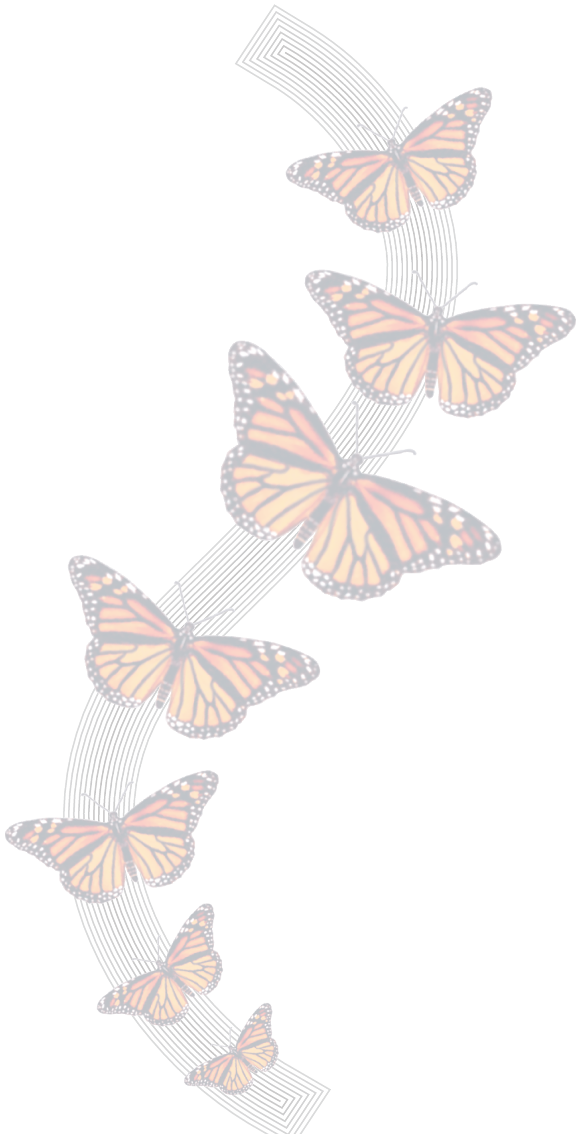
Mathematical Ideas:
History, Education, and Cognition



Cinvestav

Vol. 1

Morelia, México, 2008

PME 32
PME-NA XXX

International Group for the Psychology of Mathematics Education

Proceedings of the Joint Meeting of PME 32 and PME-NA XXX

Editors
Olimpia Figueras
José Luis Cortina
Silvia Alatorre
Teresa Rojano
Armando Sepúlveda

Morelia, México
July 17-21, 2008

International Group for the Psychology of Mathematics Education

Proceedings of the Joint Meeting of PME 32 and PME-NA XXX

Editors
Olimpia Figueras
José Luis Cortina
Silvia Alatorre
Teresa Rojano
Armando Sepúlveda

Volume 1

Plenaries, Research Fora, Discussion Groups,
Working Sessions, Seminars, National Presentation,
Short Oral Communications, Posters

Morelia, México, July 17-21, 2008

Centro de Investigación y de Estudios Avanzados del IPN
Universidad Michoacana de San Nicolás de Hidalgo

Cite as:
Figueras, O., Cortina, J.L., Alatorre, S., Rojano, T., & Sepúlveda, A. (Eds.). (2008). Proceedings of the Joint Meeting of PME 32 and PME-NA XXX. Vol. 1. México: Cinvestav-UMSNH

Website: http//:www.pme32-na30.org.mx

The proceedings are also available on CD-ROM and on the website

ISSN 0771-100X

ISBN 978-968-9020-06-6

Cover design: Carla Raigoza
Overall Printing Layout: David Páez
Logo: Vladimir Soto
Printing: Guevara Impresores S.A. de C.V.
Chichimecas Manz. 108 Lote 4
Col. Ajusco Coyoacán, D.F. 04300 México

Preface

Again members of the International Group for the Psychology of Mathematics Education will come to the American Continent, this time to México for their 32nd conference (PME 32). The North American Chapter decided to carry out, earlier this year, the 30th annual meeting (PME-NA XXX) to get together with colleagues from the rest of the world.

Hosted by the Centre for Research and Advanced Studies of the National Polytechnique Institute (Cinvestav) and the University of Saint Nicholas of the State of Michoacán (UMSNH) the Joint Meeting of PME 32 and PME-NA XXX will be held in Morelia, México. The theme of the conference is Mathematical ideas: History, Education, and Cognition.

Morelia is the capital of Michoacán, a state of the Occidental region of México where *Tarasco* People spread their domain. In pre-Hispanic times they created a strong empire unbeaten by the Aztecs. *P'urhepecha* or *Tarasco* was a lingua franca in the region they had influence.

Nowadays *Tarasco* People populates the Northeast part of Michoacán; they still settle Pátzcuaro and Tzintzuntzan, where *Tarascan* Kings had their seigniories. Today in Mexico around 200 000 persons speak *P'urhepecha*; many are monolingual. This language is the sole member of *Tarasca* family, one of 12 linguistic families in which 52 Mexican languages, officially recognized in 1980, are classified. This year five new languages were identified, Maya immigrants from Guatemala enriched the Mexican linguistic mosaic.

Colegio de San Nicolás de Hidalgo is situated in a historical building where core activities of the Joint Meeting of PME 32 and PME-NA XXX will be carried out. It was founded by Vasco de Quiroga around 1539, when consecrated bishop in Pátzcuaro. *Colegio de San Nicolás Obispo* constituted a main pillar of Vasco de Quiroga's evangelizing and civilizing work. When the episcopate moved to Valladolid (old name of Morelia), the college were transferred as well.

The next bishop (1566-1572) favoured the ecclesiastic careers and converted Augustinian doctrines into secular knowledge. They arrived at Tiripetío, Michoacán in 1537 and founded a Centre of Higher Studies of Arts and Theology, around 1540; the first Augustinian study centre of the New World. Teaching was structured as was done in Spanish universities – considering the seven disciplines of the *Trivium* (Grammar, Logic or Dialectics and Rhetoric) and the *Quadrivium* (Geometry, Arithmetic, Astronomy and Music). It has accumulated a wide variety of stories in its 469 years of existence.

Profound reforms were carried out in *Colegio de San Nicolás* to include Philosophy, Scholastic Theology, and Morality or to open Chairs for Civil and Canonical Law. Ideas regarding Christian modernity started to break through, the college was closed due to the independent movement of 1810. It opened again in 1832 and 13 years later became a lay college. Chemistry, physics, cosmography, mathematics, and biology were introduced, laboratories and libraries enriched, secondary education was offered again, and careers as Law, Notary, and Civil and Agricultural Engineering were set up. In 1902 the college started to function as a tertiary level school.

The University of Saint Nicholas of the State of Michoacán was founded in 1918 with various educational institutions, among them *Colegio de San Nicolás de Hidalgo* and *Biblioteca Pública Universitaria*. The latter located in a historical building of the centre of Morelia guards antique books of Michoacán, even some incunabular. Participants of the Joint Meetings of PME 32 and PME-NA XXX will have the privilege to make their Poster Presentations in this place.

The brief history narrated shows facts to sustain that *Universidad Michoacana de San Nicolás de Hidalgo* is one of the first universities in the American Continent; its foundations were laid by the first Franciscan study centre founded by Vasco de Quiroga in Pátzcuaro and the first study centre founded by the Augustinian in Tiripetío.

The International Committee agreed to introduce in the Scientific Program two modes of personal presentations: Seminars and National Presentation. Seminars are short intensive courses designed for a small number of participants ($\cong$ 35); topics and possible speakers are decided in advance. The National Presentation is an exposition of invited speakers from the country that hostesses the conference, its aim is to give participants an overview of the research work done by the national researchers as well as of their contributions.

Three Mexican researchers who are also members of PME or PME-NA were invited to make the National Presentation this year. The paper contained in this volume includes a description of the last 35 years. In this period, mathematicians, mathematics educators, and authorities showed a growing interest to study the problems related to the teaching and learning of mathematics and actions were carried out to consolidate research activities. The Mathematics Education Department (MED) was founded in Cinvestav in 1975 and relationships with other groups in universities around the country were established, in particular a first agreement was signed in 1981 between the UMSHN and Cinvestav.

It were well known mathematicians of Cinvestav, who decided to change the orientation of their work and started to think about mathematical ideas focused on mathematics education within the educational system. Their approach was based on the use of the history and development of mathematical concepts as a means to understand difficulties in the learning of mathematics ideas and epistemological analysis was done in search of a framework for curriculum design. The trying out of the materials written enabled them to have a close look to mathematics teaching.

The theme of the Joint Meeting for PME 32 and PME-NA XXX, Mathematical ideas: History, Education, and Cognition, reflects the orientation the work that has been carried out in Mexico has, but also centres the discussion in main interests of the international community, that is, on mathematical ideas considering different perspectives to understand how they have developed, how they can be taught, and how they are learnt.

In 2005, a suggestion that members of Cinvestav should make a proposal for hosting PME 32 made me consider the importance it would have for the MED and for strengthening research activities throughout the country, particularly in Michoacán. I asked Armando Sepúlveda if he was willing to work together for hosting the conference. He asked for two days to think it over, valuate the possible scenarios, and ask the members of the faculty of the Mathematics Education Area for their support.

It was a difficult time to envisage the future, in the three years to come the UMSNH would have a new Dean, Michoacán would have a new Governor, the city would have a new Major, Cinvestav could have a new Director General and possibly a new head of DME. At the end the challenge was taken. On behalf of the Mexican mathematics community our acknowledgments for those members of PME that supported the proposal. I am particularly indebted to Kathleen Hart for her advice and time devoted in the first stage when hard work broke into my life.

I thank Armando for favouring the possibility to have a conference in such a site, for getting the permission to use the university buildings located in the Centre of Morelia, for convincing Lourdes Guerrero, Roberto García, Carlos Cortés and Angel Hernández to work together in the organization of the conference. They made a good team! Their work made possible to treat participants of the Joint Meeting with a beautiful city, surrounded by the kindness of their people, and the colourful expressions of creativity; with a venue in which one can see and feel the efforts made to build up cultural and intellectual identity. The environment invites to deeply think about mathematical ideas and to imagine strategies for gradually improve the quality of mathematics education for all. Finally they will give us all a wonderful gift a concert in the Cathedral of Morelia, we will hear the sound of the Monumental Organ played by the famous performer Alfonso Vega Nuñez.

Many pages will be needed to acknowledge in an individual way the generosity of all the persons that have given time to support the activities to organize the Joint Meeting for PME32 and PME-NA XXX. I personally have a great debt with all of them, when reading this paragraph they will know that I recognize the support they gave me. Thanks to them. A special mention to Guadalupe Guevara the Conference Secretariat.

It is important to mention that a conference comes to its realization particularly with the work carried out by the members of the PME and the PME-NA communities. Their reward is having an interesting conference and proceedings with scientific merits to read carefully. Several persons were involved in getting ready the four volumes for the publisher, however a special recognition has to be made to David Páez for his commitment, dedication and time used for preparing the proceedings.

Olimpia Figueras

Mexico, July 2008

For the second time in the history of PME and PME-NA, a Joint Meeting of the two organizations takes place in Mexico. Once again, the Mexican mathematics-education community has the opportunity to show the membership of both groups not only its capability to organize such an important conference but also the quality and relevance of the research that it conducts, in a special way. And, yes, it is also an opportunity to show you the beauty of our country and the nobility of our people, about whose mathematics education we care dearly.

As the PME-NA Co-chair of this Joint Meeting, I want to thank the authorities of the Michoacán University of Saint Nicholas of Hidalgo and the Center for Research and Advanced Studies - IPN (Cinvestav) for their invaluable support in organizing this conference. I also want to thank all my colleagues in the Local Organizing Committee for their commitment and hard work. In particular, I want to thank Olimpia Figueras, Armando Sepúlveda, Silvia Alatorre, and Guadalupe Guevara. My appreciation also goes to the membership of the two organizations and to their leadership, the International Committee of PME and the Steering Committee of PME-NA, for trusting us with the responsibility of organizing this conference.

I hope everyone has a wonderful time in Morelia.

José Luis Cortina

SPONSORS

The conference received support from several sources, without it the organizing committee could not have met the expected standards. We are grateful to:

Centro de Investigación y de Estudios Avanzados del IPN
Universidad Michoacana de San Nicolás de Hidalgo
   Biblioteca Pública Universitaria
   Colegio de San Nicolás de Hidalgo
   Centro Cultural Universitario

Universidad Pedagógica Nacional
Texas Instruments Hill
H. Ayuntamiento de Morelia
Diócesis de Morelia
Secretarías de Cultura del Gobierno del Estado de Michoacán
Secretarías de Turismo del Gobierno del Estado de Michoacán

TABLE OF CONTENTS
VOLUME 1

Discussion Groups

Working Session

Seminars

National Presentation

Short Orals Communications

Poster Presentations

VOLUME 2

Research Reports

VOLUME 3

Research Reports

VOLUME 4

Research Reports

INTRODUCTION

THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME)

History and Aims of PME

The International Group for the Psychology of Mathematics Education (PME) is an autonomous body, governed as provided for in the constitution. It is an official subgroup of the International Commission for Mathematical Instruction (ICMI). PME came into existence at the Third International Congress on Mathematics Education (ICME3) held in Karlsruhe, Germany in 1976.

Its former presidents have been:

Efraim Fischbein, Israel
Richard R. Skemp, United Kingdom
Gerard Vergnaud, France
Kevin F. Collis, Australia
Pearla Nesher, Israel
Nicolas Balacheff, France
Kathleen Hart, United Kingdom
Carolyn Kieran, Canada
Stephen Lerman, United Kingdom
Gilah Leder, Australia
Rina Hershkowitz, Israel
Chris Breen, South Africa

The present president is Fou-Lai Lin, Taiwan.

The major goals of the Group are:

- to promote international contact an exchange of scientific information and the field of mathematical education;
- to promote and stimulate interdisciplinary research in the aforesaid area; and
- to further a deeper and more correct understanding of the psychological and other aspects of teaching and learning mathematics and the implications thereof.

PME Membership and Other Information

Membership is open to people involved in active research consistent with the Group's goals, or professionally interested in the results of such research.

Membership is on an annual basis and requires payment of the membership fees (NOK 520) for the year 2008 (January to December). For participants of PME32 and PME-NA XXX Conference the membership fee is included in the Conference Deposit. Other are required to contract their Administrative Manager (see page xl).

Website of PME

For more information concerning about International Group for the Psychology of Mathematics Education (PME) as an association, history, rules and regulations, and futures conferences see its home page at http://www. igpme.org/

PME Administrative Manager

Jarmila Novotná
Postal address
PME Administrative Manager
Charles University in Prague
Faculty of Education
M.D. Rettigove 4
116 39 Prague 1
Czech Republic
Email: admin@igpme.org

Ann-Marie Breen
Postal address
35 Andwind Street
Cape Town, 7945 South Africa
Work phone: 27 21 715 3559
Fax: 27 88 021 715 3559
Email: info@igpme.org

Honorary members of PME

Efraim Fischbein (Deceased)
Hans Freudenthal (Deceased)
Joop Van Dormolen (Retired)

Present Officers of PME

President	Fou-Lai Lin	National Taiwan Normal University, Taiwan
Vice-President	Marcelo C. Borba	Unesp, Brazil
Secretary	Helen Forgasz	Monash University, Australia
Treasurer	Ferdinando Arzarello	Universitá Di Torino, Italy

Other members of the International Committee of PME

Mike Askew	King's College London, United Kingdom
Olimpia Figueras	Centro de Investigación y de Estudios Avanzados, Mexico
Cristina Frade	Universidade Federal de Minas Gerais, Brazil
Zahra Gooya	Shahid Beheshti University, Iran
Aiso Heinze	University of Regensburg, Germany
Bat-Sheva Ilany	Beitberl, Israel
Hee-Chan Lew	Korea National University of Education, Korea
Peter Liljedahl	Simon Fraser University, Canada
Pi-Jen Lin	Hsin-Chu University of Education, Japan
Cynthia Nicol	University of British Columbia, Canada
Yoshinori Shimizu	University of Tsukuba, Japan
Pessia Tsamir	Tel Aviv University, Israel
Behiye Ubuz	Middle East Technical University, Turkey

THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION - NORTH AMERICAN CHAPTER (PME-NA)

History and Aims of PME-NA

PME-NA is the North American Chapter of the International Group and caters for researchers of Canada, United States of America and Mexico. It came into existence in 1978.

The governing body of PME-NA is the Steering Committee, which shall be composed of ten members, including at least one member from each of Canada, Mexico and the United States of America, who shall serve staggered three-year terms of office. The Steering Committee is responsible for managing the organization.

The major goals of the North American Chapter are:

- to promote international contact and exchange of scientific information and in the psychology of mathematical education;
- to promote and stimulate interdisciplinary research in the aforesaid area, with the cooperation of psychologists, mathematicians and mathematics teachers; and
- to further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

PME Membership and Other Information

Membership is open to people involved in active research consistent with PME-NA's aims or to those professionally interested in the results of such research. Membership is open on an annual basis and depends on payment of dues for the current year. For 2008, the memberships fees for PME-NA are included in Conference fee of the Joint Meeting of PME 32 and PME-NA XXX. For more information go http://www.pm ena.org/main/members.htm

Website of PME-NA

For more information about North American Chapter of the International Group (PME-NA) visit the PME-NA Website: http://www.pmena.org/main/constitution.htm

Steering Committee of PME-NA

José Luis Cortina	Universidad Pedagógica Nacional, Mexico	Chair
Lynn C. Hart	Georgia State Universitay, USA	Chair-Elect
Teruni Lamberg	University of Nevada, Reno, USA	Past Chair

INTERNATIONAL COMMITTEE OF THE JOINT MEETING OF PME 32 AND PME-NA XXX

Program Committee

Fou-Lai Lin	National Taiwan Normal University, Taiwan	President of PME
Olimpia Figueras	Cinvestav, Mexico	Chair of PME 32
José Luis Cortina	Universidad Pedagógica Nacional, Mexico	Chair of PME-NA XXX
Silvia Alatorre	Universidad Pedagógica Nacional, Mexico	
Marcelo C. Borba	Unesp, Brazil	
Jo Clay Olson	Washington State University, USA	
Teresa Rojano	Cinvestav, Mexico	
Marianna Tzekaki	Aristotle University of Thessaloniki, Greece	

Executive Local Committee

Olimpia Figueras, Cinvestav
Armando Sepúlveda, UMSNH
José Luis Cortina, UPN
Carlos Cortés, UMSNH
Roberto García, UMSNH
Lourdes Guerrero, UMSNH
Ángel Hernández, UMSNH
Eugenio Filloy, Cinvestav
Teresa Rojano, Cinvestav
Tenoch Cedillo, UPN
Silvia Alatorre, UPN

Conference Secretariat

Guadalupe Guevara, Cinvestav

Web and database: Oscar Jurado and Manuel López
Overall printing layout support: David Alfonso Páez, Cinvestav
Administrative support: Tania Guadalupe González, Martha Sánchez, Juan Carlos Ponce, and Consuelo Campos, Cinvestav
Technological support: Angel Vega and David Cruz, Cinvestav
Student's support: Carolina Rubí Real, Carolina Guerrero, Patricia Flores, Luis Alexander Conde, and Sandra Evelyn Parada, Cinvestav

PROCEEDINGS OF PREVIOUS PME AND PME-NA CONFERENCES

The tables include the ERIC numbers and/or the e-address of the websites of the past conference.

PME International

No.	Year	Place	ERIC number and/or URL
1	1977	Utrecht, The Netherlands	Not available in ERIC
2	1978	Osnabrück, Germany	ED226945
3	1979	Warwick, United Kingdom	ED226956
4	1980	Berkeley, USA	ED250186
5	1981	Grenoble, France	ED225809
6	1982	Antwerp, Belgium	ED226943
7	1983	Shoresh, Israel	ED241295
8	1984	Sydney, Australia	ED306127
9	1985	Noordwijkerhout, Netherlands	ED411130 (vol.l), ED411131 (vol.2)
10	1986	London, United Kingdom	ED287715
11	1987	Montréal, Canada	ED383532
12	1988	Veszprém, Hungary	ED411128 (vol.l), ED411129 (vol.2)
13	1989	Paris, France	ED411140 (vol.1), ED411141(vol.2), ED411142 (vol.3)
14	1990	Oaxtepex, Mexico	ED411137 (vol.1), ED411138 (vol.2), ED411139 (vol.3)
15	1991	Assisi, Italy	ED413162 (vol.1), ED413l63 (vol.2), ED413164 (vol.3)
16	1992	Durham, USA	ED383538
17	1993	Tsukuba, Japan	ED383536
18	1994	Lisbon, Portugal	ED383537
19	1995	Recife, Brazil	ED411134 (vo1.l), ED411135 (vol.2), ED411136 (vo1.3)
20	1996	Valencia, Spain	ED453070 (vol.1), ED453071 (vol.2), ED453072 (vol.3), ED453073 (vol.4), ED453074 (addendum)
21	1997	Lahti, Finland	ED416082 (vol.1), ED416083 (vol.2), ED4l6084 (vol.3), ED416085 (vol.4)
22	1998	Stellenbosch, South Africa	ED427969 (vol.1), ED427970 (vol.2), ED427971 (vol.3), ED427972 (vol.4)
23	1999	Haifa, Israel	ED436403
24	2000	Hiroshima, Japan	ED452301 (vol.1), ED452302 (vol.2), ED452303 (vol.3), ED452304 (vol.4)
25	2001	Utrecht, The Netherlands	ED466950
26	2002	Norwich, United Kingdom	ED476065
27	2003	Hawai'i, USA	http://onlinedb.terc.edu
28	2004	Bergen, Norway	http://emis.de/proceedings/PME28/
29	2005	Melbourne, Australia	http://staff.edfac.unimelb.edu.au/~chick/PME29/
30	2006	Prague - Czech Republic	http://class.pedf.cuni.cz/pme30
31	2007	Seoul - Korea	

Copies of some previous PME Conference Proceedings are still available for sale. See the PME web site at http://igpme.org/publications/procee.html or contact the Proceedings manager Dr. Peter Gates, PME Proceedings, University of Nottingham, School of Education, Jubilee Campus, Wollaton Road, Nottingham NG8 1 BB, UNITED KINGDOM, Telephone work: +44-115-951-4432; fax: +44-115-846-6600; e-mail: peter.gates@nottingham.ac.uk

PME-NA

No.	Year	Place	ERIC number and/or URL
1	1979	Evanston, Illinois	
2	1980	Berkeley, California (with PME2)	ED250186
3	1981	Minnesota	ED223449
4	1982	Georgia	ED226957
5	1983	Montréal, Canada	ED289688
6	1984	Wisconsin	ED253432
7	1985	Ohio	ED411127
8	1986	Michigan	ED301443
9	1987	Montréal, Canada (with PME11)	ED383532
10	1988	Illinois	ED411126
11	1989	New Jersey	ED411132 (vol.1), ED411133 (vol.2)
12	1990	Oaxtepec, Morelos, México (with PME14)	ED411137 (vol.1), ED411138 (vol.2), ED411139 (vol.3)
13	1991	Virginia	ED352274
14	1992	Durham, New Hampshire (with PME16)	ED383538
15	1993	California	ED372917
16	1994	Louisiana	ED383533 (vol.1), ED383534 (vol.2)
17	1995	Ohio	ED389534
18	1996	Panama City, Florida	ED400178
19	1997	Illinois	ED420494 (vol.1), ED420495 (vol.2)
20	1998	Raleigh, North Carolina	ED430775 (vol.1), ED430776 (vol.2)
21	1999	Cuernavaca, Morelos, México	ED433998
22	2000	Tucson, Arizona	ED446945
23	2001	Snowbird, Utah	SE065231 (vol.1), SE065232 (vol.2)
24	2002	Athens, Georgia	SE066887 (vol.1), SE066888 (vol.2), SE066889 (vol.3), SE066880 (vol.4)
25	2003	Hawai'i (together with PME27)	ED500857 (vol.1), ED500859 (vol.2), ED500858 (vol.3), ED500860 (vol.4)
26	2004	Toronto, Notario	http://www.pmena.org/2004/
27	2005	Roanoke, Virginia	http://www.pmena.org/2005/
28	2006	Mérida, Yucatán, Mexico	http://www.pmena.org/2006/
29	2007	Lake Tahoe, Nevada	http://www.pmena.org/2007/

Abstracts from some articles can be inspected on the ERIC web site (http://www .eric.ed.gov/) and on the web site of ZDM/MATHDI (http://www.emis.de/ MATH/ DI.html). Many proceedings are included in ERIC: type the ERIC number in the search field without spaces or enter other information (author, title, keyword). Some of the contents of the proceedings can be downloaded from this site. MATHDI is the web version of the Zentralblatt fur Didaktik der Mathematik (ZDM, English subtitle: International Reviews on Mathematical Education). For more information on ZDM/MATHDI and its prices or assistance regarding consortia contact Gerhard König, managing editor, fax: (+49) 7247 808 461, e-mail: Gerhard.Koenig@fiz-karlsruhe.de

THE REVIEW PROCESS OF THE JOINT MEETING OF PME 32 AND PME-NA XXX

Research Forum. The Programme Committee (PC) and the International Committee (IC) of PME accepted the 3 themes proposed for the Research Fora of the Joint Meeting of PME 32 and PME-NA XXX on the basis of the proposals sent by the co-ordinators. For each one, the proposed structure, the contents, the contributors, and their role were reviewed and agreed by the members of the IC. As can be seen in the papers included in this volume, pages 89 to 188, the contributions reflect great interest of the PME and PME-NA communities on teachers and the mathematical activities and student engagement s/he is able to promote in the classroom. The members of the PC thank the co-ordinators and contributors for their efforts in preparing such scenery to favour profound discussion of the topics.

For this conference, the International Committee of PME decided to try two different modes of individual presentations: Seminars and National Presentation. Both will have two 90-minute slots that will run parallel to the Research Forum presentations.

Seminars. This mode of presentation is set up for a small number of participants. The PC invited a researcher to design a short intensive course of study of a topic chosen by the International Committee of PME. It was also agreed that the invited researcher could ask other colleagues to collaborate in the preparation of the course and to participate in the discussion during the conference. The members of the IC thank Anne Teppo and Norma Presmeg for accepting the challenge to build up a tradition for this type of activity. Anne invited Marja van den Heuvel-Panhuizen to design the seminar Qualitative research methods: Mathe-didactical analysis of task design (see pages 205-208 in this volume) and Norma invited Ken Clements and Nerida Ellerton to design the seminar Quality reviewing of scholarly papers (see pages 209-216 in this volume). The members of the PC reviewed their proposals knowing in advanced the quality of their academic work.

National Presentation. It was also agreed by the International Committee of PME to offer the opportunity to a group of researchers from the country that hosts the conference to give participants an overall of the research activities done by the Mathematics Education Local Community, their most important results, and future trends. Ana Isabel Sacristán, María Trigueros and Lourdes Guerrero accepted the invitation of the Programme Committee. The paper Research in Mathematics Education in Mexico: Achievements and Challenges (pages 219 – 231 in this volume) written by the Mexican colleagues was reviewed by the Programme Committee most of all to comment or to make suggestions. The International Committee of PME thank Ana, Lourdes and María, for their willingness to contribute to the Scientific Programme of the conference.

Working Sessions and Discussion Groups. The aim of group activities is to achieve greater exchange of information and ideas related to the Psychology of Mathematics

Education. There are two types of activities: Working Sessions (WS) and Discussion Groups (DG). Six proposals for WS and four for DG were received this year. The PC reviewed and commented the abstracts structured by the co-ordinators. All except one proposal for Discussion Group were accepted (see pages 191-193 in this volume). The nine themes of the group activities planned for the conference covers a wide range of research areas that are relevant for mathematics education. The PC expresses recognition to the contributors of the group activities planned; it will be difficult to choose only one of them for participating in the debates that those surely will provoke.

Research Reports. The PC received 283 proposals for Research Report presentation. Each full paper was blind-reviewed by three peer reviewers, so 849 reviews were needed. The Administrative Manager of PME, Anne-Marie Breen, controlled the global process of submission of proposals, distribution of blind papers to reviewers, reception of reviews, and organization of the information for the two meetings of the Programme Committee. The assignation of reviewers to each proposal were reviewed by the Programme Committee and accepted or when necessary made a different choice. A great effort was done by the 210 reviewers to fulfil the task in the period of time allocated for the reviewing process (see page l - lii in this volume). There are not enough words to acknowledge the important contribution made by these members of PME and PME-NA communities for the Scientific Programme of the conference, in particular those members that belong to the IC of PME who had to review 8 proposals. Thanks to All.

Framed by the policy of the IGPME and sustained on the work done by the reviewers, the members of the PC accepted 174 (≅ 63%) proposals as Research Reports, recommended 73 (≅ 26) to be presented as Short Oral Communications in the conference and 30 (≅ 11%) as Poster Presentations. When needed the members of the PC reviewed the proposals. A double-crossing process was set up. In case the two colleagues could not arrive to an agreement, as a whole the PC carried out a careful examination of the information collected for the proposal. It is important to mention that Short Oral Communications and Poster Presentations were not seen as second or third class reports, they were considered valuable modes of presentation with peculiar characteristics. The recommendation to submit the proposals as either of these individual presentations was sustained in criteria as "the 8-page paper was not organized to describe the important aspects of the research willing to be informed, however it had relevant contributions for members of PME and PME-NA communities", or "the characteristics of the work done requires a visual or graphic presentation or needs the support of demonstration that is adequate for exposing it in a Poster Presentation". At the moment of writing this report, 8 Research Reports were withdrawn. Volumes 2, 3 and 4 of the proceedings contain these contributions that represent one of the main components of the conference.

Short Oral Communications and Poster Presentations. This year the PC received 83 proposals for Short Oral Communications and 44 for Poster Presentations. The PC reviewed each one-page proposal using a double-crossing process as the one aforementioned. As result of this process 58 proposals were accepted as Short Oral Communications (≅ 70%) and 29 as Poster Presentations (≅ 66%). In addition 29 researchers that had submitted a Research Report proposal agreed to include their proposal as a Short Oral Communication and 32 as Poster Presentation.

LIST OF REVIEWERS PME 32 AND PME-NA XXX

The PME 32 and PME-NA XXX Program Committee thank the following people for their help in the review process:

Dor Abrahamson, USA
Claudia Acuna, Mexico
Jill Adler, South Africa
Keith Adolphson, USA
Janet Ainley, United Kingdom
Hatice Akkoç, Turkey
Silvia Alatorre, Mexico
Lara Alcock, United Kingdom
Alice Alston, USA
Solange Amato, Brazil
Samuele Antonini, Italy
Ferdinando Arzarello, Italy
Amir Hossein Asghari, United Kingdom
Mike Askew, United Kingdom
Lynda Ball, Australia
Caroline Bardini, France
Annette Baturo, Australia
Luciana Bazzini, Italy
Joanna Rossi Becker, USA
Alan Bell, United Kingdom
David Ben-Chaim, Israel
Kim Beswick, Australia
Sarah Berenson, USA
Elizabeth Bills, United Kingdom
Erhan Bingolbali, Turkey
Janette Bobis, Australia
Marcelo C. Borba, Brazil
Jill Brown, Australia
Tânia Cabral, Brazil
Michelle Chamberlin, USA
Charalambos Charalambous, Greece
Egan Chernoff, Canada
Helen Chick, Australia
Erh-Tsung Chin, Taiwan ROC
Philip Clarkson, Australia
Jose Cortina, Mexico
Jorge T. Da Rocha Falcão, Brazil
A. J. (Sandy) Dawson, USA
Dirk De Bock, Belgium
Pietro Di Martino, Italy
Brian Doig, Australia
Willi Dörfler, Austria
Nadia Douek, France
Barbara Dougherty, USA
Jean-Philippe Drouhard, France
Tommy Dreyfus, Israel
Laurie Edwards, USA
Andreas Eichler, Germany
Theodore Eisenberg, Israel
Lyn English, Australia
Ruhama Even, Israel
Pier Luigi Ferrari, Italy
Olimpia Figueras, Mexico
Helen Forgasz, Australia
Cristina Frade, Brazil
John Francisco, USA
Michael N. Fried, Israel
Anne Berit Fuglestad, Norway
Fulvia Furinghetti, Italy
Athanasios Gagatsis, Cyprus
Hagar Gal, Israel
Aurora Gallardo, Mexico
Peter Gates, United Kingdom
Merrilyn Goos, Australia
Zahra Gooya, Iran
Angel Gutiérrez, Spain
José Guzman, Mexico
Markus Hähkiöniemi, Finland
Anjum Halai, Pakistan
Jean Hallagan, USA
Stefan Halverscheid, Germany
Markku Hannula, Finland
Örjan Hansson, Sweden
Lynn Hart, USA
Hana Haydar, USA
Aiso Heinze, Germany

Ann Heirdsfield, Australia
Rina Hershkowitz, Israel
Dave Hewitt, United Kingdom
Lynn Hodge, USA
Marj Horne, Australia
Veronica Hoyos, Mexico
Rongjin Huang, USA
Danielle Huillet, Mozambique
Paola Iannone, United Kingdom
Bat-Sheva Ilany, Israel
Matthew Inglis, United Kingdom
Kay Irwin, New Zealand
Keith Jones, United Kingdom
Maria Kaldrimidou, Greece
Carolyn Kieran, Canada
Boris Koichu, Israel
Hari Koirala, USA
Konrad Krainer, Austria
Sebastian Kuntze, Germany
Teruni Lamberg, USA
Ilana Lavy, Israel
Gilah Leder, Australia
Kyunghwa Lee, Korea
Roza Leikin, Israel
Stephen Lerman, United Kingdom
Yuh-Chyn Leu, Taiwan ROC
Allen Leung, China
Hee-Chan Lew, Korea
Peter Liljedahl, Canada
Kien Lim, USA
Fou-Lai Lin, Taiwan ROC
Pi-Jen Lin, Taiwan ROC
Yung-Chi Lin, Taiwan ROC
Graham Littler, United Kingdom
Jane-Jane Lo, USA
Zlatan Magajna, Slovenia
Ami Mamolo, USA
Mirko Maracci, Italy
Christos Markopoulos, Greece
João Filipe Matos, Portugal
Andrea Mcdonough, Australia
Kaarina Merenluoto, Finland
Vilma Mesa, USA
Christina Misailidou, United Kingdom
Takeshi Miyakawa, Japan
Modestina Modestou, Cyprus
John Monaghan, United Kingdom
Francesca Morselli, Italy
Judit Moschkovich, USA
Judith Mousley, Australia
Nicholas Mousoulides, Cyprus
Joanne Mulligan, Australia
Hanlie Murray, South Africa
Elena Nardi, United Kingdom
Dorit Neria, Israel
Cynthia Nicol, Canada
Jarmila Novotná, Czech Republic
Masakazu Okazaki, Japan
Federica Olivero, United Kingdom
Jo Olson, USA
Cécile Ouvrier-Buffet, France
Kay Owens, Australia
Mehmet Fatih Ozmantar, Turkey
Jeongsuk Pang, Korea
Marilena Pantziara, Cyprus
Erkki Pehkonen, Finland
Leila Pehkonen, Finland
George Philippou, Cyprus
Robyn Pierce, Australia
Marcia Pinto, Brazil
Demetra Pitta-Pantazi, Cyprus
Despina Potari, Greece
Giorgos Psycharis, Greecer
Luis Radford, Canada
Maria Reggiani, Italy
David Reid, Canada
Ferdinand Rivera, USA
Ornella Robutti, Italy
Ma. Teresa Rojano, Mexico
Katrin Rolka, Germany
Bettina Rösken, Germany
Tim Rowland, United Kingdom
Cristina Sabena, Italy
Ana Isabel Sacristan, Mexico
Ildar Safuanov, Russian Federation
Haralambos Sakonidis, Greece

Manuel Santos-Trigo, Mexico
Wolfgang Schlöglmann, Austria
Roberta Schorr, USA
Yasuhiro Sekiguchi, Japan
Anna Sfard, Israel
Keiichi Shigematsu, Japan
Yoshinori Shimizu, Japan
Daniel Siebert, USA
Dianne Siemon, Australia
Elaine Simmt, Canada
Martin Simon, USA
Ji-Won Son, USA
Kaye Stacey, Australia
Nada Stehliková, Czech Republic
Olof Bjorg Steinthorsdottir, USA
Sepideh Stewart, New Zealand
Rudolf Straesser, Germany
Andreas Stylianides, United Kingdom
Gabriel Stylianides, USA
Peter Sullivan, Australia
Michal Tabach, Israel
Howard Tanner, United Kingdom
Konstantinos Tatsis, Greece
Anne Teppo, USA
Michael O. J. Thomas, New Zealand
Steve Thornton, Australia
Dina Tirosh, Israel
María Trigueros, Mexico
Wen-Huan Tsai, Taiwan ROC
Pessia Tsamir, Israel
Marianna Tzekaki, Greece
Ron Tzur, USA
Behiye Ubuz, Turkey
Elizabeth Uptegrove, USA
Veronica Vargas, Mexico
Joëlle Vlassis, Belgium
Pauline Vos, The Netherlands
David Wagner, Canada
Margaret Walshaw, New Zealand
Elizabeth Warren, Australia
Gaye Williams, Australia
Kirsty Wilson, United Kingdom
Terry Wood, USA
Kai-Lin Yang, Taiwan ROC
Nikoleta Yiannoutsou, Greece
Kaori Yoshida-Miyauchi, Japan
Orit Zaslavsky, Israel
Rina Zazkis, Canada

LIST OF AUTHORS

LIST OF PLENARY SPEAKERS AND PANELISTS OF PME 32 AND PME-NA XXX

Ruhama Even
Weizmann Institute of Sceince Rehovot
ISRAEL
ruhama.even@weizmann.ac.il

Kath Hart
University of Nottingham
15 Chester Close, Kings Rd, Richmond
UNITED KINGDOM
kath.hart@nottingham.ac.uk

Aurora Gallardo
Centro de Investigación de Estudios
Avanzados del IPN
Av. Instituto Politécnico Nacional, No. 2508
Col. San Pedro Zacatenco, México D.F.
MÉXICO
agallardo@cinvestav.mx

Michèle Artigue
Université de Paris 7
UFR de Maths et IREM
15 Rue Mademoiselle, Villebon sur Yvette
FRANCE
artigue@math.jussieu.fr

Jens Høyrup
Roskilde University

Marj Horne
Australia Catholic University
Locked Bag 4115, Fitzroy, Victoria
AUSTRALIA
m.horne@patrick.acu.edu.au

Patrick W. Thompson
Arizona State University

LIST OF AUTHORS OF PME 32 AND PME-NA XXX

Wendy Rose Aaron
University of Michigan
603 Turner Park Court,
Ann Arbor, USA
wendyaar@umich.edu

Muhammad Abu-Naja
Ben Gurion University
ISRAEL

James Aczel
The Open University

Yaser Abo-Ras
Ben Gurion University
ISRAEL

Claudia Acuña
Cinvestav-IPN
Av. IPN #2508, Col.
Zacatenco, Mexico City,
Distrito Federal
MEXICO
claudiamargarita_as@hotmai
l.com

Keith Adolphson
Eastern Washington
University
USA

Einav Aizikovitsh
Bgu
Asher 28, Gedera, ISRAEL
einavai@bgu.ac.il

Hatice Akkoç
Marmara Universitesi
Merdivenkoy Yolu Sok. No. 26 Huzur Apt. Daire:3
Goztepe Kadikoy 34732, Istanbul, TURKEY
haticeakkoc@yahoo.com

Silvia Alatorre
Univ. Pedagógica Nacional
Hidalgo 111-8, Col. Tlalpan, México D. F., MEXICO
alatorre@solar.sar.net

Lara Alcock
Loughborough University
Schofield Building
Loughborough University, Loughborough, UNITED KINGDOM
l.j.alcock@lboro.ac.uk

Yu-Wen Allison Lu
University of Cambridge
112 Nuan-Jhong Road, Cambridge, UNITED KINGDOM
ywal2@cam.ac.uk

Alice Alston
Rutgers University
303 George Street, Suite 610, New Brunswick, USA
alston@rci.rutgers.edu

Solange Amorim Amato
Universidade de Brasília
Sqn 215 Bloco E, Apartamento 201. Brasilia, BRAZIL
solaaamato@hotmail.com

Miriam Amit
Ben Gurion University
38 Erez Street, Omer, ISRAEL
amit@bgu.ac.il

Shuhua An
California State University
USA

Judy Anderson
The University of Sydney
AUSTRALIA

Silvanio de Andrade
Univ. Estadual da Paraíba
Rua Des. Trindade 332, AP 603. Centro, Campina Grande, BRAZIL
silvanioandrade@ig.com.br

Glenda Anthony
Massey University
PB 11222, Palmerston North, NEW ZEALAND
g.j.anthony@massey.ac.nz

Samuele Antonini
Universita' Di Pavia
Via Ferrata, 1, Pavia, ITALY
samuele.antonini@unipv.it

Sergio Arenas Moreno
Universidad Autonoma de Zacatecas
MEXICO

Cecilia C. Arias
Rutgers University

David Arnau
Universitat de Valencia
Apt. 22045, Valencia, SPAIN
david.arnau@uv.es

Ferdinando Arzarello
Università Di Torino
Via Carlo Alberto 10, Torino, ITALY
ferdinando.arzarello@unito.it

Leslie Aspinwall
Florida State University
209 Carothers Hall. Middle and Secondary Education, Tallahassee, USA
aspinwal@coe.fsu.edu

Chryso Athanasiou
University of Cyprus
Aspasias 17, Aradippou, Larnaca, CYPRUS
chrathan@cytanet.com.cy

Patrick Barmby
Durham University

Roberto Behar Gutiérrez
Universidad del Valle

Michal Ayalon
Weizmann Institute of Science
Rehovot, ISRAEL
michal.ayalon@weizmann.ac.il

Richard Barwell
University of Ottowa
145 Jean-Jacques Lussier, Ottawa, CANADA
richard.barwell@uottawa.ca

Ellina Beliaeva Longuinenco
Tecnológico de Monterrey
MEXICO

Tamara Ball
University of California Santa Cruz

Eduardo Basurto
Cinvestav
Av. Inst. Pol. Nacional, No. 2508, C.P. 07360, México, D. F., MEXICO

Bastian Benz
Technische Universität Darmstadt

Rakhi Banerjee
Tata Institute of Social Sciences
V. N. Purav Marg, Mumbai, INDIA
rakhi.banerjee@gmail.com

Annette R. Baturo
Queensland University of Technology/Centre for Learning Innovation
Victoria Park Road, Kelvin Grove, Q4059, Brisbane, AUSTRALIA
a.baturo@qut.edu.au

Ole Kristian Bergem
University of Oslo
Skogfaret 29, Oslo, NORWAY
o.k.bergem@ils.uio.no

Jonei Cerqueira Barbosa
State University of Feira de Santana
Rua João Mendes Da Costa Filho, 299, Ap. 202-C – Armação, Salvador, BRAZIL
joneicb@uol.com.br

Ibrahim Bayazit
Erciyes University
Adnan Menderes Universitesi, Egitim Fakultesi, Kayserý, TURKEY
bayindiroglu@yahoo.com

Margot Berger
University of Witwatersrand
School of Mathematics
University of Witwatersrand
Private Bag 3, Wits, SOUTH AFRICA
Margot.Berger@wits.ac.za

Ruthi Barkai
Tel-Aviv University, Kibbutzim Teacher College
Tel-Aviv, Remez 36, Tel-Aviv, ISRAEL
ruthi11@netvision.net.il

Nermin Bayazit
Florida State University

Kim Beswick
University of Tasmania
Faculty of Education, Locked Bag 1307, Launceston, AUSTRALIA
kim.beswick@utas.edu.au

Liz Bills
University of Oxford

Laura Bofferding
Stanford University

Leicha A. Bragg
Deakin University
106-155 East 5th
StreetNorth Vancouver,
North Vancouver, CANADA
leicha.bragg@deakin.edu.au

Erhan Bingolbali
Gaziantep University
Gaziantep Universitesi
Egitim Fakultesi Ilkogretim
Bolumu, Gaziantep,
TURKEY
erhanbingolbali@yahoo.co.uk

Francisco José Boigues
Univ. Politécnica de Valencia
Carretera Nazaret Oliva s/n,
Gandia, SPAIN
fraboipl@mat.upv.es

Andrew Brantlinger
Brooklyn College
CUNY Graduate Center

Irene Biza
University of Athens
Dimocratias, 52, Athens,
GREECE
empiza@math.uoa.gr

A. Boileau
UQAM

Barbara M. Brizuela
Tufts University
Department of Education12
Upper Campus Rd. Paige
Hall 206, Medford, USA
barbara.brizuela@tufts.edu

Raymond Bjuland
University of Agder

Megan Bomer
Illinois Central College

Karin Brodie
University of the
Witwatersrand
PO Wits 2050, Johannesburg
SOUTH AFRICA
Karin.Brodie@wits.ac.za

Sigrid Blömeke
Humboldt University of
Berlin

Marcelo C. Borba
Unesp
Av 24a, 1515, Bela Vista,
Rio Claro, Sp., BRAZIL
mborba@rc.unesp.br

Laurinda Brown
University of Bristol
35 Berkeley Square, Bristol,
UNITED KINGDOM
laurinda.brown@bris.ac.uk

Janette Bobis
University of Sydney
Faculty of Education
A35, Sydney,
AUTRALIA
j.bobis@edfac.usyd.edu.au

Hans Erik Borgersen
University of Agder

Stacy Brown
Pitzer College
3326 Duke, Avenue
Claremont, CA 91711,
Claremont, USA
Stacy_Brown@pitzer.edu

Priscilla Brown-Lopez
University of Belize/
Durham University

María Luz Callejo
Universidad de Alicante
Campus San Vicente del
Raspeig, Ap. 99, E-
03080, Alicante, SPAIN
luz.callejo@ua.es

Alonso del Castillo
Universidad Nacional
Autónoma de México
MEXICO

Regina Bruder
Technical University of
Darmstadt
Darmstadt, GERMANY
bruder@mathematik.tu-
darm stadt.de

Matias Camacho M.
Universidad da la Laguna
5 Planta 107, Av.
Astrofisico Fco. Sanchez
s/m, La Laguna –
Tenerife, SPAIN
mcamacho@ull.es

Encarnación Castro
University of Granada
SPAIN

Lingguo Bu
Florida State University
USA

Consuelo Campos
Cinvestav
Av. IPN 2508, México
City, MEXICO
mcampos@cinvestav.mx

Enrique Castro
University of Granada
SPAIN

Ma. Guadalupe Cabañas
Cinvestav
Av. Inst. Pol. Nacional,
No. 2508, México, D. F.,
MEXICO
gcabanas52@hotmail.com

Tania M. M. Campos
UNIBAN-SP

Michael Cavanagh
Macquarie University

Marcelo L. Caffe de O.
State University of Feira
de Santana
BRAZIL

Ricardo Cantoral Uriza
Cinvestav
Av. IPN 2508, México
City, D. F.
MEXICO

Gabrielle Cayton
Tufts University
Dept of Education, Paige
Hall, Medford, USA
gabrielle.cayton@tufts.edu

Gunhan Caglayan
University of Georgia
Department of
Mathematics, Athens,
USA
sezen@uga.edu

María C. Cañadas S.
Zaragoza University
Camino de Ronda, 202,
4ºC, Granada, SPAIN.
mconsu@unizar.es

Michele Cerulli
I.T.D., C.N.R. of Genova

Maria Luiza Cestari
Agder University College
Gimlemoen, Service Box
422, Kristiansand,
NORWAY
maria.l.cestari@hia.no

Chia-Ling Chen
University of Michigan
1647 Beal Ave. #6, Ann
Arbor, USA
chialc@umich.edu

Kuan-Jou Chen
National Chunghua
University of Education

Ching-Kuch Chang
National Chunghua
University of Education
TAIWAN

Ching-Shu Chen
Taiwan Tainan University
of Technology
F13-1, No170, Ho-Der
R., Ku-San District,
Koahsuing, TAIWAN
tg0002@mail.tut.edu.tw

W. J. Chen
University Ming Dao
University

Peichin Chang
University of Michigan
USA

Chun-Yu Chen
Mei-Ho Institute of
Technology

Yen-Ting Chen
Chung Hwa College of
Medical Technology
Tainan Hsien,
TAIWAN ROC
clief000@ms34.hinet.net

Yu Liang Chang
Ming Dao University
5f-1, No. 6, Lane 5, Da
Tung Road, Wufeng,
Taichung County,
TAIWAN ROC
aldy.chang@msa.hinet.net

Ing-Er Chen
Fooyin University
3f, 10 Szwei 2nd Road,
Kaohsiung, TAIWAN
ROC
ivoryer@mars.seed.net.tw

Yuan Chen
National Kaohsiung
Normal University
TAIWAN
g27000@yahoo.com.tw

**Charalambos Y.
Charalambous**
University of Michigan
610 East University
Avenue, Room 2610, Ann
Arbor, USA
chcharal@umich.edu

Jia-Huang Chen
Kun Shan University
Pingtung City Fon Nain
St, Lane 173, #17,
Pingtung, TAIWAN ROC
c0924@mail.ksu.edu.tw

Diana Cheng
Boston University
335 Pearl St Boston,
Cambridge, USA
dianasc@alum.mit.edu

Jill Cheeseman
Monash University
401 St Kilda Street,
Brighton,
AUSTRALIA
jillcheeseman2@aol.com

Ju-Chen Chen
National Taiwan Normal
University

Ying-Hao Cheng
China Univ. of Technology
9f., No.97, Cheng-Tou St,
Taipei, TAIWAN ROC
yhjeng@cm1.hinet.net

Egan J. Chernoff
Simon Fraser University
8888 University Drive,
Burnaby, Bc, CANADA
egan_chernoff@sfu.ca

Ji Young Choi
Korea National
University of Education
Cheongwongun,
Chungbuk 363-791,
Cheongwongun, KOREA
ji2006@empal.com

Doug Clow
The Open University

Helen Chick
University of Melbourne
Level 7 Doug Mcdonell
Building, University of
Melbourne, AUSTRALIA
h.chick@unimelb.edu.au

C. Christou
University of Cyprus
CYPRUS

Paul Cobb
Vanderbilt University

Vu-Minh Chieu
University of Michigan
USA

Marta Civil
The Univ. of Arizona
617 N. Santa Rita Road,
Tucson, USA
civil@math.arizona.edu

Nitsa Cohen
David Yelin College of
Education
16 Hafkir Street,
Jerusalem, ISRAEL
nitza_nm@netvision.net.il

Chien Chin
National Taiwan Normal
University
TAIWAN

David Clarke
University of Melbourne
International Centre For
Classroom Research, 109
al Barry Street, Carlton,
AUSTRALIA
d.clarke@unimelb.edu.au

Christina Collet
Technical University of
Darmstadt
64289, Darmstadt,
GERMANY
collet@mathematik.tu-
darms tadt.de

Erh-Tsung Chin
National Changhua
University of Education
TAIWAN ROC

Ellen Clay
Drexel University

Anna Marie Conner
University of Georgia
105 Aderhold Hall, Athens,
USA
aconner@uga.edu

Kimiho Chino
Kokushikan University
1-1-1, Hirohakama,
Machida-City, Tokyo,
JAPAN
chinok@kokushikan.tsuk
uba.ac.jp

McKenzie Clements
Illinois State University
Department of
Mathematics Campus
Box 4520Normal, Illinois
61790-4520, Normal, USA
clements@ilstu.edu

Tom J. Cooper
QUT
Kelvin Grove Campus -
Victoria Park Road,
Kelvin Grove, Brisbane,
AUSTALIA
tj.cooper@qut.edu.au

Dolores Corcoran
St Patrick's College,
Dublin City University
Dublin, IRELAND
dolores.corcoran@spd.dc
u.ie

Francisco Cordero O.
Cinvestav
Av. IPN No 2508,
México, D. F., MEXICO
fcordero@cinvestav.mx

Jose Carlos Cortés
Universidad Michoacana
Virrey de Mendoza 1153,
Col. Ventura Puente,
Morelia, MEXICO
jcortes@umich.mx

Viviana Cortes
Iowa State University

Jose Luis Cortina
Univ. Pedagógica Nacional
Carreteraco 41-7 Parque
San Andres, Coyoacan,
D. F., MEXICO
jose.luis.cortina@mac.com

Matthew Crosby
Acadia University
37 Bigelow Street,
Wolfville, CANADA
matthew.crosby@acadiau.
ca

Carlos Armando Cuevas
Cinvestav-IPN
Ave. IPN, 2508, México,
D. F., MEXICO
ccuevas@cinvestav.mx

Annalisa Cusi
Università di Modena e
Reggio Emilia
Reggio Emilia, ITALY
annalo@tin.it

Dirk De Bock
University of Leuven
Vesaliusstraat 2, Leuven,
BELGIUM
dirk.debock@avl.kuleuve
n.be

Vera H. G. De Souza
UNIBAN-SP

Rafael Del Valle
Universidad Nacional
Autónoma de México
MEXICO

Eleni Deliyianni
University of Cyprus
23 Pireos Street
Strovolos, Nicosia,
CYPRUS
sepged1@ucy.ac.cy

Eva DeVries
Independent Schools
Queensland

Juan José Diaz
Universidad Aunoma de
Zacatecas
Zacatecas, MEXICO

Jennifer DiBrienza
Stanford University

David S. Dickerson
State Universtiy of New
York at Cortland
2305 East Lake Road,
Skaneateles, NY, USA
dickersond@cortland.edu

Lydia Dickey
Florida State University

Carmel M. Diezmann
Queensland University of
Technology
Victoria Park Road,
AUSTRALIA
c.diezmann@qut.edu.au

Brian Doig
Deakin University
AUTRALIA

Antony Edwards
Loughborough University
Room A0.45Schofield
Building Loughborough
University, Loughborough,
UNITED KINGDOM
a.w.edwards@lboro.ac.uk

Nerida F. Ellerton
Illinois State University
Box 4520Normal, Illinois
61790-4520, Normal
USA
ellerton@ilstu.edu

Thérèse Dooley
St. Patrick's College,
Dublin and University of
Cambridge

Laurie D. Edwards
St Mary's College
POB 4350, Moraga, USA
ledwards@stmarys-
ca.edu

Lyn English
Queensland University of
Technology
Victoria Park Road,
Kelvin Grove, Brisbane,
AUSTRALIA
l.english@qut.edu.au

Maria Doritou
University of Nicosia
School of the Deaf,
CYPRUS
doritou.m@unic.ac.cy

Tammy Eisenmann
The Hebrew University of
Jerusalem Ruhama Even
Weizmann Institute of
Sceince
Rehovot, ISRAEL
ruhama.even@weizmann.
ac.il

Vicente Estruch
Univ. Politécnica de
Valencia
Department of Applied
Mathematics
Valencia, SPAIN

Tommy Dreyfus
Tel Aviv University
P. O. Box 39040,
ISRAEL
tommyd@post.tau.ac.il

Iliada Elia
University of Cyprus
CYPRUS

Ronda Faragher
ACU National

P. Drijvers
Utrecht Univ
Utrecht

Esteban Levi Elipane
Saitama University
105 Hakkokuso, 2-30-21
Sakawa, Sakura-ku,
Saitama City, JAPAN
levielipane@yahoo.com

Solange Hassan A. A. Fernandes
Colégio Nossa Senhora
do Rosário
Rua Agudos, 03 ap 72,
São Paulo, BRAZIL
solangehf@gmail.com

Elizabeth Duus
Queensland University of
Technology
Victoria Park Road,
AUSTRALIA

David Ellemor-Collins
Southern Cross University
Lismore NSW,
AUSTRALIA
dcollins@scu.edu.au

Ceneida Fernández
Universidad de Alicante
Campus de Sant Vicent
apartado 99 E-03080,
Alicante, SPAIN
ceneidafv@hotmail.com

Ma. del Pilar Fernández
University of Barcelona
Barcelona, SPAIN
pfernandez@ub.edu

Josep Maria Fortuny
UAB
Dpto. Didáctica de la Matemática, Bellaterra, Barcelona, SPAIN
josepmaria.fortuny@uab.es

Anne Berit Fuglestad
University of Agder
Mathematics, Servicebox 422, Kristiansand
NORWAY
anne.b.fuglestad@uia.no

Olimpia Figueras
Cinvestav
Av. Inst. Pol. Nacional 2508, México, D.F., MEXICO
figuerao@cinvestav.mx

Cristina Frade
Universidade Federal de Minas Gerais
Av. Antônio Carlos 6627, Pampulha, Belo Horizonte
BRAZIL
frade.cristina@gmail.com

Athanasios Gagatsis
University of Cyprus
CYPRUS

Eugenio Filloy
Cinvestav
Av. Inst. Pol. Nacional 2508, México, D.F.
MEXICO
smmeef@aol.com

John Francisco
University of Massachusetts
10 Tyler PL, Apt 2SAmherst, MA 01002
USA
jmfranci@educ.umass.edu

F. Gamboa
National Autonomous University of Mexico
MEXICO

Patricia Flores
Universidad Pedagógica Nacional
Distrito Federal
MEXICO

Janete Bolite Frant
UNIBAN

Montserrat García C.
Cinvestav
Av. Inst. Pol. Nacional 2508, México, D.F.
MEXICO

Vicenc Font
Unknown
Passeig de la Vall d'Hebron, 171, Barcelona, SPAIN
vfont@ub.edu

Martha Fuentes
Universidad Pedagógica Nacional Unidad Morelos
4 Cda. de Narciso Mendoza No.3, Santa María Ahuacatitlán, Cuernavaca, MEXICO
marthafu@yahoo.com.mx

Susan Gerofsky
University of British Columbia
2125 Main Mall
CANADA
susan.gerofsky@ubc.ca

Helen J. Forgasz
Monash University
Wellington Road, Clayton
AUSTRALIA
helen.forgasz@education.monash.edu.au

Mariana Fuentes
Autonomous University of Barcelona
Passeig de la Vall d'Hebron, 171, Bellaterra
SPAIN
Mariana.Fuentes@uab.cat

Hope Gerson
Brigham Young University
TMCB 179, Provo
USA
hope@mathed.byu.edu

Soheila Gholamazad
Ministry of Education
#50 Dehghan Street,
Sat'tarkhan Ave, Tehran
IRAN
soheila_azad@yahoo.com

Nadia Gil
Cinvestav
Av. Inst. Pol. Nacional
2508, México, D.F.
MEXICO

Michael Gilbert
University of Hawaii
2640 Dole St. Apt E-153,
Honolulu,
USA
michael.gilbert@hawaii.edu

Camilla Gilmore
University of Nottingham

Howard Goldberg
University of Illinois at
Chicago
USA

Gerald A. Goldin
Rutgers University
118 Frelinghuysen Road,
Piscataway, NJ, USA
geraldgoldin@dimacs.rut
gers.edu

Edna González
Universitat de Valencia
Explorador Andrés 29.
pta. 9, Valencia, Valencia
SPAIN
ednagq@hotmail.com

Alejandro S. González M.
Université de Montréal
Bureau D-522; Pav.
Marie Victorin;
Université de Montréal;
CP 6128, suc. Centre-Ville
CANADA
a.gonzalez-martin@umon
treal.ca

Simon Goodchild
University of Agder

Kristy Goodwin
Macquarie University
125 Parkes Road,
Collaroy Plateau
AUSTRALIA
kristygoodwin@mac.com

Merrilyn Goos
The Univ. of Queensland
Teaching and Educational
Development Institute
St LuciaAUSTRALIA
m.goos@uq.edu.au

Zahra Gooya
Shahid Beheshti Univ.
#4-16th Street Gisha Ave,
Tehran, IRAN
zahra_gooya@yahoo.com

Sarah Green
Vanderbilt University
Department of Teaching
and Learning0230 GPC
230 Appleton Place
Nashville, TN 37203-
5721, Nashville, USA
sarah.green@vanderbilt.edu

Brian Greer
Portland State University

Roxana Grigoras
University of Bremen
Bibliothekstrasse 1,
Bremen, GERMANY
roxana@math.uni-brem
en.de

Susie Groves
Deakin University

Élgar Gualdrón
University of Pamplona
Km 3 via a Bucaramanga
Pamplona. Pamplona
COLOMBIA
elgargualdron@yahoo.es

María J. González
University of Granada

Eddie Gray
University of Warwick

Erhan S. Haciomeroglu
Univ. of Central Florida
4000 Central Florida
Blvd, Orlando
USA
erhansh@mail.ucf.edu

Lynn Hart
Georgia State University
426 Glenn Circle, Decatur
USA
lhart@gsu.edu

Carolina Guerrero
Cinvestav
AV. IPN 2508, México,
Distrito Federal,
MEXICO

Markus Hähkiöniemi
University of Jyväskylä
Harapaisentie 16 A 20,
Lappeenranta, Finlandia
markus.hahkioniemi@ed
usaimaa.fi

Mathias Hattermann
Justus-Liebig University
Giessen
Karl-Glöckner Straße 21
C, Gießen, GERMANY
Mathias.Hattermann@ma
th.uni-giessen.de

Lourdes Guerrero M.
Universidad Michoacana
de San Nicolás de Hidalgo
Luis G. Urbina, 41,
Morelia
MEXICO
gmagana@umich.mx

Stefan Halverscheid
Universitaet Bremen
Bibliothekstr 1, D-28359,
Bremen
GERMANY
sth@math.uni-bremen.de

Reinhold Haug
Paedagogische
Hochschule Freiburg
Kunzenweg 21, Freiburg
GERMANY
reinhold.haug@ph-
freiburg.de

Gregoria Guillén
Universidad de Valencia
Explorador Andrés 29.
pta. 9, Valencia, Valencia
SPAIN

Markku S. Hannula
University of Helsinki
FINLAND

Lulu Healy
UNIBAN- São Paulo
Rua Capit?Pinto Ferreira,
62 Apt 84, Jardim Paulista,
São Paulo, BRAZIL
lulu@pq.cnpq.br

Hülya Gür
Balikesir University
Necatibey Educational
Faculty, Maths Education
Department, Balikesir
TURKEY
hgur@balikesir.edu.tr

Tony Harries
University of Durham
Leazes Road, Durham
UNITED KINGDOM
a.v.harries@durham.ac.uk

John Hedberg
Macquarie University

José Guzmán
Cinvestav
Av. Instituto Politécnico
Nacional 2508, Col. San
Pedro Zacatenco, D.F.
MEXICO

Kathleen Hart
University of Nottingham
15 Chester Close, Kings
Rd, Richmond
UNITED KINGDOM
kathleenhart@o2.co.uk

Aiso Heinze
University of Regensburg
Olshausenstrasse 62, Kiel
GERMANY
heinze@ipn.uni-kiel.de

Beth Herbel-Eisenmann
Michigan State University

Siew Yin Ho
National Institute of Education, Nanyang Technological University
1 Nanyang Walk
SINGAPORE
siewyin.ho@nie.edu.sg

Cheng-Jung Hsu
National Changhua University of Education
TAIWAN ROC

Patricio Herbst
The Univ. of Michigan
610 East University Ave. #1302C, Ann Arbor
USA
pgherbst@umich.edu

Markus Hohenwarter
Universitat Autónoma of Barcelona

Hsiu-Ling Hsu
Department of Mathematics, National Taiwan Normal University

Ann Heirdsfield
Queensland University of Technology
Victoria Park Rd, Kelvin Grove, Brisbane
AUSTRALIA
a.heirdsfield@qut.edu.au

Marj Horne
Australian Catholic University
Locked Bag 4115, Fitzroy, Victoria
AUSTRALIA
m.horne@patrick.acu.edu.au

Xu Li Hua
University of Melbourne

Paul Hernandez M.
University of Manchester

Anesa Hosein
Open University
Walton Hall, Milton Keynes
UNITED KINGDOM
A.Hosein@open.ac.uk

Rongjin Huang
Texas A&M University
USA

Kate Highfield
Macquarie University
c/of 3 Rembrandt Drive, Middle Cove, Sydney
AUSTRALIA
kate.highfield@aces.mq.edu.au

Alena Hošpesová
University of South Bohemia
Jeronymova 10, Ceske Budejovice
CZECH REPUBLIC
hospes@pf.jcu.cz

Shiang-ting Huang
Chung Cheng Elementary School
Taichung City,
TAIWAN

Kenji Hiraoka
Nagasaki University
JAPAN

Chia-Wei Hsiao
National University of Tainan
TAIWAN ROC

Pi-Hsia Hung
National Univ. of Tainan
33, Sec.2, Shu-Lin St, Tainan
TAIWAN ROC
hungps@mail.nutn.edu.tw

Jodie Hunter
Massey University
6 Joan Street, Point Chevalier, Auckland
NEW ZEALAND
jodiehunter@slingshot.co.nz

Roberta Hunter
Massey University
Albany Campus, Building 52, Private Bag 102904, Auckland
NEW ZEALAND
r.hunter@massey.ac.nz

Kai-ju Hsieh
National Taichung University
National Taichung University Department of Mathematics Education, Taichung City
TAIWAN ROC
khsieh@mail.ntcu.edu.tw

Tee-Yong Hwa
Universiti Teknologi MARA Sarawak

Paola Iannone
University of East Anglia
Newmarket rd Norwich Norfolk
UNITED KINGDOM
p.iannone@uea.ac.uk

Bat-Sheva Ilany
Beitberl
27b Agnon Street
ISRAEL
bat77@013.net

Matthew Inglis
University of Nottingham, Wollaton Road, Nottingham
UNITED KINGDOM
matthew.inglis@nottingham.ac.uk

Noriyuki Inoue
University of San Diego
1840 East Pointe Ave, Carlsbad
USA
inoue@sandiego.edu

Núria Iranzo
Universitat Autónoma Barcelona
Edifici G5, Despatx 138, Facultat de Ciències de l'Educació, Cerdanyola del Vallès, Barcelona
SPAIN.
niranzod@yahoo.es

Shinya Itoh
University of Tsukuba
Tennodai 1-1-1, Tsukuba, Ibaraki, 305-8572, Tsukuba
JAPAN
shinya@human.tsukuba.ac.jp

Hideki Iwasaki
Hiroshima University
1-1, Kagamiyama, 1chome, Higashi-Hiroshima, JAPAN
hiwasak@hiroshima-u.ac.jp

Andrew Izsák
The Univ. of Georgia
105 Aderhold Hall, Athens, USA
izsak@uga.edu

Eva Jablonka
Luleå University of Technology

Jennifer Jacobs
University of Colorado Boulder

Elizabeth Jakubowski
Florida State University

Hyungog Jeon
Korea National University of Education
Hanbit apartment 103-304, Tabyeon-ri, Cheongwon-gun, Chungbuk, Korea,
KOREA
antree44@naver.com

Brenda A. Jiménez
Tecnológico de Monterrey
MEXICO

Christine Johnson
Brigham Young University
PO Box 5, Manti
USA
cj76@byu.edu

Jennifer Jones
Rutgers University

Sonia Jones
Swansea Institute of Higher Education
Townhill Road, Swansea
UNITED KINGDOM
sonia.jones@sihe.ac.uk

Coral Kemp
CRIMSE, Macquarie University
Sydney
AUSTRALIA

A. Klothou
Democritus University of Thrace, Greece

Leslie H. Kahn
University of Arizona
6261 N. Cinnamon Drive
USA
lkahn1@mindspring.com

Carolyn Kieran
Université du Québec à Montréal
C.P. 8888, Centre-Ville, Montréal
CANADA
kieran.carolyn@uqam.ca

Andrea Knapp
University of Georgia
454 Hickory Lane
Griffin, GA 30223, Griffin
USA
akknapp@uga.edu

Gabriele Kaiser
University of Hamburg
Von-Melle-Park 8, Hamburg
GERMANY
gabriele.kaiser@uni-hamburg.de

Gooyeon Kim
University of Missouri-St. Louis
361 Marillac Hall, One University Blvd, St. Louis, MO 63121
USA
gkim@umsl.edu

Christine Knipping
Acadia University

Maria Kaldrimidou
University of Ioannina
POGONIOU 4, Ioannina
GREECE
mkaldrim@uoi.gr

Hyewon Kim
Florida State University

Jillian M. Knowles
Endicott College
376 Hale Street, Beverly, MA 01915, Massachusetts
USA
jknowles@endicott.edu

Wilfred Kaleva
University of Goroka, Goroka
Papua New Guinea
kalevaw@uog.ac.pg

Rae-Young Kim
Michigan State University

Karen Koellner
University of Colorado At Denver
7721 East 28th place
USA
karen.koellner@cudenver.edu

Karen Allen Keene
North Carolina State University

Kirsti Klette
University of Oslo, Oslo
NORWAY
kirsti.klette@ped.uio.no

Boris Koichu
Technion. Institute of Technology
Boris Koichu, Haifa.
ISRAEL
bkoichu@technion.ac.il

Eugenia Koleza
Department of Primary
Education of Ioannina
Ioannina
GREECE
ekoleza@cc.uoi.gr

Teruni Lamberg
Univ. of Nevada, Reno,
College of Education
USA
terunil@unr.edu

Shian Leou
National Kaohsiung
Normal University

Heidi Krzywacki-Vainio
University of Helsinki
P. Box 9, Helsinki
FINLAND
heidi.krzywacki-
vainio@helsinki.fi

Paul Ngee-Kiong Lau
Universiti Teknologi
MARA Sarawak

Henry Leppäaho
University of Jyväskylä
P.O. Box 35, Jyväskylä
FINLAND
henry.leppaaho@edu.jyu.fi

Sebastian Kuntze
University of Munich
Theresienstr 39,
Muenchen
GERMANY
kuntze@math.lmu.de

Zsolt Lavicza
University of Cambridge

Yuh-Chyn Leu
National Taipei
University of Education
134, Sec.2, Ho-Ping
E.Road, Taipei
TAIWAN ROC
leu@tea.ntue.edu.tw

NaYoung Kwon
University of Georgia
106 college station rd.
#G-206, Athens
USA
rykwon@uga.edu

Kyong-Hee Melody Lee
Univ. of Southern Indiana
8600 University
Boulevard Evansville,
Indiana 47712, Evansville
USA
kmlee1@usi.edu

King-Man Leung
University of East Anglia
Flat D, 4/F, Yee Hoi
Mansion, Lei King Wan,
Hong Kong
HONG KONG
kmleungx@yahoo.com

Minsung Kwon
University of Michigan
#607 University Towers,
536 S. Forest Ave., Ann
Arbor
USA
mskwon@umich.edu

Kyunghwa Lee
Korea National
University of Education
Gangnaemyeon,
Cheongwongun
Chungbuk, KOREA
khmath@knue.ac.kr

Shuk-Kwan S. Leung
National Sun Yat-Sen
University

Janeen Lamb
Australian Catholic
University
AUSTRALIA

Roza Leikin
University of Haifa
ISRAEL

Hee-Chan Lew
Korea National
University of Education
Gangnaemyeon
Cheongwongun,
KOREA
hclew@knue.ac.kr

Yeping Li
Texas A
MS 4232, TAMU Texas,
College Station.
USA
yepingli@yahoo.com

Chih-Yen Liu
National Changhua
University of Education
10F, 177 Da-Ming Road,
Fongyuan
TAIWAN ROC
unique.cs@msa.hinet.net

Maria Dolores Lozano
Cinvestav
Periférico Sur 4118 Piso
7, Mexico City
MEXICO
lolis_l@yahoo.es

Fang-Chi Lin
Department of
Mathematics, National
Taiwan Normal
University

Yueh Mei Liu
Singapore Ministry of
Education

Man Wai Lui
The University of Hong
Kong

Fou-Lai Lin
National Taiwan Normal
University
88, Sec. 4, Ting-Chou
Road, Taipei,
TAIWAN ROC
linfl@math.ntnu.edu.tw

Salvador Llinares
University of Alicante
Facultad de Educación
Campus de San Vicente
Del Raspeig, Alicante
SPAIN
sllinares@ua.es

Fenqjen Luo
Univ. of West Georgia
418 Old Mill Dr,
Carrollton
USA
fluo@westga.edu

Miao-Ling Lin
Yong Ping Elementary
School, Taiwan

Jane-Jane Lo
Western Michigan
University
4425 Everett Tower,
Kalamazoo
USA
jane-jane.lo@wmich.edu

Nicolina A. Malara
Università di Modena e
Reggio E.

