequalities and the deepening of the rule of capital in all areas of social life" (Rikowski & McLaren, 2002, p. 3). We believe, however, borrowing from Lather (2006), that both the seductions of and resistance to postmodern theory can assist us in getting smart about the limits of critical theory. Or, said in another way, the synergy between critical theory and postmodern theory is found in the "interplay between the praxis of the critical and the radical uncertainty of the postmodern" (Kincheloe & McLaren, 1994, p, 144). By integrating critical theory and postmodern theory, CPT cautiously uses the activist praxis of critical theory to restore hope—and therefore, action—to the (too often) inaction of postmodern theory.

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CONTRASTING CHARACTERIZATIONS OF CHANGE AMONG PROMINENT THEORETICAL PERSPECTIVES IN MATHEMATICS EDUCATION

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In this paper, we problematize an ontological characterization of change within a complex system by illustrating how epistemological premises of interactionist, individualist, and collectivist theoretical perspectives reveal only specific aspects of a changing system. Methodological considerations resulting from our recognition that change is characterized subjectively within various theoretical perspectives are made.

Keywords: Research Methods; Measurement

Introduction

A central focus of work in mathematics education research is the characterization of change—which we view as the entailment of three processes: (1) identifying whether or not change has occurred, (2) identifying the amount of change that has occurred, and (3) identifying the potential causes of change. One may be interested in characterizing change in students' ways of understanding as they progress through a particular instructional sequence. Another may focus on explaining how the norms of a classroom environment evolve over the course of a semester and how individual students' perceptions of, and activity within, these norms change in tandem. Still another may focus on describing how institutions change in response to educational policies or reform initiatives. Change is everywhere, is occurring all the time, and whether one is studying students, teachers, or institutions, the characterization of change is an integral part of contemporary mathematics education research.

If one seeks to characterize change in a complex system, a key consideration must be to recognize the system within which that change occurs and to make assumptions about how that system can undergo change. The identification and explanation of change within a complex system depends largely on the epistemological assumptions one makes about knowledge and about learning as well as the ways in which agents of change are characterized within the system. While there have been articles that focus on describing the assumptions and practices of various theoretical perspectives (e.g., Cobb, 2007), none have explicitly focused on how specific theories of learning characterize change. The focus of the present article is to explain the basic assumptions regarding learning and knowledge, and its growth held by prominent theoretical perspectives in mathematics education research. In addition, we show how the assumptions of various learning theories serve as a lens through which change is perceived in order to illustrate how change within a single system can be characterized in different ways depending on the premises of the theoretical perspective that a researcher assumes.

In our view, any characterization of change will necessarily entail limiting the scope of analysis to specific aspects of the changing system. As a result, the epistemological assumptions made by the researcher as well as the unit of analysis chosen, constrain the type of change that can be characterized within a system, such as a classroom. Hence, we use the word *comprehensive change* to convey all aspects of the changing system, including all *anticipated* representations and causal conditions of change, that are hypothesized to collectively comprise some measurable change in a system that occurs over some interval of time.

A major focus in this article will be to problematize the notion that empirical mathematics education research characterizes change as if it is an ontological and agreed upon construct. Our review of the literature suggests that taken as a whole, current work has not rigorously addressed or defined comprehensive change. As a result, there have been few efforts that examine change in a learning system by conscientiously defining the learning system and its boundaries in order to characterize the mechanisms by which different variables within the system interact. We believe that any attempt to study change requires one to characterize a learning system and the interactions that take place within the learning system based on the assumptions of particular theoretical orientations. We propose that the theoretical perspective that one assumes serves as a lens through which one attempts to “control” specific aspects of the changing system to construct a viable characterization of change with respect to specific features of the changing system. We argue that the specific features of the changing system that one attends to, and their hypothesized effects on other aspects of the system, are largely determined by the epistemology of one’s theoretical orientation.

Survey of Theoretical Perspectives

Mathematics education researchers currently use a plethora of theoretical perspectives that originate from fields such as cognitive science, sociology, anthropology, and psychology. Such perspectives include but are not limited to, radical constructivism, behaviorism, sociocultural theory, situated cognition, cognitive information processing theory, cognitive psychology, experimental psychology, social cognitive theory, and social constructivism. To efficiently contrast the characterization of change among these perspectives, we have organized them into three general categories that account for their historical treatment of learning, and the unit of analysis by which these traditions assume learning can be understood: *interactionism*, *individualism*, and *collectivism*. Our rationale for this partition resulted from recognizing that the epistemological tenets in the theoretical perspectives prevalent in mathematics education research are not distinct, but instead are championed, shared, and modified by researchers in a learning system. This partition is helpful to understand the ways in which the field is conflicted in its message about change, the subsequent claims we can make about how change occurs, and the recommendations we can make regarding potential levers for enhancing growth in mathematical teaching and learning.

We begin by outlining the characteristics of interactionism, individualism, and collectivism; providing brief summaries of select theoretical perspectives that are encompassed within each class of perspectives relative to their characterization of individual agents in a learning system as well as their epistemological stance. This discussion highlights the potential conflicts arising in our field from people speaking the same words, but not meaning the same things regarding the study of change in teaching, learning, and policy. We conclude with methodological considerations resulting from our recognition that change is characterized differently within different theoretical perspectives.

Interactionism

Interactionism encompasses theoretical perspectives that consider cognizing agents as subjective interpreters situated in social and societal contexts. In interactionism, individual behavior is dictated by subjective interpretations of social experiences that cannot resemble an objective existence. To the interactionist, individual agents create their experiential world and act within their own experiential world. However, interactionism does not disregard an outside world, but at the same time makes no claims about the existence of a single ontological reality. This is because perspectives in interactionism assume that we cannot step outside of ourselves to observe a “real world”, as the world is a subjective reality. Thinking about the external world as a subjective reality allows interactionists to describe students’ learning as taking place within an experiential world that they are simultaneously organizing as they learn and create new knowledge. Perspectives that comprise interactionism include situated cognition, radical constructivism, and social constructivism.

Situated cognition. Situated cognition (Brown, Collins, & Duguid, 1989) frames the individual as a component of a reasoning *system* that is comprised of the individual’s *immediate* social, physical, and psychological environment. The external influence on one’s cognition is immediate in the sense that there is a consistent interaction between the individual and the reasoning system. Moreover, the external influence of one’s cognition is also dynamic in the sense that the reasoning system within which one participates is amenable to rapid change (i.e., is responsive to feedback from the environment). Learning, then, is characterized by the extent to which an individual is able to effectively coordinate elements of their immediate social, physical, or psychological environment as a reasoning aid. In other words, learning in the situated cognition perspective is characterized by an individual's ability to become a productive component of the reasoning system by using their immediate external resources productively.

Further, situated cognition assumes that knowing or understanding is inseparable from doing. Knowledge is characterized as competence with respect to norms of the setting in which one operates and accrues as one gains experience working within the constraints of a particular context. It is the individual's interaction with social norms, however, that takes precedence over group dynamics. Learning is characterized, then, as increasingly effective performance and higher levels of competency across situations (Wenger, 1998). As a result of these assumptions, a situated cognitivist might describe change as the level of familiarity one has working in a particular context or job, or the degree to which one is able to skillfully manipulate tools and representations in a discussion in the classroom. This is because change is not characterized by an accumulation of associations but instead is the attunement of actions between the agent and its environment, and that dynamic is necessary for the characteristics of learning and knowledge to be made manifest.

In the interest of full disclosure, we recognize that aspects of the situated cognition perspective lend themselves to the collectivist paradigm. However, as mentioned above, the analytical unit within the situated cognition perspective is the cognitive behavior of individuals situated within a reasoning system. Hence, we find it more appropriate for situated cognition to be considered principally a subset of interactionism rather than collectivism.

Constructivism. Radical and social constructivism are variants of a more general learning theory of constructivism. Constructivism is an epistemology asserting that humans construct knowledge and meaning from their perceptions of their own interactions with the experiential world. Formalization of constructivism is typically attributed to Piaget, who focused on the mechanisms by which learners internalize knowledge. Piaget's genetic epistemology emphasized how a cognitive organism, such as a human, becomes a cognizing agent. Piaget described adaptation and organization as the key principles to biological development. These principles of adaptation and organization are the key components of constructivism, but are interpreted in different ways by radical and social constructivists. For the sake of brevity, we detail only radical constructivism in this article.

Radical constructivism. Radical constructivism is a philosophical perspective on learning based on Piaget's more general notion of constructivism, which is concerned with the paradox of how one comes to "know" an ontological reality when one cannot step outside of his or her own ways of thinking and ways of perceiving reality. Radical constructivism posits that one perceives a subjective reality through adaptation and organization of ways of thinking, which von Glasersfeld (1995) operationalized as assimilation and accommodation.

von Glasersfeld (1995) and others (e.g., Thompson, 2000), often based on Piaget's genetic epistemology, have focused on the constructs of assimilation, accommodation, and equilibration to explain how one comes to create, refine, and evaluate a viable mental model of the world around them by focusing on conceptual analysis. Conceptual analysis is the construction of a scheme of meanings and ways of understanding that make one's actions sensible and coherent. In short, conceptual analysis allows an observer, who cannot observe another's subjective reality, to nonetheless create a viable model that makes ones actions coherent. The developmental of this mental model allows one not only to describe and explain, but also predict one's actions based on the model of one's ways of understanding a particular idea. Thus, a model can be a viable representation of the assimilation, accommodation, and equilibrium states of the student. At the same time, radical constructivists are constrained in explaining the thinking of another because they are dependent on making inferences about one's mental model from the language and actions. Thus, a characterization of change within the radical constructivist tradition depends on tracking the changes in the mental model of one's ways of understanding and ways of thinking.

Individualism

Individualist theoretical perspectives assert that individual's comprise the primary unit of reality and that societies emerge as a consequence of individual behavior. However, individualists contend that societies do not determine the identity, or govern the behavior of individuals within societies. Theoretical perspectives that comprise interactionism include cognitive information processing theory and experimental psychology.

Cognitive information processing theory. The essence of cognitive information processing is that human thought and cognition are treated as computational in nature. This theory assumes that existing mental structures process stimuli, and that knowledge is structured *in memory* as an association between concepts that have numerous branches to other concepts.

Cognitive information processing theory holds that attention is the primary mechanism by which knowledge is developed. Since individuals maintain the inherent propensity to organize information obtained from sensory input, the stimuli that individuals attend to among many potential inputs necessarily determines what information has the potential to be stored in working memory.

Information processing theory treats the processing of stimuli much like a computer program. In particular, our nervous system registers a sensory input, which is perceived and filtered through attention and interpretative structures into working memory. As a consequence, learning can be thought of as the process where new information is "fitted" into existing cognitive structures, often characterized as long term memory. Thus, the development of existing networks of understandings stored in long-term memory characterizes change within the cognitive information processing paradigm (Gagne, 1985).

Experimental psychology. Research within experimental psychology aims to develop a *collective abstract individual.* A collective abstract individual is collective in the sense that it is devised from a statistical aggregate of quantifiable attributes, and abstract in the sense that the individual need not correspond to the attributes of any particular individual in the group that comprised the statistical aggregate (Cobb, 2007). In the experimental psychology perspective, measurable characteristics of individual students are perceived to consist of discrete, isolatable attributes that can be measured with some fidelity and aggregated using quantitative methods. Thus, the amount that one has learned is measured by the extent to which one deviates from the statistical aggregate that comprises the collective abstract individual. More specifically, an aim of experimental psychology is to determine one's discrete, isolatable attributes at two or more moments in time and compare these attributes to those of the collective abstract individual. As a result of these assumptions, experimental psychology allows one to make probability estimates in the population regarding student thinking, motivation, or reactions. A decrease in deviation over time, which can be quantified, serves as evidence of learning within this perspective. However, experimental psychology does not explicitly define a lens through which causal factors for change of an individual within a learning system are identified. Rather, educational research within the experimental psychology paradigm has traditionally assumed a process-product orientation in which desired learning outcomes are attributed to observable teaching behaviors with an inattention to the cognitive or affective causal factors of learning. Hence, experimental psychologists limit the potential causal factors of change by considering only the independent variables that are hypothesized at the outset of an experiment. Therefore, causal factors of change do not have the opportunity of manifesting themselves throughout the conduct of research as a consequence of experimental psychology methodology.

Collectivism

Collectivist perspectives consider individual behavior and cognition to be fundamentally influenced by their situation within social and societal contexts. Accordingly, the analytical unit within collectivist theoretical perspectives is the activity of the culture or *collective.* Individuals serve as contributing agents in the collective as they participate in established cultural practices. As a complex system, the collective activity is an emergent property of the individual actions of its members and their interaction (Cobb &

Yackel, 1996). Norms and other social behaviors form the basis for understanding learning. Sociocultural theory is the predominant collectivist theoretical perspective.

Sociocultural theory. Sociocultural theory situates the individual within a general social environment and considers the individual's cognition inseparable from their more general social circumstances. Accordingly, many sociocultural theorists consider the individual-as-situated-in-a-cultural-practice as the appropriate analytical unit. Hence, learning in the sociocultural perspective is evidenced by "changes that occur in people's activity as they move from relatively peripheral participation to increasingly substantial participation in the practices of established communities" (Cobb, 2007, p. 24). That is, sociocultural theorists hold that cognitive behavior and participation in cultural practices co-participate in each other's evolution. This perspective differs from that of situated cognition in that situated cognitive theorists consider the relationship between cognizing subject and external environment to remain fixed. It is the recognition that intellectual development and cultural participation co-evolve that characterizes sociocultural theory as a collectivist perspective.

Sociocultural theorists identify change of an individual within a learning system by whether or not a social participant's activity is modified as they increase their participation in established cultural practices. The interaction between a participant and their social and cultural environment always serves as the causal factor for change within sociocultural theory.

Methodological Implications of Studying Change

We have thus far described major theoretical perspectives through the lenses of individualism, interactionism, and collectivism, and in doing so have shown that if one seeks to describe change within a complex system, the boundaries and assumptions about interaction of variables within the system constrain the type and amount of change that one can characterize.

We believe individualist, interactionist, and collectivist paradigms are uniquely powerful for characterizing various aspects of change within a complex learning environment, and claim that problematizing comprehensive change has important methodological implications. It is critical to understand the type of change at play, and we believe the individualist, interactionist, and collectivist perspectives are helpful in making this distinction. In this section, we consider methodological implications that one must consider in order to rigorously study change in a learning environment.

Research Question

Since a variety of aspects of a complex system are changing in tandem, and as we have argued, they cannot all be characterized simultaneously, researchers must assume the responsibility to explicate the ways in which the theoretical perspective they assume imposes a limit on the nature of change they are able to characterize. Demonstrating the recognition that one's theoretical orientation imposes conceptual blinders on specific aspects of the changing system in the statement of one's research questions is an essential aspect of communicating one's research in a way that promotes intersubjectivity among author and reader.

Because it is impossible to simultaneously characterize every type of change, a research question must address three issues. First, it must be specific enough so that the unit of analysis is unambiguous. Second, it must characterize the system within which the unit of analysis is to be studied. Third, it must specify a particular aspect of the complex system to be studied, including relevant variables and their interactions. These three considerations permit the researcher to specify what is to be studied, to determine at what grain level it is to be studied, and to demarcate boundaries and constraints within which the unit of analysis operates. These considerations not only confirm epistemological and theoretical coherence, but also allow the researcher to classify their characterization of change as individualist, interactionist or collectivist. This classification accordingly results in the recognition of changing aspects of a complex system that are not recognized by the researcher's method.

A research question that clearly identifies the unit of analysis and demarcates the boundaries within which the unit of analysis operates constrains the type of change that one can claim to characterize.

Constraining the type of change under consideration allows the researcher to identify a theoretical framework composed of descriptive and explanatory components that can characterize change in the unit of analysis. We do not claim that any of these frameworks are more appropriate than another. Instead, the usefulness of the framework in a study focused on characterizing arises from its ability to describe, explain, and even predict aspects of the complex system under study while fitting within the constraints of the boundaries of the system.

We recognize that in many cases the theoretical framing may constrain the development of the research question instead of the research question constraining the theoretical framework. In this case, one might start with the desire to characterize change using a collective, interactionist, or individualist perspective. Whatever research question develops from these constraints must still meet our three proposed specifications. This promotes the theoretical coherence of the framework and research question.

Design of Experiment: Data Collection

We believe the focus of experimental design must address the type of data that should be collected to adequately address the proposed research questions. Addressing this concern is critical to generating a data corpus that allows the researcher to characterize change within a particular component of a complex system. Accordingly, we describe the types of data collection crucial to characterizing change within the individualist, interactionist and collectivist paradigms.

Individualism recognizes change as a modification of an individual's behavior independent of their social practices and attributes the change to an individual's orientation to focus on behavior without regard to social influence. The amount of change can be measured by the displacement in alignment between an individual's behavior and idealized behavior between two or more moments in time. Thus, any data collected within the individualist paradigm must allow the researcher to make inferences about student's behavior patterns to generate a working model of those behavior patterns. Development of this working model is crucial to identifying any robust changes in behavior. Behavior patterns can be documented by tracking verbal cues, gestures, and written work as the student reasons through a particular problem, in a group of students, or with a computer program. A shift in verbal cues or gestures can suggest a change in behavior, which can then be studied in more detail. Whatever the setting in which the data is collected, when the focus of the data is on the student's individual actions, the data corpus can support characterizing change in an individualist paradigm.

Interactionism considers change as a modification of one's interpretation of experiential reality and attributes this change to a reorganization of cognitive structures initiated by an interaction with external stimuli. The amount of change is given by a displacement between one's interpretation and an intended interpretation between two or more moments in time. Any data collected within the interactionist paradigm must allow the researcher to make inferences about a student's model of the experiential world because change cannot be identified and explained without an initial working model. As with individualism, verbal cues, written text and gestures are most useful. In order to create a model of the student's experiential world, the researcher must create situations in which the student experiences constraints on their perception or thinking. It is not until the researcher experiences the constraints of the student that he or she can make a claim about the boundaries of a student's experiential world. Change then, can be characterized when the boundaries of the student's mental model of the world or a particular mathematical idea begins to shift. By focusing on the boundaries of a student's thinking, the researcher can continually generate and test hypotheses in order to create an increasingly viable and explanatory model of a student's mental model of the world.

Collectivism considers change as a modification of a social participant's activity as they participate in established cultural practices and attributes this change to the interaction between a participant and his or her culture. Accordingly, the amount of change is measured by the displacement in alignment between a social participant's activity and the established cultural practices between two or more moments in time. Data collected within the constraints of the collectivist paradigm must allow the researcher to characterize the social participant's activity as well as the cultural practices and the social participant's perception of those cultural practices. The collectivist paradigm requires the researcher to think about the social participant's perception and interaction as part of a collective, which might be the classroom in which they participate. The researcher must document the actions, including verbal cues, gestures, discussions, and written work of not only the individual student, but also the classroom as a whole. In collectivism, the classroom, not the individual, defines the boundaries of the system. The individual works within the boundaries of this larger system, but is not the focus within the collectivist paradigm. Thus, the researcher must be systematic about creating situations in which he or she can experience the boundaries of the classroom as a collective. The researcher can, at best, create a model that describes and explains the boundaries of the classroom as a collective, and this model can only come from the actions of the classroom as a whole. As the model of the classroom as a collective becomes more viable, just as in interactionism, the researcher is able to identify more subtle shifts (change) in the system.

Design of Experiment: Microgenetics and Density of Observations

Assuming that one has specified a type of data that adequately attends to the research question, how do we know if the amount of data is sufficient for creating a viable model of the individual or collective? The density and duration of time over which the observations are taken is critical. Siegler and Crowley (1991) addressed this issue with microgenetics, which has three properties. First, observations span the period from the initiation of a change to the end of a change, marked by the stability of a system under study. Second, the density of observations is high relative to the rate of change of the phenomenon. In short, the rate of change of number of observations with respect to time increases if one anticipates the system to be at a point of a critical change. Third, observed behavior undergoes trial-by-trial analysis with the goal of attributing causal agents to particular aspects of change in a system. (Siegler & Crowley, 1991, p. 606).

For example, suppose that a researcher is attempting to create a mental model of a student's thinking as he or she participates in a two-week long instructional sequence. The researcher believes that the major shifts in student's thinking will occur on days 1, 4 and 9 based on analysis of the instructional sequence. Thus, the researcher may increase the density of observations (i.e. number of documented actions, verbal cues, or gestures) on days 1, 4 and 9 relative to the other days in the instructional sequence. These observations take place at the moment the researcher anticipates a major shift to begin occurring and ends when the researcher's model of the student's thinking becomes relatively stable.

Discussion

In this paper, we have problematized an ontological characterization of change within a complex system by illustrating how epistemological premises of interactionist, individualist, and collectivist theoretical perspectives reveal only specific aspects of a changing system. Moreover, methodological considerations resulting from our recognition that change is characterized subjectively within various theoretical perspectives were made. The methodological recommendations advanced in this paper intend to support the intersubjective interpretation of research findings by promoting researchers' clarification of the ways in which their theoretical orientation constrains their recognition of various aspects of the changing system under study.

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QUANTITATIVE REASONING IN ENVIRONMENTAL SCIENCE: THE DEVELOPMENT OF A LEARNING PROGRESSION

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This qualitative study documented students' use of quantitative reasoning (QR) skills in an environmental science context. Student participants (N=39) were given one of three content assessments (carbon, water, or biodiversity) in a clinical interview setting. Themes emerged across the three categories of QR, quantitative literacy (QL), quantitative interpretation (QI), and quantitative modeling (QM). For QL, students attempted to utilize proportional reasoning, numeracy and measurement. Across QI, interpretations of tables, graphical representations, and science models were used to answer science content questions. Quantitative modeling was not utilized as frequently as QL or QI. Further development of assessments has taken place and a new data collection period will begin in May of 2012.

Keywords: Quantitative Reasoning; Environmental Literacy; Learning Progressions

Purpose

Challenges that face today's society are often centered around environmental issues. In order to understand these environmental issues, citizens are required to wade through *data* drenched political and scientific arguments as well as political commentary saturated with buzz words and slogans such as "global warming" and "drill, baby, drill." As a result, scientific discourse becomes muddled and often misleading in social commentary. In a democratic society, citizens must be able to participate alongside scientists and policy makers to solve problems and make informed decisions regarding environmental challenges (Steen, 2001). To do so, citizens must have a fundamental understanding of scientific principles as well as mathematics and statistics within an environmental context.

Quantitative reasoning (QR) is mathematics and statistics applied in real-life, authentic situations. QR problems are context dependent, interdisciplinary, open-ended issues that require critical thinking and the capacity to communicate a course of action (Author). The National Research Council (NRC) has stated that people will need to be able to apply quantitative approaches to predictions, analyzing and interpreting evidence, and developing models in order to make informed decisions as citizens (NRC, 2003, 2009). Therefore, people today need a fluency in quantitative tools in order to analytically think about how problems in the context of environmental science have an impact on an individual's life as a constructive, concerned, and reflective citizen.

Theoretical Framework

This research study is based on the theoretical frameworks of learning progressions and quantitative reasoning. Quantitative reasoning (QR) has four components: (a) act of quantification: mathematical process of conceptualizing an object and its measurable attributes with corresponding units, which entail a proportional relationship (linear, bi-linear, or multi-linear); (b) quantitative literacy (QL): sophisticated use of fundamental mathematical concepts in sophisticated ways; (c) quantitative interpretation (QI): ability to use models to make predictions and discover trends, which is central to a person being a literate citizen; and (d) quantitative modeling (QM): ability to create representations to explain a phenomena.

These components interact within a QR cycle when engaged in the process of science as model-building. The individual may begin with a qualitative science account of the phenomena based on their theory of the world, called force dynamic discourse. They might also respond in a school science discourse based on acquired knowledge, and then potentially progress to a full scientific discourse that uses science principles that explain phenomena. A quantitative science account is sought to provide support for the qualitative account. First the individual engages in the act of quantification by identifying objects, their attributes, and assigning measures. This provides variables that can be operated on mathematically or statistically. Second, depending on both the query of interest to the individual and the data they access, they engage in QR through one or more of the three processes of QL, QI, or QM. These three processes are interconnected and typically engaging in one requires elements of another.

Learning progressions are based on research in science education and cognitive psychology, foundational and generative disciplinary knowledge and practices, and strive for internal conceptual coherence. They are defined as being “hypothesized descriptions of the successively more sophisticated ways students think about an important domain of knowledge or how practice develops as students learn about and investigate that domain over an appropriate span of time” (Corcoran, Mosher, & Rogat, 2009, p. 37). The five essential characteristics of learning progressions are: (a) upper anchors which target performance or leaning goals that are the end points of learning progression and are defined by societal expectations, analysis of the discipline, and requirements for entry into the next level of education; (b) progress variables: dimensions of understanding, application, and practice that are being developed and tracked over time; (c) levels of achievement: intermediate steps in the developmental pathway(s) traced by a learning progression; (d) learning performances: tasks students at a particular level of achievement would be capable of performing; and (e) assessments: specific measures used to track student development along the hypothesized progression. The proposed QR learning progressions will build on these characteristics, incorporating mathematical and statistical frameworks.

Methodology

Study Participants and Recruitment

Four counties in the state of Wyoming were contacted via email and phone and invited to participate in this study. Three of the counties agreed to participate. Within the three counties, seven rural schools agreed to allow researchers access to students. An $N = 39$ students were interviewed, 18 females and 21 males. Students ranged in grade level from 6th grade to 12th grade and all self-identified as Caucasian.

Research Design

Establishing learning progressions requires conceptual coherence, compatibility with current research, and empirical validation (Anderson, 2009). Conceptual coherence means the learning progression tells a comprehensible and reasonable story of how naïve students develop mastery. Compatibility with current research requires learning progressions to adhere to research on science and mathematics content, pedagogy, and cognition. The learning progression identified from the data in this study will be treated as hypotheses that are to be empirically tested and validated.

There are five general approaches for hypothesizing progressions: (1) extrapolation from current and conventional teacher and curriculum practice (i.e., standards); (2) cross-sectional sampling of student performance using assessments, observations, or interviews; (3) longitudinal samples of student work over time; (4) closely observed classroom interventions; and (5) disciplinary understanding of the structure of the key concepts in the discipline (Corcoran, Mosher, & Rogat, 2009). This study utilized the second approach of cross-sectional sampling of student performance using assessments and interviews.

Clinical semi-structured interviews situated in the completion of activities and formative assessments were the initial method of gathering data to establish, validate and refine frameworks for developing the QR hypothetical learning progression. Guiding principles for developing the clinical interviews (Anderson, 2009) are that they are based on the progression framework, built around practices, linking processes, and standard representations, with branching probes to explore discourses, principles, and themes. There were three different interview assessments given; one on biodiversity, carbon, and water.

Data Analysis

The interviews were audio recorded and transcribed. Interviews were examined within and across interview questions for patterns in the participants' responses. Three researchers divided the transcripts evenly and analyzed 40% of the interviews. Using modified grounded theory (Glasser & Strauss, 1967) a list of themes by questions was developed after an initial read through of the transcripts. Discrepancies were discussed and resolved prior to the analysis of the remaining transcripts. After level one coding we found the frequencies of responses were similar across content areas (carbon, water, and biodiversity) and subsequent analyses were conducted on the individual content strands.

Learning progression matrices were created by cross tabulating achievement levels (rows in matrix) with progress variables (columns in matrix). The achievement levels were then linked with learning performances that are exemplars drawn from the clinical interviews and written assessments. The learning performances demonstrated student responses at different achievement levels. This provided for use of the learning progression as a means to classify student understanding.

Results

Within the three categories of QI, QL, and QM, the following themes were present across content assessments. For QL, students attempted to utilize proportional reasoning, numeracy, and measurement. In regards to proportional reasoning, students across grades 6 through 12 had difficulty with percent magnitude. They could not correctly identify part to whole concepts in order to answer the questions posed. With regards to numeracy, students in grades 6 through 8 avoided calculation even when prompted by researchers. Students in the upper secondary grades were able to attempt calculations, though there were variable misconceptions. Across the grade levels represented, students were unable to accurately use volume measurements and at time avoided discussing the concept of volume completely.

Table interpretations were used more often by students to determine the minimum and maximum values. This was a consistent observation across grade levels as well as across all three content strands. Graphical representations were interpreted with a variety of errors, except for those students at the upper secondary level. Students in 11th and 12th grade were able to not only correctly interpret information from graphs, but they were also able to create graphical representation to demonstrate their ability to predict trends in data. Student ability to interpret science models was inconsistent across grade levels.

Quantitative Modeling (QM) was not utilized as frequently as QL and QI although students did attempt to extend existing models in order to predict outcomes as well as create their own.

Discussion

Quantitative literacy (QL) is the ability to use basic mathematical and statistical skills in sophisticated ways. Environmental science information provided in a narrative or presented orally often has quantitative accounts embedded within it, accounts which may require the citizen to extract the information from the context, quantify it, manipulate it, and then interpret it within the science context. The mathematics and statistics required to do this are often simple arithmetic concepts, but applying them within the environmental context can be quite challenging for students.

Quantitative interpretation (QI) is an essential understanding for an environmentally literate citizen. Scientific data, findings, and models are often displayed in tables, graphs, visual science models, and equations. For a citizen to make informed decisions they must interpret information provided in these formats. We will explore 6th to 12th grade students' ability to interpret environmental science data and models represented in tables, graphs, analytic equations, and science models. Interpreting models requires a number of quantitative skills, including identifying variables and their correlation to a predicted variable, interpreting variables represented on axes of a graph, interpreting a model for a selected point or case, determining trends in a model, making predictions of future events, and translating between different models of the same phenomena. A learning progression for QI might use these skills as indicators of a student's level of understanding within the learning trajectory.

One could argue that QM is not an essential understanding for an environmentally literate citizen. That a citizen will likely not be building models from raw data, but interpreting models developed by others. While this may be the case, we believe that a student traversing the 6th to 12th grade should learn to view science as model building and testing, so that they have a better sense of the inherent strength and weaknesses of models they will interpret as citizens. The reluctance of students to question the discrepancies in the science models found in our assessments are examples of not understanding that global models are based on extensive estimations.

Future Research

The data from this study will be used to further inform researchers on quantitative reasoning learning progression development within an environmental science context. The assessments used for clinical interviews have evolved and now include six variations on: (a) carbon cycles, (b) carbon storage, (c) water cycles, (d) water transportation, (e) biodiversity communities, and (f) biodiversity extinction. Clinical interviews and written assessments will take place in May of 2012 to collect data using the new assessments.

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EXPANDING THE "DYNAMICAL THEORY FOR THE GROWTH OF MATHEMATICAL UNDERSTANDING" TO THE COLLECTIVE

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In this paper we offer an account of how we are extending and adapting the Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding to the collective domain. Through employing elements from our earlier work on collective understanding as an improvisational process we introduce and explore the new constructs of Collective Image Making, Collective Image Having, and Collective Folding Back and show how these contribute to the growth of understanding of a group of students working on a mathematical problem.

Keywords: Learning Theory; Cognition; Classroom Discourse

Purpose of the Study

The research reported in this paper forms part of our ongoing research program concerned with the nature of mathematical understanding and how it might be theorized and characterized. In recent years our specific focus has been the phenomenon of collective mathematical understanding—the kinds of learning and understanding we may see occurring when a group of learners work together on a piece of mathematics. We have characterized the growth of collective mathematical understanding as a creative and emergent improvisational process and illustrated how this can be observed in action (Martin, Towers, & Pirie, 2006; Martin & Towers, 2007, 2009; Towers & Martin, 2009). In this paper we offer an account of how we are extending and adapting the Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding to the collective domain.