Pi-Jen Lin
Hsin-Chu University of
Education
521, Nan-Dah Road,
Hsin-Chu City
TAIWAN ROC
linpj@mail.nhcue.edu.tw

Priscilla Lopez
Univeristy of Belize
P.O Box 1137 Belize,
BELIZE
prislopez_99@yahoo.com

Marcus V. Maltempi
São Paulo State
University at Rio Claro

Su-Wei Lin
National Hualien
Universtity of Education
123 Hua-Hsi Rd, Hualien
TAIWAN ROC
swlin@mail.nhlue.edu.tw

Tom Lowrie
Charles Sturt University

Ema Mamede
University of Minho

Ami Mamolo
CANADA
amamolo@sfu.ca

Bruria Margolin
Levinsky College

Jennifer Marston
Macquarie University

Andrew Martin
The University of Sydney

Ignacio Martinez G.
Universidad Autónoma
de Zacatecas
Zacatecas
MEXCIO

Magally Martinez Reyes
Univ. Autónoma del
Estado de México
Estado de México,
MÉXICO
magallymartinezreyes@g
mail.com

Marta Massa
Rosario National
University

Yuki Masuda
University of Tsukuba
2-2-17 Fukagawa,
Koutou-ku, Tokyo
JAPAN
yuki580321@yahoo.co.jp

Nanae Matsuo
Chiba University
1-33, Yayoi-cho, Inage-
ku, Chiba
JAPAN
matsuo@faculty.chiba-
u.jp

Christopher Matthews
Griffith University
Faculty of Environmental
Sciences, Nathan
AUSTRALIA
c.matthews@griffith.edu.au

Alasdair McAndrew
Victoria University

Raven McCrory
Michigan State
University

Douglas McDougall
University of Toronto

Kerry McKee
New Mexico State Univ.
2953 Valle Vista, Las
Cruces
USA
Kerry_L_McKee@hotma
il.com

Michael Meagher
Brooklyn College
School of Education,
Brooklyn College, 2111
James Hall, 2900 Bedford
Ave, Brooklyn
USA
mmeagher@brooklyn.cun
y.edu

Cynthia Medina
Universidad Michoacana
de San Nicolás de
Hidalgo

Kyong Hee Melody Lee
University of Southern
Indiana

José M. Menéndez
The University of
Arizona – CEMELA
726 E 9th St Apt 23,
Tucson
USA
jmenendez@math.arizona
.edu

Vilma Mesa
University of Michigan
1360F SEB, Ann Arbor
USA
vmesa@umich.edu

Nikolaos Metaxas
University of Athens
Thermopilon 16 Athens
10435, Athens
GREECE
nkm1012gr@yahoo.com

Michael Meyer
TU Dortmund
Vogelpothsweg 87,
Dortmund, GERMANY
michael.meyer@math.uni-dortmund.de

Isaias Miranda
Cinvestav-Laurentian University
Av. Inst. Pol. Nacional
2508 Col. San Pedro,
Zacatenco, D. F.
MÉXICO
isamiran@yahoo.com

Christina Misailidou
University of Stirling
The Stirling Institute of
Education, Stirling
UNITED KINGDOM
christina.misailidou@stir.ac.uk

Michael Mitchelmore
Macquarie University
AUSTRALIA
mike.mitchelmore@mq.edu.au

Marta Molina
University of Granada
Nuestra Señora de la
Salud 4, 3 Q, C.P, 18014,
Granada, SPAIN
martamg@ugr.es

Cynthia Moore
Illinois State University

Kaitlin Moore
Queensland University of
Technology

Deborah Moore-Russo
State University of New
York at Buffalo
566 Baldy Hall State
University of New York
at Buffalo, Buffalo
USA
dam29@buffalo.edu

Christian Morasse
Collège de Montréal

Manolis Moroglou
University of Ioannina

Judit Moschkovich
University of California,
Santa Cruz Education
Department, 1156 High
Street, California, USA
jmoschko@ucsc.edu

Eduardo Mosqueda
University of California
Santa Cruz
14821 Hope St.,
Westminster, USA
mosqueda@ucsc.edu

Judith Mousley
Deakin University
Geelong, AUSTRALIA
judym@deakin.edu.au

Nicholas Mousoulides
University of Cyprus
Department of Education,
POBOX 20537, Nicosia
CYPRUS
n.mousoulides@ucy.ac.cy

Patricia Moyer P.
George Mason University
4085 University Drive
MS 4C2, Washington
USA
pmoyer@gmu.edu

Joanne Mulligan
Macquarie Univ. Sydney
Australian Centre for
Educational Studies
Macquarie University,
Sydney, NSW Australia
joanne.mulligan@mq.edu.au

Charles Munter
Vanderbilt University
Box 230 GPC, Nashville
USA
c.munter@vanderbilt.edu

Elena Nardi
University of East Anglia
UEA-EDU, NORWICH
NR4 7TJ, UNITED
KINGDOM
e.nardi@uea.ac.uk

A. Nizam
Democritus University of
Thrace

Aki Murata
Stanford University
Stanaford University
School of Education, 520
Galvez Mall, Stanford,
CA. USA
akimura@stanford.edu

Nirmala Naresh
Illinois State University

Rosana Nogueira de L.
Uniban
Rua Guaipá, 758 ap 116
Ipê, São Paulo
BRAZIL
rlima@uniban.br

Carol Murphy
Exeter
Heavitree Road,
EXETER, Devon,
UNITED KINGDOM
c.m.murphy@ex.ac.uk

Dorit Neria
Ben Gurion University of
the Negev
Tamar 5 Omer
ISRAEL
neria@012.net.il

Guri A. Nortvedt
University of Oslo
Moellefaret 0, 0750 Oslo
NORWAY
gurin@isp.uio.no

Sandra Musanti
Univ. of New Mexico
1329 Glorieta NE,
Albuquerque
USA
simusanti@gmail.com

Michael Neubrand
University of Oldenburg
Carl-Von-Ossietzky-Str.,
Oldenburg, GERMANY
michael.neubrand@uni-
oldenburg.de

Jarmila Novotná
Charles Univ. in Prague
M.D. Rettigove 4, Praha 1
CZECH REPUBLIC
jarmila.novotna@pedf.cu
ni.cz

Margrethe Naalsund
University of Oslo
PB. 1099 Blindern, Oslo
NORWAY
margrethe.naalsund@ils.u
io.no

Cynthia Nicol
University of British
Columbia
2125 Main Mall,
Vancouver, CANADA
cynthia.nicol@ubc.ca

Terezinha Nunes
University of Oxford

Shweta Naik
Tata Institute of
Fundamental Research
Near Anushakti Nagar
Depot, V. N. Purav
Marg,, Mankhurd,
Mumbai, INDIA
shweta@hbcse.tifr.res.in

Hiro Ninomiya
Saitama University

G. Eréndira Núñez
Universidad Michoacana

Minoru Ohtani
Kanazawa University
Kakuma, Japan
mohtani@kenroku.kanaza
wa-u.ac.jp

Chandra Hawley Orrill
University of Georgia

Jeongsuk Pang
Korea National
University of Education
Chungbuk, Korea, JAPAN
jeongsuk@knue.ac.kr

Claire Okazaki
University of Hawaii
Curriculum Research and
Development Group
1776 University Avenue,
Honolulu, USA
cokazaki@hawaii.edu

Kay Owens
Charles Sturt University
Locked Bag 49, Dubbo,
NSW, AUSTRALIA
kowens@csu.edu.au

Mabel Panizza
Univ. de Buenos Aires
Montañas 1910 2° 15,
Buenos Aires,
ARGENTINA
mpanizza@mail.retina.ar

Masakazu Okazaki
Joetsu University of
Education
1-1, Naka 3, Tsushima,
Okayama, JAPAN
masakazu@juen.ac.jp

Aynur Özdaş
Anadolu University

Stavroula Patsiomitou
University of Ioannina
Agiou Meletiou 2, 11361,
Athens
GREECE
spatsiomitou@sch.gr

John Olive
The University of
Georgia

Mehmet F. Ozmantar
University of Gaziantep
Guvenevler M. 71 Nolu
C. Pembegul Apt No
33/5, Gaziantep
TURKEY
mfozmantar@yahoo.co.uk

Louis Pedrick
Rutgers University

Melfried Olson
University of Hawaii
411 Hobron Lane Apt
3502, Honolulu, USA
melfried@hawaii.edu

Maria Pampaka
The University of
Manchester

Ildikó Judith Pelczer
Instituto de Ingeniería
Circuito Escolar, Ciudad
Universitaria, D. F.
MÉXICO
IPelczer@iingen.unam.mx

Francisco Olvera
Universidad Pedagógica
Nacional
Juanacatlan y El caracol.
Infonavit Los Sauces,
Tepic
MÉXICO
frajaol@prodigy.net.mx

Areti Panaoura
Frederick University

Jorge Peralta Sámano
Universidad Autónoma
Estado de México

Pamela Perger
University of Auckland
Private Bag 92601
Symonds Street,
Auckland
NEW ZELAND
p.perger@auckland.ac.nz

Kathleen Pitvorec
University of Illinois
1318 W. Sherwin,
Chicago, USA
kapitvor@uic.edu

Dave Pratt
University of London

Susan Peters
The Pennsylvania State University
158 Birchtree Court, State College, PA, USA
sap233@psu.edu

Nuria Planas
Universitat Autónoma de Barcelona
Despatx 134, Edific G5,
Bellaterra, Barcelona,
SPAIN
nuria.planas@uab.cat

Anne Prescott
University of Technology, Sydney
PO BOX 222, Lindfield
AUSTRALIA
anne.prescott@uts.edu.au

Marilena Petrou
University of Cambridge
Afrodites 13
Makedonitissa, Nicosia,
CYPRUS
mp417@cam.ac.uk

François Pluvinage
Cinvestav
Av. IPN No. 2508,
México, D. F.,
MÉXICO
pluvin@math.u-strasbg.fr

Norma Presmeg
Illinois State University
2811 Polo Road, Illinois
USA
npresmeg@msn.com

Robyn Pierce
University of Melbourne

Svetlana Polushkina
Technische Universitaet Darmstadt,
Schlossgartenstr 764289
Darmstadt
GERMANY
polushkina@mathematik.tu-darmstadt. de

Theodosia Prodromou
University of Warwick
Centre for New Technologies Research in Education, Coventry
UNITED KINGDOM
prodromou.theodosia@googlemail.com

Marcia Pinto
Universidade Federal de Minas Gerais
Rua Trento, 160, Belo Horizonte,
BRAZIL
fusaro@ufmg.br

Juan Carlos Ponce
Cinvestav
Av. IPN No. 2508,
México, D. F.,
MÉXICO
poncejcar@yahoo.com.mx

Jerome Proulx
University of Ottawa
145 Jean-Jacques Lussier,
Ottawa, CANADA
jerome.proulx@uottawa.ca

D. Pitta-Pantazi
University of Cyprus
CYPRUS

Despina Potari
University of Patras
Aristoteles Street, Patras,
GREECE
potari@upatras.gr

Luis Puig
Universitat de Valencia
Apto. 22045, Valencia,
SPAIN
luis.puig@uv.es

Humberto Quesada
University of Alicante

David Reid
Calz. Misiones Mz 122
lote-64, Ojo de Agua
Estado de México,
MÉXICO

Luis Rico
University of Granada

Beatriz Quintos Alonso
University of Arizona
4804 Calvert Rd. Número
1 College Park, MD,
USA
bquintos@email.arizona.e
du

Hilary Reid
Durham University

Mirela Rigo
Cinvestav
Av. IPN 2508, Col. San
Pedro Zacatenco, México
City, D. F.
MÉXICO
mrigo@mail.cinvestav.mx

Ali Akbar Rabanifard
Shahid Beheshti
University

Kristina Reiss
University of Munich

Ferdinand Rivera
San Jose State University
1 Washington Square,
USA
rivera@math.sjsu.edu

Luis Radford
Laurentian University
P3E 2CC6, Sudbury,
Ontario,
CANADA
lradford@laurentian.ca

Mani Rezaie
Shahid Beheshti
University
Noori highway, Tehran
IRAN
mani.rezaie@gmail.com

María Teresa Rojano
Cinvestav
Av. IPN 2508, San Pedro
Zacatenco, México City
MÉXICO
trojano@cinvestav.mx

Chris Rasmussen
San Diego State
University
5500 Campanile Drive,
San Diego, USA
chrisraz@sciences.sdsu.e
du

Sebastian Rezat
Justus-Liebig-
Universitaet Giessen
Karl-Glöckner-Str. 21c,
Gießen,
GERMANY
sebastian.rezat@math.uni
-giessen.de

Mauricio Rosa
State University of SÃ£o
Paulo at Rio Claro
Av. Farroupilha, 8001Sao
Jose - Canoas (RS)92450-
900
BRAZIL
mauriciomatematica@gm
ail.com

Rubí Real
Cinvestav
Av. IPN No. 2508,
México, D. F.
MÉXICO

John T.E. Richardson
The Open University

Iris Rosenthal
Beit-Berl Academic
College

Joanne Rossi Becker
San José State University
95192-0103, San Jose
USA
becker@math.sjsu.edu

Filip Roubíček
Institute of Mathematics,
Academy of Sciences of
the Czech Republic
Zitna 25, Prague 1,
CZECH REPUBLIC
roubicek@math.cas.cz

Guillermo Rubio
Univ. Nacional
Autónoma de México
México City, D. F.
MÉXICO
willirubio@yahoo.com.mx

Polina Sabinin
Boston University

Jacqueline Sack
Rice University
4915 Valkeith Drive,
Houston
USA
jsack@rice.edu

Ana Isabel Sacristan
Cinvestav
Av. Inst. Pol. Nacional
2508, Col. San Pedro
Zacatenco, México City,
MÉXICO
asacrist@cinvestav.mx

Mariana Saiz
Univ. Pedagógica Nacional
Av. Iman 580. Islas Fidji-
303. México City, D. F.
MÉXICO
saizmar@yahoo.com

Haralambos Sakonidis
Democritus University of
Thrace
N. Chili, Alexandroupolis,
GREECE
xsakonid@eled.duth.gr

Martha Sánchez
Cinvestav
Av. IPN No. 2508,
México, D. F.,
MÉXICO

Eurivalda Santana
Universidade Estadual De
Santa Cruz
Rua Ana Moura, 75.
Bairro Novo Itamarati,
BRAZIL
eurivalda@hotmail.com

Marco A. Santillán V.
Universidad Nacional
Autonóma de México
México City, D. F.
MÉXICO

Manuel Santos-Trigo
Cinvestav
Av. Inst. Pol. Nacional
2508, Col. San Pedro
Zacatenco, México City,
MÉXICO

Bernhard Schmitz
Technische Universität
Darmstadt

Roberta Schorr
Rutgers University
14 Sweney Ct
USA
schorr@rci.rutgers.edu

Evelyn Seeve
Rutgers University

Yasuhiro Sekiguchi
Yamaguchi University
677-1 Yoshida,
Yamaguchi-Shi, JAPAN
ysekigch@yamaguchi-
u.ac.jp

Annie Selden
New Mexico State
University
3304 Solarridge Street,
Las Cruces, USA
aselden@math.nmsu.edu

John Selden
New Mexico State
University
3304 Solarridge Street,
Las Cruces, USA
js9484@usit.net

Armando Sepúlveda
Univ. Michoacana de San Nicolás de Hidalgo
Fray Jacobo Daciano 192, Colonia Ampliación Ocolusen, Morelia,
MÉXICO
asepulve@umich.mx

Dianne Siemon
RMIT University
PO Box 71, Bundoora
AUSTRALIA
dianne.siemon@rmit.edu.au

Kum-Nam So
Korea National University of Education

Diana Itzel Sepúlveda
Univ. Michoacana de San Nicolás de Hidalgo
MÉXICO

Harry Silfverberg
University of Tampere
PL 607, Tampere
FINLAND
harry.silfverberg@uta.fi

Armando Solares
Cinvestav
Av. IPN 2508, México,
MÉXICO
asolares@cinvestav.mx

Natalia Sgreccia
Univ.Nacional de Rosaario
Av. Pellegrini 250,
Rosario,
ARGENTINA
sgreccia@fceia.unr.edu.ar

Jason Silverman
Drexel University
3141 Chestnut Street,
Philadelphia,
USA
silverman@drexel.edu

Cruz E. Sosa Carrillo
Instituto Tec de Monterrey
Tlalpan, México City,
MÉXICO
evsosa@itesm.mx

Emily Shahan
Stanford University

Adrian Simpson
University of Durham
Principal, University of Durham, Durham,
UNITED KINGDOM
adrian.simpson@durham.ac.uk

Michael Spector
Florida State University

Gayle Spry
Australian Catholic University

Yoshinori Shimizu
University of Tsukuba

Florence Mihaela Singer
Institute for Educational Sciences
Stirbei Voda 37,
ROMANIA
mikisinger@gmail.com

Bharath Sriraman
The University of Montana

Yusuke Shinno
Hiroshima University
1-1-1, Kagamiyama,
Higashi-Hiroshima
JAPAN
shinno@hiroshima-u.ac.jp

Hannah Slovin
University of Hawaii
1776 University Ave. Cm 130, Honolulu,
USA
hslovin@hawaii.edu

Susan Staats
University of Minnesota
128 Pleasant St SE,
Minneapolis
USA
staats@umn.edu

Olof B. Steinthorsdottir
Univ. of North Carolina
Cb 3500 Peabody Hall,
Chapel Hill,
USA
steintho@email.unc.edu

Sepideh Stewart
The Univ. of Auckland
PB 92019 Auckland
NEW ZEALAND
stewart@math.auckland.a
c.nz

Shari Stockero
Michigan Technological
University
1400 Townsend Drive,
Houghton
USA
stockero@mtu.edu

Nadia Stoyanova K.
Stony Brook Univ. Building
Office 4-104 Stony
Brook, Ny 11794-3651
USA
nadia@math.sunysb.edu

Despina Stylianou
City College, The City
University of New York
USA
dstylianou@ccny.cuny.edu

Liliana Suárez Téllez
Cinvestav
Av. IPN 2508, Col. San
Pedro Zacatenco, México
MÉXICO
lsuarez@cinvestav.mx

K. Subramaniam
Homi Bhabha Centre for
Science Education

Jennifer Suggate
Durham University

Jennifer Suh
George Mason University
4085 University Drive,
200A, Fairfax, USA
jsuh4@gmu.edu

Jeong Suk Pang
Korea National
University of Education

Peter Sullivan
Monash University
6 Arbor St Alphington
3078 Australia
peter.sullivan@education.
monash.edu.au

Hsin-jung S. Sung
National Sun Yat-sen
University

Michal Tabach
Tel Aviv University

Pamela Tabor
Southern Cross Univ.,
AU/Harford County
Public Schools
801 S. Stokes St. Havre
de Grace Maryland 21078
USA
pamela.tabor@hcps.org

Caibin Tang
Hangzhou Education
Research Center on
Elementary Mathematics

D. Tanguay
Université du Québec à
Montréal

Dilek Tanışlı
Anadolu University
Education Faculty,
Primary Education
Department, Eskiþehir.
TURKEY
dtanisli@anadolu.edu.tr

Howard Tanner
Swansea Institute of
Higher Education
Townhill Road Townhill
UNITED KINGDOM
howard.tanner@sihe.ac.uk

Mourat Tchoshanov
University of Texas
4813 Excalibur Drive,
El Paso
USA
mouratt@utep.edu

Nermin Tosmur-Bayazit
Florida State University
209 Carothers Bld.
Florida State University,
Tallahassee,
USA
nt04@fsu.edu

Chien-Hsun Tseng
National Kaohsiung
Normal University

Kip Tellez
University of California
Santa Cruz
1156 High St., Santa Cruz
USA
ktellez@ucsc.edu

Chrissavgi Triantafillou
University of Patras
3 Plateia Diakoy, Lamia
GREECE
akallio@otenet.gr

Kuo-hung Tseng
Meiho Inssitute of
Technology

Anne Teppo
P.O. Box 570, Livingston
USA
arteppo@theglobal.net

María Trigueros
Instituto Tecnológico
Autónomo de México
Eugenia 26, México D.F
MÉXICO
trigue@itam.mx

Nail Tuktamyshov
Building University
Pr. Pobedy, 40-41,
Tatarstan. Kazan.
RUSSIA
nail54@pisem.net

Michael O. J. Thomas
The University of
Auckland

Barbara Trujillo
Univ. of New Mexico
301 Solano NE,
Albuquerque
USA
barbt@comcast.net

Fay Alison Turner
University of Cambridge
184 Hills Rd, Cambridge
UNITED KINGDOM
fat21@cam.ac.uk

Dina Tirosh
Tel Aviv University

Wen-Huan Tsai
National Hsinchu
University of Education
521, Nan-Dah Road,
Hsinchu City.
TAIWAN ROC
tsai@mail.nhcue.edu.tw

Marianna Tzekaki
Aristotle University of
Thessaloniki
Der. of Early Childhood
Education, Thessaloniki
GREECE
tzekaki@nured.auth.gr

Germán Torregrosa
Universidad de Alicante
Campus Sant Vicent del
Raspeig Ap. Correus
99E-03080, Alicante
SPAIN
German.Torregrosa@ua.es

Pessia Tsamir
Tel Aviv University

Ron Tzur
Purdue
2906 Cashel Lane
USA
rontzur@verizon.net

Behiye Ubuz
Middle East Technical University
Ankara
TURKEY
ubuz@metu.edu.tr

Marja van den Heuvel P.
FI Utrecht Univ. & IQB,
Humboldt University

Kim Vleugels
University of Leuven

Stefan Ufer
Theresienstrasse 39,
Munich, GERMANY
ufer@math.lmu.de

Laura Van Zoest
Western Michigan Univ.
Postal Code 5248,
Kalamazoo,
USA
laura.vanzoest@wmich.edu

Cristian Voica
University of Bucharest
14, Academiei Str.,
Bucharest,
ROMANIA
voica@gta.math.unibuc.ro

Petrina Underwood
Queensland University
of Technology

Irma Vazquez
University of St. Thomas
1950 Eldridge Parkway,
No.12105, Houston
USA
irmavazquez@sbcglobal.net

Maike Vollstedt
University of Hamburg

Sonia Ursini
Cinvestav
Av. IPN 2508, Col. San
Pedro Zacatenco, México
MÉXICO

Linda Venenciano
University of Hawaii
Av. Honolulu, HI 96822,
Honolulu
USA
lhirashi@hawaii.edu

David Wagner
Univ. of New Brunswick
Box 4400, Fredericton,
NB
CANADA
dwagner@unb.ca

Colleen Vale
Victoria University
243 Victoria Rd,
Northcote.
AUSTRALIA
colleen.vale@vu.edu.au

Lieven Verschaffel
University of Leuven

Philip Wagreich
Univ. of Illinois at Chicago
322 SEO, m/c 249, 851 S.
Halsted St.Chicago IL
60607-7045, Chicago
USA
wagreich@uic.edu

Julia Valls
University of Alicante

Jana Visnovska
Vanderbilt University
Gpc Box 330, Nashville,
USA
jana.visnovska@vanderbilt.edu

Geoff Wake
University of Manchester
Ellen Wilkinson Building,
Oxford Road
UNITED KINGDOM
geoff.wake@manchester.ac.uk

Fiona Walls
James Cook University
School of Education,
Townsville, QLD 4810
AUSTRALIA
fiona.walls@jcu.edu.au

Anne Watson
University of Oxford
15 Norham Gardens,
Oxford, United Kingdom
anne.watson@education.o
x.ac.uk

Paul White
Australian Catholic Univ.
25 a Barker Road,
Strathifield.
AUSTRALIA
paul.white@acu.edu.au

Janet G. Walter
Brigham Young
University

Kelly Watson
Australian Catholic
University

Michelle Wilkerson
Northwestern University
Annenberg Hall 2242120
Campus Drive,
USA
m-milkerson@northwest
ern.edu

Chih-Yeuan Wang
National Taiwan Normal
University
2F. No.845, Sec. 4, Bade
Rd., Songshan District,
Taipei,
TAIWAN
wcyeuan@gmail.com

Jenni Way
The University of Sydney

Annika Wille
University of Bremen
Bibliothekstrasse 1,
Bremen, Germany
awille@math.uni-bremen.
de

Juei-Hsin Wang
National Chiayi
University

Lyn Webb
Nelson Mandela
Metropolitan University
6 Fairford Ave, Mill Park,
Port Elizabeth
SOUTH AFRICA
lyn.webb@nmmu.ac.za

Gaye Williams
Deakin University
7 Alverna Grove,
AUSTRALIA
gaye.williams@deakin.ed
u.au

Lisa B. Warner
Rutgers University

Paul Webb
Nelson Mandela
Metropolitan University
SOUTH AFRICA

Sue Wilson
ACU National

Elizabeth Warren
Australian Catholic Univ.
PO BOX 456, Virginia,
AUSTRALIA
e.warren@mcauley.acu.e
du.au

Shih-Chan Wen
An-ho Elementary School

Robert J. Wright
Southern Cross
University

Chao-Jung Wu
National Taiwan Normal University
162, Sec. 1, Ho-Ping E. Road, Taipei
TAIWAN ROC
cog@tea.ntue.edu.tw

Jun-De Wu
National Changhua University of Education

Su-Chiao (Angel) Wu
National Chiayi University
5f-1, No. 6, Lane 5, Da Tung Road, Wufeng, Taichung County
TAIWAN ROC
aldy.chang@msa.hinet.net

Zhonghe Wu
National University
45 Almador
USA
zwu@nu.edu

Tracy Wylie
Kingsfield School, South Gloucestershire
Ground Floor Flat, 11 Westbury Road.
UNITED KINGDOM
tracy_wylie@hotmail.com

Kai-Lin Yang
National Changhua University of Education
No. 1, Jin-De Road, Changhua
TAIWAN ROC
kailin@cc.ncue.edu.tw

Gabriel Yáñez
Universidad Industrial de Santander
Calle 50 25-15 Ap 501, Bucaramanga.
COLOMBIA
gyanez@uis.edu.co

Shirley Yates
Flinders University
GPO Box 2100
NEW ZEALAND
shirley.yates@flinders.edu.au

Oleksiy Yevdokimov
The University of Southern Queensland
Baker Street Toowoomba
QLD 4350Australia, Toowoomba,
AUSTRALIA
yevdokim@usq.edu.au

Kaori Yoshida M.
Nagasaki University
Bunkyo-Machi 1-14, Nagasaki
JAPAN
mkaori@nagasaki-u.ac.jp

Janelle Young
Australian Catholic University

Shih-Yi Yu
National Chunghua University of Education
No.1, Jinshan 2nd St., Dali City, Taichung County 412, Chunghua
TAIWAN ROC
sheree318@yahoo.com.tw

Theodossios Zachariades
University of Athens
Panepistimiopolis, Athens
GREECE
tzaharia@math.uoa.gr

William Zahner
University of California, Santa Cruz
USA

Ricardo Zalaya
Universidad Politécnica de Valencia
SPAIN

Michelle Zandieh
Arizona State

Rina Zazkis
Simon Fraser University
University Drive
CANADA
zazkis@sfu.ca

Qing Zhao
Vanderbilt University

Orit Zaslavsky
Technion - Israel Institute of Technology
Technion City
ISRAEL
orit@tx.technion.ac.il

Fay Zenigami
Univ. of Hawaii at Manoa
1776 University Ave.
UHS 3-218, Honolulu
USA
zenigami@hawaii.edu

PLENARY ADDRESSES

Jens Høyrup
Aurora Gallardo
Patrick W. Thompson
Ruhama Even

THE TORTUOUS WAYS TOWARD A NEW UNDERSTANDING OF ALGEBRA IN THE ITALIAN ABBACUS SCHOOL (14^{TH}–16^{TH} CENTURIES)

Jens Høyrup

Roskilde University

Algebra as we encounter it in Stifel (1544) or Descartes (1637) looks wholly different from what we know from al-Khwārizmī and Fibonacci. Indeed, early Modern algebra did not build on these: its foundation was the algebra of the Italian Abbacus school. The paper follows the development of this tradition from 1307 onward, in particular the appearance of abbreviations, the naming of powers and roots, formal calculations, schemes, and the solution of higher-degree equations.

THE TRANSFORMATION

Ancient Babylonian and Ancient Egyptian mathematics were powerful calculational tools for the solution of scribal tasks - accounting, planning of resources, measurement of land; they were developed and taught for that purpose. What else was achieved by them – e.g., the impressive feats of Old Babylonian "Algebra" – was derivative and secondary to that purpose.

Classical Ancient mathematics had many components:[1]

- "Practical mathematics" of the scribal kind.
- "Liberal-Arts"-mathematics, the kind of mathematics which a well-bred person ought to know about – which was generally very little.
- What is mostly thought of as "Greek mathematics", the theoretical geometry of Euclid, Archimedes, Apollonios etc.

The latter type (though only a minor segment of it) turned out to be a powerful tool in Ptolemaic astronomy and in theoretical static and optics; Hero was also able to apply a small part to mensuration of the "scribal" type.[2] However, this was not the main purpose for which it was created, and until the late Renaissance it did not significantly broaden the range of applications it could serve.

Modern mathematics as it has unfolded since around 1600 has turned out to be an immensely more powerful tool for an ever-increasing range of practical objectives. What enabled it to go beyond the limits of ancient theoretical mathematics was the introduction of symbolic, formal calculation - first in algebra, then in *analysis infinitorum*, then in the calculus of probabilities and theoretical statistics, etc.

[1] See [Cuomo 2001]. The occasional lack of precision of this book does not prevent it from being an excellent introduction to the diversity of Classical mathematics.

[2] A little bit, though even less, also crept into for instance *Geometrica* and *Stereometrica*, pseudo-Heronian compilations closer to scribal traditions.

Algebra was thus the decisive stimulus – yet not the kind of algebra which Europe had known from al-Khwārizmī and Fibonacci. This kind, indeed, could never have transformed mathematics as a whole. What was the difference? And what had happened to algebra?

NESSELMANN'S CATEGORIES, AN ONLY PARTIAL ANSWER

A first approach to the difference would make use of the three-stage scheme which Nesselmann proposed in his *Algebra der Griechen* [1842: 302]. A "first and lowest" stage in the development of algebra should be that of "rhetorical algebra", which expresses everything in full words.[3] Nesselmann's second stage is "syncopated algebra"; here, standard abbreviations are used for certain recurrent concepts and operations, even though "its exposition remains essentially rhetorical" – that is, the whole exposition can be expanded into full words. The third stage is "symbolic algebra"; here, "all forms and operations that appear are represented in a fully developed language of signs that is completely independent of the oral exposition".

It is obvious that al-Khwārizmī's and Fibonacci's algebras are rhetorical, and no less obvious that Descartes' algebra is symbolic. However, Nesselmann's notion of symbolic algebra is broader than we might at first expect. He does indeed take European mid-17th-century algebra to be symbolic, but also counts the Indian use of schemes to the same category. He shows no examples of this, but we may borrow one from Bhāskara II as transcribed in [Datta & Singh 1962: II, 32]

yâ gha 8	*yâ va* 4	*kâ vayâ.bhâ* 10
yâ gha 4	*yâ va* 0	*kâ vayâ.bhâ* 12

corresponding to our $8x^3+4x^2+10y^2x = 4x^3+0x^2+12y^2x$, which is excellent for reducing the equation and may also be an adequate means to express a resolving algorithm once such an algorithm is known; but it does not allow, for instance, that *yâ* (the first unknown) to the third power be replaced by *P gha*, where *P* is itself a polynomial. In other words, the notation does not allow embedding, the replacement of a simple mathematical object by a different, complex object – *the* essential feature, if any exists, of the change which affected mathematics so thoroughly after 1600.

As we shall see, schematic notations also developed in European (and Maghreb) algebra, but they were eventually abandoned as a main means of expression. On the other hand, elementary embedding began independently of the use of abbreviations. Rather than stages, we should therefore speak of aspects of the expression of algebraic thought, aspects which only to some extent are sequentially ordered.

AL-KHWĀRIZMĪ'S ALGEBRA

Al-Khwārizmī's algebra was purely rhetorical. It dealt with a quantity called *māl* (literally a "possession" or "amount of money", becoming *census* in Latin), its square root (*jidhr*) and *number* (treated as a number of *dirham*, becoming *dragmas* in Latin).

[3] Here and in what follows, all translations into English are mine if nothing else is indicated.

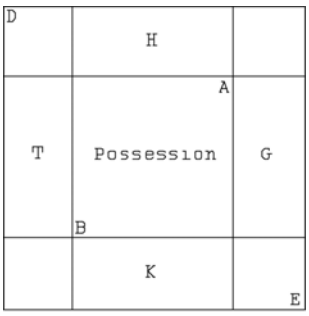
Already in al-Khwārizmī's treatise, however, the *māl* is explained as a number multiplied by itself, and the *jidhr* is identified with the *šay*'/"thing".[4] These terminological complications are traces of a complex prehistory, which does not concern us here, and which can anyhow only be reconstructed hypothetically.

Al-Khwārizmī's algebra proper[5] contains rules for solving reduced first- and second-degree problems ("cases" in what follows), geometric proofs for the correctness of the rules for the mixed cases ("possession and roots are made equal to number" etc.), rules and proofs for the calculation with square roots and binomials, and examples showing how to reduce other problems. The rule for the first mixed case runs as follows:[6]

> But possession and roots that are made equal to a number is as if you say, "A possession and ten roots are made equal to thirty-nine dragmas". The meaning of which is: from which possession, to which is added ten of its roots, is aggregated a total which is thirty-nine? The rule of which is that you halve the roots,[7] which in this question are five. Then multiply them by themselves, and from them 25 are made. To which add thirty-nine, and they will be sixty-four. Whose roots you take, which is eight. Then subtract from it half of the roots, which is five. There thus remain three, which is the root of the possession. And the possession is nine.

In modern symbols: if $y+10\sqrt{y} = 39$ (or $x^2+10x = 39$), then $\sqrt{y} = \sqrt{39+(\frac{10}{2})^2} - \frac{10}{2}$.

Two geometric proofs are given for the correctness of the rule. The first [Hughes 1986: 236*f*] runs as follows:



> A possession and ten roots are made equal to thirty-nine dragmas. Make therefore for it a quadratic surface with unknown sides, which is the possession which we want to know together with its sides. Let the surface be *AB*. But each of its sides is its root. And each of its sides, when multiplied by a number, then the number which is aggregated from that is the number of roots of which each is as the root of this surface. Since it was thus said that there were ten roots with the possession, let us take a fourth of ten, which is two and a half. And let us make for each fourth a surface together with one of the sides of the surface. With the first surface, which is the surface *AB*, there will thus be four equal surfaces, the length of each of which is equal to the root of *AB* and the width two and a half. Which are the surfaces *G*, *H*, *T* and *K*. From the root of a surface with equal and also unknown sides is lacking that which is diminished in the four corners, that is, from each of the corners is lacking the multiplication of two and a half by two and a half. What is needed in numbers for the quadratic surface to be completed is thus four times two and a half multiplied by itself. And from the sum of all this, twenty-five is aggregated. [...].

---

[4] I shall italicize the word "thing" when it is used as an algebraic unknown; below, when discussing the Italian material, also other powers and "number" when occurring as "power 0".

[5] This leaves out the chapters on the rule of three, on geometry and on inheritance calculation. The twelfth-century Latin translations also left out the latter two.

[6] I translate (as literally as possible) from Gherardo of Cremona's Latin translation [ed. Hughes 1986: 234*f*], arguably a better witness of the original text than the extant Arabic manuscripts – see [Høyrup 1998] and [Rashed 2007: 86–89].

[7] That is, the number of roots – in our terms, their coefficient.

In consequence, the argument continues, the area of the completed square *DE* is $39+25 = 64$, and its side 8. Subtracting $2\cdot2\frac{1}{2} = 5 = {}^{10}/_{2}$ we find that the side of *AB* is $8–5 = 3$.

Al-Khwārizmī's illustrates the use of the technique (here the rule for "possession and number is made equal to roots") by this example:

> "Divide ten in two parts, and multiply each of them by itself, and aggregate them. And it amounts to fifty-eight". Whose rule is that you multiply ten minus a *thing* by itself,[8] and hundred and a possession minus twenty *things* results. Then multiply a *thing* by itself, and it will be a possession. Then aggregate them, and they will be one hundred, known, and two possessions minus twenty *things*, which are made equal to fifty-eight. Restore then one hundred and two possessions with the *things* that were taken away, and add them to fifty-eight. And you say: "One hundred, and two possessions, are made equal to fifty-eight and twenty *things*". Reduce it therefore to one possession. You therefore say: "Fifty and a possession are made equal to twenty-nine and ten *things*". Oppose hence by those, which means that you throw twenty-nine out from fifty. There thus remains twenty-one and a possession, which is made equal to ten *things*. Hence halve the roots, and five result. [...].

In symbols (replacing the *thing* by x): Given is $10 = x+(10–x)$ and $(10–x)^2+x^2 = 58$. Therefore, stepwise, $100+x^2–20x+x^2 = 100+2x^2–20x = 58$; $100+2x^2 = 58+20x$; $50+x^2 = 29+10x$; and finally $21+x^2 = 5x$, the reduced equation for which we have a rule. As far as al-Khwārizmī's technique goes, it thus agrees with what *we* would do; but as we see, the composite expression $(10–x)^2$ has to be expanded before it can be inserted into the equation, there is room for no other way to operate with it.[9]

THE BEGINNING OF ABBACUS ALGEBRA

In 1202, with revision in 1228, Leonardo Fibonacci wrote his *Liber abbaci*, which contains a final section on algebra. As I have argued elsewhere [Høyrup 2005], Fibonacci must have known (and drawn part of his material from) an environment somewhat similar to the Abbacus school as we know it from Italy from the later 13th century onward (see imminently), located probably in the Western Islamic region (the Maghreb and Islamic Spain), Catalonia and Provence. However, his *algebra* is quite different from what we find in Italian Abbacus writings and close to al-Khwārizmī in style (though wider in range, being also influenced by Abū Kāmil).

The earliest traces of the Abbacus school turn up in the sources around 1265. It was primarily frequented by merchant and artisan youth for c. two years (around the age of 11), who were taught the mathematics needed for commercial life: calculation with the Hindu-Arabic numerals; the rule of three; how to deal with the complicated metrological and monetary systems; alloying; partnership; simple and composite

[8] The previous example – also of type "divided 10" – has already made the position that one part is represented by a *thing*, whence the other must be 10 minus 1 *thing*.

[9] Al-Khwārizmī thus would have had great troubles to make his reader follow the calculation $(10–x)^2+(10–x)x = (10–x)\cdot(10–x+x) = (10–x)\cdot10 = 100–10x$, so easy when symbols allow us to treat $10–x$ as a simple entity.

discount; the use of a “single false position”; and area computation. Smaller towns might employ a master, in towns like Florence and Venice private Abbacus schools could flourish. In both situations Abbacus masters had to compete, either for communal positions or for the enrolment of students.

Algebra was not part of the school curriculum, but from the early 14th century it turns up (together with other techniques like the “double false position” that were too difficult for normal students) in a number of abbacus texts. Such matters may have been meant for the education of apprentices working also as assistants, but at least algebra functioned as a token of professional aptitude and therefore also enjoyed high prestige.

The earliest extant abbacus book containing a presentation of algebra is Jacopo da Firenze's *Tractatus algorismi*, written in Montpellier in 1307, in Tuscan Italian in spite of its Latin title.

It is *not* derived from Fibonacci's algebra, nor from the “scholarly” level of Arabic algebra – that of al-Khwārizmī, Abū Kāmil, al-Karaji and Ibn al-Bannā’ – but probably from a level integrated with commercial teaching. However, the total absence of Arabicisms shows that the direct source must have been located in a Romance-speaking region – the best guess appears to be a Catalan environment of Abbacus-school type.[10]

Jacopo's algebra is also purely rhetorical, but it differs that of al-Khwārizmī in several ways: whereas the second power is referred to as *censo* (now with all connotations of money forgotten), the first power is never the “root” but invariably the *cosa*/*thing*, and the number term is always spoken of as *numero*, never as an amount of money.[11] Half of the examples (all for the first and second degree) also deal with (varied but invariably sham) commercial problems, which are almost absent from al-Khwārizmī and Fibonacci,[12] and uses the rule of three as a tool in certain algebraic arguments. As if he were conscious of introducing a new field, Jacopo avoids all abbreviations of algebraic core terms (even though non-algebraic words are often abbreviated, as habitual in manuscripts from the epoch).

Al-Khwārizmī only treats problems of the first and second degree. Problems of higher degree turn up in Abū Kāmil and Fibonacci but are not treated systematically. Jacopo instead gives rules for such basic “cases” of the third and fourth degree as are homogeneous or can be reduced to the second degree, forgetting only two biquadratics.[13] Examples accompany rules for the first and second degree only.

---

[10] For this and what follows about the beginning of abbacus algebra, see [Høyrup 2006] or [Høyrup 2007a: 147–182]

[11] Both *roots* and *dragmas* used in this way turn up (together with geometrical proofs) in a few 15th-century abbacus manuscripts of encyclopedic character, whose authors show explicit interest in the founding fathers of the field. But even in their case this pious service is isolated from their own use of algebra.

[12] Actually, they deal with only one type: A given amount of money is divided first among an unknown number (say, x) of persons, and afterwards between $x+N$ persons (N given). The sum of or the difference between the shares in the two cases is also given.

[13] Since al-Karajī, such problems had been solved routinely and systematically in Arabic algebra.

We may look at the first and simplest of the two examples for the first-degree case, "*things* are equal to *number*":[14]

> make two parts of 10 for me, so that when the larger is divided by the smaller, 100 results from it. Do thus, posit that the larger part was a thing. Hence the smaller will be the remainder until 10, which will be 10 less a *thing*. And thus we have made two parts of ten, of which the larger is a *thing*, and the smaller is 10 less a *thing*. Now one shall divide the larger by the smaller, that is, a *thing* by 10 less a *thing*, from which shall result 100. And therefore one shall multiply 100 times 10 less a *thing*. It makes 1000 less 100 *things*, which equal one *thing*. [...].

This draws on the same cognitive resources as al-Khwārizmī's text (without the proofs).

THE IMMEDIATE SUCCESSORS

During the following decades, algebra turns up in a number of abbacus books, sometimes in more or less general expositions, sometimes as isolated problems. The most interesting early exposition is in Paolo Gherardi's abbacus treatise (Montpellier, 1328).[15] Most striking here is the appearance of irreducible third-degree cases, solved by means of false rules – glaringly false indeed for anybody understanding the matter.[16] These false rules survived for more than 200 years (they are still in Bento Fernandes' *Tratado da arte de arismetica* from 1555 [Silva 2006]). They probably served to outdo colleagues in the competition for positions and students; their survival is strong evidence that few abbacus teachers understood much of algebra. Whether Gherardi understood is doubtful; indirect evidence shows that he did not invent the wrong rules.

Less conspicuous but also of importance is the earliest use of a diagram for a formal calculation (missing in the actual manuscript, which is a copy, but described unambiguously in the text)

```
100 \  / 1 cosa
     ><
100 /  \ 1 cosa piu 5.
```

It turns up in a pure-number-version of the problem described in note 12, which we may translate $\frac{100}{x}+\frac{100}{x+5}=20$. It implies an understanding of the operations (cross-multiplication etc.) needed to add the formal fractions[17] $\frac{100}{1\,cosa}$ and $\frac{100}{1\,cosa\ piu\ 5}$.

In a *Trattato dell'alcibra amuchabile* from c. 1365, we find such formal fractions written out repeatedly - for instance, in the same problem, $\frac{100}{by\ a\ thing}\ \frac{100}{by\ a\ thing\ and\ plus\ 5}$ [ed. Simi 1994: 42], explained to be performed "in the mode of a fraction" and explained

[14] [Høyrup 2007a: 304*f*].

[15] The complete text is in [Arrighi 1987], the algebra chapter with English translation in [Van Egmond 1978].

[16] For instance, the case "*cubes* equal to *things* and *number*", solved according to the rule for "*censi* equal to *things* and *number*". For the mathematically thoughtful this should imply that the *cube* is equal to the *censo*, and by division (another rule given by Gherardi) that the *thing* equals 1. Direct easy check was barred by the appearance of radicals in the solution.

[17] These are "formal" in the sense that the form of the fraction is taken not to express an actual broken number but the ratio between algebraic expressions.

in analogy with $\frac{24}{4}+\frac{24}{6}$. We are thus presented with a rudimentary example of *symbolic operation (including embedding) without abbreviation*.[18]

The expression "by a *thing* and plus 5" (*per una cosa e più 5*) is mirrored elsewhere (p. 50) in a similar fraction, where the denominator is "by two *things* and less 6" (*per due cose e meno 6*). They show that the author operated with a notion of additive and subtractive numbers, and that a subtraction is understood as the addition of a subtractive number. We should not identify the subtractive numbers with negative numbers, since they cannot occur as results; but the idea was close at hand (and soon grasped).

We also finds schemes for the multiplications of binomials (consisting of number and irrational root), for instance (p. 18) for (5+√20)·(5–√20):

5 and plus ℜ of 20

times

5 and less ℜ of 20

Sometimes, crossing lines showing the cross-multiplication replace the word "times" (*via*) – or both occur. The same lines are used in earlier abbacus manuscripts when the multiplication of mixed numbers is shown.

The *Trattato dell'alcibra amuchabile* copies Jacopo's algebra verbatim, but also has most of Gherardi's false solutions in a version which appears to predate Gherardi; all of this material must thus go back to before 1330 and hence precede Giovanni di Davizzo's algebra (from 1339, and known only from a fragment included in a manuscript from 1424) and the *Aliabraa argibra*, written by one Dardi of Pisa in 1344.

Though independent of Jacopo, Giovanni gives almost the same rules (and one false rule, almost fully illegible in the manuscript but not one of Gherardi's). However, he also gives correct examples for calculation with square roots and binomials consisting of rational numbers and roots – mostly roots of square numbers, but treated as if the roots were irrational, and not taking advantage of the possibility which this choice offers for checking (edition and translation of the relevant part in [Høyrup 2007c: 479–481]). Even more striking, he teaches the multiplication of powers (which allows us to see how these are labelled) and the division of lower by higher powers.

The powers are composed multiplicatively – the *censo of cube* is the fifth power, the *cube of cube* the sixth, etc. This is wholly traditional, both Diophantos and al-Karajī do the same. In Greek and Arabic, no linguistic problem inheres in this, but the Italian (and corresponding Latin) genitive construction soon became a challenge by suggesting embedding instead of multiplication: the *cube* of 2 is 8, and the *cube* of 8 is 512 – but the *cube of cube* of 2 is 64!

[18] The idea was borrowed from Maghreb mathematics – [Djebbar 2005: 93] shows in facsimile an equation in a manuscript containing a fraction $\frac{48}{\frac{1}{2}\ t}$.

Giovanni did not see the problem, and made worse in his division of lower by higher powers; here the power $-n$ is replaced by the nth root (by *number* if $n = 1$), and even roots are composed multiplicatively ("dividing *number* by *cube of cube* gives *cube root of cube root*", etc.). Giovanni is likely to have invented this system himself – there are no traces of it except in writings which repeat it wholesale; such wholesale repetitions, on the other hand, circulate until Bento Fernandes. We may conclude, firstly, that the ambition to extend the reach of algebra (whether intellectual or career ambition we may leave aside) was not restricted to the production of false solutions; and secondly, that few Abbacus masters had the least need for higher roots, and that most of them therefore did not need to discover the problem. We may also observe that Giovanni's extension, a dead-end as it is, was guided by an intuitive idea that mathematics (but unfortunately the mathematics he already knew about) must be coherent.[19]

Dardi's treatise is at a different level; anybody with some mathematical training who reads it will feel that here a genuine mathematician is speaking. What first strikes one is that he solves 194 cases correctly[20] – a number he reaches by involving radicals (square and cube roots of numbers as well as algebraic powers). He also gives rules for solving four "irregular" cases of the third and fourth degree, rules which only hold under particular circumstances (as he points out), but which may still serve (namely in a competition, we may add). The rules had been guessed (apparently not by Dardi) through a change of unknown in homogeneous problems;[21] deriving them requires a good understanding of the algebra of polynomials (see [Høyrup 2007b: 6*f*]); even Dardi's own elimination of radicals requires good insight.

However, Dardi's work is interesting not only as evidence of level. He uses abbreviations consistently not only for *radice* ("root", which I shall render $\mathfrak{R}$) but also for the *thing* and the *censo* – *c* and *ç*, respectively. At the same time, he uses the fraction model in a way which bars the development of formal calculations – seeing $/_4$ in $^3/_4$ as a *name* ("fourths") and not as an operation, he generalizes and writes, e.g., 10 *things* as $\frac{10}{c}$. In spite of his having schemes similar to those of the *Trattato di alcibra amuchabile*, Dardi's style is thus a good example of *syncopation not pointing toward symbolic calculation*.

Seen under a different angle, his treatise agrees more thoroughly than most abbacus algebras with the idea that mathematics should be built on arguments. He gives geometric proofs, ultimately based on those of al-Khwārizmī but as different from these in details as if he had seen them once and then reconstructed them from memory; he certainly did not copy directly. He uses the rule of three to show how to divide by a binomial ($3+\sqrt{4}$ – Dardi also uses rational roots "as if they were surds", and is indeed the one who uses this phrase; even he takes no advantage of the choice). Finally, he

[19] In the mid-15th-century encyclopedias mentioned in note 11 we find a better system, drawing on formal fractions; they speak of "fraction denominated by *censo*" etc.

[20] Or almost so, cf. [Van Egmond 1983: 417]. One solution asks for a fifth and one for a seventh root. Having no adequate terminology Dardi replaces them by "cube root" and "root of root", respectively, although he understands the embedding of root taking perfectly in other places.

[21] For instance, regarding a capital which grows in three years from 100 £ to 150 £, taking as *thing* not the value of the capital after one year but the rate of interest.

gives an intuitive proof of the sign rule "less times less makes plus", based on the calculation $(10–2)\cdot(10–2) = 10\cdot10–2\cdot10–2\cdot10+(–2)\cdot(–2)$,[22] arguing that it should be 64, which it is indeed if $(–2)\cdot(–2)$ is 4 but not if it is (–4) or 0.

In the end we may take note that Dardi does not like the ambiguous expression "the *censi*" when he wants to refer to their coefficient; instead he speaks of "the quantity of *censi*". This step toward terminological precision is likely to be his own invention; it had no perceptible impact.

ALTERNATIVES TO THE FALSE RULES

I have neither time (in my presentation) nor space (in writing) to discuss more treatises in detail. Instead I shall arrange the discussion according to select themes, beginning with the false rules. A manuscript from the outgoing 14th century [ed. Franci & Pancanti 1988: 98] speaks of the existence of particular roots beyond the square and cube roots, and explains one called "cube root with addition". The "cube root of 44 with addition of 5" is told to be 4, because $4^3 = 44+5\cdot4$ – in general, the "cube root of n with addition α" is t if $t^3 = n+\alpha t$. This is one of the equations provided with a false solution by Gherardi; the inventor of this roo+t thus knew that Gherardi's solution was false, and wanted to do better. The author of the present manuscript is not impressed; he observes that this root mostly does not exist (as an integer). He points out, however, that the cases $t^3+\beta t^2 = m$, $t^3 = \beta t^2+m$ and $\beta t^2 = t^3+m$ can be reduced to the form $t^3 = n+\alpha t$ and thus be solved by means of the same root – showing also that solutions may exist even if n is "a debt", i.e., a negative number. The way he expresses the coefficients of the transformed equations shows that he went through exactly the same change of variable as we would.

The manuscript does not identify the other particular roots, but one of them is probably the "pronic root" which we encounter in a number of sources. If $t^4+t = N$, then some sources (e.g. Pacioli [1494: I, 115^v] identify t^2 as the pronic root, others [e.g. Pierpaolo Muscharello [ed. Chiarini et al 1972: 163]) state it to be t. Benedetto da Firenze [ed. Pieraccini 1983: 26] mentions it in 1463 in connection with the equation $x^2+\sqrt{x} = 18$ but does not make it clear whether x or $t = \sqrt{x}$ should be the pronic root. What he does make clear is that even this root served to "solve" irreducible equations.

Pacioli [1494: I, 150^r] states that so far only equations where the three powers involved are "equidistant" had been correctly solved. He may have known about the solution of other equations by means of these particular roots (he admits that certain other equations can be solved *a tastoni*, "by feeling one's way"), but if so he did not see them as genuine solutions. With hindsight we would say that he was right – but with the proviso that the *transformations* that go together with the "cube root with addition" were exactly those which permitted Cardano to solve cubic equations *in general* after having solved cases with no second-degree term.

[22] "–2" is still to be understood as a subtractive, not a negative number. When repeating the same proof, Luca Pacioli [1494: I, 113^r] instead thinks of genuine negative numbers. He finds them "absurd" *but necessary* – the quest for coherence had enforced expansion of the number concept.

NAMES FOR POWERS AND ROOTS

The contradiction multiplication/embedding in the construction of names for powers and roots was eventually productive, but at first a cause for confusion.

Changes in the terminology for *roots* set in first, perhaps because the problem was most obvious here. Dardi, as mentioned, understood the embedding of roots but stumbled on the ensuing lack of terminology for the fifth and seventh root. Our earliest evidence for the term *radice relata*, "related root" for the fifth root is Antonio de' Mazzinghi, a mathematically brilliant abbacus teacher who probably died rather young in 1385.[23] Later, this became "first related root", the "second related root" being $^{7}\surd$, the "third related root" being $^{11}\surd$, etc.[24] Other roots were then named by embedding. As we see, the system is consistent, but quite unhandy.

The earliest evidence for (ambivalent) naming of *powers* by embedding is in the manuscript which speaks of particular roots, and which starts by presenting the powers until the sixth, including products which remain within this limit [ed. Franci & Pancanti 1988: 3–5]. The author is aware that the powers are in continuous proportion and uses this in his arguments, but apart from that the explanation is confusing (but not necessarily confused) – perhaps because the author is moving on unfamiliar ground. The *thing* multiplied by it self is said to be

> a root which is called a *censo*, so that it is the same to say a *censo* as to say a quantity which has a root, born from a number multiplied by itself, so as it would be to say that if the *thing* produces 4 in number, the *censo* should produce the square of the *thing*, that is, what 4 multiplied by itself makes, that is that the value of the *censo* will be 16, so that, seen that 4 is the root of 16, it therefore comes that the *thing* is said to be the root of the *censo*, so that it is as much to say *censo* as root of number.

The mixing-up of *having* and *being* a root goes through the whole discussion, but the consistently correct numerical examples suggest that the confusion is merely or principally in the words, not in the underlying thinking.[25] The product of a *thing* and a *censo* is called a *cube* (and "a cubic root of a given number"), the product of a *thing* and a *cube* is a "*censo of censo*, which is to say the root of the root of a given quantity"; the explanation of the numerical example suggests that the name is understood through embedding. *Thing* times *censo of censo* is said to be

> *cube of censi*, which is as saying a root born from a square quantity multiplied by a cube quantity [...]; and some call this root related root. So that it would be the same to say *cube of censo* as related root of a given quantity.

[23] [Ulivi 1996: 109–115]. We know his writings through extracts in the encyclopedic works referred to in note 12.

[24] This terminology, though used for a particular purpose, is in [Pacioli 1494]; cf. presently.

[25] The underlying idea *may* be that since a *thing* is also called a *root*, the higher powers must also be "roots" of some kind. If this explanation is correct, we may understand "cubic root of a given number" (etc.) as "cubic «root» *on* a given number".
Jean Peletier [1554: 5], somehow knowing the usage, explains it by speaking of the powers as "*nombres radicaux*, that is, which have in themselves some root to extract".