Theoretical Framework

In our work we have predominantly drawn on ideas from improvisational theory and applied these to the context of mathematical thinking and learning. However, our work has also continued to be informed by the Dynamical Theory for the Growth of Mathematical Understanding developed by Pirie and Kieren (Pirie & Kieren, 1994) and their characterization of mathematical understanding not as a static state to be reached or achieved, but as a dynamical, growing, and ever-changing process. While Pirie and Kieren's work offers a powerful theoretical framework for observing the growth of mathematical understanding, their model is still primarily one for the growth of *personal* understanding. Nevertheless, the framework proposes a series of modes of understanding that we suggest can, with elaboration, also offer a powerful language through which to talk about the growth of collective mathematical understanding. In this paper we take two of the modes of mathematical understanding from the Theory (Image Making and Image Having), and the key construct of Folding Back and extend their definitions to the domain of the collective. Full definitions and specific examples of these actions at the individual level can be found in Kieren, Pirie, and Gordon Calvert (1999), Pirie and Kieren (1994), and Martin (2008) and here we offer only brief descriptions of each.

When Image Making, learners are engaging in specific activities aimed at helping them to develop particular initial conceptions and ideas for the meaning of a mathematical concept. Image Making often involves drawing of diagrams, working through specific examples or playing with numbers. By the Image Having stage learners are no longer tied to actual activities; they are now able to carry with them a general mental plan for these specific activities and use it accordingly. Folding Back describes the way in which a learner might, while working at Image Having (or another layer) encounter a situation, problem, difficulty for which their current image is insufficient and it is thus necessary for them to return to more localized ways of working—perhaps through more Image Making. The metaphor of folding highlights the key notion that the learner takes his or her existing understandings back to the inner layer and uses these to build a "thicker" understanding.

Methods and Data Source

We illustrate our thinking through considering extracts of video data collected during a multi-year study whose key research question is, "In what ways can the Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding be adapted to respond to contemporary concerns with the phenomenon of collective action?" In this paper we provide a preliminary response. The larger study has collected data from a range of school mathematics classrooms as well as from smaller group problem-solving sessions. In this paper we draw on data from one problem-solving session involving three students, aged between nine and ten years old, in which they were working on the well-known 'painted cube' problem. The students were given a printed sheet with the following question, and had available a large number of small interlocking cubes. "Imagine a large cube made up from 27 small cubes. Imagine dipping the large cube into a pot of yellow paint so the whole outer surface is covered, and then breaking the cube up into its small cubes. How many of the small cubes will have yellow paint on their faces? Will they all look the same? Now imagine doing the same with other cubes made up from small cubes. What can you say about the number of small cubes with yellow paint on?" The students worked on the problem for a period of about forty-five minutes. The third author acted as a participant observer—asking occasional questions to clarify the thinking of the students and/or to prompt further action if necessary.

Data analysis was conducted using the method described by Pirie (1996) as "sit, look, think, look again" (p. 556). Drawing on this method, the entire video set for each session was viewed multiple times in full and then subsequent times in smaller chunks to identify critical incidents relating to the students' growing understanding (as identified in relation to the layers of the Dynamical Theory for the Growth of Mathematical Understanding). This process was carried out independently by two of the authors, who then compared analyses for agreement. Once agreement was reached, the relevant excerpts of video data and the analyses were provided to the third author for verification and any discrepancies discussed and resolutions agreed by the whole team.

Results

In this section we offer brief definitions of Collective Image Making, Having and Folding Back and illustrate these with reference to the "painted cube" session. We deliberately choose here to present data extracts in a descriptive form as this more powerfully captures the collective nature of the interaction than would transcript (in our presentation we will illustrate our ideas with a short piece of video).

Collective Image Making—No single learner recognizes or is able to identify or engage in an appropriate action necessary for the making of a useful and appropriate image for the mathematical concept. Instead, what is seen in Collective Image Making is the offering, by individuals, of partial fragments of ideas and understandings, which are then picked up, elaborated, and acted on by others in the group. This process of interweaving individual contributions to create a coherent shared idea or representation is what gives the Image Making its collective nature. In the case of the "painted cube" the students begin the task by building a 3×3 cube from the smaller cubes. They decide to only build one model and actually all contribute to the physical making of the model. As they are constructing this they also start to collectively hypothesize about the solution to the problem, with individual students offering thoughts or ideas around how many cubes will have a particular number of sides painted. These ideas are reacted to, built upon, and acted on through ongoing interaction. For example as they are building the model (but before it is complete) one student says, "There are going to be four with three painted faces" and this is picked up by the others in the group. It is pointed out there are more than four corners, an answer of six is briefly suggested, but the group then quickly agree on eight (and verify with reference to their model). Here, they are making an image that "whatever size the larger cube there will always be eight small cubes with three painted faces." However, they do not state this—which would suggest a shift to

Collective Image Having. It is important to note here that no one student instantly tells the others a correct answer. Their contributions instead take the form of offering of an idea into the collective.

Collective Image Having—The group is at a point where they have a useable and workable idea, and nothing new is being introduced—in other words, an idea has been initiated, followed, and built upon by the interweaving contributions of the group members and now emerges as something useable by the group in the context of the task at hand. With the painted cube, the group reaches a point in their working where they are confident that they have correctly determined a solution for the 3 × 3 cube. They are then posed the question of what would happen if the cube were larger. Here they articulate a number of their images, including that there will always be eight small cubes with three sides painted, and that for the other numbers in a larger cube they simply need to multiply their solutions from the 3 × 3 case by some appropriate number. At this point they do not see the need to build any further models, as they believe they can simply predict and calculate answers from what they already know. That is, they have an image they can use without recourse to specific actions (e.g., building a 4 × 4 cube and counting).

Collective Folding Back—At points in mathematical activity an image is sometimes no longer viable or useable (often because it is too local or specific in nature). Thus, the group needs to return to Collective Image Making and to remake, rework, or rebuild their image. To do this, there needs to be agreement around the need to Fold Back and also as to what the new Image Making actions will involve. This willingness and capacity to build on a better idea and to alter the current way of acting is a collective process—often occurring through an awareness of the "group mind" (Martin & Towers, 2009, p. 14) and in response to tiny cues that suggest a new direction for appropriate mathematical action. In the context of the painted cube session, although the group believe that their collective image is a correct one, the observer knows it is not generalizable and that, although they are correct to be seeking a pattern, their current thinking will not give correct solutions for larger cubes. She therefore asks the group to build a larger cube "to show me" and they start to do this, still confident that their predictions will work. However, having built the larger cube (again something done by all three students) and then trying to apply their rules they come to realize they are not correct. They state that "it doesn't make sense" and then one student comments, "Let's count." At this point their actions shift from being a process of using and demonstrating their image (working at Collective Image Having) to needing to rework it (thus folding back to Collective Image Making). In once again counting numbers of small cubes they have collectively folded back to work with a specific case in order to then be able to say something more general (which would be evidenced through returning to Collective Image Having). When the suggestion is made to count cubes, the group collectively sees it as appropriate—there is a sense that this is an appropriate way forward as their image is no longer viable—and all participate in the counting process. What makes this new act of Collective Image Making different (thicker) from that when working with the 3 × 3 cube initially is that they are now purposefully looking for a pattern; they are looking not to discard totally their existing images, but to modify these, through finding which elements of their more general ideas are valid and which require modifying.

Conclusion and Significance

In this paper we are advancing a theoretical development of the Dynamical Theory that enables it to be used to analyze and interpret collective action. Our work emphasizes the significance of individual action *in context* and draws attention to those moments when such individual actions and statements interweave constructively to build understanding for a group of learners. We draw attention, in particular, to the way in which ideas are taken up, built upon, developed, re-developed, and shared within and by a group. In doing this we use the language and notions of the Dynamical Theory in order to focus specifically on what different kinds of collective action look like—and how collective growth occurs through shifting between different modes of working and acting. A strength of the Dynamical Theory, and our adaptation of it to the collective domain, is the way in which we are able to observe, recognize and talk about a group, such as the three students here, making and modifying a single shared image. This contrasts with other groups we have observed where actions remained individual in nature—with each student making and having his or her own image. We believe that such noticing and valuing of the complex, recursive and lengthy process of growing mathematical understanding is important to emphasize, particularly for teachers who are searching for ways to recognize their students' growth and to understand the value of collaborative work in classrooms.

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DEVELOPING FRAMEWORKS FOR PREPARATION OF ALGEBRA TEACHERS: CHALLENGES AND OPPORTUNITIES

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Algebra has long been considered the foundation for higher-level mathematics and a gatekeeper for post-secondary opportunities. In recent decades, increasing enrollments in algebra have placed additional demands on algebra teachers for reaching a more diverse population of students. With the 2010 release and adoption of the Common Core State Standards for Mathematics (CCSSM) by all but a few states, questions arise regarding preparing teachers to teach the standards and practices described in CCSSM. In this paper, we describe the processes used to develop frameworks to study opportunities provided by secondary teacher preparation programs to learn about algebra, algebra teaching, achieving equity in algebra learning, and the algebra, functions, and modeling standards and mathematical practices described in CCSSM. In particular, we discuss opportunities and challenges encountered in our work.

Keywords: Algebra and Algebraic Thinking; Teacher Education–Preservice

Theoretical Perspective

For more than a century high school algebra courses have served as the foundation for higher mathematics and as gatekeepers for entrance to college mathematics (e.g., Moses, Kamii, Swap, & Howard, 1989). In recent decades, desires for equity and for higher achievement have resulted in suggestions that algebra be required for all students. Many states currently mandate completion of Algebra I or Algebra II for high school graduation (Teuscher, Dingman, Nevels, & Reys, 2008). However, failure rates in algebra are high (Loveless, 2008). The increasing numbers of students taking algebra and the corresponding high failure rates in algebra beg the question of how algebra teachers are prepared. With the 2010 release and adoption of the Common Core State Standards for Mathematics (*CCSSM*) by all but a few states, specific questions arise regarding preparing teachers to teach the standards and practices described in *CCSSM*.

Recent research highlights the importance of both strong content knowledge and pedagogical content knowledge for pre-service mathematics teachers. Specifically, teachers need preparation that "covers knowledge of mathematics, of how students learn mathematics, and of mathematical pedagogy that is aligned with recommendations of professional societies" (National Research Council [NRC], 2010, p. 123). The National Council on Teacher Quality also developed recommendations for mathematics teacher education programs, emphasizing the importance of making connections between the content in the mathematics courses taken by pre-service teachers and their methods courses and fieldwork (Greenberg & Walsh, 2008). The National Council of Teachers of Mathematics has, over the past several decades, made recommendations for changes in *what* mathematics should be taught and *how* it should be taught (e.g., NCTM, 2000, 2009). For example the NCTM emphasizes functions as a central topic in algebra; and highlights processes such as problem solving, reasoning and proof, and using multiple representations. But little is known about how such recommendations are incorporated into programs of study in mathematics for teachers (NRC, 2010).

Objectives

Preparing to Teach Algebra: A Study of Teacher Education (PTA), an NSF-funded collaborative project involving two universities, focuses on the following research question:

What opportunities do secondary mathematics teacher preparation programs provide to learn about algebra, algebra teaching, issues in achieving equity in algebra learning, and the algebra, functions, and modeling standards and mathematical practices described in the *Common Core State Standards for Mathematics* (*CCSSM*)?

The 2012 PME-NA theme, *Navigating Transitions Along Continuums*, is particularly salient in this study as we examine the transition of pre-service secondary mathematics teachers from students of algebra to teachers of algebra. Currently the *PTA* study is in its first year; so in this brief report we will share the processes and initial findings of our framework development and our plans for pilot work to be carried out in spring 2012. At the session in November 2012 we will be able to share results from the pilot study and our revised frameworks and protocols.

Methods of Inquiry used to Develop Frameworks

PTA is a mixed-method study that consists of a national survey of secondary mathematics teacher preparation programs and case studies of four diverse programs. In order to answer our research question, we developed frameworks to serve as the foundation for all aspects of our work, including constructing items for the survey, examining instructional materials, and preparing protocols for instructor interviews and focus groups of pre-service teachers. Given the four aspects of our research question, we developed four closely related frameworks. Here, we describe the process of developing the frameworks and the challenges encountered.

Algebra in *CCSSM* Framework

The initial *Algebra in CCSSM Framework* was developed based on the descriptions of the high school algebra, functions and modeling standards in *CCSSM*; relevant standards about algebra, functions and modeling in grades 6, 7 and 8; and the Standards for Mathematical Practice. Although the Standards for Mathematical Practice were not specifically written for algebra, they are recommended across all mathematical strands; therefore, we included them in the framework. The format of the framework followed the *CCSSM* format with three levels: (1) domains, (2) clusters, and (3) standards. The domains and clusters in the *Algebra in CCSSM Framework* were taken literally from *CCSSM*. However, the standards listed in *CCSSM* were edited and condensed for the framework.

Algebra Content in College Framework

The initial *Algebra Content in College Framework* was developed using recommendations from: (1) mathematicians (e.g., CBMS, 2001), (2) organizations charged with licensing teachers and accrediting teacher preparation institutions (e.g., NCTM, 2003), and (3) experts in teaching specific topics related to high school algebra (e.g., Cooney, Beckmann, Lloyd, & Wilson, 2010). First, statements about mathematical knowledge for teaching were extracted literally and listed as individual items. Second, items were organized by field (e.g., linear algebra) and practices (e.g., reasoning and proof). Finally, duplicated information was combined and items already included in *CCSSM* were deleted. The resulting framework had two main categories (each with multiple sub-categories): (a) Mathematics Content, and (b) Process, Practices and Perspectives.

Algebra Teaching Framework

The components of the initial *Algebra Teaching Framework* were developed using three primary sources: (1) algebra teaching and research literature (e.g., Kieran, 2007), (2) algebra teaching recommendations from professional organizations (e.g., NCTM, 2003), and (3) sample secondary mathematics education syllabi. The development process involved addition, deletion, and consolidation of categories and sub-categories in an iterative cycle. This led to a framework with two levels (and multiple categories): (1) Teaching Algebra, and (2) Teaching Mathematics.

Equity in Algebra Framework

The initial *Equity in Algebra Framework* was developed by consulting literature about equity issues in algebra and mathematics more generally (e.g., Moses, Kamii, Swap, & Howard, 1989; NCTM, 2000) and the ways in which equity issues in algebra/mathematics were addressed in sample syllabi. This led to a three-level framework: (1) Equity in Teaching and Learning Algebra, (2) Equity in Teaching and Learning Mathematics, and (3) Equity in Education.

Recommendations for Revision

The *PTA* Advisory Board is a multidisciplinary team, including two mathematics educators, a mathematician, and an expert in survey research. The initial frameworks and the following questions were sent to the Advisory Board two weeks prior to our first meeting with them:

- Are we missing important sources related to any of the frameworks?
- Are we missing important categories in the frameworks? Would you eliminate any categories?
- What advice do you have about sorting out learning algebra vs. teaching algebra?
- Is it reasonable to code syllabi from mathematics courses using only the two content frameworks and education courses using only the teaching and equity frameworks?

The Advisory Board members provided detailed feedback related to these questions and all frameworks at the meeting. They also had the opportunity to use the draft frameworks to code sample syllabi from relevant courses (e.g., Linear Algebra, Teaching High School Mathematics) offered at universities that offer teacher preparation programs.

Overall, the advisors made the following recommendations:

1. Streamline the frameworks; the initial drafts were long and cumbersome. For example, rather than list all recommended topics in mathematics related to teaching algebra, focus on a few important "big" ideas that might distinguish one program from another.
2. Maintain the focus of *PTA* on algebra. Especially in the *Algebra Teaching* and the *Equity in Algebra Frameworks*, delete the components that relate to mathematics or education in general, but not specifically to algebra.

Based on these recommendations, substantial revisions were made to all but the *CCSSM* framework. Based on the first recommendation, eight big ideas in algebra were identified and used to revise the *Algebra Content in College Framework* and the *Algebra Teaching Framework*; the pre-service teacher is a learner of the former and is in transition to become a teacher of the latter. These eight ideas are: (1) Reasoning and proof; (2) Contexts, applications, and modeling; (3) Treatment of functions; (4) Structure of algebra; (5) Nature of school algebra; (6) History of algebra; (7) Use of tools and technology in algebra classes; and (8) Connections between high school algebra and college mathematics courses related to algebra. Based on the second recommendation, the *Equity in Algebra Framework* is no longer focused on equity in mathematics education or education more broadly, but solely on the equity issues directly related to the teaching of algebra.

Challenges and Opportunities

Despite the importance of using carefully constructed frameworks for analysis in studies in mathematics education, researchers seldom discuss the process of developing their frameworks. In this paper, we endeavor to begin this conversation by highlighting the challenges and opportunities encountered in the process.

The lack of clear distinction between algebra curriculum, on the one hand, and algebra pedagogy on the other, proved challenging, particularly given our goal to avoid repetition across frameworks. In addition, accounting for the wide variation across recommendations and syllabi found on the Internet resulted in initially unwieldy frameworks that likely could not capture important differences among secondary mathematics teacher preparation programs.

However, the opportunity for critical collaboration in which colleagues ask challenging questions of one another's work with the goal of improving research and mathematics education figured critically into the revision of the frameworks. The *PTA* advisory board proved to be invaluable in this process as they were able to provide multiple perspectives and expertise related to each aspect of the study very early in framework development. In particular, they reminded us to keep our focus on algebra. The collaboration among the *PTA* team members (three faculty and six graduate and two undergraduate students) in weekly meetings also provided another set of perspectives that enhanced the quality of the resulting frameworks.

In spring 2012, these frameworks will be piloted to analyze instructional materials and develop protocols for interviewing instructors and students nearing the end of their mathematics teacher preparation programs. The results of these pilot studies will be used to make final revisions for the main data collection to take place during the 2012–13 academic year. Results from the pilot study and final revisions will be shared during the brief report session.

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SURVEY DESIGN FOR SECONDARY MATHEMATICS TEACHER EDUCATION PROGRAMS: CHALLENGES AND OPPORTUNITIES

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Since the 1980s, desires for equity and higher achievement in mathematics have given rise to changes in policies regarding enrollment in algebra. The federally funded project, Preparing to Teach Algebra (PTA), is designed to investigate opportunities for pre-service secondary mathematics teachers to learn about algebra, algebra teaching, equity issues in algebra learning, and Common Core State Standards for Mathematics. A mixed-method approach is used to integrate data using a web-based national survey of approximately 400 randomly selected secondary teacher preparation programs, case studies of four purposefully chosen secondary teacher preparation programs, and focus group interviews with eight pre-service teachers at the case study institutions. This brief report provides an overview of our sampling and survey development work, highlighting the challenges encountered throughout the process.

Keywords: Mathematics Education; Teacher Education; Algebra; Survey

Objectives

The project, *Preparing to Teach Algebra (PTA)* explores opportunities for pre-service secondary mathematics teachers to learn about algebra, algebra teaching, and equity issues in algebra learning and *Common Core State Standards for Mathematics (CCSSM).* A web-based national survey of secondary mathematics teacher education programs across the United States is utilized as a part of this project to provide a snapshot of the current conditions of preparing pre-service teachers to teach algebra, including attention to the recommendations of professional societies and *CCSSM*. The goal of this brief report is to provide an overview of the sampling process and the development of the survey, and to share the issues and challenges that have arisen in this process and the progress that we have made in addressing them.

Theoretical Framework

In developing the survey, we built upon the work of other scholars who have studied the characteristics of mathematics teacher education programs and how these programs were impacting pre-service teachers. We reviewed previous national surveys of mathematics education programs that included: The Mathematics Teaching in the 21st Century (MT21) Study (Schmidt et al., 2007), TEDS-M Institutional Program Survey (Tatto et al., 2008), Secondary Mathematics Teacher Education Programs in Iowa Survey (Murdock, 1999) and 2000 National Survey of Science and Mathematics Education School Mathematics Program Questionnaire (Horizon, Inc., 2000). Below we outline the goals and results of these studies.

The *2000 National Survey of Science and Mathematics Education Questionnaire* (Horizon, Inc., 2000) gathered information about K–12 mathematics teachers' beliefs, pre-service preparation and their teaching practice from a nationally representative sample of 5,728 mathematics and science teachers in schools across the United States. The survey revealed that a greater proportion of the teachers majored in mathematics than in mathematics education, high school mathematics teachers have relatively strong mathematics content backgrounds and teachers in the higher grades are better prepared to teach mathematics than those in the lower grades.

On the other hand, Murdock (1999) attempted to study all teacher education institutions in Iowa that granted licenses to teach secondary mathematics and provided a detailed description of these programs. The questionnaire he utilized in the study was designed based on the recommendations of the National

Council of Teachers of Mathematics (NCTM) for mathematics teacher education and a similar survey done on science education. The key findings of Murdock's study include: the larger universities had higher graduation rates of mathematics education teachers than the smaller private colleges, the larger universities also had a greater proportion of staff dedicated to mathematics education, and secondary mathematics methods was the most frequently required course by all universities that participated in the study.

Graham, Li, and Buck (2000) conducted an exploratory study of K–12 teacher education programs across 28 institutions. In their study, the survey was conducted to give an overall picture of the status of mathematics teacher education programs in light of the vision and recommendations provided in the following reform documents: *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), *Professional Standards for Teaching Mathematics* (NCTM, 1991), *A Call for Change: Recommendations for the Mathematical Preparation of Teachers* (MAA, 1991), and the *Principles and Standards for School Mathematics* (NCTM, 2000). The survey results from 28 institutions indicated that instructional and assessment practices of methods courses were consistent with recommendations, but mathematics content courses were not and the overall structure of teacher preparation programs was minimally impacted by reform. However, because no sampling design was described in the report, it is difficult to judge the generalizability of these findings.

Recently, international comparative research has been investigating how pre-service teachers acquire mathematical and related pedagogical content knowledge. The Mathematics Teaching in the 21st Century (MT21) Study (Schmidt et al., 2007) examined the backgrounds, course taking and other program activities, and knowledge relevant to teaching mathematics among pre-service teachers preparing to teach Grades 7 or 8 in six countries (Bulgaria, Germany, Mexico, South Korea, Taiwan, and the United States). The results indicated that U.S. pre-service teachers performed near the bottom among the six countries on the algebra and functions subtest.

The Teacher Education and Development Study [TEDS-M] (Tatto et al., 2008) extended the MT21 study by including additional countries to provide further evidence about the knowledge and beliefs of mathematics teachers, as well as the opportunities they had to learn and teach mathematics content and pedagogical knowledge for primary and secondary pre-service teachers. Although the final report of the study has not yet been released, it is worth noting that TEDS-M used a stratified sampling method with primary and secondary level and concurrent or consecutive routes for stratification.

Although these studies provided information about the current state of mathematics education programs in general, little is known about the preparation that pre-service teachers receive to teach algebra. In addition, the generalizability of the findings in most of these studies is rather limited. Thus, by focusing on algebra learning and teaching, the survey utilized by *PTA* is designed to gather information about teacher preparation across a nationally representative sample of secondary mathematics teacher education programs in the U.S.

Methods

Sampling Mathematics Education Programs

A stratified random sampling method was employed to select 400 secondary mathematics education programs within the fifty states. The *Carnegie Foundation's Basic Classification* of baccalaureate, master's and doctoral degree granting institutions was used as a sampling frame. The number of 400 was determined by a priori power analysis of possible statistical analyses which will be utilized on survey responses, and an expected non-response rate (i.e., 50%). With the use of appropriate representative percentages of degree granting institutions, 176, 160, and 64 colleges and universities were, respectively, included in the sample. The websites of the selected schools were then visited to determine the existence of a secondary mathematics education program and a contact person for the program. If no program existed, resampling occurred until 400 institutions with secondary mathematics education programs were found.

Item Development for the National Survey

The first draft of the survey was developed by adapting items selected from the previously mentioned surveys. The first draft consisted of five proposed sections: general program characteristics, opportunities to learn algebra specific content, qualifications for program entry, required courses and staffing information. However, the items gained from the aforementioned surveys primarily asked about general program characteristics. Since our survey seeks to gather information on more than just general program characteristics, new items for the remaining categories were developed extensively by the *PTA* researchers. Each item on the initial draft of the survey was thoroughly discussed and evaluated by research team members for its quality (including ease of response) and alignment with the research questions. As a result, the initial draft was significantly modified by reducing the number of items, changing item formats, and adding new items. The second draft of the survey was further revised based on the suggestions of two mathematics educators on the research team. The items were also evaluated further by a survey expert, a mathematician, and two mathematics educators who serve as the project's external Advisory Board.

Results

Sampling Secondary Mathematics Education Programs

One of the difficulties that arose while ensuring a representative sample of the target population was caused by the fact that there exists no comprehensive list of secondary mathematics education programs to use as a sampling frame. Confirming the difficulty of creating such a list, through the sampling process, we found significant variation in the paths that pre-service teachers can take to become secondary mathematics teachers within an institution and across institutions.

Survey Item Development

Based on the feedback received from multiple sources, the resultant pilot survey contains two sections: (1) general characteristics of secondary school mathematics teacher education programs, and (2) opportunities to learn about algebra, algebra teaching, issues in achieving equity in algebra learning, and the algebra topics included in the *CCSSM*. Questions regarding general characteristics were limited to information about institutional and program characteristics such as the department where the secondary mathematics teacher education program was housed, and where most secondary mathematics education courses were offered. Questions regarding opportunities to learn asked about learning opportunities in a specific course. For example, a sample question utilized in the study is: "*The following courses often offer opportunities to learn school algebra, which of these courses are offered by your program?"* accompanied with a list of ten mathematics content courses (e.g., College Algebra, Linear Algebra etc.). The respondent should give an answer about whether or not the course is offered, if it is required for the degree and the number of credit hours required for the course. Another example, of a sample question utilized in the study: "*The following courses may offer an opportunity to learn about CCSSM. Please select the courses from the list that your institution offers."* The respondent is asked to indicate if the course is offered, if the course is required, if there is an emphasis on algebra and the number of credits for the course

Discussion

Multiple challenges arose in the process of both the survey sampling and survey instrument development. For the survey sampling, confirming that institutions had an established secondary mathematics education program and locating the program coordinator or contact person via the school's website were the major challenges. In cases where we were unable to obtain confirmation of a program's existence or find a contact person, phone calls and/or e-mails were used. Through the process of designing and utilizing the sampling frame as well as gathering contact information, we have received a glimpse of the diversity and complexity of mathematics teacher education in the United States. For the survey development, the lack of previous related studies on mathematics teacher education and operational definitions for the terms to be used in the survey - particularly "Opportunity to Learn Algebra (OTL)" has been particularly challenging. For example, in terms of educational policy OTL can be used to describe the quality of schooling, equal treatment and the fairness of high-stakes accountability (Floden, 2002). The *PTA* researchers have spent a lot of time and effort in determining a definition of OTL that is in alignment with the guiding principles of *PTA* project. *PTA* survey will help describe the diverse nature of mathematics education in the United States

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HYBRID LESSON STUDY: ADDED VALUE OF ONLINE COLLABORATION

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In the Japanese Lesson Study program in Japan and open research lessons in China, teachers work collaboratively to create and reflect upon *research lessons* in order to advance theory and improve on "best practices" (Lewis, 2002). Lesson study is seen to have great promise for improving U.S. math education by developing knowledge of anticipated student responses, contributing to curricular alignment, and developing teacher's self-efficacy (Sibbald, 2009). Research clearly shows that an online community can serve as an environment to support teachers' ongoing collaboration efforts (Barab, Kling, & Gray, 2004). Therefore, we developed a *hybrid lesson study*, an in-person and online collaboration to accommodate schedules and support teachers' ongoing professional development. We report on a partnership between two school districts (80 teachers in grades 3–Algebra 1) and a university in the southwestern United States. There are ten facilitators, or coaches, with prior experiences on lesson study assigned across 18 lesson study groups. Collaborative documents stored on the website include lesson plans, revisions, debriefing notes, as well as edited video clips. These educational objects can reveal aspects of and changes to teachers' *interpretive systems*. The quantitative and qualitative study answers the question, *What aspects of the hybrid model support communal interchanges, foster regular and inclusive participation, and what role do peer coaches and other outside experts play in fostering this?*

Methods

We conducted an analysis involving documenting who posts, how often, and what communal interchanges occur by monitoring and documenting the use of communication tools. Post-hoc analysis reviews thread topics across teams, characterizes topics, and analyzes whether some topics generate more posts. We administered and analyzed a Usage Survey to capture visits to the site without posts. We are conducting a final qualitative analysis, coding the actual content of discussions using grounded theory (Strauss & Corbin, 1994).

Results and Discussion

We are interested in patterns of interaction. The presence of 18 subgroups allows for an analysis of differences. Some of these differences are in the nature of postings at an explanatory level—a factual reporting versus a well-elaborated analysis, explanation, or reflection (Hakkarainen, Lipponen, & Järvelä, 2002). Our goal is to improve research lessons through an online component that evolves with the needs of the participants.

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EXPLORING CLASSROOM DISCOURSE THROUGH AN AGENT-BASED MODEL

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An important aspect of reform includes enabling mathematics teachers to support new forms of discourse that allow students to talk and do mathematics. However, supporting substantive student discourse around mathematics is very difficult for teachers (Sherin, 2002). Therefore, this poster introduces a tool to aid in the transition between teacher-directed discourse and student-directed discourse.

This poster will present a computer model of mathematics classroom discourse to help teachers learn about supporting discourse. The software is an agent-based model built in NetLogo that models different student and teacher interactions during a classroom discussion.

The model was built using several frameworks for classroom discourse from Hufferd-Ackles et al. (2004), Boaler and Brodie (2004), and Chapin, O'Connor, and Anderson (2003) to focus on the mathematical aspects of discourse and the varied teacher moves that support discourse. An agent-based model was chosen so that discourse could be modeled as an emergent phenomena resulting from interactions among students and the teacher (Wilensky, 2001). Following constructionist design principles (Papert, 1993), teachers engaging with the model can change different parameters about how the students and teacher interact with one another. Student parameters include likelihood of sharing ideas and directly responding to ideas of other students. Teacher parameters include likelihood of evaluation, asking generating questions, and asking clarifying questions. Students' ideas have varying levels of clarity, which determines the text students say when they participate. As the model is run, student parameters change based on the interactions in the classroom.

Students and the teacher are visible in the model and what is said each turn is shown above them (one person speaks each turn). The overall discourse can be seen through the transcript of the dialog, a graph of the types of student talk, and a graph of the different teacher moves. Different combinations of parameter levels will produce different types of discourse. Through experimenting with different parameter levels, teachers construct understandings of classroom discourse and the lower-level interactions that create the discourse (such as teacher moves). The model can also be switched to a mode where the teacher engaging with the model chooses the actions for the teacher agent in the model.

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A CLASSROOM OBSERVATION RUBRIC FOR MATHEMATICAL JUSTIFICATION

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Keywords: Reasoning and Proof; Classroom Discourse; Research Methods

As part of our NSF MSP project, we have developed a rubric with 3 spectra for describing levels of justification made explicit in classroom discourse. Our poster will present this rubric, its framework and rationale, and illustrative examples from our classroom observations.