Here, the thinking is obviously multiplicative. The next step, however, goes by embedding: a *thing* times a *cube of censo* is

> a *censo of cube*, which means as much as saying, taken the root of a quantity, and of this quantity taken its cube root, so that if the *thing* is 3, the *censo* will be 9, the *cube* will be 27, the *censo of censo* will be 81, the *cube of censo* will be 243, the *censo of cube* will be 729, because taken the root of 729 will make 27, whose cube root is 3 and equal to the value of the *thing*.

Then products of powers are discussed – and unfortunately it is said in the end that *cube* times *cube* is "*cube of cube*, that is cube root of cube root".

What looks like a further development of this system is described twice by Pacioli [1494: I, 67^{v}, 143$^{r–v}$]. In the interest of completeness (i.e., describing a system he has not invented and does not use) he gives in parallel the habitual sequence of names (now based on embedding) and the "root names", which are now completely arithmetized. The former are

> *number – thing – censo – cube – censo of censo – first related – censo of cube and also cube of censo – second related – censo of censo of censo – cube of cube – ...*,

ending with the 29th power, the *ninth related*. The corresponding root names are *1st root*, *2nd root*, *... 30th root*. Since *thing*$^{n-1}$ is the *n*th *root*, this arithmetization, while adequate for seeing which terms (in Pacioli's expression) are "equidistant" and thus for reducing equations, e.g., of the type $x^{2+p}+\alpha x^{1+p} = \beta x^{1+p}$, they are less useful for seeing for instance that the type $x^{2p}+\alpha x^{p} = \beta$ is of the second degree in x^p.[26]

A more adequate arithmetization came from the abbreviated writing of equations, mostly occurring in the margins of manuscripts – for instance Vatican, Vat. lat. 3129, written by Pacioli in 1478. Here, abbreviations for powers (*co* for *cosa*, □ alternating with *cen* for *censo*) are written above or as superscript following the coefficient.[27] This graphic distinction allowed first Chuquet (in 1484 [ed. Marre 1880: 632 and *passim*]) and later Bombelli [1572] to replace the abbreviation by the number of the power[28] – Bombelli with an arc below[29] to further emphasize the graphic distinction.[30] Both use the numbers we regard as exponents.[31]

[26] When needing on Fol. 182^{r} the sequence of genuine roots in problems about composite interest (and not reporting what he had found in circulation), Pacioli still uses the multiplicative system for everything except $^5\surd$ – in order, "ℜ" ($\surd$), "cube ℜ" ($^3\surd$), "ℜℜ" ($^4\surd$), "related ℜ" ($^5\surd$), "cube ℜ of cube ℜ" ($^6\surd$), "ℜℜ of cube ℜ" ($^7\surd$), "ℜ of cube ℜ of cube ℜ" ($^8\surd$), etc. One wonders how deep his understanding was.

[27] Even this vertical organization goes back to Maghreb algebra – see, e.g., [Cajori 1928: I, 93*f*] and [Djebbar 2005: 92].

[28] Chuquet's sense of system also lets him designate $^n\surd$ as ℜ.^{n} (even when $n = 2$).

[29] In the manuscript, the exponent is above the coefficient and the arc separates the two – facsimile in [Bombelli 1966: xxxiii].

[30] Tartaglia [1560: 2^{r}] has a table similar to that of Pacioli but with numbering of the *dignitates*/"powers" coinciding with our exponents. He uses the same traditional names (composed with embedding) as Pacioli. However, he is preceded by Stifel [1544: 235^{r}–237^{r} closely followed by Peletier [1554: 8–11], who speak of the numbers as *exponentes*/*exposans*.

[31] Eventually, when combined with Viète's use of letters, this led to the modern notation of variable with exponents.

SCHEMES

The use of abbreviations for frequently recurrent words or endings was common in manuscripts from the period. The abbreviation of *cosa*, *censo*, *radice* etc. thus adopted a tool which already existed.[32] So do the schemes for multiplication of binomials which we encountered in the *Trattato dell'alcibra amuchabile* and in Dardi (which borrow the cross indicating how to multiply mixed numbers) as well as the formal fractions with algebraic expressions in the denominator.

A final development exemplifying the principle of algebraic wine in non-algebraic bottles is the emergence of algebraic calculation within schemes. The manuscript Vatican, Ottobon. lat. 3307, Fol. 331^{r} (c. 1465) contains a problem $\frac{100}{1\rho}+\frac{100}{1\rho+7}=40$ (the formal fractions, without + and =, are already in the text; ρ is used for *thing*). The solution makes use of the transformation $\frac{100\rho+100\cdot(\rho+7)}{(1\rho)\cdot(1\rho+7)}=\frac{100\rho+(100\rho+700)}{1\sigma+7\rho}=40$, whence $200\rho+700=40\sigma+280\rho$ (σ is used for *censo*). In the margin, the solution is summarized as follows:

$$\begin{array}{l} 100\rho \\ \underline{100\rho\ \ 700} \\ \underline{200\rho\ \ 700} \\ \ \ 1\sigma\ \ \underline{7\rho\ \ \ \ \ }\ 40 \\ \\ 200\rho\ \ 700 \text{ ——— } 40\sigma\ \langle 280\rho\rangle \end{array}$$

("280ρ" has been forgotten but stands in the text). This emulates the way non-algebraic items can be added, combined with the fraction notation. The stroke -, seemingly an equation sign, is also used more broadly for confrontations – thus confronting (fol. 338^{r}) the contributions of two business partners. Since Regiomontanus [ed. Curtze 1902: 278] uses exactly the same scheme, it is likely to represent a common procedure.

In the late 14th-century manuscript introducing the cube root with extension we find not only the abbreviations ℜ (*radice*), p (*più*/"plus"), m (*meno*/"less"), ρ (*cosa*) and *c* (*censo*) and Dardi's diagram for the multiplication of binomials but also [ed. Franci & Pancanti 1988: 11] a scheme for multiplying longer polynomials which follows the principles of number multiplication *a chaselle* with vertical columns. Similar schemes are not only found in quite a few later abbacus algebras; they also came to play an important role in Stifel's *Arithmetica integra* [1544: Fol. 123–125 und *passim*], in Scheubel's *Algebrae compendiosa facilísque descriptio* [1551: 3^{v}*ff*], in Peletier's *L'algebre* [1554: 15–22] and in Ramus's *Algebra* [1560: A iii^{r}].

[32] As non-algebraic abbreviations, those for *cosa* and *censo* were rarely used systematically (Dardi being an exception). Only *radice* was used almost consistently in certain manuscripts.
The use of abbreviations may have received inspiration from Maghreb algebra. Here, however, single-letter abbreviations were employed, inside a fully consistent notation. *If* inspired, the Italian writers understood the Maghreb abbreviations within the framework of their own habits.

THE EFFECT OF A CHANGE OF AIR

Fraud and experiments with not immediately reducible higher-degree equations, not too consistent use of abbreviations, and schemes for the calculation with polynomials – this is more or less as far as Italian abbacus algebra went before 1500. Only on the first, somewhat dubious account did it go beyond developments that had already taken place in the Maghreb well before 1250 – developments which some abbacus authors had probably known about directly or indirectly, but which the abbacus environment had to digest before it could make them their own

The experiments with higher-degree equations led to a general breakthrough due to del Ferro, Tartaglia, Cardano and Ferrari in the years 1515–1545. The use of abbreviations and schemes also took root for good in the sixteenth century – beginning however in Germany already in the 15th century,[33] well before it happened in the Italian environment, and also soon to be seen in French writings. In consequence, writings on algebra already look very different from 15th-century Italian predecessors well before Viète.

We may ask why. Book printing *per se* is hardly the explanation – in the manuscript version of Bombelli's *L'algebra*[34], the symbolism for powers and parentheses is different from what we find in the printed edition from [1572], and actually more transparent. Neither is the mere migration to new territories likely to explain much, since the new trends can also be seen in Italy. We may notice, however, that the innovations go together with integration of the abbacus environment with environments more oriented toward university learning – del Ferro was a University professor, Cardano a most learned physician, German algebra was expressed in Latin already in the 15th century.[35] Already Chuquet, in many respects (an unsuccessful) precursor of 16th-century developments, was actually a university scholar, having completed the degrees of the arts as well as medicine in Paris. The Italian abbacus 14th and 15th-century abbacus environment, though governed by norms of precision and coherence at the levels where every abbacus master and any good student could understand what went on [Høyrup 2007b], lacked a social mechanism which could impose intellectual progress on everyone once it had been made (*vide* the survival of the fraudulent rules and Giovanni's nonsensical divisions for more than 200 years). Such mechanisms were not perfect in the 16th century (nor today), but much stronger than in the free-market teaching in Italy in the 14th and 15th centuries. Once Stifel had published his *Arithmetica integra* in [1544], it was obvious to both Peletier (who cites him) and Ramus (who pretends never to have heard about him in [1560] as well as [1569]) to draw on the inspiration he offered. Here, of course, printing was important: it was much easier to have access to the good model; who like Fernandes [Silva 2006] took his inspiration from the manuscripts he could get hold of depended on good or bad luck.

[33] The earliest evidence for fully systematic use of standard abbreviations may be the appendix to Robert of Chester's translation of al-Khwārizmī's *Algebra* [ed. Hughes 1989: 67]. Schemes come later, for instance in Christoff Rudolff's *Coss* [1525].

[34] A facsimile of a representative page is in [Bombelli 1966: xxxiii].

[35] See [Folkert 2006: XII]. In [Høyrup, forthcoming] I argue that the role Folkerts ascribes to Regiomontanus is overstated – Regiomontanus turns out to be very close to Italian models and no more systematic than these.

The 16th-century maturation and stabilization of formal fractions, names for powers and roots organized with embedding (or arithmetized), abbreviations for operations used every time and not just now and then, and schemes – all developments starting in the 14th and 15th century on the basis of existing non-algebraic writing – made possible that freer development of the algebraic language which set in with Viète and Descartes, and in the end reduced the schemes – for a while the most advanced expression of the autonomy of algebra from spoken language – to algorithmic aids or eliminated them altogether from the algebra textbooks.

References

Arrighi, G. (ed.) (1987). Paolo Gherardi, *Opera Mathematica: Libro di ragioni – Liber habaci*. Lucca: Pacini-Fazzi.

Bombelli, R. (1572). *L'Algebra*. Bologna: Giovanni Rossi.

Bombelli, R. (1966). *L'Algebra*. 1. ed. Integrale. Milano: Feltrinelli.

Cajori, F. (1928). *A History of Mathematical Notations*. 2 vols. La Salle, Illinois: Open Court.

Chiarini, Giorgio, et al (Eds.) (1972). [Pierpaolo Muscharello], *Algorismus*. Verona: Banca Commerciale Italiana.

Cuomo, S. (2001). *Ancient Mathematics*. London & New York: Routledge.

Curtze, M. (Ed.) (1902). *Urkunden zur Geschichte der Mathematik im Mittelalter und der Renaissance.* 2 vols. Leipzig: Teubner.

Datta, B. & Singh, A. N. (1962). *History of Hindu Mathematics. A Source Book.* 2 vols. Bombay: Asia Publishing House.

Djebbar, A. (2005). *L'Algèbre Arabe: Genèse d'un Art*. Paris: Vuibert.

Folkerts, M. (2006). *The Development of Mathematics in Medieval Europe: The Arabs, Euclid, Regiomontanus*. Aldershot: Ashgate.

Franci, R. & Pancanti, M. (Eds.) (1988). Anonimo (sec. XIV), *Il Trattato d'Algibra*. Siena: Servizio Editoriale dell'Università di Siena.

Høyrup, J. (1998). "`Oxford' and `Gherardo da Cremona': On the Relation between two versions of al-Khwārizmī's algebra". In *Actes du 3me Colloque Maghrébin sur l'Histoire des Mathématiques Arabes* (vol. II. pp. 159–178). Alger: Association Algérienne d'Histoire des Mathématiques.

Høyrup, J. (2005). "Leonardo Fibonacci and *Abbaco* culture: A Proposal to invert the roles". *Revue d'Histoire des Mathématiques*, *11*, 23–56.

Høyrup, J. (2006). "Jacopo da firenze and the beginning of Italian Vernacular algebra". *Historia Mathematica*, *33*, 4–42.

Høyrup, J. (2007a). *Jacopo da Firenze's Tractatus Algorismi and Early Italian Abbacus Culture*. (Science Networks. Historical Studies, 34). Basel etc.: Birkhäuser.

Høyrup, J. (2007b). "What did the abbacus teachers really do when they (sometimes) ended up doing mathematics?" *Filosofi og Videnskabsteori på Roskilde Universitetscenter.* 3. Række: *Preprints og Reprints* 2007 Nr. 4.

Høyrup, J. (2007c). "Generosity: No doubt, but at times excessive and delusive". *Journal of Indian Philosophy*, *35*, 469–485.

Høyrup, J.forthcoming. [Review af Menso Folkerts 2006]. To appear in *Aestimatio*.

Hughes, B. B. (Ed.) (1986). "Gerard of cremona's translation of al-Khwārizmī's *Al-Jabr*". *Mediaeval Studies*, *48*, 211–263.

Hughes, B. B. (Ed.) (1989). Robert of Chester's latin translation of al-Khwārizmī's *Al-jabr*. Wiesbaden: Franz Steiner, 1989.

Marre, A. (Ed.) (1880). "Le triparty en la science des nombres par maistre Nicolas Chuquet parisien". *Bullettino di Bibliografia e di Storia delle Scienze Matematiche e Fisiche*, *13*, 593-659, 693-814.

Nesselmann, G. H. F. (1842). *Versuch Einer Kritischen Geschichte der Algebra*. I. *Die Algebra der Griechen*. Berlin: G. Reimer.

Pacioli, L. (1494). *Summa de Arithmetica Geometria Proportioni et Proportionalita.* Venezia: Paganino de Paganini.

Peletier, J. (1554). *L'algebre*. Lyon: Ian de Tournes.

Pieraccini, L. (Ed.) (1983). M° Biagio, *Chasi Exenplari all Regola dell'Algibra Nella Trascelta a Cura di M° Benedetto*. Siena: Servizio Editoriale dell'Università di Siena.

[Ramus, P.] (1560). *Algebra*. Paris: Andreas Wechelum.

Ramus, P. (1569). *Scholarum Mathematicarum Libri Unus et Triginta*. Basel: Eusebius Episcopius.

Rashed, R. (Ed., Trans.) (2007). Al-Khwārizmī, *Le Commencement de l'Algèbre*. Paris: Blanchard.

Rudolff, Ch. (1525). *Behend und Hübsch Rechnung Durch die Kunstreichen Regeln Algebra, so Gemeincklich die Coss Genennt Werden*. Straßburg.

Scheubel, J. (1551). *Algebrae Compendiosa Facilísque Descriptio, qua Depromuntur Magna Arithmetices Miracula*. Paris: Gulielmum Cavellat.

Silva, M. do C. (2006). "The algebraic contents of Bento Fernandes' tratado da arte de arismetica (1555)". *Preprint*, Centro de Matematica da Universidade do Porto, April 27, 2006. Forthcoming, *Historia Mathematica*.

Simi, A. (Ed.) (1994). Anonimo (sec. XIV), *Trattato dell'Alcibra Amuchabile*. Siena: Servizio Editoriale dell' Università di Siena.

Stifel, M. (1544). *Arithmetica Integra*. Nürnberg: Petreius.

Tartaglia, N. (1556). *La Seconda Parte del General Trattato di Numeri, et Misure.* Venezia: Curtio Troiano.

Ulivi, E. (1996). "Per una biografia di Antonio Mazzinghi, maestro d'abaco del XIV secolo". *Bollettino di Storia Delle Scienze Matematiche*, *16*, 101-150.

Van Egmond, W. (1978). "The earliest vernacular treatment of algebra: The libro di ragioni of Paolo Gerardi (1328)". *Physis*, *20*, 155-189.

Van Egmond, W. (1983). "The algebra of master Dardi of Pisa". *Historia Mathematica*, *10*, 399-421.

HISTORICAL EPISTEMOLOGICAL ANALYSIS IN MATHEMATICAL EDUCATION: NEGATIVE NUMBERS AND THE NOTHINGNESS

Aurora Gallardo

Centro de Investigación y de Estudios Avanzados

Cognitive processes displayed by students as they solve tasks involved in the transition from arithmetic to algebra are analyzed from an historical epistemological standpoint that has chosen three key texts in the study of negative numbers and the zero. The historical texts are: The Jiu Zhang Suanshu (Nine Chapters on the Mathematical Arts) 1st Century, Le Triparty en la Science des Nombres by Nicolas Chuquet 15th *Century and A Treatise on Algebra by George Peacock, 19th Century.*

David Wheeler (1996) asked:

> Are we sufficiently clear about the ways in which ordinary symbolic algebra differs from the arithmetic of integers?
>
> History gives us the answer to the question: There is no such arithmetic of integers. Algebra, positive numbers, negative numbers and the zero arose simultaneously back in the most remote of times. I will do my best during this presentation to consolidate that statement.

The process of acknowledging negative numbers, as a legitimate mathematical concept, has not evolved continuously. It has rather varied from one culture to another, even demonstrating breaches or breaks and setbacks. At this point it should be noted that negative numbers do not constitute an isolated concept within the heart of mathematics. Instead they arise well beyond the concept of number, at an essential level, becoming a challenge for mathematics proper. Schubring (1988) states that *"negative numbers called into question the fundamental pillars of the philosophy of mathematics. Mathematics was conceived as the science of quantities. Negative numbers implicitly forced us to understand it in a different non empirical manner, because in the outside world no reality could be assigned to such numbers."*

I've carried out research, based on an historical perspective, in order to analyze extending the natural-number domain into integers among students undergoing the transition from arithmetic to algebra, and within the context of word problems.

The historical analysis showed the need to consider mutual interrelationships between the algebraic language and the methods of solving word problems and linear equations, for the understanding of the evolution of negative numbers. Four levels of acceptance of these numbers were extracted from the historical texts. The empirical analysis showed that the first three levels were observed among 35 students of 12-13 years old as well (Gallardo, 2002).

These levels of acceptance are the following:

1. Subtrahend, where the notion of number is subordinated to magnitude (for example, in a – b, a is always greater than b where a and b are natural numbers);
2. Relative or directed number, where the idea of opposite quantities in relation to a quality arises in the discrete domain and the idea of symmetry appears in the continuous domain;
3. Isolated number, that of the result of an operation or as the solution to a problem or equation;
4. Formal negative number, a mathematical notion of negative number within an enlarged concept of number embracing both positive and negative numbers (today's integers).

How was I able to find the evidence in support of this research?

I've based my work on the theoretical elements reported in the long-term work undertaken by Filloy, E. (1990); Filloy, E. and Rojano, T. (1989); and Filloy, E., Rojano, T. and Puig, L (2008) which show how the theory of local models has enabled a deeper study of phenomena in the field of acquiring algebraic language, considering aspects that are relevant to learning, teaching and research. To start with they propose going back to history to study the evolution of algebraic ideas, analyzing historical texts as cognitions in the same in way that they analyze students' productions, which in turn constitute mathematical texts.

This *rapprochement* of historical research and research into mathematics education, allows them to state that the work falls into the category of research in mathematics education, and that it is characterized by recurrent movements between historical texts and school systems. Filloy (1991) introduces two central terms: "mathematical sign systems" (MSS) and "local theoretical models" (LTM). With respect to the latter term, one should note that the local nature is due to the fact that the model is produced to explain the phenomena exhibited in the teaching learning process for certain concrete mathematics content that refer to concrete students; moreover the LTM is only intended to be appropriate for the phenomena observed under those circumstances. Hence the LTM does not exclude the possibility of describing, explaining and predicting those very same phenomena differently, using another model. As regards the MSS, the authors warn of the need to use a sufficiently broad notion of this term. It had to serve as a tool to analize the texts produced by students when they are taught mathematics in school systems, and those texts are conceived as the result of processes of production of sense, as well as to analyze historical mathematical texts, taken as processes of cognition belonging to an episteme.

Besides working with children and teachers in schools, they have also used other sources: semiotics, epistemological analysis (history of mathematical ideas), phenomenological analysis (Freudethal's approach to curriculum development), formal mathematics and cognitive theories, to mention just a few. This semiotics approach emphasizes the pragmatic perspective of meaning in use rather than formal meaning, which has led to focusing attention on the user's performance with the MSS. The claim is that grammar (the formal abstract system) and pragmatics (the principles of language usage) are complementary domains in this work.

It is noteworthy to mention that the concepts of meaning and sense are central in any semiotic treatment of algebraic language. Luria (1995) stated: […] *"if the meaning of a word is the objective reflection of the system of ties and relations, then sense is the contribution of the subjective aspects of the meaning, in correspondence with a given time and situation."*

Also within the perspective of semiotics and historical analysis, the work of Radford (2000, 2004) should not be omitted. In those works, Radford takes an anthropological standpoint in addressing the emergence of algebraic thought, as well as the appearance of algebraic symbolism throughout history.

I have recently continued my research into the matter under discussion here, concentrating on the number zero in the emergence of negative numbers (Gallardo and Hernández, 2006). There cannot be, of course, a negative number without the presence of a zero; however, in Europe, the mathematicians have had the zero since the 14th century. Another line along which I have continued my research has been the construction of number, variable and linear function meanings when students aged 15 to 17 are faced with continuous variation problems (Rubio, G.; Del Castillo, A.; Del Valle, R. and Gallardo, A. (2008). In the latter work, the students produce "intermediate senses in use" for the negative as subtrahend, relative number, isolated number, ordered number and negative parameter, thus broadening the senses that eventually lead to appropriating the meaning of integer. Based on the foregoing study, I have adopted the term "intermediate sense in use" rather than the term "level of acceptance" that arose in Gallardo (2002).

With this backdrop in mind, in this paper I will only be addressing very specific portions of three crucial texts pertaining to my historical analysis.

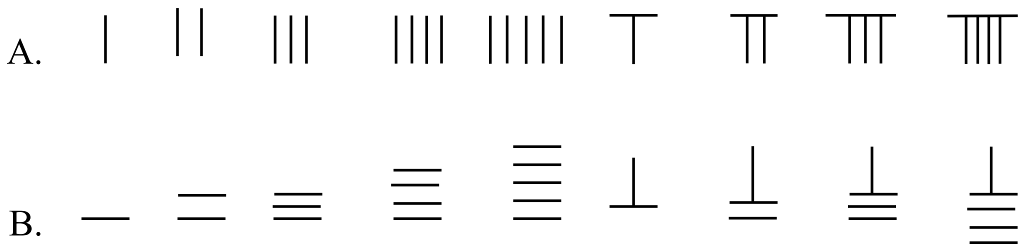
Why did I choose these texts that are centuries apart within the history of ideas?

Because fundamental issues of negative numbers and nothingness are dealt with on the pages of those texts. To wit, the Chinese text –1st century- shows the emergence of such numbers linked to an algebraic method and an MSS, both of which emerged many centuries later in the West.

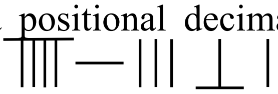
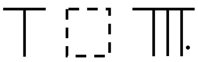
While in the French text, written in the 15th century, word problems are solved and, for the first time, that very process leads to negative and nil solutions expressed in an MSS (syncopated language) by way of an algebraic method that enables verification and interpretation of the solutions. And finally, in the British text from the 19th century the author warns how important it is to consider the possible subtraction for all cases. He uses the current algebraic MSS and reveals the complex syntax of quantities, terms and symbols –where they can all take on positive, negative or nil values- that will later give rise to integers.

THE CHINESE TEXT (1ST CENTURY)

The Fiu Zhang Suanshu (Nine Chapter of the Mathematical Art) (Lay – Yong, L. Se, A. T., 1987), is one of the earliest mathematical books in China. Let us examine the eighth chapter entitled Fang Cheng. Just like the other chapters in the text, the present version of the Fang Cheng chapter contains a number of problems together with their respective solutions. Firstly, we find senses of use of negative numbers, showing that the Chinese were able to apply the negatives in mathematical problems as we would do nowadays. Secondly, the Fang Cheng chapter shows the formulations and solution of simultaneous linear equations of up to five unknowns. Thirdly, the Fang Cheng chapter introduces the methods of solving equations by tabulating the coefficients of unknowns and the absolute term in the forms of a matrix on the counting board, thereby facilitating the elimination of the unknowns, one by one. It must be emphasised that ancient civilisations had no ready made symbolic MSS. Conceptualisations were in a verbalised form, though the Chinese took a forward step when they used rod numerals (concrete mathematical signs) to convert concepts onto the counting board. There are two types of rod numerals as shown below:



The A type numerals is for representing units, hundreds, ten thousands, etc., while the B type is for tens, thousands, etc.

One can see that the Chinese had a positional decimal system. For instance, the number 91 361 was represented as:  and the number 608 as: .

In the latter number representation, the empty place is consistent with the zero in the positional system.

The term Fang Cheng is defined as the arrangement of a series of things in columns for the purpose of mutual verification. The number of columns to be set up is determined by the number of things involved. In modern notation each column has two sections; the top section consists of the quantities a_{ij} (i,j = 1, 2,...n) of the various things while the bottom one shows the absolute terms b_i (i = 1, 2,... n). Such an arrangement on the counting board can be shows as follows:

	Left		Right	
Top	a_{n1}	a_{21}	a_{11}	Thing 1
	a_{n2}	a_{22}	a_{12}	Thing 2
	a_{nn}	a_{2n}	A_{1n}	Thing n
Bottom	b_n	b_2	b_1	Absolute terms

The whole process of operation is done on the counting board using the rod numerals to represent the various quantities. The unique place-value feature of this method of computation renders the use of symbolic MSS unnecessary. In each column of things on the counting board, the space between a_{ij} and b_i, has the implicit function of an equal sign. The former matrix arrangement is transformed in such a way that all numbers in the upper side of the main diagonal are equal to zero (only columns are operated on). This transformed matrix corresponds to a diagonal set of equations, from which all the unknowns are successively determined.

One can see that this method is essentially the usual method in present day algebra. Since the process of the Fang Cheng solution is the successive elimination of numbers through mutual subtraction of columns, there could be cases when a number to be subtracted from one column is smaller than the corresponding one in the other column. The opposite result obtained has to be indicated and certain rules have also to be established in order to continue the eliminating process. This gives rise to the creation of names: the term **fu** to indicate the resulting opposite amount to the term **zheng** for the normal difference.

It is important to point out that the Fang – Cheng method in essence contains the duality of the zero. This is characterized by "the nil zero", that is to say the nothingness considered to be the creation of a void in the representation space (the empty place on the counting board), as well as "the total zero" made up of infinite pairs of opposites that eliminate each other when columns are subtracted in the matrix.

The concepts of **zheng** and **fu** seems to have evolved from such ideas as 'gain' and 'loss' as clearly shown in Problem 8 which reads:

"By selling 2 cows and 5 goats to buy 13 pigs, there is a surplus of 1000 cash. The money obtained from selling 3 cows and 3 pigs is just enough to buy 9 goats. By selling 6 goats and 8 pigs to buy 5 cows, there is a deficit of 600 cash. What is the price of a cow, a goat and a pig?" The text considers the selling price **zheng** because of the money received and the buying price **fu** because of the money spent. The surplus amount is considered **zheng** and the deficit **fu.** These terms are merely names to indicate the nature of numbers. For the purpose of computation, numbers described by these terms have to be transcribed into a concrete mathematical sign system. There are two ways of doing this with rod numerals. If different coloured rods are used,

then red ones represent **zheng** and black ones represent **fu.** Alternatively, if the rods are of one colour only, the **fu** numeral is indicated by an extra rod placed diagonally across its last non-zero digit.

Problem 8 involves selling and buying which equate to the concept of positive and negative respectively. The corresponding set of equations in tabulated form becomes:

	(3rd equation)	(2nd equation)	(1st equation)
(cows)	– 5	3	2
(goats)	6	– 9	5
(pigs)	8	3	–13
	–600	0	1000

One can see that the concept of positive and negative, which initially evolved from opposing entities such as 'gain and loss', 'add and subtract' and 'sell and buy', is now detached from linguistic associations.

Its development has resulted in subtrahends and relative numbers (senses in use of negatives) with properties which are connected with the group of 'normal' or positive numbers. These properties are defined by rules which correspond to the modern ones. Suppose $A > B > 0$, then 'the method of positive and negative' may be represented in modern symbols as follows:

subtraction: $\pm A - (\pm B) = \pm(A - B)$,

$\pm A - (\mp B) = \pm(A + B)$,

$0 - (\pm A) = \mp A$

addition: $\pm A + (\pm B) = \pm (A + B)$,

$\pm A + (\mp B) = \pm (A - B)$,

$0 + (\pm A) = \mp A$

This historical analysis shows the algebraic emergence of the negatives and the zero that arose as intermediate results in the process of solving linear equation systems that model word problems.

The Fang – Cheng method did not spread to the West until the 19th century. It can be found in our textbooks generally known as the method of the triangular form, the name being due to the fact that after the elimination process, the remaining non – zero numbers form a triangle of the matrix.

Whereas the concrete mathematical sign system known as the rod numerals arose in our research literature on mathematics education by way of a teaching model for integers. It is called the "equilibrium model" (Janvier, 1983) and has been widely used until now. The fact that we had lost the purely algebraic historical trail of rod numerals and that they appeared in the West alone and devoid of the Fang – Cheng method has resulted in the inadequate use of the equilibrium model in the field of pure arithmetic. We must recover our long-lost memory and return the algebraic nature to these numbers.

THE FRENCH TEXT (15^{th} CENTURY)

In the appendix of Le Triparty on la science des nombres, (Marre, A., 1881) Chuquet exhibits problems in which negative solutions are accepted and those are interpreted according to the context of the problem. Chuquet introduced a MSS (syconpated language) very similar to modern symbolisation. He writes the numbers with a zero exponent, for example 12 as 12^0, the linear term **x** as 1^1, the square term $\mathbf{x}^2$ as 1^2 and so on. He abandoned any geometric referent that was associated with **x,** $\mathbf{x}^2$ in ancient times. Also he used the symbols **p** for the addition sign and **m** for the subtraction sign.

Introduced as "La règle des premiers" ("doing the same on both sides of the equation"), it enabled him to recognize the senses in use of the negatives, the duality of the zero and to accept the nil solution.

The following is an example of one of Chuquet's problems:

> A merchant has bought 15 pieces of cloth at the price of 160 crowns, one kind of which costs 11 crowns a piece and the other 13 crowns. We must determine the respective quantities of cloth purchased.

Chuquet's solution was very similar to the following in modern notation.

$$x_1 + x_2 = 15$$

$$11x_1 + 13x_2 = 160$$

Take $x = x_1$ as the unknown. Thus $x_2 = 15 - x$, and the second equation becomes

$$11x + 13\,(15 - x) = 160$$

whence

$$x = 17\frac{1}{2}$$

Having obtained $17\frac{1}{2}$ for one unknown he said: *"Now subtract $17\frac{1}{2}$ from 15; there remain $-2\frac{1}{2}$ pieces at the price of 13 crowns a piece".*

After verifying that the second equation is satisfied, Chuquet remarks that such problems are considered impossible. The impossibility (i. e. the occurrence of the negative result) is due, he observes, to the fact that $\frac{160}{15}$ equals to $10\frac{2}{3}$ (crowns) does not fall between 11 and 13 crowns, the given prices.

Chuquet proposes the following interpretation. "The merchant bought $17\frac{1}{2}$ pieces at 11 crowns per piece with cash, thus paying $192\frac{1}{2}$ crown. He then took $2\frac{1}{2}$ pieces at 13 crown per piece on credit to, the amount of $32\frac{1}{2}$ crowns. Thus he has a debt of $32\frac{1}{2}$ crowns, the subtraction (!) of which from $192\frac{1}{2}$ gives 160. Following the same reasoning, Chuquet considers that the $2\frac{1}{2}$ pieces bought on credit must be subtracted from the $17\frac{1}{2}$ pieces purchased, and that the merchant has only 15 pieces which are properly his'.

This problem was posed to 20 students of the research study being discussed in Gallardo (2002).

Following are the methods used by students when solving Chuquet's problem.

Arithmetic Method (used by 15 students). The students look for multiples of 11 and 13 that add up to 160. When the students do not find the multiples needed to solve the problem, that is 11 x 11 + 13 x 3 = 160, they use an additional interpretation to explain their results, for example,

- Student 1. He writes 66 + 91 = 157, and says: "he bought 6 pieces costing 11 coins and he had 3 coins left over".
- Student 2. He writes 154 + 0 = 154, and explains "he bought 14 pieces costing 11 coins each and none costing 13 coins".
- Student 3. He writes 154 + 13 = 167, and says: "he owned 7 coins".

Additive Method (used by one student). The problem of the purchase of goods is modified such that the figures are smaller in order to facilitate solution. The equations which model the problem in this case are: $x + y = 3$; $2x + 3y = 40$. The student respected the number of pieces reported by the problem, but considered 3 prices. He found the new price by means of a subtraction. He wrote, 1x2 + 1x3 = 5; 40 – 5 = 35. He confirmed, "such person bought 3 pieces: 1 for 2 coins, another one for 3 and another for 35".

Sharing Method. (used by one student). This is also found in the modified version ($x + y = 3$; $2x + 3y = 40$). A student divides the total price, 40, by two. The result of the division, 20, is used with the other data of the problem 2, 3, and he formulates the sums: 18 + 2 = 20; 17 + 3 = 20. His answer is 'he bought 18 pieces worth 2 coins and 17 pieces worth 3 coins each'. When the interviewer told him that there are 3 pieces in total, the student clarified: "I thought there were two" and wrote: $40 \div 3$. He added "$40 \div 3$ equals 13, more or less". He wrote again: $13 \div 3$ = 6.5; 6.5+2= 6.7; 6.5 +3= 6.8. "He bought one piece of 6.5, another one of 6.7 and another one of 6.8".

It is important to point out that, contrary to what might be expected, the modified version of the statement (with small numbers) renders the problem impossible for many students. The conflict is accentuated since the solution is sought in the positive domain and the lack of adjustment between the data of the problem is more notorious than in the previous version ($x + y = 15$; $llx+13y=160$) where the magnitude of the numbers tends to hide the conflict. This obstacle disappears when it is suggested to the student that he uses algebra to solve the problem.

Algebraic Method (used by two students). Spontaneous formulation of a system of equations to solve the problem. Let us now look at the case of a student who, by using the process of substitution of the solution in a system of equations, managed to solve the problem which at first he had thought impossible. The student formulates the equations $11x+13y= 160$; $x+y= 15$. He obtains the solution, $x = 17.5$. The following dialogue then ensued:

S: Totally impossible

I: And now, how are you going to find y?

S: It can't be done, totally impossible.

I: Let's try anyway (student finds $y = - 2.5$. Spontaneously he substitutes the values in the equations).

S: It worked!

I: What happened then?

S: Instead of buying, he gave 2 and a half pieces of cloth to the person he was going to sell them to, and bought 17.5 of the other kind. That is, the buyer gave the seller 2 and a half pieces of cloth and the seller sold 17.5 pieces to the buyer. It’s like an exchange.

I: Why did you say before it was impossible?

S: Because it’s impossible with positive numbers.

From the analysis of the problem exhibited in the study, the following was concluded.

- In the solution of this problem, student’s sense in use of negative numbers emerge: subtrahend, relative number and isolated number.
- When faced with a problem with negative solution the student makes changes or adjustments to the data of the problem statement as well as constructing meanings which allows him to give plausible interpretations to the solution obtained.
- A problem which can appear impossible to solve with arithmetic methods, is thought of as possible using algebra, once the negative solution is validated by being substituted in the corresponding equations.

Once again as in the Chinese text, the historical analysis shows that the emergence of the negatives belongs to algebra. From the methods used by the students in Chuquet’s problems, it was concluded that the use of algebraic language becomes essential if a negative solution is to have the possibility of arising. In fact, the extension of the numerical domain of natural numbers to that of integers becomes a crucial element for achieving algebraic competence in solving problems.

BRITISH TEXT (19th CENTURY)

A Treatise on Algebra by George Peacock (1845) is the result of Peacock's desire to draft a text with which his students –while Peacock himself was as a tutor at Trinity College in Cambridge- could make sense of the emergence of the algebraic sign system. His proposal led to the creation of what he called Arithmetical – Algebra, conceived as a bridge between Arithmetic and Symbolic Algebra. For mathematics education, this text is essential because it was written in a gradual and meticulous manner, carefully indicating how one arrives at algebraic syntax beginning with digital arithmetic (our current positive numbers and the zero).

Peacock states that verbal language does not use operation signs, but that written language needs them. He notes the change in the placement of expressions from vertical (Arithmetic) to horizontal (Arithmetical Algebra). When substituting letters for numbers, the letter can appear in the result leaving traces of the operations carried out.

Peacock places emphasis on demonstrating that the subtraction operation that belongs to arithmetic becomes an undefined operation with the emergence of letters. In the expression $a - b$, the condition $a>b$ must be included in order for the familiar subtraction to continue to be valid. This constraint rules all of the proposals raised in Arithmetical Algebra. The author states that once the students have understood this body of knowledge, they can gain access to the Symbolic Algebra that he constructs, which enables $a - b$ to be valid in all cases. In Symbolic Algebra, this fact leads to an acknowledgment of the "senses in use" of the subtrahend, relative number and isolated number, albeit not to extending the concept of number. In other words, in Peacock's work symbols represented as $-a$ do arise, but negative integers do not.

Peacock said that we are perpetually encountering in Arithmetical Algebra examples of operations which cannot be performed or of results which cannot be recognized, consistently with the definitions upon which that science is founded. It is very difficult in innumerable cases, to discover the impossibility of the operation or the inadmissibility of the result, before the operation is performed or the result is obtained. For example, it is required to subtract from **$7a + 5b$**, the several subtrahends **$a + 3b$, $3a - 2b$** and **$3a + 7b$**. We apply the general rule of subtraction which would give us **$7a + 5b - a -3b - 3a +2b - 3a -7b =7a - a - 3a - 3a + 5b - 3b + 2b - 7b = 7b - 10b$**, a result which indicates that the final operation is impossible in Arithmetical Algebra.

Peacock moved from Arithmetical to Symbolical Algebra as follows:

> The assumption ... of the independent existence of the signs + and – ... renders the performance of the operation denoted by – equally possible in all cases: and it is this assumption which effects the separation of Arithmetical and Symbolical Algebra, and which renders it necessary to establish the principles of this science upon a basis of their own: for the assumption in question can result from no process of reasoning from the principles or operations of Arithmetic, and... it must be considered therefore as an

> independent principle, which is suggested as a means of evading a difficulty which results from the application of arithmetical operations to general symbols. (Peacock 1830, viii-ix)

Alternately, later in the same work, Peacock wrote:

> If, however, we generalize the operation denoted by –, so that it may admit of application in all cases, we shall then find the independent existence of this sign which follow as a necessary consequence, and we shall thus introduce a class of quantities, whose existence was never contemplated in Arithmetic or Arithmetical Algebra ... This generalization of the operation denoted by – , is in reality an assumption, inasmuch as it is not a consequence deducible from the operation of subtraction as defined and used in Arithmetic and Arithmetical Algebra. (Peacock 1830, 70-71)

Peacock had therefore introduced the symbol –a into Symbolical Algebra by assumption and without definition.

It is in the transition from Arithmetical to Symbolical Algebra, when the symbols or the conditions of their use, cease to be arithmetical, that the meaning of the operations and the quantities must be determined, not by definition, but interpretation. Because the results of symbolical addition and subtraction are obtained from an assumed rule of operations and not from the definition of the operation itself, it will be necessary to resort to an interpretation of their meaning. For example, the addition of a symbol preceded by a negative sign is equivalent to the subtraction of the same symbol preceded by a positive sign and inversely. Thus

$$\mathbf{a + (-b) = a - b = a - (+b);}$$

$$\mathbf{a - (-b) = a + b = a + (+b)}.$$

It appears, therefore, that in the case of negative symbols, the operation of addition is no longer associated with the fundamental idea of increase, nor that of subtraction with that of decrease. However, numerous are the cases in which negative quantities admit of a consistent interpretation. One first example of the existence of qualities of magnitudes will be in expressing the opposite directions on lines in geometry, and which constitutes one of the most extensive applications of Symbolical Algebra. Another example is the symbolization of property possessed and owed. If a merchant possesses **a** pounds and owes **b** pounds, his substance is therefore **a - b**, when **a** is greater than **b**. But since **a** and **b** may possess every relation of value, we may replace **b** by **a - c** or by **a + c**, according as **a** is greater or less than **b**. In the first case we get

$$\mathbf{a - b = a - (a - c) = c}$$

and in the second,

$$\mathbf{a - b = a - (a + c) = -c}$$

if **c** therefore express his substance or property when solvent, -c will express the amount of his debts when insolvent. And if from the use of + and – as signs of affections or qualities in this case, we pass to their use as signs of operation, then

$\mathbf{a + (-c) = a - c}$

and $\mathbf{a - (-c) = a + c}$.

It will follow, that the addition of a debt **(-c)** is equivalent to the subtraction of a property (**c**) of an equal amount. It consequently appears that the subtraction of a debt, in the language of Symbolical Algebra, is not its obliteration or removal, but the change of its affection or character, from money or property owed to money or property possessed. Peacock added, "the preceding examples of the interpretation of the meaning of negative quantities and the operations to which they are subjected, will be sufficient to show the student that the Symbolical Algebra is not unreal and imaginary", (Peacock, 1845).

The author states that the interpretation of operations must be extended to the equal sign, which connects the primitive expression and the result derived from it. This view of its general meaning will include, as a consequence, arithmetical equality or algebraic equivalence, accordingly as either one or the other may be shown to exist. In mathematics education research, Kieran (1981) is one of the first authors to note that duality of the equal sign.

Once the historical-critical analysis of Peacock's work is concluded, the challenge to be pursued in the educational setting can be to analyze the convenience to "create an Arithmetical Algebra" looking forward to the construction of a transition bridge between Arithmetic and Symbolical Algebra nowdays.

In the three historical texts cited, we have been able to see that the MSSs have characterized the senses in use produced by the authors in the solution of the problems raised. In turn, these facts have contributed to the analysis of the cognitive processes displayed by present-day students as they solved tasks involved in the transition from arithmetic to algebra.

References

Filloy, E. (1990). PME algebra research. A working perspective. *Proceedings of the 14th Annual Conference for the Psychology of Mathematics Education*. Vol. 1. PII 1 – PII 33. México.

Filloy, E. (1991). Cognitive tendencies and abstraction processes in algebra learning. *Proceedings of the 15th Psychology of Mathematics Education*, Vol. 2, pp. 48 – 55. Italy.

Filloy, E. & Rojano, T. (1989). Solving equations: The transition from arithmetic to Algebra. *For the Learning of Mathematics* 9(2), 19 – 25, Canada.

Filloy, E., Rojano, T., & Puig, L. (2008). Educational algebra. A theoretical and empirical approach. *Mathematics Education Library*. Vol. 43. Springer.

Gallardo, A. (2002). The extension of the natural-number domain to the integers in the transition from arithmetic to algebra. *Educational Studies in Mathematics,* 49, 171-192. Kluwer Academic Publishers.

Gallardo, A. & Hernández, A. (2006). The zero and negativity among secondary school students. *Annual Meeting International Group for the Psychology of Mathematics Education* (PME-30), pp. 153-160. Prague, Czech Republic.

Janvier, C. (1983). The understanding of directed numbers. *Proceedings of the 15th Annual Conference of the North American Chapter of PME*, 295 – 300 Montreal.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, 12, 317 – 326.

Lay-Yong, L. & Se, A. T. (1987). The earliest negative numbers: how they emerged from a solution of simultaneous linear equations. *Archives Internacionales d'Histoire des Sciences,* 37, 222 – 269.

Luria, A. R. (1995). *Conciencia y Lenguaje. Aprendizaje Visor.* España.

Marre, A. (1881). *Appendice au Triparty en la Science des Nombres de Nicolas Chuquet Parisien*. 14, 413 – 460.

Peacock, G. (1830). *A Treatise on Algebra*. London.

Peacock, G. (1845). A treatise on algebra. Second version. *Reprinted from the 1842 eddition. Scripta Mathematica*. New York, 1940.

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis. *Educational Studies in Mathematics,* 42(3), 237 – 268.

Radford, L. (2004). *The Cultural – Epistemological Conditions of the Emergence of Algebraic Symbolism*. Paper presented at the 2004 History and Pedagogy of Mathematics Conference, Uppsolo, Sweden.

Rubio, G., Del Valle, R., Del Castillo, A., & Gallardo, A. (2007). Producción de sentidos para los objetos algebraicos de número, variable, y función al resolver problemas de variación continúa. Evidencias empíricas sobre nuevos sentidos de uso del número negativo. *Investigación en Educación Matemática XI*. SEIEM 2007, pp. 237-247. La Laguna, Tenerife, España.

Schubring, G. (1988). Discussions Epistémologiques sur le Statut des nombres Négatifs et leur Représentation dans les Manuels Allemands et Français de Mathématique entre 1795 et 1845. Actes du premier colloque franco-allemand de didáctique des mathématiques et de l'informatique. Editions La Pensée Sauvage.

Wheeler, D. (1996). Rough or Smooth? The transition from arithmetic to algebra in problem solving. *Approaches to Algebra*. 147 – 149. Kluwer Academic Publishers. Printed in The Netherlands.

CONCEPTUAL ANALYSIS OF MATHEMATICAL IDEAS: SOME SPADEWORK AT THE FOUNDATION OF MATHEMATICS EDUCATION[1]

Patrick W. Thompson

Arizona State University

Mathematics during the late 18th century through the early 20th century experienced a period of turmoil and renewal that was rooted in a variety of attempts to put mathematics on solid conceptual footing. Taken-for-granted meanings of concept after concept, from number to function to system, came under increasing scrutiny because they could not carry the weight of new ways of thinking. In a very real sense, that period of time can be characterized as mathematicians' search for broad, encompassing coherence among foundational mathematical meanings. Part of the resolution of this quest was the realization that meanings can be designed. We can decide what an idea will mean according to how well it coheres with other meanings to which we have also committed, and we can adjust meanings systematically to produce the desired coherence. Mathematics education is in the early stages of a similar period. Competing curricula and standards can be seen as expressions of competing systems of meanings--but the meanings themselves remain tacit and therefore competing systems of meanings cannot be compared objectively. I propose a method by which mathematics educators can make tacit meanings explicit and thereby address problems of instruction and curricula in a new light.

My apologies to non-U.S. readers of this article. What I say here is focused very much on problems that exist in the United States. My only excuse is that the problems are so great in U.S. mathematics education that vetting some of them publicly might provide useful insights for others to avoid similar problems elsewhere.

With that said, I start with three observations. The first is that students' mathematical learning is the reason our profession exists. Everything we do as mathematics educators is, directly or indirectly, to improve the learning attained by anyone who studies mathematics. Our efforts to improve curricula and instruction, our efforts to improve teacher education, our efforts to improve in-service professional development are all done with the aim that students learn a mathematics worth knowing, learn it well, and experience value in what they learn. So, in the final analysis, the value of our contributions derives from how they feed into a system for improving and sustaining students' high quality mathematical learning.

The second observation is that, in the United States, the vast majority of school students rarely experience a significant mathematical idea and certainly rarely experience reasoning with ideas (Stigler, Gonzales, Kawanaka, Knoll, & Serrano,

[1] Preparation of this paper was supported by National Science Foundation Grant No. EHR-0353470. Any conclusions or recommendations stated here are those of the author and do not necessarily reflect official positions of NSF.

1999; Stigler & Hiebert, 1999). Their experience of mathematics is of procedure after procedure. This is not to say that various curricula do not aim for students to learn ideas. Rather, students do not experience any that are significant. By "significant" I mean ideas that carry through an instructional sequence, that are foundational for learning other ideas, and that play into a network of ideas that does significant work in students' reasoning. Base ten numeration is a significant mathematical idea. If students have learned it well then, without being taught a procedure to do so, they can reason their way to answering the question, "How many hundreds are in 35821?" They can reason that there are 358 hundreds in 35821 because 5 thousands contain 50 hundreds, and 3 ten thousands contain 30 thousands, and therefore contain 300 hundreds. Or, they can reason that there are 358.21 hundreds in 35281 because 2 tens make two tenths of one hundred, and 1 one makes one hundredth of one hundred. This type of reasoning is rare in U. S. schools because students are not expected to develop mathematical meanings and they are not expected to use meanings in their thinking.

The third observation is that too many mathematics teachers at all levels spend too little time at the outset of teaching a topic on having students become steeped in ideas and meanings that are foundational to it. As Deborah Ball said at a recent conference, mathematicians and mathematics teachers are too eager to condense rich reasoning into translucent symbolism. They are too eager to get on to the "meat" of the topic, namely methods for answering particular types of questions.

COHERENCE AND MEANING

The issue of coherence is always present in any discussion of ideas. Ideas entail meanings, meanings overlap, and incoherence in meanings quickly reveals itself. Thus, my talk will be about issues of coherence, incoherence, and meaning as much as it will be about mathematical ideas and analyses of them.

The word "coherence" and its derivates occurs with increasing regularity in mathematics education publications. The 1989 *Curriculum and Evaluation Standards* (NCTM, 1989) raised the issue only 8 times in 364 pages. The 2000 *Principles and Standards* (NCTM, 2000) raised issues of coherence 39 times in 402 pages. The 2006 *Curriculum Focal Points* raised it 16 times in 40 pages. The final report issued by the *National Mathematics Advisory Panel* (2008) raised it 19 times in 91 pages. One would think that with the increasing emphasis on curricular coherence, everyone would be clear on how to think about it. This is, unfortunately, not the case. Of all these documents, only the NMAP final report defines coherence, and only in regard to curricula.

> By the term *coherent*, the Panel means that the curriculum is marked by effective, logical progressions from earlier, less sophisticated topics into later, more sophisticated ones. (National Mathematics Advisory Panel, 2008, p. xvii)

I must confess my disappointment in the Math Panel's definition of curricular coherence. Effectiveness might be a consequence of coherence, but it cannot define it.

Nor does coherence imply logical progression of topics, at least from a mathematical point of view (Thompson, 1995). Despite these, my disappointment might have been less had their definition been followed by examples of what the panel accepts as coherent curricula. But there were no examples. I suspect that even with examples my disappointment would have been the same, because the panel defined curricular in terms of topics, not in terms of ideas. Ultimately, coherence of a curriculum (intended, implemented, or experienced) depends upon the fit of meanings developed in it. Schmidt's (2002) example of Hong Kong's curriculum displays this very characteristic—the example, drawn from the topic of ratio, rate, and proportion, highlights the development of meanings of each and the construction of contextual inter-relationships among them. The lack of attention to meaning, I believe, is at the root of many problems that become visible only later in students' learning.

Unfortunately, to declare a shortage of ideas and meanings in mathematics teaching and curricula is not the same as saying what having them is like. The National Mathematics Advisory Panel final report calls repeatedly for balanced emphases upon conceptual understanding and procedural fluency. But it does not explain what conceptual understanding is or how one might teach for it. NCTM's Principles and Standards for School Mathematics repeatedly extols the community to have students understand the mathematics they learn so that they are better prepared to understand ideas they encounter in the future. But in not one of 857 instances in which the Principles and Standards uses "understand" and its derivatives does it explain what "to understand X" means. "Understand" is one of mathematics education's most primitive terms.

To make this point, that inattention to meaning is at the root of many problems of students' learning, I will develop three examples. The first will draw from trigonometry, the second from linear functions, and the third from exponential functions.

TRIGONOMETRY

Trigonometry is a notoriously difficult topic for U.S. middle-school and secondary-school students. I claim that the roots of students' difficulties lie in an incoherence of foundational meanings developed in grades 5 through 10. The U.S. mathematics curriculum develops two unrelated trigonometries: the trigonometry of triangles and the trigonometry of periodic functions.

In the trigonometry of triangles, students are taught SOH-CAH-TOA, which stands for Sine is Opposite over Hypotenuse, Cosine is Adjacent over Hypotenuse, and Tangent is Opposite over Adjacent. They are also taught that some triangles are special, meaning that you know their angles and their relative side lengths. The triangles are 30-60-90 degree triangles, 45-45-90 degree triangles, etc. They then are given many exercises in which they solve for the length of some missing side.

The idea of angle measure is hardly present in the standard U.S. development of triangle trigonometry. By this I do not mean references to an angle's number of degrees. Such references abound. In students' understanding sine, cosine, and tangent do not take angle measures as their arguments. Rather, they take triangles as their arguments. A question might mention a specific angle measure, but the angle having that measure is always in a triangle and the argument to the trig function involves the whole triangle. Also, angles do not vary in triangle trigonometry. In fact, students find it problematic just to imagine how an angle in a triangle might vary (Figure 1). Clearly, students who enter their study of trigonometric functions with the image of triangle variation depicted in Figure 1 will have a difficult time thinking, in any way we would find acceptable, of variable angle measures in relation to sine, cosine, and tangent.

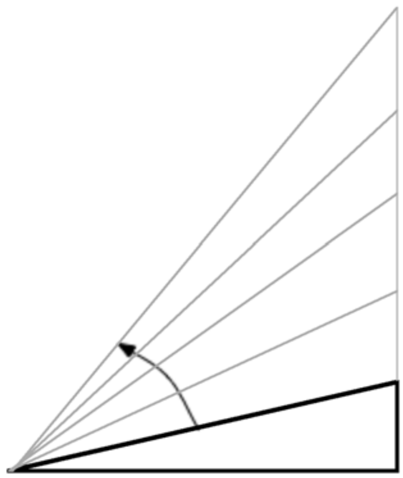


Figure 1. A common way for students to imagine varying an angle in a triangle.

Angle measure is problematic in another way. It has no clear meaning in students' thinking. This is understandable upon examining particular practices. Figure 2 shows how angle measures are often presented. The arc in the left diagram serves no other purpose than to indicate the angle upon which the text wants students to focus. It is nothing more than a pointer, which could be accomplished just as well with a different indicator (e.g., as in the right diagram of Figure 2). The right diagram highlights the shallow meaning that we unwittingly convey to students about what we are measuring when we measure an angle. "Whatever 37° means, that's what this angle is!" As an aside, textbooks say, for example, "sin A" to indicate the ratio they want students to think of, even though "A" is not a number. It is the angle's name. This is what I meant when students understand trig ratios as taking triangles as their arguments instead of numbers.

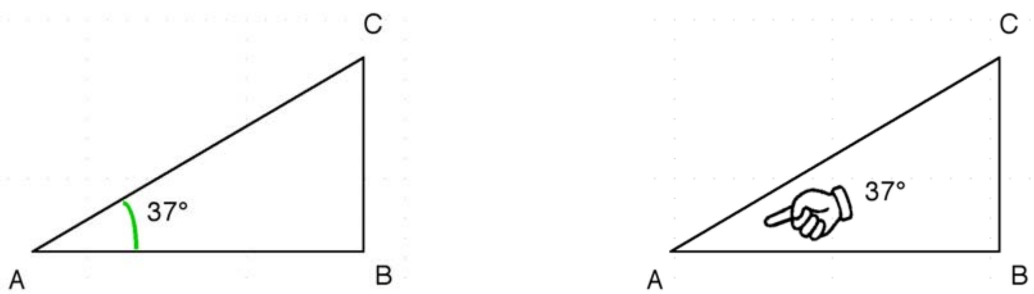


Figure 2. Conventional arc indicates the angle
to which a number of degrees is attached.

This lack of clear meaning for angle measure continues from early grades through calculus. Figure 3 shows two diagrams from a popular U.S. calculus book in its development of radian measure (a book known for its "conceptual" approach to calculus). In these diagrams, "θ" is used just as is "A" in Figure 2—to name the angle. The grammar in Figure 3's right diagram even suggests this. The phrase "… spanned by θ" makes sense only if θ is the name of the angle. It is not a number. Clearly, this is a holdover of habits established in the authors' history with triangle trigonometry.

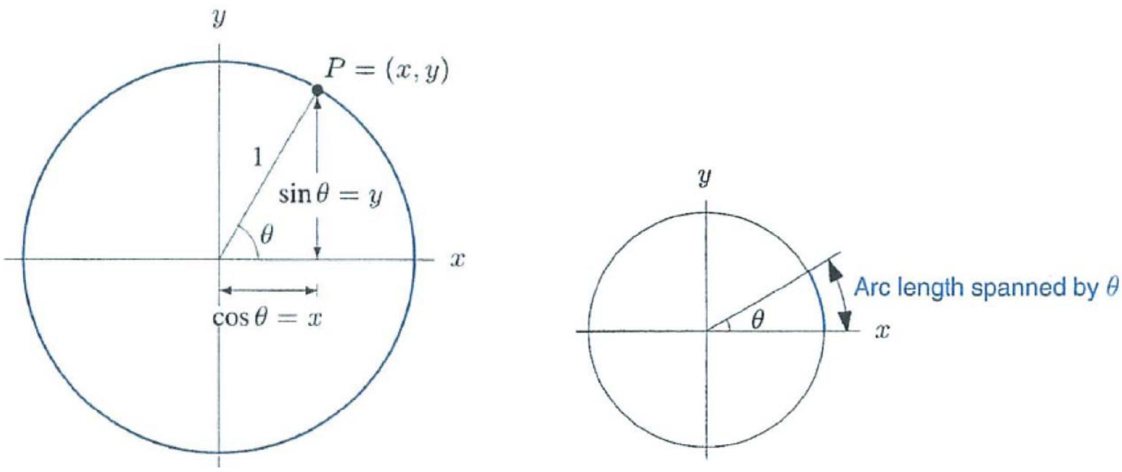


Figure 3. Figures from a popular calculus book in which θ is the name of the angle instead of representing a measure of it.

Using a letter to name an angle instead of representing its measure is embedded deeply in the school mathematics culture. In a recent professional development project, a facilitator to a school-based professional learning community suggested, in the context of the group developing a unit on trigonometric functions, that teachers ask their students to use a string to "estimate the sine of 1". The teachers looked at each other, then one asked, "What do you mean … 'the sign of 1'? That doesn't make sense." Then another teacher interjected, "Oh, you mean estimate the sine of θ where θ has a measure of 1 radian!" The facilitator asked what the difference was. The teachers responded that the second was much clearer. Videos of their implementation of this unit confirmed that they were thinking of θ as naming the angle, not representing its measure.2

Calculus texts' treatments of radian measure have the intention of measuring an angle's "open-ness" by measuring the length of the arc that the angle subtends in a circle centered at the angle's vertex. The reason that 1 radius is used as the unit of measure is so that the circle's size does not affect the angle's measure. In fact, we do not need to use a radius as our unit. We could satisfy this constraint (the circle's size cannot matter) by using any unit that is proportional to the circle's circumference (Thompson, Carlson, & Silverman, 2007). The reason that we use a radius as the unit

[2] It would be more accurate to say that they could think about it both ways, but only in the sense that θ could represent *both* the angle and its measure.

of arc length by which we measure angles is that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ when θ is a number of radii, whereas $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = \frac{\pi}{180}$ when θ is a number of degrees.

Angle measure in earlier grades, when developed at all, is taught as a fraction of a complete rotation converted into an equivalent fraction of 360. To be a measure, thought, we must say what about an angle we are measuring and the method by which we derive a measure of it. However, angle measure in degrees is taught as a procedure, it is not really taught as a measure. We need to develop angle measure in degrees so that it is a measure of something, and so that as a measure it coheres with radian measure. One way to do this is to base the idea of a degree also on arc length. One degree would then be an arc of a circle whose length is 1/360 the circle's circumference. The property being measured is the angle's "open-ness". The method of measuring that open-ness would be to draw a circle centered at the angle's vertex and measure the arc that the angle subtends in units of arc that are 1/360 the circle's circumference. In this way, degree measure and radian measure are exactly the same type of thing-a measure of subtended arc.

It is worth mentioning that I just outlined the principle by which a protractor works. Figure 4 illustrates this. Of course, for students to "see" an indicator arc as depicted in Figure 4, they must be taught, and must learn, the scheme of meanings behind it. Moreover, they must practice reasoning with these meanings so that those meanings become, indeed, their way of seeing measured angles. To develop relationships between degree and radian measure, they must first understand the conventionality of both, just as we expect them to understand that the relationship between measuring temperature in Fahrenheit and Celsius is between measuring the difference between freezing and boiling temperatures of water in 180 segments or 100 segments. The measured thing (an amount of arc) is the same either way; its magnitude is simply cut into different numbers of segments according to the system we happen to use (Figure 5).

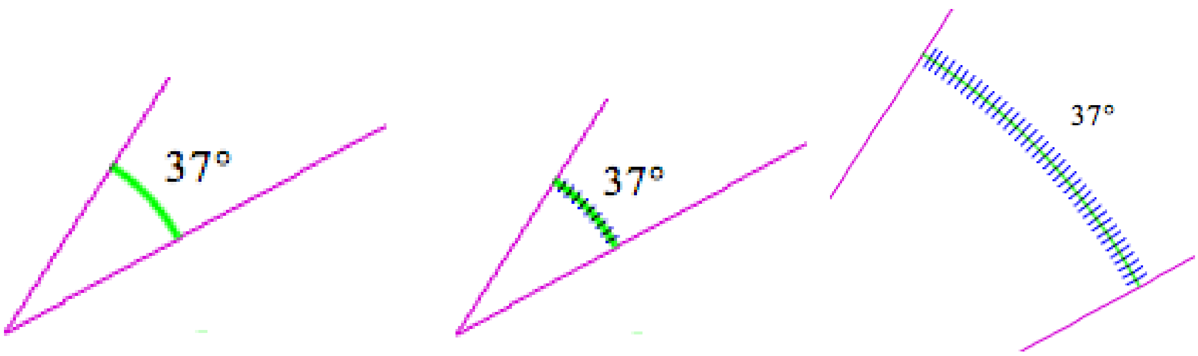


Figure 4. Seeing the "indicator arc" as a subtended arc of a circle centered at the angle's vertex, measured in a unit of arc whose length is 1/360 the circle's circumference.

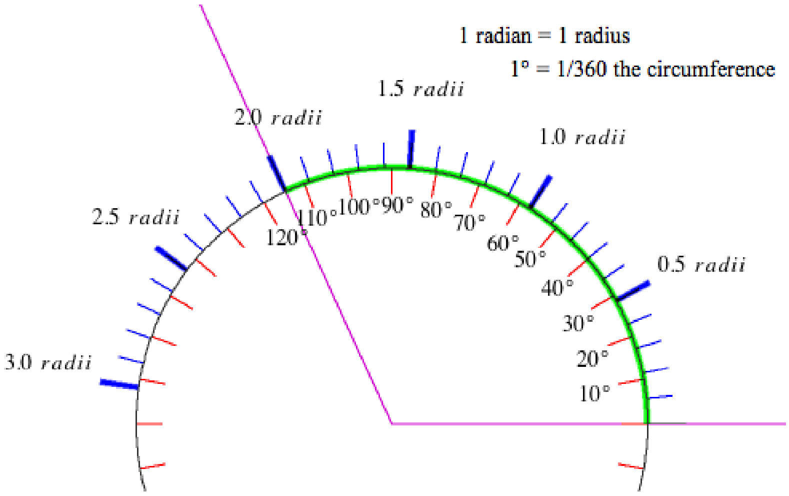


Figure 5. An angle measured simultaneously in degrees and in radians.

Curricular treatments of triangle trigonometry and of periodic functions that are coherent both within themselves and between the two developments would draw from meanings specific to each and from meanings that are common to each. Triangle trigonometry would draw from the meaning of angle measure as outlined above and would also draw from similarity –that similar triangles have the same ratios. Thus, to know the ratios between sides of one triangle will give the ratios of corresponding sides of all similar triangles. Periodic functions would draw from the meaning of angle measure as outlined here, from similarity, and would additionally highlight how one must think of varying an angle so as to systematically vary its measure. The rest is details.

I should point out that this discussion of meanings for angle measure highlights the important consideration that meanings students create at the time of learning something are highly consequential for their later learning that depends on it (Wearne & Hiebert, 1994). Students who early on learn that sinA means SOH will be at a severe disadvantage when needing to think that trig functions take angle measures as arguments. Students who learn early on that angle measures are indexical-that, for instance “90° means perpendicular” will be a severe disadvantage when angle measure needs to be thought of as a continuous variable.