Spectrum 1 describes levels for *justification of a strategy, method, or procedure*, informed by Lambert's transition of focus in classroom discourse from answer to method to justification (1990), and by Simon's account of how generalization and justification co-develop as students reflectively abstract their own goal-directed activity (2011):

<table>
<tr>
<td>0. Students produce result (answer) but do not explain, justify, or show work.</td>
<td>1. Students "show work," listing the steps of the method they used to get the answer.</td>
<td>2. Students provide a basis for the steps, such as naming admissible actions based on what has already been established in the classroom.</td>
<td rowspan="2">3. Students provide an argument for why the steps must work to provide the correct answer. Even if they later check their answer, conviction is not treated as coming from this act. This can indicate an understanding (though not yet an articulation) of the generality of the method.</td>
<td rowspan="2">4. Students provide an argument based on necessity, but also articulate the generality of the method and, if appropriate, address the domain of applicability on which the method works.</td>
</tr>
<tr>
<td colspan="3">+ Students justify their answer by gauging its reasonableness or checking to see that it satisfies the original problem. (This could accompany any of the above discourse types.)</td>
</tr>
</table>

Spectrum 2 describes levels for *justification of a specific (non-general) claim*, particularly by appealing to a general basis. **Spectrum 3** describes levels for *justification of an articulated general claim or property*, adapted largely from Harel and Sowder's proof schemes (1998): (0) No justification; (1) External source of conviction; (2) Empirical reasoning; (3) Deductive reasoning, including recognition of the necessity of the conclusion following from a basis or bases, without attention to the generality of the claim; (4) Same, but with attention to the generality of the claim; and (5) Same, with explicitly-stated basis or bases. We construe "basis" broadly; it could include any established relationship, axiom, property, definition, strategy, theorem, principle, analogous situation, or structure apparent in a particular representation.

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NAVIGATING ASSESSMENT FOR LEARNING AND FORMATIVE ASSESSMENT IN MATHEMATICS EDUCATION: ESTABLISHING A COMMON DEFINITION

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Purpose

Formative assessment (FA) and assessment for learning (AfL) are key phrases in the educational assessment community, yet the exact definitions of these terms remain unclear. Dunn and Mulvenon (2009) have called for "a clear and shared lexicon [to] be established and shared among all educational stakeholders to lead to more productive communication among teachers, researchers, policy makers, parents, and students" (p. 9). The purpose of this research (funded by NSF grant DRL-0733590) is to examine all of the major theoretical papers published on FA and AfL within the context of mathematics education and identify consistent themes to establish a common definition.

Methodology

An exhaustive search of the literature for "assessment for learning" or "formative assessment" along with "mathematics" was performed using Google Scholar and checked against ProQuest and ERIC. Resource lists of each paper were also reviewed. A theoretical sampling method was then used to narrow the sample (Marshall & Rossman, 2011).HyperRESEARCH software was used to open code each document followed by axial coding. Emergent themes were analyzed to develop a definition and model of FA and AfL (Creswell, 2007).

Results and Conclusion

As a whole, the literature suggests that there is no distinction between FA and AfL and the two phrases can be used synonymously. The coding process resulted four primary characteristics of FA and AfL (Student Involvement, Learning and Assessment Expectations, Instructional Changes, and Feedback), each with several overlapping sub-components, as depicted in our model of AfL/FA.This research began as a means to answer the call for clear definitions of the terms "assessment for learning" and "formative assessment." The resulting model is the first step toward providing this universal definition for researchers and practitioners. With this clearly defined, research-based definition of formative assessment and assessment for learning, researchers will be able to more accurately study these classroom actions and dissemination of research will be better able to influence teaching practices.

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Chapter 11

Working Groups

Van Zoest, L. R., Lo, J.-J., & Kratky, J. L. (Eds.). (2012). *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Kalamazoo, MI: Western Michigan University.

ADDRESSING EQUITY AND DIVERSITY ISSUES IN MATHEMATICS EDUCATION

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As the title suggests, this Working Group has a dual focus on issues of mathematics teaching and learning and issues of equity and diversity. We have narrowed the topics discussed at the Working Group in 2009, 2010, and 2011, to focus on mathematics teacher education that incorporates issues of race and power. We will be working on a series of manuscripts attending to social justice, race, teacher educator racial identity, and supporting prospective teachers in mathematics methods courses to integrate students' in and out-of-school mathematics. This work attempts to take a multifaceted approach, aimed at multiple levels from the classroom to broader social structures, within a variety of contexts both in and out of school, and at relationships between researcher and study participants, teachers and students, and teachers and schools.

Keywords: Equity and Diversity; Teacher Education–Inservice/Professional Development; Teacher Education–Preservice; Teacher Knowledge

Brief History

This Working Group builds on and extends the work of the Diversity in Mathematics Education (DiME) Group, one of the Centers for Learning and Teaching (CLT) funded by the National Science Foundation (NSF). DiME is a group of emerging scholars who graduated from three major universities (University of Wisconsin–Madison, University of California–Berkeley, and UCLA). The Center was dedicated to creating a community of scholars poised to address some critical problems facing mathematics education, specifically with respect to issues of equity (or, more accurately, issues of inequity).

The DiME Group (as well as subsets of that group) has already engaged in important scholarly activities. After two years of a cross-campus collaboration dedicated to studying issues framed by the question of why particular groups of students (i.e., poor students, students of color, English learners) fail in school mathematics in comparison to their white (and sometimes Asian) peers, we presented a symposium at AERA 2005 (DiME Group, 2005). This was followed by the writing of a chapter in the recently published *Handbook of Research on Mathematics Teaching and Learning* which examined issues of culture, race, and power in mathematics education (DiME Group, 2007). Further, in an effort to bring together and expand the community of scholars interested in this work, DiME, at AERA in 2008, sponsored a one-day Professional Development session examining equity and diversity issues in Mathematics Education. In addition, DiME members have joined with other scholars in joint presentations and conferences. A book on research of professional development that attends to both equity and mathematics issues has recently been published (Foote, 2010). Many DiME members as well as other scholars contributed to this volume.

Moreover, the Center historically held DiME conferences each summer. These conferences provided a place for fellows and faculty to discuss their current work as well as to hear from leaders in the emerging field of equity and diversity issues in mathematics education. Beginning in the summer of 2008, the DiME Conference opened to non-DiME graduate students with similar research interests from other CLTs such as the Center for the Mathematics Education of Latinos/as (CEMELA), as well as graduate students not affiliated with an NSF CLT. This was initially an attempt to bring together a larger group of emerging scholars whose research focuses on issues of equity and diversity in mathematics education. In addition, DiME graduates, as they have moved to other universities, have begun to work with scholars and graduate students including those with connections to other NSF CLTs such as MetroMath and the Urban Case Studies Project in MAC-MTL whose projects also incorporate issues of equity and diversity in mathematics education. Funding for the DiME project has ceased and the PME Working Group has become a major way in which to keep the conversation going.

It is important to acknowledge some of the people whose work in the field of diversity and equity in mathematics education has been important to our work. Theoretically we have been building on the work of such scholars as Marta Civil (Civil & Bernier, 2006; González, Andrade, Civil, & Moll, 2001), Megan Franke (Franke, Kazemi, & Battey, 2007), Eric Gutstein (Gutstein, 2006), Danny Martin (Martin, 2000), Judit Moschkovitch (Moschkovich, 2002), and Na'ilah Nasir (Nasir, 2002). We have as well been building on the work of our advisors, Tom Carpenter (Carpenter, Fennema, & Franke, 1996), Geoff Saxe (Saxe, 2002), Alan Schoenfeld (Schoenfeld, 2002), and again Megan Franke (Kazemi & Franke, 2004), as well as many others outside of the field of mathematics education.

A significant strand of the work of the DiME Center for Learning and Teaching included implementing professional development programs grounded in teachers' practice and focusing on equity at each site. The research and professional development efforts of DiME scholars are deeply intertwined, and much of the research thus far produced by members of the DiME Group addresses issues of equity within Professional Development. Additionally, since the majority of the DiME graduates, as new professors are engaged in teaching Mathematics Methods courses, the integration of issues of equity with issues of mathematics teaching and learning in Math Methods has become a site of interest for research. We have learned through experience that collaboration is a critical component to our work.

We were pleased for the opportunity offered by the first three years of being a Working Group at PME 2009, PME 2010, and PME 2011 to continue working together as well as to expand the group to include other interested scholars with similar research interests. We were encouraged that our efforts were well received; more than 40 scholars from a wide variety of universities and other educational organizations took part in the Working Group each of the past three years.

Focal Issues

Under the umbrella of attending to equity and diversity issues in mathematics education, researchers are currently focusing on such issues as teaching and classroom interactions, professional development, pre-service teacher education (primarily in mathematics methods classes), student learning (including the learning of particular sub-groups of students such as African American students or English learners), and parent perspectives. Much of the work attempts to contextualize the teaching and learning of mathematics within the local contexts in which it happens, as well as to examine the structures within which this teaching and learning occurs (e.g., large urban, suburban, or rural districts; under-resourced or well-resourced schools; and high-stakes testing environments). How the greater contexts and policies at the national, state, and district level impact the teaching and learning of mathematics at specific local sites is an important issue, as is how issues of culture, race, and power intersect with issues of student achievement and learning in mathematics.

Existing research tends either to focus on professional development in mathematics (e.g., Barnett, 1998; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Kazemi & Franke, 2004; Lewis, 2000; Saxe, Gearhart, & Nasir, 2001; Schifter, 1998; Schifter & Fosnot, 1993; Sherin & vanEs, 2003), or professional development for equity (e.g., Sleeter, 1992, 1997; Lawrence & Tatum, 1997a). Little research exists, however, which examines professional development or mathematics methods courses that integrate both. The effects of these separate bodies of work, one based on mathematics and one based on equity, limits the impact that teachers can have in actual classrooms. The former can help us uncover the complexities of children's mathematical thinking as well as the ways in which curriculum can support mathematical understanding in a number of domains. The latter has produced a body of literature that has helped to reveal educational inequities as well as demonstrated ways in which inequities in the educational enterprise could be overcome.

To bridge these separate bodies of work, the Working Group has begun and will continue to focus on analyzing what counts as mathematics learning, in whose eyes, and how these culturally bound distinctions afford and constrain opportunities for students of color to have access to mathematical trajectories in school and beyond. Further, asking questions about systematic inequities leads to methodologies that allow the researcher to look at multiple levels simultaneously. This research begins to take a multifaceted approach, aimed at multiple levels from the classroom to broader social structures, within a variety of contexts both in and out of school, and at a broad span of relationships including researcher to study participants, teachers to schools, schools to districts, and districts to national policy.

Some of the research questions that the Working Group will consider are:

- What are the characteristics, dispositions, etc. of successful mathematics teachers for *all* students across a variety of local contexts and schools? How do they convey a sense of purpose for learning mathematical content through their instruction?
- How do beginning mathematics teachers perceive and negotiate the multiple challenges of the school context? How do they talk about the challenges and supports for their work in achieving equitable mathematics pedagogy?
- What impediments do teachers face in teaching mathematics for understanding?
- How can mathematics teachers learn to teach mathematics with a culturally relevant approach?
- What does teaching mathematics for social justice look like in a variety of local contexts?
- What are the complexities inherent in teacher learning about equity pedagogy when students come from a variety of cultural and/or linguistic backgrounds all of which may differ from the teacher's background?
- What are dominant discourses of mathematics teachers?
- What ways do we have (or can we develop) of measuring equitable mathematics instruction?
- What is the role of both teachers' and students' academic and mathematics identity in achievement?
- How do students' out-of-school experiences influence their learning of school mathematics?
- What is the role of perceived/historical opportunity on student participation in mathematics?

Plan for Working Group

The overarching goal of the group continues to be to facilitate collaboration within the growing community of scholars and practitioners concerned with understanding and addressing the challenges of attending to issues of equity and diversity in mathematics teacher education. More specifically, we plan to work on a series of articles sharing our journeys with pre-service and in-service mathematics teachers as they: incorporate out of school practices, explicitly examine race, and analyze broader social structures (teaching math for social justice). In addition we intend to examine educator identity around race and teaching about race. The PME Working Group provides an important forum for these scholars to come together with other interested researchers who identify their work as attending to equity and diversity issues within mathematics education in order to develop plans for future research. Our main goal for this year, then, is to continue a sustained collaboration around key issues (theoretical and methodological) related to research design and analysis in studies attending to issues of equity and diversity in mathematics education, and more specifically to begin collaborating on manuscripts that attend to these topics.

Our plans for PME 2012 we will proceed as follows.

SESSION 1:

- Review and discussion of goals of Working Group.
- Introduction of participants.
- Present and discuss manuscript ideas.

SESSION 2:

- Continue presentations and discussions.
- Begin work on manuscripts.

SESSION 3:

- Continue work on manuscripts.
- Organize for ongoing collaboration on manuscripts.
- Developing a tentative agenda for future Working Group meetings.

Previous Work of the Group

The Working Group met for three productive sessions at PME 2009, PME 2010, and PME 2011. In 2009, we identified areas of interest to the participants within the broad area of equity and diversity issues in mathematics education. Much fruitful discussion was had as areas were identified and examined. Over the past three years subgroups meet to consider potential collaborative efforts and provide support. Within these sub-groups, rich conversations ensued regarding theoretical and practical considerations of the topics. In addition, graduate students had the opportunity to share research plans and get feedback. The following are topics covered in the subgroups.

Teacher Education that Frames Mathematics Education as a Social and Political Activity

This sub-group discussed teacher education that frames mathematics as a social and political activity, including multicultural education, teaching math for social justice, funds of knowledge, ethnomathematics, issues of equity and diversity in mathematics, and so forth. The goal was to share resources to improve our own work as teacher educators and to support each other in our research. Long term goals included developing an annotated list of articles, developing an annotated collection of resources (lessons, activities, syllabi, etc.), writing a paper about our differing meanings and approaches to teacher education that frames mathematics as a social and political activity, and conducting research about doing this work across contexts.

This sub-group acknowledged tensions in our work focusing on equity and social justice in relationship to reform mathematics. Frameworks are needed to understand these issues. These can build on work in culturally relevant pedagogy (Ladson-Billings, 1995), teaching for social justice (Gutstein, 2003), funds-of-knowledge (González et al., 2001) as well as more general issues of equity, diversity, social analysis, and critical pedagogy. We need to begin by defining what we mean by these terms (e.g., reform mathematics, social activity, political activity); and how we recognize them in the classroom.

Culturally Relevant and Responsive Mathematics Education (CRRME)

The title reflects our view that mathematics education needs to be both culturally responsive and culturally relevant and a primary goal of this group was to develop a comprehensive collection of the scholarship we draw on to define these terms. We were interested in language, discourse, ethnicity, ways of interacting, family, community, experiences, generational issues, expectations (not high and low, but individual or community's expectations). We wanted to examine what we mean by social justice. Issues such as teaching (a) about social justice (the context), (b) with social justice (status and participation), and (c) for social justice (power and question) were raised. Various aspects of CRRME include (a) local contexts, (b) local associations, (c) using cultural referents, (d) ethnomathematics, (e) critical pedagogy, and (f) teaching "classical" math. We were concerned as well with how literature on culturally relevant pedagogy is grounded in existing theory and research on culture and social constructivism.

Creating Observation Protocols around Instructional Practices

This group was developing a protocol that can measure instructional practice AND be a tool to help teachers improve their instructional practice. The focus was on the importance of improving instruction for students of color; this is our goal. We recognized that protocols have limits. For example, protocols do not necessarily look at microgenesis, teacher change, structural issues, dispositions. At the past PME-NA we reviewed various protocols to examine discuss existing protocols posted on google groups to stimulate rich discussions around questions such as: What do we like? What are they missing? How might we revise, combine, and extend them? Since group members were using various measures and it was not viable to have everyone use the same one, we began to develop a dimension that could be added on to the different protocols. To focus the work we decided to develop a dimension on expanding notions of competence in the classroom.

Language and Discourse Group: Issues around Supporting Mathematical Discourse in Linguistically Diverse Classrooms

This sub-group was interested in examining language diversity in the mathematics classroom. The goal was to define this broadly to be inclusive of the perspectives of teachers, students, and parents.

A Critical Examination of Student Experiences

This sub-group was interested in examining the intersectionality of students' experiences as learners of mathematics and in mathematics classrooms. This involves considering students' mathematics identity in relationship to one or more of their racial, social, cultural, and gender identities. This also includes understanding how structural inequalities shape students' mathematics experience, particularly students from non-dominant groups.

Anticipated Follow-up Activities

Drawing on the conversations that have taken place in the working groups over the past three years, we have developed a plan of action for a series of manuscripts. These manuscripts will build on the aforementioned topics to share experiences, practices, resources, and research in mathematics methods courses that focus on equity.

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COLLABORATING TO INVESTIGATE LIVED AND LIVING MATHEMATICAL EXPERIENCES: THE DIME WORKING GROUP

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The "Developing Investigations of Mathematical Experience" (DIME) working group, initiated 2011, has focused on building a research-based understanding of the interiorized experiential world of a person's lived mathematics. Goals of this research are to characterize individual experiences in ways that acknowledge a person's active and reflective thinking efforts within mathematical contexts linked to emotive dimensions of their lived and living mathematical experiences, in order to inform mathematics teaching practices. To probe these largely unexplored complex domains, we are using both and phenomenological and constructivist theories and methods. We seek to identify and relate expressed indicators of intended and actual mathematical experiences, as they appear to occur in the complex relationships of participants involved in mathematics teaching and learning. The primary goal of the PME-NA Working Group is to involve interested scholars in the ongoing work of the DIME Research Team, through further conceptual analyses related to the phenomena of lived and living mathematical experiences, to extend our epistemological and psychological perspectives for characterizing critical elements, and to identify and address methodological issues inherent in investigating interiorized mathematical experiential phenomena.

Keywords: Affect, Emotion, Beliefs, Attitudes; Research Methods

Brief History of the Working Group

A core membership for this PME-NA Working Group was established through the initial invitational "Planning Conference for WISDOM[e]" conducted September 8–10, 2010 at the University of Wyoming. The background context for that conference was the establishment of three collaborative, interdisciplinary Research Teams connected to four new Ph.D. program research identities: quantitative reasoning, mathematical modeling, technology tools and applications, and lived/living mathematical experience. Each of the teams consists of UW mathematics education, mathematics, and educational cognate faculty, mathematics education doctoral students, and four-to-five young, active Mathematics Education researchers from a variety of national and international universities. Following four invited plenary papers prepared and presented at the conference by key active senior researchers, each team met to frame an initial program of collaborative work. The DIME (Developing Investigations of Mathematical Experience) Team currently includes six UW faculty members, six UW Mathematics Education Ph.D. students, and five faculty members from as many other universities. [Complimentary copies of Volume 1, *New Perspectives and Directions for Collaborative Research in Mathematics Education: Papers from a Planning Conference for WISDOM[e]*, in the WISDOM[e] monograph series were provided to all participants at the 2011 PME-NA conference (Chamberlin, Hatfield, & Belbase, 2011; Hatfield, 2011).]

Through the conduct of the initial Working Group sessions at the 2011 PME-NA conference, we established an important venue to continue and expand discussions of the background perspectives and issues related to investigations of lived and living mathematical experience, to share information, issues, and problems related to ongoing research, to promote interest and potential participation in furthering these and other disciplined inquiries into these phenomena, and to provide continuing support to the team members to collaborate within and across ongoing and future research.

The following individuals attended two or more of the three scheduled sessions in Reno. In addition to the eight individuals from the original WISDOM[e] DIME Research Team, this included seven new participants interested in joining, and contributing to, the collaborative DIME work.

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Following an opening presentation and overall discussion, we quickly organized into four breakout groups to address more focused interest areas framed by the following:

A. Framing "problems and questions" for research on lived/living mathematical experience—what to study?
B. Addressing "issues and concerns" for research methodologies to be used—how it is to be studied?
C. Discussing techniques for using and transform ing data to describe, analyze, and interpr et in ways that illuminate and inform—how is data to be evidentially used?
D. Discussing approaches to reporting what is found and to applying to inform improved educational practices—how results are to be reported and used?

Groups C and D quickly merged as they shared their more specific questions and methods. While interests varied across questions of studying mathematical experiences of students, teachers, and others, many common themes related to fundamental issues of purposes, constructs, methodologies, and ways of reporting emerged. Reporting back sessions led to plenary discussions, followed by subsequent breakout group discussions.

Before we adjourned, participants identified plans for continued discussions toward developing several specific collaborative research efforts, and expressed a positive intention to return to the DIME Working Group at the Kalamazoo PME-NA conference. [The following rationale presents many ideas drawn from the original Working Group proposal, and other sources and discussions being formulated for this relatively new domain of research focused on lived and living mathematical experiences (Hatfield, 2012; Hatfield, Belbase, Chamberlin, & Schnorenberg, 2011)]

Perspectives on *Lived and Living Mathematical Experience*

Today, there is surely an implicit and implied emphasis globally upon high quality mathematical experiences within the goals and expectations for a sound mathematical education. Yet, within our literatures and advocacies for curriculum, for teaching, for learning and development, and for research knowledge there seems to be very little explicit attention to the phenomenal data of an individual's lived/living mathematical experiences. This may be partially explained by the nature of what most mathematicians and mathematics educators seem to focus upon within mathematically productive activity: cognition, thinking, sense making, reasoning, solving, explaining, etc. Research to build models of such focal emphases should surely be a fundamental cornerstone of mathematical education research. But, to further our knowledge of actual lived/living mathematical experience set against such mathematically productive activity frames an important goal DIME research efforts.

Some fundamental questions that we are seeking to investigate include the following:

- What is the nature of lived and living mathematical experiences?
- What are the particular qualities (essences) of human experience that make it a "mathematical experience"?
- Are these experiential qualities that are unique to mathematics, or are they also found more broadly in other human experiences?
- How do identifiable qualities of lived/living mathematical experiences relate to learning mathematics and to psychological models to describe mathematical thinking and feeling?
- In what ways do lived/living mathematical experiences vary or differ among humans, and why?
- What factors related to the individual or the experiential context might frame or affect the actual phenomenon of a high quality "living mathematical experience"?
- In what ways do the experiences of individuals vary within mathematical contexts, such as classroom situations that are intended to engender the "same" kinds of experiences for all participants?
- How do lived mathematical experiences accumulate within differing individuals, and with what more general intellective and emotive consequences?
- In what ways do thinking and feeling aspects of mathematical experience interact within a living experiential context, and with what kinds of impacts or outcomes?
- In what ways might a mathematics educator use what can be learned about lived/living mathematical experiences of students, parents, teacher educators, mathematicians, or others in society to improve educational practices?

These are some of the questions that serve as "starting points" for DIME research. In addition, we are confronting many research methodological questions that are challenging much of our initial struggles to conceptualize viable approaches to such phenomena.

If we want educational outcomes that mirror the empowerment that a deep knowledge and proficiency of mathematics can afford, we must now seek to move beyond our current perspectives and approaches that still seem to produce (for too many of our students) much less than we seek. If we are to understand deeply why so many of our students still achieve very poor understandings of the mathematical ideas and processes after experience with our reformed curricula and our improved teaching methods, materials, and tools, we must begin to penetrate beyond more superficial indicators, such as test scores or written work, or even observed classroom behaviors into the interior world of the student's actual lived/living mathematical experiences where we may be able to identify deeper explanatory aspects of the individual's progressive growth (or deficits) toward greater mathematical knowledge and proficiency.

There is one other rationale that seems increasingly important. Too many of our citizens bear negative feelings toward their own past mathematical education (Hersch & John-Steiner, 2011), but for us educators to help our current students avoid such debilitating beliefs, feelings and attitudes we must begin to understand the nature of their experienced emotions that occur and develop within the mathematical contexts we offer. That is, not only must we understand the intellective dimensions of a person's constructions and reconstructions of their mathematical knowledge and proficiency, but we must also understand the origins and dynamics of emotional dimensions of their experiences, and how the interplay of thinking and reasoning actually functions within their live experiential feelings, emotions and affective schemas in relation to engaged, productive mathematical thinking. It is time to seek to understand the "whole mathematical life" of our students.

The goal of such new understandings is the assumed potential that from such knowledge we mathematics educators will all be helped to engage in improved forms of mathematical education in which enhanced qualities of lived/living mathematical experiences occur for all persons. We seek a pedagogy that understands and honors the experiences of the other (surely a primary intention of humanist philosophy and constructivist epistemology).

It is the goal of the DIME research program to begin to study intentionally the phenomena of lived/living mathematical experiences. Why bother? Are not the traditional views emphasized in intended curriculum, instruction, learning, assessment, evaluation, and research on learning and teaching adequate? Indeed, it is exactly because the goals and strategies of a sound mathematical education today embody a major emphasis upon stimulating, nurturing, and demonstrating high quality thinking arising from particular kinds of intended experiences that we must now seek to understand more clearly the nature of what students and teachers are actually experiencing in the "flow of their math lives" (Csikszentmihalyi, 1990).

Issues in a Psychology of Mathematical Experience

We each know the centrality of our own lived experiences. Even modest personal reflection can lead an individual to a sense of realization that it is specific experiences that shape "who they are, what we know, how we think" in powerful and fundamental ways. This includes what is experienced "outside," in the so-called "real world" that involves our interactions within our proximal environment (our sensory-based physical experiences) and within our interactions and relationships with other minds (our socially-based logical and emotional experiences). But, this also includes interior experiences that occur in our mind within our "interiorized" constructed world, where our thinking and our feeling "being" (person) is shaped and functions in thought (mind).

In DIME, we acknowledge the significance of the typical "mathematical" environment that includes students, teachers, parents, discourse, textbooks, technology tools, classrooms in schools, tasks and tests, the structured contexts of lessons, local, state and national curriculum frameworks, professional preparation and development, mathematicians, societies and cultures, governance, politics—all of the usual cultural and social elements in and around a person's mathematical education today. In doing so, we also accept that the meaning structures for any or all of these elements is a totally idiosyncratic construction for any of the multitudes of persons attending to matters of mathematical education. Now, we seek to "look into" the interior phenomenal world of lived experiences where each individual dynamically encounters and processes their idiosyncratic "mathematical experiences that lead to "who they are and become," mathematically. As educators, perhaps a most important aspect of this involves the dynamics of lived/living experiences that might characterize how experience might possibly bring about change (transformations) in an individual's thought and feeling with respect to mathematics.

Phenomenology begins in the lived world, and seeks to bring to reflective awareness the nature of the events of lived experience (Hegel, 1977; Husserl, 1970, 1982). The principle of intentionality acknowledges an inseparable connection to the world (in our focus, the world of mathematical education) wherein "…we question the world's very secrets and intimacies which are constitutive of the world, and which bring the world as world into being for us and in us" (van Manen, 1990, p. 5). In DIME, we are choosing to study thematic meanings, adopting themes and conducting thematic analyses within our particular orientation to the phenomenon of mathematical experience as "people of mathematics:" all being teachers, teacher educators, students of learning and teaching, mathematically educated, and interested is pedagogic theories. We are trying to be explicit about our individual and shared intentions and orientations as a preparatory anchoring step in our research process.

As such, phenomenology accepts the curriculum of being and becoming (paideia), pursuing understanding of the personal, the individual, set against the background of an understanding of the other, the whole, the communal, or the social. It seeks to explicate phenomena as they present themselves to consciousness—the only access humans have to the world. But, consciousness cannot be described directly (the fallacy of idealism); the world cannot be described directly either (the fallacy of realism); real things in the world are only meaningfully constituted by conscious human beings, and these constructed meanings can only be revealed by the constructing human as inferences.

In our formative research approaches we presume the nature of lived/living experience to be fundamentally an internal construction/re-construction that emphasizes a consciousness of "sense-making," attempted within the unique idiosyncratic mental operations, schemas and constructive mechanisms as they exist and function in the mind of the individual within that lived/living experience. To emphasize Piaget's (Piaget & Inhelder, 1969) theoretical conclusions that both intellective and affective aspects are involved within experience leading to development, we seek to study both as a seamless whole, even as we reject many other dichotomies as false (such as "thinking versus feeling") typical of a strictly modernist structuralism.

Among the constructivist focal constructs we want to consider in our study of lived/living mathematical experience are representation and re-presentation, reflection and reflective analysis and abstraction, intuition and intuitive reasoning, and perturbation and equilibration, and particularly to search for where and how they may be found to function in, impact upon, and in turn be affected by particular experiential contexts. For example, in today's curricular frameworks one sees a major attention given to "representations" and representational activities; students are expected to learn to understand, use, and make various canonical mathematical representations. Yet, what from a study of lived/living mathematical experiences can we find about a student's actual conceptions and views of representations, and especially how they experientially use such images in problem contexts to re-present the conceptual ideas they are presumed to represent? Moreover, von Glasersfeld's (1991) view of cognitive functioning sees representation, re-presentation, and reflective abstraction as inseparable aspects; can we find this exhibited in lived/living mathematical experience?

Against these theoretical lenses, we have identified a variety of issues to confront as we attempt to study lived/living mathematical experiences. Some of these are identified below, and these will be addressed in the activities of the Working Group described next.

1. In what ways is it possible to study the lived/living mathematical experiences of anyone: one self? The "other"? We have found helpful literature that addresses important theoretical perspectives and methodological strategies for this question [e.g., Dewey, 1938; Hegel, 1977; Hurlburt & Akhter, 2006; McLeod, 1964; Moustakas, 1994; Petitmengin, 2006; van Manen 1990]. We have identified these guiding principles:

A. A research methodology for studying lived/living experience includes the theoretical precepts behind the methods—the values and assumptions of phenomenology and constructivism as briefly discussed above. Within a focus on actual experience, we are willing to yield upon procedures or techniques that certain methods, even in qualitative research, attempt to objectify or make more standardized.

B. Our overall framework is adapted from van Manen's (1990) structure "…seen as a dynamic interplay among six research activities:

(1) Turning to a phenomenon, which seriously interests us and commits us to the world;
(2) Investigating experience as we live it rather than as we conceptualize it;
(3) Reflecting on the essential themes that characterize the phenomenon;
(4) Describing the phenomenon through the art of writing and rewriting;
(5) Maintaining a strong and oriented pedagogical relation to the phenomenon;
(6) Balancing the research context by considering parts and whole." (p. 30-31)

C. Some views of phenomenology aim at being "presuppositionless," warding off a tendency to construct or enact a predetermined set of fixed procedures or techniques that would rule-govern the research. We will engage our observations and analysis with general acceptance of this view, while also making efforts to stipulate and articulate, a priori, as many of our individual values, assumptions, beliefs, and attitudes about the phenomenon of "mathematical experience as we can and seems relevant. One view (van Manen, 1990)—*our problem is not that we know too little about it, but that we know too much!* (p. 46). As such, we are predisposed to interpret the nature of the phenomenon before we have even come to grips with the phenomenological questions.

D. In those research contexts where our aim will be to stimulate new experiences within real-time, unfolding events, we adopt the dynamics of constructivist orientations but set aside "teaching or learning" aims, per se (e.g., where questions from the researcher-teacher would seek to provoke particular kinds of mathematical thinking or productions). Rather, within an initial problematic situation posed to provoke or engender lived mathematical experiences, we honor the paths as determined and taken by the person.

E. We seek to address the phenomenon of mathematical experience in a variety of ways. In doing so, we will seek attentively to orient to the phenomenon as we strive to deepen our formulation of the phenomenological questions.

F. We are building a team approach with a purposeful aim of including a variety of perspectives and voices, and this brings opportunities beyond research conducted by one, or even two collaborating scholars. As such, we have adopted views and tactics to mirror what we perceive as formative, developmental research, and to be alert to elements of our inquiry that includes aspects of team building, per se.