LINEAR FUNCTIONS (CONSTANT RATE OF CHANGE)

The idea of constant rate of change is foundational to understanding linear functions and subordinate ideas such as average rate of change, proportionality, and slope. However, as Lobato has shown, many students do not see ideas of rate of change, average rate of change, proportionality, and slope as being interconnected (Lobato, 2006; Lobato & Siebert, 2002; Lobato & Thanheiser, 2002). They see them as separate sets of procedures and see them as associated with unrelated contexts. Many teachers have similar disconnections. Coe (2007) modeled three high school algebra and calculus teachers’ meanings using semantic maps gained from interviews over six months. He found that all three had few connections between their meaning of slope and their meaning of constant rate of change, and they had effectively no connections among their meanings of constant rate of change, average rate of change, instantaneous rate of change, slope, and proportionality.

The problem with students and teachers' understanding of the foundations of linear functions is deeper than not having connections among meanings. The meanings they have of constant rate of change cannot provide those connections. Hackworth (1995) studied the effect of first semester calculus on calculus students' understandings of rate of change. She found, using a test-retest method, that the vast majority of 90 students (enrolled in multiple sections) had very weak understandings of rate of change at the beginning of the course and even weaker understandings at the end of the course. Average rate of change meant the arithmetic mean of the rates. Constant rate of change meant that the instantaneous rate of change did not change. But instantaneous rate of change was the number that a speedometer is pointing at were you to freeze time. In other words, students understanding of constant rate of change rarely involved two quantities changing, and certainly did not involve them changing in such a way that all changes in the value of one quantity, no matter how small or large, are proportional to corresponding changes in the value of the other.

In conversations with teachers and future teachers in many convenience samples (i.e.., students and teachers with whom I've worked), I ask them to explain the idea of average speed as if to someone who did not already understand it. The most common answer by far is, of course, "distance divided by time". I point out that this might calculate a value for an average speed, but it is not the meaning of average speed. Eventually, often with considerable support from me, they come to the meaning of average speed as entailing these aspects:

- It involves a complete trip or the anticipation of a complete trip (i.e., having a start and an end).
- The trip takes or will take a path which involves moving a definite distance in a definite amount of time.
- The average speed for that trip is the constant speed at which someone must travel to cover the same distance in the same amount of time.

But this does not answer the question, "why divide the number of distance units by the number of time units to calculate an average speed?" To answer this question requires that we have a powerful meaning for constant speed.

One meaning of constant speed is that all amounts of distance (say, number of feet) traveled in any amount of time (say, number of seconds) are proportional to the number of seconds in which you traveled that distance. To travel at a constant speed of 88 ft/sec means that in any period of 1/1000 second you will have traveled 1/1000 of 88 feet. It means that in any period of 1/107 seconds you will have traveled 1/107 of 88 feet. Therefore, to say that you traveled d feet in t seconds at a constant speed means that in any one second (1/t of t seconds) you will have traveled 1/t of the time in which you traveled d feet, and therefore you will have traveled (1/t) of d feet in one second.[3] But (1/t) of d feet is the same number of feet as d ÷ t. Therefore, the constant speed, in feet per second, at which one must travel to move d feet in t seconds is calculated by d ÷ t. That is, you divide distance by time because of the proportional relationship between distance traveled and time taken to travel that distance when traveling at a constant speed.

When students understand the ideas of average rate of change and constant rate of change with the meanings described here they see immediately the relationships among average rate of change, constant rate of change, slope, secant to a graph, tangent to a graph, and the derivative of a function. They are related by virtue of their common reliance on meanings of average rate of change and constant rate of change.

Another way to view the relationship between quantities that change together at a constant rate is that there is a homogeneous relationship between the two. Kaput and West (1994) noticed this in their investigation of students' understanding of rate of change. They noted that to understand constant rate of change entails the same mental operations as understanding uniform concentration or uniform density. Harel (1994) saw the same relationship as students came to conceptualize constancy of taste with regard to anticipating differences in "oranginess" of different sized sips of a mixture in which orange pulp and water are mixed thoroughly. The idea of uniform concentration is that if you mix m units of substance A thoroughly with n units of substance B, then to say they form a uniform concentration means that any sample of the mixture will contain the two substances in the same proportion as any other sample (including the entire mixture). Similarly, if a substance has uniform density, then any part of that substance will have its volume and mass in the same proportion as any other part (including the entire amount).

This way of thinking about constant rate of change, that corresponding changes in two quantities are homogeneous, supports thinking about continuous variation of one quantity and concomitant continuous change in the other. If one quantity changes by some extremely small amount, then the other must change accordingly in the same proportion. Thus, if we work with students so that they develop a rich meaning of constant rate of change, we will at the same time support them in coming to conceptualize functional relationships as entailing continuous variation and as entailing a relationship between values that remains the same even as the values themselves vary. Boyer (1946) anticipated this when he outlined an approach to functions that starts with proportional reasoning. Piaget and colleagues (Piaget, Blaise-Grize, Szeminska, & Bang, 1977), thought uninterested in issues of teaching or curriculum, also saw proportionality at the root of ideas of function relationship. In a recent teaching experiment (Thompson, McClain, Castillo-Garsow, Lima, in preparation), a teacher who based her Algebra I instruction on ideas of constant rate of change and continuous variation led her students to think with remarkable sophistication about the behaviors of linear, quadratic, and polynomial functions and their analytic properties.

[3] If you travel 288 feet in 7 seconds at a constant speed, then in any 1 second (1/7 of 7 seconds) you will travel 1/7 of 288 feet.

In regard to the homogeneity it is worth mentioning that homogeneity is not characteristic of Confrey's notion of constant rate of change. She characterizes constant rate of change as a unit-per-unit comparison (Confrey, 1994; Confrey & Smith, 1994, 1995) and includes comparing successive values of a function (when there are such things) as the units being compared. I will elaborate upon this in my discussion of exponential functions, but it is worth mentioning now that her notion of constant rate of change, which she devised largely so that she can say that exponential functions have a "multiplicative" constant rate of change-evaluated by $f(x+\Delta x)/f(x)$.

The value of $f(x+\Delta x)/f(x)$ for any exponential function is dependent upon the size of Δx. If the underlying function is f(x) = 2x, then 2 is its constant multiplicative rate of change when $\Delta x = 1$, $\sqrt{2}$ is its constant multiplicative rate of change when $\Delta x = 0.5$, and $\sqrt[10]{2}$ is its constant multiplicative rate of change when $\Delta x = 0.1$. Different values of Δx produce a different constant rate of change for the same underlying function. Thus, constant multiplicative rate of change for exponential functions is not homogenous in Confrey's scheme. The same exponential function has different constant multiplicative rates of change depending on the granularity with which you examine changes.

Confrey's notion of rate of change has another consequence that has not been mentioned. Homogenous rate of change entails the characteristic that constant rate of change entails change and accumulation simultaneously (Thompson, 1994a; Thompson & Silverman, 2008). This was the foundation of Newton's approach to calculus and is the root idea of the Fundamental Theorem of Calculus. However, Confrey's notion of rate of change entails only the idea of change, it does not entail the idea of accumulation. As such, Confrey's system has no Fundamental Theorem of Calculus. There are no Taylor series in Confrey's system. There is no way to systematically calculate an approximate value of 2x for non-integral values of x.4

Finally, continuous variation in Confrey's way of thinking about multiplicative change is very hard to imagine. If the underlying idea is that all multiplicative change happens by a split, then I do not know how to imagine the value of 2x varying smoothly as I smoothly vary the value of x. Again, I'll return to this under exponential functions.

EXPONENTIAL FUNCTIONS

All teachers of calculus know that a defining characteristic of exponential functions is that the rate at which an exponential function changes with respect to its argument is proportional to the value of the function at that argument. A natural question is how to have this property emerge meaningfully in students' thinking. A well known approach to developing ideas of exponential function is by developing the idea of

[4] Put another way, there are no calculators in a splitting world.

splitting (Confrey, 1994; Confrey & Smith, 1995; Smith & Confrey, 1994), where growth happens by a constant multiplier (e.g., in a geometric sequence). But the idea that an exponential function's rate of change is proportional to the value of the function does not arise easily in the splitting approach, especially if we hope to develop this idea in the context of continuous variation. I propose another approach: Start with simple interest.

The amount of money in a deposit account earning simple interest and starting with P dollars grows at a constant rate with respect to time. If the interest rate is 8% per year, then the value of the account after x years is v(x) = P + 0.08Px dollars. The formula P + 0.08Px makes it clear that the account's value grows at a rate that is proportional to the initial value of the deposit.

If the bank compounds interest at the end of each year, the conventional practice, and the conventional way of thinking about growth in the account's value, is that the bank adds earned interest only at the end of each compounding period, in this case at the end of each year. By this method, the account's value over time is given by the formula

$$v(x) = P(1.08)^{\lfloor x \rfloor}, 0 \leq x \tag{0.1}$$

where "$\lfloor x \rfloor$" means "floor x", or the greatest integer less than or equal to x. Thus, after 2.3 years the account's value will be $v(2.3) = P(1.08)^{\lfloor 2.3 \rfloor}$, or P(1.08)2. That is, for every value of x between 2 and 3 (meaning, at every moment in time during the third year), the account's value will be P(1.08)2. Figure 6 shows the resulting step function (the vertical segments in Figure 6 are an artifact of the graphing program's "calculator drool").

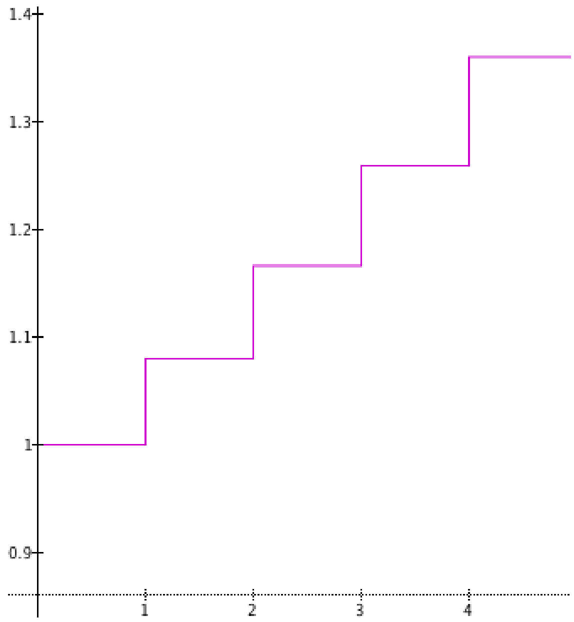


Figure 6. Graph of account value that has an initial value of \$1.00 and which earns interest at 8% per year compounded annually (Vertical line segments are just "calculator drool").

However, we need not think of the account value that the bank might report when you examine it online during a compounding period. It would just show the account's value at the beginning of the compounding period. Instead, we can imagine that, during each compounding period, interest accrues as simple interest. By this scheme, the accounts value grows within any compounding period at a rate of change that is proportional to the account's value at the beginning of that compounding period. The function giving the account's value by this scheme at each moment in time can be defined piecewise, as in

$$v(x)=\begin{cases} P+(0.08P)x, 0<x<1 \\ P(1.08)+P(1.08)^2(x-1), 1\le x<2 \\ P(1.08)^2+P(1.08)^3(x-2), 2\le x<3 \\ \ldots \\ P(1.08)^n+P(1.08)^{n+1}(x-n), n\le x<(n+1) \end{cases} \tag{0.2}$$

The difference between functions (1.1) and (1.2) is that (1.1) is a special case of (1.2). It is as if, in the case of simple interest, we are in an infinitely long compounding period.

The graph of v(x) as defined in (1.2) appears in Figure 7. (I used an interest rate of 80% rather than 8% to accentuate the constant rate of change within compounding periods.) The function's linearity within each compounding period is a result of interest accruing at a constant rate – highlighting that the rate of change within any compounding period is proportional to the function's value at the beginning of that period.

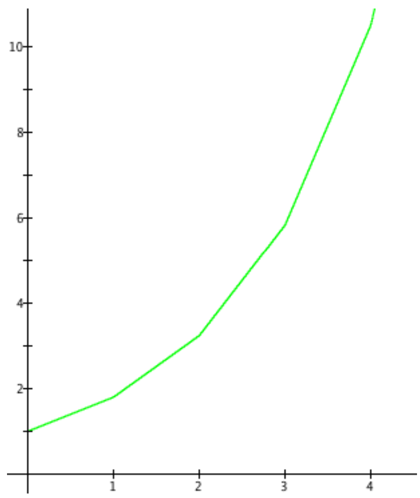


Figure 7. Graph of $v(x)$ as defined in (1.2).

Finally, we can easily adjust the definition of v(x) to accommodate any number of compounding periods per year. But regardless of the number of compounding periods, within any one of them, the function changes at a rate of change that is proportional to the value of the function at the start of that period.

Figure 8 shows three graphs, the first for 2 compounding periods per year, the second for 4 compounding periods, and the third for 12000 compounding periods (1000 times monthly)-again using a yearly rate of 80% to accentuate the change. I hope you attend to the fact that, for each graph, the function giving it increases within each interval of length 1/n at a rate that is proportional to the function's value at the beginning of that interval. The intent is that students come to see that, for very large n (a very large number of annual compounding periods and thus a very small amount of time), the function's value at the beginning of each period is "nearly equal to" the function's value at every point within the period. Thus, the characteristic property of exponential functions, that an exponential function always changes at a rate that is proportional to the function's value, emerges naturally from the idea of compound interest.

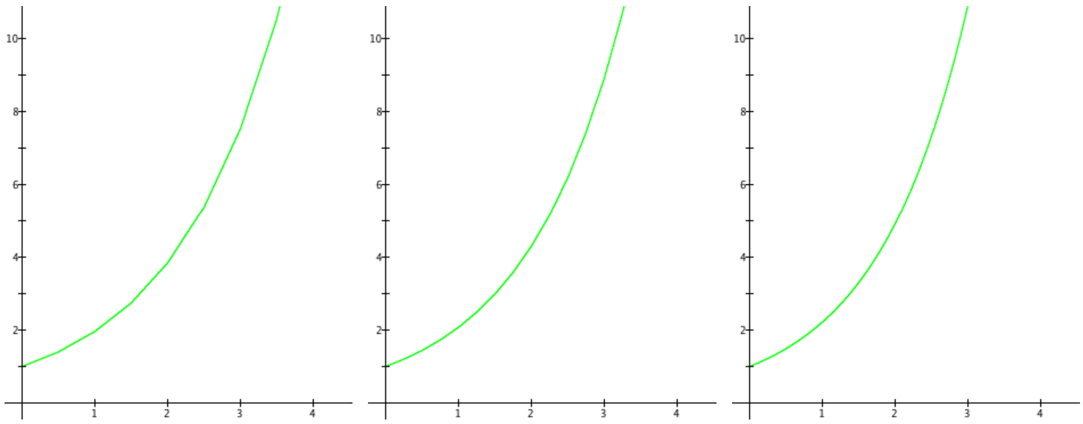


Figure 8. Graphs of v(x) for 2 compounding periods per year, 4 compounding periods per year, and 12000 compounding periods per year.

Two observations are worth noting about this development of exponential function. The first is that the idea of r in ert being like a constant rate of change does not arise from anything special about ert. Rather, r having a meaning like constant rate of change arises metonymically by virtue of the fact that simple interest during any compounding period is a constant rate of change and because, regardless of the number of compounding periods, we always refer back to the annual (simple) interest rate.

The second observation is that this development accentuates the characteristic property of exponential functions (rate of change being proportional to the value of the function) at the expense of the intuition of doubling, tripling, etc. that comes from the idea of splitting. They both rest on a multiplicative conception of comparison and growth, but the two do not tie together neatly.

CONCEPTUAL ANALYSIS

The examples given above each entailed a conceptual analysis of a mathematical idea. Two issues arise immediately:

- What is conceptual analysis (and how does one do it)?
- What use is conceptual analysis for mathematics education?

We often need to describe what students might understand when they know a particular idea in various ways. Glasersfeld (1995) calls his method for doing this conceptual analysis. As Steffe (1996) notes, the main goal of conceptual analysis is to propose answers to this question: "What mental operations must be carried out to see the presented situation in the particular way one is seeing it?" (Glasersfeld, 1995, p. 78).

Glasersfeld first introduced me to conceptual analysis when he wondered how to convey the concept of triangle to a person who is congenitally blind and does not know the word already. His example went like this (if you are sighted, close your eyes).

Imagine that you:

- Are in some location, facing in some direction.
- Walk, straight, for some distance;
- Stop. Turn some amount.
- Walk straight for another distance.
- Stop. Turn to face your starting position.
- Walk straight to it.

Your path is a triangle.5 (Glasersfeld & Czerny, 1979)

Glasersfeld employed conceptual analysis in two ways. The first was to generate models of knowing that help us think about how others might know particular ideas. Glasersfeld's meaning of model is very much like Maturana's (1978) notion of scientific explanation.

> As scientists, we want to provide explanations for the phenomena we observe. That is, we want to propose conceptual or concrete systems that can be deemed intentionally isomorphic to the systems that generate the observed phenomena (p. 29).

Glasersfeld's operationalization of "triangle" was more than a way to define it to a blind person. It was also an attempt to develop one hypothesis about the operational aspects of imagining a triangle. I find this approach especially powerful for research on mathematics learning. For example, in research on students' emerging concepts of rate it has been extremely useful to think of students' early understanding of speed as, to them, speed is a distance and time is a ratio (Thompson, 1994b; Thompson & Thompson, 1992, 1994). That is, speed is a distance you must travel to endure one time unit; the time required to travel some distance at some speed is the number of speed-lengths that compose that distance. Upper-elementary school children bound to this way of thinking about speed will often use division to determine how much time it will take to travel a given distance at a given speed, but use guess-and-test to determine the speed required to travel a given distance in a given amount of time. Their employment of guess-and-test is not a change of strategy. Rather, it is an attempt to assimilate the new situation into their way of thinking about speed – that it is a distance. Guess-and-test is their search for a speed-length that will produce the desired amount of time when the given distance is actually traveled.

[5] For readers who recall Logo, the similarity between Glasersfeld's operationalization of a triangle and a turtle-procedure for drawing one is striking. However, his example predates the general availability of Logo, and neither of us had heard of it anyway.

There is a second way to employ Glaserseld's method of conceptual analysis. It is to devise ways of understanding an idea that, if students had them, might be propitious for building more powerful ways to deal mathematically with their environments than they would build otherwise. For example, I was working with high school students on the idea of sampling distributions. We were discussing opinion polling, and they were having difficulty distinguishing between the ideas of population parameter and sample statistic, and I began to suspect that their main problem was that they were unable to conceive a population parameter. I found myself saying this:

> Suppose we are like Mork[6] and can stop time for everyone but ourselves. Imagine freezing everyone in our target population. At that moment, each person in the population has an answer (yes, no, or no opinion) to the question we will ask, even if we happen not to ask him or her the question. So, the population as a whole, at that moment in time, has a percent of it who would say "yes" to our question were they to be asked.

In other words, in order to talk about population parameters, students needed to think of populations as having characteristics whose measures have specific values at each moment in time. This is not to say that this example's population really had a characteristic whose measure had specific values at each moment in time. For the purpose of building a concept of sampling distribution, it is merely useful to think that it does. However, this was my realization – that it was merely useful to think of a population having a particular measurable characteristic. Students needed to believe that populations can have measurable characteristics, or else they would have been unable to conceive of sampling distributions as arising from repeatedly drawing samples of a given size from that population. They also would have been unable to consider how the set of sample statistics clusters around the population parameter. To coordinate all these aspects of sampling distributions, population parameters needed to be real to them.

Steffe and Tzur (Steffe, 1993; Tzur, 1999) have employed this use of conceptual analysis to guide their instruction in teaching experiments on rational numbers of arithmetic. Confrey and her colleagues have employed conceptual analysis in similar ways to convey how one might think about multiplication so that it will simultaneously support thinking about exponential growth (Confrey, 1994; Confrey & Smith, 1994, 1995). Thompson & Saldanha (2003) employed conceptual analysis to show how a person's understandings of multiplication, division, measurement, and fraction could each be expressions of a core scheme of conceptual operations, all entailed by multiplicative reasoning. As Steffe (1996) noted, conceptual analysis (the conjoining of a theory of mathematical understanding and radical constructivism as an epistemology) emphasizes the positive aspect of radical constructivism – that knowledge persists because it has proved viable in the experience of the knower. Knowledge persists because it works.

Conceptual analysis can also provide a technique for making operational hypotheses about why students have difficulties understanding specific situations as presented in

[6] Of Mork and Mindy, a television program of the 1970's, starring Robin Williams, about an alien living on earth. Many students had watched reruns of this program.

specific ways. For example, standard fractions instruction often proposes fractions as "so many out of so many" (e.g., 3/5 of 10 apples is "three parts out of five equal-sized parts of the 10 apples". When students understand fractions, in principle, as "so many out of so many", they understand fractions as an additive part-whole relationship. Fractional relationships like "7/5 of 10" apples make no sense whatsoever to students who understand fractions additively, because they would have to understand it as specifying "seven parts out of five equally-sized parts of 10 apples".

Finally, as illustrated in this papers first part, conceptual analysis can be employed to describe ways of understanding ideas that have the potential of becoming goals of instruction or of being guides for curricular development. It is in this regard that conceptual analysis provides a method by which to construct and test a foundations of mathematics education in the same way that people created a foundation of mathematics.

In summary, conceptual analysis can be used in four ways:

(1) in building models of what students actually know at some specific time and what they comprehend in specific situations,
(2) in describing ways of knowing that might be propitious for students' mathematical learning, and
(3) in describing ways of knowing that might be deleterious to students' understanding of important ideas and in describing ways of knowing that might be problematic in specific situations.
(4) in analyzing the coherence, or fit, of various ways of understanding a body of ideas. Each is described in terms of their meanings, and their meanings can then be inspected in regard to their mutual compatibility and mutual support.

I find that conceptual analysis, as exemplified here and practiced by Glasersfeld, provides mathematics educators an extremely powerful tool. It orients us to providing imagistically-grounded descriptions of mathematical cognition that capture the dynamic aspects of knowing and comprehending without committing us to the epistemological quagmire that comes with low-level information processing models of cognition (Cobb, 1987; Thompson, 1989). Conceptual analysis provides a technique for making concrete examples, potentially understandable by teachers, of the learning trajectories that Simon (1995) calls for in his re-conceptualization of teaching from a constructivist perspective, and which Cobb and his colleagues employ in their studies of emerging classroom mathematical practices (Cobb, 2000; Gravemeijer, 1994; Gravemeijer, Cobb, Bowers, & Whitenack, 2000). In addition, when conceptual analysis is employed by a teacher who is skilled at it, we obtain important examples of how mathematically substantive, conceptually-grounded conversations can be held with students (Bowers & Nickerson, in press). Teachers in the U. S. rarely experience these kinds of conversations, and hence they have no personal image of them. Having positive examples of such conversations will be very important for mathematics teacher education.

References

Bowers, J. & Nickerson, S. (in press). Development of a collective conceptual orientation in a college-level mathematics course. *Educational Studies in Mathematics*.

Boyer, C. B. (1946). Proportion, equation, function: Three steps in the development of a concept. *Scripta Mathematica, 12*, 5-13.

Cobb, P. (1987). Information-processing psychology and mathematics education: A constructivist perspective. *Journal of Mathematical Behavior, 6*, 3-40.

Cobb, P. (2000). Constructivism in social context. In L. P. Steffe & P. W. Thompson (Eds.), *Radical Constructivism in Action: Building on the Pioneering Work of Ernst von Glasersfeld* (pp. 205-252). New York: Falmer.

Coe, E. (2007). *Modeling Teachers' Ways of Thinking about Rate of Change.* Unpublished doctoral dissertation, Arizona State University, Tempe, AZ.

Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 293-330). Albany, NY: SUNY Press.

Confrey, J. & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics, 26*(2-3), 135–164.

Confrey, J. & Smith, E. (1995, January). Splitting, covariation and their role in the development of exponential function. *Journal for Research in Mathematics Education, 26*(1), 66-86.

Glasersfeld, E. v. (1995). *Radical Constructivism: A Way of Knowing and Learning*. London: Falmer Press.

Glasersfeld, E. v. & Czerny, P. (1979). *A Dynamic Approach to the Recognition of Triangularity*. Unpublished manuscript, Department of Psychology.

Gravemeijer, K. P. E. (1994). *Developing Realistic Mathematics Education*. Utrecht, The Netherlands: Freudenthal Institute.

Gravemeijer, K. P. E., Cobb, P., Bowers, J., & Whitenack, J. (2000). Symbolizing, modeling, and instructional design. In P. Cobb & K. McClain (Eds.), *Symbolizing and Communicating in Mathematics Classrooms: Perspectives on Discourse, Tools, and Instructional Design* (pp. 225-274). Hillsdale, NJ: Erlbaum.

Hackworth, J. A. (1995). *Calculus Students' Understanding of Rate.* Unpublished Masters Thesis, San Diego State University, Department of Mathematical Sciences. Available at http://pat-thompson.net/PDFversions/1994Hackworth.pdf.

Harel, G., Behr, M., Lesh, R. A., & Post, T. (1994). Constancy of quantity's quality: the case of the quantity of taste. *Journal for Research in Mathematics Education, 25*(4), 324–345.

Kaput, J. J., & West, M. M. (1994). Missing value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 237–287). Albany, NY: SUNY Press.

Lobato, J. (2006). Alternative perspectives on the transfer of learning: History, issues, and challenges for future research. *Journal of the Learning Sciences, 15*(4), 431-449.

Lobato, J. & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *Journal of Mathematical Behavior, 21*(1), 87-116.

Lobato, J. & Thanheiser, E. (2002). Developing understanding of ratio as measure as a foundation for slope. In B. Litwiller (Ed.), *Making Sense of Fractions, Ratios, and Proportions: 2002 Yearbook of the NCTM* (pp. 162-175). Washington, DC: National Council of Teachers of Mathematics.

Maturana, H. (1978). Biology of language: The epistemology of reality. In G. A. Miller & E. Lenneberg (Eds.), *Psychology and Biology of Language and Thought* (pp. 27–63). New York: Academic Press.

National Mathematics Advisory Panel. (2008). *Foundations for Success: Final Report of the National Mathematics Advisory Panel*. Washington, D. C.: U. S. Department of Education. Available at http://www.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf.

NCTM. (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

NCTM. (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Piaget, J., Blaise-Grize, J., Szeminska, A., & Bang, V. (1977). *Epistemology and Psychology of Functions*. Dordrecht: D. Reidel.

Schmidt, W. H., Houang, R., & Cogan, L. S. (2002, Summer). A coherent curriculum: The case of mathematics. *American Educator*, 1-17. Available at http://www.aft.org/pubs-reports/american_educator/summer2002/curriculum.pdf.

Simon, M. A. (1995, March). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Smith, E. & Confrey, J. (1994). Multiplicative structures and the development of logarithms: What was lost by the invention of function. In G. Harel & J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 333-360). Albany, NY: SUNY Press.

Steffe, L. P. (1993, February). *Learning an iterative fraction scheme*. Paper presented at the Conference on Learning and Teaching Fractions. Athens, GA.

Steffe, L. P. (1996). Radical constructivism: A way of knowing and learning [Review of the same title, by Ernst von Glasersfeld]. *Zentralblatt für Didaktik der Mathematik [International reviews on Mathematical Education], 96*(6), 202-204.

Stigler, J. W., Gonzales, P., Kawanaka, T., Knoll, S., & Serrano, A. (1999). *The TIMSS Videotape Classroom Study: Methods and Findings from an Exploratory Research Project on Eighth-Grade Mathematics Instruction in Germany, Japan, and the United States* (National Center for Education Statistics Report No. NCES 99-0974). Washington, D.C.: U. S. Government Printing Office.

Stigler, J. W. & Hiebert, J. (1999). *The Teaching Gap*. New York: Free Press.

Thompson, P. W. (1989). Artificial intelligence, advanced technology, and learning and teaching algebra. In C. Kieran & S. Wagner (Eds.), *Research Issues in the Learning and Teaching of Algebra* (pp. 135-161). Hillsdale, NJ: Erlbaum.

Thompson, P. W. (1994a). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics, 26*(2-3), 229–274. Available at http://pat-thompson.net/PDFversions/1994Rate&FTC.pdf.

Thompson, P. W. (1994b). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 179–234). Albany, NY: SUNY Press.

Thompson, P. W. (1995). Constructivism, cybernetics, and information processing: Implications for research on mathematical learning. In L. P. Steffe & J. Gale (Eds.), *Constructivism in Education* (pp. 123–134). Hillsdale, NJ: Erlbaum.

Thompson, P. W., Carlson, M. P., & Silverman, J. (2007). The design of tasks in support of teachers' development of coherent mathematical meanings. *Journal of Mathematics Teacher Education, 10*, 415-432. Available at http://patthompson.net/PDFversions/2007-JMTETasks.pdf.

Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. Martin, & D. Schifter (Eds.), *Research Companion to the Principles and Standards for School Mathematics* (pp. 95-114). Reston, VA: National Council of Teachers of Mathematics.

Thompson, P. W. & Silverman, J. (2008). The concept of accumulation in calculus. In M. P. Carlson & C. Rasmussen (Eds.), *Making the Connection: Research and Teaching in Undergraduate Mathematics* (pp. 43-52). Washington, DC: Mathematical Association of America.

Thompson, P. W. & Thompson, A. G. (1992, April). *Images of Rate*. Paper presented at the Annual Meeting of the American Educational Research Association. San Francisco, CA. Available at http://pat-thompson.net/PDFversions/1992Images.pdf.

Thompson, P. W. & Thompson, A. G. (1994). Talking about rates conceptually, Part I: A teacher's struggle. *Journal for Research in Mathematics Education, 25*(3), 279–303. Available at http://patthompson.net/PDFversions/1994Talking%20about%20Rates%201.pdf.

Tzur, R. (1999, July). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education, 30*(3), 390-416.

Wearne, D. & Hiebert, J. (1994). Place value and addition and subtraction. *Arithmetic Teacher, 41*(5), 272–274.

OFFERING MATHEMATICS TO LEARNERS IN DIFFERENT CLASSES OF THE SAME TEACHER

Ruhama Even

Weizmann Institute of Science

The research program Same Teacher – Different Classes investigates the complex interactions among teachers, curriculum and classrooms. The methodology comprises of multiple case studies. Each case includes a teacher who teaches the same mathematics curriculum program, syllabus, or topic in two classes. Illustrations are given from case studies of teaching the same probability syllabus in high-school classes of different matriculation levels, and from case studies of teaching the same algebra curriculum program in 7th grade classes in different schools.

INTRODUCTION

In the last decades the focus of research in mathematics education started to extend from the individual student's cognition and knowledge to include also socio-cultural aspects of mathematics education, and aspects related to teaching and teachers. This is reflected, for example, in the rapidly growing number of research studies presented at the International Group of Psychology of Mathematics Education (PME) meetings on these topics, marking a major distinction between current and past work of the PME Group. Whereas the first milestone PME book (Nesher & Kilpatrick, 1990) was devoted solely to cognitive research related to student learning of various mathematical topics and concepts, one of the five main research domains of current interest to the PME Group, as presented in the second milestone PME book (Gutiérrez & Boero, 2006), is socio-cultural aspects of teaching and learning mathematics; and another is teaching and teachers. Whereas researchers in mathematics education had tended in the past to conduct their studies on student knowledge and learning in universities or laboratories, in the last decades they began to study contexts explicitly and situate their inquiries within schools, in order to capture, rather than eliminate, the complexity of teaching and learning in the classroom. Mathematics education researchers began to focus on student and teacher participation in classroom activities and on different kinds of interaction.

Scholarly work on offering mathematics to learners has changed in accordance with the development of the field of mathematics education. In the past it has been associated with curriculum development, conducted mainly at universities (e.g., the "new math" massive curriculum development projects). This kind of work continues today in sophisticated forms of developmental research and teaching experiments that involve classroom research and attention, not only to learning, but also to various aspects of teaching. Today, researchers, who are often also members of the curriculum development team, observe classrooms of teachers who agree to try a preliminary version of a new curriculum, and the information gathered is used by the curriculum developers for anticipating students' ways of dealing with the materials, for estimating the time needed to work on the materials in class, and for constructing a conjectured learning trajectory (e.g., Hershkowitz et al., 2002).

In recent years, scholarly work on offering mathematics to learners began to focus also on the complexity of the interactions among teachers, curriculum and classrooms, detached from the immediate goal of curriculum development or evaluation. This line of research yields important information about ways teachers use curriculum materials (Remillard, 2005), showing that different teachers enact the same curriculum materials in different ways (Manouchehri & Goodman, 2000; Tirosh, Even, & Robinson, 1998). Studying different classes of the same teacher, however, has only now started to be the focus of research studies. In one such study, Herbel-Eisenmann, Lubienski and Id-Deen (2006) studied the instructional practices of one teacher who taught two eighth-grade mathematics classes using different curricular materials in each of the classes. Lloyd (in press) studied a high school mathematics teacher's decisions about classroom organization and interactions during his first two years using a new curriculum. These studies highlight contextual factors that contribute to teacher's enacted curricula (e.g., student/parent expectations).

The research program *Same Teacher – Different Classes* belongs to this line of research. Its overarching aim is to gain insights about the interactions among teachers, curriculum and classrooms. To achieve that we compare teaching and learning mathematics in different classes of the same teacher and of different teachers, examining the enacted curricula (e.g., the mathematical ideas offered to learners), the teaching practices (e.g., teacher response to, and use of, students' talk and action), the classroom culture (e.g., nature of argumentation), etc. Current research studies focus on teaching probability, algebra, analysis and geometry in secondary school. In the following I illustrate the nature of work and initial findings from work in probability (conducted in collaboration with Tova Kvatinsky) and algebra (conducted in collaboration with Tammy Eisenmann).

TEACHING THE SAME PROBABILITY SYLLABUS IN CLASSES OF DIFFERENT LEVELS[1]

Mathematics teaching that aims to develop understanding is frequently associated with devoting considerable class time to solving problems, proposing and justifying alternative solutions, critically evaluating alternative courses of action, leading to different methods of solving problems, not necessarily anticipated by the teacher ahead of time (e.g., Cobb, Stephan, McClain, & Gravemeijer, 2001; Even & Lappan, 1994). This way of teaching is often contrasted with teaching that aims to help students reach correct answers with no attention to developing understanding. The latter is commonly associated with devoting considerable class time to performing fragmentary, individual, small rituals that are practiced until they can be executed accurately, and emphasis is put on mechanistic answer-finding. For a matter of

[1] More information on this work can be found in Even and Kvatinsky (2007).

convenience, in the following I refer to these two stereotypical descriptions of teaching approaches as *teaching for understanding* and *teaching for mechanistic answer finding*, respectively.

In classes characterized by the *teaching for mechanistic answer finding* approach students hardly talk in class. Rather it is the teacher who provides explanations, asks questions, and evaluates students' short answers, often using a discourse pattern of the form of Funnel Pattern (Bauersfeld, 1988; Wood, 1994), where the teacher's questions are aimed at directing students to a "predetermined solution procedure preferred by the teacher" (p. 155). In contrast, in classes characterized by the *teaching for understanding* approach students are expected and encouraged to make conjectures, explain their reasoning, validate their assertions, discuss and question their own thinking and the thinking of others, and argue about what is mathematically true. A prevalent discourse pattern in *teaching for understanding* classes is the Focusing Pattern (Wood, 1994), where the teacher's questions are aimed at helping students focus on the important aspects of the mathematics problem but leave the actual solution of the problem to the students. Thus, in *teaching for understanding* classes, the students have a significant and influential role in the class discourse and in solving problems in class.

The literature suggests that teachers tend to adopt the *teaching for mechanistic answer finding* approach, more when teaching in classes of lower-achieving students than in classes of higher-achieving students (e.g., Raudenbush, Rowan, & Cheong, 1993; Zohar, Degani, & Vaaknin, 2001). However, this finding is based mainly on teachers' self-reports (questionnaires and interviews). Missing are studies that analyze in detail teaching practices in high- and in low-achieving classes. Research also shows that usually the more competent teachers teach classes of high-achieving students whereas the less competent ones teach the low-achieving students (Yair, 1997). Thus, it is not clear whether the differences reported in the literature between mathematics teaching in high- and in low-achieving classes are related to differences between the teachers teaching in the respective classes. Consequently, there is a need to study the same teacher's ways of teaching in high- and in low-achieving classes.

This study examines actual practices of teaching mathematics and of classroom interactions in classes having different levels taught by the same teacher – investigating two of the main features that differ in *teaching for understanding* and *teaching for mechanistic answer finding* classes: students' opportunities to have a significant and influential role in the class discourse, and the nature of decision making about ways of solving problems in class.

Methodology

Participants are Betty and Gloria (pseudonyms), two high school teachers teaching in the same school. Both teachers had a reputation of being competent and responsible teachers. Each teacher taught probability in two classes – one class of 3-unit level, the other of 4-unit level (in Israel, the Matriculation Examination in mathematics is offered in three levels: 3-, 4-, or 5-units; 3 being the lowest). All four classes used the same syllabus and were preparing for the Matriculation Examination in mathematics at the time of the study. The two 3-unit classes used the same two textbooks intended for the 3-unit level matriculation exam. Similarly, the two 4-unit classes used the same two textbooks intended for the 4-unit level exam.

The main data source was observation of all probability lessons in each class during one school year – total of 46 lessons (15 lessons in each 3-unit class; 8 in each 4-unit class). After all observations were completed an individual semi-structured interview was conducted with each teacher focusing on the teachers' views of teaching probability in different-level classes, and whether they thought there were differences between their ways of teaching in the two classes.

Detailed data analysis of the lessons included the talk during whole class work. The interviews and observations of the seatwork were used to support or downplay interpretations and to provide additional information about the teachers' views. Two units of analysis were used: one is utterance; the other is activity (i.e., the whole-class work on one probability problem). Using utterance as the unit of analysis we examined students' opportunities to have a significant and influential role in the class discourse. We first employed, to a large extent, the coding system developed in the TIMSS-Video Study (Hiebert et al., 2003), with some modification. Statistical analysis was performed to compare the work on the same six problems in same-level classes, and also the work during randomly selected two full lessons in each class, substantiating the validity of the sampled activities. Then qualitative and quantitative analyses were conducted, using an activity as the unit of analysis, to examine the nature of decision making about ways of solving problems in class. All problems solved during the whole-class work in the 46 observed lessons were analysed, a total of 193 activities.

Differences in class discourse and ways of solving problems in class

The four classes covered by and large the same sub-topics, following the same teaching sequence. All used the objective approach to probability, focusing on the classical approach. Still, although both Betty and Gloria mentioned in class the fundamental characteristic of probability, namely uncertainty, Gloria focused on that significantly more than Betty. Yet, the main differences between the teachers lied in their teaching approaches, as described in the following.

Utterance analyses of the relative share and nature of classroom talk revealed that in all four classes most of the teachers' talk was devoted to asking students mathematical questions. However, Betty focused almost entirely on the final answer to a mathematical problem (e.g., "Then what is the answer?") – 94% of Betty's elicitation utterances in her 3-unit class (B3) and 92% in her 4-unit class (B4). Unlike Betty, Gloria devoted a considerable extent of her questions to probing students about how they got their answers, encouraging students to explain their reasoning (e.g., "How did you get that?"); a little more so in Gloria's 3-unit class (G3) than in her 4-unit class (G4) – 52% of Gloria's elicitation utterances in G3 and 38% in G4.

When students did not respond to Gloria's questions, she often rephrased her questions, and expected the students to answer them. For example, G4 worked on a problem that involved a gambling game in which one can win 1000 Shekels (the Israeli currency), 500 Shekels, or nothing at all. Working with the class on finding the probability of winning exactly 500 Shekels, Gloria asked: "How can you win exactly 500?" After a long pause she rephrased her question: "He played twice. It's like two tosses or two draws. Or an arrow you shoot twice. What can it be?"

In contrast, Betty frequently answered her own questions, and did not wait for students to offer an answer. For example, Betty asked B3: "When do we do 'plus'?" After a short pause, before students attempted to respond, she answered: "If it is 'either this or that'". Betty also often ignored students' suggestions and answered her own questions when students did respond but not in the way she expected. For example, Betty instructed her students to use the probability of the complement of an event when solving a probability problem that includes the phrase "at least". Later, B3 worked on solving a probability problem that involved finding the probability that at least one of two students succeeds in a test,

B: At least one succeeds?

S: Either one succeeds and the second [student] fails, or one fails and the second succeeds, or both succeed.

B: What did we say, if we want this combination of 'at least one'? [Pause] I asked you to remember this. When I see, 'at least one something', I do this: One minus the probability [of] none.

Analysis reveals a statistically significant difference between the two teachers. Whereas 18% and 11% of Betty's utterances in B3 and in B4 (respectively) were answers to her own questions, only 3% and 4% of Gloria's utterances in G3 and in G4 (respectively), when working on the same problems, were of this form. Moreover, Betty answered her own questions more often in B3 than in B4 (not statistically significant).

Analysis of students' opportunities to make decisions about ways of solving problems suggested that, in general, Gloria allowed, and even encouraged, students to solve the problems she assigned in any way they chose. In contrast, Betty decided how to solve the problems she gave. For example, Betty often required students to use rules based on semantic hints when solving problems in class: "Then now remember: When there is an 'and' you multiply probabilities. If there is 'this *and* this' you multiply probabilities." Even when students suggested several times different (correct) ways of solving a problem, Betty ignored them and insisted that they follow "the rule", as the "at least" example above illustrates. In contrast, Gloria did not require students to use rules based on semantic hints. Accordingly, significant differences between the classes of the two teachers existed. Both of Betty's classes used verbal hints explicitly in more than half of the problems solved in class (54% in B3 and 58% in B4) and only rarely Gloria's classes, more so in G4 (0% in G3 and 13% in G4).

Moreover, Betty always decided what representations to choose when solving problems, and when students suggested choosing a different representation, Betty rejected it. As with the case of semantic hints, Betty gave students rules to use when choosing representations to solve problems, and emphasized the importance of following these rules. For example, Betty introduced the representation of a two-dimensional table, and immediately told students: "If you don't make a table, it is very easy to make mistakes. Therefore, when you have two dice, or one die twice, or a die and a spinner, you start with a table". Later, when the students worked on solving similar problems, Betty repeated the rule: "The first thing when throwing two dice or one die twice, you make a table. Who does not remember what table I am talking about?" In contrast, when a student suggested solving a problem using a representation different from the one suggested by the teacher or by another student, Gloria responded to the student's suggestion and, as a result, the problem was often solved in two different ways, using the two representations.

In B3 Betty went even further in not letting students make decisions about ways of solving problems, and required students to switch to decimal fractions whenever a problem dealt with percentages or simple fractions. For example, after a B3 student told her that she solved a problem with simple fractions and not with decimals, Betty said: "No. No. A decimal fraction... Always switch to a decimal number." In contrast with her behavior in B3, Betty allowed B4 to use any form of number (i.e., decimals, simple fractions, percents) they wanted, and so did Gloria in both of her classes. Thus, in B3 89% of the problems solved in class were solved using decimal numbers, but in B4 and in Gloria's classes the percentages were significantly lower: 38% of the problems in B4; 44% of the problems in G3 and 41% of the problems in G4.

In addition to differences in opportunities to choose methods of solving problems, there were differences between students' opportunities to present and discuss their own solutions with the whole class. The utterance analysis showed that, unlike Gloria, Betty seldom asked students to explain how they solved problems. Correspondingly, in Betty's classes the students had fewer opportunities than Gloria's students to present and share their solutions with the rest of the class, still fewer in B3 than in B4. Betty's students reported and shared their solutions of only 9% (in B3) and 21% (in B4) of the problems solved in class compared with Gloria's students in both classes presenting and sharing their solutions of 66% of the problems solved in class. Work on the remaining problems did not include students reporting or sharing their solutions, but rather the teacher leading the whole class in solving the problem. The following excerpt from B4 illustrates Betty leading a whole-class solution of a problem.

B: Problem 2 [reading out loud]. There are 50 passengers in a bus. Twenty-five are from Tel-Aviv, 15 from Jerusalem, and the rest are from various other places in the country.

What did they tell us? That 25 are from Tel-Aviv, 15 from Jerusalem. How many are the rest?

S: 10.

B: Right, a total of 50 passengers. You choose randomly one passenger. What is the probability that the passenger is from Tel-Aviv? How many outcomes are good for us? Desired by us? All those from Tel-Aviv, right? We have 25 of those, out of?

S: 50.

B: 50, yes. And each of them has the same probability to be chosen. So we have 50 possible outcomes. We have 25 desired outcomes. The probability is 25/50. [Writes on the board] P(from TA) = 25/50.

In this illustrative example, we see how trivial and insignificant the students' contribution to the whole-class solution of the problem was. It was mainly Betty who solved the problem. The students' contribution is characterized mostly by answering small fragmented teacher's questions. In contrast, the following excerpt from G4 illustrates a different role for students' solutions during whole-class work.

S: I couldn't solve it.

G: Did you do it with a tree? I am interested in knowing.

S: No.

G: Come show me on the board.

S: Shall I make its tree?

G: Show me what you did. Copy it [from your notebook] to the board so we can see.

[The student copies his partial solution on the board.]

G: [Approaches the whole class] Now, what do you think?

This excerpt illustrates a central and significant role for students in contributing to the whole-class solution of a problem. Not only did students present their work – Gloria also asked other students to comment on it, often basing the whole-class solution of a problem on students' methods and analysis, resulting in solving the same problem in different ways.

As the above excerpt illustrates, in addition to encouraging students to solve problems any way they wanted and to discuss their methods, Gloria often invited students, who did not reach correct solutions, to present their work to the whole class for discussion. In contrast, on the rare occasions when students in Betty's classes presented their solutions to the whole class, only correct solutions were presented. Betty stopped students' talk whenever they started to say something wrong, and invited students to the board only after she made sure they received correct solutions. For example,

S: Can I do it on the board?

B: Did you get a correct answer?

S: I didn't finish yet, but

B: Then finish. Get a correct answer, and then come to the board.

Ways of offering mathematics to students who encounter more difficulties

The findings above show that Betty discouraged students' talk and sharing of their own ways of solving problems. Instead, she dictated to students how to solve problems, emphasizing rule following. She seldom invited students to present their work to other students, and she never discussed unsuccessful attempts – typical features of *teaching for mechanistic answer finding* approach. Moreover, Betty did not always behave similarly in the two classes she taught. Whenever there were differences (either statistically significant or not), Betty exhibited common characteristics of *teaching for mechanistic answer finding* approach more in B3 than in B4. In B3 Betty gave even less opportunities to students to have a significant role in the class mathematics discourse (e.g., she answered her own questions more often in B3), and she demanded more of her B3 students to follow rules in order to solve problems (e.g., Betty required her B3 students to switch to decimal fractions whenever a problem dealt with percentages or simple fractions, while allowing her B4 students to use any form of number they wished).

Unlike Betty's students Gloria's students had numerous opportunities to have a significant and influential role in the class mathematics discourse, to solve problems in different ways, to choose methods of solving problems, and to present and discuss their own methods as well as their unsuccessful attempts with the whole class – typical features of *teaching for understanding* approach. Gloria also did not always behave similarly in the two classes she taught. Whenever there were differences (either statistically significant or not), she exhibited common characteristics of *teaching for understanding* approach more in G3 than in G4. Gloria gave her G3 students more opportunities to have an influential role in the class discourse, and to choose and discuss alternative methods of solving problems (e.g., she asked students to explain how they solved problems more often in G3).

In other words, both Betty's and Gloria's teaching approaches were amplified to some degree in their lower level class. Betty's amplified teaching approach in the lower-achieving class fits the picture portrayed in the literature, which suggests that teachers tend to adopt the *teaching for mechanistic answer finding* approach, more when teaching in classes of lower-achieving students than in classes of higher-achieving students (e.g., Raudenbush, Rowan, & Cheong, 1993; Zohar, Degani, & Vaaknin, 2001). However, Gloria's amplified teaching approach in the lower-achieving class, of adopting more extremely the *teaching for understanding* approach, is contrary to this prevalent view.

How may this inconsistency be resolved? One way is to argue that Gloria is the exception. However, there is another way to resolve this inconsistency. Although so different from each other, both Betty and Gloria were considered skilled and caring teachers in their school. As caring teachers, they drew on their preferred instructional strategies – their teaching approach – to meet the learners' demands that they perceived to be more challenging. Hence, it may not be surprising that each teaching approach was amplified to some degree in the lower-level class. In their own way, each teacher aimed at helping more those students who encountered more difficulties – the low-achieving students – and they did so by using the resources available to them: enhancement of their teaching approaches. The following excerpts, first from Betty's interview and then from Gloria's support this interpretation. Betty said,

> It is very important, especially for the 3-unit class, to provide clear-cut tools, so that they know what to do in each case. Here it is very clear and simple. If it is 'this or this' then it is addition. If it is 'this and this' then it is multiplication. And I also give them 'at least one something' is the complementary rule. It makes order for the students, especially for the 3-unit. Otherwise they need to think what rule it is, what to do, and they make a mess. 4-unit students also like it and it is easy for them.

And Betty added,

> 3-unit students - all they know well is to substitute in a formula. It is important for them to have a formula.

Gloria also explained in her interview how she tried to help more the low-achieving students:

> It is important to hear their [the 3-unit level class] thinking, to hear their ways, to see their mistakes, to relate to each of them. The 4-unit class - they handle it, they don't need me as much.

And she added,

> In general I don't give many formulas… I do everything in an intuitive way… The 3-unit [students] don't like formulas. They want to feel what's going on. Therefore, they keep asking questions and try to understand. The 4-unit [students] – they manage, they see what needs to be done, and ask much less, because they see what needs to be done. In [the] 3 [unit class] they want to know why, neither tricks nor rules. Therefore, in [the] 3 [unit class] you saw that I explained why this is multiplication, and why this is so. And I don't just give rules that they don't remember later what suits what.

It is quite astonishing how different, yet at a deeper level similar, the two teachers talked about the lower-achieving students. Both displayed a real desire to help the students. Still, for Betty it meant to give them formulas and "to give clear-cut tools, so that they know what to do in each case". In contrast, for Gloria it meant to be attentive to the students, "to hear their thinking", not to give formulas and rules, but instead to focus on explanations and understanding.

USING THE SAME ALGEBRA TEXTBOOK IN DIFFERENT CLASSES[2]

Recently, Kieran (2004) developed a model of algebraic activity that is useful as a framework for organizing school-level algebra activities. The framework distinguishes among three types of school algebra activities:

- *Generational* activities. These activities involve the forming of expressions and equations that are the objects of algebra (e.g., writing a rule for a geometric pattern). The focus of generational activities is the representation and interpretation of situations, properties, patterns, and relations. A lot of the initial meaning making of algebra (i.e., developing meaning for the objects of algebra) occurs within generational activities.

[2] More information on this work can be found in Eisenmann and Even (in press).

- *Transformational* activities. These include 'rule-based' algebraic activities (e.g., collecting like terms, factoring, substituting). Transformational activities often involve the changing of the form of an expression or equation in order to maintain equivalence. It is important to note that meaning building is not related solely to generational activities, as transformational activities involve meaning building for equivalence, and for the use of properties and axioms in the manipulative processes.
- *Global/meta-level* activities. These are activities that are not exclusive to algebra. They suggest more general mathematical processes and activity. In those activities algebra is used as a tool. They include problem solving, modeling, generalizing, predicting, justifying, proving, and so on.

Algebra textbooks have traditionally centered on the transformational aspects of algebraic activity. In contrast, the innovative curricula developed in recent years focus on all three types of algebraic activity. Illustration for the three types can be found in Eisenamm and Even (this volume).

Obviously, students that use different curriculum materials may experience different types of algebraic activity. For example, studying from a traditional textbook that focuses primarily on transformational work would tend to result with more emphasis on transformational activities in the classroom than in the case of using a contemporary textbook that focuses also on generational work and includes global/meta-level activities as well. Moreover, because, as mentioned earlier, different teachers enact the same curriculum materials in different ways, students in different classes that use the same curriculum materials may also experience different types of algebraic activity when taught by different teachers. But do students in different classes that use the same curriculum materials experience the same types of algebraic activity when taught by the same teacher? The literature provides little information about the enacted curriculum in different classes of the same teacher, and even less information about the mathematical ideas enacted in different classes of the same teacher. This study addresses this deficiency of current research. It examines the enactment of the three types of algebraic activity (i.e., generational, transformational and global/meta-level) by two teachers; each of them used the same curriculum materials in two different classes.

Methodology

Participants are two teachers, Sarah and Rebecca (teachers' and schools' names are pseudonyms), each taught two 7^{th} grade classes, each class in a different school. The two teachers used the same curriculum materials (i.e., textbook and teacher guide) in both classes (one of the innovative 7th grade mathematics curriculum programs developed in the 1990's in Israel). Classes differed from each other. For example, most students in Sarah's class in Carmel School cooperated with the teacher, worked on assigned tasks, shared and discussed their mathematical work, and responded to Sarah's questions, whereas Sarah's class in Tavor School was noisy and there were many disciplinary problems. Students in Rebecca's class in Gamla School were active, enthusiastic, and often challenged their peers' and the teacher's thinking, whereas students in Rebecca's class in Arbel School seemed to care more about "getting it right" than about doing challenging mathematics.

The main data source included video-taped observations of the teaching of units 1-15 from the beginning of the topic *equivalent algebraic expressions* – 19 lessons in Carmel School, and 15 in Tavor School; 16 lessons in Gamla School, and 17 in Arbel School. In addition, an audio-taped interview was conducted with each of the teachers after all observations in her classes were completed.

The data were analyzed both quantitatively and qualitatively. After analyzing the types of algebraic activity in the written curriculum materials, we analyzed the types of algebraic activity enacted in the four classes. Using a Chi-square test, we then compared between the distributions of algebraic activity types: a) in the curriculum materials and in the enacted curriculum, for each of the four classes; and b) in the enacted curricula in the two classes taught by the same teacher. Then, we compared the suggested and enacted sequence of the three types of algebraic activity. Finally, we examined the nature of the class activity and the realization of the potential of the suggested algebraic types as well as Sarah's and Rebecca's views on that.

Differences in emphasis on the three types of algebraic activity

The four classes covered by and large the same sub-topics, following the same teaching sequence. Neither Sarah nor Rebecca enacted all the units, assignments and tasks suggested in the curriculum materials. Sarah rarely used tasks that were not from the curriculum materials; Rebecca added quite a few tasks not from the curriculum materials.

An analysis of the sequence of activities in the two classes showed that all three types of algebraic activity were enacted in the four classes, and in a similar succession. Reflecting the structure of the curriculum materials, in each of the four classes, most enacted generational activities appeared in the first part of the teaching sequence, the last part of the teaching sequence included mainly transformational activities, and global/meta-level activities were assigned in both classes from the beginning of the teaching of the topic.

Although there were some differences in the emphases on generational and transformational activities between Rebecca's two classes, in both cases, that of Sarah and that of Rebecca, the main difference found between the two classes taught by the same teacher was different emphases on global/meta-level activities. Whereas in line with the structure of the curriculum materials, global/meta-level activities continued to be assigned throughout the teaching of the topic in Sarah's Carmel School and in Rebecca's Gamla School, in Sarah's Tavor School and in Rebecca's Arbel School the last part of the teaching sequence included almost no global/meta-level activities. Furthermore, in Gamla School Rebecca often modified activities, which originally did not include a global/meta-level component, into global/meta-level activities; in Arbel School Sarah occasionally omitted the global/meta-level component from global/meta-level activities. The latter is illustrated below.

In both Sarah's classes the students investigated in small groups the relationship between the number of matches and the length of a "train" for different numbers of matches and trains (see examples of "trains" in Figure 1).

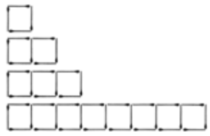


Figure 1. Examples of "Match trains".

Then the classes worked on the following task (Robinson & Taizi, 1997, p. 10), in which students are asked to examine a common student mistake,

> Doron said: "For the number of matches required to build a train with r squares, the algebraic expression 4+3*r is suitable." Is this algebraic expression suitable? Use substitution to check. How many numbers need to be substituted to determine that this algebraic expression is not suitable?

The classes substituted a specific number in Doron's expression to enable a comparison between the numerical result of the substitution and the result of the actual counting of the number of matches in the corresponding train, showing that Doron's suggestion was inappropriate. They also examined the situation, analyzed the hypothetical process Doron used to form his algebraic expression and formed suitable expressions (e.g., 4+3*(r-1)). However, only in Carmel School did Sarah connect the class work to the method of counter example as an important method of refutation in mathematics,

> When we want to prove that something is incorrect, I can give a counter example. Counter example means that I, it is enough that I provide one example where this is not correct, in this case what Doron says, then it is sufficient for saying that it does not work out.

In contrast, in Arbel School the class activity did not include a global/meta-level aspect. Neither Sarah nor the students mentioned the role of examples in mathematical proof and refutation or incorporated other general mathematical processes and activity.

Offering different algebras to students in different classes

Generational and transformational activities are often considered to be the heart of school algebra and are the main focus of school algebra textbooks. Thus, it may seem that the fact that the main difference between two classes of the same teacher was less opportunities for students in one class to engage in global/meta-level algebraic activities implies that Sarah and Rebecca exposed students in their two classes to similar algebraic ideas. However, global/meta-level algebraic activity is an integral component of algebra (Kieran, 2004). Knowledge about mathematics (i.e., general knowledge about the nature of mathematics and mathematical ways of work) is not separate from but rather is an essential aspect of knowledge of any mathematics concept or topic (Even, 1990). Thus, Sarah's Tavor School and Rebecca's Arbel School students were learning a different algebra than Sarah's Carmel School and Rebecca's Gamla School students; algebra that, in contrast with Carmel's and Gamla's algebra, included less generalizing, hypothesizing, justifying, and proving.

The difference in emphasis on global/meta-level activities between Sarah's and Rebecca's two classes seemed to be related to the different characteristics of the two classroom environments. Discipline problems and lack of student cooperation with Sarah at Tavor School, caused Sarah to change her instructional strategy to implement less thinking-related activities and more basic and practice activities during whole class work, as she explained in her interview:

> In Tavor I chose a more concrete direction. Later on. It was not like this at the beginning. But when I realized what is going on there… Less the direction of thinking and new things in the same topic, but more to strengthen what they have learned already… I knew that not everything could work there… Because of the problems that, discipline problems, problems of students' cooperation.

In Rebecca's Gamla School students often initiated global/meta-level work by making generalizations in cases where it was not originally part of the assigned work. But in Arbel School students encountered difficulties whenever activities asked for generalization, as Rebecca explained in her interview,

> What I don't see in Arbel is the ability to generalize. Even if a student reached a generalization he doesn't spell it out. And if we do spell it out, or I give the generalization, then it looks as if the shades go down. You see that the eyes become, they say: 'not clear'.

FINAL REMARKS

The two examples above from the research program *Same Teacher – Different Classes* illustrate how comparing different classes of the same teacher and of different teachers are useful for developing insights about the complex interactions among teachers, curriculum and classrooms. The mere fact, as found in the probability study, that different teachers offer mathematics to learners in different ways, even when using the same curriculum materials, is not entirely surprising, and has been documented by empirical research (e.g., Manouchehri & Goodman, 2000; Tirosh, Even, & Robinson, 1998). Nonetheless, the nature of the differences is important because what people know is defined by ways of learning, teaching practices and classroom interactions, as documented clearly by Boaler (1997). Thus, mathematical knowledge is closely connected to, and inseparable from, the processes that produced it through classroom practices or in other contexts. Consequently, Gloria's and Betty's students did not only studied probability differently, but also studied different probability ideas. Betty's *teaching for mechanistic answer finding* approach emphasized the learning of ideas, such as, "When there is an 'and' you multiply probabilities", whereas Gloria's *teaching for understanding* approach emphasized the learning of ideas, such as, uncertainty. The contrast between Betty's teaching practices and classroom interactions and those of Gloria's highlights the prominent and indispensable role that teachers play in curriculum enactment and their influential role in the nature of learning experiences provided to students, as well as the mathematical ideas students learn – a role that no curriculum program by itself can fulfill.

Furthermore, important information is revealed when, instead of focusing solely on the comparison between teachers, different classes taught by the same teacher are also compared. In the probability study such a focus showed small, but consistently of the same nature, differences in the teaching practices of teachers who taught the same syllabus in classes of different levels. The detailed information about actual teaching practices and classroom interactions in classes of the same teacher allowed us to detect a rather surprising finding, which is contrary to the prevalent view portrayed in the literature nowadays about teaching low-achieving classes, and lays the groundwork for follow-up studies.

The algebra study further illustrates how the comparison between different classes of the same teacher enables us to see that not only students of different teachers may learn different mathematical ideas, but also students of the same teacher who uses the same textbook. In each case, that of Sarah and that of Rebecca, one class was introduced mainly to the ideas of forming algebraic expressions and changing their forms, whereas the other class was engaged also with the ideas of generalizing, predicting, justifying, proving, and so on.

As the illustrations above suggest, attending to the interactions among teachers, curriculum and classrooms has a great potential to contribute to our understanding of teaching and learning in the classroom. The unique methodology of the research program *Same Teacher – Different Classes*, which examines teaching and learning mathematics in different classes of the same teacher and of different teachers, enables us to understand classroom mathematics teaching and learning in new ways.

References

Bauersfeld, H. (1988). Interaction, construction, and knowledge: Alternative perspectives for mathematics education. In D. A. Grouws & T. J. Cooney (Eds.), *Effective Mathematics Teaching* (pp. 27-46). Reston, VA: National Council of Teachers of Mathematics and Lawrence Erlbaum.

Boaler, J. (1997). *Experiencing School Mathematics: Teaching Styles, Sex and Setting*. Buckingham, UK: Open University Press.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *The Journal of the Learning Sciences, 10*(1&2), 113-163.

Eisenmann, T. & Even, R. (in press). Similarities and differences in the types of algebraic activities in two classes taught by the same teacher. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), *Teachers' Use of Mathematics Curriculum Materials: Research Perspectives on Relationships Between Teachers and Curriculum*. New York: Routledge.

Eisenmann, T. & Even, R. (this volume). *Teaching the Same Algebra Topic in Different Classes by the Same Teacher.*

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics, 21*, 521-544.

Even, R. & Lappan, G. (1994). Constructing meaningful understanding of mathematics content. In D. B. Aichele & A. F. Coxford (Eds.), *Professional development for teachers of mathematics, 1994 Yearbook* (pp. 128-143). Reston, VA: NCTM.

Even, R. & Kvatinsky, T. (2007, August). *Teaching Mathematics to Low-Achieving Students: Enhancement of Personal or Traditional Teaching Approach?* Paper presented at the 12th Biennial Conference of the European Association for Research on Learning and Instruction (EARLI), Budapest, Hungary.

Gutiérrez, A. & Boero, P. (Eds.) (2006). *Handbook of Research on the Psychology of Mathematics Education*. Rotterdam, The Netherlands: Sense Publishers.

Herbel-Eisenmann, B.A., Lubienski, S. T., & Id-Deen, L. (2006). Reconsidering the study of mathematics instructional practices: the importance of curricular context in understanding local and global teacher change. *Journal of Mathematics Teacher Education*, *9*(4), 313-345.

Hershkowitz, R., Dreyfus, T., Ben-Zvi, D., Friedlander, A., Hadas, N., Resnick, T., Tabach, M., & Schwartz, B. (2002). Mathematics curriculum development for computerized environments. A designer-researcher-teacher-learner activity. In L. D. English, M. Bartolini-Bussi, G. A. Jones, R. A. Lesh, & D. Tirosh (Eds.), *Handbook of International Research in Mathematics Education. Directions for the 21st Century* (pp. 657-694). Mahwah, NJ: Lawrence Erlbaum.

Hiebert, J., Gallimore, R., Garnier, H., Givvin, K. B., Hollingsworth, H., Jacobs, J., et al. (2003). *Teaching Mathematics in Seven Countries: Results from the TIMSS 1999 Video Study*. NCES. CDRom: Philadelphia, PA.

Kieran, C. (2004). The core of algebra: Reflections on its main activities. In K. Stacey, H. Chick & M. Kendal (Eds.), *The Future of the Teaching and Learning of Algebra; The12th ICME Study* (pp.21-33). Boston, MA: Kluwer.