2. Whose experiences should be studied, and why? Who are to be the subjects of the research? How are they chosen? Because mathematics as a human endeavor in society and globally is so pervasive and seemingly universal, we foresee a full range of research participants who experience mathematics in a wide array of situations and for a diverse set of reasons. Of course, one orientation we bring to this is the "enterprise" of mathematical education, and this will greatly influence our choices for whose experiences we will try to investigate, and also determine how we frame the contexts and the templates of analysis and interpretation. We want to include at least these in our sampling of lived/living mathematical experiences—ourselves, mathematicians, mathematics teacher educators, mathematics education researchers, pre-service and in-service mathematics teachers across levels of school mathematics, mathematics students across levels of school mathematics, and parents of the students. Different types of participants will allow us to address a variety of aims; specific individuals will be chosen in terms of particular aims and purposes.

3. What are sources or forms of "data of lived/living experience?" How can these be generated in ways that yield penetration into phenomena? Be seen to be accurate (true to the phenomena)? Valid? Reliable? Viable? (Steffe, 2011) We accept the following views about the nature of "data of lived/living experience." The world of lived/living mathematical experience is for us both the source and the object of our research. We each bring strong (yet varying) orientations to it. But, we share a fundamental assumption: *experiential accounts are never identical to lived/living experience itself.*

We already see that the sources and forms of our "data of lived experience can be rich and varied. Yet, we will pursue in each the step of generating written descriptions, and these may be of two kinds: (a) an immediate description of the "*life-world as lived*," or (b) an intermediate (or a mediated) description of the "*life-world as expressed in symbolic form*." While we accept that in this step there occurs interpretation, among team members we will share in an analysis of the description as produced, and engage in intentional interpretation (hermeneutic) to produce a "second-generation description" that purposefully seeks to identify and describe "essences as deeper meanings of the lived mathematical experience."

4. In what ways can research subjects be directly engaged in experiential mathematical situations while informing the researcher about what they are experiencing? It is one of the basic assumptions of phenomenology that experience will be changed within an attempt to introspect—to "rise above" and give attention to the experience while it is occurring, and that the distinctions between what is introspection and retrospection are blurred (thus our use of "lived/living"). While we accept this assumption in theory, we also want to explore this phenomenon. Petitmengin (2006) used an interview method aimed at helping a person to become aware of her subjective experience and to describe it with great precision. Hurlburt and Akhter (2006) used a "descriptive experience sampling" method to explore inner experience. Their subjects were prompted by an electronic beeper which they carried as they moved in their natural environments. When the beep sounded, they were trained to "capture their inner experience and jot notes about it"; they discussed it during a later expositional interview.

As researcher-observers interactively engaged in mathematical situations we've posed for the purposes of engendering active involvement by, say, a student, we intend *to become a part of the student's experiences*, per se—to get into "the flow" of what the student is experiencing (Csikszentmihalyi, 1990).

"Being there" can mean (to the student) that, as a part of their unfolding experience, we ask questions. While these questions will primarily focus upon their activity and their "thinking aloud" verbalizations related to it, at times we will ask a question pointing more directly to their conscious reflection upon their experiences, per se. In some sense, we anticipate that such a question can result in a kind of interruption of the flow of experiences related to the posed mathematical situation; we intend to ask the student later about the effects of such questions upon their perceptions of the flow of their experience. We anticipate there will be variable impacts reported by different persons, but we also expect that across time and successive observational interviews, individuals will develop a greater capacity for minimizing (or perception of) the disruptive effects of such questions.

5. How do we, as researchers, conduct analysis and interpretation of data to build accurate portrayals of lived/living mathematical experience? Key to phenomenological or constructivist research is analysis and interpretation as an observer, or teacher-researcher. Critical to either are the struggles to maintain, as much as possible, open thinking in which one consciously acknowledges potential biases and avoids "projecting" one's own experiences onto the situation. Intentional "bracketing" is attempted; members of our team are trying to describe, *a priori* to individual or shared acts of analysis and interpretation, our individual perspectives on what we each may "see" in the phenomena of lived mathematical experience—we will refer to these as our "initial construct views" (ICV). We will share these written ICV descriptions, and try to use these when we are subsequently engaged in analyses and interpretations of our observations and descriptions. We are unsure about exactly how we will use these, but one could be when we disagree about what we "see" in a particular protocol or description; we may be able to find reasons for a researcher's interpretations in the anchoring viewpoints they expressed in their ICV.

We believe that through a team approach in which multiple descriptions can be generated independently and then discussed and debated, we will likely achieve more sensitively accurate interpretations of the phenomena—"negotiated meanings." Across experiential episodes we will look for consistencies as well as variation, thus being attuned to elements of cross-validation of the qualities to be found in a person's lived mathematical experiences. Also, in these we will look for how the nature of experiences for each subject may change; again, as educators we seek and expect change—growth and development as a consequence of something we call "mathematical experience."

6. In what ways can we "make sense of" our study of the observed lived/living mathematical experience in relation to its implications and potential applications? This question speaks to the important intent that our research results lead beyond information about the interior "mathematical life," although such research-based information is generally lacking today. Our goals include the possible implications for such new insights and understandings about lived mathematical experience to impact upon future mathematical experiences. We foresee possible benefits to the ways that mathematics teachers seek to stimulate and engage their students to engender certain qualities of experiences they might have. Information about what is occurring in the lived mathematical experiences of students may prompt changes in curricular topics, placements, or treatments. Deeper understandings from studies of experience may raise implications for testing and assessment strategies related to mathematics learning.

Specific Plans for DIME Sessions at PME-NA 2012

Overall, the three sessions will be conducted in "workshop/working" format, structured to inform and orient new, interested participants, while allowing returning WG participants to report and share, to interact toward clarifying basic issues and challenges in conducting research on lived/living mathematical experiences, and to extend individual and collaborative activities that could occur prior to PME-NA 2013. Specifically, the following tentative plans have been developed in collaboration with prior DIME WG participants.

Session 1

Overview and background for DIME research activities (Larry L. Hatfield; 15 min.)
Aspects of "mathematical experience" implied in mathematical education literatures (Travis A. Olson; 10 min.)
Investigating mathematical caring relations within lived mathematical experiences in cross-cultural classrooms (Amy Hackenberg; 10 min.)
Plenary group discussion (15 min.)
Breakout groups working discussions—issues and questions (30 min.)

Session 2

Investigating experiential aspects of student mathematical conjectures and conjecturing (Andy Norton; 10 min.)
Using lived-experience descriptions as a mathematics learning tool for teachers and students (Yuichi Handa; 10 min.)
Examining beliefs and practices of pre-service secondary mathematics teachers in terms of mathematical experiences (Shashi Belbase; 10 min.)
Plenary group discussion (10 min.)
Breakout groups working discussions—framing specific research ideas (30 min.)

Session 3

Reporting lived/living experiences from geometric problem solving of pre-service elementary teachers: A case study example (Larry Hatfield; 10 min.)
Plenary group discussion (10 min.)
Breakout groups working discussions—developing collaborative plans (30 min.)
Closing plenary session—reporting back; looking to PME-NA 2013 (25 min.)

Feedback and input from all discussions will be collected; post-conference written summaries for each sub-group will be prepared at UW and distributed to all WG participants, and posted to the WISDOM^e website.

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DEVELOPING ELEMENTARY TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING: BUILDING ON WHAT WE KNOW

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Lack of appropriate and adequate mathematical knowledge of elementary teachers is a major concern in mathematics education. This new working group on ***Developing Elementary Teachers' Mathematical Knowledge for Teaching*** *aims to explore this issue from multiple, diverse perspectives and hopes that mathematicians and mathematics educators will come together to discuss this topic. At the initial meeting, participants will have an opportunity to share their current research, raise important questions for this work, and collectively bring together current thinking in the field. In the second matting, the group will identify possible directions for future research, and in the third meeting, collaborative groups will further specify areas of interest and plan future work together. The long-term goal is to develop an edited book that examines issues around the development of mathematics knowledge for teaching from various perspectives.*

Keywords: Mathematical Knowledge for Teaching; Elementary School Education; Teacher Education–Preservice; Affect, Emotion, Beliefs, and Attitudes

Most elementary teachers in North America are prepared as generalists during initial teacher preparation, ultimately assuming positions in schools that require the teaching of all subjects in the elementary classroom. Such all-purpose preparation has led to a corpus of elementary teachers needing improved mathematical knowledge for effectively teaching cognitively-demanding mathematics curricula. As this knowledge has been linked to student learning and achievement, the inadequate preparation of these teachers is disconcerting. Wu (2009) fittingly asserted:

> The fact that many elementary teachers lack the knowledge to teach mathematics with coherence, precision, and reasoning is a systemic problem with grave consequences. Let us note that this is not the fault of our elementary teachers. Indeed, it is altogether unrealistic to expect our generalist elementary teachers to possess this kind of knowledge. (p. 14)

This concern is pervasive in the field of mathematics teacher education (see Ball, Hill, & Bass, 2005; Hill, 2010; Ma, 1999; Rowland, Huckstep, & Thwaites, 2005), and has prompted many institutions of higher education to require specialized mathematics content courses for prospective elementary teachers. These courses, referred to here as *Math for Teachers* (MFT) courses, aim to provide a thorough understanding of elementary mathematics concepts in order to develop prospective teachers' confidence and flexibility in teaching mathematics (Kilpatrick, Swafford, & Findell, 2001; Williams, 2008). Such courses were endorsed in a report by the National Mathematics Advisory Panel (Greenberg & Walsh, 2008), which argued for increasing the number of required courses. In Canada as well, a recent policy statement constructed by a working group at the Canadian Mathematics Education Forum (Kajander & Jarvis, 2009) argued for the need for at least 100 hours of specialized mathematics content courses for prospective teachers. It is currently the case that many, although not all, MFT courses are taught in mathematics departments by mathematics faculty.

The purpose of this new working group is to examine how teacher preparation experiences, particularly those focused on developing specialized mathematical knowledge, can support the development of mathematical knowledge for teaching of elementary teachers. The group will critically consider several factors influencing this development, including: teacher mathematical preparation program and course experiences, teacher mathematical knowledge for teaching, and teacher mathematical beliefs and affect. This list is not intended to be all-encompassing, as the working group will be encouraged to initially consider any factors and issues of interest related to the broad topic. The following presents overviews of theoretical perspectives and research that are relevant to this working group.

Teacher Knowledge in Mathematics

Elementary teachers require well-developed knowledge of mathematics to be effective in their teaching (Hill, 2010), particularly to support their ability to create *standards-based learning environments* that promote classroom discourse and foster conceptual understandings of mathematics. Teachers with only procedural understandings of mathematics cannot be expected to teach at in-depth, conceptual levels that lead to the complex understandings needed for mathematical applications in today's society (Ma, 1999). Accordingly, a recent emphasis has been placed on *mathematical knowledge for teaching* (MKT) (Ball, Thames, & Phelps, 2008; Hill, 2010; Kajander, 2010a) as a necessary precursor for the effective use of knowledge of mathematics *in* teaching (Ball, Lubienski, & Mewborn, 2001).

Though there is general agreement on the need for highly developed mathematical knowledge, this knowledge is complex, and a breadth of viewpoints and assumptions complicate the discourse (Kajander et al., 2010). There have been multiple efforts to precisely define the nature of this knowledge (Ball, Hill, & Bass, 2005; Kajander, 2010b; Rowland, Huckstep, & Thwaites, 2005). Shulman (1986) defines *subject matter knowledge* (SMK) as knowledge of the discipline, including substantive and syntactic knowledge, and *pedagogical content knowledge* (PCK) as "ways of representing the subject which make it comprehensible to others" (p. 9). In Shulman's view, PCK includes teachers' knowledge of effective ways of representing concepts, particularly the most cogent examples, illustrations, explanations, or demonstrations, to make these concepts lucid to students. Foss and Kleinsasser (1996) explain that SMK and PCK, respectively, involve "knowing the content of a subject or discipline and being aware of the means by which the content is taught" (p. 430).

In recent years, researchers (Ball & Forzani, 2010; Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008; Hill 2010) have further refined Shulman's distinctions between SMK and PCK, proposing a *specialized content knowledge* (SCK). SCK is defined as "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Ball et al., 2008, p. 399). Specifically, Schilling and Hill (2007) explain that SCK "consists of mathematical tasks such as representing numbers and operations with pictures or manipulatives, examining and generalizing from non-standard solution methods, and providing explanations for mathematical ideas or procedures" (p. 78).

Teacher Affect and Beliefs

Across many years, research has established a robust relationship between teachers' affect and beliefs and their instructional practices, especially teacher thinking and behaviors such as instructional decision-making and use of curriculum materials (Buehl & Fives, 2009; Clark & Peterson, 1986; Phillip, 2007; Raymond, 1997; Romberg & Carpenter, 1986; Thompson, 1992; Wilson & Cooney, 2002). Philippou and Christou (2002) suggest that affect is on a continuum, with feelings, emotions, and anxiety at one end, characterized as short-lived and highly charged. Beliefs are at the other end, typified as more cognitive and stable in nature. Teachers' affect and beliefs develop over time (Richardson, 1996), beginning with their own experiences as students in K–12 classrooms, during what Lortie (1975) terms the *apprenticeship of observation*, and are well-established by the time they enter college (Pajares, 1996). Phillip (2007) underscores the importance of beliefs about mathematics when he asserts, "For many students studying mathematics, the feelings and beliefs that they carry away *about* the subject are at least as important as the knowledge they learn *of* the subject" (p. 257). The affect and beliefs of prospective teachers have an influence on how and what they learn and should be targets of change during the teacher preparation process (Feiman-Nemser, 2001; Richardson, 1996), though programs are constrained by the limited time available to effect changes.

Two influential teacher belief constructs include pedagogical beliefs (i.e., beliefs about teaching and learning) and teaching efficacy beliefs (i.e., beliefs about capabilities to teach effectively and influence student learning). The reform perspective on teaching recommended by the National Council of Teachers of Mathematics (NCTM, 2000) requires substantial paradigmatic shifts for many prospective elementary teachers. This perspective includes the amalgamation of mathematical content and process standards, requiring a pedagogical approach different than the traditional instruction found in many classrooms. Many of the suggestions espoused by the NCTM (1991, 2000) are grounded in a constructivist-compatible method of teaching, in which teachers provide learning tasks intended to develop students' understandings of concepts and procedures in ways that foster students' abilities to solve problems and to reason and communicate mathematically. Studies on changing the mathematical pedagogical beliefs of elementary prospective teachers have largely focused on moving them toward this reform perspective. These studies often examined change during only one course or semester and although some reported success in achieving the desired effects, others did not (Holm & Kajander, 2012; Kalchman, 2011; Philipp et al., 2007; Swars, Hart, Smith, & Smith, 2007; Swars, Smith, Smith, & Hart, 2009; Wilkins & Brand, 2004).

Moreover, teaching efficacy beliefs have been linked with classroom instructional strategies, willingness to embrace educational reform, commitment to teaching, and student achievement. Disconcertingly, many prospective teachers' mathematics teaching efficacy beliefs are built on their past experiences with traditional, behaviorist methods of mathematics instruction, thus generating a tension between the development of cognitively-oriented pedagogical beliefs and a sense of efficaciousness with respect to teaching in this way (Smith, 1996). Although there are numerous studies on generalized teaching efficacy, there has been less research specifically on the mathematics teaching efficacy of elementary prospective teachers. Most of the extant studies examined the effects of one mathematics pedagogy course and indicated significant increases in mathematics teaching efficacy upon completion of the course (Huinker & Madison, 1997; Kajander, 2010a; Kalchman, 2011; Swars, Hart, Smith, & Smith, 2007; Swars, Smith, Smith, & Hart, 2009; Utley, Moseley, & Bryant, 2005).

Prospective Elementary Teachers' Experiences in Mathematics Content Courses

There is limited research on prospective elementary teachers' experiences in mathematics content courses, including MFT courses. A search of the literature revealed that most studies involving prospective elementary teachers were largely conducted in the context of mathematics pedagogy courses rather than content courses. As described below, the few found related to content courses examined the effects of reform practices on the students.

Royster, Harris, and Schoeps (1999) investigated general mathematics content courses that had been modified to align the curriculum and instructors' practices with reform recommendations. They surveyed 182 college students from various majors at the beginning and end of semester-long courses to assess changes in dispositions. The findings revealed elementary education majors showed the greatest positive changes in dispositions toward mathematics in comparison to other majors, suggesting a positive outcome of reform-based practices with prospective elementary teachers in mathematics content courses. In another study, Lubinski and Otto (2004), operating under the assumption that if elementary prospective teachers are expected to teach mathematics for understanding then they must learn mathematics in this way, implemented a reform curriculum in a semester-long MFT course comprised of 28 elementary education majors. After conducting pre- and post-surveys and post-interviews with the students, the findings revealed the students' beliefs and attitudes about mathematics had been positively influenced by the course. In an additional study focusing on MFT courses, Philipp et al. (2007) examined the effects of field experiences and focusing on children's thinking on the mathematical content knowledge and beliefs of prospective elementary teachers. Those who studied children's mathematical thinking while learning mathematics developed more reform-oriented beliefs about mathematics teaching and learning and improved their mathematical content knowledge more so than those who did not. Hart and Swars (2009) was the only study found that specifically looked at the experiences of pre-service teachers in traditional math for teachers content courses. The preservice teacher they interviewed reported that much of the coursework was highly traditional (e.g., lecture, power-points) and unrelated to their experience in elementary classrooms. They also reported that the mathematics they were learning was not *useful* for them as elementary teachers.

Post-Secondary Mathematics Instruction

Some studies have examined the student perspective on effective teaching of mathematics at the post-secondary level (Hart, Oesterle, & Swars, 2011; Hart & Swars, 2009; Powell-Mikle, 2003; Schulze & Tomal, 2006; Weinstein, 2004). Hart and Swars (2009) interviewed 12 prospective teachers to examine their experience in MFT courses. They reported that the students found traditional pedagogical practices such as lecture and power point were not effective for learning the mathematics, and strategies such as small-group work and discussions were more productive for learning. Powell-Mikle (2003) interviewed six college students, who reported specific classroom characteristics that supported their learning in mathematics courses: adequate instructor availability, clear instructor explanations, prevalent classroom discourse, and a caring classroom environment. In a similar study, Weinstein (2004) surveyed and observed 18 college mathematics students. The students indicated the most effective mathematics instructors spent less time lecturing and more time developing student confidence. In a much larger study, Schulze and Tomal (2006) surveyed 2,042 college students, identifying factors that contribute to a *chilly* mathematics classroom climate, i.e., a climate that creates a negative atmosphere for teaching and learning. Three factors most likely to contribute to this type of climate were: (1) the difficulty level of course content, (2) the teaching style and personality of the professor, and (3) the personality styles of classmates.

Studies including mathematics instructors as participants are sparse. VanMinden, Walls, and Nardi's (1998) compared the pedagogical knowledge-structure representations of 15 participants, including three university mathematicians, three university mathematics pedagogy professors, along with three elementary, three middle, and three high school teachers. Their findings showed the university mathematicians' pedagogical knowledge was the least learner-focused and most algorithmic compared to the other participants. The mathematicians characterized teaching and learning as didactic and unidirectional or as the transmission of information to passive learners. The researchers concluded that reflective practitioners at all levels "need links between subject-matter concepts in the mathematics domain and pedagogical content knowledge" (p. 354).

Speer and Wagner (2009) examined the knowledge needed by mathematics instructors (referred to in their study as professional mathematicians) to implement inquiry-oriented curricula in post-secondary mathematics courses. They identified three areas of requisite knowledge for successful implementation of reform teaching practices. The areas were: knowledge of typical ways students' think (correctly and incorrectly) about the task or content in question, knowledge of the curriculum in use, and knowledge to support the specialized type of mathematical work teachers do when dissecting and analyzing students' expressions of their ideas. In a related study, Wagner, Speer, and Rossa (2007) examined the teaching practices of one professional mathematician and identified similar forms of knowledge beyond content knowledge that are needed in reform-oriented teaching. They also pointed out that traditional instructional practices frequently fail to support student-centered learning.

Only one study focussed specifically on mathematics faculty who teach MFT courses. Through an analysis of interviews with ten MFT instructors, Oesterle (2011) found wide diversity in their interpretations of the course, including their goals with respect to knowledge-for-teaching, beliefs about mathematics, and the attitudes/emotions of their students (Oesterle & Liljedahl, 2009). These differences influenced how they set priorities for affective and cognitive goals within the course and their teaching approaches. Furthermore, instructors were found to experience tensions (Oesterle, 2010) as they strove to meet these goals within the contexts of their own experience, knowledge, and beliefs, their perceptions of their students, and the demands of their institutions.

Proposed Features of Effective Teacher Preparation Programs in Mathematics

In order to develop mathematical knowledge for teaching and positively change mathematical beliefs and affect, several features of effective teacher preparation programs have been proposed (National

Research Council, 2001; Sowder, 2007), though their actualization is wrought with challenges. One such feature is increased collaboration between mathematicians and mathematics educators, including a unification and coordination of mathematics content and teaching methods courses. This was also recommended in a study by Hart, Oesterle, and Swars (2011) after contrasting the perspectives of students and instructors in these courses. Further, effective programs are expected to build in-depth understandings of mathematics that are useful to prospective teachers. These understandings should be developed via inquiry and problem solving, grounded in theories of how people learn. Provision of sufficient time for problem-based mathematics learning is particularly challenging during programs in which the only available courses related to mathematics are pedagogy or "methods" courses. For example, Kajander (2010a) studied the development of mathematical understanding as needed for teaching during standard mathematics methods courses which included a substantial focus on mathematics content development, and found that while significant growth was evident, much more time would ideally be required.

Specific methods for prompting prospective teacher learning in mathematics have been suggested, including studying children's thinking, using K–12 curriculum materials, examining case studies of teaching and learning, and relating coursework to K–12 classrooms (Phillip et al., 2008; Sowder, 2007; Swars, Smith, Smith, Hart, & Carothers, 2011). In addition, there has been a call for increasing the number of MFT courses required of elementary prospective teachers to include 9 semester hours focusing on elementary mathematics (CBMS, 2001; Greenberg & Walsh, 2008).

The Working Group

This new working group aims to explore these and other issues related to developing the mathematical knowledge for teaching of elementary teachers. The group is interested in multiple, diverse perspectives and hopes that mathematicians and mathematics educators will come together to explore this topic. At the initial meeting, the organizers will ask participants to: briefly share current research they are engaged in, raise important questions and directions for this work, and collectively bring together current thinking in the field. The long-term goal is to develop an edited book that examines these issues from various perspectives.

Following are statements from the four organizers of the group regarding their current work and thinking on the topic:

Lynn Hart is a Professor of mathematics education at Georgia State University in Atlanta, Georgia. Her research on teacher change in the areas of beliefs and content knowledge spans over 2 decades. She is particularly interested in how the culture of mathematics departments and mathematics education departments impact the instruction received by elementary teachers, the mathematical learning of elementary teachers, and the beliefs and attitudes of the teachers. Most recently she participated in a 4-year study of potential elementary teachers examining their beliefs and content knowledge; and, she participated in a comparison study of the perspectives of instructors and students in mathematics content courses for elementary teachers.

Susan Swars is an Associate Professor of mathematics education and STEM Coordinator at Georgia State University in Atlanta, Georgia. Her research interests include the study of elementary teacher change and learning during mathematics teacher preparation, with a particular focus on the outcomes of mathematical knowledge for teaching and mathematical beliefs. As co-director for a master's degree program in elementary education with an embedded K–5 mathematics endorsement, she is currently studying the development of Elementary Mathematics Specialists in the context of this program. She is also involved in a longitudinal, comparative study of the effects of increased MFT courses on prospective elementary teachers' knowledge and beliefs. One key finding is that increasing the number of MFT courses did not result in notable differences in mathematical knowledge for teaching. This work also involved phenomenological exploration of prospective elementary teachers' experiences in MFT courses, which resulted in a two-dimensional model for learning in MFT courses around the dimensions of caring classroom practices and curricular relevance.

Susan Oesterle is a mathematics instructor at a two-year college who has been teaching content courses for pre-service elementary teachers for over 20 years. In a recent research study, she examined the experience of teaching MFT courses from the perspectives of ten mathematics faculty who teach these courses, exposing a wide diversity in the approaches taken and providing an analysis of the tensions experienced by these instructors as they seek to understand and meet the needs and expectations of their students, their institutions, and the community. Her current research interests are focussed on the knowledge, beliefs, and attitudes characteristic of effective teachers of mathematics, and more specifically on how to support development of these skills and attributes within mathematics content courses.

Ann Kajander is an Associate Professor of mathematics education at Lakehead University in Thunder Bay, Ontario, who has been involved in research on mathematical development of preservice teachers for eight years. She currently teaches a new course in mathematics-for-teaching for elementary teacher-candidates. Prior to teaching in the Faculty of Education, she taught mathematics at the secondary level as well as teaching an introductory mathematics course for prospective elementary teachers housed in the Mathematics Department for ten years. Teachers' understanding of the construction of mathematical models, in particular those that support linking concrete classroom explorations to the development of more formal methods in an explicit manner, are a particular focus of her research. Ann is currently completing a book manuscript for a new resource for prospective teachers on mathematics-for-teaching.

Plans for Three Working Group Sessions

Three sessions are planned for the working group. The organizers anticipate addressing the following ideas and questions in the sessions. The organizers will make sure that the discussion stays related to the broad topic of the development of mathematical knowledge for teaching of elementary prospective teachers during teacher preparation. Within the frame of mathematical knowledge for teaching:

Session 1: What do we know from the research . . .

(a) about the mathematical teacher preparation of elementary teachers;
(b) about what they bring to their teacher preparation programs;
(c) about what they acquire in their programs;
(d) and, about what they encounter in the schools during internships/field placements?

In this session participants will have an opportunity to share their current work on the topic, as well as relevant research. Topics could include, but are not limited to, program experiences, coursework, teacher beliefs and affect, pedagogy, and teacher knowledge.

Session 2: What do we want to explore about providing experiences during elementary teacher preparation that develop mathematical knowledge for teaching?

(a) How can we examine the effectiveness of program experiences and course topics/approaches/learning environments?
(b) Might a specific set of topics, tasks, and questions, as well as suggestions for learning environments, be developed that would best support mathematical development?
(c) What are the views of different stakeholders about program experiences and course topics/approaches/learning environments?
(d) Is there a best learning environment for mathematics courses for elementary teachers?

In this session participants will raise areas of interest for researchers and begin to form sub-groups to explore different questions.

Session 3: Setting goals for our work together

In this session, individual research teams will set goals for their work together; and the working group will set goals for our collective work to be continued in 2013. An overall long-term goal for the working group is to produce a book on current research in the area.

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HOW CAN MODELS OF MATHEMATICAL DEVELOPMENT BE STRUCTURED, REPRESENTED, COMMUNICATED, AND USED IN FORMATIVE ASSESSMENT?

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The purpose of this working group is to identify ways in which teachers and students can use models of mathematical development (learning progressions, trajectories) productively as part of a formative assessment process. This is the second year for the group. Last year, we made progress on representing such models in ways that are relevant to instruction, and on ways to use them in the development of diagnostic tasks for use in formative assessment. The focus this year is the use of such models to organize curricula and lesson plans around student learning. In the sessions, we will build upon last year's work, and, starting from some existing "seed" examples, participants will identify and explore different ways of using these models to organize lesson plans, pacing guides, and learning activities in support of formative assessment. We will reflect on implications for organizational structures within schools, such as Professional Learning Communities. We will identify interesting research questions that may provide insight into how models of mathematical development might affect teachers' use of formative assessment. The session will balance issues in current practice and on-going research. Follow-on activities include expanding and evaluating different approaches through collaborations among participants, and convening again to learn from our collective experiences.

Keywords: Learning Trajectories (or Progressions); Teacher Education–Inservice/Professional Development; Teacher Knowledge

Models of mathematical development hold promise in driving teaching, learning, and assessment (Heritage, 2008) but significant progress needs to be made on several fronts for their effective use. Areas of exploration range from the identification of learning progressions as an essential attribute of effective formative assessment (CCSSO, 2008) to the role that learning progressions play in the articulation of standards, e.g., the Common Core State Standards (CCSSI, 2010), to their value in organizing teaching around how student learning develops. Given that this is a relatively unexplored area, but has substantial practical value in improving mathematical teaching and learning, this year's working group will focus on exploring issues related to the use of learning progressions in organizing curriculum, classroom activities, and assessment.

In previous work (Harris, Wylie, & Bauer (2011)) proposed an approach for teachers to develop teaching progressions (similar to hypothetical learning trajectories proposed by Simon & Tzur, 2004) that integrate models of student's mathematical development with broader curriculum materials to achieve targeted learning goals. In a research proposal that grew out of discussions at last year's working group, we began to refine these ideas to consider how such teaching progressions would affect and provide value to teacher professional development activities and lesson planning. While still falling into the overall scope of the working group, the intent this year is to focus on this more specific, and timely, question:

> "What are ways of using learning progressions to inform formative assessment, instructional approaches, support, and resources to help students learn?"

We will build on the interest and work completed last year with a coherent set of sessions that will address the question. We summarize last year's session to provide a context for the continued work, and then describe how the three sessions this year will be structured.

Activities and Outcomes from the 2011 PME Working Group

For the 2011 Working Group we focused on the question "How can models of mathematical development be structured, represented, communicated and used in formative assessment?" Our interest in conducting the session grew out of our work in building assessment tasks for formative purposes, and using models of mathematical development (also called learning trajectories or learning progressions) to guide development. Thus the title of the session captured the multiple aspects of our interest in developmental models: how can they be structured, represented, communicated, and used. While learning progressions can be used to inform summative assessment, our specific interest was in formative assessment, and we wanted to learn from the experiences of others to both inform our work and add to the general understanding of developmental models.

Our goal over the three sessions was to explore different ways of representing models of mathematical development with teachers and students specifically in mind, to apply some of the representation ideas to an example of a developmental model, and to consider how members of the working group might continue a dialogue around these issues beyond the conference. The first part of the first session provided set-up for what was to follow in the later sessions.

Session One

We set the stage using an example from Grant Wiggins (1998) in which industrial arts' students were able to compare their weld to a set of five welds that were characteristic of different levels of development in the skill of welding. In doing so students: assessed at what level they were performing; determined their next learning goals; and understood what specific next things they needed to do to improve. We posed the question of whether this could be done in mathematics, wondering how we could get students to compare their thinking to the different levels of development, and relatedly, how we might support their next steps in learning. Each session of the working group had a different focus, shown below:

1. Overview of project and working group goals. Focus on alternative structures/representations of the developmental models for use in formative assessment, and how to support students' use of the models.
2. Small group work on specific models and alternative representations.
3. Identify research questions and associated approaches for addressing them based on work in sessions 1 and 2.

During the first session we provided some background information on one of our current projects that was the catalyst for our interest, along with establishing common ground for the group's understanding of formative assessment. We have an IES-funded Goal 5 project (Grant Reference R305A100518) that runs from July 2010 to June 2014. The focus of this assessment development project is on the development of formative assessment, with respect to three developmental models or learning progressions, to support instruction for middle school mathematics teachers. The three developmental models focus on the following mathematical concepts: Equality and Variable; Linear Functions; Proportional Reasoning

Within this project we have two distinctly different, but complementary, approaches to formative assessment: a locator (or placement) test, and incremental tasks. These two assessment types play different roles in the instructional process. The locator test, as the name suggests, "locates" students with respect to their levels of understanding across the three models. The incremental tasks have a finer-grained focus, and target the transition between two levels, thus playing both an assessment and an instructional role. The incremental tasks can provide additional evidence of student knowledge and skills to update a teacher's understanding of where students are in their thinking, and to support student learning as they transition from one level to the next within a progression. We envision that the locator test would be used one or two times during a school year, whereas a teacher might draw on a much greater number of the incremental tasks to supplement instructional and assessment approaches in any given curriculum with a focus on topics related to these learning progressions.