Lloyd, G. M. (in press). Teaching mathematics with a new curriculum: Changes to classroom organization and interactions. *Mathematical Thinking and Learning*.

Manouchehri, A. & Goodman, T. (2000). Implementing mathematics reform: The challenge within. *Educational Studies in Mathematics, 42*, 1-34.

Nesher, P. & Kilpatrick, J. (Eds.). (1990). *Mathematics and Cognition: A Research Synthesis by the International Group for the Psychology of Mathematics Education*. Cambridge: Cambridge University Press.

Raudenbush, S. W., Rowan, B., & Cheong, Y. F. (1993). Higher order instructional goals in secondary schools: Class, teacher and school influences. *American Educational Research Journal, 30*, 523-555.

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research, 75*, 211-246.

Robinson, N. & Taizi, N. (1997). *Everybody Learns Mathematics: Mathematics for Heterogeneous Classes, on the Algebraic Expressions 1*. Rehovot, Israel: The Weizmann Institute of Science (in Hebrew).

Tirosh, D., Even, R., & Robinson, N. (1998). Simplifying algebraic expressions: Teacher awareness and teaching approaches. *Educational Studies in Mathematics, 35*, 51-64.

Wood, T. (1994). Patterns of interaction and the culture of mathematics classrooms. In S. Lerman (Ed.), *The Culture of the Mathematics Classroom* (pp. 149-68). Dordrecht, The Netherlands: Kluwer Academic.

Yair, G. (1997). Teachers' polarization in heterogeneous classrooms and the social distribution of achievements: An Israeli case study. *Teaching and Teacher Education, 13*, 279-293.

Zohar, A., Degani, A., & Vaaknin, E. (2001). Teachers' beliefs about low achieving students and higher order thinking. *Teaching and Teacher Education, 17*, 469-485.

PLENARY PANEL

Kathleen Hart (Chair)
Michèle Artigue
Marj Horne

PME HISTORY: COGNITIVE THEORIES AND SCHOOL PRACTICES

Kathleen Hart

Panel Coordinator

DISCUSSION POINTS

The three members of the panel are mathematics education researchers with many years of experience in the field and of working with teachers. We had three questions suggested to us by the committee.

What evidence has Mathematics Education Research offered teachers to help improve the teaching and learning of the subject?

The UN millennium plan requires plenary primary education for all. What advice can Mathematics Education Research offer to a new practitioner to prevent the same mistakes in teaching mathematics that have been made over the last thirty years?

A twenty year old trainee teacher says today 'I want to be a good Mathematics teacher'. What advice can you give her [him] as to philosophy, methods, book, etc?

Essentially what can we say with confidence and evidence based on research, particularly PME research, about the teaching of mathematics which might be of use to the practising teacher. The evidence has to be based in the real world if the teacher is to take notice. That world imposes restrictions and structures, some more extreme than others, depending on where you live and try to carry out your profession.

We are an international society but although our schools may look different, the mathematics taught in them has many unifying features. Otherwise international surveys such as TIMSS would not be carried out. Many countries have a national curriculum to which all the state schools adhere, indeed if there is a system of school inspectors the delivery of the written curriculum is the basis of their judgements. Who invents that curriculum and on what evidence is often shrouded in mystery. We are indebted to TIMSS for the discussion of the three versions of the mathematics curriculum: the stated, the delivered and the absorbed curriculum.

'Evidence' has various definitions in the dictionary but the legal use might apply to research: "information tending to establish fact; statements or proofs admissible as testimony in court". Evidence of use to teachers has to be something they will recognise, maybe in their own experience or because the content is that which they teach. In our panel discussion I would like us to distinguish among opinion or belief, even when we think that is backed by experience, and research evidence.

THE TRAINEE TEACHER

Those entering training for the teaching profession will often have been very recently in school themselves or have newly completed a first degree and are now acquiring skills and techniques for a job. Teacher trainers are expected to impart a considerable amount of information in a limited time. Thirty years ago in England, primary school teachers might enrol on a three or four year course for a teaching qualification. Now they would follow a one year course after taking a degree which might not be in any teaching subject, and of that year one third would be spent in a school. In the school the prevailing influence would be of the class teachers not the teacher-trainers. In 1973 'The Maths Methods Program' out of Indiana University [funded by UPSTEP of the National Science Foundation] was quite revolutionary because the students spent their 'in class' time studying material written around suitable activities for primary age children. The aim was to improve the content knowledge of the future teachers but in the "school -use context". They also went into schools to try some of this same material with pupils and then were able to return for discussion at the University.

Much of the research on students in training now chronicles the lack of mathematical knowledge of the entrants to the profession. This is not new.

Some research I did in 1970 when I was a teacher trainer revealed that of 239 first year teacher trainees [the whole intake] who were to qualify for a certificate which entitled them to teach in a primary school and therefore to teach elementary mathematics, 99 said they had 'been weak at mathematics' during their school career. This was mostly in secondary school although 13 trainees recalled being weak in the subject in the first and second grade.

A third year sample [154] when asked for their degree of agreement or rejection of the statement that there was such a thing as a 'mathematical mind' 117 said they agreed. Further 75 /154 thought seven year olds might display a 'mental block' to mathematics.[Hart,1993] To the future teacher I say that holding such a belief means you will accept lack of success in your class when you should be seeking to provide success. One either allows some children to experience a career of failure or one tailors what is taught to the "possible". Regarding as acceptable a pass mark of 38% as is the case in Zambian and Bangladeshi primary schools means the next phase of mathematics teaching is based on holes. Such a mark is found on the leaving certificates in European countries but presumably it is considered then to be a summative mark.

Reports on the lack of mathematical knowledge of future teachers have been published by Stacey [2005], Mjoli [2007] and many others .The teaching in the classroom has to take place so the teacher needs help, perhaps a textbook and teachers' guide. It was very fashionable at one time to decry the textbook and to suggest that teachers write their own material, despite this Johnson. D and Millett A. [1996] reported that many English teachers used a commercial text in their teaching for more than 50% of the time. It varied according to the grade of the child being taught, Key Stage1-33%, KS2- 59% and KS3 – 79 %. How does a teacher, group of teachers in a department, advisor or in some places the politician choose the textbook?

THE TEXTBOOK

We know very little about what makes a textbook useful and successful. Many decisions about the presentation of material in a book are taken by the publisher on economic grounds. The books may be used by children but they are selected by adults, sometimes on the doubtful criteria of colour, cost or robustness. Santos-Bernard [1997] ,investigated what first-grade Mexican children read into illustrations in their mathematics textbooks. Pictures are included often to add colour and to break-up the text but children read them for their content. The cosmetic picture is essentially there to be ignored! The book will be built around a progression of mathematical topics, even in the wild days when children learned everything from cards, the cards were ordered in some way. If there is a national curriculum the progression follows it closely BUT where did it come from in the first place. We cannot proffer from Mathematics education research a semblance of what might be considered a tried and tested progression suitable for the primary school. What we, as researchers, can do is suggest to the teacher what seems to be hard and how topics which follow in the next sentence in the curriculum might require very much more work. Take for example the syllabus which says ‘extend the decimal system to two, three and four places’ or ‘multiplication by ten, 100, 1000’ as if they did not need to build on each other and require careful teaching but were of the same order of complexity.

In 1986 I was director of ‘Nuffield Secondary Mathematics’ a curriculum development project housed at Kings College, London. At that time we had no National Curriculum although we did have a national examination at the end of compulsory schooling and through the work of subject advisors and national inspectors what was taught in the schools was very similar. One series of textbooks dominated the secondary school market.

The philosophy of Nuffield Secondary Mathematics was that every child would succeed and to that end a group of children would be working from the same topic book. In a class of 25 there might be four such groups each working from a different book. There were three sets of material. Topic books which gave a progression of difficulty in each of five topics to provide a set of mathematical skills. Each child would work on the book which matched his/her attainment. Additionally each grade level had a book of problems –solving tasks to which the child could apply the mathematical skills and through which they could demonstrate their thinking processes in mixed attainment groups. Teachers guides for both sets included assessment procedures, suggestions for class groups and reference to any research which had informed the material [Hart, 1990].

Where to start was a crucial question. The books were destined for all the pupils in a secondary school, including those called ‘low attainers’, ‘slow learners’ or with ‘special educational needs’. The plan was to match the topic book level to the child’s demonstrated attainment, there was no question of all the children in a class having the same book. At the transition from primary to secondary school [age 11-12 years] traditional textbooks assumed the child could work with four operations on whole numbers, had been introduced to Fractions and Decimals and could add and subtract them. From the research of CSMS [Hart, 1981, 2004] we knew this was not so. Many children were managing with the operations of addition and subtraction and scarcely used multiplication even with whole numbers. Greer [1992] says "A fundamental conceptual restructuring is necessary when multiplication and division are extended beyond the domain of positive integers" [p276]. We also knew that success with the fraction 'one half' was common but did not demonstrate any knowledge of other fractions. For information to start the Nuffield Number books we went to the research on 'Early Number' and looked at what was normally taught to six year-olds. From this we drew up a set of questions which we gave to the children in the last class in the primary school and in some 'special schools'. We interviewed those pupils who seemed not to manage the questions, as well as those who succeeded.

Having drawn up a list of where we might start in order to cater for even the least successful in the first year of secondary school we showed it to teachers. The majority said all their first year pupils in the secondary school would be able to cope with these suggested topics. When we had the material based on the list in three schools for trial I had daily phone calls asking could another school join as they had heard of the materials.

So what are we suggesting to teachers? Firstly find out what the child really knows and not what you hope he knows and build from there. Look at the work the pupils did in the early grades. Did they learn all that was there? Are you building your teaching of new topics on very shaky ground?

WHAT WE HAVE FOUND OUT

Each of the operations of addition, subtraction, multiplication and division has more than one meaning and yet the introduction we provide in many teaching schemes stresses just one of them. The children tend to remember one, such as "removal" for subtraction. Have we taught the other meanings or assumed that they will be 'picked up' by the pupil en route through school [Brown, 1981]. The difficulty of addition and subtraction problems varies, even in the early elementary school. Considerable work was done at the Wisconsin Center for Educational Research on analyzing the mathematical skills used in solving such problems [Carpenter and Moser, 1983]. There are 20 different ways the statement 8+5=13 can be interpreted as addition and subtraction problems. Translating a word problem into symbols is not straightforward. Do we as teachers encourage the pupils to translate symbols to words and vice versa? From CSMS we know that many children think that 12 division sign 4 is the same as 4 division sign 12, except you cannot get an answer to the second because it is impossible to divide a smaller number by a larger. [Kerslake, 1986]

Fractions are introduced in the primary school in most countries, but why?

In the new Mathematics Framework [2007] issued by the Department of Education and Skills of the UK and meant to be followed closely by the teacher, prescribed for

Year 1 is that the pupil should be able to read and write numerals from 0 to 20, use knowledge of place value to position these numbers on a number line and further use the vocabulary of halves and quarters in context. In year 2 they have to find one half, one quarter and three quarters of shapes and sets of objects. I suggest that we have evidence that 0 is a difficult concept and that writing any number with two digits is of much greater difficulty than giving a name to a collection. We know from CSMS [secondary school pupils] that naming part of a shape is easier than labelling a collection. The impression given to the teacher is of the same demand and that is not true.

There has been a lot of research on the teaching and learning of Fractions ,using manipulatives , sharing pizzas, looking at multiple meanings of a/b but in almost all cases we find the success rate is about 65%, leaving 35% who fail to cope. The results are very similar for the learning of Decimals [Stacey, 2005]. Many teachers think Fractions are not numbers [Kerslake, 1986] and continue to use the pie/pizza model, forgetting that you cannot multiply parts of pizzas

I hope during the Panel that members of the audience will be able to supplement what we have presented.

References

Brown, M. (1981). Number operations. Place value and decimals. In K. Hart (Ed.). *Children's Understanding of Mathematics: 11 to 16.* John Murray, London

Carpenter. T. & Moser, J. (1983). In Romberg, T. & Collis, K. *Journal of Research in Mathematics Education. Monograph 2. Learning to Add & Subtract*. NCTM. Reston.

Department for Education & Skills (2006). *Primary Framework for Literacy and Mathematics.* Norwich, UK

Greer, B. (1992). Multiplication and division as models of situations. In D. Grouws (Ed.) & L. Vygotsky (1978), *Handbook of Research on Mathematics Teaching and Learning.* New York. Macmillan.

Hart, K. (1981). *Childrens Understanding of Mathematics: 11-16*. John Murray, London.

Hart, K. (2004). *Childrens Understanding of Mathematics: 11-16*. Antony Rowe, Eastbourne, UK.

Hart, K. (1990). *A Research–Based Mathematics Scheme*. Australian Association of Mathematics Teachers 13th Biennial Conference. Hobart, Tasmania.

Hart, K. (1993). *Confidence in Success*. Plenary Lecture. PME 17, Tsukuba, Japan. Hirabayashi, Nohda, Shigematsu and Fou Lai Lin (Eds).

Johnson, D. & Millett, A. (Eds.), Implementing the Mathematics National Curriculum. *Policy, Politics and Practice*. BERA Dialogue. Paul Chapman, London. UK.

Kerslake, D. (1986). *Fractions: Children's Strategies and Errors.* NFER Nelson, Slough.

Mjoli, B. (2007). Problems Experienced by the Teachers in Using Mathematics Learning Material In Distance Education. Unpublished Ph.D. thesis. University of Nottingham.

Santos-Bernard, D. (1997). The Use of Illustrations in School Mathematics Textbooks: Presentation of Information. Unpublished Ph.D. thesis. University of Nottingham.

Stacey, K. (2005). *Travelling the Road to Expertise. A Longitudinal Study of Learning*. Plenary lecture PME 29. In H. Chick & Vincent (Eds), Proceedings PME 29. University of Melbourne.

PLENARY PANEL PAPER

Michèle Artigue
Université Paris

INTRODUCTION

Mathematics education has been developing as a research field for more than four decades now and has accumulated results, but the usefulness of these for practice remains a debated issue. Many authors point out the gap between research and practice, reflect on its possible sources, and try to learn from the successful examples that nevertheless exist. This was for instance the purpose of Boaler (Boaler, 2008) in the lecture she gave in Roma last March at the Symposium organized for celebrating the centennial of the International Commission on Mathematical Instruction (ICMI). Moreover, the conflicts generated in many countries by curricular changes evidence that even when research is considered influential, the positive character of this influence is frequently a matter of discussion. Making clear what research offers teachers and student teachers for enhancing teaching practices remains thus a non trivial enterprise.

In this contribution, reflecting on my personal experience, I will try to contribute to this panel, discussing what didactic research can offer teachers to help improve mathematics teaching and learning. I will focus on a specific mathematical topic: elementary algebra. Since the early nineties, I have been involved in different research projects about algebra (Artigue & al., 2003). I have organized training sessions for student teachers and teachers when professor at the IUFM (Institut Universitaire de Formation des Maîtres) of Reims, and then in the frame of my activities at the IREM (Institut de Recherche sur l'Enseignement des Mathématiques) of the University Paris Diderot - Paris 7. I hope that this reflection will offer valuable insights for the discussion, illustrating different ways through which research can impact practice as well as the potential and limits of these. In order to situate this contribution, I feel necessary to briefly describe the French educational context.

THE FRENCH EDUCATIONAL CONTEXT

In France, compulsory education is 10 years long and there is a national curriculum. The composition of the commissions in charge of its elaboration varies from one period to the other according to political changes. Generally these commissions mix different expertises: mathematicians, didacticians, teachers and teacher educators, regional or national inspectors. Curricular changes are quite frequent. The current curriculum for junior secondary school (the period when students are introduced to algebra) began to be implemented in 2005.

The ordinary route for becoming secondary mathematics teachers is to get a License in mathematics, then to pass a selective national competition called CAPES and to have one year of professional training in an IUFM. CAPES mainly tests the students' academic knowledge even if, in the oral part, they are asked to propose a lesson, comment and produce exercises on given themes. During the year of professional training, student teachers have one or two classes in full responsibility and practice at another level of schooling. They have professional preparation at the IUFM linked to this practice and helping them reflect about it. They prepare a professional essay about a question raised by their practice. If no major problem is detected, at the end of that year, they become civil servant and get a permanent position.

Another characteristic of the French educational system is the existence of the IREMs. These institutes, part of universities, were created at the time of the new math period with three different missions: developing innovation and research, contributing to teacher education, producing educational resources for teachers. They are networked through inter-IREM commissions. Different categories of actors contribute part time to the IREM activities: university mathematicians, teachers, teacher educators, didacticians. They are organized in mixed working groups which carry out innovative and research projects, prepare training sessions and resources based on these projects, and contribute to the national network. Together with IUFM, IREMs play an important role in the connection between research and practice.

THE SUBSTANTIAL OUTCOMES OF RESEARCH IN ALGEBRA

No one can deny, I think, that the didactic research carried out in algebra has produced a substantial amount of results. One of its first important contributions has been to evidence characteristics of the transition between arithmetic and algebra that make the entrance in this domain especially difficult: change in the status of the equal sign and lack of closure for algebraic expressions, relationships to letters and symbolism, change in reasoning modes, change in modes of control and validation. It has also shown the difference between a process and a structural view of algebraic expressions and the negative effect of teaching strategies focusing too early on structural views. It has evidenced the complexity of the historical development of algebraic symbolism and has used these historical and epistemological studies for analysing students' difficulties and behaviour, and for proposing learning trajectories. It has attracted the attention on the multiplicity of semiotic registers that are or can be involved in algebraic work and the importance of establishing adequate connections between these. It has shown the limits of approaches to algebra limiting it to some kind of generalized arithmetic. Technological research in that area has contributed to attract the attention on the dialectic relationship between concepts and techniques in the development of algebraic knowledge and to show the negative effects of discourses opposing techniques and concepts for disqualifying teaching strategies too much restricted to skill learning. It has also evidenced the potential that technology offers for linking semiotic registers, working on syntactic issues, smoothing the transition between arithmetic and algebra and supporting modelling approaches to this domain. International comparisons have also shown that different learning trajectories can be envisaged in algebra, and that a first contact with this domain can be established quite early. Such results are today reasonably well established and many documents synthesizing them are accessible (see for instance (Bednarz, Kieran & Lee, 1996), (Stacey, Chick & Kendal, 2004), (Kieran, 2007)). A priori, they could efficiently support the improvement of mathematics teaching and teachers practices. Is this really the case? In the next part, I address this question looking at the French educational system.

THE IMPACT OF RESEARCH ON THE FRENCH EDUCATIONAL SYSTEM

As is the case in many other countries, in France, algebra teaching too much focuses on its object and syntactic dimension, on the learning of techniques for manipulating algebraic expressions and solving equations. Algebraic gestures quickly loose their mathematical roots and become meaningless for many students. The algebraic discourse is a discourse of rules, of legality more than a discourse of meaning, of truth. Conventions and mathematical rules tend to be given the same status. Algebra is introduced at junior high school level rather abruptly, and mainly through word problems whose solving does not necessarily require the use of algebra. This corresponds to the cultural tradition, and it is not especially successful.

This reality does not exactly reflect the intended curriculum which, for more than one decade, has tried to organize a more progressive entrance in algebra and promote a better balance between its *tool* and *object* dimensions (Douady, 1986). The influence of the results of didactic research summarized above is particularly clear in the current version, as for instance attested by the accompanying document entitled “Du numérique au littéral” published in 2006 by the Ministry of Education (EduSCOL, 2006). Starting from the emblematic situation of the “framed square”, this document explains the different possible status of letters, how the production of formulas can be used for motivating a meaningful introduction of letters and algebraic symbolism, makes explicit the respective characteristics of arithmetic and algebraic resolution modes using examples taken from research, introduces the distinction between the procedural and structural dimensions of an algebraic expression and proposes activities for helping students cope with these two dimensions, and finally points out the interest of using algebra for developing proof competences. It concludes by a synthetic view of the expected progression along the four years of junior high school, and an illustration of the role that spreadsheet can be given.

The influence of didactic research, and of the way its results have been synthesized and disseminated through different channels: the IREM and IUFM channels already mentioned, but also the INRP (National Institute for Pedagogical Research) (Combier, Guillaume & Pressiat, 1996) and the report on computation produced by the CREM (Commission de Reflexion sur l’Enseignement des Mathématiques) (Kahane, 2001), on those in charge of piloting the educational system is evident in this text. Can this make a difference and how?

FROM CURRICULAR CHOICES TO TEACHERS' PRACTICES

Taking curricular decisions, inspired by didactic research, and carefully presenting the rationale for these in official documents is not enough for having research influence practice in a substantial and productive way, as we all know. Even when curricular documents are reasonably detailed, there remains a big distance between the level of description that such documents can provide and the decisions that a teacher has to take when trying to implement these curricular decisions, designing and managing classroom progressions and situations. Adequate formation and support has to be organized for teachers.

The situation in France is from this point of view not ideal. The didactical formation that the IUFMs can provide is strongly limited by the constraints that the concentration of professional training on one year introduces. Regarding in-service teacher education, whatever be the quality of the sessions organized by the IREMs and the IUFMs, their impact is limited by the fact that this form of in-service teacher training is neither compulsory nor valued in terms of careers. Recent enquiries show that more than 50% of teachers never attend such formations. Moreover, reductions in funding have resulted in the concentration of the sessions offered on the most pressing issues and in a reduction of the length of most sessions to two to three days.

In spite of these limitations, research and innovative actions develop, trying to move the existing constraints, and to think about realistic dynamics for professional development. I would like to mention some of these. For her doctoral thesis, Lenfant (Lenfant, 2004) has followed during two successive years student teachers at the IUFM of Reims, analysing how they moved from a student to a teacher position regarding algebra, what sense they made of the formation and how it impacted their practices. The results she has obtained show that, even within the strong constraints of the IUFM, a formation using research for empowering the reflection on practice can have an interesting impact. This research also shows that not all the dimensions of the formation have the same impact and allows us to better understand what can reasonably be achieved in such a first year of professional training, what can be made operational and what will certainly remain simple awareness. Regarding in-service teacher training, the more and more systematic use made of videos for connecting research and the analyse of effective practices has also proved its usefulness for making teachers aware of the characteristics of the algebraic discourse in the classroom mentioned above, for addressing issues of didactic contract and for discussing realistic and productive sharing of the mathematics responsibility in the classroom between students and teacher. Analysis of programs aiming at the collaborative development of resources by teachers and researchers such as (Grugeon & al., 2008) or the SFODEM project (Guin, Joab & Trouche, 2007) show both the potential offered by new visions of the relationships between teachers, teacher educators and researchers, and by new technologies for going on supporting teachers at the distance when the official time of the formation has ended. Research projects developed in connection with AIED research such as that the Pepite and Lingot projects (pepite.univ-lemans.fr/) or the Aplusix project (aplusix.imag.fr) show how teacher professional work and teacher formation in that area can be supported by the development of digital tools directly inspired by didactic research.

There is no doubt that these actions and similar ones which are developing in many countries show that we have at our disposal new and potential means for making didactic research in algebra accessible and useful for teachers, but they also show that making this potential reality for most teachers and not only for some privileged ones needs much more than the sole good will and engagement of researchers.

References

Artigue, M., Assude, T., Grugeon, B., & Lenfant, A. (2001). Teaching and learning algebra: Approaching complexity through complementary perspectives. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study Conference: The Future of the Teaching and Learning of Algebra*, pp. 21-32. Melbourne: The University of Melbourne.

Bednarz, N., Kieran, C., & Lee, L. (Eds.) (1996). *Approaches to Algebra: Perspectives for Research and Teaching*. Dordrecht: Kluwer Academic Publishers.

Boaler, J. (to appear). Bridging the gap between research and practice: International examples of success. *Lecture Given at the ICMI Centennial Symposium.* Roma, March 2008.

Combier G, Guillaume J.C, & Pressiat, A. (1996). *Les Débuts de l'Algèbre au Collège. Au Pied de la Lettre*, INRP, Didactique des Disciplines, Paris.

Douady R. (1986). Jeux de cadres et dialectique outil-objet. *Recherches en Didactique des Mathématiques*, 7(2), 5-32.

EduSCOL (2006). Projet de document d'accompagnement. Du numérique au littéral. http://www.eduscol.education.fr/D0015/

Grugeon, B., et al. (2008). *Le Calcul Algébrique: Des Pistes pour une Progressivité des Apprentissages de l'École au Lycée*. Brochure n°92, IREM Paris 7.

Guin, D., Joab, M., & Trouche, L. (2007). *Conception Collaborative de Ressources pour l'Enseignement des Mathématiques, l'Expérience du SFoDEM (2000-2006)*, INRP et IREM. Université Montpellier 2.

Kahane, J.P. (Coord.) (2001). *L'Enseignement des Sciences Mathématiques.* Paris: Odile Jacob.

Kieran, C. (2007). Learning and teaching algebra. In, F.K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning*, pp. 707-762. Information Age Publishing.

Lenfant A. (2002). *De la Position d'Étudiant à la Position d'Enseignant: l'Évolution du Rapport à l'Algèbre de Professeurs Stagiaires*. Doctoral thesis. Université Paris 7.

Stacey, K., Chick, H., & Kendal, M. (Eds.) (2004). *The Future of the Teaching and Learning of Algebra. The 12th ICMI Study.* Dordrecht: Kluwer Academic Publishers.

USING EDUCATION RESEARCH TO INFORM MATHEMATICS TEACHING IN A SCHOOL

Marj Horne

Australian Catholic University

Research has provided information about children's understanding and misconceptions in number and algebra. The process of teachers collecting data to inform themselves about their students' understanding through interview and other approaches, as well as providing a challenge to assist them in developing their own understanding of mathematics, can motivate and assist them to make decisions about their teaching and the approaches they use.

Teachers care about the learning of the children in their classes. They want to be good mathematics teachers and make a difference. Teachers overwhelmingly have a propensity to be caring teachers although caring does not necessarily lead to good teaching (Cooney, 1999). This caring is the main motivator of teachers in professional learning. It also means that engagement of teachers in particular professional learning is often contingent on their perception that the learning will improve their teaching thus making a difference to children's learning.

Working in a school to effect improvement

During the last three years I have spent some days consulting with a large grade 0-12 school to assist them in improving mathematics. The school has four sections: 0-3, 4-6, 7-8 and 9-12. I was brought in by senior management on acceptance by the senior mathematics coordinator. The mathematics staff within the school were initially ambivalent about the project. Staff have been given about four days of release time for the rewriting of the mathematics curriculum and some professional learning each year. Most staff were unconvinced that any change was needed.

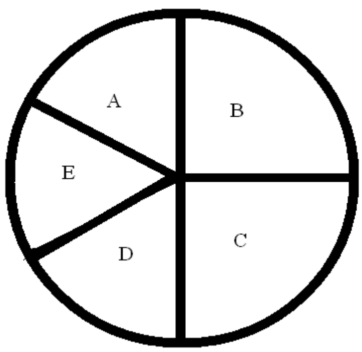
A lesson learnt from the Early Numeracy Research Project (ENRP)

A report on an extensive professional development program (Horne, Cheeseman, Clarke, Gronn, & McDonough, 2002), mentioned four knowledge domains. The suggestion was that to teach effectively the teacher needed four strands of knowledge: knowledge of mathematical content, including specialised mathematical knowledge connected to student learning and mathematical context; knowledge of children's thinking and learning of mathematics; pedagogical content knowledge, including aspects such as representations, examples, explanations, typical student misunderstandings, curriculum, curriculum materials and instructional strategies; and knowledge of the particular children they are teaching. The knowledge required for teaching has been extensively discussed recently. Sowder (2007) notes that there is now recognition that a number of aspects of knowledge are required. These include the first three knowledge domains listed above with the addition of three other knowledge domains: the development of a shared vision, equity issues and a sense of self. There is much cross-over between these knowledge domains. For teachers that knowledge also needs to connect with the children they are teaching.

The professional development in the ENRP approached the four key knowledge domains identified in the report tackling the fourth by providing the teachers with training and a tool of an interview protocol which enabled them to gain greater knowledge of the students they were teaching. This use of interview not only assisted the teachers to improve their knowledge of the children in their classes but also modelled questioning techniques, raised the importance of children talking about their mathematics learning and highlighted aspects of mathematics itself. The increased knowledge about the children in their classes challenged their thinking about their teaching practices. They discovered both expertise and gaps in thinking about which they had not known. This new knowledge contributed to their approach to their professional learning focussing their attention and providing motivation.

Collecting data to inform teaching through written and interview assessments

The decision was made in the school to collect data on the mathematical understanding in the areas of number and algebra and attitudes towards mathematics of students from grades 4-10. This was done initially using a pen and paper test with the questions based on research findings of children's common misconceptions in those domains. For example the fraction question in Figure 1 has been used by Clarke, Sukenik, Roche and Mitchell (2008) in task-based interviews.



a) What fraction of the circle is B?

b) What fraction of the circle is D?

Figure 1. A fraction question used in both assessments.

Some questions were common across all the assessments with others more limited. For example questions on relational understanding in grade 4 were strictly numerically based as in question 1 in Figure 2, while in grades 7-10 some symbolically based relational understanding questions from Küchemann (1981) were added as shown in questions 2 and 3 in Fgure 2.

The results from the pen and paper test indicated areas of weakness and focussed attention on those basic understandings. Concerns were raised for management and the mathematics coordinator or grades 9-12 when the grade 9 and 10 students showed greater weaknesses in basic arithmetic, fractions and decimals than the grade 7 and 8 students. For example for question 4 in Figure 2, the percentage correct at each grade level from grades 7-10 were 59, 69, 48 and 61 raising some concerns about what was happening at years 9 and 10, particularly since students did answer the question.

1. If $542 + 38 = 580$ then $544 + 38 = ?$

2. If $n - 346 = 762$ then $n - 347 = ?$

3. If $e + f = 8$ then $e + f + g = ?$

4. A piece of ribbon 17cm long has to be cut into four equal pieces.

Which answer is most accurate for the length of each piece?

A 4 cm remainder 1 piece

B 4 cm remainder 1 cm

C $4\frac{1}{4}$ cm

D $\frac{4}{17}$ cm

Figure 2. Further examples of questions.

As a result of concerns about student understanding and many staff questioning the validity of any results, the core team of mathematics staff responsible for the project decided in the following year to use an interview with a sample rather than a pen and paper test with the whole cohort. One condition for this was that each mathematics teacher interviewed at least three students with the rest of the interviews being conducted by teacher education students from the University. The interview contained mathematics questions, many of which had been used on the pen and paper assessment the preceding year and covering the same areas of mathematics, and a short set of questions to elicit aspects of attitude towards mathematics. Question 17 in Figure 3 (from the ENRP interview), illustrates the interview protocol with the not so happy face indicating to move on to question 18 if an incorrect answer is given.

Teachers' Professional Identity

17. MISSING NUMBER

Show the orange card with 54 × __ = __ __ 2

a) The answer to 54 × ? ends in 2. What can you tell me about this missing number? ***(pointing to the space after the multiplication sign).***

😐 → 18

Figure 3. Part of the interview.

Teachers commented, as they also had in the ENRP, on the surprises they had, both with student's understanding and with their lack of understanding, as students attempted to explain their thinking. Teachers reserved judgement about the usefulness of the exercise prior to the interviews but expressed the value of the experience of listening to the students in a one-on-one interview subsequently.

The teachers initially did not agree on the meaning of the symbols nor on the correct answers to the questions in figure 4 (from Fujii, 2003). This created a cognitive dissonance which led to considerable discussion about meaning in algebra and strategies for teaching that would lead to relational understanding rather than just procedural knowledge. Through discussions like this the use of the questions led, not only to the teachers gaining a greater knowledge of their students, but also to a greater knowledge of the mathematics and the related pedagogical content.

1. Amara had the following problem to solve "Find the value(s) for x in the following expression: $x + x + x = 12$ " She answered in the following manner A. 2, 5, 5 B. 10, 1, 1 C. 4, 4, 4 Which of her answer(s) is (are) correct? Circle the letter(s) for each correct answer.	2. Chu had the following problem to solve "Find the value(s) for x and y in the following expression: $x + y = 16$ " He answered in the following manner A. 6, 10 B. 9, 7 C. 8, 8 Which of his answer(s) is (are) correct? Circle the letter(s) for each correct answer.

Figure 4. Questions which challenged the teachers' knowledge of mathematics.

Apart from the interviewing process and the data thus collected assisting the teachers to reflect on student understanding and motivating them to follow up on some of the specific mathematical topics the collated information gained from the attitude section, shown in figure 5, has led to a new approach to the teaching at grade 9.

A. How do you feel about maths?

If they seem unsure of the question ask Do you like maths?

Negative answer	*Positive answer*
Have you ever liked it?	What do you like most about it?

If yes: When? What did you like (or enjoy)?

If no: Are there any things you like about it?

B. What things help you most when you are learning maths?

C. Are there any suggestions you would like to make about how we could make learning maths better?

Figure 5. Collecting attitude information.

What appeared to make a difference for the teachers at year 9 and 10 was that the students in year 7 and 8 were more positive about mathematics and when describing what they liked about mathematics collectively mentioned every topic with algebra, fractions and decimals being regarded very positively. The year 9 and 10 students confined their comments more to statements about teachers, teachers answering student questions and students being given clear notes and explanations. The apparent decrease or lack of increase in mathematical understanding for students from grade 8 through to grade 10 combined with a clearly different view of mathematics teaching and learning has contributed to a willingness to consider new approaches and new curriculum organisation. The teachers have asked for assistance in assessing students again to gain data on the effectiveness of the changes that they are implementing and to guide future planning at both grade 9 and grade 10.

Concluding comments

The use of interview assessment which enables teachers to listen to students and gain a greater knowledge of their thinking and conceptual understanding can provide motivation to teachers to seek further knowledge in other areas. The professional learning for teachers needs to be extended, providing them with opportunities to interact with colleagues discussing issues raised through their data collection and to become more informed through reading connected research findings. This requires research findings to be presented in a form easily accessible for teachers. All areas of knowledge need to be addressed in professional learning in an integrated fashion that maintains the relevance and connection with the teachers' interests. It is not enough for research just to study student learning and increase knowledge of children's developing mathematical understanding, language and misconceptions. Studies of different ways of presenting particular mathematical concepts with analysis of the effectiveness of the variety of possible approaches are needed as well.

References

Clarke, D. M., Sukenik, M., Roche, A., & Mitchell, A. (2006). Assessing fraction based understanding using task-based interviews. In J. Novotná, H. Moraová, M. Krátká, & N. Stehliková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (2, pp. 337-344). Prague, Czech Republic: PME.

Cooney, T. (1999). Conceptualizing teachers' ways of knowing. *Educational Studies in Mathematics 38* (1) 163-187.

Fujii T. (2003). Probing students' understanding of variables through cognitive conflict problems: Is the concept of a variable so difficult for students to understand? In N. A. Pateman, B. J. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of the International Group for the Psychology of Mathematics Education (PME) and PME-NA* (Vol. 1, pp. 49-65). Honolulu: University of Hawai'i.

Horne, M., Cheeseman, J., Clarke, D., Gronn, D., & McDonough, A. (2002). *Professional Development and Effective Teaching: Developing Students' Mathematical Thinking in the Early Years of School through Enhancing Teachers' Knowledge.* Paper presented to the Research Presession at the National Council of Teachers of Mathematics Annual Meeting, Las Vegas.

Küchemann, D. E. (1981). Algebra. In K. M. Hart (Ed.), *Children's Understanding of Mathematics,* (pp. 11-16). London: John Murray.

Sowder, J. T. (2007). The mathematical education and development of teachers. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning*. Charlotte, NC: NCTM.

RESEARCH FORA

Classroom research in mathematics education as a collaborative enterprise for the international research community: The learner´s perspective study

David Clarke and Jarmila Novotná - Coordinators

Examining teachers' use of (non-routine) mathematical tasks in classrooms from three complementary perspectives: Teacher, teacher educator, researcher

Ron Tzur, Orit Zaslavsky, and Peter Sullivan - Coordinators

Pursuing excellence in mathematics classroom instruction in East Asia

Yeping Li and Gabriele Kaiser - Coordinators

CLASSROOM RESEARCH IN MATHEMATICS EDUCATION AS A COLLABORATIVE ENTERPRISE FOR THE INTERNATIONAL RESEARCH COMMUNITY: THE LEARNER´S PERSPECTIVE STUDY

David Clarke
University of Melbourne

Jarmila Novotná
Charles University in Prague

The key features of the Learner's Perspective Study (LPS) methodology are the use of multiple video cameras to capture sequences of lessons, supplemented by post-lesson video-stimulated reconstructive interviews, and collaborative analysis of the resultant data set by an international team of researchers employing different theoretical perspectives. The multiple, parallel analyses being undertaken on the data, once generated, have fueled lively discussion regarding the complementarity and commensurability of the various theories being employed. The process and outcomes of undertaking international classroom research as a collaborative enterprise are evident in all the following contributions.

BACKGROUND

The Learner's Perspective Study was designed to examine the practices of eighth grade mathematics classrooms in a more integrated and comprehensive fashion than had been attempted in previous international studies. The project was originally designed to complement research studies reporting national norms of student achievement and teaching practices with an in-depth analysis of mathematics classrooms in Australia, Germany, Japan and the USA. The project was initiated by David Clarke, Christine Keitel and Yoshinori Shimizu. Since its inception, research teams from other countries have continued to join the Learner's Perspective Study. The title of the project (The Learner's Perspective Study) was intended to complement teacher-focused studies by foregrounding the learner's perspective. As the project grew, its purpose was progressively reinterpreted and expanded. Students, teachers and researchers can all be considered to be learners: partners in an international collaboration to develop new knowledge and to understand and improve the practices and outcomes of our classrooms. It is an essential thesis of the Learner's Perspective Study (LPS) that international comparative research offers unique opportunities to interrogate established practice, existing theories and entrenched assumptions.

General structure and topics

The Research Forum is structured around the following topics:

i. The challenge of international comparative classroom research
ii. Approaches to researching lesson structure
iii. Connecting the Learner's and the Teacher's perspectives

iv. Contrasting theoretical approaches to the analysis of classroom data (including the reflexive association between theory and methodology)
v. The results of international classroom research: Finding structure in diversity
vi. The capacity of international classroom research to inform practice

This choice of topics reflects the progression in the activities of the international LPS research community over the past nine years, but also sets out the challenges confronting anyone contemplating international comparative classroom research, through the essential methodological, technical and theoretical considerations, to the production of substantive research findings and the consequent question of how such international research might inform classroom practice.

THE CHALLENGE OF INTERNATIONAL COMPARATIVE CLASSROOM RESEARCH

Logistical and technical challenges in international research collaborations

David Clarke

It is imperative that research in mathematics education makes optimal use of available technology. International comparative classroom research, in particular, poses methodological and technical challenges that are only now being adequately addressed through advances in:

- techniques and equipment for the generation of audio-visual data in classrooms;
- tools for the compression, editing and storage of digitised video and other data;
- storage facilities that support networked access to large complex databases;
- data distribution systems that support secure, remote access for data entry and retrieval on an international scale; and
- analytical tools capable of supporting sophisticated analyses of such complex databases.

The LPS community has addressed each of these challenges.

All too often it is forgotten that any use of technology in a research setting implies the existence of an underlying theory on which the type of data, the means of data generation, and the anticipated method of analysis are all predicated. Clarke (2001 and 2006) has argued that since a classroom takes on a different aspect according to how you are positioned within it or in relation to it, our research methodology must be sufficiently sophisticated to accommodate and represent the multiple perspectives of the many participants in complex social settings such as classrooms.

The LPS data have been generated for sequences of at least ten consecutive lessons occurring in the "well-taught" eighth grade mathematics classrooms of three teachers in fourteen of the participating countries (Australia, China, the Czech Republic, Germany, Israel, Japan, Korea, Norway, The Philippines, Portugal, Singapore, South Africa, Sweden and the USA). This combination of countries gives good representation to European and Asian educational traditions, affluent and less affluent school systems, and mono-cultural and multi-cultural societies.

Each participating country used the same research design to collect videotaped classroom data for at least ten consecutive math lessons and post-lesson video-stimulated interviews with at least twenty students in each of three 8th grade mathematics classrooms. The three mathematics teachers in each country were identified for their locally-defined 'teaching competence' and for their situation in demographically diverse government schools in major urban settings. The three lesson sequences were spread across the academic year in order to gain maximum diversity of local curricular content. Post-lesson student interviews were conducted, in which a split-screen video record was used as stimulus for student reconstructions of classroom events. Students were given control of the video replay and asked to identify and comment upon classroom events of personal importance. Each teacher was interviewed at least three times using a similar protocol.

The *Learner's Perspective Study* is committed to (i) adequate recognition of the perspectives of all participants and specific embodiment in the data generation of those perspectives, (ii) deliberate utilisation of both primary and secondary analyses to provide a wide range of theoretical perspectives on the social setting and situations being studied, (iii) the synthesis of the subsequent primary and secondary analyses into an integrative amalgam of interrelated complementary accounts (Clarke, 2006), and (iv) the development of "practical explanatory theory" (Nuthall, 2004, p. 295) by which classroom activity is connected to learning outcomes.

THE QUESTIONABLE LEGITIMACY OF INTERNATIONAL COMPARATIVE CLASSROOM RESEARCH

Eva Jablonka

Luleå University of Technology

The questionable legitimacy of comparison based on common sense approaches to culture

The ICMI launched its 13th study with the title "Mathematics education in different cultural traditions: A comparative study of East Asia and the West". The discussion document (ICMI, 2001) stated: "For this study, culture refers essentially to values and beliefs, especially those values and beliefs which are related to education, mathematics or mathematics education." The West is, in this ICMI-study, identified with the Greek, Latin and Christian tradition. Such a framing of comparative research runs the risk of oversimplifying the situation by appearing to assume that school systems or classrooms can easily be aligned with one of these traditions. Wong and Wong (2002) point to the fact that the "examination culture", which was designed for governance purposes in China, might have a much greater impact on achievement orientation than Confucianism.

There might be a lot of other appropriate definitions of regions to compare. The content of school curricula is linked to political and economic characteristics, including colonial history. A case could be made for grouping former British colonies or for grouping Islamic countries. In some studies a simplistic interpretation of "culture" as synonym for nation is adopted. This identification of culture with nation or geographical region is only reasonable if it refers to the use of the same language or of socially significant different forms of it, or to the commonalities of the institutional setting and its tradition. Categorizing school systems and the environment in which they are embedded by country or by geographic location, does in any case not take into account that in many countries, classrooms are inhomogeneous in terms of ethnic affiliation. In addition, taking countries as units of analysis conceals differences within provinces or states, for example in countries like Germany and the United States. It has to be acknowledged that cultural phenomena do not occur in a social and economic vacuum.

Representativeness versus typification

One goal of comparative classroom research has been typification of elements of practices that are interpreted as being representative of mathematics teaching in a distinct cultural context. Such a research can produce valuable insights about differences between contexts and the amount of variation within one context. Based on a representative sample of mathematics lessons from Finland and Iceland, Savola (2008), for example, found a common lesson structure in the lessons from Finland. In Iceland, on the other hand, half of the lessons exhibited variations of a Review-

Lesson-Practice structure, whereas the other half followed a totally different pattern. While the notion of representativeness refers to statistical practice, typification is linked to ethnographic methodologies and phenomenology. However, describing data from classroom observations in terms that make the data comparable is a shared goal. For the LPS, the identification of patterns that are representative of the teaching in a nation was not the goal, but rather to compare and contrast different elements of classroom practices in a variety of school systems. The selection of classrooms was based on identifying competent teachers based on local criteria of what being a competent teacher might mean. Whether any national pattern identified on a base of a representative sample of lessons in a country manifests itself in the classrooms chosen for the LPS, has been analysed for Germany, Japan and the United States (Clarke et al, 2007). It turned out that the variation between the lessons of different teachers in each country was considerable. There was no evidence of the lesson patterns reported for these nations by Stigler and Hiebert (1999). The LPS classrooms cannot be seen as representative in a statistical sense. Still, the classrooms are typical for practices, in which some pedagogical values that define the notion of "competency" of a mathematics teacher are in operation.

The data produced in the LPS have the potential of identifying the lesson elements not only on the base of classroom observation, but include the insiders' perspective with the help of the post-lesson student interviews. Unfamiliar forms can be more easily identified by the outsider, but are harder to be interpreted. In turn, it is harder for the insider to look for alternative interpretations. This remains a general issue for qualitative research in cross-cultural settings (cf. Hoonaard, 1997; Udy, 1964).

APPROACHES TO RESEARCHING LESSON STRUCTURE

Instructional Units for Cross-Cultural Analyses of Classroom Practice

David Clarke

University of Melbourne

The contention of Stigler and Hiebert was that at the level of the lesson, teaching in each of Germany, Japan and the USA could be described by a "simple, common pattern" (Stigler & Hiebert, 1999, p. 82). This proposal was based on analysis of a "nationally representative" sample of single lessons. By contrast, the Learner's Perspective Study (LPS) conducted a fine-grained study of sequences of ten lessons, informed by the reconstructive accounts of the participants. Such a study has the capacity to identify any recurrent pedagogical elements in a teacher's classroom practice and any evidence of regularity in the sequencing of those elements. Such regularities and recurrent elements have the potential to serve as the basis for comparative analysis.

Lesson structure can be interpreted in three senses:

i. At the level of the whole lesson – regularity in the presence and sequence of instructional units of which lessons are composed;
ii. At the level of the topic – regularity in the occurrence of lesson elements at points in the instructional sequence associated with a curriculum topic, typically lasting several lessons;
iii. At the level of the constituent lesson events – regularity in the form and function of types of lesson events from which lessons are constituted.

In terms of international comparison, it is useful to consider which of these three forms of lesson structure are likely to prove useful as units of comparative analysis. The same three alternatives are available for the purposes of national typification, but the optimal unit of international comparison need not be the same as the optimal unit for national (or cultural) typification. We can conceive of the possibility of an idiosyncratic practice that might typify the classrooms of a nation, but be so unusual as not to constitute a legitimate basis for international comparison.

In terms of lesson structure, it might be that for one nation or culture there is no nationally characteristic structure to the lesson as a whole, but that particular types of idiosyncratic lesson events offer the most appropriate typification. For another nation or culture, there could be a high degree of regularity to the composition of lessons, or in the sequencing of particular types of instructional activity in the delivery of a topic. Such differences in the form of typification provide a basis for international comparison that reflects something more essential to each than the imposition of the same structural level as the basis for the comparison.

Incommensurability of the emergent typifications becomes relevant if the comparison is intended to be evaluative. However, in the case of the LPS, the identification of idiosyncratic practices, identified in one or a few classrooms but absent entirely in other classrooms, offers the teachers of those other classrooms entirely new pedagogical tools, potentially valuable, since they derive from the practices of competent teachers elsewhere.

The teachers whose classrooms we had documented showed little evidence of a consistent lesson pattern, but instead appeared to vary the structure of their lessons purposefully across a topic sequence. The evident differences in the manner in which teachers structured their lessons, suggested that another unit of analysis was needed: one that corresponded more closely to the decisions made by each teacher regarding the structure of any particular lesson. Our analysis of the LPS lessons focused therefore on the form and function of recognizable activity conglomerates that we called 'lesson events.'

Each individual lesson event had a fundamentally emergent character, suggested by the classroom data as having a *form* (visual features and social participants) sufficiently common to be identifiable within the classroom data from each of the countries studied. In each classroom, both within a culture and between cultures, there were idiosyncratic features that distinguished each teacher's enactment of each lesson event, particularly with regard to the *function* of the particular event (intention, action, inferred meaning and outcome). The teacher and student post-lesson interviews offered insight into both the teacher's intentions in the enactment of a particular lesson event and the significance and the meaning that the students associated with that event type.

Each lesson event required separate and distinct identification and definition from within the international data set. Lesson events included: Kikan-Shido (between-desks-instruction), beginning the lesson, the learning task, student(s) at the front, putting into practice, and Matome (summing up), and detailed analyses related to these lesson events can be found elsewhere (Clarke, Emanuelsson, Jablonka & Mok, 2006; Clarke, Keitel, & Shimizu, 2006; Clarke, Mesiti, O'Keefe, Xu, Jablonka, Mok & Shimizu, 2007).

AN ANALYSIS OF JAPANESE LESSONS ON LINEAR FUNCTIONS

Minoru Ohtani

Kanazawa University

INTRODUCTION

Recent research studies have a common and persuasive vision of mathematics classroom as socioculturally mediated milieu. Different classroom cultures mediate different beliefs, attitudes, and contracts with respect to classroom interaction, and with respect to mathematical activity (Novotná & Hospesová, 2007). In everyday classroom practice, teacher and students coordinate the extent to which they participate in a particular mathematical activity, their role in accomplishing it, and the extent to which they take direct responsibility for accomplishing it (Clarke, 2003).

The purpose of this paper is to investigate "mathematical task structure" and "participation structure" in a Japanese (J1) classroom. By "mathematical task structure", we mean the way the teacher elaborately organizes mathematical tasks throughout the unit. By "participation structure", we mean the way the teacher coordinates the extent to which students participate in accomplishing mathematical tasks (Ohtani, 2002). The focus is on patterns of distribution of participation rights allocated for teacher and the students in accomplishing mathematical task. Transcripts of video-audio records of ten consecutive J1 lessons on linear equation were analysed.

Task Structure and Participation Structure

For "mathematical task structure", we find three ordered components; "contextual tasks" (Day1-3), "transitional tasks" (Day 4-7), and "general tasks" (Day 8-10). Each component performs a unique role: setting mathematical motive; guided use of symbolic devices or cultural tools, and appropriation of mathematical object, respectively.

The first component involved establishing the motive for functional thinking. Contextual tasks in concrete situations serve as a continuous reference and model of quantitative relations. The teacher poses an open-ended contextual task and expects students with different perspectives to find many kinds of dependent and independent variables.

The second component consisted of the progressive transition from contextual task to referential tasks. The role of transitional tasks is to guide students to use symbolic devices such as table, graph, and algebraic expressions as cultural tools. During initial use, the symbolic devices have an operational aspect for computing particular values. The principal function of symbolic devices is making the transitory constantly present and, at the same time, providing tangible means of communicating their idea and conjecture in particular concrete situations to others. This function is called "intermental" (Vygotskii, 1984).

In the third component, the teacher proceeds from referential to general tasks. The role of general tasks is to employ "linear function" as an abstract object, where the symbolic devices function not only as a means to solve decontextualized problems but also as objects representing the linear function itself. The teacher introduces problem conditions, which contain defining characters, and the mathematical terminology of linear functions. Students engage in solving these problems using symbolic devices in order to find invariant properties of linear functions. Such a use of symbolic devices is called "intramental" (Vygotskii, 1984).

Teacher's Strategy for Organising Lesson Structure

For describing the teacher's strategy for organizing "task structure" and "participation structure", I draw on the concept of "revoicing" (O'Connor & Michaels, 1996). By "revoicing" we mean a particular kind of re-utterance of one's contribution by another participant in a discussion.

Analysis of the data showed that two kinds of revoicing were extensively and exclusively used by the teacher during classroom interaction. One was "public revoicing" and the other was "measured revoicing". For public revoicing, the teacher not only replied to nominated individual students, but also addressed all the students. This means that the teacher capitalized on particular students' contributions to address the whole class in order to promote collective reflection. "Publicity of revoicing" was obvious during student independent work. For "measured revoicing", the teacher expected a variety of student responses to the assigned task and had a plan to capitalize on their contribution in order to formulate challenging problems and elaborate their solutions through collective argumentation.

CONNECTING THE LEARNERS' AND TEACHER'S PERSPECTIVES

Capturing Complex Classroom Interactions

Gaye Williams

Deakin University

Background of participation in the LPS

Participating in the Learners' Perspective Study (LPS) during PhD research provided research opportunities that would otherwise not have been available. The research design captured multiple perspectives on classrooms interactions that enabled study of what supported students during their creation of new (to the student) mathematical ideas. Taking a role in ensuring across country consistency of application of the design protocols gave a broad perspective of what it could mean to teach and learn mathematics. My studies were enriched by these opportunities.

The LPS team has at least one international meeting per year and in some years a Learners' Perspective Conference in Melbourne, and a retreat to Wilson's Promontory, which is a peaceful sanctuary in an isolated beach side area in Victoria, Australia. This provided many opportunities for researchers to get to know each other and discuss ideas. This study commenced with researchers from different countries sharing their research perspectives. During the study, we alerted each other to aspects of our country's data that could be relevant to another's focus. In addition, we probed interview responses further where we recognised the relevance of a response to another team member. Thus, I had a 'research team' who willingly alerted me to data that might be relevant, and sometimes even generated such data (Williams, 2005).

Team discussions about consistency of application of the study design across countries, and my participation in initiatives to gain this consistency, helped me to appreciate how differently a research design can be interpreted without such initiatives.

Visiting Year 8 classrooms in four countries (Germany, the USA, the Philippines and South Africa), focusing intently on classes in two other countries, discussing teaching and learning with research teams, and sharing my own observations about these classrooms from the perspective of my study, broadened my perspective, and helped me to communicate and crystallise my ideas.

LPS Team Membership: developing research rigor

In 2003, when I presented my findings at an LPS conference, I intended to show the importance of 'spontaneity' in the creative development of new knowledge. I was surprised to find this group of experts, who were empathetic to my research, did not understand what I was trying to communicate. The questions they asked me, and the intensity of the subsequent discussion, helped me to realise I needed to develop the construct of spontaneity more rigorously and illustrate it empirically.

The power of the LPS research design

In addition to the multiple perspectives enabling triangulation, the data collection techniques provided opportunities to ‘retrieve’ data when its significance later became apparent. Take the example where the student (Leon) stated:

> When you look around the classroom and see how everyone else is doing it and you are doing it a completely different way- … and you think ooh! maybe my method isn’t the best and … you think about everyone’s ... and then you think about your own and they all sort of piece together and you just sort of go oh! and it pops into your head (Williams, 2006a, p. 227).

What had Leon seen? Had other students already found what “popped into” Leon’s head? Or did he really develop this idea for himself by integrating what he could see on the pages with others with his own developing ideas? A search of the whole class, teacher, and focus-student videos suggested other students were undertaking the problem in less sophisticated ways, but not all student pages were visible on the videos. In her interview, the teacher confirmed that other students had used less sophisticated approaches. Knowing what was on other students’ pages assisted my analysis of how Leon developed his insight. Without access to another data source, the conclusions could not have been held with the same strength. Multiple secondary analyses of the data were employed to support interpretations made. The research design contributed to the insights developed.

THE USE OF WORK PLANS IN SIX NORWEGIAN 9TH GRADE MATHEMATICS CLASSROOMS

Ole Kristian Bergem and Kirsti Klette

University of Oslo

Our empirical material was collected from six 9th grade classrooms in Norway. In all these classrooms, work plans were used as an organisational and didactical tool. The work plan is a document that describes what the students are supposed to do in the different subjects over a certain period of time, often two or three weeks. The idea behind work plans is to empower the students and give them the opportunity to make decisions related to their own work at school: *what to do, how to do it, when to do it, and with whom*. Work plans are in this way supposed to stimulate and facilitate self-regulated learning by making the students assume responsibility for their own learning processes (Klette 2007). In all the six mathematics classes, the content of the work plan in mathematics was decided by the mathematics teacher only. The students did not participate in composing the plan.

The work plans were not individual, but did contain some sort of level differentiation, usually three. It was up to the individual students to decide which one of these three levels to follow. The students could choose different levels from one work plan period to another. The three levels would usually cover the same mathematical themes, but were differentiated either by the amount of tasks connected to each level, by task difficulty, or by a combination of these two criteria. Common to all the schools was the practice of allocating time for the students' handling of their work plans.

Observation of student behaviour during math lessons, especially the study/guidance-lessons, revealed that different strategies were being used in relation to the handling of the assignments on the periodical work plan. This was to a large extent confirmed in the interviews; through the students' own explanations of how they strategically positioned themselves in the handling of this plan. Basically these strategies seemed to fall into three categories:

1. To postpone the work in mathematics to the end of the work plan period.
2. To finish the work in mathematics in one or two days at the beginning of the work plan period.
3. To apportion the work in mathematics throughout the work plan period

Students' reasoning for choosing these strategies varied quite a bit.

The first position is characterized by students who try to postpone the work until the very end of the work plan period. Especially at two of the schools this was a strategic positioning that the majority of the boys seemed to embrace. At these schools we observed that, while the majority of the girls were able to apportion their work and disperse it throughout the whole work plan period, nearly all the boys waited until the end of the last week to put any effort into completing their assignments.

The second strategic positioning involved students completing their math-assignments for the whole work plan period in just one or two days at the beginning of the period. The students presented two reasons for the choice of this strategy. The first one was connected to the pronounced wish of finishing the math assignments as fast as possible, because it was boring to work on. The second reason was that since mathematics was their favourite subject, they just couldn't wait to work on the new assignments. Even if these stated reasons differed quite a lot, the consequences of both were quite similar; all the students in this group would finish their math assignments in just a couple of days at the beginning of the work plan period.

The third strategic positioning that seemed to attract certain students was to disperse the work throughout the period. Most of the students that consciously chose this strategy, and could account for it in the interviews, were high achievers. They were quite articulate in arguing that this was the best way of securing high grades. Many of them also had quite high ambitions for their future careers.

Summary

A central characteristic of the LPS study is the documentation of the teaching of sequences of lessons, rather than just single lessons. Using this research design, we have been able to document that the use of work plans in mathematics gives the students the opportunity to choose strategies that mean they will only work with mathematics one or two days during a work plan period of two/three weeks. For these students the consequences seem to be that there is little continual work in mathematics, it is all about completing a certain number of tasks. This is usually not regarded as an optimal way of working with mathematics. On the contrary, several theories of learning and instruction emphasize regular, step-by-step learning opportunities.

The results of our study can be used as a basis for discussion of the practice of using work plans as an organisational and didactical tool in mathematics classrooms and to help us to understand discrepancies between student and teacher perspectives.

CONTRASTING THEORETICAL APPROACHES TO THE ANALYSIS OF CLASSROOM DATA

Studying Students' Creative Development of New Mathematical Knowledge

Gaye Williams

Deakin University

Creative activity accompanied by high positive affect ('flow') can occur when people spontaneously set challenges and develop new skills in order to overcome them (Csikszentmihalyi, 1992). Creative thinking during flow specific to mathematical problem solving can occur when students discover a mathematical complexity of which they were previously unaware, and decide to explore it. Flow conditions include spontaneously setting an intellectually challenging question about the mathematical complexity, and exploring this question using non-routine mathematics (Williams, 2005). The thinking framework to study cognitive activity (Williams, 2005) was formulated by integrating aspects of thought processes identified by others (Krutetskii, 1976; Dreyfus, Hershkowitz, & Schwarz, 2001). The construct of student spontaneity was elaborated by subcategorising social elements of the abstracting process (Dreyfus, Hershkowitz, & Schwarz, 2001) into those from internal and external sources (Williams, 2005). Undertaking such activity involves moving from what is known to what is unknown, and there can be many failures before success is achieved. Some students are not inclined to undertake such activity (Seligman, 1995).

Flow activity during mathematical problem solving is illustrated through the activity of a Year 8 student, Eden (Williams, 2007), who found he could not position linear functions to 'shoot' 'globs' on a Cartesian plane in a computer game. Eden was not aware of connections between algebraic forms of linear functions and their positions as graphs. He observed and reflected on a dynamic visual display as it was generated by another student. This display showed a family of parallel lines appearing one after another on the screen as the student undertook a trial and error process to try to hit globs. Eden identified a pattern between the x and y values of co-ordinates of each point on the same line. He thought he saw a link between this pattern and the algebraic equation at the bottom of the screen. He returned to his own computer, and experimented. After seven minutes of intense activity, he left his computer screen and exclaimed softly to himself. He had confirmed that his patterns were expressed by the algebraic equation. He had developed new conceptual understanding that linked algebraic, numerical, verbal, and graphical expressions for linear functions (conceptual knowledge novel to Eden). Eden recognized patterns, and built with them by expressing them verbally and algebraically, and then experimenting to see whether there was always a link between the pattern and the equation. He synthesised to gain insight when he realised that all of the information he had found about the graph could be 'held' in the linear equation and 'unpacked' as needed.

In his interview, Eden described how he problem solved in mathematics. He perceived failure to understand as temporary and able to be overcome with effort.

> You just have got to sort of think out the answers in your head (pause) occasionally you have gotta- got to write down on paper what you are thinking about (pause) and eventually get the answer (Williams, 2006b, p. 397).

My insights into Eden's creative thinking relied upon analysis of the teacher video to find what the students were told at the start of the lesson, the focus student video to find whether other students provided mathematical input to Eden's exploration, the whole class camera to determine that Eden was not interacting with others during his seven minutes of exploration, and all three videos in the next lesson to make sure others had not contributed to Eden's understanding prior to his interview after that lesson. The video-stimulated interview assisted Eden to remember detail, and communicate his thoughts in class. This interview drew attention to the parts of the lesson when Eden had developed new knowledge, and provided indicators of his inclination to explore. The LPS data collection processes were crucial to this study.

THEORY OF DIDACTICAL SITUATIONS IN MATHEMATICS

Jarmila Novotná — Charles University Prague
Alena Hošpesová — University of South Bohemia

One of the important questions currently discussed in the LPS community is "What are the related existing mathematics education research methodologies that can serve as the theoretical framework for drawing new results from the LPS resources?" In this contribution, the relationship between the LPS and the Theory of Didactical Situations in Mathematics (Brousseau, 1997) is presented.

Why just this theory? Laborde and Perrin-Glorian (2005, p. 2) state that "...(classroom) is the place of social interrelations between the teacher and students shaped by the difference of position of the two kinds of actors with respect to knowledge and giving rise to *sociomathematical norms* (Yackel & Cobb, 1996) or to a *didactical contract*" (Brousseau, 1997). Our analyses of sets of videotaped lessons (Binterová, Hošpesová, & Novotná, 2006; Novotná & Hošpesová, 2007; Novotná & Hošpesová, 2008) are based on the theory of didactical contract. The implicit nature of Brousseau's concept of "didactical contract" is fundamental when explaining environment effects on learning mathematics (Sarrazy & Novotná, 2005).

Elements from the Theory of Didactical Situations in Mathematics (TDSM)

In TDSM, a learning process is characterized as a sequence of identifiable situations (natural or didactical), reproducible and leading regularly to the modification of a set of behaviours of the students, modifications that are characteristic of the acquisition of a particular collection of knowledge (Brousseau, 1975). Brousseau considers the conditions of a particular use of a piece of mathematical knowledge to form a system, which he calls a *"didactical situation"*. (In *non-didactical situations*, the evolution of the learner is not submitted to any didactical intervention whatever.) Didactical situations which are partially liberated from direct teacher's interventions are called *a-didactical situations.* In TDSM, situations are classified according to their structure (action, formulation, validation, institutionalization, etc.), which determines different types of knowledge (implicit models, languages, theorems, etc.).

The process by which the teacher manages a didactical situation by putting the learner in the position of a simple actor in an a-didactical situation is called *devolution*. Devolution does not only propose a situation to the learner which should provoke him/her to an activity not previously agreed, but also makes him feel responsible for obtaining a proposed result, and that the solution depends only on the use of knowledge which he/she already has.

Environmental effects on learning mathematics are explained using *didactical contract*, i.e. the set of the teacher's behaviours (specific to the taught knowledge) expected by the student and the set of the student's behaviour expected by the teacher. This contract is not a real contract; in fact it has never been “contracted” either explicitly or implicitly between the teacher and students and its regulation and criteria of satisfaction can never be really expressed precisely by either of them. One of the fundamental teacher’s tasks in a didactical situation is i*nstitutionalisation*, i.e. the passage of a piece of knowledge from its role as a means of resolving a problem or proof to a new role that of reference for future personal or collective uses.

TDSM and LPS

A significant distinguishing characteristic of LPS is its documentation of the teaching of *sequences* of lessons, rather than just single lessons. The main goal is to produce empirical analysis of pedagogical phenomena based on well recorded “reality”. In the following overview, the topics and related clusters of questions in the framework of TDSM that we tried to answer are summarized:

<u>Didactical contract (Binterová, Hošpesová, & Novotná, 2006; Novotná & Hošpesová, 2007)</u>: Can we trace hidden didactical contract established in a particular classroom and illustrate it by suitable teaching episodes? What influence of the didactical contract on students' mathematical knowledge can be presupposed? How does it support or constrain learning? How does the teacher create a secure, confident work environment for the students in the classroom?

<u>Topaze effect (Novotná & Hošpesová, 2007)</u> How does Topaze effect reflect teacher’s beliefs? How does Topaze effect influence students’ work? What types of Topaze effect we can find in Czech lessons?

<u>Linking in mathematics lessons (Novotná & Hošpesová, 2008):</u> What are the prerequisites that the teacher refers to when solving new problems, developing new domains of school mathematics (these teachers’ actions are called “linking”)? What types of linking are used by teachers and how does their use influence students’ behaviour in the classroom and their understanding of mathematics?

Endnote

This research was partially supported by project GACR 406/08/0710.

THE RESULTS OF INTERNATIONAL CLASSROOM RESEARCH: FINDING STRUCTURE IN DIVERSITY

Similarities in students' perspectives, classroom discourse and lesson elements

Eva Jablonka

Luleå University of Technology

Students' motivations and the meanings they attribute to classroom activities

How students view their learning environment and why they choose to (or not to) participate in classroom activities has an impact on knowledge development. For the analysis, post-lesson interviews with 109 students referring to 60 lessons in LPS classrooms from Germany, Hong Kong and the U.S.A. were used. The findings have been compiled with a focus on similarities (Jablonka, 2005). The similarities found reflect how the students attribute meaning to distinct aspects of the classroom practices, which they regard as constitutive for their learning. The students' motives were linked to the expectations they held and corresponded to the possibilities the classroom practice offered. Classroom practices obviously shape behavior and thought and can be taken as the premises on which the students' (and the teachers') ways of trying to succeed are based.

In total, 52 students talked about passing tests and examinations or about their grades in the interviews. A total of 13 students made statements in which they associated mathematical activities with "*thinking*". (HK1: 8, HK3: 3; US1: 1, G3: 1). Some students referred to acquiring *knowledge for everyday and professional practices*. Everyday practices comprise managing a bank account, shopping, dealing with rents, salaries, fees, taxes, or buying and renovating a house, computer use, uncovering cheats. However, these examples do not refer to the topics of the lessons videotaped. Much more students from US2 than from the five other classrooms (that is 11 compared to 1-3) make reference to understanding why, finding patterns and establishing connections. This is entirely in accordance with the teacher's goals. A total of 35 students from all six classrooms employed the metaphor of carrying out steps, 'doing things' and obtaining results.

Mathematical reasoning

Instances of a mathematical reasoning discourse have been identified on the basis of the lesson transcripts (Jablonka, 2004). It turns out that these were rare in all classrooms. This is not to say that interactive involvement of students was infrequent. Much of the interaction in which the teachers addressed the whole class involved the students interactively by posing questions. The questions often were a request to provide a reason. If counted as single events, the teacher asked the students to provide reasons many times in each lesson in US1. Similarly, in G1 the teacher asked the students to provide reasons a couple of times in most of the lessons. In US2 this happened only occasionally in half of the lessons. It is even less frequent in the Hong

Kong classrooms and in G3. Self-initiated reasoning on the part of the students in public talk was rare in all the classrooms except G3.

Lesson structure

In all the six classrooms teacher and students were frequently engaged in a pattern of interaction that was labeled ‘questioning-developing classroom talk’. This is a mode of whole class interaction in which the teacher interactively involves the class by asking a series of connected questions. The definition employed was intended to include a variety of forms (distinguished in terms of degrees of openness and closeness) and functions (setting a new task, review, application or developing new content). When the teacher’s questions are connected and aim at collectively developing new knowledge, these episodes can be interpreted as a form of a reasoning process. The teacher and the students collectively provide a chain of (minor) premises (“reasons”) and implicit inferences, though the discourse contains at most a few utterances that can be interpreted as a request for, or a provision of, a reason. The teacher pre-structures the discourse by breaking down a chain of inferences into a series of questions, which - if answered accordingly - in sum warrant the resulting conclusion, that is the solution of the task. It can be argued that this form of interaction implies a systematic transformation of mathematical reasoning as found in the context of knowledge production, that is, the mathematician’s mathematics or ‘inquiry mathematics’. This transformation is linked to other features of classroom practice that are considered by the students as important for their learning.

CODING CATEGORIES AND TIME SCALES AS ANALYTICAL DEVICES

Kirsti Klette and Ole Kristian Bergem

University of Oslo

The benefits of video data are obvious (in terms of the tool's ability to freeze, recapture and visualize learning situations). The analytical challenges and efforts this poses for the researchers are equally obvious. There is an obvious risk to get lost in the details in video data. In our analyses of the LPS data (LPSoslo) we have used coding categories and time scales as analytical tools for getting an overview of the material. In addition individual researcher(s) have made more specified and focused analyses based on delimited and fixed research interests.

Coding categories/ Coding schemes as complexity reduction strategies

We used coding categories/coding schemes operating on two levels as analytical devices for complexity reduction. The levels differ in focus. The first level (level 1) aims at capturing some central features of how the teachers design the classrooms as learning sites (in terms of instructional format, grouping arrangement, subject matter involved etc.) while the second level of analyses (level 2) concentrates on content driven teaching activities and interaction patterns across the same sites and classrooms (such as language features, dialogue initiatives and teachers' use of different didactic tools, such as reviewing, summarizing, developing new content knowledge, 'going over the do now' etc).

Scales as analytical devices

Levels of time scales are equally another important factor that impregnates the conclusions that can be drawn from our analytical endeavors. Interpreting meanings involved in the classrooms observed is closely linked to time scales of interpretations. For these interpretations the researcher, for example, analyzes short time interactions in order to then situate them within longer time segmentations such as episodes and themes, which themselves subsequently could be situated in larger scales like sessions, the whole teaching sequence etc. Lemke (1990) makes a distinction between macro, meso and micro analyses as three levels of time scale for analyzing classrooms activities. In our analyses, we have used meso level (teaching sessions) and micro level (teaching segments) as two units of analyses. By using a combinations of meso and micro analyses, different time scales open for multiple interpretations and findings.

Findings

In all classrooms, students were seated in groups or pairs. The instructional format was organised around teacher-led whole class instruction and individual seat work. Group activities were rare in the observed classrooms, and the students were given few opportunities to discuss, analyse and talk mathematics with their peers, despite physical arrangements that could fuel such collaboration.

Teacher-led whole class instruction and individual seat work were the two most frequently used instructional activities in all observed math classrooms. The content-focused analyses of teacher activities (level 2) supported and elaborated on these findings and revealed “developing new canonical knowledge” and “offering seat work” as the two most recurrent teacher content driven activities.

There was ample room for student initiative in the mathematics lessons we observed. Despite a prevalence of teacher-led whole class instruction, students were given abundant space for initiatives and questions in the observed classrooms and, consequently, teacher-led whole class instruction did not equate to teacher monologues and recitation patterns.

Looking across categories and scales - Findings taken together

- The teacher repertoire was rather narrow, dominated by the two codes “developing new canonical knowledge” and “offering seat work”. This made the mathematics lessons quite monotonous.
- Students frequently initiated classroom discourse in the mathematics classroom. Across the six schools the percentage of classroom discourse that was student initiated varied between 35 and 65, which indicates that students' active participation was granted a significant space in terms of speech initiative and turn-taking. Compared with former studies addressing the issue of participation in classroom discourse, our studies indicate a significant change in classroom practice, from teacher dominated instruction (i.e. teacher monologues) to teacher-led discourses characterized by vast opportunities for student initiative.

AN ETHNOMETHODOLOGICAL ANALYSIS OF TEACHER'S STRATEGY FOR MANAGING LEARNERS' DIFFERENT IDEAS

Minoru Ohtani

Kanazawa University

INTRODUCTION

Recent research has a common and persuasive vision of mathematics classrooms as a discursive practice. This ethnomethodological study investigated how a Japanese mathematics teacher (J2) used strategy for managing learner's different ideas.

Ethnomethodology investigates members' accounting practices to attain the factual character of the social reality. The facticity of the sense is maintained by interpretive work. However, expressions are vague and equivocal, lending themselves to several meanings. The sense of these expressions is a product of the very way we look at something and talk about it. There is often a competition over the correct, appropriate or performed way of representing objects, events, or people. Proponents of various positions in conflicts waged in and through discourse attempt to capture or dominate modes of representation. The competition over the meaning of ambiguous events, people, and objects in the world has been called the "politics of representation" (Mehan, 1993: 241). It is tenable that a similar competition over the meaning of events and objects is played out in mathematics classroom discourse.

UNIT OF ANALYSIS

If the social formation of mathematical practice in the classroom is to be analyzed, *social* and *mathematical* dimensions as a whole have to be taken into account (Ohtani, 2000). Thus, we need a "unit of analysis" (Vygotskii, 1982). In his *Proofs and Refutations*, Lakatos (1976) portrays historical debates within mathematics about what a proof of a theorem represents by constructing a argumentation among a group of students that contains mixed within it many conceptual horizons among mathematicians through the last several centuries. In the midst of an argumentation, revised definitions and conditions are progressively introduced in light of refutations. It seems that formation and revision of definitions and introducing conditions are indispensable and essential components that constitute mathematical discourse.

RESULTS AND FINDINGS

My analysis of the Japanese (J2) data corpus found that mathematical definition and condition operate in classroom interaction as social and multi-consequential devices to coordinate and sustain interaction. Telling definitions and introducing conditions function as social resources widely used in order to negotiate certain representation of a problem situation rather than as cognitive resources used to analyze and describe the problem situation and to construct mathematical dialogue. Definitions and conditions are characterized by a monologic suppression by an authoritarian voice, rather than by a dialogicality of voices. It is as if an invisible barrier has been placed around the topical space that is eligible for discussion. The formulation to which the student has privileged access is not motivated by the needs of an instructional activity. The condition functioned to regulate the student's mathematical activity in ways that were appropriate for the classroom setting. The condition was characterized as directive. And by introducing the condition, the student was constrained to engage in a process sanctioned and regulated by the teacher.

CONCLUDING REMARKS

In sum, telling mathematical definitions and conditions involves the following social functions: to sanction and defend unexpected or insignificant interactions with students; as a means to defeat students' ideas and proposals; to justify the teacher's control over students; to show that the student's proposal is unrealistic; to terminate a student's request and to attain a degree of uniformity of what it transmits. Thus, in place of diversity or heterogeneity, the act of telling mathematical definitions is designed to get the student to participate in formulating the problem in particular way. It serves as a method of managing the teacher's asymmetrical relationships with his students; and it tempers the teacher's obligation to be knowledgeable about the affairs of students.

THE CAPACITY OF INTERNATIONAL CLASSROOM RESEARCH TO INFORM PRACTICE

Connecting International Research and Teacher Professional Development: A Personal Experience in China

Rongjin Huang

Texas A&M University

I have made some contribution to math teacher education in China through sharing LPS methodologies and findings, conducting in-service and pre-service teacher education programs. In this presentation, I would like share my personal experience in the following aspects: (1) disseminating LPS methodology and findings; (2) comparing and contrasting teaching strategies in pre-service teacher education program; (3) demonstrating effective teaching in school-based teaching research projects.(4) complementing teaching ideas in summer courses; (5) sharing Chinese mathematics instruction internationally. I give a selection of examples below.

Comparing and Contrasting Teaching Strategies in Pre-Service Teacher Education Program

In my secondary prospective teacher education program, I use LPS methods and materials to compare the characteristics of mathematics classroom instruction in different cultures. On this basis, we encourage students to figures out what effective mathematics teaching looks like in China.

Shaping School-Based Teaching Research Project

Heavily influenced by LPS, I have participated in and led some school-based in-service teacher professional development.

Since 2003, I have participated in a national wide in-service professional development, called as *Xingdong Jiaoyu* (a teacher education programme using action research methods), in Shanghai, China. The LPS methodology has directly influenced the project (Huang & Bao, 2006). Later, the project was developed into Hypermedia Video Case Study (VCS) and popularized around China (Bao & Huang, 2007).

During 2006-2007, I led a school-based teacher professional development project in Macau. The aim of the project was to pursue effective teaching through reflecting and improving classroom instruction practice. This project included the following phases: Theory learning, classroom observation, field investigation, exemplary lesson study, and experience sharing and reflection. As a product of this project, we developed a multiple media Video Case. The LPS study has influenced this project in two ways: First, through showing some video clips from LPS and sharing some research findings, it helps teachers to reflect on characteristics of effective teaching; Secondly, the method of conducting the project was influenced by LPS methodology.

Items	Mean
Changing perspective on teaching and learning	3.4
Enhancing the understanding of mathematics content and pedagogical methods.	3.5
Fostering teaching design ability	3.4
Advancing classroom instruction skill	3.4
Enhancing awareness of reflection and improvement	3.2
Enhancing analyzing classroom from multiple perspectives.	3.0
Enhancing cooperation and exchanges among colleagues	3.1
Overall, enhancing mathematics professional development	3.3

Table 1. Teacher's perception of importance of relevant aspects

Once completing this project, we conducted a survey on the effectiveness of the project. We asked participating teachers (17 teachers in one school) to rank the importance of the project for teacher' professional development as not important, important, and very important. A numerical value (2, 3, 4) was then assigned in our data analysis for each scale, respectively. The means of the teachers' responses are shown in Table 1. From participating teachers' perspectives, this project has quite a positive impact on their professional development.

Concluding Remark

As described above, it is exciting and effective to adopt new ideas and findings from international research to inform normal teaching and teacher professional development. Thus, broad and updated views will benefit local teachers' professional development. On the other hand, when we examine local classroom instruction and teacher professional development, we also identity some Chinese features to share with international audiences (Huang & Bao, 2006; Huang & Leung, 2004; Huang, Mok, & Leung, 2006).

INFORMING PRACTICE: WHAT CAN WE DO IN LPS THAT COULD NOT BE DONE IN OTHER TYPES OF CLASSROOM PRACTICE OBSERVATIONS?

Alena Hošpesová
University of South Bohemia

Jarmila Novotná
Charles University Prague

The methodology of LPS can be successfully employed in cooperation with in-service teachers and in teacher training. The quality of students' statements is to a great extent influenced by the student's trust in the experimenter and his/her willingness to reflect on the preceding lesson and talk about it. Our experience with students' post-lesson interviews from the Czech Republic has not been very favorable. What could be observed was certain students' unwillingness to occupy their mind with an already-ended lesson they had already "successfully survived", and perhaps also the novelty and even their fear of critical consideration of the teacher. Students seemed to regard themselves as the passive objects of pedagogical activity and did not appear to feel responsible for their own results.

Interviews with *teachers* were carried out simultaneously. What makes these interviews even more valuable is the fact that they are based on watching a video recording of the analysed lesson. This offers the researcher an uncommon insight into the teacher's self-reflection. It can be presupposed that self-reflection is, on the intuitive level, present in all human activities. However, qualified pedagogical reflection is different; it considers description and analysis of key elements, evaluation or revaluation, ways of explanation, accepting decisions and determining a new strategy (Tichá & Hošpesová, 2006). In this sense, Jaworski (2003) speaks of reflection-in-action, reflection-on-action, and reflection-for-action. Reflection carried out within LPS naturally belongs to the second type of reflection. Nevertheless the use of video recordings brings in some features of the remaining two categories.

The data gained by the LPS methodology can be subjected to the methods of qualitative research; their structure is close to LPS methods comprising "direct" observation, in-depth interviews, and analysis of documents and materials. This data structure enables us to perceive classroom reality as socially constructed, to study its various complex variables that can otherwise be measured only with great difficulty, without at the same time neglecting the perspective of the subject matter. Our goal was in-depth understanding of the teachers' and students' behaviour and the motives that guide it.

Within our investigation, we compared a video recording of a lesson with its perception by the teacher and the students expressed in post-lesson interviews. Let us illustrate this idea with an episode from the Czech data (grade 8, topic: parametric equations, this episode has already been discussed from a different point of view in Binterová, Hošpesová, & Novotná, 2006). For homework, the students were asked to solve the equation $dx + 1 = 2(4x + 1) - 5x$ by substituting for d the day of their birth. The teacher expected that some pupils would substitute 3. Then the equation would have no solution. In the lesson, she tried to show why this had happened and she together with her students adjusted the equation to the form $x(d - 3) = 1$ (CZ1-L05, 00:16:42-00:17:56). In the post-lesson interviews with the pupils, the experimenter asked the student and the teacher how they perceived this episode.

(CZ1-L05, interview with Michal, 00:01:03):

Exp. 1: Would you like to tell me anything about the homework?

Michal 1: ... First of all, I didn't know why the teacher had assigned it to us, but when we went through it together, I knew the result $\frac{1}{3-d}$. We tried it before the lesson and one boy – Adam – showed it to me.

Exp.2: You knew why you had been given that homework?

Michal 2: I knew.

Exp 3: And you knew the explanation?

Michal 3: He (Adam) didn't tell us. We knew the solution, but we couldn't explain it.

(CZ1-L05 post-lesson interview with the teacher, 00:11:21):

T 1: Well, it's certainly not easy. My feeling is that they didn't quite cope with it. Every equation will be an exception. I will have to go over it again.

Exp 2: Those who substituted 3 got the result. They will have understood.

T 2: It seems they discussed it before the lesson.

Michal's statement shows that he did not find the homework difficult. On the other hand, the teacher in her utterance (T1) seems to be expressing the belief that she is the one who is responsible for "passing knowledge over to her students" and seems to perceive any students' difficulties as her own failure. This conclusion has not been drawn only from our observation of this episode. We have come to this conclusion after having analysed a sequence of lessons – which is another advantage of LPS methodology.

The LPS video materials are not only used in research. Selected interesting episodes are processed for joint-reflection in courses of in-service teacher training. The success of joint-reflection largely depends on the selection of the teaching episode. Discussion is best provoked by a short episode which involves a particular problem. When the teachers watch the episode together and jointly reflect upon it, they confront their ideas, which often results in changes in their beliefs. It is often very useful if the discussion is followed by a recording of the reflection of the teacher who had given the lesson.

Endnote

This research was partially supported by the grant GACR 406/08/0710.

References

Bao, J. & Huang, R. (2007). Hypermedia video cases for in-service mathematics teacher professional development in China (Regular Lecture). In C.S., Lim, S. Fatimah, G.

Munirah, S. Hajar, & M.Y. Hashimah (Eds.), *Proceedings of 4th East Asia Regional Conference on Mathematics Education*, (47-53). Penang: Universiti Sains Malaysia.

Binterová, H., Hošpesová, A., & Novotná, J. (2006). Constitution of the classroom environment: A case study. In D. J. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics Classrooms in Twelve Countries: The Insider's Perspective.* Rotterdam: Sense Publishers, 275-288.

Brousseau, G. (1975). *Etude de l'Influence des Conditions de Validation sur l'Apprentissage d'un Algorithme*. Bordeaux: IREM de Bordeaux.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics 1970-1990*. Translation from French: M. Cooper, N. Balacheff, R. Sutherland & V. Warfield. Kluwer Academic Publishers.

Clarke, D.J. (Ed.) (2001). *Perspectives on Practice and Meaning in Mathematics and Science Classrooms*. Dordrecht, Netherlands: Kluwer Academic Press.

Clarke, D. (2003). Whole class patterns of participation and the distribution of responsibility for knowledge generation in the classroom. Paper presented at the 10th Biennial Conference of the European Association for Research on Learning and Instruction, Padova, Italy.

Clarke, D.J. (2006). The LPS research design. In D. J. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics Classrooms in Twelve Countries: The Insider's Perspective.* Rotterdam: Sense Publishers, 15-37.

Clarke, D. J., Emanuelsson, J., Jablonka, E., & Mok, I.A.C. (Eds.) (2006). *Making Connections: Comparing Mathematics Classrooms around the World*. Rotterdam: Sense Publishers.

Clarke, D.J., Keitel, C., & Shimizu, Y. (Eds.) (2006). *Mathematics Classrooms in Twelve Countries: The Insider's Perspective*. Rotterdam: Sense Publishers.

Clarke, D., Mesiti, C., O'Keefe, C., Jablonka, E., Mok, I., Xu, L. H., & Shimizu, Y. (2007). Addressing the Challenge of Legitimate International Comparisons of Classroom Practice, *International Journal of Educational Research*, *46*(5), 280-293.

Csikszentmihalyi, M. (1992). The flow experience and its significance for human psychology. In M. Csikszentmihalyi & I. Csikszentmihalyi (Eds.), *Optimal Experience: Psychological Studies of Flow in Consciousness*. New York: Cambridge University Press, 15-35.

Dreyfus, T., Hershkowitz, R., & Schwarz, B. (2001). The construction of abstract knowledge in interaction. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education*. Utrecht: PME. Vol. 2, 377-384.

Hoonaard, W. van (1997). *Working with Sensitizing Concepts: Analytical Field Research*. Thousand Oaks, California: Sage Publications. (Qualitative Research Methods Series 41).

Huang, R, & Bao, J. (2006). Towards a model for teacher's professional development in china: Introducing keli. *Journal of Mathematics Teacher Education*, 9, 279-298.

Huang, R. & Leung, F.K.S. (2004). Cracking the paradox of the Chinese learners: Looking into the mathematics classrooms in Hong Kong and Shanghai. In L. Fan, N. Y., Wong, J.

Cai, & S. Li (Eds.), *How Chinese Learn Mathematics: Perspectives from Insiders*. Singapore: World Scientific, 348-381.

Huang, R., Mok, I., & Leung, F.K.S. (2006). Repetition or Variation: "Practice" in the mathematics classrooms in China. In D.J. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics Classrooms in Twelve Countries: The Insider's Perspective*. Rotterdam: Sense Publishers, 263-274.

International Commission on Mathematical Instruction (ICMI) (2001). *Discussion Document: Mathematics Education in Different Cultural Traditions: A Comparative Study of East Asia and the West.* http://www.inf.fu-berlin.de/inst/ag-bg/ev/icmi13/dd.3.html.

Jablonka, E. (2004). *Structure in Diversity: Initiation into Mathematical Practice in Classrooms from Germany, Hong Kong and the United States*. Habilitation Thesis: Freie Universität Berlin, Germany.

Jablonka, E. (2005). Motivations and meanings of students' actions in six classrooms from Germany, Hong Kong and the United States, *Zentralblatt für Didaktik der Mathematik*, 37 (5), 371-378.

Jaworski, B. (2003). Research practice into/influencing mathematics teaching and learning development: Towards a theoretical framework based on co-learning partnerships. *Educational Studies in Mathematics*, 54 (2-3), 249-282.

Klette, K. (2007). Bruk av arbeidsplaner i skolen – et hovedverktøy for å realisere tilpasset opplæring? (The use of work plans in school – a tool to realise individually adapted learning). *Norsk Pedagogisk Tidsskrift,* Vol. 91, 344-358.

Krutetskii, V. (1976). *Psychology of mathematical abilities in schoolchildren*. (J. Kilpatrick & I. Wirzup (Eds.), J. Teller, Trans.). Chicago: University of Chicago Press. (Original work published in 1968).

Laborde, C. & Perrin-Glorian, C. (Eds.) (2005). Teaching Situations as Object of Research. Empirical Studies within Theoretical perspectives. *Educational Studies in Mathematics*. Special Issue. 59, 1-3.

Lakatos, I. (1976). *Proofs and Refutations: The Logic of Mathematical Discovery*. Cambridge University Press.

Lemke J. (1990). *Talking Science: Language, Learning and Values.* New Jersey: Ablex Publishing Corporation.

Mehan, H. (1993). Beneath the skin and between the ears: A case study in the politics of representation. In S. Chaiklin & J. Lave (Eds.), *Understanding Practice: Perspectives on Activity and Context*. Cambridge University Press, 241-268.

Novotná, J. & Hošpesová, A. (2007). What is the price of Topaze? In Woo, J.-H., Lew, H.-Ch., Park, K.-Si., & Seo, D.-Y. (Eds.), *Proceedings of 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, 25-32). Seoul: PME.

Novotná, J. & Hošpesová, A. (2008). Types of linking in teaching linear equations. In O. Figueras & A. Sepúlveda. (Eds.), *Proceedings of the Joint Meeting of the 32nd Conference of the International Group for the Psychology of Mathematics Education, and the XX North American Chapter*. Morelia: PME. Submitted.

Nuthall, G. (2004). Relating classroom teaching to student learning: A critical analysis of why research has failed to bridge the theory-practice gap. *Harvard Educational Review*, 74(3), 273-306.

O'Connor, M. & Michaels, S (1996). Shifting participant frameworks: Orchestration thinking practices in group discussion. In D. Hicks (Ed.), *Discourse, Learning, and Schooling*. Cambridge University Press, 63-103.

Ohtani, M. (2000). High context and hidden agency in Japanese mathematical discourse: A Vygotskian Perspective. In T. Nakahara et al. (Eds.), *Proceedings of the 24th Annual Conference of the International Group for the Psychology of Mathematics Education, Vol. 1*. Hiroshima University, 213.

Ohtani, M. (2002). *Gakkou Suugaku ni okeru Sugakuteki Katsudou no Shyakaiteki Kusei* (*Social Formation of Mathematical Activity in the Classroom*). Kazama Shobo.

Sarrazy, B. & Novotná, J. (2005). Didactical contract: Theoretical frame for the analysis of phenomena of teaching mathematics. In J. Novotná (Ed.), *SEMT 05*. Prague: Charles University in Prague, Faculty of Education, 33-45.

Savola, L. (2008). *Video-based Analysis of Mathematics Classroom Practice: Examples from Finland and Iceland*. Doctoral Thesis: Teachers College, Columbia University.

Seligman, M. (with Reivich, K., Jaycox, L., & Gillham, J.) (1995). *The Optimistic Child*. Adelaide: Griffin Press.

Stigler, J. W. & Hiebert, J. (1999). *The Teaching Gap: Best Ideas from the World's Teachers for Improving Education in the Classroom*. New York, NY: Free Press.

Tichá, M. & Hošpesová, A. (2006). Qualified pedagogical reflection as a way to improve mathematics education. *Journal of Mathematics Teacher Education, special issue: Inter-Relating Theory and Practice in Mathematics Teacher Education,* (9) 2, 129-156.

Udy, S.H., Jr. (1964). Cross-cultural analysis: A case study. In P. Hammond (Ed.), *Sociologists at Work*. New York: Basic Books, 161-183.

Vygotskii, L.S. (1982). Myshlenie i rech': Psikhologicheskie issredovaniya. *Sobranie sochinenii. Tom Vtoroi. Porblemy Obshchei Pshikhologii* (Thinking and speech: Psychological investigation. *Collected Works. Vol. 2. Problems of General Psychology*). Moskva: Izdatel'stvo Pedagogika. 5-361.

Vygotskii, L.S. (1984). Orudie i znak v razvitii rebenka. *Sobranie Sochinenii. Tom Shestoi. Nauch'noe Nasletstuvo* (Tool and sign in child development. *Collected Works. Vol. 6. Scientific Legacy*). Moskva: Izdatel'stvo Pedagogika. 5-90.

Williams, G. (2005). *Improving Intellectual and Affective Quality in Mathematics Lessons: How Autonomy and Spontaneity Enable Creative and Insightful Thinking.* Unpublished doctoral dissertation, University of Melbourne, Melbourne, Australia. Accessed at http://eprints.infodiv.unimelb.edu.au/archive/00002533/

Williams, G. (2006a). Autonomous Looking-In to support creative mathematical thinking: Capitalising on activity in Australian LPS classrooms. In D. J. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics Classrooms in Twelve Countries: The Insider's Perspective.* Rotterdam: Sense Publishers, 221-236.

Williams, G. (2006b). Impetus to explore: Approaching operational deficiency optimistically. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education*. Prague, Czech Republic: PME. Vol. 5, 393-400.

Williams, G. (2007). Abstracting in the context of spontaneous learning. In M. Mitchelmore & P. White (Eds.), *Abstraction, Special Edition, MERJ*, 19(2), 69-88.

Wong, N.Y. & Wong, W.Y. (2002). The "Confucian Heritage Culture" learner's phenomenon. *Asian Psychologist*, 3(1), 78-82.

Yackel, E. & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458-477.

EXAMINING TEACHERS' USE OF (NON-ROUTINE) MATHEMATICAL TASKS IN CLASSROOMS FROM THREE COMPLEMENTARY PERSPECTIVES: TEACHER, TEACHER EDUCATOR, RESEARCHER

Ron Tzur — Purdue University
Orit Zaslavsky — Technion
Peter Sullivan — Monash University

This Research Forum (RF) offers three complementary perspectives for examining how mathematics teachers use non-routine tasks in their classrooms. Following an introduction to the entire RF, Patricio Herbst presents a perspective of the teacher as a stakeholder in the symbolic economy of mathematics classrooms. Next, Peter Sullivan presents a perspective of a mathematics teacher educator through a model of task use and its implications for working with teachers. Then, Ron Tzur presents a perspective of a mathematics education researcher that focuses on how teachers' epistemological stances impact their management of tasks. Finally, Anne Watson discusses aspects of the three perspectives and highlights additional considerations, including the benefit of engaging mathematics teachers in task design.

In the last three decades, along with the shift to reform-oriented approaches to teaching, a growing body of research has paid close attention to the design and implementation of tasks—problem situations, questioning methods, and activities for promoting student learning of mathematics (e.g., Ainley, Pratt, and Hansen, 2006; Henningsen and Stein, 1997; Hiebert and Wearne, 1993; Houssart, Roaf, and Watson, 2005; Simon and Tzur, 2004). This focus is a sound extension of the two dominant theories of learning, socio-cultural (Leont'ev, 2002; Lerman, 2006; Vygotsky, 1978) and constructivist (Confrey and Kazak, 2006; Piaget, 1985; von Glasersfeld, 1995), as both contend that learners' goal-directed activity is the source for conceptual advance. That is, in reform-oriented approaches, tasks play the key role of interface between teacher intentions and student activities and attainments. Such an interface is needed because, as Pirie and Kieren (1992) contended, a teacher can occasion students' learning only indirectly, through engaging them in situations that prompt non-linear progressions toward intended mathematical understandings.

Kilpatrick et al. (2001) maintained that the quality of teaching depends "on whether teachers select cognitively demanding tasks, plan the lesson by elaborating the mathematics that the students are to learn through those tasks, and allocate sufficient time for the students to engage in and spend time on the tasks" (p. 9). This, in a nutshell, highlights the challenges teachers face in their choices, design, and implementations of instructional tasks. Although there are teachers who generate tasks on their own, most interpret and implement tasks generated by others (math educators, curriculum designers, etc.). Consequently, there are often discrepancies between designers' intentions and the actual implementation. Our

Research Forum offers novel ways and considerations for examining, from three complementary perspectives, how and why teachers interpret, alter, and use non-routine, inquiry-promoting mathematical tasks in their classroom:

1. A mathematics teacher's perspective that considers constraints within which teachers operate, their goals and beliefs, and their degree of confidence and flexibility (Herbst);
2. A mathematics teacher educator's perspective that considers the task as a way for conveying desirable teaching goals, promoting students' learning, and providing mathematics teachers with feedback and guidance that may help them transform their teaching (Sullivan);
3. A mathematics education researcher's perspective that analyzes characteristics of tasks and ways in which tasks unfold in the classroom, particularly focusing on epistemological assumptions that underlie teachers' use and alteration of tasks (Tzur).

At times, these perspectives may seem inseparable, just as mathematics educators may hold several roles, i.e., teacher, teacher-educator, and/or researcher. Yet, each of the presenters in the following contributions highlights a particular perspective. The discussion and synthesis of key issues that emerge from all three perspectives (offered by Watson) provides insight into different aspects of task design and implementation, and adds to the bridging between theory and practice as well as to the identification of aspects that require additional scholarly attention (e.g., teachers' sequencing of tasks). Consequently, the significance of this RF lies in the coordination among different perspectives, and the broader and more complex picture they present in terms of understanding discrepancies between intended and implemented classroom activities and norms.

In the group discussions that will follow each presentation, we will address the following questions, as well as other questions that the audience will raise:

- In what ways are (non-routine) tasks that teachers use in their classroom similar to or different from the intended tasks suggested by mathematics teacher educators and curriculum developers? What kinds of discrepancies between the intended and the implemented tasks can be identified? What is lost/gained by teachers' modifications of tasks?
- What explanations can we offer to account for such discrepancies? Can regularities in characteristics of tasks that teacher alter be identified/explained?
- How might teacher modification of tasks serve in inferring into and promoting their pedagogies? What can mathematics teacher educators and researchers offer teachers to support and enhance their engagement in task adaptation that promotes student learning?

References

Ainley, J., Pratt, D., & Hansen, A. (2006). Connecting engagement and focus in pedagogic task design. *British Educational Research Journal, 32*(1), 23-38.

Confrey, J. & Kazak, S. (2006). A thirty-year reflection on constructivism in mathematics education. In A. Gutiérrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present, and Future* (pp. 305-345). Rotterdam, The Netherlands: Sense.

Henningsen, M. & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education, 28*(5), 524-549.

Hiebert, J. & Wearne, D. (1993). Instructional tasks, classroom discourse, and students' learning in second-grade arithmetic. *American Educational Research Journal, 30*(2), 393-425.

Houssart, J., Roaf, C., & Watson, A. (2005). *Supporting Mathematical Thinking*. Independence, KY: David Fulton.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.) (2001). *Adding it Up: Helping Children Learn Mathematics*. Washington, DC: National Academy Press.

Leont'ev, D.A. (2002). Activity theory approach: Vygotsky in the present. In D. Robbins & A. Stetsenko (Eds.), *Voices within Vygotsky's Non-Classical Psychology: Past, Present, Future* (pp. 45-61). New York: Nova Science.

Lerman, S. (2006). Socio-cultural research in PME. In A. Gutiérrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present, and Future* (pp. 347-366). Rotterdam, The Netherlands: Sense.

Piaget, J. (1985). *The Equilibration of Cognitive Structures: The Central Problem of Intellectual Development.* T. Brown & K. J. Thampy (Trans.). Chicago: The University of Chicago.

Pirie, S.E.B., & Kieren, T.E. (1992). Creating constructivist environments and constructing creative mathematics. *Educational Studies in Mathematics, 23*(5), 505-528.

Ross, J. A., McDougall, D., & Hogaboam-Gray, A. (2003). A survey measuring elementary teachers' implementation of standard-based mathematics teaching. *Journal for Research in Mathematics Education, 34*(4), 344-363.

Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning, 6*(2), 91-104.

Von Glasersfeld, E. (1995). *Radical Constructivism: A way of Knowing and Learning*. Washington, DC: Falmer.

Vygotsky, L. S. (1978). *Mind in Society: The Development of Higher Psychological Processes*. Cambridge, MA: Harvard University Press.

THE TEACHER AND THE TASK

Patricio Herbst
University of Michigan

Why would a teacher make changes to a task? What is at stake for a teacher in a task? I propose to consider the work of teaching as one of effecting academic transactions in which the teacher is accountable to both the mathematical meanings represented in a task and the opportunity to learn that the task offers to students.

I examine the role of tasks from the position of the teacher, eventually coming around to the question of why a teacher might adapt or change a task designed by developers or researchers on learning. I take my charge as a researcher of teaching who endeavors to understand the rationality of teachers, making it clear that inasmuch as this rationality is often unspoken and tacit (see Herbst and Chazan, 2003), I am producing a theoretical model rather than relaying a testimony. In my own work (Herbst, 2006), I make a distinction between two frequent uses of the word task referring to one as *problem* and to the other as *task*. I use *problem* to refer to the mathematical statement of the work to do. For example, I consider "given two intersecting lines and one point on each of them (but not on the intersection), draw a circle tangent to both lines, so that the given points are its points of tangency" as a problem. I use *task* to refer to the (anticipated or observed) deployment of one such problem over time, in the actions and interactions of particular people (say in a US high school geometry class), doing particular operations with particular resources.

The first notion ('problem') echoes Brousseau's (1997, p. 79) epistemological notion of problem as a counterpart of a specific mathematical idea. In the example, the problem is the counterpart of a theorem (the "tangent segments theorem") that specifies the necessary and sufficient conditions on which such construction is possible: a circle exists which is tangent to two intersecting lines at two given points if and only if the two given points are equidistant from the intersection. The second notion ('task') builds on Doyle's (1983) proposal to study the curriculum by describing the work done in classrooms. From Doyle we get the observation that the work done (and thus the opportunity to learn) may be different depending on the overt goal proposed (e.g., to produce a circle tangent), the resources available, and the viable operations (all of which echo Brousseau's notion of the *milieu*). Doyle also noted that a task plays a role in the accountability system in the class, where accountability refers to how much value a task had for students (i.e., in terms of grades). I explore accountability from the perspective of a teacher. What is at stake for a teacher in a task?

The notion of task as the deployment of work on a problem over time and in an institutional space is a common ground for each of the three vertices of the instructional triangle: mathematics, the students, and the teacher (see Cohen, Raudenbush, and Ball, 2003). Mathematically, insofar as tasks are segments of social practice, one can see tasks as embodiments or representations of mathematical ideas, much in the same way that performances in art or dance embody ideas (see Herbst and Balacheff, in press; Lakatos, 1976). As far as the students are concerned, tasks are not performances to contemplate but opportunities for students to become, to come to know more or differently. As the introduction to the Research Forum argued, after Pirie and Kieren (1992), "a teacher can occasion students' learning only indirectly, through engaging them in situations that prompt non-linear progressions toward intended mathematical understandings." That is, tasks are opportunities for students to act and possibly learn.

If one accepts those as descriptions of how mathematics and the student have a stake in a task, it also follows that there is a kind of tension between the two conceptualizations. One could imagine, for example, a scripted classroom discussion where students and teacher elegantly acted out the emergence of a solution to a problem. And one could contrast that image with another classroom, where long silences extend while students struggle with that same problem, the teacher resists giving away hints, some students solve a different problem while others give up, etc. While the first scenario might illustrate how *the work on the problem* (the task as performance) embodies a piece of mathematical knowledge, the second one illustrates what the room could look like when students are given the opportunity to progress nonlinearly "toward intended mathematical understandings," which surely has to include at least as a possibility that such nonlinearity might take them to unintended places. None of the two scenarios is realistic or desirable, but they help make the case for a teacher who acts rather than withdraws (Smith, 1996), and introduce their role and stake vis-à-vis the task.

WHAT IS AT STAKE FOR A TEACHER

A teacher is responsible to manage the tension that a task presents in those two senses. She is responsible for the task as a representation of the *mathematics* to be learned and for the task as an opportunity to *study* and *learn* that mathematics. I conceive of classrooms as symbolic economies: Classrooms are places where transactions take place between the work that people do and the mathematics that they lay claim on. The teacher manages this economy-she manages transactions between work done and knowledge acquired. The teacher also has a stake in a task.

When teaching mathematics in school, a teacher is bound to mathematics and to students by a didactical contract. Any didactical contract gives the teacher a privileged position in organizing the work that the class will do over the duration of the course of studies. Thus, the teacher is *entitled* to decide what will be done, when, and for how long; and, for the same reason, she is also *accountable* for that work. From the teacher's perspective, a task is a bid to fulfill some of his or her contracted responsibilities. In engaging her class in a task a teacher exercises her entitlements and also submits to the responsibilities that those entitlements carry. It is not a risk-free venture for a teacher, at the very least because it entails an investment of time, a scarce, non-renewable resource in the duration of a course of studies within a school.

The work of developers creating mathematical tasks, and of researchers who focus on student learning through tasks can serve as resources for a teacher. Such work helps argue for the goodness of investing class time on some tasks. But they don't relieve the teacher from the responsibility to account for the time spent on one such task and to manage the process by which such task will deliver what it has promised. Moreover, what is at stake is not only time (invested, wasted, or unused). The learning opportunity to be experienced by students in that time and the mathematics to be produced with students in that experience are at stake as well: They are not automatic earnings derived from the decision to engage in a task, they could be earned, shortchanged, or even lost depending on what happens in action. The choice to spend a certain amount of time working on a problem might be a defensible investment initially. But management has to be active during its deployment to make the investment work out. And, among other things, active management might recommend second-guessing that investment, suggesting that new things must be done in order to sustain the soundness of the investment. The point is that a teacher who honors her or his professional responsibility in the didactical contract is accountable for attending to whether and how a task fulfils its promise as it develops over time.

EXPLAINING CHANGES IN TASKS

Those problems of accountability and management are proposed here as an explanation for why practitioners may change the task in ways that puzzle researchers on learning or curriculum developers. Accountability and management are not necessarily conscious problems for a practitioner, so one might not be able to elicit them as declarations of belief or goals. The specifics of how they are handled are likely dependent on individual teacher knowledge and beliefs, but their existence as problems is a characteristic derived from the institutional position of the teacher and the rationality of practice. The problems may not apply equally to teaching outside of schooling. The problems are proposed as elements of a theoretical model of the role of the teacher, but they can be confirmed empirically.

Let me illustrate this argument with a concrete example. The example is a set of possible classroom episodes that include a task that could unfold as a class works on the tangent circle problem (see above). As part of our study of practical rationality of mathematics teaching (Herbst and Chazan, 2006; Herbst and Miyakawa, in press; grip.umich.edu), we have produced an animated story of cartoon characters (The Tangent Circle) and comic book variations of that story to represent possible ways in which that task could unfold. In the 11-minute animation, a teacher reminds students that on the previous day they had learned that the tangent to a circle is perpendicular to the radius at the point of tangency. The teacher asks them to draw a circle tangent to two given lines at two given points that appear not to be equidistant from the point of intersection. Some students draw a circle without a compass, forcing it to be tangent at the expense of making it look unlike a circle (see Figure 1a), whereas other students draw circles with a compass at the expense of not achieving any one of them that looks tangent (see Figure 1b).

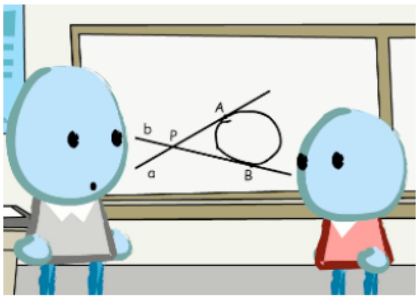
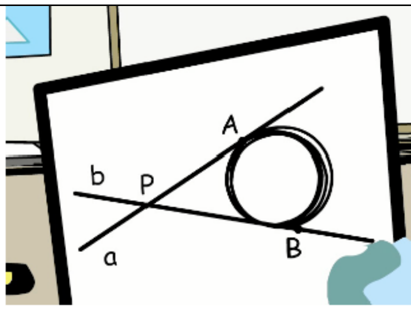
	

Figure 1a. Alpha's circle.	Figure 1b. Rho's circle.

One student (Lambda) claims early on that it is impossible to solve the problem and suggests moving the points to be able to do it; other students scorn her for changing the problem and the teacher lets her claim of impossibility fade out. The teacher poses another problem to the whole class: Given two intersecting lines and a point on one of those lines, where should we plot the other point in order to construct the tangent circle? A 5-minute dialogue ensues that over time elicits viable and unviable ideas from students and implements those in constructions until an idea appears to choose points equidistant from the point of intersection and draw perpendiculars to find the center. The teacher then says, "what we just did is, we discovered a theorem" and writes on the board "if two intersecting lines are tangent to a circle, the points of tangency are equidistant from the point of intersection." We have created alternative representations of this story. Among these we have varied the initial conditions of the problem (giving no points of tangency, giving 1 point, or giving 2 points that appear to be equidistant) and we have also created a "short version" where two non equidistant points are given but the teacher moves to state the tangent segments theorem immediately after Lambda says that the problem is impossible. We use these representations of teaching as prompts for experienced teachers to comment on the decisions made by the cartoon teacher. Analysis of that commentary is ongoing, to document whether and how teachers perceive these problems of accountability and management. In what follows I illustrate how the media can prompt teacher commentary that confirms the existence of those problems.

The problem of "accountability" is at play in the decision whether and why to make time, between the installation of the radius-perpendicular-to-tangent theorem and the statement of the "tangent segments theorem," to work on a variation of the tangent circle problem. If so, this period will engage students in doing something that they might or might not be able to do (depending on the choice of the givens); yet the success or failure doing the construction does not correlate with the success or failure of getting the new theorem on the table. What will that time count for? The teacher has to justify this investment a priori as well as to monitor its efficiency leading to the intended theorem. Epistemological and learning arguments might (and often times do) persuade a teacher that time would be well spent in this task: These arguments could draw, for example, on the value of creating in students a sense of intellectual need for the new theorem (see Harel and Sowder, 2005). The difference between the animation and the "short version" highlights how this problem is an open problem for the teacher. From the developer's or the learning theorist's perspectives it may be clear that more could be done after Lambda says that one could construct the circle if one moved the points. But for the teacher the issue is whether what has already happened suggests that it is time to cash the investment made. In other scenarios the need to account for the investment of time might press the teacher to write off a loss (of time) before it is too late. The problem is grounded in the assumption that sooner or later the teacher will be accountable for the time spent and the hypothesis that a teacher experiences larger investments of time as deserving larger "cash" value. The "cash" value alluded consists of claims that the teacher can lay on the class's knowledge of mathematics. This takes me back to the problem of management which requires the teacher to manage two senses of task noted above—the task as representation of mathematics and the task as opportunity for students to learn.

The problem of "management" is really a multitude of problems and it refers, to be quite clear, to the management of knowledge and learning, rather than just to the management of behavior. A teacher is an observer of the activity that exists in the classroom and can therefore attest to its mathematical value. But a teacher is also an actor in sustaining that activity with the students and can attest to its cost. The teacher needs to manage tensions that arise from that double identity (Herbst, 2003; also Ball, 1993). In the story described above it may be apparent to the reader that when the teacher states the theorem that they "discovered," that is not quite an appropriate assessment of what they actually did. Even if one ignores for a moment that they only verified perceptually that the circle is actually tangent, the construction really asserts the possibility to find a tangent circle as long as the points of tangency are equidistant from the point of intersection. In contrast, the theorem stated assumes the tangency and claims that the points are equidistant. A more accurate reading of what they actually achieved is that they have a better action model of what is good to have in order to do the construction. It would probably take more time and more tasks (an adidactical situation of formulation and later an adidactical situation of validation, in the sense of Brousseau's, 1997, p. 65) to claim that the aggregate work actually has the mathematical value of "discovery of the tangents theorem" (as a statement validated by a mathematical theory).

But to hold off effecting that transaction (i.e., to expect more work before claiming that the theorem has been discovered) might incur in extra costs as regards to the nature of the opportunity to learn that the teacher needs to sustain. These costs might include, for example, exacerbating the individual differences among students in regard to what they understand the goal of the task to be, what they think the resources or the operations needed are. All of these differences are part and parcel of what a learning theorist looks after, to understand and to document. But for the teacher of a class, who is responsible to teach the same curriculum to all students, these differences presage management nightmares. In the story one can see a glimpse of that by comparing how much earlier than her peers Lambda came to the realization that one had to move the points: The teacher makes the choice of giving another task rather than "cashing" the theorem on account of Lambda's comment, but the learning cost that the teacher has to manage in that case is one of sustaining attention to at least two interpretations of the goal of the task (to draw a tangent circle in the given conditions, to find the conditions on which one can draw a tangent circle) that span the learning environment.

UNDERSTANDING THE WORK OF TEACHING

The curriculum developer and the researcher on learning are task stakeholders just as the teacher is. They may be frustrated with the changes that the teacher makes on the moment. They may think that those changes come from ill will or poor knowledge. While some times those reasons may aggravate matters, I hope I have made the case that if changes do occur, those can be explained by understanding better the work of teaching. The teacher is a stakeholder of the task in that she needs the task to be instrumental to the institutional goals of teaching, which quite often mean communicating some specific mathematical ideas to all of a diverse group of children within a particular space and a set amount of time. These goals are related but not reducible to the goals of representing mathematics or occasioning individual learning. The teacher needs to use the task to fulfil those goals. The need to handle the problems of accountability and management may explain why at times such use of the task may run against the expectations of developers or learning theorists and why some other times a teacher may just choose not even to try. Researchers and developers could be more deferential, accepting that the teacher is really busy solving her own work problems. Both curriculum development and the study of learning in classroom settings need to be better informed by descriptive (rather than normative) theories of teaching.

References

Ball, D.L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal, 93*(4), 373-397.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. Dordrecht, The Netherlands: Kluwer

Cohen, D., Raudenbush, S., & Ball, D. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis, 25*(2), 1-24.

Doyle, W. (1983). Academic Work. *Review of Educational Research, 53*, 159-199.

Harel, G. & Sowder, J. (2005). Advanced mathematical-thinking at any age: Its nature and its development. *Mathematical Thinking and Learning, 7*, 27–50.

Herbst, P. (2003). Using novel tasks to teach mathematics: Three tensions affecting the work of the teacher. *American Educational Research Journal, 40*, 197-238.

Herbst, P. (2006). Teaching geometry with problems: Negotiating instructional situations and mathematical tasks. *Journal for Research in Mathematics Education, 37*, 313-347.

Herbst, P. & Balacheff, N. (in press). Proving and knowing in public: What counts as proof in a classroom? In M. Blanton, et al. (Eds*.), The Teaching and Learning of Proof Across the Grades*. Hillsdale, NJ: Lawrence Erlbaum.

Herbst, P. & Chazan, D. (2003). Exploring the practical rationality of mathematics teaching through conversations about videotaped episodes. *For the Learning of Mathematics, 23*(1), 2-14.

Herbst, P. & Chazan, D. (2006). Producing a viable story of geometry instruction. In S. Alatorre, J.L. Cortina, S. Mariana, & A. Méndez (Eds.), *Proceedings of the Twenty Eighth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, 213-220). Merida, Yucatán, Mexico: Universidad Pedagógica Nacional.

Herbst, P. & Miyakawa, T. (in press). When, how, and why prove theorems? A methodology for studying the perspective of geometry teachers. To appear in Zentralblatt für Didaktikk der Mathematikk.

Lakatos, I. (1976). *Proofs and Refutations: The Logic of Mathematical Discovery*. Cambridge, UK: Cambridge Press.

Pirie, S.E.B., & Kieren, T.E. (1992). Creating constructivist environments and constructing creative mathematics. *Educational Studies in Mathematics, 23*(5), 505-528.

Smith III, J.P. (1996). Efficacy and teaching mathematics by telling: A challenge for reform. *Journal for Research in Mathematics Education, 27,* 387–402.

DESIGNING TASK-BASED MATHEMATICS LESSONS AS TEACHER LEARNING

Peter Sullivan

Monash University

One of the challenges in all educator-led mathematics teacher learning is to create awareness of opportunities afforded by non-routine tasks, while at the same time fostering appreciation of the constraints in implementing such tasks in classrooms. The intention is that when teachers have opportunities to use non-routine tasks and associated pedagogies in classrooms, they will be aware of the advantages and potential of such tasks as well as anticipating potential barriers to implementation.

INTRODUCTION

I am assuming that: both prospective and practising teachers are interested in talking about tasks and lessons; it is a substantial and under-recognised challenge to convert potentially engaging tasks into productive learning through particular teacher actions; and, by studying the processes of constructing lessons, teachers can come to see the potential in non-routine tasks, and also the constraints that they might experience in using them.

This process of converting documented tasks to student learning was described by Stein, Grover, and Henningsen (1996), who analysed 144 tasks in terms of their features and cognitive demands, and studied the implementation of the tasks in classrooms. Their process has the task-influenced by the teacher goals, subject matter knowledge, and knowledge of students-informing the lesson (meaning the task in the classroom) which, influenced by classroom norms, task conditions, teacher instructional habits and dispositions, and students' learning habits and dispositions, creates the potential for student learning. The process they outline can assist prospective and practising teachers to appreciate the importance of theories of learning mathematics, and the ways that theories can inform classroom practice. Of course, the implications of theories for practice need to be made explicit, and one approach to this can be to link tasks and pedagogies to classroom practice. Essentially the intention is to offer practically experimented exemplars of non-routine tasks, and to study the implementation of those tasks in classroom, even over some iterations.

The following draws on results from a project researching the implementation of a particular type of non-routine tasks in classrooms, and a resulting recommended model for planning and teaching mathematics. This model can form the basis of collaborative approaches such as learning study (Runesson, in press), study groups (Arbaugh, 2003), coaching (Fullan, 2000), and Japanese lesson study (e.g., Stigler, and Stephenson, 1994). The advantage of incorporating the planning and teaching model into those formats being that there are aspects of using non-routine tasks in classrooms that are far from obvious. For example, unless teachers are aware of processes such as managing post-investigation discussions to facilitate making connections and forming generalisations, or adapting tasks to support learners experiencing difficulties, or ways of building a culture of collaboration, then key affordances of using non-routine tasks may be missed.

THE *OVERCOMING BARRIERS* PROJECT

The model that can be used as the basis of structured teacher learning was an outcome of research that identified and described particular aspects of classroom teaching that may act as barriers to mathematics learning for some students. The project first drew on responses from focus groups of teachers and academics to suggest strategies for overcoming such barriers (see Sullivan, Zevenbergen, and Mousley, 2002). Next, the project analysed some partially scripted experiences taught by participating teachers (see Sullivan, Mousley, and Zevenbergen, 2004). This analysis allowed reconsideration of the emphasis and priority of the respective teaching elements. Eventually the project researched ways teachers adapted the model to their classrooms.

There are five key elements of the model: the tasks and their sequence; *enabling* prompts that offer a particular approach to supporting students experiencing difficulty; *extending* prompts that can be used to challenge students who have completed the set work; making implicit pedagogies explicit; and the building of mathematical community.

The tasks and their sequence

Many commentators (e.g., Christiansen and Walther, 1986; Brousseau, 1997) have argued that the choice of tasks is a key element of any planning. As implied by Vygotsky's (1978) zone of proximal development, one aspect of the teacher's task is to pose to the class problems that most students are not able to do. While this model is applicable to any non-routine tasks, the project was based on a particular type of open-ended task, which can be illustrated by an example:

> After 5 games, the mean number of points that a basketballer had shot was 6, and the median number of points was 4. What scores might the basketballer have shot in each game?

This task is non-routine in that it is not readily solved by the application of a formula, and students must consider the meaning of the concepts of mean and median. Assuming that students have met mean and median, it has an easy entry in that students can choose possible scores with which they are familiar, and there are obvious and ready extensions possible for students who find a few responses quickly. Such tasks are content-specific in that they address the type of mathematical operations that form the basis of textbooks and the conventional mathematics curriculum.

Connected to the choice of task is what Simon (1995) described as a hypothetical learning trajectory made up of three components: the *learning goal* that determines the desired direction of teaching and learning, the *activities* to be undertaken by the teacher and students, and a *hypothetical cognitive process*, "a prediction of how the students' thinking and understanding will evolve in the context of the learning activities" (p. 136). In the case of the example task, this might involve selected preliminary experiences such as, for example, posing tasks exploring mean and median separately, and illustrating trial-and-error processes, and also planning what might come after this task such as transfer to different contexts, practice to fluency, introducing the concept of mode, and even box plots.

Enabling prompts to support students experiencing difficulty

If the teacher chooses tasks that most students are not able to do, as is desirable, there is a need to consider processes for supporting students who may not be able to complete the tasks even with adult guidance. The model suggests that teachers offer *enabling prompts* to allow those experiencing difficulty to engage in active experiences related to the initial goal task. Enabling prompts can involve slightly lowering an aspect of the task demand, such as the form of representation, the size of the numbers, or the number of steps, so that a student experiencing difficulty can proceed at that new level, and then if successful can return to the original task. This approach can be contrasted with the more common requirement that such students (a) listen to additional explanations; or (b) pursue goals substantially different (less demanding) from the rest of the class. In the project, the use of enabling prompts generally resulted in students experiencing difficulties being able to start (or restart) work at their own level of understanding and allowed them to overcome barriers met at specific stages of the lessons. As an example of an enabling prompt for the task above, the teacher might invite a student to work out what might be scores if there are 5 games and the mean number of points shot is 6. The effect of this enabling prompt is to reduce the variables from two to one, while preserving the open-ended nature of the exploration. The teacher might also say that the mean is 6 and the median is 4, without specifying the number of games. This removes one of the constraints, and so the task is one step easier.

Extending prompts for students who complete the initial task readily

If the task is at the appropriate level of challenge for most students, there may well be students who complete the task quickly. Teachers can pose prompts that extend students' thinking on the initial task in ways that do not make them feel that they are getting more of the same or being punished for completing the earlier work. Students who complete the planned tasks quickly are posed supplementary tasks or questions that extend their thinking. An example of an extending prompt for the above task could be:

> What if I told you that the mode was 7 as well?

The effect of this prompt is to examine the impact of the additional constraints. Another example, which introduces the case of median when there is an even number of scores, could be:

> What if there had been 6 games?

Explicit pedagogies

Especially with non-routine tasks, the model assumes that it is critical that teachers make explicit for all students the usual practices, organisational routines, and modes of communication that impact on approaches to learning. These include ways of working and reasons for these, types of responses valued, views about legitimacy of knowledge produced, and responsibilities of individual learners. As Bernstein (1996) noted, through different methods of teaching and different backgrounds of experience, groups of students receive different messages about the overt and the hidden curriculum of schools. Sullivan et al. (2002) listed a range of particular strategies that teachers can use to make implicit pedagogies more explicit and so address aspects of possible disadvantage of particular groups. An example in the case of this type of task is for the teacher to explain to students that not only are multiple solutions possible, but they are desirable. Likewise, for example, students can be invited to be creative, to consider the appropriateness of trial and error methods, and to discuss the role of basketball in the question.

Learning community

A deliberate intention in the model is that all students progress through learning experiences in ways that allow them to feel part of the class community and contribute to it, including being able to participate in reviews and class discussions about the work. It is assumed that all students will benefit from participation in at least some core activities that can form the basis of common discussions and shared experience, both social and mathematical. It was also clear from the research that teachers can take particular actions that can support or inhibit the building of community. Teachers, in observed reviews of student work, for example, would often invite students to contribute randomly and so would not be aware of the nature of the contribution that the particular students would make. Further, it was common for teachers to fail to interrogate students, or encourage other students to do so.

The net result was that there was little sense of a learning community developed. In the case of the example above, it is assumed that teachers would want to ensure that a student who found an answer by random trial-and-error would be invited to describe their responses, and perhaps another student who had systematically determined a range of responses, and another who had sought a generalised response could also be called on.

USE IN TEACHER EDUCATION

The *Overcoming Barriers* project demonstrated that both primary and secondary teachers are able to implement the planning and teaching model in everyday classrooms. The model can be adapted to the methods of studying tasks and lessons that are used as the basis of many teacher education programs. Some of the key elements could include teachers:

- studying the nature of tasks, and especially ways in which non-routine tasks are different mathematically and pedagogically from conventional tasks;
- considering the affordances and constraints in using non-routine tasks;
- demonstrating the planning and teaching model through a mathematics "lesson" delivered to the participants at teacher learning sessions;

- collaborative planning of other hypothetical lessons, with no intention that they be taught, with critical review of those plans;
- forming small groups to plan and then teach a lesson, incorporating iterative processes for review; and
- creating opportunities for review and reflection not only on the teaching and planning model but also on the teacher learning process itself.

References

Arbaugh, F. (2003). Study groups as a form of professional development for secondary mathematics teachers. *Journal of Mathematics Teacher Education, 6*(2), 139-163.

Bernstein, B. (1996). *Pedagogy, Symbolic Control, and Identity: Theory, Research, Critique*. London: Taylor and Francis.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. Dordrecht: Kluwer.

Christiansen, B. & Walther, G. (1986). Task and activity. In B. Christiansen, A. G. Howson, & M. Otte (Eds.), *Perspectives on Mathematics Education* (pp. 243–307). Dordrecht, The Netherlands: Reidel.

Fullan, M. (2000). The return of large-scale reform. *Journal of Educational Change, 1*(1), 5-28.

Runesson, U. (in press). Learning to design for learning. In T. Wood (Series Ed.) & P. Sullivan (Vol. 1 Ed.), *International Handbook of Mathematics Teacher Education.* Rotterdam, The Netherlands: Sense.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114–145.

Stein, M.K., Grover, B., & Henningsen, M. (1996). Building students' capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*(2), 455-488.

Stigler. J.W. & Stephenson, H.W. (1994). *The Learning Gap: Why Our Schools Are Failing and What We Can Learn from Japanese and Chinese Education.* New York, NY: Simon and Schuster.

Sullivan, P., Mousley, J., & Zevenbergen, R. (2004). Describing elements of mathematics lessons that accommodate diversity in student background. In M.J. Joines & A. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 265–272). Bergen, Norway: Bergen University College.

Sullivan, P., Zevenbergen, R., & Mousley, J. (2002). Contexts in mathematics teaching: Snakes or ladders? In B. Barton, K.C. Irwin, M. Pfannkuch, & M. Thomas (Eds.), *Mathematics Education in the South Pacific* (pp. 649–656). Auckland: MERGA.

Vygotsky, L.S. (1978). *Mind in Society: The Development of Higher Psychological Processes*. Cambridge, MA: Harvard University.

A RESEARCHER PERPLEXITY: WHY DO MATHEMATICAL TASKS UNDERGO METAMORPHOSIS IN TEACHER HANDS?

Ron Tzur

Purdue University

The central argument of this essay is that, in order to understand different ways in which mathematics educators (MEs) and mathematics teachers (MTs) interpret and use instructional (non-routine) tasks to promote student learning, it is necessary to account for their epistemological stances. I identify three ways by which MTs alter tasks and propose three plausible sources for such alteration. The third source, MTs' epistemological assumptions and considerations, is examined through their conceptions of the goals of teaching and the activities used to accomplish those goals. Using the distinction between perception- and conception-based perspectives as a lens, I address the different use of tasks for linking students' extant conceptions with those to be learned, and propose three key implications of such differences.

As a researcher interested in better understanding how mathematics teachers develop learner-empowering pedagogy I am perplexed by the phenomenon that is the focal-point of this Research Forum. Why do teachers enact mathematical tasks designed by mathematics teacher educators (METs) in ways that substantially deviate from how those MTEs (a) would enact the tasks themselves or (b) anticipate the tasks to unfold in a teacher's classroom? Addressing this problem is important because of the key role assigned to mathematical tasks—the tool by which teachers can nurture student learning (Sullivan et al., 2004; Zaslavsky, 2007). MTEs' scholarship repeatedly demonstrates how non-routine tasks, when enacted adeptly, succeed in fostering the desired quality of mathematical understandings whereas, for too many students, current practices fail. When the alternative tasks undergo metamorphosis the quality of student mathematics may be harshly compromised, hence the necessity to examine the issue.

Like Watson (this forum), the premise of my argument is that a teacher is always responsible for tailoring tasks to the unfolding of students' work in a mathematics classroom. Likewise, I assume that a MT alters tasks in service of genuine, best intentions to foster student learning while coping with constraints she or he faces (Herbst, this forum). Thus, I turn to articulating differences inherent to MTs' and MTEs' teaching, that is, assumptions and considerations that underlie their use of tasks. This includes three common ways in which teachers seem to alter a task and three plausible sources for these alterations, theoretical constructs for analyzing the third source (epistemological stance) that in my judgment is most resistant-to-change, and implications of this analysis.

WAYS AND SOURCES OF TASK ALTERATION

My work with MTs pointed to three characteristic ways in which they modify tasks. First, when planning, teachers often conclude that a task, as designed, is not likely to accomplish their goal for student learning and they adjust it before the lesson to fit with their own anticipation of how student learning unfolds. Second, a teacher may begin a lesson with a task enacted as intended, but, sooner or later, interpret students' work on the task to indicate lack of progress, which leads to task renegotiation. Third, teachers may plan and enact a task believing that their teaching corresponds to how they saw it used by a ME and/or to the designer's expectations, whereas a MTE who observes the lesson cannot but notice striking differences. Regarding all three, Stein et al. (1996) found, for example, teacher tendencies to reduce the demand/challenge level of a task. As much as those three types of task alteration differ, an underlying feature common to all three is the relationship between the teachers' goal and the regulation of their actions with the task as a tool to accomplish their goal. Let me illustrate the point with an observation from Dan's lesson.