We introduced three possible ways in which learning progressions play an important role within formative assessment. First, they can be used to inform assessment developers about important content and conceptual understanding to be assessed. Second, teachers can use learning progressions to better understand how student thinking can develop over time with respect to key disciplinary concepts. In addition, tying assessment evidence to the learning progressions helps teachers think how students cluster together in terms of similar learning patterns. The third role that learning progressions may play is informing students about their own mathematical thinking, in the same way that the examples of the welds with various problems helped the students assess their own welds and identify weaknesses. This is the area in which we had done the least exploration and so it seems like a fruitful topic to pursue with the working group. We finished the first session by asking the group to consider the following question: “What information should be included in a model and how should that be represented to developers, teachers and/or students to support learning?” Critical ideas and suggestions from the first session were as follows:

1. It is important to not only identify individually important learning progressions within mathematics, but also to show how they are interconnected to support teachers’ thinking across progressions. Within a progression, connection points to other progressions should be identified.
2. Connections should also be made to curriculum concepts to support teacher use of the learning progressions, for example, where in mathematics would a naïve understanding of variable present significant difficulties for students.
3. Multiple representations, such as concept maps, may be useful
4. It is critical to identify misconceptions within the levels of the progressions, in addition to how a student’s understanding of a concept evolves from novice to expert.
5. While the levels within a progression are presented as a linear sequence, it is important to clearly present the idea that we do not expect all students to progress similarly in an “upward” linear fashion: students’ level may vary with context, they may regress at times, and they may skip entire levels.
6. Example tasks with responses—or examples of student discourse—could be used to illustrate what student thinking looks like—or sounds like—at each level.

Session Two

During the second session, we presented one Linear Functions task to the group to provide a concrete example on which to base the discussion. The task was one of the incremental tasks that focused on the transition from level 3 to level 4 in the Linear Functions model. The group was asked to imagine a teacher and class of students using the learning progression and the task and to consider what they would change, and what additional elements or materials would be helpful for students and teachers. Ideas and suggestions from the second session were as follows:

- There needs to be a tool that helps the teacher evaluate student work—not just a single solution path, in other words a more extended rubric.
- As a supplement to the rubric, provide help with “distractor analysis” for relevant questions, and note things for a teacher to listen for, which might be evidence for particular student understandings or misunderstandings. Suggest appropriate probes or comments that a teacher might make.
- Consider ways to use the idea of student traffic lighting (signaling whether understanding is red (almost none), yellow (some confusion), or green (clear)) to support self-assessment through the task?
- Use the language of the transitions to write learning intentions for each task.
- Set up PowerPoint so that questions could be used one at a time, without revealing the entire task.
- Create an “overarching question” that is task specific that could be used as a pre/post question for students. For example, for the task reviewed, students could consider just the context provided with an overarching question, “Which fair costs more?” The question is quite open-ended, but if students track their thinking via journal entries, they should be able to recognize changes in their thinking.
- Support teachers’ information processing by giving them a flow chart to show branching points with the questions and suggest either specific actions or points at which action might be needed if students were struggling.

Session Three

During the third session, we had a broader conversation regarding next steps, although we had fewer attendees at this final session than we had at previous ones. There was definite interest in maintaining the relationships and continuing the work next year.

What We Have Done Since the Conference

Since the conference we have established a research partnership with one of the group members and submitted an NSF MSP grant proposal to further explore how learning progressions can be used as an organizing framework to support professional development for both pre-service and in-service teachers. It had become clear at the conference that we had developed complementary resources for mathematics teachers, and that both groups would be stronger together as a result of exploring these ideas.

Plan for This Year's Sessions

This year's working group sessions will focus on another important aspect of the use of learning progressions in supporting teaching and learning: organizing instruction that guides and is informed by teachers' formative assessment practices. Wiliam (2004) defines three steps for assessment to function formatively: "it needs to identify where learners are in their learning, where they are going, and how to get there" (p. 5). A learning progression provides both a longer term view to guide instruction along with more immediate goals. The full progression presents teachers with a view of the nature of expertise with respect to the particular concept or concepts that is the focus of the progression. In addition, the progression provides a description of more immediate learning goals. A learning progression can be structured as a series of transitions to identify important "starting and landing" places in learning to support the first two steps of formative assessment. Using learning progressions to organize material and activities across a series of lessons can enable teachers to gather evidence of students' level of understanding, help decide next learning goals, and then connect to new learning activities to move students toward those goals.

Figure 1 below presents a schematic of a family of approaches teachers may use to organize, carry out, and revise instructional plans. This process includes (1) identifying learning goals and curricular components, (2) identifying the learning progressions and other materials that inform patterns of student learning, (3) gauging students' current levels of understanding, (4) revising lesson plans to meet current student needs, (5) carrying out math activities that support learning, (6) updating understanding of student learning to continue the process. Layered on top of this cycle is a larger cycle of long term planning. Over the course of the three sessions we will engage in lesson plan design activities to develop and reflect on some possible ways of organizing instructional plans around learning progressions.

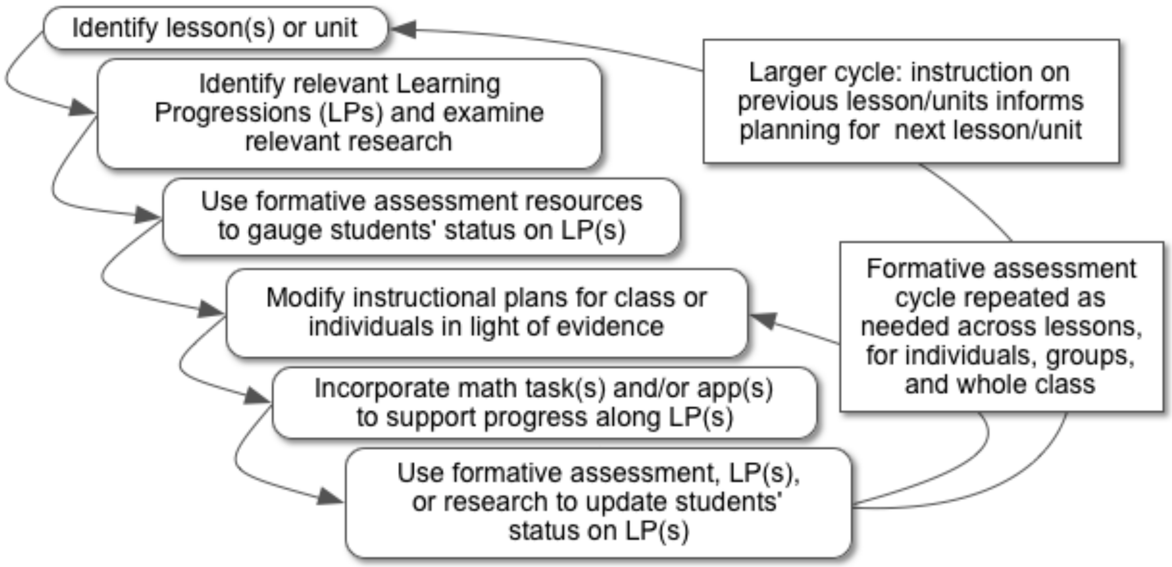

Figure 1: Cycles of formative assessment using learning progressions

In session one we will: (a) identify attendees' background and interests with respect to learning progressions and lesson planning; (b) review and discuss prior work of the group and advances in the field of learning progressions, with respect to lesson planning, identifying student thinking, and supporting next instructional steps; (c) identify one or more progressions of interest to the group, and one or more parts of curricula on which to focus the design activities in the remaining two sessions. We will also discuss the formative assessment process, with a focus on using evidence to take "next instructional steps" to ensure that lesson planning incorporates the idea of contingency planning into the initial development, in other words anticipating a range of student responses and planning instruction for those various categories of response.

In session two we will work in small groups to develop a lesson plan for a short series of lessons that would draw on one or more learning progressions, and some of the additional resources presented (Entries from ATP encyclopedia, incremental tasks and online manipulatives). As we debrief from the experience of organizing materials and activities, attending to curriculum while still drawing on the learning progressions, we will synthesize our experiences into a set of steps or routines that a teacher might follow in order to incorporate learning progressions into her lesson planning.

In session three we will reflect on the plans, discuss how to support teachers incorporating learning progressions into their instruction (what are the hindrances and affordances), and identify opportunities for tryouts and research questions.

As a result of the three sessions we expect to have (1) worked out examples of instructional plans organized with respect to learning progressions, (2) a set of approaches for creating instructional approaches based upon learning progressions, (3) research questions to further explore as a group, (4) ideas and specific plans for collaborative projects in 2013.

Follow-On Activities

Throughout the year, participants may explore the use of the representations of models and ideas for use in formative assessment in their research or in their practice. It is the hope of the organizers that early pre-proposal explorations are carried out and one or more research proposals may be generated. The work will also inform an aspect of an existing grant on which the organizers are working. As noted earlier the organizers have submitted an NSF grant that will provide opportunities to explore how best to support early career teachers which will provide some opportunities to pilot the emerging ideas from this working group session and will provide feedback to the working group participants on their ideas. The intent is to reconvene at the next PME-NA to share experiences on specific collaborations that developed, to discuss progress on aspects of the work, and to plan continued efforts in this area.

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MATHEMATICS LEARNING DISABILITIES

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The purpose of this working group is to examine how students with mathematics learning disabilities (MLDs) can be effectively taught and assessed in mathematics. Traditionally, research and instruction related to students with MLDs has focused on procedural skills. However, this working group is rooted in a twofold premise: (a) students with disabilities are capable of and need to develop conceptual understanding and mathematical reasoning skills, and (b) special education instruction and assessment needs to transition towards this focus. Participants will discuss pairing conceptual diagnosis (assessment) with instructional interventions, in order to achieve better mathematical learning for students with disabilities.

Keywords: Equity and Diversity; Instructional Activities and Practices; Assessment and Evaluation

Brief Overview of the Working Group

A need exists to promote increased conversation, research, and practice in relation to the intersection of mathematics teaching and learning and students with MLD. Substantial work exists that focuses on cognition, learning, and development of students in general education. Yet, much of the available research related to students with MLD has historically focused on rote memorization of basic skills. As a result, much less is known about the mathematical thinking, learning, and development of students with MLD. Our Working Group is designed to create an opportunity for researchers and practitioners interested in disability and mathematics for collaborating in the examination of perspectives on this important dimension of the psychology of mathematics education.

Moving towards the active study of how students with MLD develop mathematics concepts and skills has several implications for both research and practice. First, practitioners in both general and special education can gain essential knowledge of how to differentiate instruction related to alternative pathways of understanding mathematics concepts evidenced by diverse learners. Secondly, by studying atypical development, researchers gain a richer understanding of how cognitive processes involved in learning essential mathematical concepts evidence themselves. As mathematical cognitive changes in MLD occur over a long period of time, research of these changes can provide insights into processes that may be hard to detect and document in general education. Finally, active study of the development of mathematics concepts and skills for students with MLD provides both researchers and practitioners mechanisms for moving toward a methodological focus on pedagogy rooted in assessment of what these students' *are* capable of learning. For the purposes of starting our conversations, "Disability" as it relates to students with MLD includes:

- learning disabilities specific to mathematics
- students with cognitive differences in how they understand and process number
- students with language-based learning disabilities who struggle with mathematics
- students who are placed in special education either through traditional assessments or are at Tier 3 of a Response-to-Intervention (RTI) learning disabilities identification program.

We plan to focus discussions around several central themes, including: (a) conceptually rich teaching and learning for students with MLD, (b) unique learning challenges facing students with MLD along with historical and alternative perspectives on disability, and (c) methodologies for assessing student's difficulties and evaluating the effectiveness of interventions. We invite researchers and educators interested in issues of disability and mathematics to participate.

Issues Relating to Psychology of Mathematics Education

A Case for Conceptually Rich Teaching and Learning for Students with MLD

Currently, special education researchers and teachers focus almost exclusively on students' mastery of procedural skills, such as basic number combinations and ability to execute mathematical algorithms (Jackson & Neel, 2006; Fuchs et al., 2010; Geary, 2010; Swanson, 2007; Kameenui & Carnine, 1998). A recent literature review comparing instructional domains for students with disabilities found that the majority of research conducted in the field of special education addressed basic computation and problem solving, with the primary focus placed on mnemonics, cognitive strategy instruction, or curriculum-based measurement (Van Garderen, Scheuermann, Jackson, & Hampton, 2009). Instructional practices, including task analysis (breaking up skills into steps), flash cards (Cole & Washburn-Moses, 2010), and turning complex problems into decontextualized steps that need to be memorized and followed, have been advocated for in order to increase the proficiency in problem solving and higher order mathematical skills of students with MLD. Yet the focus on primarily procedures-driven instruction and rote memorization of skills seems to result in students' incomplete and inaccurate understanding of fundamental mathematics concepts as well as a lack of retention and/or transfer (Baroody, 2011). The inability of students with MLD to transfer, retain, and fundamentally understand mathematical concepts continues to plague intervention research efforts in the field of special education (Rittle-Johnson & Alibali, 1999).

Crucial for rich mathematical understandings that enable retention and transfer of fundamental concepts is the iterative development of conceptual understanding along with procedural proficiency (Rittle-Johnson, Siegler, & Alibali, 2001; Rittle-Johnson & Koedinger, 2005). Rittle-Johnson and Alibali (1999) noted that conceptual knowledge constrained procedural generalization. In particular, conceptual knowledge could aid children in mindfully avoiding the use of procedures that fail to work in novel situations. Additionally, an ability to understand and manipulate different mathematical representations to conceptually navigate a mathematical context contributes to conceptual understanding and procedural skill (Ball, 1993; Kaput, 1987; Rittle-Johnson et al., 2001). It seems that any investigation into mathematical cognition, whether related to disability or not, must fundamentally engage with issues of conceptual understanding.

A focus on procedural skills limits students with disabilities' access to the general education curriculum, which is a requirement of the Individuals with Disabilities Educational Improvement Act (Maccini & Gagnon, 2002). In mathematics, access to the general education curriculum means addressing problem-solving, mathematical reasoning, and communication of mathematical thinking as advocated by the National Council of Teachers of Mathematics (NCTM) Standards (2000). To accomplish these Standards, mathematics educators need to actively engage students in making conjectures, justifying and questioning each other's ideas, and operating in ways that lead to deep levels of mathematical understanding (Kazemi & Stipek, 2001; Lampert, 1990; Martino & Maher, 1999; Yackel, 2002).

Unique Learning Challenges Facing Students with MLD: Historical and Alternative Perspectives on Disability

We are advocating that students with disabilities be given access to mathematically significant curriculum through a new emphasis on conceptual understanding and mathematical reasoning. We are not claiming that access to such a curriculum can be obtained by using the same techniques that have been used with the general education population. One important reason is that students with disabilities often have different activation patterns in their brains when solving mathematics problems than typically developing students (Butterworth, Varma, & Laurillard, 2011; Davis et al., 2009). Therefore, we should anticipate that they will have different ways of solving problems, will rely upon different resources, and also might need alternative instructional approaches.

Students with disabilities are often viewed by their teachers as less developed than their peers, as opposed to "differently developed." Such a deficit ("less") lens seems to underlie a view of these students' learning as including the same skills and happening in essentially the same ways as their peers, only several years behind. In turn, this lens seems to orient prescriptive pedagogical approaches focused on addressing only the lowest order basic skills. Reliance on a "deficit" model is fundamentally problematic because it involves defining a person in negative terms instead of what she or he *is able* to learn. Identifying an absence of something necessitates that researchers identify what an individual *should* have (or be "fixed"). Both the presumption of a "norm" and the practice of classifying some individuals as "deficient," have been widely criticized by scholars of disability studies (Gallagher, 2004). Davis (2006) argues that the concept of "normalcy" has long been used as a way of classifying some individuals as deficient with respect to race, class, and gender, and currently it is employed to "create the 'problem' of the disabled person" (p. 3). We argue that it is more productive to conceptualize disability in terms of natural biological differences, and will employ a positive lens on ability as the root for pedagogy and research.

Such a lens is supported by a Vygotskian perspective, which provides a productive alternative for conceptualizing disability. Vygotsky claimed that a student who has a disability "is not simply a child less developed than his peers but is a child who has developed differently" (Vygotsky, 1929/1993, p. 30). Thus, individuals' biological differences result not in deficient development but in a different path of development. The research supporting this theoretical orientation is strongest in the field of specific learning disabilities (SLD). Strong converging evidence supports the validity of the concept of SLD, and its subcategory MLD. This evidence is compelling because it converges across different indicators and methodologies. The central concept of specific learning disabilities (SLD) involves disorders of learning and cognition that are intrinsic to the individual. SLD affect a relatively narrow range of academic and performance outcomes. SLD may occur in combination with other disabling conditions, but they are not primarily due to other conditions, such as mental retardation, behavioral disturbance, lack of opportunities to learn, or primary sensory deficits.

Recently, Johnson, Humphrey, Mellard, Woods, and Swanson (2010) conducted a meta-analysis of 32 studies to examine the cognitive processing differences between students with SLD and typically achieving peers. With regard to mathematics, their main finding was that despite having intelligence scores within the average range students with MLD were severely impaired in mathematics ability. These students also had difficulties with executive functioning, processing speed, and short-term memory, which suggests that these processes may be implicated in their brains' execution of mathematics.

There is some evidence that MLD stems from neurological disorders that affect the brain's ability to receive, process, store, and respond to information (Dehaene, 1997; Dehaene & Akhavein, 1995). Recent neuroimaging studies have supported the neurological basis of MLD by demonstrating that students with MLD tend to have less activation in the parietal lobe when processing numbers, have less gray matter in the parietal lobe, and have fewer connections between the various parietal regions than typically developing students (Butterworth, Varma, & Laurillard, 2011). Davis et al. (2009) utilized functional magnetic resonance imaging (fMRI) to explore the brain patterns of activation associated with different levels of performance in exact and approximate calculation tasks. They found significant differences between well-defined cohorts of children with mathematical calculation difficulties (MLD) and their typically developing (TD) peers. Both groups of children activated the same network of brain regions; however, children in the MLD group had significantly increased activation in parietal, frontal, and cingulate cortices during calculation tasks. Most of the differences occurred in anatomical brain regions associated with cognitive resources such as executive functioning and working memory, which are known to support higher level arithmetic skills but are not specific to mathematical processing. These findings provided evidence that children with MLD use the same types of problem solving strategies as TD children, but their weak mathematical processing system causes them to employ a more developmentally immature and less efficient form of the strategies. As students with mathematical disabilities have different activation rates in their brains as they solve mathematics, we should anticipate that they may have different ways of solving problems, will rely upon different resources, and also might need alternative instructional approaches.

Recent research in the area of MLD has also made advances in understanding the cognitive and academic profiles of students with MLD (Compton, Fuchs, Fuchs, Lambert, & Hamlett, 2012; Davis et al., 2009). Compton et al. (2012) explained how students with MLD experience unexpected pockets of strengths and weaknesses across cognitive dimensions or academic domains. The idea, referred to as the specificity hypothesis, is that MLD involves specific rather than generalized learning difficulty and that for each individual neurological functions selectively impair some but not other areas of cognitive functioning. For example, a student with difficulties in retrieving math facts, may have a strength in understanding arithmetic properties (Geary et al., 2007). As we learn more about the relative strengths and weakness associated with MLD, we can build on a student's strengths to promote learning of weak areas of academic performance.

Common Characteristics of MLD

In spite of individual differences, there are some common characteristics of students with MLD. They tend to have a delayed adoption and understanding of efficient counting strategies (Geary, Bow-Thomas, & Yao, 1992), due to inadequate construction of one-to-one correspondence and an over-reliance on adjacency. Thus, students with MLD are slow to grasp and use retrieval strategies for adding basic facts, using back-up strategies such as counting all and counting from the first number throughout elementary school (Ostad, 1997). Their reliance on counting tangible objects means that they continue counting on their fingers long beyond the duration in typical peers (Butterworth, Varma, & Laurillard, 2011). Students with MLD also seem to have problems in many aspects of basic number sense, such as comparing magnitudes of numbers by quickly visualizing a number line or transforming simple word problems into simple equations (Jordan, Hanich, & Kaplan, 2003; Fuchs et al., 2005). In addition, two studies (DiPerna, Lei, & Reid, 2007; Fuchs et al., 2005) have found that teachers' ratings of a child's attention span and task persistence are good indicators of the student's subsequent problems in learning mathematics. Each student with MLD may exhibit all or a subset of these difficulties, and the precise nature of how they affect learning vary across students. This makes any profile of MLD difficult to validate, as every student may have qualitatively different strengths and weaknesses, and accordingly respond differently to instruction.

Conceptual Diagnosis Based Pedagogy—Assessing Students' Mathematics and Evaluating Effectiveness of Interventions

A pedagogical approach to be explored and advanced during this Working Group's meetings is one that focuses on promoting conceptual learning (cognitive change) in MLD. This approach is rooted in a constructivist stance (Piaget, 1985; von Glasersfeld, 1995), particularly the notion of assimilation, which stresses the need to build instruction on what students *already know* and are *able* to *think/do*. That is, teaching needs to be sensitive, relevant, and adaptive to students' available ways of operating mathematically (Steffe, 1990). To this end, teachers must learn how to: (a) diagnose students' available conceptions, and (b) design and use learning situations that both reactivate these conceptions and lead to intended transformations in these conceptions.

Building on Simon's core idea of hypothetical learning trajectories, Tzur (2008) has articulated such an adaptive pedagogy, which revolves around the *Teaching Triad* notion: (a) students' current conceptions, (b) goals for students' learning (intended math), and (c) tasks/activities to promote progression from the former to the latter. Key here is that in designing every lesson one proceeds from conceptual diagnosis of the mathematics *students are capable of thinking/doing*. That is, assessment methods need to focus on dynamic (formative) inquiry into student understandings, as opposed to on testing correct and incorrect answers per se. This day-to-day diagnosis, obtained via engaging students in solving tasks and probing for their reasoning processes, gives way to selecting goals for students' intended learning. Building on this diagnosis, a mathematics lesson begins with problems that students can successfully solve on their own, which Vygotsky (1978) referred as the *Zone of Actual Development* (see also Tzur & Lambert, 2011). Recent studies of mathematics teaching in China (Jin, 2012; Jin & Tzur, 2011) revealed a strategic, targeted method, *Bridging*, which is geared specifically toward both: (a) reactivating mathematical conceptions the teacher supposes all students know, and (b) directing their thinking to the new, intended ideas. While working with students with MLD, Tzur et al. (McClintock et al., 2011; Tzur et al., 2009; Woodward et al., 2009; Xin et al., 2009; Zhang et al., 2009) have been piloting and studying this adaptive approach with high levels of success in promoting substantial conceptual advances (e.g., concept of number, multiplicative reasoning). The adaptive pedagogy approach seems to assist in moving students out of previous 'disable' labeled grouping (Tzur et al., 2010) and in reducing such labeling due to inadequate teaching practices. Our Working Group will include brief presentations and further explorations (and theorizing) of the adaptive pedagogy (conceptual diagnosis based) approach, as we believe it can become a core for teaching and studying MLDs' conceptual understandings.

Plan for Working Group

The aim of this working group is to facilitate collaboration amongst researchers and educators concerned with mathematics education for students with disabilities. The main goal is to address the transition of instructional and assessment practices from an emphasis on (deficient) procedural knowledge to a more balanced and constructive focus on conceptual understanding and enhanced mathematical reasoning skills. Engaging in scholarly discussions of theoretical and pedagogical concerns as they pertain to both special and mathematics education will facilitate collaborative efforts to further support instructional needs of students with disabilities. This working group intends to accomplish the following: (a) examine evidence of students with MLD engaging in mathematical tasks, (b) discuss student engagement across varying perspectives (e.g. mathematics education, special education), (c) discuss means of assessing mathematical reasoning of students with disabilities, and (d) development of a research agenda addressing outcomes of the working group.

These goals are further outlined across sessions as follows.

Session 1: Mathematics Learning Disabilities—Mapping the Terrain

- Discuss goals for the working group
- Introduction of participants, including their interests in participating
- Presentation of video clips of students with MLD engaging in mathematics
- Small group discussion:
  - Varying perspectives/interpretations of the student work
  - Comparison of approaches for students with and without MLD
  - Challenges/differences of teaching students with MLD
  - Produce concept map
- Large group discussion:
  - Share out important points from small group discussions

Session 2: Assessment and Instruction

- View the same video clips as on Day 1, but look at it with a diagnostic pedagogy lens
- Small group discussion:
  - What did you learn about this student's mathematical thinking from this video?
  - What other questions would you ask the student to gain a deeper understanding of their mathematical thinking?
  - What other tasks would you pose to the student?
  - What could your next instructional move be in order to help the student develop a more sophisticated level of understanding?
  - How does this video clip fit in with current research? How does it challenge current research?
- Large group discussion:
  - Share out important points from small group discussions

Session 3: Developing a collaborative plan for research and teaching

- Develop research and teaching agendas based on discussions from session 1 and 2

Anticipated Follow-up Activities

This working group will continue working on research problems of common interest and will prepare/propose a monograph/special issue to a leading journal in the field.

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MEASURING INSTRUCTION IN RELATION TO CURRICULUM USE

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This working group explores tools for measuring instructional activities and practices in relation to curriculum use. This work involves examining kinds of teacher capacities (knowledge, abilities, ways of understanding and acting) needed to design and enact lessons by using mathematics curriculum materials effectively. This working group consists of three projects that have investigated teacher knowledge and capacities by developing tools to measure instructional activities and practices. Each of the three subgroups presents their work regarding the tool they developed to measure instruction, in order to facilitate discussions and reflections on different approaches to measuring instructional activities and practices, and teacher capacities needed for effective design and enactment of a lesson. The goals of the working group are to generate interaction about instructional activities and practices and the relationship between written and enacted curricula, and to examine and critique tools developed to measure this relationship.

Keywords: Instructional Activities and Practices; Curriculum; Teacher Knowledge

This working group explores tools for and approaches to measuring instructional activities and practices in connection to curriculum material use. The goals of the working group are to generate interaction about instructional activities and practices and the relationship between written and enacted curricula, and to examine and critique tools developed to measure this relationship.

History of the Working Group

This working group focuses on the overarching question: *How can instructional activities and practices be measured to investigate the kinds of teacher capacities needed for effective curriculum use?* This question emerges from our ongoing explorations of the relationship between written and enacted curricula.

This working group consists of three subgroups: (1) Learning Mathematics for Teaching (LMT) project begun at the University of Michigan and continued in part at Harvard University, (2) CME Project Mathematical Practices Implementation (MPI) Study by the Education Development Center (EDC), and (3) Improving Curriculum Use for Better Teaching (ICUBiT) project by the University of Pennsylvania and Western Michigan University. All of these subgroups have investigated teacher knowledge and capacities that are required to teach mathematics effectively. The LMT project focuses on measuring the mathematical knowledge for teaching and mathematical quality of instruction revealed in actual teaching; the other two projects examine teacher knowledge and capacities that are required for effective curriculum use to design and enact instruction. Each subgroup has developed a tool to measure instructional activities and practices with slightly different emphases. The MPI study focuses on the secondary level and targets a particular curriculum program developed at EDC for high school students, whereas the other two examine the elementary/middle school level and do not have any particular type of curriculum in mind. In fact, the LMT and the ICUBiT projects intend to develop a tool to measure teacher knowledge and capacities regardless of the types of curriculum programs used in instruction. The LMT project has completed the development of their tool, the Mathematical Quality of Instruction (MQI) instrument. In contrast, tools by the MPI and the ICUBiT teams are still under development.

These three projects have a history of collaboration and exchange of ideas. Members of each team have served as advisors to and/or commentators on the projects of others. For example, the ICUBiT team referenced the MQI in their work to develop a tool, and members of the LMT project helped the ICUBiT team shape their initial approach; in turn, a member of the ICUBiT project helped LMT project members' multi-case studies investigating the relationship among teacher knowledge, MQI, and curriculum use. Moreover, the MPI and the ICUBiT teams organized a working group at the Research Presession of the annual meeting of the National Council of Teacher of Mathematics in April 2012. This NCTM session aimed at presenting tools in the process of development from the two subgroups and receiving feedback from the audience for further refinement. Building on our previous distinct yet related and collaborative work, we will organize this PME-NA working group to further elaborate teacher capacities required for instructional design and enactment, and issues involved in measuring instruction in terms of teacher capacities.

Issue: Measuring Instructional Activities and Practices

The issue that this working group explores in the psychology of mathematics education is measuring instructional activities and practices in relation to curriculum use. This work involves examining kinds of teacher capacities (knowledge, abilities, ways of understanding and acting) needed to design and enact lessons by using mathematics curriculum materials effectively.

Published curriculum programs, past and present, are relied upon heavily to improve instruction and increase student learning. This reliance is based on the assumptions that curricular programs are designed with a set of intentions, and that teachers will use curriculum programs in ways that are aligned with those intentions. Yet, there is substantial variation in how teachers use mathematics curriculum materials. These variations are functions of teachers' personal characteristics, such as knowledge and dispositions, as well as their pedagogical choices and interactions with students during curricular enactment (Stein, Remillard, & Smith, 2007). Curriculum materials designed to foster ambitious math instruction place even greater demands on teachers than conventional materials (Stein & Kim, 2009) and require different kinds of capacity to use them. Brown (2009) used the term *pedagogical design capacity (PDC)* to refer to "an individual teacher's ability to perceive and mobilize curricular resources in order to design instruction" (p. 29). This capacity is at the heart of how teachers use and enact the written curriculum. We see PDC as an interaction between teacher capacity and the resource available in the particular curriculum. Consider the following vignette that illustrates aspects of PDC.

This vignette is taken from a fourth-grade classroom in which students learn the concept of mean and the procedure to find the mean of a given data set. After a warm-up activity on median and mode, MR, the teacher, distributes square tiles to each group: five groups of 4 and one group of 5 students. MR asks each of the students in each group of four to count out 12, 8, 17, and 3 tiles, respectively. MR asks each of the five students in the last group to count out 12, 8, 17, 3, and 10 tiles, respectively. Then, MR asks each group to make one big pile of tiles in the center of their table and share/divide them evenly. A few minutes later, MR convenes a whole group discussion. MR summarizes student strategies. "Some students directly used paper and a pencil to do calculation without using the tiles [they remember this procedure from a science class]. Some students distributed tiles to each student until there were no more tiles left. And, some did both." Then, she collects tiles back from students and asks students to open their textbooks. She explains the computational approach of figuring out the mean, i.e., finding the sum of the numbers and dividing the sum by the number of those numbers. Then, she says that both strategies (i.e., computation and using tiles) work well. She emphasizes several times that the divisor is the number of data points. This is also done through several practice problems provided in student workbook pages of the lesson. Finally, students do practice problems individually.

The curriculum program (Charles et al., 2008) that this class uses includes in the student textbook the following statements:

> Like the median and the mode, the mean tells what is typical of the numbers in a set of data. The mean is sometimes called the average. To find an average, all the items are combined and divided equally. The diagram at the right shows that 5 is the average of 7, 4, and 4 (p. 404).

In the diagram (see Figure 1), there are three different-colored cube towers, representing 7 (blue), 4 (red), and 4 (green), respectively. To the right of these towers is a set of 3 cube towers of 5 cubes each, illustrating the redistribution of the cubes. The color coordination indicates two green cubes are moved from the tall green tower to the other shorter towers in the process of leveling out. Right below this diagram, the computational procedures to find the mean are introduced with two different example data sets. The guides/notes for teachers for this diagram include one sentence: "Explain that an average levels off or evens out the numbers in the data set so that all the numbers are the same" (p. 404). In the lesson overview at the beginning of the written lesson, there is a similar description: "Averaging involves distributing numerical data evenly across a set of numbers and provides a single number to describe what is typical in that set of numbers" (p. 404A). However, the rest of the guidance focuses on the computational procedures. Moreover, the cube towers presented were just an illustration, not suggested as an activity for students.

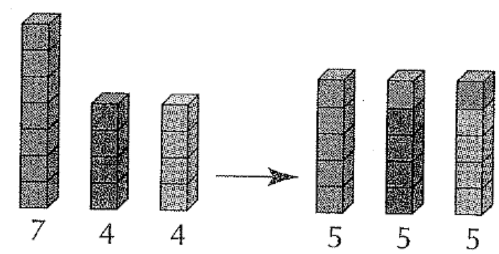

Figure 1: Diagram representing mean

A closer look at the vignette and the written lesson along with a follow-up interview with the teacher reveals a few things about MR's knowledge and capacities. On the one hand, she follows what is included in the book. For example, she uses tiles to help students explore mean, even though cubes, not tiles, are shown in the textbook and the way tiles are used does not clearly reflect the way the cube towers are shown in the textbook. She also highlights computational procedures as presented in the textbook. On the other hand, she creates her own data set for students to think about the mean using the tiles (i.e., 12, 8, 17, 3 instead of 7, 4, 4). She carefully chooses the four numbers that are easy to work with (e.g., $12 + 8 = 20$, $17 + 3 = 20$, and $40/4 = 10$) and includes a relatively large number, i.e., 17, and a very small number, 3, in the data set. Her intention is that by working with numbers that are easy for adding and dividing, students think about the mean rather than computation itself. She also wants her students to see that small and large numbers are leveled out to the mean. She recognizes the importance of "leveling out" in the meaning of the mean from the parts she read in the curriculum. In fact, she points out this right away when she is asked, "Did this [parts she indicated that she read, including the two quotes on the meaning of mean from pages 404 and 404A] help your planning or teaching? How? If not, why not?" She elaborates how important the meaning of mean is and how visual the tile activity is in illustrating the meaning.

Despite such recognition and careful planning, what students experience with the tile activity does not help them see the meaning of mean explicitly. This activity is not even clearly connected to the computational procedures in the observed lesson. The tile activity is hands-on and visual, and yet hardly accomplishes what the curriculum may intend. Students do not level the tile towers out; rather they make one big pile of tiles and divide them equally among four or five smaller piles as they are told to do so. The meaning of mean mentioned in the curriculum and recognized by MR (i.e., leveling out) is not visible or explored in any of the six student groups and in the subsequent whole group discussion. The focus is on the procedures to find the mean by using addition and division appropriately.

This vignette illustrates the centrality of teacher knowledge and capacities in designing and enacting lessons using curriculum materials. MR certainly exhibits her knowledge about mean, demonstrating her understanding of the conceptual meaning of mean. She also develops a thorough plan to teach the lesson by using the hands-on activity and carefully choosing a sample data set. Her capacity to teach the lesson that connects the conceptual representation and the procedures is, however, somehow limited. Her knowledge of the meaning of mean is not mobilized in the enacted lesson. Or, possibly her knowledge is not well formed or not sufficiently developed to teach mean conceptually as well as procedurally. Overall, her instructional emphasis is on the procedures, demonstrating and attending to them step by step. This approach might help students get the correct answers in the textbook. Also, the guidance given in the curriculum does not support MR's teaching to the extent that she needs. Other than the two sentences cited above, everything else in the curriculum is about combining the data set and dividing them evenly, reinforcing the procedure to find the mean. The diagram in Figure 1 is not accompanied by appropriate explanations in the textbook. What would her instructional activities and practices look like, if she were equipped with required knowledge and capacities to enact the lesson with the meaning of mean? In turn, what knowledge and capacities does it take for her to enact this lesson effectively? How can her instructional activities and practices be measured? Such questions need to be answered beyond this particular vignette.

As illustrated in the vignette above, examining instructional activities and practices is critical in our investigations of the kinds of teacher capacities that are needed for teachers to transform written to enacted curricula. Previous instruments for analyzing teachers' use of curriculum materials tend to focus on assessing fidelity to the curriculum, with a focus particularly on the aspects of the resources that teachers use (O'Donnell, 2008; Tarr, Chávez, Reys, & Reys, 2006). Each of the three subgroups has been investigating teacher capacities needed for mathematics instruction and ways in which enacted lessons are examined. Through such investigations, one project represented in this working group has developed a tool to measure teaching practice. The other two projects are developing new tools to measure teaching practice as it relates to the curriculum being used and the role that teacher capacity plays. The work being done by all three subgroups involves creating and adapting measures of instructional practice to get a broader and richer sense of the ways in which teachers make use of curriculum and learn from curriculum, and interactions among teacher knowledge and capacities, instructional practice, and curriculum use. One of our aims is to develop measures of curriculum use that are tied to central ideas in the curriculum being used and that capture the complex work of curriculum enactment in the classroom. Below, each subgroup's work is described along with their specific focus in measuring instruction. During the working group sessions, the participants will have an opportunity to work with and critique tools developed by the three subgroups.

Project 1 (Elementary and Middle School): Learning Mathematics for Teaching (LMT)

This project developed instruments to measure teachers' Mathematical Knowledge for Teaching (MKT), or the "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Ball, Thames, & Phelps, 2008, p. 399). These measures are multiple-choice and indicate teachers' MKT in a number of knowledge domains. To validate these measures, researchers used a wide variety of data sources including cognitive interviews with teachers, mathematicians and laypersons focusing on the different domains of content knowledge used in the interview; data correlating teachers' scores and their students' achievement; Item Response Theory and factor analysis; and a videotape validation study that linked paper-and-pencil scores with quality of instruction. When researchers sought to measure the quality of videotaped instruction, there were no instruments available that appraised teachers' mathematical knowledge for teaching, rather than just straight mathematical knowledge. Thus the project also developed its own observational measure of observed teaching, the Mathematical Quality of Instruction (MQI). Because other researchers were also looking for measures that could be used to evaluate programs and professional development, among other purposes, the MQI has become a widely used measurement instrument in its own right.

One of LMT's most recent large-scale videotape studies using the MQI investigated middle school teachers' mathematical quality of instruction in combination with their value-added scores, or student achievement (Hill, Umland, & Kapitula, 2011; Lewis, under review). Fortuitously, and not by design, several sets of teachers were videotaped teaching the same or very similar lessons from the same curriculum materials (Connected Mathematics Project [CMP] or lessons on similar content but from different curriculum materials). This serendipitous happening allowed researchers a unique opportunity to investigate the relationship among these teachers' MKT, MQI and their use of curriculum materials. In a recent multiple-case study (Charalambous & Hill, in press; Hill & Charalambous, in press a, in press b; Hill, Charalambous, & Mitchell, in press; Lewis & Blunk, in press; Sleep & Eskelson, in press), researchers simultaneously attend to teacher knowledge and curriculum materials to explore how they separately and jointly contribute to instructional quality. The pairs or triads of teachers featured in three of the case studies differed in their knowledge level and/or in the support their curriculum materials provided for lesson enactment. This allowed examining the respective contributions of MKT and the curriculum materials to instructional quality in the context of teaching a rich fractions problem that admits several solution approaches (case study 1), teaching integer operations for conceptual understanding and procedural fluency (case study 2), and teaching a linear algebra lesson (case study 3). In case study 4 (a lesson on adding fractions) teachers' differing dispositions toward their curriculum suggest yet another factor besides MKT or the materials themselves that may also account for the differences in the mathematical quality of instruction. LMT researchers will share findings from these case studies in the working group.

Project 2 (Secondary): CME Project Mathematical Practices Implementation Study (MPI)

This project explores teachers' use of a coherent high school mathematics curriculum organized around mathematical habits of mind, or ways of thinking central to the discipline of mathematics (Cuoco, Goldenberg, & Mark, 1996, 2010). The substantive focus on habits of mind mirrors the focus on mathematical practices in the Common Core State Standards for Mathematics, where standards for practice are as important as standards for mathematics content to be taught (Common Core State Standards Initiative, 2010). The habits of mind focus in the curriculum and professional development are intended to be educative for teachers on two levels: in supporting their instruction and in providing opportunities to learn mathematical practices and content. To test this theory, researchers have developed and adapted instruments to examine the role the curriculum plays in the design, selection, and modification of mathematical tasks, and in particular, the ways in which teachers make use of the mathematical habits of mind approach in their classroom. Our goal is to refine a set of instruments that can be used in future large-scale research studying the relationships among use of a principled curriculum and teacher learning, teaching practice, and student achievement in high school mathematics.

Specifically, the MPI study measures three aspects of teacher implementation around the CME Project Algebra 1 curriculum in the 9th grade. The Academic Rigor scales of the Instructional Quality Assessment (IQA) are used to evaluate the extent to which instruction maintains, diminishes, or increases the cognitive demand of the mathematical tasks in the curriculum (Mastumura et al., 2006). The IQA is being implemented with a full sample of 41 teachers using student work packets, and with a subsample of 20 teachers through live and videotaped classroom observations. The Mathematical Habits of Mind observation tool (MHoM) (Matsuura et al., 2011) assesses the extent to which the habits of mind which are the focal points of the curriculum, are developed and emphasized in the classroom, and by whom (students, teachers, or both). Finally, a battery of curriculum use instruments is used to identify components and design principles, including the mathematical approaches of the curricular materials that are and are not enacted by teachers in their first and second years of using the curriculum. Together, this set of tools provides a multifaceted view of curriculum implementation that moves beyond simple fidelity and curriculum use surveys, and links pedagogical and mathematical practices, curricular design principles, and curricular artifacts in a rich characterization of what it means to teach and learn from a high school mathematics curriculum.

Project 3 (Elementary): Improving Curriculum Use for Better Teaching (ICUBiT)

This project investigates teachers' curriculum use and the capacities and supports critical to it. After undertaking analyses of five elementary curriculum programs, ranging from reform-oriented to commercially-developed, researchers collected data on how teachers used the teacher's guides to plan lessons and also video-recorded enacted lessons. Analysis of video data focuses on *design moments,* which refer to instances during a lesson when the teacher makes decisions that are not specified in the book or his/her plans. Examples of design moments include instances where the teacher makes choices about who to call on, how to respond to student errors or questions, and even when to move on with the lesson. Our aim is to develop a coding scheme for analyzing teacher moves in these design moments that can be used with any curriculum programs and can be used to assess teacher capacity and to consider the role that features in the curriculum play in supporting teachers' designs during curriculum enactment.

Currently, the coding scheme has four categories in terms of teacher moves that seem particularly significant in design moments: (1) curriculum use moves, (2) teacher-generated mathematical moves, (3) teacher moves in response to students, and (4) moves to reinforce sociomathematical norms. Teacher curriculum use moves are coded by examining whether the teacher uses, changes, omits, or adds to what appears in the written curriculum, such as tasks, questions, and models/strategies. This category also examines whether the teacher changes the sequence of the entire lesson. The category of teacher-generated mathematical moves targets examining teacher decision-making and capacities when the teacher initiates certain mathematical moves. This includes whether the teacher exhibits a confusion or error, and, if so, whether the teacher corrects or clarifies it. Also, moments in which the teacher emphasizes certain mathematical ideas/concepts and moments in which the teacher clarifies student understanding are coded in this category. The third category is different from the second in that it is intended to capture instances where teacher moves are aimed at *responding to students* and are initiated by students' correct or incorrect responses or student-generated ideas or questions. The final category identifies instances in which the teacher's move appears to be aimed at communicating or reinforcing a sociomathematical norm. In these cases, the teacher might ask other students to respond to a student's answer or ask a student to justify his/her solution. The coding scheme is being used to analyze the classroom practices of 25 elementary teachers, using five different curriculum programs.

Plan for the Working Group Sessions

The three sessions of the working group will be organized around the tools the three subgroups developed, with one tool in each session. Each session will consist of a 20-minute presentation about each study, including main aspects that are focused on in the analysis of instructional activities and practices, and the preliminary tool developed to measure those aspects. Then, the participants will be asked to analyze a classroom video clip using the tool shared in the presentation and engage in discussion on this approach. Finally, the participants and organizers will discuss the issues related to measuring instruction using the following guiding questions:

1. What does it mean to measure instruction, especially in relation to curriculum use?
2. What considerations should guide the design of tools that aim to measure instruction?
3. What is the relationship between curriculum fidelity and high capacity curriculum enactment?
4. To what extent do the tools presented measure aspects of instruction that the studies intend to capture? What elements are missing?
5. Are the elements of the tools clearly described in order to produce consistent results? In what ways can they be improved?
6. What are the limitations of the tools? What additional tools can be used with them?

In the first session, members of the LMT project will present their work on the development of the Mathematical Quality of Instruction (MQI), guide the participants in terms of using MQI, and lead the discussion on measuring instruction. In the second session, a group from the MPI study will present their work to orchestrate the ongoing discussion on measuring instruction. The last session will be led by the

ICUBiT project team. They will share their work on the development of a tool to measure instruction and build on the two previous sessions to orchestrate discussion. On the last day, the working group will also summarize the ongoing discussion and generate future activities to further refine the tools shared and investigate ways in which written and enacted lessons are examined. This way of organizing the working group will help both the participants and the organizers examine different approaches to measuring instructional activities and practices, and elaborate teacher capacities needed for effective design and enactment of a lesson.

Anticipated Follow-up Activities

The three sessions of the working group will not only help each subgroup refine their thinking about the tool they developed and ways in which it measures instructional activities and practices, but also facilitate discussions on the relationship between written and enacted lessons and the role that the teacher plays in this relationship. The sessions will also help generate research interests among participants in relation to the work presented and issues discussed. Anticipated follow-up activities of the working group include:

- Each subgroup refines the tools, when needed, based on results of the working group.
- Each subgroup receives further feedback from the other subgroups and working group participants on the tool they refined.
- The three subgroups and working group participants interested in this work organize collaborative work and discussions on measuring instruction.

The work presented in this working group, which seeks to examine the relationship between written and enacted curricula, represents an important area of research that is currently underdeveloped in the literature. The anticipated follow-up activities of the working group will contribute to this research by exploring conceptual and methodological issues related to curriculum use and teacher capacity through measuring instructional activities and practices.

Relationship to the Conference Theme

Overall, this working group will facilitate discussions and reflections on *navigating transitions along professional learning continuum*. Identifying teacher knowledge and capacities needed for effective design and enactment of a lesson will help better design mathematics teacher preparation and professional development programs. It will also help curriculum developers create and organize “educative” (Davis & Krajcik, 2005) resources for teachers.

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REPRESENTATIONS OF MATHEMATICS TEACHING AND THEIR USE IN TRANSFORMING TEACHER EDUCATION: THE ROLE OF APPROXIMATIONS OF PRACTICE[1]

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Brief History of the Working Group

This is the third meeting at PMENA of this Representations of Mathematics Teaching (RMT) working group. The idea of this working group emerged during a series of three-day conferences on representations of mathematics teaching held in Ann Arbor, Michigan in 2009 and 2010 (and earlier workshops in 2007 and 2008) organized by ThEMaT (Thought Experiments in Mathematics Teaching), an NSF-funded research and development project directed by Pat Herbst at the University of Michigan and Daniel Chazan at the University of Maryland. ThEMaT originally created animated representations of teaching using cartoon characters to be used for research, specifically to prompt experienced teachers to share the rationality they draw upon while teaching. The workshops were conceived to begin creating a community of researchers and teacher educators who were interested the use of representations of teaching and the analysis of data collected in response to these representations. The RMT conferences in 2009, 2010, and 2011 gathered developers and users of all kinds of representations of teaching (including video, written cases, dialogues, photographs, comic strips, and animations) to present their work and discuss issues that might be common to using these representations in teacher education and education research. A fourth conference took place on June 6–8, 2012. In proposing a continuation of the working group for PMENA 2012 we'd like to continue the discussion and work we had in recent PMENA working groups at Columbus in 2010, and Reno in 2011 around the elaboration and investigation of a pedagogical framework for teacher development that makes use of representations of teaching, and work toward an edited volume on the subject.

Issues in the Psychology of Mathematics Education that Will Be the Focus of the Work

The use of representations of practice, particularly those that are maintained in a digital form, calls for specialized pedagogical practices from teacher educators. They also open new areas for investigation of how future professionals learn teaching and the role that various technologies play in scaffolding that learning. In the 2010 PMENA discussion paper, Herbst, Bieda, Chazan, and González (2010) briefly reviewed the literature on the use of video records and written cases in teacher education. We noted that classroom scenarios sketched as cartoon animations have begun to be utilized for those purposes and argued that they have affordances that are distinct from those of video and written cases (see also Herbst, Chazan, Chen, Chieu, & Weiss, 2011). We also noted existing literature on the use of written and video cases in teacher education and cited examples that concern mostly face-to-face facilitation. We argued that the increased capabilities of information technologies for creating, manipulating, and collaborating over multimedia point to a promising future for teacher development assisted by representations of practice. A special issue of the *ZDM* journal dedicated to the theme of representations of teaching added new articles to this literature. In particular Ghousseini and Sleep (2011) and Nachlieli (2011) describe the facilitation of face-to-face discussions around representations of practice and provide two views on what makes these effective for studying practice. Yet the features of novel media and their use with digital technologies, for example in online or blended (face-to-face and online) interactions, may require other pedagogical strategies for teacher education that have not been sufficiently identified and explored.

In the discussion document for the working group meeting in 2011, we complemented the previous year's review by briefly accounting for three areas of emerging scholarship: (1) information technologies that support teachers' learning from representations of practice; (2) the particular challenge of helping prospective teachers understand students' thinking; and (3) research and theory about what is important or possible to achieve in having prospective teachers look at or work with representations of teaching.

In this document we come back to the central goal of the working group: What are the components of a pedagogy for teacher education assisted by representations of practice? Through an example we demonstrate how that question is anchored in scholarship and activities of interest to the PMENA community. Just as considerations of the subject matter being taught are key in examining pedagogy in K–12 mathematics education, a better grounding of the questions related to the use of representations of teaching in teacher education can benefit from specifications of what is being taught. In the case of mathematics educators, the content being taught to preservice teachers includes mathematics, students' thinking, and instructional practices. To anchor the need for a pedagogy assisted by representations of practice, we examine the use of representations to teach instructional practices. We focus this examination on a generic learning activity called *approximations of practice* and consider how these can be used to teach instructional practices and the kind of questions that arise from that use and that are of interest to mathematics educators gathered at PMENA.

Approximations of Practice

The expression *approximations of practice* was introduced by Grossman and colleagues (2009) to allude to "learning opportunities provided to novices" in which they can get actively involved in "the authentic practices they will be expected to enact." Grossman notes, "Students may be asked to simulate certain aspects of practice through activities such as role-plays. Simulating certain kinds of practice within the professional education classroom can allow students to try piloting the waters under easier conditions. Providing support and feedback while novices learn to paddle may better equip them to navigate the rapids of real practice." We argue that creating and supporting approximations of practice is critical for teaching instructional practices to novices. Two lines of argument contribute to this. On the one hand, Lampert's (2010) argument that teaching needs to be learned in, from, and for practice recommends the creation of opportunities to engage in aspects of the work of teaching that reproduce at least some, if not all, of the complexity of actual teaching practice. The more that the teacher candidate can be engaged in doing the practice, the more this learning will be "in" practice; and while learning "from" practice does not necessarily require representing practice in all its complexity, the more this complexity is represented the better it will ground the teacher education curriculum in actual practice. On the other hand, there is a long tradition of recommending active learning, or learning by doing, across educational levels (Bonweil & Eison, 1991). This approach to teaching seems to concurrently argue for learning activities for teacher candidates where they learn from their own experience engaging with authentic problems of practice and where reference materials such as readings play the role of supporting resources rather than focus. To mathematics teacher educators, approximations of practice offer opportunities and challenges that concern the creation of approximations, activities utilizing these approximations, and the investigation of teacher candidates' learning from these. Below we illustrate how representations of practice can feature in these approximations.

What form do approximations of practice take? One could argue that approximations of practice have always been included in teacher education programs that contain a practice curriculum—practicum experiences and student teaching are in fact approximations of practice. The literature on student teaching is vast enough to discourage a review, and it has been well represented in handbooks (e.g., McIntyre, Byrd, & Fox, 1996). Beyond practicum experiences in actual classrooms, university classes on teaching methods classes have also contained approximations of practice.

Microteaching (Allen & Eve, 1968; Cruikshank & Metcalf, 1990; McLeod, 1987), or the enactment of short lessons in front of peers, is an example of how approximations of practice have taken shape in the teacher education curriculum. While popular for some time, microteaching has not produced the desired results in supporting teacher candidates in learning to teach (McIntyre, Byrd & Fox, 1996). Over the years teacher educators have worked to improve this technique though the inclusion of targeted guidance and feedback as well as coupling microteaching with the observation of competent performance of the teaching practice being studied (for an example, see Chazan, Herbst, Sela, & Hollenbeck, 2011). A related variation has developed to match the medical education practice of the "standardized patient" (Stillman et al., 1991). Dotger, Harris, and Hansel (2008) have proposed a standardized-patient-type approximation for training teachers to talk to parents.

Yet a more recent incarnation of microteaching is that of teaching rehearsals (Lampert et al., 2010). In the process of engaging teacher candidates in performing targeted aspects of the work of teaching, some rehearsal cycles also rely on the use of video recordings of the teaching practice (Kazemi, Franke, & Lampert, 2009; Lampert & Graziani, 2009). In these activities the teacher educator guides the teacher candidates in observing live or pre-recorded exemplars of the practice that they will rehearse or watching and debriefing video of the novices' rehearsals.

Another recent approach to engaging teacher candidates with approximations of practice is through the construction and enactment of instructional dialogues, also called "lesson plays" (Crespo, Oslund, & Parks, 2011; Ghousseini, 2008; Zazkis, Liliejdahl, & Sinclair, 2009). These activities provide teacher candidates with opportunity to imagine how a classroom scenario might unfold and the specific consequences of the word choices of both the teacher and students.

Historically, teacher preparation programs have also engaged teacher candidates with approximations of teaching through lesson planning and lesson anticipation. Lesson planning, or asking teacher candidates to create a timeline for a lesson, has long been used as a tool for preparing teacher candidates for the work of teaching and scholars have begun to examine the techniques commonly used with teacher candidates and their effectiveness (Harris & Hofer, 2009; John, 1991, 2006; Mutton, Hagger, & Burn, 2011; Rusznyak & Walton, 2011). As Americans became more knowledgeable about Japanese lesson study (Hiebert & Stigler, 2000), some educators have begun to use lesson study in preservice education (Fernandez, 2002; Hiebert, Morris, & Glass, 2003; Parks, 2008.). Pre-service lesson study teams may be comprised of a group of teacher candidates, although some work has involved teams of mentor teachers and teacher candidates collaborating together (Burroughs & Lubeck, 2010). These teams may work in microteaching or lab-type settings (Fernandez, 2005), as well as in the context of actual classrooms. Distinctions in who contributes to planning, observing, and debriefing the lesson, as well the context of the lesson, determine how closely lesson study approximates actual teaching practice. For instance, when lesson study teams work on developing a lesson to be taught in an actual classroom, the school curriculum, norms, and student characteristics must be taken into consideration and heighten the authenticity of the lesson planning when compared to a lesson study for fellow teacher candidates in a microteaching setting. But approximations of teaching may also be deployed in virtual settings.

Approximations of teaching in virtual settings: A use of *Depict*. Herbst and Chieu (2011) introduced the *Depict* tool (a component of the *LessonSketch* environment in www.lessonsketch.org). *Depict* enables users to create a classroom scenario using text, inscriptions, and graphics (see also Herbst et al., 2011). Chen (2012) has shown that when preservice teachers were asked to anticipate a lesson using Depict they were able to think through the tasks they would propose in more detail than when they merely talked through a lesson plan they had written before. As a result it is possible to envision a new kind of approximation of practice that moves above and beyond activities in which novices construct dialogues to show how they would handle problems of practice. The teacher educator can depict the beginning of a classroom scenario, using text and graphics, and leave it to the novice to complete the scenario and submit it to the teacher educator, who in turn might insert comments or alternatives and return that to the novice. Thus approximations of practice can be deployed and transacted through the use of multimedia representations. We exemplify this use below.

Teaching the instructional practice of "explaining a concept" using approximations of practice. One thing the first author does as part of his methods course is teach novices how to explain concepts.

Explaining concepts can be a teacher-centered activity in that the teacher may "provide" all the explanation, but it may also be a blend, using discussion or brief explorations as parts of the explanation. In teaching novices how to explain concepts, however, the goal is not so much to identify the best activity type for them to use, but to teach novices about what things need to be included in an explanation of a concept. As Leinhardt and Steele (2005) have shown, dialogue-based lessons can be constructed to share features of instructional explanations found in the instructional explanations documented of expert teachers (Leinhardt, 1989, 2001). To support the teacher candidates in learning how to explain concepts, Herbst uses a decomposition of practice for the practice of "explaining concepts and propositions" (Herbst, 2011). This decomposition of practice builds on Leinhardt's work on instructional explanations by describing and exemplifying the components of an explanation in text form. Components such as problematizing the concept, exemplifying the concept, and so on are described and illustrated in documents such as Herbst (2011). Until 2011 students in Herbst's methods course would practice what they learn in the context of approximations of practice like the problem shown on Figure 1.

4. Abilene Clark has been teaching her Algebra II class the exponential function and properties about the multiplication and division of exponents. She wants to make sure students use those properties well so she points out possible errors related to operations with exponents. Write a dialogue in which you show how Abilene could

a. demonstrate one of those errors, and
b. explain why it is an error

Figure 1: A dialogue based approximation of the practice of explaining a concept (co-designed by Gloriana González, Pat Herbst, and Adam Poetzel)

In fall 2011, Herbst and his team started using an illustrated version of the decomposition for explaining concepts and created graphic approximations of practice using *Depict*. The problem in Figure 1, which gives novices the opportunity to practice examining common errors, which is one aspect of explaining concepts, was then posed using the depiction shown in Figure 2 and in the context of an online homework assignment including three problems like it. Novices were told that the depiction shows how what Ms. Clark has done so far and asks, "What are some of the conceptual errors that are at the root of common mistakes students could make when working with the logarithmic function? Write your comments in the box below." Then it prompts them, "Now, please press View and then Edit to edit the slideshow, so that you can complete what Ms. Clark should say to the class to point out common errors that students make when using the logarithmic function. Your edited slides should demonstrate one common error and Ms. Clark's explanation to the class of why it is an error (and how they might avoid it). [The red text on the whiteboard and the text between brackets in the speech bubbles] shows where you can fill in what the teacher should say and write on the board. Feel free to add more slides…."

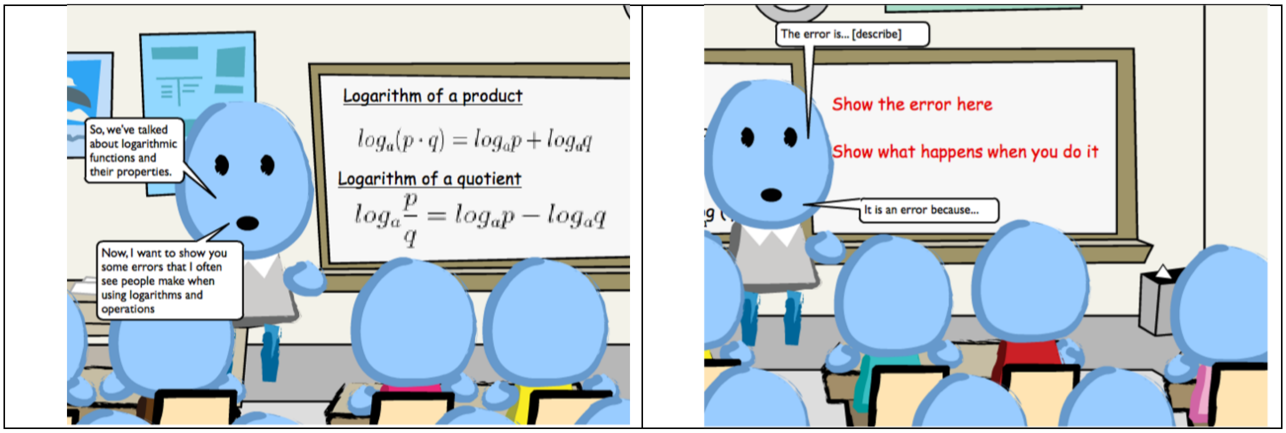

Figure 2: The context given for an approximation of practice using graphics

Figure 3 shows parts of the depiction that one novice (D) made continuing the provided slides. D's depiction contained five original frames that followed the provided ones. D's work shows not only how the approximation got novices involved in practicing but also, as Chen (2012) had found, it made them think of the details of tasks and multimodal student involvement.

A teacher educator could provide feedback to that depiction, or perhaps sketch an alternative scenario. In Herbst's class, teacher candidates not only created the depictions that showed what a teacher could say and do when providing an explanation, but they also rehearsed them in front of their peers on the following class. Peers and instructors could then provide constructive criticism about choices made in planning and about the qualities of their performance.

Through depicting the continuation of scenarios like those, teacher candidates can develop and display their knowledge of instructional practices. The decomposition of practice provides teacher candidates with a framework for thinking about the essential aspects of the practice while the work of depicting the scenario provides teacher candidates with the opportunity to explore the possibilities for what the practice will look like when it is enacted in the classroom.

As this example shows, approximations of practice can be used to create opportunities to learn a practice (explaining concepts, and its component of examining common errors) *in* practice (by providing an explanation in a depicted classroom) and *from* practice (where the feedback addresses alternative choices and ways of enacting them). A more sophisticated case of using approximations of practice has been proposed by Chieu and Herbst (2011), who describe the features of a teaching simulator, where teacher candidates practice teaching by choosing what the cartoon teacher would do and the simulator provides the ensuing events in a simulated class.

Clearly, *Depict* is only one example of how technology can support learning from approximations of practice (in the case shown, homework problems based on the work of teaching are used to support teacher candidates in learning how to preform an instructional practice). The same approximation could be realized using video; the teacher educator could record the initial scenes of the scenario, while playing the role of teacher, and could ask the teacher candidates to record themselves doing the ensuing actions. Feedback could come in the form of an annotation of the video or a video response, where the teacher educator demonstrates how he would modify what the teacher candidate did. In both cases one can see how technology-supported representations of practice can be used to create approximations of practice than blend active learning with learning in and from practice. This activity can involve novices in figuring out what to do in particular circumstances, while blended with microteaching it can also address the development of skills in performing the work of teaching; that is, learning *for* practice.