Dan was a mid-career, enthusiastic, grade-6 teacher whose lesson on division of whole numbers I studied. He was <u>not</u> math anxious, deeply appreciated its importance and beauty, adhered to reform-oriented methods, enjoyed solving challenging problems, and hoped for his students to develop similar dispositions. While planning, he thoroughly revisited the mathematical concepts he himself learned the previous summer in a reform-oriented workshop. He clarified for himself the big idea (Schifter et al., 1999) of division as a crucial component of multiplicative reasoning, and worked out examples of long-division algorithm with Base Ten Blocks. In the previous lesson (Friday), the class solved partitive division problems using the blocks. For Monday, then, he planned to recap Friday's lesson, have three students solve a long-division problem on the board, then engage the class in understanding each step in the algorithm. The first two parts took about 20 minutes (as planned). However, as Dan turned to the third part, which he anticipated to be a straightforward work on linking two known processes, he realized students were 'lost.' Several times he mentioned what they did on Friday, growing frustrated with their inability to connect it with the algorithm, then asked them to bring out and use the blocks for solving the problem. They did, but could not yet 'see' the link to the algorithm that was painfully obvious to him. Extremely frustrated, Dan shifted to the traditional method (show-and-tell) that *he despised*, directly pointing out step-by-step correspondences between the activity with the blocks and the algorithm.

An example of the second type of task alteration, Dan's lesson sheds light on three plausible sources for this phenomenon. A teacher's prime goal is to foster student learning of particular mathematical ideas within an environment that presents many real and/or perceived constraints (Sullivan, this forum). The goal a teacher sets and the activities she or he takes to accomplish it depends on her or his assimilatory conceptions, namely, understanding of mathematics and pedagogy when planning, implementing, and adjusting situations (Ball, 2000; Shulman, 1987; Tsamir, 2005), as well as institutional norms and practices (see Herbst's excellent discussion of some critical constraints, this forum). Consequently, three plausible sources may effect task management: (a) the teacher's conceptions of the mathematical learning goal that a task is designed to promote, (b) her or his facility with using the task as a pedagogical tool, and (c) the teacher's implicit or explicit epistemological stance as to how a person comes to understand a mathematical idea she or he does not yet know and the role a task plays in this process.

Dan's example is telling because it demonstrates the logical status of the first two sources—they are necessary but insufficient. Dan's very strong mathematical understanding was clearly on par with what the MTE community yearns. He was also highly competent in using reform-minded problem solving processes and questioning techniques, small group and whole class discussions, and technological tools. Yet, his reasoning about the planned tasks as well as his reflections on the growing dissatisfaction from the impact of task modification on students' progress indicated epistemological commitments that markedly differed from the MTE's stance. The work with numerous teachers like Dan convinced my colleagues and me of the need to seriously examine this third source.

TASK MANAGEMENT AND TEACHER EPISTEMOLOGICAL STANCES

Because a mathematical task is a strategic means for accomplishing the teacher's goal of student learning, a teacher's understanding and managing of a task (in the sense articulated by Herbst, this forum) depends on her or his idea of learning, that is, one's implicit or explicit epistemological stance (Tzur, in press). Apart from the traditional show-and-tell perspective, a research team of which I was part (see Simon et al., 2000; Tzur et al., 2001) postulated that practices and thinking of teachers like Dan who attempt to adopt reform-oriented pedagogies could be rooted in *perception-based* or *conception-based* perspectives. A perception-based perspective is marked by a noticeable transformation in a teacher's traditional practice due to adopting a view of learning as an active process (e.g., heavy use of manipulatives). This change, however, is not accompanied by a change in the teacher's view of the epistemological status of mathematical knowledge and what in students' activities enable its formation. Like in the traditional perspective, the underlying premise of a perception-based perspective is that the mathematics a teacher came to know/understand has existence of its own independent of the person who knows and how she or he came to know it.

Such a view makes sense if one considers, for example, how Dan formed his understanding of the long division algorithm, including his excitement for gaining it. Obviously (to Dan), the instructors who led the workshop knew about it before he ever had a chance to encounter the new, meaningful interpretation. Moreover, once he formed this deep understanding he could 'see it everywhere' (textbooks, Base Ten blocks organization, worked-out long-division algorithm examples). All these experiences supported a sensible conclusion: the mathematical knowledge (e.g., algorithm for efficiently dividing numbers of any magnitude) is independent of the knower (e.g., Dan, see Steffe, 1990). More often than not the teacher may not be aware of this epistemological stance or of its implication that anyone, hence one's students, can perceive ('see') the mathematics the teacher came to perceive.

Typically, teachers whose practices seem to be grounded in a perception-based perspective appreciate the difficulties involved in coming to 'see' abstract mathematical concepts. However, for these teachers the process of learning is essentially not problematized. Rather, those teachers conceive of learning as a straightforward transition from not 'seeing' to 'seeing' the mathematics the teacher now 'sees.' Fostering such a transition becomes the teacher's goal; a task is a tool for accomplishing that goal—clearly and most efficiently pointing to and revealing the piece of mathematics to students. This is a key reason why such teachers embrace reform-oriented, child-centered methods, which lead to a classroom ecology that differs markedly from traditional classrooms in terms of *how* learning is fostered. Yet, the knower-independent epistemological stance common to both traditional and perception-base perspective entails analogous response to the teacher's ongoing question, "*What* should I teach next?"

Consider a teacher who has robust understandings of the mathematical terrain to be learned by students. She or he may also understand developmental landmarks that researchers found to underlie student progress (e.g., Dan knew that quotitive division is conceptually less advanced than partitive division, whereas partitive division was more compatible with the long-division algorithm). The teacher assesses that a group/class of students is (a) yet to understand or (b) already understands concept "X." In the former case the teacher intuitively teaches concept "X" because students do not 'see' it. In the latter case the teacher moves to fostering students' 'seeing' of the next-in-sequence "Y" concept. The intuition for doing so is sensible if one assumes that students already see concept "X" and do not yet see "Y." In short, within traditional or perception-based perspective pedagogies one intuitively focuses not on showing students what they already 'see' but rather on teaching (revealing) what they don't.

Two interrelated reasons seem to be at the root of this intuitive tendency (see von Glasersfeld, 1995). First, it is rooted in a deep presumption about human communication, where people customarily assume that the sense others make of what they utter is compatible with one's own sense. Second, when people form mathematical conceptions that underlie their 'seeing' of the world in a certain way they most often cannot 'return' to 'seeing' it without those conceptions. Coupled with the knower-independent epistemological stance they naturally attribute to fellow humans the unproblematic capacity for the same 'seeing.' Consequently, in spite of the seemingly different methods, instructional tasks are used in both traditional and reform-oriented as a means for showing students the mathematics in equivalent way to how the teacher 'sees' it.

Traditional and perception-based perspectives were distinguished from an epistemologically different approach termed conception-based perspective, which draws on Piaget's (1985) key notion of assimilation and the implied, learning-problematizing notion of the *learning paradox* (LP, Pascual-Leone, 1976). If assimilation is determined by a person's extant conceptions, how can anyone ever form more advanced conceptions? In particular, how can students assimilate tasks/activities in which a teacher engages them to promote learning of a new (to them) mathematical idea unless they somehow have already established conceptions that afford this assimilation?

Teaching rooted in a conception-based perspective draws on Piaget's explanation of how reflective processes (specifically, reflective abstraction) enable construction of new mathematical ideas as transformation (accommodation) in learners' assimilatory conceptions (Steffe and Wiegel, 1992). This explanation underlies an examination of an epistemological stance needed for successfully teaching mathematics (and mathematics teachers) that I recently introduced, termed *Profound Awareness of the Learning Paradox* (*PALP*) (Tzur, in press). In-depth discussion of *PALP* goes beyond the scope of this paper. However, it suffices to stress that teaching rooted in *PALP* begins with engaging students in tasks and activities that encourage them to independently use mathematics they already know (i.e., concept "X"). To foster learning of the intended piece of mathematics (concept "Y"), a teacher uses tasks as a means to (a) let students use their available conceptions for setting a goal and initiating an activity to accomplish this goal, (b) orient their attention to effects of the activity that differ from what students anticipated, and (c) relate the newly noticed effects with the activity in an anticipatory way (see Simon and Tzur, 2004; Tzur and Simon, 2004). That is, a task enables student construction of a new regularity (conceptual invariant) as transformation in their previously available conceptions (Steffe, 2002). This approach entails not only that tasks do not have agency (Watson, this forum), but also that tasks do not directly and straightforwardly reveal the new idea to students; rather, tasks indirectly occasion their mental constructive processes (Mason, 1998; Pirie and Kieren, 1992).

From an epistemological standpoint, then, I distinguish two cases of how interpretations of MTs and MTEs may differ regarding task enactment. In the first case, a MT's traditional perspective is incompatible with the MTE's perception-based perspective, that is, both are unaware of the learning paradox. When interpreting MTs' task management the MTE is likely to focus on shifts in student and teacher (inter)activity. In the second case, the MTE's conception-based (*PALP*-rooted) perspective is incompatible with either a MT's (2a) traditional or (2b) perception-based perspective. When interpreting MTs' task management the MTE is likely to focus on both the nature of student activities and the limitations of teacher attempts to straightforwardly engender student 'seeing' of the intended mathematics. Thus, the MTE analyzes how a teacher's plan and implementation of tasks reflect a host of teacher anticipations regarding how students' work on the task might (or might not) bring forth their learning. Most importantly, the MTE can apply the *PALP* to the teachers' work and potential growth, that is, consider how teachers' anticipations structure (afford and constrain) their assimilation and interpretation of events that demonstrate the extent to which a task enabled students' progress (Tzur, 2007). Thus, the MTE's analysis will regularly focus on how (a) the interplay between anticipated and actual teaching-learning events and (b) the teacher's regulation of her or his anticipation-explain task modification.

IMPLICATIONS

The analysis presented above, regarding *one of the important reasons* for task transformation (teacher epistemological stance), has three important implications. First, researchers who study teacher development can greatly benefit from being cognisant of their own epistemological stance relative to teachers' stance (e.g., case #1, 2a, or 2b). This provides researchers with a tool for inferring into teachers' assimilatory conceptions of how/why they use insightful tasks (see Krainer, 1999). For example, when working with Dan I was able to not only avoid denouncing his desperate shift to traditional methods, but also to figure out what could be a continual assimilatory barrier to his sense making of my co-teaching and co-planning with him. Articulating a teacher's epistemological stance, when coupled with the MTE's application of *PALP* to *teacher learning*, informs the design of mathematics teacher education tasks that are likely to promote teacher progress from a perception-based to a conception-based perspective. For example, I found Dan's questioning to be rooted in conceptions that could be transformed into novel separation between his own mathematical models and his models of student thinking, which thus became my goal for his learning. This implication is relevant to Herbst's (this forum) emphasis on teachers as stakeholders accountable for the task being instrumental to the institutional goals of teaching. I argue that a teacher's sense of accountability necessarily includes an implicit or explicit view of (a) what constitutes learning and (b) why a particular way of managing a task, in the specific here-and-now of an unfolding mathematics lesson, is likely to foster it.

Second, this analysis can assist researchers in identifying the necessary minimum shift in teachers' epistemological stance so that task modifications are not detrimental to the quality of student mathematics. Key here is the postulation of a continuum along which teacher epistemological stances may emerge. While certainly desired, fostering teacher progress to the higher end of the continuum can prove very difficult (Tzur et al., 2001). Focusing on conceptually feasible shifts is likely to require articulation of individual teachers' epistemological stances, but it will assist the MTE in finding a sound starting point for the desired shift. Moreover, it can contribute to a scholarly examination of how might teacher development of intended epistemology be promoted. For example, Watson's (this forum) proposal to engage teachers in the design of tasks seems to be conducive to teacher shift from the middle to the upper end of the continuum, because of the need to use the task as an explicit link between the intended mathematics and assumed student extant conceptions.

Last but certainly not least, this paper pointed to a critically needed shift in MTEs' stance. Case #1 above indicates that a MTE may identify a task modification without being aware of epistemological stances. In this sense, my analysis sheds light on a profound awareness that we as a community of MTE need to develop and embrace. To borrow from Steffe's (1995) distinction between first and second order models of mathematics, I ask: How can we promote MTEs' progress toward a conception of teacher task management that clearly distinguishes between the MTE's own (first) order model of task characteristics/pedagogy and the MTE's second order model of mathematics teachers' models? In this regard, I agree with Sullivan's (this forum) suggestion to engage MTs in analysis of non-routine and conventional tasks, particularly because it can foster the MTEs' reflection on and comparison between what makes specific task pedagogies easier/harder for the MTs (and for students).

References

Ball, D.L. (2000). Bridging practices: Interweaving content and pedagogy in teaching and learning to teach. *Journal of Teacher Education, 51*(3), 241-247.

Krainer, K. (1999). Teacher growth and school development. *Journal of Mathematics Teacher Education, 2*(3), 223-225.

Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. *Journal of Mathematics Teacher Education, 1*(3), 243-267.

Pascual-Leone, J. (1976). A view of cognition from a formalist's perspective. In K.F. Riegel & J.A. Meacham (Eds.), *The Developing Individual in A Changing World: Historical and Cultural Issues* (Vol. 1, pp. 89-110). The Hague, The Netherlands: Mouton.

Piaget, J. (1985). *The Equilibration of Cognitive Structures: The Central Problem of Intellectual Development* (T. Brown & K. J. Thampy, Trans.). Chicago: The University of Chicago.

Pirie, S.E.B. & Kieren, T.E. (1992). Creating constructivist environments and constructing creative mathematics. *Educational Studies in Mathematics, 23*(5), 505-528.

Schifter, D., Russell, S.J., & Bastable, V. (1999). Teaching to the big ideas. In M.Z. Solomon (Ed.), *The Diagnostic Teacher: Constructing New Approaches to Professional Development* (pp. 22-47). New York: Teachers College.

Shulman, L.S. (1987). *Knowledge and Teaching: Foundations of the New Reform*. *Harvard Educational Review, 57*, 1-22.

Simon, M.A. & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning, 6*(2), 91-104.

Simon, M.A., Tzur, R., Heinz, K., Kinzel, M., & Smith, M.S. (2000). Characterizing a perspective underlying the practice of mathematics teachers in transition. *Journal for Research in Mathematics Education, 31*(5), 579-601.

Steffe, L.P. (1990). Mathematical curriculum design: A constructivist's perspective. In L.P. Steffe & T. Wood (Eds.), *Transforming Children's Mathematics Education* (pp. 389-398). Hillsdale, NJ: Lawrence Erlbaum.

Steffe, L.P. (1995). Alternative epistemologies: An educator's perspective. In L.P. Steffe & J. Gale (Eds.), *Constructivism in Education* (pp. 489-523). Hillsdale, NJ: Lawrence Erlbaum.

Steffe, L.P. & Wiegel, H.G. (1992). On reforming practice in mathematics education. *Educational Studies in Mathematics, 23*, 445-465.

Stein, M.K., Grover, B.W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*(2), 455-488.

Sullivan, P., Mousley, J., & Zevenbergen, R. (2004). Describing elements of mathematics lessons that accommodate diversity in student background. In *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 265–272). Bergen, Norway: Bergen University College.

Tsamir, P. (2005). Enhancing prospective teachers' knowledge of learners' intuitive conceptions: The case of Same A–Same B. *Journal of Mathematics Teacher Education, 8*(6), 469-497.

Tzur, R. (2007). What and how might teachers learn via teaching: Contributions to closing an Unspoken Gap. In J.-H. Woo, H.-C. Lew, K.-S. Park, & D.-Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 142-150). Seoul, Korea: The Korean Society of Educational Studies in Mathematics.

Tzur, R. (in press). Profound awareness of the learning paradox: A journey towards epistemologically regulated pedagogy in mathematics teaching and teacher education. In T. Wood (Ed.) & B. Jaworski (vol. 4 Ed.), *The International Handbook for Mathematics Teacher Education*.

Tzur, R. & Simon, M.A. (2004). Distinguishing two stages of mathematics conceptual learning. *International Journal of Science and Mathematics Education, 2*, 287-304.

Tzur, R., Simon, M., Heinz, K., & Kinzel, M. (2001). An Account of a teacher's perspective on learning and teaching mathematics: Implications for teacher development. *Journal of Mathematics Teacher Education, 4*(3), 227-254.

von Glasersfeld, E. (1995). *Radical Constructivism: A way of Knowing and Learning*. Washington, D.C.: Falmer.

Zaslavsky, O. (2007). Mathematics-related tasks, teacher education, and teacher educators. *Journal of Mathematics Teacher Education, 10*(4), 433-440.

TASK TRANSFORMATION IS THE TEACHER'S RESPONSIBILITY

Anne Watson

University of Oxford

There is a resurgence of interest in task design as an important factor in mathematics teaching. Design has to be taken seriously not only for extended, multi-stage, authentic assessment tasks but also for the very ordinary things we ask students to do day-to-day in classrooms. For example, in lesson study, microanalysis of those aspects of the object of learning emphasised by teachers shows that, even when using very similar tasks, affordances for learning can be quite different.

I distinguish between task and activity, and claim that it is the teacher's professional task to adapt and select tasks. I compare two lessons to show how task-specific-pedagogy can make a difference to learning, even when there is shared design and agreement about the nature of mathematics and learning. Thus teachers are positioned as designers, and task design needs to be a focus of mathematics teacher education.

WHAT DO I MEAN BY TASK?

While some authors see 'task' as referring only to complex, multi-stage, exploratory problems, such as a problem situation (Brousseau 1997, p. 214), designed over time, I include any statements, materials, questions, incidents which are expected to impel certain kinds of activity in the classroom. This includes deliberately designed situations, and also the prompts and questions constructed in lessons by teachers and learners. It is natural then to look at task sequences because the way small tasks are strung together structures activity just as much as the interactive moves associated with longer tasks. I choose this distinction because it allows me to focus on task as a tool (see also Tzur, this forum), and to talk of the mathematical activity that ensues in classrooms as both influenced by and influencing the tasks (see Christiansen and Walther 1986).

A task has no agency. It is a tool alongside other tools, designed for a hypothetical purpose, but which only becomes purposeful when it is used and adapted by a teacher and also by the learners. A task, like any tool, on its own does not have purpose, except latently in its design. It becomes purposeful in activity through human agency. Thus the task becomes transformed by classroom activity and can also transform classroom activity through affording particular kinds of engagement. It is possible through principled design to make different kinds of activity possible, yet it is also possible to imbue a casual task (such as a question made up on-the-fly) with rich mathematical purpose through teaching. Purpose is as much a feature of pedagogy as it is of task, so analysis of what happens has to conjoin task and pedagogy. The relationship between task and task-specific-pedagogy is most informative about how the relationship between task, teaching and learning is seen and enacted.

This activity-theoretical analysis explains why purposes of activities with tasks are transformed by teachers, and also that this must be so. Cases where no transformation takes place can now be seen to be special, and may even demonstrate insensitive teaching, silencing learners' voices. But this analysis does not show the nature of these transformations, nor even how tasks can manifest goals, only that tasks afford certain kinds of activity.

Herbst (this forum) distinguishes between task as representation of mathematics and the task as opportunity for students to learn. This is a helpful distinction because it can be used to question the assumed hegemony of designers' intentions. A designer may know more mathematics, or have a more research-informed understanding of how task and learning might relate but, as Herbst shows, there is a complex management task to be done including attention to institutional factors such as examinations, timetables, and broader factors such as establishing norms. Herbst therefore champions descriptive rather than normative theories of teaching.

TEACHING GEOMETRICAL LOCI

Several teachers in the same school were teaching groups of 12 year-olds who were more or less similar in previous attainment. All teachers agreed to use a similar approach to teaching loci using a combination of straight-edge-and-compass construction tasks and the physical whole class activity of 'acting out' loci, They would ask students to work on paper to 'find all the points which satisfy a given rule' and to follow physical instructions to 'find a place to stand so that ….' (e.g. 'find a place to stand so that you are the same distance from these two points'). Teachers agreed that all classes would construct, both physically and with conventional tools, circles, perpendicular bisectors of line segments, and angle bisectors and some other loci. Teachers also agreed not to use the word 'locus' until students had a sense of what it meant as the set of points that 'follow' or 'satisfy' the given rules. They also agreed that 'angle-bisector' should be introduced as 'points which are the same distance from these two lines' so that angle-bisection was seen as a result. An indoor open space was available to do the physical task, and teachers used this at different points during their lessons. The teachers discussed and agreed the overall aim that students would relate their physical experience of standing according to rules to the processes of geometrical construction. This relationship is partially obscured by the affordances of the physical task: it is possible to take a 'gap-filling' role without constructing a personal interpretation of the instructions, and hence not to have an experience of being a point in relation to other points to refer to when reproducing the locus on paper. It is also worth mentioning that these students had little experience of geometry beyond some knowledge of angles, and the naming of polygons – no classical, formal, geometry at all.

Five lessons taught by five teachers as a result of this co-planning process were filmed. The mediational devices (words, artefacts, actions, images) and instructions used by the teacher and other students, whether intentional or not, shape the learners' experience of the lesson. In interactive lessons such as these, the mediational tools of language and purposeful tool use are also shaped by the learners. For example, while a ruler affords measuring and line-drawing activity, learners' take-up of these affordances is different in different tasks. We even saw three students using two rulers to create an ad hoc, set square to 'test' whether a particular angle might be 90 degrees.

I shall describe similarities and differences between two lessons to show how task-specific-pedagogy contributes to learners' different experiences. Both teachers used a mixture of asking, prompting, telling, showing, referring students to other students' work and so on. They focused on getting students to explain their choices and actions. All students had to work out as much as they could themselves about how to do the constructions, by reasoning and by listening to each other's reasons. The tasks were presented in remarkably similar ways offering similar variation in similar ways in terms of language, gestures, and statements of aims. Teachers' intentions, as reported to each other and to us before the lessons, were similar. Written work was similar, a combination of rough sketches and accurate diagrams; all teachers praised accurate constructions. In one class they also had to write statements describing similarities between the tasks. One could loosely describe these lessons as being models of good modern mathematics teaching practice, with respect to both classical mathematical validity and current ideas about social norms to enable learning.

Analysis of variation, situational norms, questions and prompts, and the demands on learners provided very similar results. Yet as a mathematical observer I knew that the mathematical affordances of the lessons varied; they provided different kinds of intellectual and mathematical engagement. Teachers offered the components of the tasks in different orders; teachers emphasised different things to students at different times; there was a range of different patterns of participation for individual students in each lesson; there were different kinds of tool use. I do not have space to describe all of these but will focus on critical differences.

Lesson A

In this lesson, the physical activity took place first, with teacher *A* emphasising 'same distance' throughout the various loci. Some students observed the action from a balcony to have an overview of the final shapes achieved. Students then returned to the classroom and were asked to construct the same loci as had been acted out physically. Throughout her small-group interactions the teacher repeatedly used the phrase 'same distance'. The physical activity happened first so that students were expected to use their memory of the physical actions when they came to make constructions in pencil and paper. No public instructions for constructing were given; instead students were asked to work out how to do them. The teacher worked with small groups of students asking them what they remembered and how they could reproduce it. In general she said 'you can use the compasses' when students needed to join points, sometimes showing them how to do it and then asking them to do it again for themselves. The emphasis was on collections of points, each of which has a particular property to do with 'same distance', and on joining them using the compasses.

Lesson B

The task sequence started with students working out, as a class, how to use a pair of compasses to construct circles, perpendicular bisectors and angle bisectors. Teacher *B* repeatedly referred to compasses as the tool for reproducing equal lengths: he said this himself, and also asked students 'what can we use to get equal lengths?' and 'what do compasses do for us?' and 'why would I use the compasses?' He invited students to demonstrate their ideas on the board, and also used the strategy of placing 'wrong' points to encourage students to understand the role of constraints. The teacher reinforced the power of the tool by comparing its role in constructing the two different bisectors, so that students were looking at the positions of, and relationships between, equal lengths in the constructions. By taking this approach, learners were able to talk about relationships within the diagrams as if they were caused by the equal lengths, rather than equal lengths merely being a drawing method. It was possible for them, by this focus, to get a sense of classical geometry. Then the physical activity took place with all students. After that they had to produce statements about the connection between the physical and construction activities.

COMPARING DIFFERENCES

In lesson *A* the emphasis had been on sets of points and 'same distance', in *B* the emphasis was on constrained trajectories and the comparison of activities. I interviewed the teachers a year after these lessons, having triggered their memory with videos. Each teacher was still teaching the same class. In case *A* they had recently returned to the concept of loci. Students remembered the lesson and some time during the intervening year had connected the physical and constructing experiences for themselves. The teacher believed this to be due to her use of similar language throughout to link tasks. She talked about how hard it is for students to understand the two-way implications of loci, that all points following a rule can be spatially represented (in these cases by lines) and that any point on these lines therefore followed the rule. Rather than being explicit about this she had chosen to emphasise 'same distance' in each context. Teacher *B*'s overriding memory of the lesson was the difficulty students had in reproducing individually the constructions they had developed as a group, compared to the strong qualities of their statements about the relationship between the tasks.

From a mathematical content viewpoint both lessons were successful in promoting significant and lasting learning about loci. Each invited learners to shift from obvious, intuitive visual and physical responses to the more formal responses required for mathematics. In each of these lessons there were emphases on relationships between variables, properties, reasoning about properties and relationships among properties, so hierarchies based on assumptions about complexity do not identify difference.

Task/activity differences

In these lessons, interpretations of the task have been made by individual teachers, after team planning. There is no reduction of challenge, in the terms offered, such as that reported by Stein and her colleagues about adoption of research-informed tasks (Stein et al., 1996).

What differed was what was emphasised by the teacher, but I am not saying that this was merely talk. Rather, the difference was, I claim, due to the underlying general relationships within which the teacher saw the task as being embedded. Because teachers see these differently they therefore use different language, different sequencing and different emphases so that different comparisons and connections can be made. This sense of different, but equally valid, mathematical activity around the same concepts does not, for me, appear to be captured totally in Tzur's reasons for different task adaptation. Tools were used differently, but we do not know if this was due to deliberate choice or not. Views of how students come-to-know mathematics were similar. The institutional and management issues are similar. But in mathematical terms Teacher *A* talked about a two-way relationship between points and lines, and how this is also an issue with graphs as representations of functions. Meeting this duality with loci would make it easier to recognise the duality with other graphical representations. Teacher *B* saw the comparison between tasks as being an example of looking for similar structures in disparate experiences. These two groups of students would therefore be differently prepared for future mathematical activity.

Working with teachers

Sullivan (this forum) uses 'non-routine' tasks with teachers to think about sequencing, prompts to enable and extend mathematical activity, explicitness about desirable aspects of activity, and the development of a community in which it is habitual to compare and reflect on methods and results as new objects of study.

So far I avoided the assumption that the teacher is somehow deficient in relation to the designer-researcher, but in Sullivan's paper the focus is explicitly on how teacher educators can work with teachers on incorporating explicit kinds of designed tasks into their teaching. The description 'non-routine' has well-understood implications for task-type, yet what we have found in the UK is that all task-types can become routinised by reducing engagement to a sequence of instructions for action. Pre-service teachers often have embedded assumptions of what it means to 'do' 'non-routine' tasks because in school they followed limited rubrics for assessment purposes. I would extend Sullivan's definition to include tasks 'not readily solved by the application of a familiar process'.

Watson and Mason (2007) list aspects of task-use in teacher-education settings that are common throughout the world. This includes asking teachers to: work on mathematics; then use similar tasks in practice (extended, comparative, multi-stage, realistic, or exercise tasks); analyse task structures and observe lessons using them; observe, analyse, compare teaching and learners' work using similar tasks. From these practices, it is the comparisons that are most likely to expose different conceptualisations of mathematical ideas because, as shown above, variation shows up best against a background of similar practices (Watson and Mason 2005, Tzur this forum). The role of the task in these educational practices combines Herbst's distinctions between task as representation of the mathematics and task as tool for fostering learning. In all these cases, and in Sullivan's, the task itself is given.

Another way to engage teachers with tasks is to involve them in the design process, an approach taken by Swan (2006) in his study of improving mathematics teaching. Swan's study encompasses all stages of the design process, from theoretical and experimental design, through systematic trials of tasks and pedagogy with teachers, and re-design. He then designed training for teachers to use the tasks, and researched the effects of using the tasks in teaching. He identified changes in 'typical' teaching while using these 'new' tasks and found that the nature of change depended strongly on the teacher's previous practice. While nearly all teachers changed from less to more learner-centred approaches, their practice and their students' experiences still differed significantly. A major difference in the final teaching was whether teachers were able to move from a desire to control students' engagement with content, to giving free rein to processes of 'conceptualisation'. Teachers were using the same task in the similar ways, but ultimately what appeared to make the most difference was whether the teacher believed that learning happens by making sense of confused and conflicting experiences. This difference influenced whether they used such tasks a lot or a little, whether they adapted the task types to other topics, and how they managed the 'closure' of such tasks. Keitel (2006) sees explicitness of purpose and value as a crucial ingredient in teachers' use of tasks, and Sullivan (this forum) sees explicitness about pedagogy as also important. For example, in Watson and Sullivan (in press) we point to the importance of discussion after a task has been completed to enable learners to relate their experience to their developing mathematical repertoire, and to the conventional canon. To do this convincingly a teacher has to believe that a task as she sets it affords learning of an appropriate kind, even where tasks are the product of rigorous design research, as Swan's were.

TEACHER AS DESIGNER

Anecdotally, we hear people saying that 'teachers get in the way' and that the aim is to create 'teacher-proof materials', yet the above examples question whether seeing task designers as custodians of mathematical meaning makes any sense in practice.

Institutional and cultural components of teachers' decision-making have to influence task adaptation (Herbst, this forum); prevailing classroom norms also make a difference to learners' experiences, even when teachers are trying to change. These factors alone guarantee that teachers will adapt and design tasks for their own purposes, and Tzur (this forum) draws attention to change in such adaptation due to epistemological differences. I have added further differences, namely teachers' conceptualisations, how they see mathematical ideas embedded in relation to other ideas, and how these emerge in task-specific pedagogy.

A non-teacher-designer has to ensure that tasks afford the most possible intellectual challenge, such as reasoning about properties and relationships, or adaptations of skills and techniques used in unfamiliar contexts. However, as the authors in this forum show, the teacher is not a neutral conduit for tasks but is also a designer.

It makes sense, therefore, to work with teachers on task design rather than only on task implementation. Karp (2007) reports how task design was a major aspect of teacher preparation in his work in the former Soviet Union. Prestage and Perks (2007) do this with pre-service teachers, turning the issue on its head, showing how teachers can identify the limitations of designed tasks and use them to develop richer teaching through task-specific-pedagogy.

References

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. Dordrecht, The Netherlands: Kluwer.

Christiansen, B. & Walther, G. (1986). Task and activity. In B. Christiansen, A. Howson, & M. Otte (Eds.), *Perspectives on Mathematics Education* (pp. 243-307). Dordrecht, The Netherlands: Reidel.

Karp, A. (2007). 'Once more the quadratic trinomial': On the formation of methodological skills. *Journal of Mathematics Teacher Education*, 10(4-6), 405-414.

Keitel, C. (2006) 'Setting a task' in German schools: Different frames for different ambitions. In D. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics Classrooms in Twelve Countries: The Insider's Perspective* (pp. 37-58). Rotterdam/Taipei: Sense.

Prestage, S. & Perks, P. (2007). Developing teacher knowledge using a tool for creating tasks for the classroom. *Journal of Mathematics Teacher Education, 10*(4-6), 381-390.

Mason, J. & Johnston-Wilder, S. (2006). *Designing and Using Mathematical Tasks*. London: QED Publishing.

Stein, M. K., Grover, B. W., & Henningsen, M. A. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*(2), 455-488.

Swan, M. (2006). *Collaborative Learning in Mathematics: A Challenge to Our Beliefs and Practices.* London: National Institute of Adult Continuing Education.

Watson, A. & Mason, J. (2005, August). *Enriching Student Learning: Structuring Tasks so as to Limit Possible Generalisations*. Symposium paper presented at EARLi, the University of Cyprus.

Watson, A. & Mason, J. (2007). Taken-as-shared: a review of common assumptions about mathematical tasks in teacher education. *Journal of Mathematics Teacher Education, 10*(4-6), 205-215.

Watson, A. & Sullivan, P. (in press). Teachers learning about tasks and lessons. In T. Wood (Series Ed.) & D. Tirosh (Vol. 2 Ed.), *International Handbook of Mathematics Teacher Education*. Rotterdam, The Netherlands: Sense.

PURSUING EXCELLENCE IN MATHEMATICS CLASSROOM INSTRUCTION IN EAST ASIA

Yeping Li
Texas A&M University

Gabriele Kaiser
University of Hamburg

As it is commonly perceived that mathematics classroom instruction in East Asia has a high quality and unique culture, this Forum aims to take an in-depth examination of the mathematics classroom instruction excellence nurtured and valued in East Asia. By focusing on four selected education systems, this Forum is organized not only to examine features of excellent classroom instruction valued and possible approaches undertaken in these education systems in East Asia, but also to provide a platform for cross-examining and discussing their similarities and differences. Possible socio-cultural values underlying mathematics classroom instruction excellence and its pursuit in the East are also probed. The final paper takes a culturally different point of view to discuss how mathematics classroom instruction may be viewed in the West, in comparison to what is practiced and valued in the East.

INTRODUCTION

Worldwide efforts in improving students' mathematics learning have led to the contention that the quality of classroom instruction matters. In particular, accumulated research over the past decade has contributed to our understanding about differences in mathematics teaching and learning between East and West, such as mathematics classroom instruction (e.g., Li, 2007; Stigler & Hiebert, 1999), teachers' knowledge of mathematics for teaching (e.g., An, Kulm, & Wu, 2004; Ma, 1999), and teachers' perspectives on effective mathematics teaching (e.g., Cai, Perry, & Wong, 2007). Yet, if taking a closer look at mathematics classroom instruction in the East alone, it is not clear how excellent mathematics instruction may look and what approaches are typically taken for promoting the excellence in mathematics classroom instruction. Becasue the issues of mathematics classroom instruction quality and its improvement are important to PME community, this forum is thus organized with contributions developed from ongoing research interest among PME members for reaching a better understanding of mathematics classroom teaching culture nurtured in East Asia.

Through this research forum, we ask:

- What aspects of mathematics classroom instruction are emphasized or valued in mathematics instruction of excellence in different education systems?
- What cultural values may be placed behind what can be counted as excellent mathematics instruction?
- What approaches and cultural resources are utilized for developing excellent mathematics instruction in different education systems in East Asia?

- How may the excellence of mathematics classroom instruction be similar and different across different education systems in East Asia?
- How may mathematics classroom instruction excellence and its pursuit be viewed from a Western perspective?

In particular, this research forum aims to examine the nature of mathematics instruction excellence valued in four selected education systems in East Asia (i.e., Japan, Mainland China, South Korea, and Taiwan), and all aspects of social-cultural factors that help pursue and nurture/evaluate excellence in the mathematics instruction in discussion. These four education systems were selected in light of two considerations: students' high mathematics achievement and cultural connections. The selection affords the possibility of learning the similarities and differences in the culturally-valued excellence in mathematics classroom instruction, which is often taken as a key contributing factor for students' high achievement, across several different education systems in East Asia. The importance of such learning is embedded not only in learning possible variations in what is commonly termed as teaching culture in East Asia, but also in examining what valuable practices are possible for others to adapt in a different system or culture. At the same time, because these four education systems may share some similarities in certain ways but differ in others, a selected focus on specific aspects of classroom instruction may favour one system over another. Thus, this forum is designed to be open to the aspects of classroom instruction that are to be analysed and reported.

Collectively, this research forum is not only to report specific research findings, but also to provide a platform for understanding and cross-examining the similarities and differences in mathematics instruction excellence and ways employed for its development in diverse system contexts. These are believed to provide insights necessary for reflecting on mathematics instruction excellence and its culture in East Asia.

References

An, S., Kulm, G., & Wu, Z. (2004). The pedagogical content knowledge of middle school mathematics teachers in China and the U.S. *Journal of Mathematics Teacher Education, 7*(2), 145-172.

Cai, J., Perry, R., & Wong, N.-Y. (Eds.) (2007). What is effective mathematics teaching? A dialogue between East and West. *ZDM - The International Journal on Mathematics Education, 39*(4).

Li, Y. (2007). Examining and understanding mathematics classroom instruction and possible contributing factors through a cross-national lens. *School Science and Mathematics, 107*, 308-310.

Ma, L. (1999). *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

Stigler, J. W. & Hiebert, J. (1999). *The Teaching Gap: Best Ideas from the World's Teachers for Improving Education in the Classroom*. New York: Free Press.

RESEARCH ON THE QUALITY OF MATHEMATICS CLASSROOM INSTRUCTION: AN OVERVIEW

Yeping Li

Texas A&M University

RESEARCH ON THE QUALITY OF MATHEMATICS CLASSROOM INSTRUCTION: WHAT DO WE KNOW?

It is generally recognized that the quality of classroom instruction matters the most for improving students' learning. However, there has not been a clear agreement about what can be counted as a good mathematics instruction (Krainer, 2005). The interpretation of classroom instruction as 'good' or 'bad' is a value-loaded judgment that goes beyond the description of what is going on in a classroom setting, and it clearly requires researchers to develop and/or use certain criteria. As discussed by Wilson, Cooney, & Stinson (2005), various criteria were used implicitly or explicitly by different education scholars over time in specifying the features of good teaching. The lack of a clear agreement on the criteria for what can be counted as good teaching in the past suggests both the difficulty and needs in understanding the nature of teaching.

Cross-culturally, it is even more complicated to evaluate the effectiveness of classroom instruction. Although students' performance can be taken as a possible indicator of the effectiveness of classroom instruction, students' performance itself does not spell out the nature of classroom instruction across systems. With such a consideration, the Third International Mathematics and Science Study (TIMSS) was the first large-scale international study that included a classroom video-analysis component. The study led the researchers to conclude that teaching is fundamentally a cultural activity (Stigler & Hiebert, 1999). The culturally divided values across systems can help us better understand the nature of teaching activity in different education systems from a cross-cultural perspective. The importance of understanding the underlying cultural values across systems has been exemplified in a contrast between the East and the West (e.g., Leung, 2001), as well as among education systems in East Asia (e.g., Li, 2007; Li & Ginsburg, 2006). Although the TIMSS video study did not aim to characterize what can be counted as excellent mathematics instruction within each participating education systems, it has led to further inquiries about various aspects of the teaching culture that is formed and nurtured in East Asia (e.g., Correa et al., 2008; Fan, Cai, Wang, & Li, 2004; Fernandez & Cannon, 2005).

The quality of mathematics classroom instruction in East Asia, in fact, has been mysterious to many education researchers. For example, relevant studies have revealed that Chinese teachers not only have a profound understanding of the school mathematics they teach (e.g., Ma, 1999), but also stay in an environment that fosters mutual exchanges of curriculum and instructional ideas (e.g., Paine & Ma, 1993;

Wang & Paine, 2003). Findings from these studies seem to suggest possible reasons that can lead to teachers' construction of good classroom instruction in China. Paradoxically, there are some other factors that can puzzle our thinking about how classroom instruction in China can possibly be called "good" from a Western point of view. In particular, it is well documented that Chinese classroom instruction can be characterized as a large class size, teacher's lecture-oriented, and examination-driven (e.g., Watkins & Biggs, 2001). This presents a form of classroom instruction that is not favourable in the West. Nevertheless, mathematics classroom instruction excellence does exist as valued in East Asia and may be presented as a result of instruction competition, a joint effort of group collaboration, or a master teacher's exemplary lessons. Therefore, across different systems and cultural contexts, what can be counted as good mathematics classroom instruction may share some similarities in certain aspects but not in others. An examination of excellent mathematics instruction across several education systems should provide an opportunity for understanding possible similarities and differences in the instructional process and specific cultural values that may be placed behind what can be counted as excellence in mathematics instruction in East Asia.

Rather than letting researchers take a specific stance in judging what excellent mathematics instruction is in a specific education system, this research forum is proposed with a focus on what is already being valued as excellent mathematics instruction in several selected education systems in East Asia. In particular, excellent mathematics instruction exists in culturally specific formats in many educational systems in East Asia. They are not typical classroom instruction, but embody culturally valued aspects and features for what can be counted as excellence in mathematics instruction for others to follow in that setting. While these features may be emphasized and shared within a system and cultural context, they are not transparent to outsiders of that system. An explicit examination of these culturally valued features as manifested in excellent classroom instruction should provide a unique opportunity for others to understand the nature of instructional excellence that is pursued in East Asia. We believe that a better understanding of the excellence in classroom instruction in East Asia developed through this Research Forum is important, especially when we consider the possibility of learning from high-achieving education systems in East Asia for improving mathematics instruction elsewhere.

RESEARCH ON THE QUALITY OF MATHEMATICS CLASSROOM INSTRUCTION: HOW IS MATHEMATICS INSTRUCTION EXCELLENCE DEVELOPED AND NURTURED IN EAST ASIA?

World-wide efforts to improve mathematics classroom instruction have led to increased interest in exploring not only teachers' instructional practices in high-achieving education systems in East Asia, but also ways employed to improve the quality of mathematics instruction (e.g., Stigler & Hiebert, 1999). It is now well recognized that lesson study is an important practice utilized in Japan to improve the quality of mathematics instruction. In fact, there are various approaches developed and used in the pursuit of excellence in mathematics instruction in different education systems in East Asia. However, much remains unknown to outsiders about other approaches used in many education systems. For example, the model of exemplary lesson development is developed and used in mainland China (Huang & Bao, 2006). Instructional contests are organized to identify and promote excellent mathematics instruction in several educational systems. Master teachers are also an important part of the teaching culture in some education systems in East Asia, and play an important role in nurturing that culture (Li & Huang, 2008). Excellent mathematics instruction may be made possible and recognized via different approaches in different education systems, including discussion and evaluation with certain procedures in place in a system, instructional contest, and/or high quality lessons taught by master teachers.

As classroom instruction mirrors culturally valued teaching and learning activity in a specific education system (e.g., Stigler, Fernandez, & Yoshida, 1996), further understanding of the development of mathematics instruction excellence should benefit from an examination of underlying educational philosophies and cultural context (e.g., Kaiser, 2002; Li, 2007). It is generally recognized that Confucianism is highly influential to educational practices in East Asia and needs special attention (e.g., Leung, 2001; Leung, Graf, Lopez-Real, 2006). Moreover, different approaches employed to foster excellence in mathematics instruction constitute the cultural niche that supports the creation of excellent mathematics instruction in that context. An exploration of approaches and cultural resources utilized can not only provide others a better understanding of the mechanism, existing in a system and culture context, that supports the generation and valuation of mathematics instruction excellence, but also highlight possible restrictions in simply adapting certain instructional practices from one context to another. Thus, this Research Forum also tends to serve as a window through which mathematics educators can gain a glimpse of various approaches and possible cultural resources utilized for achieving excellence in mathematics instruction across several selected education systems in East Asia.

References

Correa, C.A., Perry, M., Sims, L.M., Miller, K.F., & Fang, G. (2008). Connected and culturally embedded beliefs: Chinese and US teachers talk about how their students best learn mathematics. *Teaching and Teacher Education*, 24, 140-153.

Fan, L., Wang, N., Cai, J., & Li, S. (Eds.) (2004). *How Chinese Learn Mathematics – Perspectives from Insiders.* Singapore: World Scientific Publishing.

Fernandez, C. & Cannon, J. (2005). What Japanese and U.S. teachers think about when constructing mathematics lessons: A preliminary investigation. *Elementary School Journal, 105*, 481-498.

Krainer, K. (2005). What is "good" mathematics teaching, and how can research inform practice and policy? *Journal of Mathematics Teacher Education, 8*, 75-81.

Huang, R. & Bao, J. (2006). Towards a model for teacher professional development in China: Introducing keli. *Journal of Mathematics Teacher Education*, 9, 279-298.

Kaiser, G. (2002). Educational philosophies and their influence on mathematics education – an ethnographic study in English and German mathematics classrooms. *Zentralblatt fur Didaktik der Mathematik, 34*(6), 241-257.

Leung, F.K.S. (2001). In search of an East Asian identity in mathematics education. *Educational Studies in Mathematics, 47*, 35-51.

Leung, F.K.S., Graf, K.-D., & Lopez-Real, F. J. (Eds.) (2006). *Mathematics Education in Different Cultural Traditions*. New York: Springer.

Li, Y. (2007). Curriculum and culture: An exploratory examination of mathematics curriculum materials in their system and cultural contexts. *The Mathematics Educator, 10*(1), 21-38.

Li, Y. & Ginsburg, M. (2006). Classification and framing of mathematical knowledge in Hong Kong, Mainland China, Singapore, and the United States. In F.K.S. Leung, K.-D. Graf, & F. J. Lopez-Real (Eds.), *Mathematics Education in Different Cultural Traditions*. (pp. 195-211). New York: Springer.

Li, Y. & Huang, R. (2008). *Learning about and From a Master Mathematics Teacher in China*. Research symposium presented at Research Pre-session of National Council of Teachers of Mathematics (NCTM) annual meeting, Salt Lake City, Utah; April 7-9, 2008.

Ma, L. (1999). *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

Paine, L.W. & Ma, L. (1993). Teachers working together: A dialogue on organizational and cultural perspectives of Chinese teachers. *International Journal of Educational Research, 19*, 667-778.

Stigler, J.W., Fernandez, C., & Yoshida, M. (1996). Traditions of school mathematics in Japanese and American elementary classrooms. In L.P. Steffe, P. Nesher, P. Cobb, G.A. Goldin, & B. Greer (Eds.), *Theories of Mathematical Learning*. (pp. 149-175). Mahwah, NJ: Lawrence Erlbaum.

Stigler, J.W. & Hiebert, J. (1999). *The Teaching Gap: Best Ideas from the World's Teachers for Improving Education in the Classroom*. New York: Free Press.

Wang, J. & Paine, L. W. (2003). Learning to teach with mandated curriculum and public examination of teaching as contexts. *Teaching and Teacher Education, 19*, 75-94.

Wilson, P.S., Conney, T.J., & Stinson, D.W. (2005). What constitutes good mathematics teaching and how it develops: Nine high school teachers' perspectives. *Journal of Mathematics Teacher Education, 8*, 83-111.

Watkins, D. A. & Biggs, J. B. (Eds.) (2001). *Teaching the Chinese Learner: Psychological and Pedagogical Perspectives*. Hong Kong: Comparative Education Research Center, The University of Hong Kong.

EXPLORING INDISPENSABLE ELEMENTS OF MATHEMATICS INSTRUCTION TO BE EXCELLENT: A JAPANESE PERSPECTIVE

Yoshinori Shimizu

University of Tsukuba

This paper examines aspects of mathematics classroom instruction that are valued and emphasized by Japanese teachers to explore the elements of mathematics instruction to be excellent and cultural values behind what can be considered as excellent. Then, key features of Japanese approach to develop and maintain quality mathematics instruction through a particular form of activity called "lesson study" are discussed with a focus on the opportunity for developing teachers' pedagogical content knowledge. The importance is emphasized of recasting the "excellence" as discussed in the current paper in the light of international comparison for identifying a culturally specific character to be found in mathematics classrooms.

INRODUCTION

The findings of large-scale international studies of classroom practices in mathematics include aspects of instruction as identified with a resemblance among participating countries as well as the uniqueness of Japan (Stigler & Hiebert, 1999; Hiebert, et al., 2003; Clarke, Emanuelsson, Jablonka & Mok, 2006). Japanese mathematics teachers, for example, appeared to spend more time on the same task in one lesson than their counterparts in the other countries by having students work on a challenging problem and discuss alternative solutions to it. Also, experienced teachers in Japan typically highlighted and summarized the main points at some particular phases of lessons to have their students to reflect on what they have learned (Shimizu, 2006). These striking characteristics can be regarded as indicating some indispensable elements of mathematics classroom instruction that are valued and emphasized by Japanese teachers.

The current paper examines aspects of mathematics classroom instruction that appear to make Japanese lessons different from the other countries and explores the elements of mathematics instruction to be considered as excellent. Cultural values behind what can be considered as excellent are also explored. Particular attention is given to how lessons are structured and delivered with an emphasis on presenting and discussing alternative solutions to a problem in the teaching and learning processes. The selected findings of the international studies mentioned above are used for setting the contexts for discussion of the uniqueness of Japanese lessons. Second, key features of the approach by Japanese teachers to develop and maintain quality mathematics classroom instruction through a particular activity called "lesson study" are discussed.

ELEMENTS OF LESSONS TO BE EXCELLENT

The video component of the Third International Mathematics and Science Study (TIMSS) was the first attempt ever made to collect and analyse videotapes from the classrooms of national probability samples of teacher at work (Stigler & Hiebert, 1999). Focusing on the actions of teachers, it has provided a rich source of information regarding what goes on inside eighth-grade mathematics classes in Germany, Japan and the United States with certain contrasts among three countries. One of the sharp contrasts between the lessons in Japan and those in the other two countries relates to how lessons were structured and delivered by the teacher. The structure of Japanese lessons was characterized as "structured problem solving", while a focus was on procedures in the characterizations of lessons in the other two countries. The following sequence of five activities was described as the "Japanese pattern": reviewing the previous lesson; presenting the problems for the day; students working individually or in groups; discussing solution methods; and highlighting and summarizing the main point.

In the Learner's Perspective Study (LPS, Clarke, Keitel & Shimizu, 2006), an analytical approach was taken to explore the form and functions of the particular lesson event with a focus on "highlighting and summarizing the main point", or "*Matome*" in Japanese, in eighth-grade well-taught mathematics classrooms in Australia, Germany, Hong Kong, Japan, Mainland China (Shanghai), and the USA (Shimizu, 2006). For the Japanese teachers, the event "*Matome*" appeared to have the following principal functions: (i) highlighting and summarizing the main point in the lesson, (ii) promote students' reflection on what they have done, (iii) setting the context for introducing a new mathematical concept or term based on the previous experiences, and (iv) making connections between the current topic and previous one. For the teachers to be successful in highlighting and summarizing the main point of the lesson, the goals of lesson should be very clear to both the students and themselves, the lesson as a whole should be coherent, and the students need to be involved deeply in the process of learning and teaching. The results suggest that identifying and achieving the goals of the lesson, coherence of the entire lesson, and students' involvement in the lesson are all to be noted for the excellence of lessons.

As for the goal of lessons, teachers' responses on the questionnaire to the question, "What was the main thing you wanted students to learn from today's lesson?", were analysed in the TIMSS Video Study. There was a significant difference between the reported goals of teachers in Japan and those in the other two countries. A majority of Japanese teachers reported that fostering mathematical thinking was the main goal for their lessons, while 55 percent of German teachers and 61 percent of U.S. teachers reported that development of skills was the main thing to be learned (Stiglar et al., 1999, p.46). It should be noted that an underlying assumption of "structured problem solving" for lessons is that it enables a teacher to give students opportunities for working on problem by themselves or in a group, and for communicating ideas with their classmate. Thus, teachers need to plan a lesson by trying to allow mathematics to be problematic for students, to focus on the methods used to solve problems.

Associated with such a description of the "structured problem solving" approach to mathematics instruction, several key pedagogical terms are shared by Japanese teachers. "*Hatsumon*", for example, means asking a key question for provoking students' thinking at a particular point in a lesson. The teacher may ask a question for probing or promoting students' understanding of the problem at the beginning of the lesson. "*Yamaba*", on the other hand, means a highlight or climax of a lesson. The point here, from a Japanese perspective, is that all the activities, or some variations of them, constitute a coherent system called a lesson. Further, a lesson is often regarded as a drama, which has a beginning and leads to a climax and a conclusion, among Japanese teachers. The idea of "KI-SHO-TEN-KETSU", an idea that originated in a Chinese poem, is often referred by Japanese teachers when they plan and implement a lesson. It is suggested that Japanese lessons has a particular structure of a flow moving toward the end ("KETSU", summary of the whole story).

If we take a story or a drama as a metaphor for considering an excellent lesson, a lesson needs to have a highlight or climax with a summing up to achieve the goal, fostering students' mathematical thinking, in a coherent way. Stigler and Perry (1988) found *reflectivity* in Japanese mathematics classroom. They pointed out that the Japanese teachers stress the process by which a problem is worked and exhort students to carry out procedure patiently, with care and precision. Given the fact that the schools are part of the larger society, it is worthwhile to look at how they fit into the society as a whole. The reflectivity seems to rest on a tacit set of core beliefs about what should be valued and esteemed in the classroom. As Lewis noted, within Japanese schools, as well as within the larger Japanese culture, *Hansei*---self-critical reflection---is emphasized and esteemed (Lewis, 1995).

PURSUING EXCELLENCE THROUGH "LESSON STUDY"

There are opportunities for teachers to learn with and from their experienced colleagues to pursue excellent lesson. Lesson study, "*Jugyo Kenkyu*" in Japanese, is an approach to develop and maintain quality mathematics instruction through a particular form of activity (Fernandez & Yoshida, 2004; Shimizu, 2002). Workshops of particular style are regularly held at each school level or at the other levels for both beginning and experienced teachers.

Generally a lesson study consists of the following events: the actual classes taught to pupils, observation by others, followed by intensive discussion called the study discussion. Designing, enacting, and analysing are the three stages of lesson study that evolve before, during, and after the lesson. There is extensive preparation made before the class, and there will be extensive work to be done after the lesson study as well, which will be used as a follow up and as a preparation for the next lesson studied. These events form a cycle or iterative process.

In the process of a lesson study, lesson plans are used as "vehicles" with which teachers can learn and communicate about the topic to be taught, anticipated student approaches to the problem presented, and important teacher roles at various phases of lessons. Japanese teachers usually do not write any lesson plan for their daily practices. However, writing lesson plans is a critical exercise for pre-service teachers. They are intensively taught how to write lesson plans. All pre-service teachers teach at one or more schools under the supervision of experienced teachers. During this period of education, prospective teachers learn through intensive coaching to write and polish up their lesson plan.

"*Kyozai-kenkyu*" in Japanese means analysing the topic carefully in accordance with the objective(s) of a lesson. It includes analyses of the mathematical connections both between the current topic and previous topics (and forthcoming ones in some cases) and within the topic, anticipation for students' approaches to the problem to be presented, and planning of instructional activities based on them.

Knowledge used and developed in the process of “Kyozai-Kenkyu” includes pedagogical content knowledge (Shulman, 1986: Ball and Bass, 2000). In addition to general pedagogical knowledge, teachers need to know things like what mathematical topics students find interesting or difficult, or what representations are most useful for teaching a specific content idea, and so on. The success of a lesson depends on the appropriate interpretation of mathematics topic in relation to the psychological aspects of students. Thus, Kyozai-kenkyu is a crucial part of the lesson planning for Japanese teachers. This kind of analysis is heavily emphasized in pre-service teacher training courses at the university. Through such processes, they learn educational values of teaching mathematics as well as the goals and method of it. Educating teachers about lesson plans is an important opportunity for the professional development of teachers.

In sum, the key features of the approach by Japanese teachers to develop and maintain quality mathematics instruction through a particular form of activity called "lesson study" include its potentials to offer the opportunity for developing teachers’ pedagogical content knowledge. Also, the important role of lesson plans should be noted as "vehicles" with which teachers learn and communicate about the topic to be taught.

CONCLUDING REMARKS

The ultimate goal of any study of classroom instruction is to improve teaching practice for enhancing students' learning, even if its major focus would be on, for example, constructing a theoretical model of learning in classroom or comparing teaching methods among countries. For this goal, various approaches and methodologies can be adapted to capture the "excellence" of mathematics instruction in each culture. We need to recast the discussion in the current paper in the light of international comparison for exploring a culturally specific character to be found in mathematics classrooms.

References

Ball, D.L. & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple Perspectives on Mathematics Teaching and Learning*. Weatport, CT: Ablex Publishing.

Clarke, D., Keitel, C., & Shimizu, Y. (2006). The Learners’ Perspective Study. In D. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics Classrooms in Twelve Countries: The Insider’s Perspective.* Rotterdam: Sense Publishers.

Fernandez, C. & Yoshida, M (2004). *Lesson Study: A Japanese Approach to Improving Mathematics Teaching and Learning*. Mahwah, NJ: Lawrence Erlbaum Associates, Hiebert, J. (2003). Signposts for teaching mathematics through problem solving. In F.K. Lester (Ed.), *Teaching Mathematics through Problem Solving: Grades PreK-6*, National Council of Teachers of Mathematics. Reston. V.A.: The Council.

Hiebert, J., Gallimore, R., Garnier, H., Givvin, K.B., Hollingsworth, H., Jacobs, J., Chiu, A.M.-Y., Wearne, D., Smith, M., Kersting, N., Manaster, A., Tseng, E., Etterbeek, W., Manaster, C., Gonzales, P., & Stigler, J. (2003), *Teaching Mathematics in Seven Countrie: Results From the TIMSS 1999 Video Study*. U.S. Department of Education. Washington, DC: National Center for Education Statistics.

Lee, S.Y., Graham, T., & Stevenson, H.W. (1996). Teachers and teaching: Elementary schools in Japan and the United States. In T.P. Rohlen & G.K. Letendre (Eds.), *Teaching and Learning in Japan*. New York: Cambridge University Press.

Lewis, C. (1995). *Educating Hearts and Minds: Reflections on Japanese Preschool and Elementary Education*. New York: Cambridge University Press.

Shimizu, Y. (1999). Aspects of Mathematics Teacher Education in Japan: Focusing on Teachers' Role. *Journal of Mathematics Teacher Education. 2*(1), 107-116.

Shimizu, Y. (2002). Lesson study: what, why, and how? In H. Bass, Z. P. Usiskin, & G. Burrill (Eds.), *Studying Classroom Teaching as a Medium for Professional Development: Proceedings of a U.S.-Japan Workshop*. Washington DC: National Academy Press, 53-57, 154-156.

Shimizu, Y. (2006) How do you conclude today’s lesson?: The form and functions of "Matome" in mathematics lessons. In D. Clarke, J. Emanuelsson, E. Jablonka, & I. Ah Chee Mok (Eds.), *Making Connections: Comparing Mathematics Classrooms Around the World*. Rotterdam: Sense Publishers.

Stigler, J. W. & Hiebert, J. (1999). *The Teaching Gap: Best Ideas from the World’s Teachers for Improving Education in the Classroom*. New York: NY, The Free Press.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15, 4-14.

PURSUING EXCELLENCE IN MATHEMATICS CLASSROOM INSTRUCTION TO MEET CURRICULUM REFORM IN TAIWAN

Pi-Jen Lin

National Hsinchu University of Education

This paper describes a general picture of learner-centred approach as recommended instruction and one of the features of good teaching shaped by classroom teachers who participated in a teacher professional development program.

INTRODUCTION

Much has been written about the fact that Asian students show superior performance in international mathematics assessments compared with their non-Asian counterparts, such as PISA, TIMSS 2003 (Mullis, Martin, Gonzalez, & Chrostowski, 2004; OECD, 2003). Students' performance can be taken as a possible indicator of the effectiveness of instruction. Stigler and Hiebert's analysis of the TIMSS video study collected from Japan, America, and Germany indicates that students in Japan spent more time inventing mathematical concepts while students in the United States spent more time on routine practice (Stigler, Gallimore, & Hiebert, 2000). A similar pattern is found regarding mathematics concepts that were developed by "the teacher with students' participation" (77% in Japan and 22% in the United States) vs. simply "stated by the teacher" (23% in Japan and 78% in the United States) (Stigler and Hiebert, 1999). Results suggest that there are cultural differences between Asian and Western countries in expectation for student achievement in mathematics and instructional strategies.

LEARNER-CENTERED APPROACH EMPHASIZED IN CURRICULUM

Instruction involved in a complex process is shaped by the interaction of teachers and curriculum materials. The implementation of curriculum materials varies considerably as teachers make different interpretations. Engagement with particular curricular features can impact teachers' pedagogical understanding, and then shape mathematics instruction (Remillard & Bryans, 2004). This suggests that various instruction approaches are driven by different curricular reforms. For instance, teacher-oriented instruction emphasized in the Curriculum Standards of Elementary Mathematics (CSEM) issued in 1975 is much different from learner-centred approach emphasized in the CSEM reissued in 1993 (Ministry of Education, 1993). Traditionally, most teachers begin teaching with textbook and teacher's guide from the beginning of each semester, following it lesson by lesson. The teachers' focus is helping students passing a quiz after another. Most teachers characterize a most effective teaching as offering well-organized teacher-directed instructions. As a result, memory and drilled practice are emphasized, while important mathematics education goals such as meaningful understanding of concepts and the skills of communicating, problem solving, reasoning, and connecting tend to be overlooked.

Conversely, the philosophy underpinning the 1993 version reflects that knowledge should be constructed actively rather than passively. Learning mathematics is viewed as an integrated set of intellectual tools for making sense of mathematical situations instead of as accumulating facts and procedures. Mathematics classrooms are expected to become as mathematical communities instead of classroom as simply a collection of individuals. The right answer is verified by logic and mathematical evidence, instead of being determined by the teacher. Excellent teaching includes that teachers know how to ask critical questions and plan lessons that reveal students' prior knowledge, teachers create mathematics tasks and analyse student learning in order to make ongoing instructional decisions, and teachers stimulate classroom discourse so that the student are clear about what is being learned. The role of teachers is shifted into a problem-poser and a facilitator from a problem-solver and a knowledge constructor. Students become a problem-solver and knowledge constructor instead of a knowledge copier.

In order for the implementation of learner-centred approach to be successful, teachers need to be committed to the vision of the reform and to be more versatile in using instructional strategies to facilitate students' mathematical power. Therefore, to move the reform ahead, various strategies, techniques, and activities are developed in different professional development programs of Taiwan. For instance, a training master teacher program supported by the MOE since year 2003 aims in training teachers to be masters of mathematical instruction toward learner-centred approach. Fifty teachers participating in the program each year are recruited from different school districts distributed in different areas. They receive a series of institutes or workshops at the beginning and the end of school semester.

The rationale underpinning the curriculum is the preliminary courses of the workshops. It is followed by the courses related to mathematical instruction and assessment. During the school year, they are assisted in classroom practices by a pool of teacher educators from different universities. The teachers to be master teachers are asked to teach a lesson for teachers who are not participating in the program to learn to teach effectively. There are about 6-10 master teachers in each county as a pool of consultants for school teachers to improve their mathematics teaching. They are frequently invited by schools to deliver a lecture, to write textbooks, and to do professional work with respect to mathematics instruction. Their professional work can be an indicator of the effect of the training master teacher program.

One of the development programs supported by the National Science Council supporting teachers in developing teachers' high quality of instruction has been run for ten years. The goals of the teacher education program include 1) enhancing the rethinking of mathematics teaching in classrooms; 2) fostering teachers' awareness of children's learning; 3) supporting teachers as they begin to put into practice their new vision of a learner-centered approach to teaching mathematics. Social constructivism dealing with the construction of knowledge through interactions between humans and social worlds is drawn on as the basis for the professional program. Each year, a collaborative school-based professional team consisting of a teacher educator and 6 to 8 primary school teachers from a school or across schools is set up for providing teachers with professional dialogues based on classroom practices. Reflection, social interaction, and cognitive conflict are considered as three mechanisms of improving teachers' teaching. Mathematics classrooms and school-based professional team are two contexts for supporting teachers. Various strategies for improving teachers' understanding of students' learning including assessment integrated with instruction, analysis of students' various solutions, and the use of teaching cases were inquired in the studies (Lin, 2002, 2005, 2006). Within the space constrains of this paper, I mainly focus on the design of high cognitive demand of tasks as a kernel part of teaching a lesson, as one of the features of learner-centered approach, since it plays an important role in student learning.

MAINTAINING HIGH LEVEL COGNITIVE DEMAND TASKS AS A FEATURE OF GOOD TEACHING

Different tasks require different levels of student thinking. The cognitive demands of tasks can be changed during a lesson. Starting with a high-level task does not guarantee student engagement at a high-level. Thus, teachers participating in the professional program need support to maintain high level cognitive demands at the implementation stage. T1, a sixth grade teacher, is one of the teachers involved in the development program. The nature of high-level cognitive demands she maintains in teaching a lesson relevant to ordering fractions is displayed here as an example.