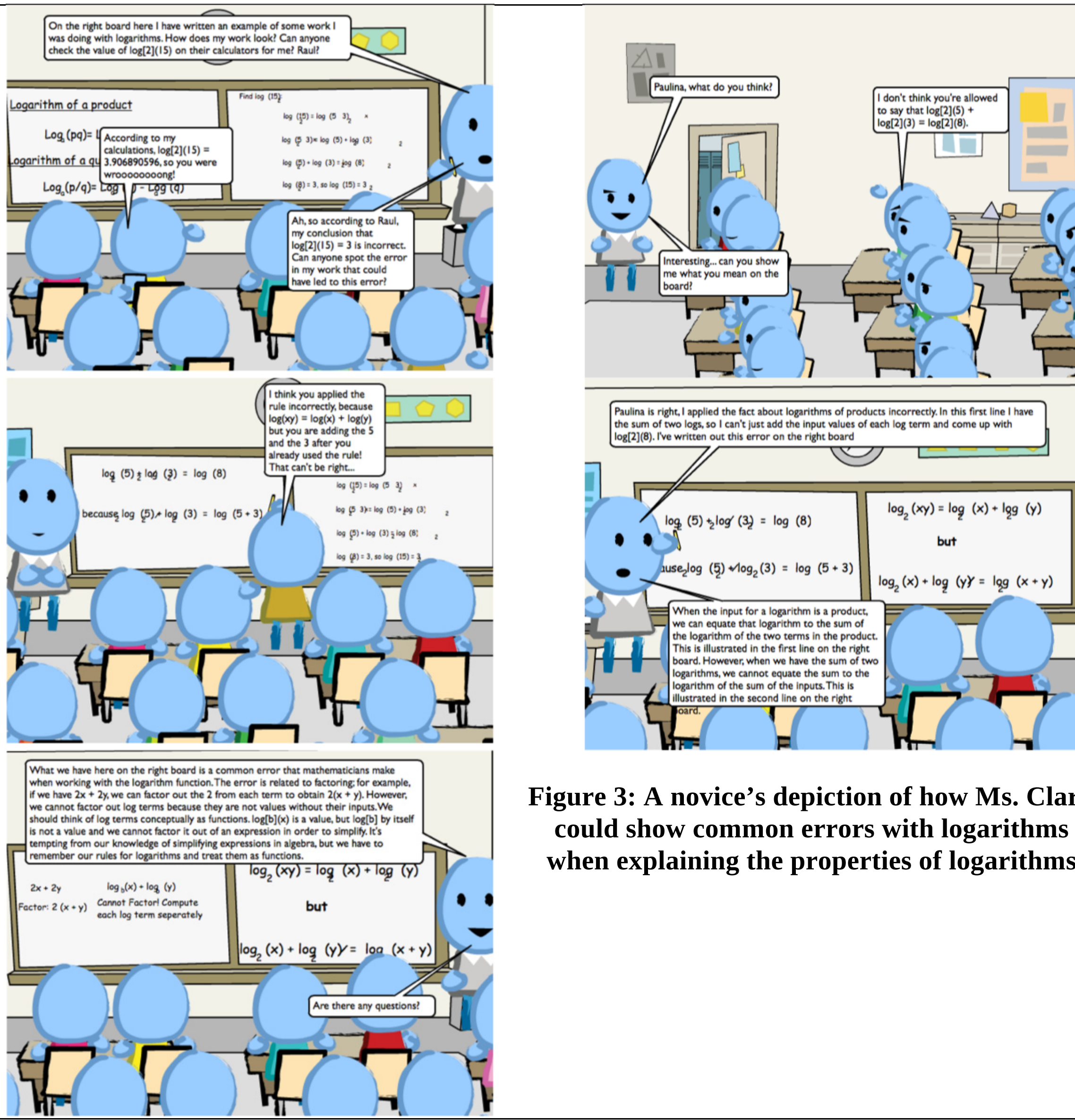

Figure 3: A novice's depiction of how Ms. Clark could show common errors with logarithms when explaining the properties of logarithms

The prior discussion of approximations of practice suggests that a generic learning activity in teacher education could involve teacher candidates in studying a practice (e.g., viewing illustrations that decompose a practice) then doing problems of practice where they engage in virtually enacting those practices. The learning materials and the problems could be posed in some information technology based environment such as LessonSketch, and the teacher candidates would produce their responses using tools integrated in that environment (in this case *Depict*). This generic learning activity helps raise questions that are of interest to the PMENA community and that illustrate why the working group fills a gap in the community. Some questions are about the novices' learning: What do novices learn by engaging with representations of teaching in the context of activities that approximate practice? Do different kinds of representations afford different learning opportunities when similar approximations of practice are used (e.g., text only vs. *Depict* vs. video)? In other words, are there cognitive or performance changes in novice teachers that go along (conceptually or statistically) with different kinds of representation-based activities or different kinds of representations? On the other hand the extent to which this kind of approximation of practice involves media and communication technologies provides a snippet of how much pedagogical innovation is possible and needed in order to handle representations of practice with novices. Teacher educators need to choose what their novices will learn indeed. But they also need to design how they are going to create opportunities for them to learn. Approximations of practice illustrate the complexities involved in this work. Teacher educators are not limited to selecting media artifacts; they can also produce them. This requires making choices of symbol systems, content, and form. Beyond producing or selecting representations to use, teacher educators need to design or choose activities in which novices will engage with those representations, and they need to design how to propose those activities to novices. Teacher educators need to identify the medium within which to share those activities and the representations associated with them. Face-to-face group encounters with a projector screen are only one of the many choices available, which include notably, online environments like *LessonSketch* that can be used at distance (e.g., when students are doing homework at home) or co-located (e.g., when students browse through a representation in class, using their own laptops). Teacher educators also need to design the tasks they pose to the novices—these tasks may be "what do you notice?" but they may also be "what would you do or say next?" In sum, teacher educators need to design learning environments for the learning of teaching—recent improvements in Internet broadband speed, web-based software, user experience standards, and graphics technologies have made the choices available for that design much greater, and more diverse, than ever before at least in terms of their possibilities. We argue that this presents the challenge and the opportunity of developing a pedagogical framework with which to conceive these learning environments. The case of approximations of practice and its application in the teaching of how to explain a concept in secondary mathematics methods exemplifies how the pedagogy of teacher education may be expanding in response to existing technologies. A framework can help us direct technology development as well.

Toward a Pedagogy for Teacher Development Assisted by Representations of Practice

Building on the proposals from previous years, the working group's purpose is to design a pedagogical framework for teacher development. The framework is aimed at assisting teacher educators who want to help teacher candidates actively learn teaching *in*, *from*, and *for* practice by taking advantage of representations of practice and new technologies. This enterprise may require conceptual developments, for example in articulating connections between theories of teaching and the design of a curriculum for teacher education. The enterprise also requires the creation of pedagogical templates or generic learning activities/environments that can be particularized for the specific goals of individual teacher educators, the needs of their students, and the media artifacts or software tools that are available. Thus far, the working group has proposed a framework articulated by a number of categories of things that are involved in different ways in the process of teaching with the assistance of representations of teaching. These categories include *boundary objects*, *activity types*, *technology tools*, *problem types*, and *teacher education goals*. Each of those categories contains elements from which choices can be made to design learning activities for novices. The working group has been operationalizing those categories by using *Plan*, a software tool included in *LessonSketch*. *Plan* allows teacher educators to design a learning module for their clients, putting together media artifacts, tools, and tasks and to sequence them in a desired order that may include individual or group work. In the discussion document for last year (Herbst et al., 2011) we described the framework in considerably more detail. Our work this year will include asking questions like: Given that a teacher educator has a specific learning goal in mind for her students, such as learning how to probe student thinking or learning how to demonstrate the subtraction algorithm, what are appropriate representations of teaching, activity structures, problems types, and technology tools to use to reach that goal? The group will work on articulating goals of teacher education and a pedagogy of teacher education.

The convenors of this working group are particularly interested in exploring how cartoon-based representations of practice facilitate teacher learning. Over the past few years teacher educators have begun to use *LessonSketch* in content and methods courses for teachers. This working group is an opportunity for users to share their experiences and insights. These contributions are invaluable to the life of the working group and, more generally, to the development of the knowledge base for use of these resources. We expect the continued use and development of *LessonSketch* will help improve the framework of the pedagogy and further develop specifications for yet other technologies that respond to the needs of the field.

We have proposed that a pedagogy of teacher preparation assisted by representations of practice needed at least four categories of elements: boundary objects (or open ended expressions), activity types, problem types, and technology tools or screens. This year we add the category of *teacher education goals* that one needs to consider when planning educative experiences for teacher candidates around representations of practice—since different resources and tasks may be needed depending on those goals. These teacher education goals can include having novices learn instructional practices such as "explaining concepts" or other, even "high-leverage practices" (Hatch & Grossman, 2009) such as "facilitating classroom discussions." The goals can be student-centered too (such as having novices develop capacity to notice, describe, and explain students' errors). Also, they can be mathematical, such as when one wants novices to map the terrain of a given problem (Lampert, 2001). We refer the reader to the discussion document of last year's working group for a detailed description of the other categories.

Plan for Active Engagement of Participants and Anticipated Follow-up Activities

The plan includes starting with a brief exposition by the authors of the structure and contents of the present framework for which we will illustrate how the use of approximations of practice narrated here maps onto elements of the framework. We will engage the audience in creating learning activities they would like to use to engage their clients. The idea is to use the collective planning of these sessions to probe the framework and possibly enrich it by adding more items to the lists considered, possibly also adding new categories of elements. Participants will then form groups and spend the second half of the first session and the first half of the second session creating exemplars. Then the second half of the second session and the closing session will be dedicated to sharing these exemplars and improving the framework, including discussions about the questions raised earlier in this document, paving the way for an edited publication.

By the time this working group meets we will have had the fourth conference on Representations of Mathematics Teaching in Ann Arbor (June 6–8, 2012). We will be proposing a session slot at the AMTE Annual Meeting in 2013 to continue this work. We plan to use that slot to mirror the work done at the PMENA meeting and to engage in further work on (1) improving the exemplars, and (2) using the exemplars to improve the taxonomies. We hope we will be able to use those products to continue this working group at next year's PMENA.

Endnote

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STEWARDSHIP OF SCHOLARSHIP: INCREASING ACCESS TO RESEARCH

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This new Working Group is designed to engage members of PMENA in the conceptualization and development of a monograph series that will fill a unique niche in the set of publishing outlets for mathematics education researchers. Participants will engage in identifying and characterizing the hindrances to timely publication, the affordances of new technologies, and the resources of existing conferences and professional society organizations, and focus this conversation on building an infrastructure for the support of (particularly junior) faculty into creating records of their work that are publishable, and ultimately published—in a rapid time frame. The outcomes of this Working Group will be a set of recommendations for the design and management of such a monograph series, and an outline of initial volumes in the series, where participant teams are authors of articles in each volume.

Keywords: Research Reporting; Junior Faculty; Scholarly Publishing; Informal Education

The Nature of Scholarly Reporting

Since at least the advent of natural philosophy in the Renaissance period, scholars have depended upon each other to share their ideas and research findings, and to provide critique of these ideas and findings for their mutual advancement of knowledge. It was during this period in the West, coinciding with the invention of the printing press, that academic publishing became a universal criterion for membership in the academy. The Elzevir brothers opened an academic publishing house in the Netherlands 432 years ago to support (and benefit financially from) the knowledge boom afforded by moveable type.

The importance of scholarly writing and its dissemination cannot be overestimated in the advancement of knowledge in mathematics education. Both as a record of what has taken place, and as a continuing scholar dialogue about the nature of mathematics teaching, learning, policy and practice, the journal article, yearbook chapter, edited volume, and authored text constitute the primary means by which ideas become common knowledge. Without these media, we have neither cultural memory, nor a means of transcending our time and space limitations to engage in this larger conversation. We face unique problems now, as the number of scholars in our field grows, and as our paradigms and methods become more diversified, in vetting and compiling all this work into useful, organized media. Moreover, due to this diversification of perspectives, it is our opinion that our ability to understand each other is becoming compromised, thus limiting our ability to provide practical solutions to the problems of mathematics education.

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Current Problems in Academic Publishing

We have anticipated that there are a number of constraints that exist in the current landscape of academic publishing that hinder many of our colleagues from publishing their work. We briefly outline a few of these constraints to emphasize the need for a new model of publication support that would, in part, solve some of these issues and provide more opportunity for the equitable sharing of knowledge.

First, for the researcher, one of the problems facing education today is the length of time to publication of most studies. Traditional journals may have a backlog of manuscripts of greater than a year. If we factor in the review and acceptance process, revision and resubmission, and subsequent editorial work, it can take an average of at least 30 months for empirical work to find its way to the field where it can be used. Such a long process can have a number of negative impacts:

- Individual submission of work to a journal involves an extended review process, eliminating from contention works that have potential, but that are not yet ready for dissemination.
- These backlogs, delays, and restrictive attitudes work in a manner that is counterproductive for rapid deployment of ideas for the purpose of innovation in research and development.
- Further, for junior faculty, whose research is often the most innovative, their careers are dependent upon publication to receive tenure. With only 5 years to develop standing in the field, a huge bottleneck exists, preventing many promising scholars from publishing in venues that will actually be read.

Second, as a case in point, only seven English-language mathematics education research journals are listed in Social Science Citation index.

- *Educational Studies in Mathematics*
- *For the Learning of Mathematics*
- *International Journal of Science and Mathematics Education*
- *Journal for Research in Mathematics Education*
- *Journal of Mathematical Behavior*
- *Journal of Mathematics Teacher Education*
- *Mathematical Thinking and Learning*

Of those, an average of about 16 articles per journal are published each year (~100 each year in total). Data from 2003 show that there are about 80 PhDs awarded from U.S. institutions each year (Reys et al, 2008). Given that about 80% of these graduates move into the professoriate, that means between 80 and 160 papers each year must be published from this pool of candidates alone! If that number is multiplied by 5, the average number of years before going up for tenure, we have approximately 400 junior faculty, competing with at least as many experienced researchers, for only about 100 openings for their work in high quality journals. Of course, there are other outlets for much of our work: General education journals (e.g., *American Educational Research Journal*), and discipline specific journals (e.g., *Journal of Educational Psychology*). In addition, there is a current proliferation of online journals and other new venues that have not yet been fully acknowledged as “legitimate,” in terms of promotion and tenure decisions.

Despite these additional outlets, there still exists a significant bottleneck, particularly for junior faculty. Such a bottleneck impacts time to publication immensely, and it further prevents important ideas to reach the field where they can be shared, thus inhibiting innovation among researchers and practitioners.

Third, the need for rigorous empirical work in mathematics education is becoming more and more acute. The problems of education require that substantive changes be made in teaching, learning, curriculum, and policy. But the pace of innovation is rapidly outstripping the ability of the research community to provide disciplined guidance to the field. A means of organizing this knowledge proactively may be able to get the right ideas to the right people so as to make a difference, as opposed to having to wait until search engines and other indices catch up with current publications.

Brief History of the Working Group

This is a new Working Group intended to develop and help shepherd a new monograph series in mathematics education. The leaders, Jinfa Cai and James Middleton have met on several occasions, working out the need for group and its deliverables. We have consulted with Springer publishers who are interested in publishing a monograph series that meets the needs we have outlined in this paper. The series would be representative of the latest research in our field, and would be geared towards highlighting the work of bright, graduate students and junior faculty, working in conjunction with senior scholars. The audience will be the intersection between researchers and mathematics education leaders—people who need the highest quality research, methodological rigor, and potentially transformative implications.

The purpose of creating a Working Group at PMENA is to engage our colleagues in the design and development of this series, and to develop the structure, topics, and outline for papers in an initial volume.

Issues in the Psychology of Mathematics Education in the Proposed Working Group

Following up on the work of Reys et al. (2008), the needs of the mathematics education community, particularly those needs of junior faculty who are working towards tenure, and the practitioner-researcher who needs timely, quality research with which to improve their education system will be discussed, and a plan for addressing these needs, at least in part, through the design of a new monograph series will be developed.

Plan for Active Engagement of Participants in Productive Reflection on the Issues

Jigsaw (First Session): Problem Identification and Solution Finding

On the first day of discussion, we will use the issues outlined in the introductory sections of this paper as a starting point for participants. We will first provide the participants with the issues as we currently see them, and then solicit new issues and ask the audience to clarify and provide examples of the issues that both prevent and support them in their scholarly writing. Following this initial conversation, we will divide participants into discussion groups, each focusing on a particular subset of the identified problems. Discussion groups will be tasked with characterizing their problem(s), defining the key issues that need to be addressed in each area, and providing a set of creative potential solutions to the problem(s).

Following the discussion groups, we will divide participants up into "Design Teams." Design Teams will be tasked with sharing the problem characterizations and potential solutions from the discussion groups, and then to create a set of design specifications for the development of a monograph series that would optimally solve the set of problems identified.

After Design Teams have developed their solutions, the group will meet as a whole to winnow through the issues and create a collective design matrix upon which the design of an initial volume in the series, and its editing process, can be constructed.

Homework for participants is to create a 2-minute elevator speech to be delivered to the Working Group in the next meeting time.

The **first working group session** will also provide an introduction of the vision of the Springer monograph series, as well as the logistics for publishing each volume in the monograph series.

Volume Design (Second Session)

The second day of discussion, participants will deliver their elevator speech where they propose a topic of research that they are currently working on, that they would like to collaborate on with other members of the group. Each participant will give their speech, and the topics will be recorded on large sheets chart paper. Participants will then "vote" on the topics they would like to work on (one per participant) by signing their name to a piece of chart paper that has their intended topic listed.

We will then divide the audience up into small Writing Groups, to prepare a volume proposal, as well as an outline and work schedule for the production of possible papers in a volume. Participants will also read through the Proceedings of the conference to choose potential collaborators for the volume based on the research they have presented at PMENA.

Finalizing Volume Proposal

As a result, by the end of PMENA 2012, we will have support and management criteria for the series, an outline for a set of papers that will constitute a volume in the series. We anticipate at least one volume proposal from the working group, but it may have more than one volume proposal, depending on participants' research interests.

Anticipated Follow-up Activities

Following this initial meeting, the Working Group leaders will meet and develop an editorial board for the publication, develop the submission, editing, and support guidelines from the set of criteria developed in the Working Group, and begin to nurture the development of the initial volumes of the series. The anticipated publication of the volume resulting from PMENA is 12 months from the conference.

Reference

Reys, B., & Dossey, J. (Eds). *U. S. doctorates in mathematics education: Developing stewards of the discipline*. Washington, DC: Conference Board of the Mathematical Sciences.

WHAT IS THE CONTENT OF METHODS? BUILDING AN UNDERSTANDING OF FRAMEWORKS FOR MATHEMATICS METHODS COURSES

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This discussion group will focus on exploring the use of conceptual frameworks in building mathematics methods courses for prospective mathematics teachers. Participants will consider (a) frameworks, (b) activities, (c) relationships between frameworks and activities, (d) the residue of activities and how they contribute to learning to teach, (e) research literature and attempts to explore these questions, and (e) development of a research agenda. Dialogues and collaboration among working group members will be encouraged by the development of teams to address facets of the emerging research agenda.

Keywords: Teacher Education–Preservice; Instructional Activities and Practices

Focuses and Aims for the Working Group

In light of the improved ability to track and compare student performance, mathematics teachers' impact on that performance has drawn increased scrutiny. Additionally, national accountability movements have begun to turn a lens toward mathematics teacher preparation in order to identify why some teachers are able to impact student performance, while others struggle. Although mathematics teacher educators (MTEs) have always examined and sought to justify their practices, studies have identified broad differences in emphases and instructional approaches (Harder & Talbot, 1997; Taylor & Ronau, 2006; Watanabe & Yarnevich, 1999) employed in teaching preservice teachers (PSTs). To begin to build descriptions, understanding, and theory about the work of MTEs in methods courses, we posed the question "What is the content of mathematics methods courses?" This question includes the idea of curriculum in a broad sense. We view learning opportunities and MTE's enactment of them with PSTs as content, but also realize that discussions during the working group will help us explore notions of content. The central question that emerged through our discussions was whether and how MTEs use research-based frameworks to build and explore their work and the impact of that work on our PSTs' learning and teaching. The exploration and development of frameworks used by MTEs will enable the field to build "a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof," one of the three main goals of PME-NA.

In the remainder of this proposal we interweave the focus of the working group with what we hope to accomplish during our working group sessions and beyond. First, we position ourselves as MTEs in the role as researchers who, collectively, must begin developing records of our practice in methods courses. Next, we summarize studies conducted on the content of methods courses. This prior research is then linked to a survey administered by our research team that asked MTEs to consider how they frame their methods courses. Finally, before outlining the working sessions, we discuss the goals of methods courses and the potential impact of selected methods course activity-types on PSTs' learning and practice. In each section we pose a few questions that may be of interest to working group participants.

Scholarly Inquiry in Teaching Methods Courses

In 2005, Mewborn identified confusions regarding frameworks and perspectives, or worldviews, in the reporting of research. In this work she called for the development of frameworks for individual researchers and at the level of the field of mathematics teacher education. In 2010, Arbaugh and Taylor (2008), drawing from Hiebert, Gallimore, and Stigler (2002), called for the development of "professional knowledge" they described as "knowledge that the research community establishes" or knowledge developed from empirical studies (p. 2). This knowledge was contrasted with "practical knowledge" that is built up by MTEs as they do the work of teaching and reflect on that work. Building from this view, in 2009, Lee and Mewborn, citing Richlin (2001), emphasized the significance of the development of scholarly inquiry and scholarly practices. Scholarly inquiry was described as explorations of "issues and practices through systematic data collection and analysis that yields theoretically grounded and empirically-based findings" (p. 3). This work could in turn be used to develop scholarly practices, "practices adapted from empirical studies of the teaching and learning of mathematics and the preparation of mathematics teachers" (p. 3). Within the tangle of terminology lies the idea that MTEs build practices for and through their interactions with PSTs and engage in the work of teacher as researcher by reflecting on and modifying the practices they enact. Such practical knowledge contains facets of scholarly inquiry, but is not often recognized as research. Yet MTEs know that they are doing what researchers might call a personally powerful form of action research. These pieces of work are powerful to individual MTEs and close colleagues, who are collaborators or confidants, but are often not shared more widely in the form of peer-reviewed articles in mathematics teacher education literature. In this working group we hope to build a collaboration that develops methods for communicating and synthesizing these personally powerful practices and comparing these practices to what we can find in the research literature.

Practices may be shaped by worldviews or frameworks as defined by Mewborn (2005). One example of this direction is provided by Kazemi, Franke, and Lampert (2009), who describe their work to develop "pedagogies of practice" and generate activities for prospective mathematics teachers using a view of practice from social practice theory. They assert,

> The future viability of professional teacher preparation requires that we systematically pursue appropriate ways to develop, fine tune, and coach novice teachers' performance across settings. These activities must find their way into university coursework rather than be relegated solely to field placements (Lampert & Graziani, 2009). Our hypothesis is that organizing professional education in mathematics education around core instructional activities and building links from the activities to student outcomes will enable us to support ambitious teaching. (p. 12)

Efforts such as this begin to build a perspective for the work of MTEs in which frameworks and goals for teacher practice are used to build instructional activities. As Kazemi et al. point out, we have empirical evidence of teacher practices that impact students in various ways; what is less clear is how MTEs' practices builds the sorts of teacher practices that can be sustained in the varied contexts of schools. Identifying and hypothesizing about pedagogical practices and their links to frameworks (not just worldviews) that enable the subtle modification of activity and analysis of associated evidence is critical for our field.

Studies of Methods Courses

Members of the mathematics education field have recognized both the lack of communication and the lack of consistency across methods instructors and courses in the United States, and studies have been conducted documenting these inconsistencies. For example, Harder and Talbot (1997) collected methods course syllabi from members of the Association of Mathematics Teacher Educators (AMTE), and examined the instructional approaches and assignments in the syllabi. The most commonly reported instructional approaches were whole class and group discussions, lab experiences (software, manipulatives, graphing calculators), student presentations, micro-teaching or peer-teaching, field-experience, lecture or direct instruction, and cooperative learning. The five categories of assignments reported were writing assignments, planning, presentations, participation, and resource files.

Watanabe and Yarnevich (1999) gathered survey data from elementary mathematics methods instructors, mathematics supervisors, inservice elementary school teachers, and preservice teachers. It is noteworthy that the survey used for this study asked respondents to rate pre-determined topics on a Likert Scale (1—topic should not be included, up to 4—topic must be included). This context is important when interpreting the findings of the study, given that participants were not able to say what was important to them in methods courses from their own perspective—they were limited by the choices on the survey, and they were also limited by their understanding of the meaning of the topics on the survey. The authors found substantial agreement between methods instructors, mathematics supervisors, and inservice teachers that mathematics methods courses should include current trends, doing mathematics, teaching a lesson, curriculum resources, manipulatives, problem-centered teaching, and questioning techniques. The inservice teachers and supervisors felt that demonstration lessons, lesson plan analysis/critique, writing in mathematics, lesson plans, authentic assessment, and performance assessment were important topics for elementary mathematics methods courses.

Taylor and Ronau (2006) approached this line of inquiry differently, examining methods course syllabi and identified seven common categories of goals and objectives: pedagogical skill, knowledge of content, dispositions, professionalism/leadership, pedagogical content knowledge, human development, and pedagogical knowledge. The authors noted,

> the most remarkable result is the surprising level of variability between mathematics methods courses … Syllabi that are clearly different from the de-facto consensus with respect to what they chose to include, or perhaps more strikingly, what they do not include, may offer quite different experiences for their students. We do not know if their students benefit from these differences or if they miss something crucial. (pp. 14–15)

These studies illustrate that there is substantial variation between mathematics methods courses in general. What is less understood is the source of such variation. Do MTEs draw from different frameworks as they develop activities? If different frameworks are drawn upon and a common activity is used, what are the impacts on PSTs?

Working Toward Frameworks

To explore the notion of framing in methods courses, we solicited MTEs on two listserves to respond to the following questions. Seventy-nine MTEs responded.

1. If you already have a frame for your mathematics methods course briefly describe how you organize/frame it.
2. If you do not have an existing framework, look through your syllabus and posit a framework to describe it.
3. Briefly describe the most impactful activities you engage in with your methods students.

We intentionally did not describe what we meant by “frame” to see how members of the field would interpret the term. In our conversations, we had come to view frames as more a way of *orienting* students to our course than of *organizing* our course. What we found in our categories, however, was more of an organizing structure. The categories ended up sounding more like the *topics* of the course more than the *framework* for the course. Responses of MTEs who named a framework and activities they shared are summarized in Table 1.

Table 1: Frameworks and Activities from Survey

Frameworks	Activities
1. Importance of knowing the learner 2. NCTM Process Standards and CCSS Standards for Mathematical Practice 3. Addressing the needs of all learners 4. Task selection and analysis 5. Understanding how students learn mathematics 6. Emphasis on students' mathematics 7. Manipulatives and concrete models 8. Cognitive or developmental stages and learning trajectories 9. Motivation and engagement 10. Curriculum 11. Modeling best practices for teaching 12. Reflection on mathematics teaching and learning practice 13. Integration of content and pedagogy/mathematical knowledge for teaching	1. General lesson planning 2. Manipulatives and technology 3. Making sense of PST's own mathematics 4. Microteaching 5. Interviews and interventions with K–12 students about their mathematical thinking 6. Assessment 7. Discourse-focused activities 8. State and national standards 9. Problem solving 10. Reading reflections 11. Facilitating lessons or tasks with undetermined audiences 12. Analyzing student work and error analysis 13. Unit planning

We hope to encourage participants to move beyond thinking about organizational structures and to think about frameworks as structures for orienting methods courses. We also hope to consider frameworks as something more specific than worldviews. How are framework(s) used in planning and exploring impacts of mathematics methods course? How do the results of activities help inform frameworks? To answer these questions, we need to consider goals for methods courses and the impact on our PSTs' learning and eventual teaching practice.

Goals for Methods Courses

NCTM (2007) outlines standards for the education and professional growth of mathematics teachers, focused around five issues: (a) teachers' mathematical learning experiences, (b) knowledge of mathematical content, (c) knowledge of students as learners, (d) knowledge of mathematical pedagogy, and (e) participation in career-long professional growth (p. 109). Although these standards do not constitute a framework as described by Mewborn (2005), they do provide a structure around which to identify specific goals that inform the development of frameworks for methods courses.

Despite the recommendations and vision statements from NCTM over the last few decades, much of the teaching in the United States focuses on helping students get through courses and pass standardized tests. If our PSTs experienced this type of mathematics, expecting them to teach in other ways can lead to cognitive dissonance for the PSTs and frustration for the MTE. However, this does not mean that MTEs should abandon expectations that new teachers teach in ways outlined by NCTM's vision documents. Instead, developing PSTs' proficiency "in designing and implementing mathematical experiences that stimulate students' interests and intellect" (NCTM, 2007, p. 5) might be addressed by many activities in methods courses, including task/lesson/unit planning and equity/diversity activities. Teachers must also be able to "orchestrate classroom discourse in ways that promote the exploration and growth of mathematical ideas" (p. 5) and "assessing students' existing mathematical knowledge and challenging students to extend that knowledge" (p. 6). To work toward becoming proficient in these areas, PSTs need to be provided opportunities to analyze student work—both written and verbal, to formatively and summatively assess student reasoning and understanding, and to interact with students in an effort to understand their thinking and learn to ask good questions.

Although methods courses might implicitly, or even explicitly, address the teachers' mathematical learning experiences and content knowledge, one might argue that these courses should launch PSTs into their careers with some knowledge of learners' mathematics, knowledge of pedagogical strategies, and a disposition toward continual growth and collaboration. We hope to generate discussion of these, and other, goals for the education of mathematics teachers, particularly at the preservice level. What additional goals might help us build frameworks for our methods courses? How can we design activities that support our goals?

Residue of Activities

In the process of envisioning and designing effective methods courses, MTEs must consider their goals and outcomes for PSTs and the activities or experiences they believe will be useful in helping PSTs reach these goals. One facet that is often neglected or unknown, however, is the ultimate impact methods courses experiences have on PSTs once they leave campus and enter their own classrooms. In a mathematics course, the term *residue* refers to the mathematics retained by students as a result of solving problems or completing a specific task (Davis, 1992). In considering an approach to the framing, content, and design of methods course activities, we posit it is crucial to consider, understand, and empirically examine the residue the methods course activities have on PSTs. A search of the *Journal of Mathematics Teacher Education* revealed approximately 70 articles about activities in methods courses; however, those articles do not paint a coherent picture of what is valued in methods courses or of the long-term residue of such activities on the PSTs' eventual teaching practice. Similarly, in examining our survey results, we realized that although some of the activities most commonly used in methods courses are supported by empirical evidence, we would do well as a field to engage in further study of the implementation and outcomes of specific activities and to findings about residue to systemically inform the design of methods courses.

Lesson planning activities. A few empirical studies of commonly used methods course activities have documented the residual effects, albeit short-term, of those activities. For example, in recent years, MTEs have used lesson study or modified versions of lesson study in their methods courses and have reported that the experiences help PSTs learn to become collaborative, reflective practitioners (Matthews, Hlas, & Finken, 2009; McMahon & Hines, 2008; Suh & Parker, 2010). McMahon and Hines also noted that the PST's post-lesson reflections were focused on student learning rather than on the role of the teacher. However, the PSTs indicated reluctance to instigate lesson study cycles with their collaborative teachers, citing concerns about inconveniencing the teachers. If the PSTs are not likely to engage in lesson study in the future, we might question the lasting residue from such activities. Matthews et al. found that PSTs benefitted from the ideas developed during lesson study in the short-term. They discussed the value in the 4-column lesson plan common in lesson study to focus PSTs' attention on how to build and support student understanding rather than to focus merely on what the teacher does during the class. Finally, Suh and Parker found that engaging PSTs with inservice teachers in the lesson study process helped develop the PSTs' mathematical knowledge for teaching, revealed gaps in the PSTs' mathematical knowledge, and increased the PSTs' awareness of the complexity of teaching and the importance of reflective practice. For each of the last two studies, because no post-methods course data were reported, MTEs are left wondering about the residue of using this 4-column lesson plan format, or lesson study in general, in methods courses.

Discourse activities. A synthesis of the types of discourse and associated research demonstrates the powerful impact of mathematical discourse on student learning (Franke et al., 2007). This impact was also reflected in the beliefs of our survey respondents; a number of MTEs reported inclusion of activities focusing around helping PSTs learn strategies and approaches to facilitate mathematical discourse. Specific examples reported in the survey include use of the texts, *Classroom discussions: Using math talk to help students learn* (Chapin, O'Connor, & Anderson, 2003) and *5 practices for orchestrating productive mathematics discussions* (Smith & Stein, 2011). These specific frameworks for discourse are readily used in professional development. Of interest would be activities used to support understandings of discourse and the impact of these activities on PSTs facilitations of mathematics discourse.

Understanding and extending students' current mathematical thinking. According to our survey results, another outcome valued by MTEs is the development of PSTs interpretations of and utilization of students' current ways of thinking. Research in this area has demonstrated the power of teachers' use of children's thinking as a basis for mathematics instruction (Franke et al., 2007; Koehler & Grouws, 1992). Recently, Jacobs, Lamb, and Philipp (2010) have introduced the *professional noticing of children's mathematical thinking* construct as a means to make sense of teacher actions in the classroom. Their study involved a group of PSTs as well as three other groups of increasingly more experienced teachers. The authors concluded that the constructs of noticing were much less developed in PSTs than would be expected. For MTEs the question might be how to provoke noticing that leaves residue useful in interactions with children. More importantly, the authors posit these skills are not a part of the typical knowledge of adult learners and presented evidence that with experience and training, teachers can become much more effective at attending, interpreting and responding to student thinking.

Equity and social justice activities. The Equity Principle of the NCTM (2000) calls for high expectations and support for all students to learn challenging mathematics. This call "challenges a pervasive societal belief in North America that only some students are capable of learning mathematics" (p. 12). NCTM also states that accomplishing equity, including providing the kind of accommodations, resources, and differentiation needed for all students to be successful requires teachers "to understand and confront their own beliefs and biases" (p. 14). Researchers have highlighted the difficulties of preparing PSTs to teach in ways that support a vision of equity described by Gutiérrez (2002) and NCTM (Garii & Rule, 2009). Yet work has begun to identify instructional activities (de Freitas & Zolkower, 2009; Rodriguez & Kitchen, 2005) and learning trajectories (Turner et al., 2012) that help MTEs understand learning to teach "effectively in diverse classrooms" (p. 67). Such work builds from existing research on noticing (Jacobs et al., 2010) and transformation. But will any of the short-term residue of these activities remain with the PSTs as they meet the daily challenges of their teaching contexts? How do differences in field experiences impact residue of such activities?

With the notion of residue in mind, it is crucial that we tie our activities closely to our goals and frameworks. Is residue the reflection of goals? Which activities are most likely to result in residue? Are there clusters of activities that might lead to residue? We also may need to consider how our goals and activities are supported or challenged by the teacher education programs in which our methods courses exist. How can we promote residue within the existing culture? Which activities have the most potential for residue within the existing culture?

Exploration of the content of methods, to build understanding of MTEs' use of frameworks to inform activities and activities to inform or develop frameworks, supports the PME-NA conference theme of navigating transitions in professional learning. While learners in this context are both MTEs and PSTs, they work together to navigate career transitions and understandings of the academic and school settings in which they work. Explorations of frameworks and activities will further allow MTEs to leverage their considerable practical knowledge to build lines of scholarly inquiry supportive of the development of scholarly practices.

Outline of Working Group Sessions

Our first session will include an introduction and overview of the working group. We will begin with a presentation describing the background and goals of the group. We plan to present the disparate knowledge base about methods courses and prior efforts to explore the content of methods courses. To orient the participants to the discussion of frameworks and activities, we will present one activity used by authors of the proposal and explore how it links to the framework for the course. In addition, participants will discuss how the framework can be used to gather evidence of residue that engagement in an activity might precipitate. Next, we will introduce two threads of inquiry with respect to frameworks, activities, and residue for the working group. The first thread is Framework-Activities-Residue and the second thread is Activity-Frameworks-Residue. Pictorial representations are provided in Figures 1 and 2, respectively.

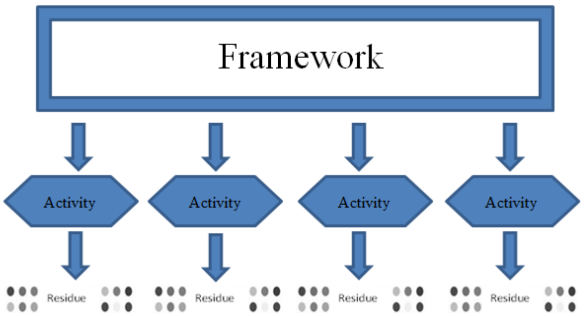

Figure 1: Framework-Activity-Residue

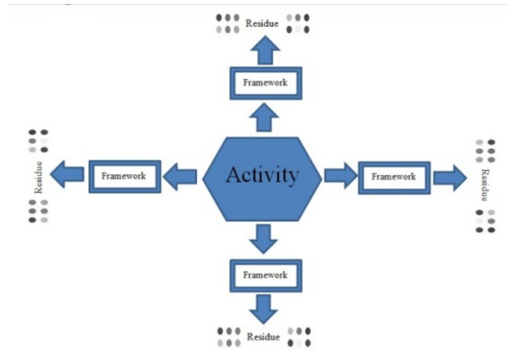

Figure 2: Activity-Framework-Residue

In the first thread, a particular framework is selected as a starting point. For example, consider the Task Analysis Framework (Stein, Smith, Henningsen, & Silver, 2009) for the cognitive demand of mathematical tasks. Authors have used the Task Analysis Framework to build activities for methods courses (e.g. Rutledge & Norton, 2008; Norton & Kastberg, 2012). We are interested in the residues of the activities. The question for the first thread of inquiry is:

> For a particular framework, what are the activities for which we have empirical evidence of residue, and what is the nature of that residue?

The second thread of inquiry takes as its starting point an activity. Here, let us take as an example video case analysis. Alsawaie and Alghazo (2010) used van Es and Sherin's (2002) Learning to Notice Framework. Stockero (2008) adapted van Es and Sherin's (2008) attributes of reflection by replacing their levels of reflection with those from Manouchehri (2002). The question for the second thread of inquiry is:

> For commonly used activities in methods courses (or novel ones that have been reported), what frameworks are supported, what empirical evidence of residue is available, and what is the nature of that residue?

Looking across both threads of inquiry, we seek to identify and describe existing findings. This inquiry will also reveal gaps in the literature, for which scholarly inquiry can be designed. In general we seek to describe gaps in the literature and answer the broader question:

> What does the research literature reveal about mathematics methods courses in terms of frameworks, activities, and residues with respect to mathematics methods courses?

The answer to this question sets a research agenda for scholarly inquiry into the practice of methods instruction.

In our second session, participants will start by unpacking frames and activities from courses. Participants will focus on looking at potential residue and relationship to frameworks. A series of presentations will serve to launch our conversations. One presenter will share activities and findings illustrating residue from the activities. Another presenter will share her study of MTEs frameworks, goals, and activities. Participant discussions of the work of the presenters will focus on connections between frameworks, activities, and residue and relationships to the two threads. Following the presentations, participants will break into smaller groups to work on the two threads of inquiry to propose syntheses of existing literature and research questions to be explored by the group as part of a developing research agenda. A database of articles, compiled by the authors of the proposal, exploring facets of MTEs' work with PSTs in mathematics methods will be used as a resource.

In our third session we will focus on building a plan for follow up activities.

1. Small groups will present a summary of discussions and plans for scholarly inquiry. This work will likely not be completed during the conference, but will be continued throughout the year electronically.

2. We will develop potential collaborations and plans to move forward on proposals for scholarly inquiry and opportunities to present at national conferences such as the national meetings of NCTM and AMTE.
3. We will discuss the use of Skype and Google Group for working as a group throughout the year.

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WISDOM^e: QUANTITATIVE REASONING AND MATHEMATICAL MODELING

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The Quantitative Reasoning and Mathematical Modeling (QRaMM) working group is one of three research strands initiated by the Wyoming Institute for the Study and Development of Mathematical Education (WISDOM^e). QRaMM brings together researchers from multiple universities across the country to share ideas and conduct research on quantitative reasoning and modeling within an interdisciplinary context. In collaboration with the National Science Foundation MSP Pathways in Environmental Literacy Project, the QRaMM group has engaged in research collaborations resulting in presentations at national conferences, an International STEM Research Symposium: Quantitative Reasoning in Mathematics and Science Education 2012, and publications addressing QR. Focusing issues include: (1) middle and high school students' development of quantitative reasoning and mathematical modeling, (2) creation of QR learning progressions to explain such development, and (3) the impact and interplay of QR and modeling on students' development in mathematics and science. Participants in the QRaMM working group will join members of the QRaMM research team in interpreting data on QR in mathematics and science, critiquing QRaMM learning progressions, vetting QRaMM assessment items, and debating the QRaMM theoretical framework.

Keywords: Cognition; Learning Trajectories (or Progressions); Modeling; Mathematical Knowledge for Teaching

QR Research Collaborative

A core membership for this continuing PME-NA Working Group was established through the initial invitational **Planning Conference for WISDOM^e** conducted September 2010 at the University of Wyoming. During that conference three collaborative, interdisciplinary research teams were established: Quantitative Reasoning and Mathematical Modeling (QRaMM), Developing Investigations of Mathematical Experience (DIME), and Technology Tools and Applications in Mathematics Education (TTAME). The QRaMM and DIME research teams went on to host working groups at PME-NA in 2011. The QRaMM team consists of researchers focused on the cognitive act of quantifying in mathematics, which includes examining how students quantify an object through conceiving a measurable attribute of the object. Research on quantification has explored concepts such as function, covariation, multiple representations, continuity as smooth or chunky, and conceptions of angle measure. Another aspect of interest to the research team is the concept of QR within context; the ability to interpret quantitative models especially in the sciences and mathematics and the ability to create, test, and refine models. Since its inception the QRaMM team has been working collaboratively with the NSF Pathways in Environmental Literacy Project, that focuses on QR in environmental sciences. The Pathways Project is developing learning progressions for three content strands that are fundamental to the development of environmentally literate citizens from sixth to twelfth grades: carbon cycle, water cycle, and biodiversity. The Pathways Project includes a QR Theme team that is exploring the impact of quantitative reasoning on the development of environmental literacy. The outcome of the partnership has been a series of QR research collaborations including sessions at PME-NA 2011, NCTM National Conference 2012, an International STEM Research Symposium: Quantitative Reasoning in Mathematics and Science Education 2012, and publications on QR. The research papers and presentations are informing the field on the development of QR and its impact on the learning and teaching of mathematics, on the ability of students to apply QR within context including interpreting and creating models, on the development of QR learning progressions, including creation of QR assessment items for clinical interviews and written assessments of QR abilities in the context of environmental science.

The opportunity to conduct a second Working Group at PME-NA provides an important venue to continue and expand discussions of the background perspectives and issues related to investigations of quantitative reasoning and modeling. The Working Group will create a setting where participants can share information, issues, and problems related to ongoing research, to promote interest and potential participation in furthering these and other disciplined inquiries into these phenomena, and to provide continuing support to the team members to collaborate within and across ongoing and future research.

A Program of Research for QR

The central question for the QRaMM research team is: How do students reason quantitatively and what impact does that reasoning have on their learning in mathematics and science? For the Pathways project this question is focused on the impact of QR in understanding environmental science. The purpose of the Pathways QR Theme is to determine the quantitative reasoning aspects of the learning progression leading to the primary QR research question: What are the essential quantitative reasoning abilities that are required for the development of environmental literacy? The QRaMM research team incorporates these questions into a broader QR research program with the following research questions:

1. How does quantitative reasoning integrate and interact across mathematics and the sciences?
2. What is the role of learning progressions for quantitative reasoning in fostering interdisciplinary learning in mathematics and the sciences?
3. How do learning progressions for quantitative reasoning inform professional development in ways that support interdisciplinary mathematics and science teaching and learning?

Learning Progressions Supporting Students' Development of Environmental Literacy: Work of the Pathways Project

Quantitative reasoning and learning progressions underpin the work of QRaMM. The Consortium for Policy Research in Education (CPRE) defines learning progressions as follows:

> Learning progressions are hypothesized descriptions of the successively more sophisticated ways student thinking about an important domain of knowledge or practice develops as children learn about and investigate that domain over an appropriate span of time. (Corcoran, Mosher, & Rogat, 2009, p. 37)

A number of learning progressions in science are currently under development including: tracing carbon in ecological systems (Mohan, Chen, & Anderson, 2009), particle model of matter (Merrit, Krajcik, & Swartz, 2008), modeling in science (Schwarz, Reiser, et al., 2009), genetics (Duncan, Rogat, & Yarden, 2009), chemical reactions (Roseman, et. al., 2006), data modeling and evolution (Lehrer & Schauble, 2002), explanations and ecology (Songer, Kelcey, & Gotwals, 2009), buoyancy (Kennedy & Wilson, 2006), atomic molecular theory (Smith, Wisner, Anderson, & Krajcik, 2006), and evolution (Cately, Lehrer, & Reiser, 2005). Examples of three of these learning progressions are provided in (Corcoran, Mosher, & Rogat, 2009). However, QR learning progressions have not been developed. We argue that without quantitative accounts environmental literacy cannot fully develop.

The Pathways learning progressions for environmental literacy are based on research in science education and cognitive psychology, foundational and generative disciplinary knowledge and practices, and strive for internal conceptual coherence. The QR in environmental literacy frameworks builds on these characteristics, incorporating mathematical and statistical tools essential for science. Research on the act of quantification is essential to the progression, because quantifying is the prerequisite to QR and modeling.

The CPRE panel identified essential elements of learning progressions to be:

1. Upper Anchor: target performance or leaning goals which are the end points of learning progression and are defined by societal expectations, analysis of the discipline, and requirements for entry into the next level of education.
2. Progress Variables: dimensions of understanding, application, and practice that are being developed and tracked over time.
3. Levels of Achievement: intermediate steps in the developmental pathway(s) traced by a learning progression.
4. Learning performances: tasks students at a particular level of achievement would be capable of performing.
5. Assessments: specific measures used to track student development along the hypothesized progression.

The Pathways learning progressions have a lower anchor, which is the typical accounts of environmental issues given by students at the upper elementary and middle school level (Anderson, 2009). These accounts are empirically tested through a cyclic research process of clinical interviews informing the learning progression and leading to the development of written assessments given on a large scale. The Pathways learning progressions upper anchor is based on experts views of what a scientifically literate citizen should know and be able to do by the 12th grade. The upper anchor is much like a NSTA or NCTM standard, but learning progressions differ from standards in that the lower anchor and intermediate achievement levels are research-based, reflecting the actual trajectory of student learning. A limited number of achievement levels (4 or 5) are identified as plateaus in students' development of more sophisticated ways of thinking about enduring understandings, concepts, and processes.

The progress variables address both environmental literacy and QR. The environmental literacy progress variables for the Pathways project are the carbon cycle, water cycle, and biodiversity, which are considered areas in which students must develop conceptual understanding if they are to become environmentally literate citizens. The progress variables for QR include the act of quantification, quantitative literacy (QL) which consists of arithmetic understandings supporting science, quantitative interpretation (QI) which is the process of interpreting scientific models to determine trends and make predictions, and quantitative modeling (QM) which is the creation of models by the student. The fundamental process is the act of quantification, the cognitive processes essential to mathematizing within a context. Our goal of scientifically literate citizens necessitates the second progress variable of quantitative interpretation. Given a data table, graph, equation, or scientific model, the citizen must interpret the model to make data informed decisions. Finally, the quantitative modeling progress variable supports current approaches to have model building, testing, and refinement as drivers for science education.

Within the QL, QI and QM progress variables are hypothesized mathematical and statistical tools critical for science (Table 1). The Pathways project focuses not on the mastery of these tools, but on the act of quantification and the process of quantitative reasoning within science contexts.

Table 1: Tools Supporting QR in Science Contexts

Tools	Quantitative Literacy	Quantitative Interpretation	Quantitative Modeling

Component	Numeracy
	• Number Sense
	• Small/large Numbers
	• Scientific Notation
	• Logic
	Measurement
	• Accuracy
	• Precision
	• Estimation
	• Units
	Proportional Reasoning
	• Fraction
	• Ratio
	• Percents
	• Rates/Change
	• Proportions
	• Dimensional Analysis
	Basic Prob/Stats
	• Empirical Prob.
	• Counting
	• Central Tendency
	• Variation
		Representations
		• Tables
		• Graphs/diagrams
		• Equations
		    • Linear
		    • Quadratic
		    • Power
		    • Exponential
		• Statistical displays
		• Translation
		Science diagrams
		• Complex systems
		Statistics & Probability
		• Randomness
		• Evaluating Risks
		• Normal Distribution
		• Statistical Plots
		• Correlation
		• Causality
		• Z-scores
		• Confidence Intervals
		Logarithmic Scales
			Logic
			Problem Solving
			• Problem Formulation
			Modeling
			• Normal Distribution Model
			• Regression Model
			    • Linear
			    • Polynomial
			    • Power
			    • Exponential
			    • Logarithmic
			• Logistic Growth Model
			• Multivariate Model
			• Simulation Model
			• Scientific Diagram
			• Table & Graph Models
			Inference
			• Inference
			• Hypothesis Testing
			• Practical Significance

The development of learning progressions is an iterative process typical of design-based research. The Pathways environmental literacy learning progressions began with hypothetical frameworks based on theories about reasoning about processes in socio-economic systems, including discourse, practices, and knowledge, as well as linking processes between lower and upper anchors. The QR Pathways Theme is using the learning progressions to research the impact of QR on students' development of environmental literacy. The overarching goal is to study the capacity of students to understand and participate in evidence-based discussions of socio-ecological systems.

A framework for the Pathways learning progression (Table 2) consists of progress variables (matrix columns), levels of achievement (matrix rows), learning performances (content of matrix cells), and linking processes which are common processes that are recognized by students at all levels of achievement (Anderson, 2009). Learning performances are exemplars drawn from the clinical interviews and written assessments that demonstrate student responses at different achievement levels. Learning progression matrices cross-tabulate achievement levels (rows in matrix) with progress variables (columns in matrix).

Table 2: QR Learning Progression Framework

	QR Progress Variable		
Achievement Level	**Quantification Act**	**Quantitative Interpretation**	**Quantitative Modeling**
Level 4 (Upper Anchor)	**4a** Variation: reasons about covariation of 2 or more variables; comparing, contrasting, relating variables in the context of problem **4b** Quantitative Literacy: reasons with quantities to explain relationships between variables; proportional reasoning, numerical reasoning; extend to algebraic and higher math reasoning (MAA) **4c** Context: situative view of QR within a community of practice (Shavelson); solves ill-defined problems in socio-political contexts using ad-hoc methods; informal reasoning within science context (Steen & Madison; Sadler & Zeidler) **4d** Communication: capacity to communicate quantitative account of solution, decision, course of action within context	**4a** Trends: recognizes and provides quantitative explanations of trends in model representation within context of problem, including linear, power, exponential trends **4b** Predictions: makes predictions using model with covariation and provides a quantitative account which is applied within context of problem **4c** Translation: translates between different models, at least categorically (ie this graph looks exponential) **4d** Revision: revise models theoretically without data, evaluate competing models for possible combination (Schwarz) **4e** Authority: question model by challenging quantitative aspects as estimates or due to measurement error, especially when contrasting models	**4a** Create Model: ability to create a model representing a context and trace through model correctly **4b** Refine Model: test and refine a model for internal consistency and coherence to evaluate scientific evidence and explanations; results; extend model to new situation (Duschl) **4c** Model Reasoning: construct and use models spontaneously to assist own thinking, predict behavior in real-world, generate new questions about phenomena (Schwarz) **4d** Methods: demonstrate ability to use variety of methods to construct model within context; least squares, linearization, normal distribution, logarithmic, logistic growth, multivariate, simulation models; **4e** Statistical: conduct statistical inference to test hypothesis (Duschl)
Level 3	**3a** Variation: recognizes correlation between two variables but provides a qualitative or isolated case account; lacks covariation **3b** Quantitative Literacy: manipulates quantities to discover relationships; measure, numeracy, proportional, statistical procedures **3c** Context: display confidence with and cultural appreciation of mathematics within context; number sense, practical computation skills (Steen)	**3a** Trends: expand recognition of patterns in models of one variable to recognizing linear vs. curvilinear growth **3b** Predictions: interprets models where one variable is categorical, identifying trends and making predictions with strong quantitative accounts; make predictions using model with covariation but only provide qualitative account **3c** Translation: attempts to translate between models if prompted but fails to relate variables between models	**3a** Create Model: create simplistic models for covariation situations that lack quantitative accounts; fail to trace model correctly **3b** Refine Model: test and refine model based on supposition about data; extend model without verifying fit to new situation **3c** Model Reasoning: construct and use multiple models to explain phenomena, view models as tools supporting thinking, consider alternatives in constructing models (Schwarz)

	QR Progress Variable		
Achievement Level	**Quantification Act**	**Quantitative Interpretation**	**Quantitative Modeling**
	3d Communication: capacity to communicate qualitative account of solution, decision, course of action within context; weak quantitative account **3e** Variable: mental construct for object within context is identified, conceptualized so that the object has attributes that are measurable (Thompson -act of quantification); uses variable in context	**3d** Revision: revise model to better fit evidence and improve explanatory power (Schwarz) **3e** Authority: question differences between models, but use erroneous qualitative accounts not error or approximations	**3d** Methods: demonstrate ability to use two different methods to model a situation **3e** Statistical: use descriptive statistics for central tendency and variation; make informal comparisons to address hypothesis
Level 2	**2a** Variation: sees causation in relationship between two variables, provides only a qualitative account; lacks correlation **2b** Quantitative Literacy: poor QL interferes with manipulation of variables; struggle to compare or operate with variables; ability to manipulate and calculate with one variable to answer questions of change, discover patterns, and draw conclusions; **2c** Context: lack confidence with or cultural appreciation of math within context; practical computation not related to context **2d** Communication: provides elements of account, but lacks capacity to communicate solution, decision, course of action within context; weak qualitative account **2e** Variable: object within context is identified, but not fully conceptualized with attributes that are measurable; object is named creating a variable (Thompson)	**2a** Trends: identify and explain single case (point) in model within context; recognize increasing/ decreasing trends but not relating to change in both variables (covariation lacking) **2b** Predictions: makes predictions for models with one variable but provides only qualitative arguments **2c** Translation: indicate preference for one model over another but do not translate between models **2d** Revision: revise model based on authority rather than evidence, modify to improve clarity not explanatory power (Schwarz) **2e** Authority: acknowledge quantitative differences in models but does not provide an explanation **2f** Interpret: identify variables in the model (i.e., graph axes, table headings, equation unknowns); provide qualitative account, avoiding quantities; form correct mental image to conceive problem; difficulty with models that embed variable or have more than two interrelated variables	**2a** Create Model: creates visual models to represent single variable data, such as statistical displays (pie charts, histograms) **2b** Refine Model: extends a given model to account for dynamic change but provides only a qualitative account **2c** Model Reasoning: construct and use model to explain phenomena, means of communication rather than support for own thinking (Schwarz) **2d** Methods: constructs a table or data plot to organization information but does not use as model **2e** Statistical: calculates descriptive statistics for central tendency and variation but does not use to make informal comparisons to address hypothesis

	QR Progress Variable		
Achievement Level	**Quantification Act**	**Quantitative Interpretation**	**Quantitative Modeling**
Level 1 (Lower Anchor)	**1a** Variation: does not compare variables **1b** Quantitative Literacy: struggles to manipulate and calculate with even one variable to answer questions of change, discover patterns, and draw conclusions; **1c** Context: does not relate quantities to context **1d** Communication: discourse is force-dynamic; avoids quantitative account, ignoring quantities providing weak qualitative account **1e** Variable: objects within context are not identified, no attempt to conceptualize attributes that are measurable	**1a** Trends: do not identify trends **1b** Predictions: avoid making predictions **1c** Translation: fail to acknowledge quantitative difference in models **1d** Revision: view models as fixed, test to see if good or bad replicas of phenomena (Schwarz) **1e** Authority: does not acknowledge difference in models **1f** Interpret: fail to relate model to context; avoid using model	**1a** Create Model: does not view science as model building and refining so does not attempt to construct models; forced dynamic or low level school science discourse, expect to receive facts and memorize processes **1b** Refine Model: no model created to refine **1c** Model Reasoning: construct and use models that are literal illustrations, model demonstrates for others not tool to generate new knowledge (Schwarz) **1d** Methods: no evidence of knowledge of methods for building models **1e** Statistical: does not use statistics; no calculation of even descriptive statistics

Aspects of the Pathways learning progression framework are highly quantitative in nature. For example, one such aspect is scaling. Students at the lower anchor often function at a macroscopic or "individual" scale in which they view the world from their sensory purview. Hence, their accounts of environmental issues are based within their daily perceptions at this scale. Moving students to the upper anchor requires that they scale up to a global view, as well as scale down to a microscopic-atomic view. Regardless of the scale, moving up or down the scale becomes intensively quantitative.

Challenges for the QR Theme researchers include determining which theories about reasoning carry over to QR aspects of environmental literacy and to discover potential theories that are more QR-centric. For example, students achieving the upper anchor in the environmental literacy learning progression should function as informed decision makers within the socio-economic system at three levels: discourse, practices, and knowledge. "Knowledge is embedded in practices, which in turn are embedded in discourses" (Anderson, 2009). We all participate in multiple discourses that associate us with communities of practice (Gee & Green, 1998) and as we gain understanding of a phenomena as our discourse around it matures. The lower anchor discourse for the environmental literacy progression is force dynamic, relying on students' theory of the world with a focus on actors, enablers, actor's purposes, conflicts between actors, and settings for the actions (Pinker, 2007; Talmy, 2003). At the upper anchor scientific discourse is essential, with students moving away from actors in settings to laws that govern the work of systems. Moving from knowing mathematical or statistical algorithms to understanding of enduring concepts can allow one to analyze scientific problems quantitatively.

QR: Perspective and Context

To support consideration of situations from a mathematical/scientific perspective one could have students work on tasks designed to promote QR. Often tasks designed to support QR describe situations that could be considered to have "real-life" contexts. As such, asking whether or not a student is familiar with a context seems a natural question. However, students' familiarity with the situation described by a task is only one aspect of the multifaceted interactions that can occur as students work on a task. Van Oers

(1998) asserted "What counts as context depends on how a situation is interpreted in terms of activity to be carried out" (p. 481). By considering individuals' perspective on mathematical tasks, context is no longer objective and external to an individual, rather it is steeped within an individual's interaction with a task.

When investigating students' QR, it is important to consider students' perspective of the problem situation posed by the task. Such tasks might include situations involving quantities that could change. For example, a task could involve a bottle of varying width filling with liquid being dispensed at a constant rate. As part of the filling bottle task, one could prompt students to consider how the volume of liquid would be changing as the height of the liquid in the bottle is increasing.

Secondary students working on the filling bottle task may not necessarily engage in QR. Students employing QR might consider relationships between the varying quantities of volume and height. Two types of relationships include (1) making comparisons between associated amounts of volume and height, and (2) considering variation in the intensity of the change in volume as related to the change in height (Johnson, 2011). Students engaging in (1) considered amounts of change in volume and height that *occurred* on particular intervals. The student engaging in (2) considered change in volume *as occurring* with respect to the changing height.

Despite the filling bottle task's design supporting QR, students may not employ QR when working on the task. A student could appeal to the sound of a filling bottle, as did one student in Johnson's (2010) study, who responded: "I'm thinking as the water bottle, you are filling it up and as the rim comes, you can hear it filling up faster." Although this student was working on the same printed task as students employing QR, she seemed to be wrestling with the problem of actually filling a bottle rather than considering quantities involved in the task.

Taken together, these student responses suggest that a student's perspective on the nature of the task to be solved might afford or constrain his or her quantitative reasoning. Further, a printed task itself does not indicate the quantitative reasoning in which a student might engage. Such considerations are not limited to secondary students with limited mathematical coursework. In working with college students solving a task involving constructing an open box from a sheet of paper, Moore and Carlson (2012) found that students' images of the problem context influenced the quantitative reasoning in which students' engaged. When engaging students in tasks designed to support QR, students' interpretation of the problem context and quantities involved is nontrivial and should not be taken for granted.

QRaMM Reflection Tasks

The three QRaMM working group sessions will actively engage the participants in discussion of the research questions, variables, and framework provided above. Each session will begin with a 15-minute plenary presentation by a member of the QRaMM research team to pose research issues and problems for discussion. Seminal readings related to the presentation will be provided to all participants in the first session, serving as support materials for discussions in later sessions. Breakout sessions for working subgroups will be organized around task-oriented discussions. The last session will provide plenary presentations summarizing subgroup work and report out discussions by the subgroups.

The sub-groups will focus on the following domains of discussion related to our research questions:

1. How does quantitative reasoning integrate and interact across mathematics and the sciences?
    a. How does the act of quantification play out within a mathematics and science context?
    b. How do alternative conceptions of science and mathematics concepts influence the use of QR?
    c. What is the role of context in QR?
    d. Context for whom—secondary students, collegiate students, or perspective teachers?
    e. What is the connection of QR to the common core mathematical practices?

2. What is the role of learning progressions for quantitative reasoning in fostering interdisciplinary learning in mathematics and the sciences?
   a. How do we best elicit information on learning progressions in QR from students (interviews, large scale quantitative assessments, etc.)?
   b. What role does discourse analysis and scale have in QR learning progressions?
   c. How do science learning progressions and QR learning progressions interact?
3. How do learning progressions for quantitative reasoning inform professional development in ways that support interdisciplinary mathematics and science teaching and learning?
   a. What role can the assessments related to QR learning progression development play in classroom assessment of QR?
   b. What should interdisciplinary teaching experiments focused on QR in STEM look like?

For each of the research questions, the subgroup will address the following related topics as well:

1. Address issues and concerns for research methodologies to be used.
2. Discuss techniques for using and transforming data to describe, analyze, and interpret in ways that will illuminate and inform the concept of quantitative reasoning.
3. Discuss approaches to reporting what is found and to applying it to inform improved educational practice related to interdisciplinary aspects of quantitative reasoning and mathematical modeling.
4. Discuss instruction, student learning, curriculum, policy, and future research with respect to quantitative reasoning.

QRaMM Follow-up Activities

We will sustain the efforts of the working group through the WISDOM^e research initiative, a proposed QR monograph targeting researchers in QR in STEM, collaboration with the NSF Pathways Project, and a second WISDOM^e Research Conference.

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