Before this lesson, students have learned ordering fractions with like denominator fractions. In the lesson including setup phase and implementation phase, T1 gave students four pairs of fractions to decide which is greater. The four pairs ($\frac{1}{5}$ *vs.* $\frac{1}{7}$; $\frac{5}{16}$ *vs.* $\frac{5}{9}$; $\frac{4}{9}$ *vs.* $\frac{8}{12}$; $\frac{11}{12}$ *vs.* $\frac{14}{15}$) have four basic types: unit fractions, fractions with like numerator or denominator, and fractions with unlike numerators and denominators.

The setup phase includes T1's communication to her students regarding how they were expected to decide which of the fractions is greater and how they were expected to compare them. The four pairs of fractions identified as the cognitive demands at the level of "procedure with connection" were based on the following three reasons: (1) The four pairs of fractions develop mathematical understanding; (2) T1 purposely changed the tasks with different types of fraction from the textbook for developing students' multiple strategies. (3) T1 intentionally designed the numerals of numerator and denominator between two fractions for developing students' various strategies rather than relying on the algorithm.

The implementation phase starts as soon as students began to work on the task and continued until T1 and students turned their attentions to a new task. Five different strategies were used by the TI's students for the four problems. Students used two strategies to compare $\frac{1}{5}$ *vs.* $\frac{1}{7}$. One is referred to unit fraction. They realized that there is an inverse relation between the number of parts into which the whole is divided and the resulting size of each part, so that $\frac{1}{5} > \frac{1}{7}$. It was then followed by the problem "comparing $\frac{5}{16}$ vs. $\frac{5}{9}$". Students still used two previous strategies, partitioning and finding a same denominator. They also developed a new strategy by finding a referent point ($\frac{1}{2}$ or 1). TI attempted to reduce the use of common denominator, since the product of 16x5 is too big to getting correct answer. TI expected students to learn various strategies and each strategy can be applied in a suitable situation.

Moving on to the third problem "Order $\frac{4}{9}$ vs. $\frac{8}{12}$", students focused only on the numerator or only on the denominator and as a result made incorrect conclusions. T1 encouraged students solved successfully the problem by either using reference point $\frac{1}{2}$, or finding a common denominator requires finding $\frac{4x4}{9x4}$ equivalent to $\frac{4}{9}$ and $\frac{8x3}{12x3}$ equivalent to $\frac{8}{12}$ with the like denominator 36 or finding the same numerator 4 requires finding $\frac{4}{6}$ equivalent to $\frac{8}{12}$ and then ordering $\frac{4}{6}$ and $\frac{4}{9}$, or finding the same numerator 8 requires finding $\frac{8}{18}$ equivalent to $\frac{4}{9}$ and then ordering $\frac{8}{18}$ and $\frac{8}{12}$.

During the implementation phase, both T1 and her students were viewed as important contributors to how tasks were carried out. T1 questioning to students or asking follow-up questions was relied on what her students worked on the task. The ways and extent to which T1 supported students' thinking was a crucial ingredient of maintaining high-level tasks at the level of procedure with connections. These tasks evolved during the lesson involved multiple strategies, required an explanation, and connected procedures to meaning. Part of the lesson shown in a 5-minute video will be presented in the Research Forum.

REMARKS

Maintaining high quality of cognitive demand of tasks is orchestrated by the teachers participating in a teacher professional program for pursuing excellence in mathematics classroom teaching to meet the innovation of curriculum. The features of excellent mathematics teaching to be achieved are characterized as contextual problems to be posed, multiple representations for a given problem, coherence and progression from one activity to next, students' problem solving to be encouraged, students' various solutions and explanations to be articulated. However, learner-centered focusing on students' speaking mathematics does not constitute the mainstream of mathematics classrooms in Taiwan. The teaching of instructor-centered whole class organization in Taiwanese mathematics classroom is supported by the international TIMSS 2003 study (Mullis, et al. 2004). The learner-centered approach recommended in the curriculum is not popularly implemented into classrooms is based on the following possible reasons. First, it is a challenge work for teachers who are used to teaching with a teacher-centered approach. Second, it is not supported by the mathematicians who are concerns with mathematics teaching. Third, the learner-centered approach replaced by instructor-centered approach is not coherently recommended in the newly curriculum.

References

Lin, P.J. (2002). On enhancing teachers' knowledge by constructing cases in classrooms. *Journal of Mathematics Teacher Education*. Vol. 5, Issue 4. 317-349.

Lin, P.J. (2005). Supporting teachers' learning to teach mathematics in primary teacher professional development program. In B. Linda (Ed.), *Contemporary Teaching and Teacher Issues* (175-196). NY: Nova Science publisher.

Lin, P.J. (2006). Conceptualizing teachers' understanding of students' mathematical learning by using assessment tasks. *International Journal of Mathematics and Science Education*. *4*(4), 545-580.

Martin, M. O., Mullis, I.V.S., & Chrostowski, S.J. (2004). *TIMSS 2003 Technical Report : Findings from IEA's Trends in International Mathematics and Science Study at the Fourth and Eighth Grades*. Chestnut Hill, MA: TIMSS International Study Center, Boston College.

Ministry of Education (1993). *Curriculum Standards for Elementary Mathematics*. Taiwan: Taipei.

Mullis, I.V.S., Martin, K.D., Gonzalez, E.J., & Chrostowski, S.J. (2004). *TIMSS 2003 International Mathematics Report: Findings from IEA's Trends in International Mathematics and Science Study at the Fourth and Eight Grades*. Chestnut Hill, MA: TIMSS International Study Center, Boston College.

Organization for Economic Co-operation and Development (2003). *The PISA 2003 Assessment Framework-Mathematics, Reading, Science, and Problem Solving Knowledge and Skills*. Paris: Author.

Remillard, J.T. (2004). Curriculum materials as tools: A sociocultural analysis of teachers' use of curriculum: Paper presented at the annual meeting of the *American Educational Research Association*, San Diego, CA.

Stigler, J.W., Gallimore, R., & Hiebert, J. (2000). Using video surveys to compare classrooms and teaching across cultures: Examples and lessons from the TIMSS Video Studies. *Educational Psychologist, 35*, 87-100.

Stigler, J.W. & Hiebert, J. (1999). *The Teaching Gap: Best Ideas from the World's Teachers for Improving Education in the Classroom*. New York: Free Press.

GOOD MATHEMATICS INSTRUCTION AND ITS DEVELOPMENT IN SOUTH KOREA

Jeong Suk Pang

Korea National University of Education

This paper describes diverse aspects of good mathematics instruction perceived in Korean context, and then probes factors contributing for good mathematics instruction and their implications in mathematics education.

INTRODUCTION

Mathematics instruction illuminates not only students' conceptual development but also the nature and effects of their participation in socially situated activities. Educational leaders have sought to implement "good" mathematics instruction at the classroom level. Diverse aspects of high-quality teaching and learning are articulated along with the curriculum standards for school mathematics (NCTM, 2007). However, what counts as good mathematics instruction is not always manifested, partly because it is closely connected to cultural values and norms. For instance, Li and Yang (2007) show that master teachers in China keep the good tradition of teaching basic knowledge and basic skills while they attempt to embrace student-centered principles from the Western perspective.

Given this background, this paper introduces typical and recommended mathematics teaching practices in Korea, followed by diverse aspects of good mathematics instruction perceived in Korean contexts. This paper then illustrates one specific master teacher's mathematics instruction in order to look closely at what is high-quality teaching and learning as implemented at the classroom level. This paper finally probes factors contributing for good mathematics instruction and their implications in mathematics education.

TYPICAL AND RECOMMENDEND TEACHING PRACTICS

Typical Korean teachers orchestrate their lessons more systematically, coherently, completely, and progressively than their U.S. counterparts do. In comparing Korean mathematics instruction with other Asian classrooms, Grow-Maienza, Hahn, and Joo (1999) found some similarities such as the pattern of instruction/practice/evaluation, the placement of problems in real-world contexts, the representation of one problem using several modes, the use of concrete demonstration or manipulative materials, and the coherence and progression of the lesson. Nevertheless, they underlined that Korean instruction focuses primarily on procedures to solutions of given problems. Korean mathematics instruction is indeed teacher-centered in that the teachers' explanations and directions constitute the mainstream of mathematical practices.

Countering the common teacher-centered pedagogy in mathematics, many characteristics of student-centered teaching methods are consistently recommended. The most recent national curriculum stresses giving students opportunities to study mathematics based on their individual learning capacity, aptitude, and interest (MEHRD, 2007). Specifically, the curriculum urges a teacher to (a) provide students with meaningful questioning on the basis of their cognitive development and experience, (b) teach mathematical concepts and principles through students' concrete manipulative activities and inquiries, (c) foster mathematical thinking and reasoning ability on the basis of students' own justification and explanation, (d) help students communicate mathematical ideas with multiple means such as symbols, tables, and graphs, (e) facilitate students' problem solving ability by letting them explore problem situation on their own and emphasizing the solution processes, (f) be sensitive to students' interest and confidence in mathematics, and (g) employ appropriate educational resources and technology.

DIVERSE ASPECTS OF GOOD MATHEMATICS INSTRUCTION

Despite the recommended teaching practices described above, it is not easy to articulate what really counts as "good" mathematics instruction in Korea. Instead of seeking uniform characteristics of effective mathematics instruction, diverse aspects are provided. First, there are instruction-research contests for teachers organized by the educational offices of each province. The criteria to select good mathematics instruction include creativity and appropriateness of lesson design, students' understanding of contents and their participation, accuracy of contents and promotion of creativity and thinking, extension and synthesis of students' thinking, adequate and diverse levels of questioning, and timely use of instructional materials.

Second, Choe (2002) selected six specific cases of good mathematics instruction throughout Korea and solicited the common characteristics of such cases in the following five dimensions: 1. With regard to curriculum and mathematics contents, the teachers re-constructed textbooks on the basis of their students' abilities and local conditions; 2. With regard to instructional methods, multiple techniques were implemented such as differentiated approaches tailored to students' individual differences, motivation-evoking methods in terms of students' real-life contexts, and appropriate use of information and communication technology; 3. With regard to the understanding of students, the teachers attempted to improve students' mathematical attitudes by being sensitive to their aptitudes and interests, and established good relations with their students; 4. With regard to assessment, the teachers monitored students' progress during their instruction and employed performance assessment; and 5. With regard to professional development, the teachers were actively involved in self-directed re-training courses as well as multiple professional activities among teachers.

Third, the Korea Institute of Curriculum and Evaluation announced detailed criteria of assessing mathematics instruction, articulating learning environment and actual teaching practice. The former includes the establishment of a physical environment for effective instruction, mutual respect and interaction between the teacher and students, and management of students by consistent norms and procedures. The latter includes examination of students' prior knowledge, motivational strategies adequate to the contents to be covered, diverse and adequate modes of instruction in terms of contents and students, encouragement of students' active participation, effective use of small-group and whole-class formats, encouragement of students' confidence and ability, diverse and effective questioning, adequate feedback, flexible improvisation against unexpected events, and multiple assessment strategies and timely feedback.

GOOD MATHEMATICS INSTRUCTION IMPLEMENTED IN THE CLASSROOM

The data used in this session are from a one-year project of understanding the culture of Korean mathematics classrooms in transition (Pang, 2005). Ms. K was identified as the most successful 6^{th} grade teacher by the other participant teachers as well as by the researcher. The overall characteristics of Ms. K's instruction can be summarized by the following five aspects.

First, each lesson consisted of a brief review of the previous lesson, the teacher's introduction of new mathematical contents or activities, students' individual or small-group activities, and whole-class discussion and summary. As Ms. K encouraged students to explain and justify what they discovered during the whole-class discussion phase, they tended to be actively engaged in the previous activities.

Second, Ms. K was very skillful in re-constructing the learning sequence and the activities in the textbook on the basis of mathematical significance. For instance, the teacher added a full lesson emphasizing the meaning of division of fractions before exposing students to many activities geared at finding the common algorithm for the division.

Third, mathematical concepts or principles were introduced not by the teacher but by students' mathematical activity. In a typical classroom, a mathematical concept is usually introduced by the corner of "definition" in the textbook, and explained by the teacher. In contrast, Ms. K did not explain such a definition as it is in the textbook. She rather encouraged students to define mathematical concepts or principles on the basis of classroom activities.

Fourth, Ms. K encouraged students to find different solution methods for a given problem. In this way, students had many opportunities to explore the meaning behind algorithms and to connect visual representations with numerical equations. Students also compared and contrasted multiple solution approaches in terms of mathematically significant ideas.

Finally, Ms. K tended to provide a detailed explanation of the main task before students' own activities. This led students to consider what mathematical thinking was called for, instead of simple completion of the given task. Similarly, with regard to important mathematical contents, the teacher re-stated in detail or insisted on clear explanation and justification to the presenter (student), so that the whole class examined the crucial contents.

MOTIVATING FACTORS FOR GOOD MATHEMATICS INSTRUCTION

Good mathematics instruction in Korea is motivated by multiple factors such as a mathematics teacher with enthusiasm and deep content knowledge, diverse instructional resources, and a professional community. Such factors can be divided into teacher factors and cultural factors.

In general, education is culturally valued and a teacher is highly respected in Korea. Because of the competitive process of being a teacher, the overall quality of teachers is high. In addition, the teacher preparation program emphasizes mathematically sound knowledge (Leung & Park, 2002; Li, Ma, & Pang, in press). Considering that teacher educators in Korea still seek to provide even more training in school mathematics, the content-oriented aspects of good mathematics teaching practices are not unexpected. Another important element is the teacher's own willingness and effort to develop good mathematics instruction. The teachers described above implemented good teaching, despite multiple unfriendly local conditions, on the basis of their consistent effort to make their lessons meaningful for students.

Multiple cultural factors may be related to the development of good mathematics instruction. Such factors include (a) the development of mathematics textbooks and their related resources such as student's workbook and teacher's lesson guide, (b) various instructional resources ready to be used such as teaching materials, teaching tips, and teaching episodes, (c) a professional community established by teachers, (d) instruction-research contests among teachers, and (e) diverse modes of supervision of mathematics instruction among teachers. For instance, instruction-research contests as a whole promote participant teachers' ongoing commitment towards good mathematics instruction throughout the year rather than through a single-use performance. The teachers who get good scores in contests are respected as "research teachers for improving instruction" or "consulting teachers for instruction" and are obligated to open their lessons to other teachers periodically. This leads to other typical teachers to see good instruction practices implemented at the classroom level.

CONCLUSION

Traditionally, good mathematics instruction was equal to effective instruction, which is mainly evaluated by students' achievement as the product of learning. This perspective has been changed to consider not only the product but also the process of learning. Given the complexity of teaching and learning, it is difficult to summarize what a "Korean version" of good mathematics teaching is. Nevertheless, the two salient features are mathematics content and students. The former is related to the degree by which the main mathematics topics are taught in a meaningful way, whereas the latter is connected to the degree by which instruction considers students' prior knowledge, interests, and attitudes.

Another issue to be discussed is insufficient models of good mathematics instruction. Teachers have opportunities to observe others' instruction but such opportunities do not guarantee observing a high-quality lesson, which in turn may help them implement a similar lesson at their next class. Considering that the current recommended teaching practices reflect diverse theoretical perspectives, the mathematics education community should offer various teaching approaches so that the teacher considers the strengths and weaknesses of the approaches with regard to his or her own pedagogical intentions in the specific classroom situation (Kirshner, 2002). Alternative models of good mathematics instruction allow teachers to examine, to reflect on, and to develop their own teaching philosophy. In this respect, multiple models of high-quality mathematics instruction need to be developed and seen by teachers.

References

Choe, S.H. (2002). *Improving the Quality of School Education in Korea: A Qualitative Case Study on Good Mathematics Teaching in Secondary Schools* (in Korean). Research Report to Korea Institute of Curriculum and Evaluation (RRC 2002-4-3).

Grow-Maienza, J., Hahn, D-D., & Joo, C-A. (1999, April). *Mathematics Instruction in Korean Primary Schools: A Linguistic Analysis of Questioning*. Paper presented at the Annual meeting of the American Educational Research Association. Montreal, Canada.

Kirshner, D. (2002). Untangling teachers' diverse aspirations for student learning: A crossdisciplinary strategy for relating psychological theory to pedagogical practice. *Journal for Research in Mathematics Education, 33*(1), 46-58.

Leung, F.K.S. & Park, K.M. (2002). Competent students, competent teachers? *International Journal of Educational Research, 37*(2), 113-129.

Li, S. & Yang, Y. (2007). Insight into the development of mathematics teaching: A comparative study of two videotaped lessons. In *Proceedings of National Meeting of Korea Society of Mathematical Education* (pp. 1-10).

Li, Y, Ma, Y, & Pang, J.S. (in press). Mathematical preparation of prospective elementary teachers. In T. Wood (Series Ed.) & P. Sullivan (Ed.), *International Handbook of Mathematics Teacher Education: Vol. 1. Knowledge and Beliefs in Mathematics Teaching and Teaching Development*. Rotterdam, The Netherlands: Sense.

Ministry of Education and Human Resources Development (2007). *Revision of the 7th Mathematics Curriculum* (In Korean). Seoul, Korea: the Author.

National Council of Teachers of Mathematics (2007). *Mathematics Teaching Today* (2^{nd} Ed.), Reston, VA: The Author.

Pang, J.S. (2005). Transforming Korean elementary mathematics classrooms to student-centered instruction. In H.L. Chick & J.L. Vincent (Eds.), *Proceedings of the 29^{th} Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 41-48). Melbourne: PME.

DEVELOPING EXEMPLARY LESSONS TO PURSUE MATHEMATICS CLASSROOM INSTRUCTION EXCELLENCE IN CHINA

Rongjin Huang and Yeping Li

Texas A&M University

To develop a better understanding of mathematics classroom instruction in China, this paper will first provide a general picture of mathematics classroom instruction in China based on previous empirical studies. Then, we describe how a master teacher developed an exemplary lesson in teaching a newly added content topic in the new curriculum, in a nationwide research project. Finally, the characteristics of the exemplary lesson are analysed.

INTRODUCTION

Mathematics classroom instruction and teacher education in China have experienced tremendous changes and confronted many new challenges, since the release of new mathematics curriculum standards in 2001 (Ministry of Education, 2001, 2003). To cope with the challenges of developing quality classroom instruction as valued in the new curricular conception, various approaches have been developed and utilized. In particular, exemplary lesson development has been undertaken as an important approach by many in-service teacher professional programs to explore effective teaching with innovative teaching ideas and/or to teach some newly added content topics (e.g., Huang & Bao, 2006; Zhang, Huang, Li, Qian, & Li, 2008). In this paper, we first provide a brief summary of the characteristics of Chinese mathematics classroom instruction. We then report a case study to illustrate the process of developing exemplary lessons, along with a detailed analysis and discussion of the exemplary lesson being developed.

GENERAL CHARACTERISTICS OF MATH INSTRUCTION IN CHINA

Based on the observation of around 800 elementary mathematics classes in China, Japan, and the United States, Stigler and Perry (1988) found a number of differences between East Asian and American mathematics classrooms. For example, East Asian students were more involved in mathematics tasks posed by the teacher than were American students; the frequency of East Asian students offering their ideas was significantly higher than those for American students. Also based on extensive classroom observations, Leung (1995) outlined the general structure of mathematics lessons in Beijing. That includes: (1) revising work that students had learnt in the previous lesson; (2) introducing the topic of the lesson and developed the topic; (3) demonstrating and discussing classroom exercise on the black board; and (4) summarization and assignment of homework.

Recently, based on the TIMSS 1999 video study and the LPS study data, more features of Chinese mathematics classroom were identified as follows: (1) lecturing and explaining dominated as the form of a whole classroom instruction, (2) introducing a new content topic through reviewing and solving problems; (3) explaining and illustrating the new topic carefully; (3) unfolding the lesson coherently; (4) emphasizing mathematics reasoning; (5) emphasizing knowledge construction and development; (6) emphasizing internal mathematical connections among problems and variation exercises; (7) summarizing and assigning homework (e.g., Huang, Mok, & Leung, 2006; Leung, 2005).

The new curriculum standards advocate some innovative notions of effective teaching as follows: (1) building on students' learned knowledge and existing experience, and cognitive development levels; (2) using multiple teaching methods and measures such as self-exploration, cooperation and exchanges to guide students active learning through mathematics activities; (3) understanding and mastering basic knowledge, skills, and the underlying mathematical ideas and methods; (4) developing students mathematical application and creativity awareness, enhancing mathematics quality and positive attitudes towards to mathematics, providing a profound foundation for further study and development (Ministry of Education, 2001, 2003). However, to what extent do the practicing teachers have similar notions of effective teaching as suggested by the new curriculum standards? Several studies investigating master teachers' perspective on effective teaching in China found that an effective lesson should have the following features: (1) comprehensive and feasible teaching objective (knowledge, skill, mathematics thinking and attitudes); (2) scientific and reasonable lesson design such as the connections and development of content; (3) students' participation, self-exploratory learning, independent thinking, collaboration and exchange; (4) teacher's sound subject knowledge and apt teaching skill, and good personality; (5) providing proper classroom exercise and homework as well as high-order thinking opportunities (Huang, Chen, & Zhao, 2005; Huang & Li, in press). These findings seem to suggest that master teachers have tried to make a balance between traditional features and innovative ones in order to pursue effective teaching. In the following section, we will provide a case to demonstrate how a master teacher developed an exemplary lesson within a nationwide research project.

THE CASE STUDY OF DEVELOPING AN EXEMPLARY LESSON

The Case Context

A longitudinal and national-wide project, entitled "Structuring Mathematics with Core Concepts at Secondary School Level and Its Experimental Implementation", has been in action since early 2006 (Zhang et al, 2008). More than three hundreds team members in different fields from more than seven representative provinces in Mainland China participated in this project. A team headquartered in a south-eastern city, with about ten members, developed a lesson on Algorithms, a newly added content topic in high school mathematics curriculum. Ms. Chen was responsible for designing and teaching this selected topic. She was a senior mathematics teacher at a key school in the city. She had bachelor's degree and masters' diploma in mathematics with about 20 years of teaching experience, and had participated in several teacher professional development programs such as, the new curriculum training, provincial key teacher training. Ms. Chen also won the first-class award of junior teacher instruction competition at the municipal level. Developing an exemplary lesson included the following phases: individual and collaborative instructional designs of the experimental teaching, implementing experimental teaching, reflecting and improving the instructional design and implementation, implementing the revised design, and forming a case of the exemplary lesson (video, lesson design, and the reflection of its development) for nation-wide exchanges (Zhang, et al. 2008).

Features of the Exemplary Lesson

The teaching procedure of the exemplary lesson included the following phases: (1) instruction of the topic; (2) introduction to the concept of algorithms with a problem-solving approach; (3) forming the concept through analysis and synthesis; (4) fostering the understanding of the concept and learning to express with daily language and solving deliberately selected problems; (5) classroom exercise and home work.

Firstly, through showing pictures of counting chips, an abacus and a computer from the textbook, it was aimed to induce a common method underlying those instruments, namely, algorithms.

Secondly, the teacher presented one problem below: Can you find out the procedures to solve system of linear equations with two unknowns $\begin{cases} x-2y=-1 & (1) \\ 2x+y=1 & (2) \end{cases}$? After that, two problems on the system of linear equations with two unknowns were presented to students to solve and discuss. It was aimed to help students know that the algorithm is a method for solving a group of problems which can be generalized.

Thirdly, the teacher asked the following questions: What is meant by algorithms? How to express an algorithm? The intention of these problems was to let students get a preliminary knowledge about algorithms.

Fourthly, more abstract problems were posted to students. One was to judge whether 7 is a prime number. The other was to find out the approximate solution of an equation. The intention of these problems was to review relevant methods and demonstrate the sequence and operation clearly, further realize the logical structure of algorithms, comprehend the algorithmic thinking and characteristics, and further consolidate how to express algorithm with normal language.

Finally, through questioning and answering, the content taught was reviewed and summarized, and some exercises and homework were assigned.

In order to capture the features of this lesson, we adopted a framework by Carpenter and Lehrer (1999), which suggested that the following five forms of mental activities are conducive to developing mathematics understanding in a classroom. They are: (a) constructing relationships, (b) extending and applying mathematical knowledge, (c) reflecting about experiences, (d) articulating what one knows, and (e) making mathematical knowledge one's own. Through discourse analysis, we identified some features of this exemplary lesson such as (1) building the new concept through reviewing and problem solving progressively; (2) consolidating the concept through applications of the concept systematically; (3) clarifying the concept through encouraging students' articulation; (4) reflecting on the concept through summarizing. For example, the teacher paid a close attention to building the new concept on previously relevant concepts and methods, and students' contribution as shown below:

T : Please read the expression on the blackboard, what are the salient features of this expression compared to your own?

S : Sequent and reasonable.

T : What does it mean?

S : Uh …

T : Can you express it in other words?

S : Sequence and order.

T : Express in sequence and order procedures, is it right ？

S : Right

T : Thus, based on the previous problem, we solve a group of problems, is not it? We project the procedure of solving this group of problems by following sequent and ordered steps, and these steps consist of an algorithm, is not it?

After presenting the definition of the algorithm, the teacher re-emphasized its characteristics (definite, finite, and sequent) and its relationship with methods to solve a particular system of equations.

DISCUSSION AND CONCLUSION

In China, mathematics curriculum reform has brought not only new thinking about what to teach, but also how to teach effectively. This exemplary lesson demonstrates a problem-based teaching approach: through solving a series of deliberately selected problems to stimulate learning interest, connect the new topic to previous knowledge, form the new concept, clarify and consolidate the concept, and finally apply the concept in different contexts. These features still reflect both some Chinese mathematics instruction traditions as well as some innovative notions. Moreover, as expressed by the practicing master teacher, she has benefited from this participation in many ways, such as advancing her understanding of the new curriculum, fostering her understanding of the textbook and ability in dealing with the textbook properly, obtaining insights from comparing different designs, observing lessons, and listening to experts' comments. We are certain that participating teachers have benefited from developing exemplary lessons in terms of gaining a better understanding of the content taught as well as the process of developing an effective lesson (Huang & Bao, 2006).

Endnote

We would like to thank Dr. Jianyao Zhang for allowing us to use the exemplary lesson from their project. Thanks also go to Mr. Xuejun Li for his help in data collection.

References

Carpenter, T.P. & Lehrer, R. (1999). Teaching and learning mathematics with understanding. In E. Fennema & T.A. Romberg (Eds.), *Mathematics Classroom that Promote Understanding* (pp.19-32). Mahwah: Lawrence Erlbaum Associates, Publishers.

Huang, R. & Bao, J. (2006). Towards a model for teacher professional development in China: Introducing keli. *Journal of Mathematics Teacher Education*, 9, 279-298.

Huang, R., Chen, Y., & Zhao, X. (2005). How do master teachers evaluate mathematics lessons? [In Chinese] *Journal of Mathematics Education, 14*(1), 52-56.

Huang, R. & Li, Y. (in press). Examining the nature of effective teaching through master teachers' lesson evaluation in China. In J. Cai, G. Kaiser, B. Perry, & N.-Y. Wong (Eds.), *Effective Mathematics Teaching from Teachers' Perspectives: National and Cross-National Studies.* Rotterdam: Sense Publishers.

Huang, R., Mok, I., & Leung, F.K.S. (2006). Repetition or Variation: "Practice" in the mathematics classrooms in China. In D.J. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics Classrooms in Twelve Countries: The Insider's Perspective* (pp. 263-274). Rotterdam: Sense Publishers.

Leung, F.K.S. (1995). The Mathematics classroom in Beijing, Hong Kong and London. *Educational Studies in Mathematics,* 29, 197-325.

Leung, F.K.S. (2005). The implications of TIMSS 1999 Video Study for Chinese mathematics curriculum reform [in Chinese]. *Journal of mathematics education, 14*(1), 7-11.

Ministry of Education, P.R. China (2001). *Mathematics Curriculum Standard for Compulsory Education Stage (experimental version)* [in Chinese]. Beijing: Beijing Normal University Press.

Ministry of Education, P.R. China. (2003). *Mathematics Curriculum Standards for Normal Senior High Schools* [in Chinese]. Beijing: People's Educational Press.

Stigler, J.W. & Perry, M. (1988). "Mathematics learning in Japanese, Chinese, and American classrooms". *New Directions for Child Development*, 41, 27-54.

Zhang, J., Huang, R., Li, Y., Qian, P., & Li, X. (2008). *Improving Mathematics Teaching and Learning with a Focus on Core Concepts in Secondary School Mathematics: Introducing a New Nation-Wide Effort in Mainland China*. Paper presented at TSG4, the ICME-11, Monterry, Mexico; July, 2008.

PURSUING EXCELLENCE IN MATHEMATICS CLASSROOM INSTRUCTION IN EAST ASIA - A PERSONAL COMMENTARY FROM A WESTERN PERSPECTIVE

Gabriele Kaiser and Maike Vollstedt

University of Hamburg

The preceding papers of the research forum describe what is meant by the term *excellence* in mathematics classroom instruction and how this excellence is achieved in different East Asian systems, namely Japan, Taiwan, Korea, and China. We now provide a commentary on these descriptions from a Western perspective. For this commentary we specifically refer to the positions and perspectives from Continental Europe in contrast to Anglo-Saxon Europe and western countries influenced by the Anglo-Saxon approaches. First, we will concentrate on the question of what the different East Asian systems as presented in the papers mean by excellence in the mathematics classroom: Which aspects are valued as being especially important? Then, we will discuss the meanings and how this excellence is supposed to be reached. Finally, commonalities and differences to the Western perspective on excellence in mathematics classroom instruction are demonstrated.

(1). DESCRIPTIONS OF EXCELLENCE AND CRITERIA FOR EXCELLENT MATHEMATICS INSTRUCTION

The papers of the research forum show a great spectrum of what is perceived as excellence in mathematics instruction in the East Asian education systems named above.

In Japan, the approach is characterised in general by structured problem solving. This basically means that the lesson has a special structure of flow moving towards the end with several activities in between. These activities play a special role within the lesson. Therefore, the lesson needs a highlight or climax and a summing up as central features. Furthermore reflectivity is an important factor of mathematics lessons and is therefore emphasised in them.

In Taiwan, excellent mathematics teaching is characterised by maintaining a high level of the cognitive demand of tasks, i.e. contextual problems to be posed, multiple representations for a given problem, coherence, and progression from one activity to the next. The teachers encourage students' problem solving, students' quest for various solutions, and the articulation of explanations.

In Korea, excellent mathematics teaching is characterised by a careful orchestration of the lessons by being systematic, coherent, complete, and progressive. The emphasis on content (i.e. meaningful teaching of mathematical topics) and on students (i.e. consideration of the students' prior knowledge, interests, and so on) can be seen as salient features of good Korean mathematics teaching.

In China, a whole set of criteria is formulated for excellent mathematics lessons, such as comprehensible teaching objectives, reasonable and scientific lesson design, as well as sound subject knowledge and teaching skills of the teacher.

These descriptions clearly show the central role of the lesson composition and structure with introduction into the lesson, introduction of new concepts, exercise, and homework in establishing high quality instruction in these four East Asian education systems. Especially in Japan, a very strong emphasis is put on the careful composition of a lesson. Two very specific elements are stressed in particular, namely the necessity of a climax and a summing-up as methodical resource to ensure excellence. In a certain contrast to this, the description of Korea places the mathematical content in the foreground. High importance is attached to the orientation on content to gain excellent teaching. Taiwan and China somehow take a position in between these two poles, as a greater number of criteria for good teaching are named. However, we the indications about the lesson composition and mathematical contents are mentioned, too.

(2). MEANS TO ACHIEVE EXCELLENCE IN MATHEMATICS INSTRUCTION

The means applied to achieve excellence in mathematics instruction also vary to a great extent within the four East Asian education systems.

In order to achieve excellent mathematics lessons, the lesson study approach is emphasised in Japan. This means that teachers observe a lesson being given to students. Afterwards, the teachers discuss the lesson and revise the lesson plan together. The development of good lesson plans is an extensive iterative process. Particular attention is given to the development of teachers' pedagogical content knowledge.

In order to achieve excellence in mathematics teaching, Taiwanese teachers shall participate in a teacher professional programme, wherein teachers are trained. One of these programmes is established as mentoring programme. There, teachers are trained to become masters in mathematics instruction. Afterwards, these master teachers serve as consultants, i.e. they teach lessons for other teachers in order to improve their teaching.

The development of excellent textbooks, accompanying teaching materials and diverse modes of supervision are important in Korea, e.g. instruction-research contests, where teachers are graded according to their teaching performance. These teachers are obliged to open their lessons for other teachers.

Two central features are common in order to achieve excellence in instruction within in-service teacher professional programmes in China. One is the concept of master teachers, a ranking of senior teachers who share their professional experience with other teachers. The second feature is exemplary lessons shown to average teachers by the master teachers.

As can be seen, a great spectrum of positions can also be found concerning the means by which excellence in mathematics instruction is to be achieved. On the one hand, Japan basically focuses on strategies to improve lesson designs. On the other hand, the main focus in Taiwan is on mentoring programmes for teachers. In China, both can be found: strategies which aim at awarding excellent teachers who are supposed to act as role models, as well as the development of exemplary lessons. Awards for excellent teachers, who then are to serve as role model for others, can also be found in Korea. However, the idea of competition is also stressed here, stronger than in other East Asian countries.

(3). COMMONALITIES AND DIFFERENCES TO WESTERN PERSPECTIVES

When comparing the different perceptions of excellence in mathematics teaching and the means to achieve this excellence in mathematics lessons to a Western perspective, we can detect a crucial difference. While there is a discussion about good and high quality mathematics teaching in the Western European scientific community, there is no such discussion about excellence. Solely the term excellence aims at creating an elite, which is contrary to the rather egalitarian attitude of Continental European countries. An explanation to this might be found in the different cultural background. As Li (2004) points out, one of the Confucian values concerning learning is a belief in human self-perfection, which results in the attitude of lifelong learning. Teachers believe that they always can learn from other teachers. In contrast, there is no such belief in Western Europe. The Anglo-Saxon discussion, on the other hand, comprises the idea of elite as it is not too different from the character of the spirit of society, which also embraces competition as an element.

The descriptions of excellence in mathematics classroom instruction as given in the papers of the research forum are not far from Western European characteristics for good and high quality lessons. Similar descriptions of the lesson structure can also be found within the German-speaking scientific community. Also, the great importance of mathematical content is a characteristic for the understanding of mathematics instruction, at least in Continental Europe. But these approaches operate under the name of different aims and premises due to the orientation of lessons in the direction of all pupils or the average pupil (cf. Kaiser, 1999).

This certain distance towards the concept of excellence in mathematics classroom instruction is accompanied by the fact that the East Asian means to achieve or ensure excellence are hardly found in Western European countries (for details see Kaiser & Vollstedt, 2007).

The concept of master teachers cannot be found in Western cultures, amongst others, due to the egalitarian spirit of their societies, as mentioned above. This makes it difficult for societies to rank teachers according to their teaching practice and their achievements. There are concepts of the best teacher in a state or a city, especially in the US, but that only has a minor influence on teacher education and training.

Joint efforts put into the development of the quality of teaching practice are rare and does not have an established tradition in Western Europe. Teachers might jointly develop teaching materials in groups, but there is no established tradition to teach these materials, observe the teaching experiment jointly, or to improve it afterwards. In Germany there is, however, something like a lessons study approach during the time of practical teacher training, but it is abandoned after this time. Teachers then do usually not reflect on their way of teaching, at least not in an institutionalised way. There might be many reasons for this lack of tradition, e.g., a missing tradition of observing and criticising lessons in a constructive way, the usage of teachers to teach behind a closed classroom door in isolation without any supervision.

Until now many teacher education institutions and schools in Europe or North America are reluctant to discuss the quality of mathematics instruction. Student-centred approaches that refuse to discuss teaching quality are still prominent. This is due to the attitude that the student is the centre of instruction and not an abstract quality. In addition, results of empirical studies on, amongst others, the role of classroom management are still not discussed in schools or many teacher training institutions. The latter would have the power to make clear that strong indicators for quality in teaching lie in the role of classroom management.

To conclude, two aspects seem to play a decisive role in the context of excellent mathematics instruction in Western or Continental Europe, respectively: a strong individual orientation of the lessons with a focus on the individual students and his/her individual development on the one hand, and the rather egalitarian orientation of the Western societies (at least in principle) on the other hand. This egalitarian orientation judges the awarding of high achieving teachers, at that in a competition, as intrusion into the pedagogical domain which is supposed to serve the students free from competition and orientation towards excellence.

References

Kaiser, G. (1999). *Unterrichtswirklichkeit in England und Deutschland. Vergleichende Untersuchungen am Beispiel des Mathematikunterrichts*. Weinheim: Beltz Verlag.

Kaiser, G. & Vollstedt, M. (2007). Teachers' views on effective mathematics teaching: commentaries from a European perspective. *ZDM – The International Journal on Mathematics Education, 39*, 341-348.

Li, J. (2004). A Chinese cultural model of learning. In L. Fan, N.-Y. Wong, J. Cai, & S. Li (Eds.), *How the Chinese Learn Mathematics. Perspectives from Insiders* (pp. 124-156). Singapore: World Scientific Publishing.

DISCUSSION GROUPS

ONLINE MATHEMATICS EDUCATION

Marcelo C. Borba — UNESP-São Paulo State University
Salvador Llinares — University of Alicante

THE MAIN SECTION HEADING STYLE IS PME HEADING 2

Until 2002, it was very rare to find research on online mathematics education on PME Proceedings. This was not surprising, since the Internet was very different until the mid 90s when the WWW interface became available. It took a little bit more time for it to become a means for education, and more time to become the object of research. A difference can be observed when one looks at the last two proceedings of PME, which include research reports, short orals and parts of research forums dedicated to different issues of online education, even though they are not very numerous.

This discussion group recognizes such a change and proposes to focus on this issue. From the research presented in previous PMEs and elsewhere, some questions have arisen about student and teacher learning: What are the differences in learning mathematics and learning knowledge for teaching in online environments, when compared to face-to-face environments? How can we develop *b-learning* methodologies (blended methodologies, face-to-face instruction complementing online work) in mathematics teaching and mathematics teacher education? Does the nature of mathematics change as it is expressed through *interfaces* such as "chat rooms" or videoconferences when compared to the blackboard or a projector? How does a teacher deal with the usual lack of mathematical symbolism in online environments? What are the different models of organizing online courses, and what are the consequences for learning mathematics, learning knowledge for teaching and constituted *communities of learning*? What differences do online courses bring to pre-service and continuing teacher education? What role do the interactions in the on-line learning play? How are communities of learning constituted in *online interactions*?

The discussion group will emphasize small group work, which will be formed based on interest in the above questions as well as others that participants will bring to the initial whole group discussion. In the second period of the discussion group, some time will be allocated for participants to show some virtual environments to participants. At the end of the session, a small amount of time will be dedicated to discussion of future projects such as publications and continuation of the Discussion group in the next PME.

RESEARCHING MATHEMATICS TEACHERS' KNOWLEDGE AND BELIEFS

Michael Neubrand | Helen Chick | Roza Leikin
University of Oldenburg | University of Melbourne | University of Haifa

Teacher education - pre-service and in-service as well - needs to provide a sound basis of knowledge and to give opportunities to cultivate beliefs about mathematics as a subject and about teaching and learning mathematics. This task requires theoretical analysis, but should not leave out the ties to practice. Three factors could account for the apparent complexity of the task: The multiple aspects of the knowledge required for teaching, the interconnectedness of all those knowledge facets, and the fact that teachers' knowledge comes from different and, in certain cases, even contradictory sources. Consequently, there is still a lack of comprehensive and categorical descriptions that frame teachers' knowledge and beliefs, particularly from a decisive content-oriented viewpoint.

Nevertheless, in the last years several working groups have started intensively to research the mathematical and the pedagogical knowledge of teachers of mathematics. To name only a few, one can refer to LMT (Learning Mathematics for Teaching), KAT (Knowledge of Algebra for Teaching), TEDS (Teacher Education and Development Study), COACTIV (Cognitive Activation in the Mathematics Classroom - Professional knowledge of Teachers), etc. These and the other studies are loosely organized around issues of content and pedagogical knowledge, and on various aspects of teachers' beliefs. However they use different conceptions, different methods, different scopes, stretching from narrative case studies to gaining structural overview data.

The Discussion Group aims to provide a forum of exchange for the various existing research groups and their modes of studying the domain. Guiding questions could be:

How are teachers' knowledge and beliefs conceptualized?

What are the methods and instruments used in the studies? What is distinctive about the different forms? What information do they provide?

Is the research aimed to improve teacher education by a better understanding of the structure of knowledge and beliefs needed for teaching?

The structure of the Discussion Group will allow short glimpses into each of the studies, and than an informed discourse should start. By considering the differing questions, methods, and outcomes of the studies we hope to move towards a level of understanding that allows comparisons among them and some (perhaps tentative) over-arching conclusions.

COORDINATING PSYCHOLOGICAL AND SOCIAL ASPECTS OF CLASSROOM LEARNING

Chris Rasmussen — San Diego State
Michelle Zandieh — Arizona State
Andrew Izsák — University of Georgia

We propose to discuss recent complementary lines of research that examine interactions between psychological and social aspects of classroom learning. For over 15 years, mathematics educators have recognized that both individual and social aspects are central to mathematical thinking and learning. A main challenge has been to respond to two theoretical positions on learning that can appear to be in direct opposition. One position, often traced back to Piaget, gives priority to individual psychological processes. A second, often traced back to Vygotsky, gives priority to social and cultural processes.

The *emergent perspective* (e.g., Cobb & Yackel, 1996) has been one of the most visible theoretical perspectives that seeks to transcend past divisions between individual and social accounts of classroom learning. This perspective emphasizes reflexive relationships between the learning of classroom communities—characterized in terms of social norms, sociomathematical norms, and classroom mathematical practices—and the learning of individuals—characterized in terms of beliefs and understandings that are psychological correlates of norms and practices.

We will discuss the theoretical and methodological challenges of conducting research that investigates such questions as (a) how is the learning trajectory for a class related to the learning trajectory of various individuals in the class; (b) how can one determine the emergence of a particular norm or a taken-as-shared mathematical practice in classrooms where there is little student debate; (c) how do the teacher and students in a given classroom interpret lessons in which they participate together; (d) what is the relationship between classroom mathematical practices and larger disciplinary practices such as defining, symbolizing, and proving; (e) how is an individual's participation in particular mathematical practices related to his or her acquisition of knowledge; (f) how are notions of classroom mathematical practices related to other notions of practice used both within the math education research community and within the larger social science research community? The presenters will offer brief examples from their own research to initiate discussions of the challenges one encounters when investigating relationships between psychological and social aspects of classroom learning.

References

Cobb, P. & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*(3/4), 175-190.

WORKING SESSION

MATHEMATICS AND GENDER: DISCOVERING NEW VOICES IN PME

Joanne Rossi Becker
San José State University

Helen Forgasz
Monash University

Kyunghwa Lee
Korea National University of Education

Olof Bjorg Steinthorsdottir
University of North Carolina

In 2005 and 2006 we had lively discussion group sessions centered on several areas of interest related to gender and mathematics. Noting that this area of research differed greatly by country, we focused on intervention strategies that might be used in countries such as South Korea with large extant gender differences in achievement; how to study linkages among gender, ethnicity and socio-economic status; and, setting a research agenda for future work on gender and mathematics. We discussed the policy issues that influence the collection of data necessary for the study of gender differences/similiarities, and focused on possible new methodological approaches and theoretical frameworks that would enable us to investigate difficult and unresolved issues concerning gender, especially as they relate to ethnicity and socio-economic status.

In 2007, the discussion group was turned into a working group at which we had several short presentations from which we derived discussion. There followed planning and future assignments related to ICME 11, opportunities for reviewing for a special upcoming issue of ZDM on gender and mathematics, and a special volume on gender and mathematics from an international perspective.

At this point we have circulated a call for papers for the special volume, have received 17 proposals, and are in the process of seeking a publisher. For PME 2008 we plan to have fully developed outlines or drafts of papers for this special issue which can be presented and critiqued.

ACTIVITIES

Using the PME newsletter and listserve, as well as our mailing list, we will determine who would like to informally present some work, fully or partially developed, to the working group. In particular, we will solicit presentations from those who have submitted proposals for book chapters.

Beginning with brief introductions, we will break up into smaller groups on Day 1 around interest areas pre-determined by the organizers. These groups will discuss and critique and offer suggestions to participants who have brought work to share.

On Day 2, main ideas from the smaller groups will be shared with the whole group, and the Working Group will strategize about how to organize and move forward with the book proposal. We will collect participants' email addresses so that all may keep in contact to continue collaboration after the conference.

EMBODIMENT, LANGUAGE, GESTURE AND MULTIMODALITY IN MATHEMATICS EDUCATION

Janete Bolite Frant — UNIBAN
Laurie Edwards — St Mary's College
Ornella Robutti — Torino University

The goal of this Working Session is to increase our understanding of mathematical thinking and learning by considering the variety of modalities involved in the production of mathematical ideas. We plan to examine how basic communicative modalities such as gesture and speech, in conjunction with the symbol systems and social support provided by culture, are being used to construct mathematical meanings. In addition, the role of unconscious conceptual mappings such as metaphors and blends will be investigated in relation to gesture, language and the genesis of mathematical concepts.

Our plan is to work in small groups based on the specific interests of the participants. Possible questions for small groups include:

A. On Metaphor and Conceptual Blends:

1. How can we study the connections between metaphors and students' misunderstandings?
2. How to help students in constructing and using powerful metaphors? What does powerful mean?

B. On Gesture in General:

1. Are we using gestures to internalize and or interpret, and is interpretation a whole body experience?
2. How does the grain or time range you are using influence the study of gesture?
3. Are gestures different for ideas-objects that are new, vs already understood?
4. Awareness of compression of time & space?
5. Are gestures a tool of memory?
6. How are gestures co-constructed (between two people)?
7. What is the relationship between directions and gestures and metaphors?
8. What is the role of gesture for thinking?

C. On Gestures and Mathematics:

1. Are gestures a way to pass to another level of mathematization?
2. Are there mathematical gestures, specifically?

D. On Gestures and Inscriptions and Artifacts:

1. Do gestures come first when exploring an inscription or phenomenon?
2. What is the effect of an artifact on the same task, in terms of gesture?

E. On Gestures in Teaching and Learning:

1. How to make students and teachers aware of importance of gesture?
2. What is the influence of teachers' gestures and metaphors on those of students?

F. What the role of Language in exploring mathematics cognition?

LESSON STUDY WORKING GROUP

Lynn C. Hart — Georgia State University
Alice Alston — Rutgers University
Aki Murata — Stanford University

While some literature suggests that Lesson Study can facilitate greater reflection and more focused conversations about teaching, there is a need for researchers to propose and carry out rigorous studies focusing on all aspects of Lesson Study. In particular, we need to look at teachers' growth in content and pedagogical knowledge and have specific and authentic conversations about student learning and the impact of significant and subtle changes in lesson design. The body of knowledge about lesson study is growing, but it remains elusive and composed of discrete and disconnected research endeavours. The proposed working group will bring researchers together to create a context for communication and collaboration.

AIMS AND PLANNED ACTIVITIES

At the previous meetings of the Lesson Study Discussion Group (2006 & 2007), we initiated plans for putting together an edited book where individuals conducting research in Lesson Study could share their findings, questions, and issues. The group agreed this would be useful to others with interest in implementing a Lesson Study program or for reviewing the state of research on Lesson Study. Preliminary ideas for the format and conceptual organization of the proposed book were discussed. The organizers contacted Springer Publishing and they were encouraged to put together a proposal. Therefore, prior to the Working Group meetings in Morelia, participants from the previous meetings will exchange drafts of chapters to review. At the 2008 meeting these drafts will be discussed. We encourage newcomers with an interest in Lesson Study or conducting research on Lesson Study to participate in the discussions and propose additional chapters from their own work.

Reference

Hiebert, J., Gallimore, R., & Stigler, J.W. (2002). A knowledge base for the teaching profession: What would it look like and how can we get one? *Educational Researcher 31*(5), (3-15).

SHIFTS IN GENERATING PEDAGOGICAL THEORY IN UNIVERSITY LEVEL MATHEMATICS EDUCATION RESEARCH

Elena Nardi and Paola Iannone
University of East Anglia

Irene Biza
University of Athens

Alenjandro S. González-Martin
University of Montreal

Marcia Pinto
Federal University of Minas Gerais

> It is possible to conceptualize educational (and other) research in such a way that "pure" and "applied" work are not in conflict, but so that contributions to basic knowledge and contributions to practice can be seen as compatible and potentially synergistic dimensions of our work (Schoenfeld 1999, p5).

University-level mathematics education research is a relatively young research area that over the last twenty or so years has started to embrace an increasingly wider range of theoretical frameworks (cognitive/developmental, sociocultural, situated etc.), methodologies (experimental, basic, developmental/design research etc.) and methods (quantitative, ethnographic, narrative etc.). In this Working Session we will take a close and intertwined look at the substantive and methodological developments that define the present of this research area and are likely to propel its future. In particular we will examine the ways in which university-level mathematics education research currently contributes to pedagogical knowledge both in terms of theory and practice and we will debate possibilities of collaborative research design, implementation, evaluation and generation of theory that optimise the link between theory and practice. For this purpose we will scrutinise the contribution made by a range of recent projects and, using them as our example-basis, we will aim to answer the question on the 'potentially synergistic' link between theory and practice. Specifically we envisage exemplifying from four types of research as follows (two in each of the two 90-minute sessions, accompanied with discussion):

RME→M: Researchers in Mathematics Education (RME) produce pedagogical recommendations; mathematicians (M) apply them; RME evaluates/modifies.

RME//M: As above but RME is also M with teaching responsibilities.

RME←M: M participates in research designed by RME and in evaluation/theory generation.

RME↔M: Collaborative research design, implementation, evaluation and theory generation.

As a starting point of the discussion of each type participants will be invited to consider a brief research sample (e.g. of data, findings etc) prepared in advance by the co-ordinating team.

References

Schoenfeld, A.H. (1999). Looking toward the 21st century: Challenges of educational theory and practice. *Educational Researcher, 28*(7), 4-14.

TEACHERS RESEARCHING WITH UNIVERSITY ACADEMICS

Jarmila Novotná — Charles University in Prague
Laurinda Brown — University of Bristol
Merrilyn Goos — The University of Queensland

In 2007, the Working Session with the same name took place as the follow-up of PME 30 Research Forum. It aimed to develop the collaboration of teachers and university academics – with a broader, international dimension.

At that Working Session a framework was developed for analysing ways in which university academics and teachers might conduct research together. At the end, participants decided to continue the cooperation in the field. Several of them prepared and started projects that could be included in the perspective of the WS.

After one year of work on the projects, there is a lot of experience, materials and proposals for improvements that could be of interest to a broad mathematics education community.

During the two WS sessions, these examples of research collaborations between teachers and university academics will be presented and discussed. The experiences gained will be shared not only among the project authors but with all WS participants. Strong and weak points will be analysed in order to multiply the benefits of such cooperation.

During the first session, participants will work in groups where each will be centred on one of the projects that emerged from the PME31 WS. This will allow each group to become familiar with the data and analyses of one project.

In the second session, new groups will be formed. They will represent a mixture of projects. This organisation will offer the space for comparison and identification of common themes.

References

Novotná, J., Zack, V., Rosen, G., Lebethe, A., Brown, L., & Breen, Ch. (2006). RF01: Teachers researching with university academics. In Novotná, J. Moraová, H., Krátká, M., & Stehlíková, N. (Eds.). *Proceedings PME 30* (Vol. 1, pp. 95-124). Praha: PME.

Novotná, J. – Goos, M. (2007). Teachers researching with university academics. In: Woo, J-H., Lew, H.-C., Park, K.-S., & Seo, D.-Y. (Eds.), *Proceedings of PME 31* (Vol. 1, pp. 188). Seoul: PME.

Goos, M. (2008). Critique and transformation in researcher-teacher relationships in mathematics education. *Symposium on the Occasion of the 100th Anniversary of ICMI*. http://www.unige.ch/math/EnsMath/Rome2008/partWG3.html

TEACHING AND LEARNING MATHEMATICS IN MULTILINGUAL CLASSROOMS

Richard Barwell — University of Ottawa
Judit Moschkovich — University of California
Susan Staats — University of Minnesota

Multilingualism is a widespread feature of mathematics classrooms around the world. The nature of this multilingualism and the relationship between multilingualism and mathematical discourse are complex phenomena. Often, the "problem" for students who are learning mathematics in a multilingual classroom is framed in terms of developing "academic language" in mathematics in the language(s) of instruction. Research in linguistics suggests that learning academic language is a challenging process for both teachers and students. It can be difficult to identify the nature of academic language in mathematics, or the features of this language that are relevant for teaching or learning mathematics in multilingual settings. The aim of the working group, therefore, is to explore the following questions:

- What is academic language in mathematics? In which languages?
- How can teachers support the development of academic language in mathematics in multilingual settings?
- What are the multiple relationships and connections between the languages students speak and academic language in mathematics?

ACTIVITIES

Over the two working sessions, our exploration will be stimulated by student work, transcripts, and video data. We will analyse: the use of online Somali language videos for immigrant students in the U.S.; a video and transcript of bilingual Latino/a students in the U.S discussing algebra problems; and official guidance for teachers from the UK and Canada. Analysis will be informed by various ideas from discourse analysis and socio-linguistics.

SEMINARS

QUALITATIVE RESEARCH METHODS: MATHE-DIDACTICAL ANALYSIS OF TASK DESIGN

Anne R. Teppo — Livingston

Marja van den Heuvel-Panhuizen — FI Utrecht Univ. & IQB, Humboldt Univ

This seminar is focused on the exploration of and further characterization of the use of mathe-didactical analyses in qualitative research task design. Participants will employ a set of criteria to unpack the underlying mathematical structure and pedagogical intentions of several actual research tasks. Evaluation of the ways that other tasks are presented and justified in a selection of published research reports will by used to examine the contribution that this type of analysis makes towards the development of good research practice.

Boaler, Ball and Even (2003) describe researcher activity as *disciplined inquiry* (Shulman, 1997), characterizing it as "the attentive and rigorous care with which scholars frame problems, design ways to work on them, consider results, and make claims" (pp. 492-493). Amplifying this list, careful attention must also be paid to a principled consideration of task design. While specifically addressing the creation of task-based clinical interviews, Goldin's (1998) descriptors have wide application across research methodologies employed in mathematics education. He calls for "the examination, analysis, and communication to others" of the processes of design and implementation of the research task (p. 42). Serpinska (2004) explicitly stresses the central role of mathematical tasks in research design, regarding the tasks "as tools of research on a par with methodological tools such as statistics or coding schemes for qualitative data analysis" (p. 25). She highlights the complexity of task design and the impact of this endeavour on research outcomes.

> Different tasks are needed for different purposes. Students' responses may be very sensitive to even small changes in formulation of a task, or its mathematical, social, psychological, and didactic contexts. This is why I think it is so important to justify the choice of the mathematical tasks used in a research, not just in terms of the general goals and theoretical framework of the research, but in terms of the specific characteristics of the task. A task may be set in different contexts and formulated in different ways; it is important to be aware of the possible variants and reflect on the influence on the results of the research of the choice of one of these variants rather than another. This reflection makes explicit the boundaries of the generality of conclusions that can be drawn from the research (Serpinska, 2004, p. 25).

Whether the research is quantitative, or qualitative, the particular details of the mathematical activity directly affect the nature of the data that can be obtained (Boaler, Ball, & Even, 2003). Even if the research is purely observational (e.g., deriving data from classroom practice), such episodes are driven by the particular classroom context, of which, some form of mathematical task is integral. It is crucial, therefore, that "the developer of the tasks knows the full extent of the intricacies of the mathematical constructions under investigation," since the cognitive structures that children build reflect the nature of the tasks with which they engage (Behr et al., 1994, p. 124).

Researchers must develop "a flexible and unpacked understanding of mathematics, one that can be readily fashioned for use in looking at students' work, listening to their talk, and observing their teachers' moves" (Boaler, Ball & Even, 2003, pp. 510-511). The focus on *what* is central to the unpacking process. What is the nature of the content inherent in the given task? What is its mathematical structure? How is this structure related to other important constructs? How does the content fit within a developmental learning trajectory? Answering these and related questions can be far from trivial due to the compressibility of mathematics – a fact that leads to the disciplines' power, but impedes its teaching and learning. As Freudenthal (1983) points out, "the way back to insight is blocked by the processes of algorithmising and automatising" (p. 209).

Teppo and van den Heuvel-Panhuizen (2007) advocate the importance of incorporating *mathe-didactical* analyses as an integral component of research task design. Such an analysis attends to both the mathematical content and the learning and teaching of that mathematics within the selected research context. Not only does an unpacked understanding of the mathematical possibilities (or lack thereof) inherent in the task increase the potential of the research to probe for rich mathematical activity, but this analysis also informs the nature of the inferences that are made related to observed behaviour (Goldin, 1998). In addition, a deeper understanding of and a more carefully justified rationale for the selection of the particular mathematical and didactical aspects of the research task enhances the researcher's ability to contribute to an informed discussion of the "what" questions in mathematics education – placing decision-making about the content and goals of school mathematics at the heart of mathematics education research (van den Heuvel-Panhuizen, 2005).

SEMINAR

The seminar will focus on the use of mathe-didactical analyses in the design and evaluation of tasks used in qualitative research studies in mathematics education. Initial investigation of this type of analysis was carried out in the Discussion Group "Keeping the Mathematics in Mathematics Education Research" at PME 28 in Bergen, Norway (Teppo et al., 2004). At that time a preliminary set of criteria were developed to characterize the content analyses that were conducted during the Discussion Group's meetings.

The present seminar is intended to build on that work. While a set of analytical criteria will be provided, it is expected that this list will also be refined and elaborated as a result of the groups' participation in the seminar activities. A goal of this activity is to move the field forward with respect to the elucidation of elements of research task design.

First session

Participants will analyse the mathematical potential and appropriateness of several tasks taken from actual research studies. The focus will be on unpacking the mathematical structure of two separate task, examining how variations in the task influence student behaviour, and identifying the mathematical constructs inherent in a range of student responses to a third open-ended task. The mathematics in the research tasks will include both primary and lower secondary topics. Participants will be divided into groups that incorporate a range of mathematical expertise in order to enrich the collective experience.

Second session

Participants will investigate how mathe-didactical analyses inform task design. Attention will be paid to how these processes are made explicit and communicated in the research reporting process. Examples from actual research studies will be evaluated in terms of the criteria developed during the first session. Attention will be paid to how the mathematical aspects of the task design align within a given theoretical framework and how the actual task is justified, for example, in terms of mathematical and task structure, choice of representation, and context of presentation.

The latter part of the session will be devoted to collaboratively developing a description of good research practice related to the mathe-didactical analysis component of task design.

References

Behr, M., Harel, G., Post, T., & Lesh, R. (1994). Units of quantity: A conceptual basis common to additive and multiplicative structures. In G. Harel & J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 123-176). Albany, NY: State University of New York Press.

Boaler, J., Ball, D.L., & Even, R. (2003). Preparing mathematics education researchers for disciplined inquiry: Learning from, in, and for practice. In A.J. Bishop, M.A. Clemments, C. Keitel, J. Kilpatrick, & F.K.S. Leung (Eds.), *Second International Handbook of Mathematics Education* (pp. 491-521). Dordrecht: Kluwer Academic Publishers.

Freudenthal, H. (1983). *Mathematics as an Educational Task.* Dordrecht: Reidel Publishing Company.

Goldin, G.A. (1998). Observing mathematical problem solving through task-based interviews. In A.R. Teppo (Ed.), *Qualitative Research Methods in Mathematics Education* (pp. 40-62). Reston, VA: The National Council of Teachers of Mathematics.

Serpinska, A. (2004). Research in mathematics education through a keyhole: task problematization. *For the Learning of Mathematics, 24*(2), 7-15.

Shulman, L.S. (1997). Disciplines of inquiry in education: A new overview. In R. M. Jaeger (Ed.), *Complementary Methods for Research in Education* (pp. 3-19). Washington, DC: American Education Research Association.

Teppo, A., Speiser, R., Søndergaard, B.D., & van den Heuvel-Panhuizen, M. (2006). Keeping the mathematics in mathematics education research. In M.J. Høines & A.B. Fuglestad (Eds.), *Proceeding 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, p. 261). Bergen, Norway: PME.

Teppo, A. & van den Heuvel-Panhuizen, M. (2007). Mathe-didactical analysis: A crucial component of task design. Presentation at Mathematical Thinking: An Interdisciplinary Workshop, University of Nottingham. Retrieved February 8, 2008 from http://www.fi.uu.nl/~marjah.

Van den Heuvel-Panhuizen, M. (2005). Can scientific research answer the "what" question of mathematics education? *Cambridge Journal of Education 35*(1), 35-53.

QUALITY REVIEWING OF SCHOLARLY PAPERS

Norma Presmeg, Ken Clements, and Nerida Ellerton

Illinois State University

An important professional service that is requested of mathematics education researchers is that of reviewing papers proposed for conferences or for publication, written by others working in the field. Whether the author of the paper is a novice or a seasoned researcher, the reviews of that paper are an essential component in ensuring a quality publication or presentation. Reviews have both an evaluative and a mentoring function: they are written to help an editor or program organizing team to make decisions, and they are written to help the author to improve the paper in various ways. Criteria for high quality reviewing of research reports and theoretical papers are addressed.

INTRODUCTION

It seems apparent that if one has been asked to review a paper, the context of the writing of the paper needs to be taken into account; however, the point bears repeating. Scholarly papers are written for different purposes: whether papers are theoretical syntheses of extant literature on a particular topic, or research reports; whether the intent is dissemination of results or ideas, or a record of a research presentation; there is always a particular audience that the author has in mind. Different journals often have different readerships. It sometimes happens that reviewers overlook the audience for which a paper was written. An interesting example of such a case appears in a small, informative book by Gunstone and Leder (1992), in which a quantitative research report written by Gilah Leder and already published, was reviewed critically by Boris Crassini, Suzie Groves, and Richard Gunstone (after obtaining Gilah's permission), for the educational purpose of addressing quality criteria, such as coherency, fullness of reported detail for replicating the study, etc. The paper was chosen because Leder's research and reporting were known to be of high quality. However, the reviews uncovered several "defects" in the reporting, and possibly in the research itself. Leder was given the opportunity to respond, and in a very balanced and positive rejoinder, she pointed out that the journal in which the paper was published served a variety of readers, including some who were "serious practitioners wishing to keep up with current educational trends and findings, but less interested in the full technical information" (p. 39), which had been reported in detail in other publications. In this light, the style, tone, and level of detail of Leder's paper were seen to be quite appropriate. The audience for which a paper is written is part of the context.

DIFFERENT KINDS OF REVIEWS FOR DIFFERENT PURPOSES

In addition to various audiences or readerships for papers, even within the same *genre* of publication, such as journal articles or conference proceedings, there are differences that need to be taken into account in doing a review.

Conferences

Reviewers of research reports for PME are aware that there are only two choices for decisions, namely, accept or reject, because the author(s) of a paper will not have the opportunity for revision of the 8-page paper. (However, there is the choice of whether or not to recommend a rejected research report for a short oral presentation, and there is the option of earmarking an accepted paper as being of outstanding quality, and suggesting that such a paper should be expanded and submitted to *Educational Studies in Mathematics* as a manuscript.) PME-NA, however, allows authors to expand the initial "abstract" (typically 3 or 4 pages) that is reviewed, before it is published as a 6- or 7-page paper in the Proceedings of PME-NA. Although the same criteria of quality are appropriate for both types of paper, reviewers might regard PME-NA authors with a more mentoring stance, because they have the opportunity to improve their initial submissions. There are both positive and negative aspects of this situation, because PME-NA revised papers—the full papers—are not reviewed again, which raises questions when such papers are listed in some promotion and tenure processes in academia. In either case, it is helpful to reviewers to read the examples of helpful and unhelpful reviews that were worked out by a subcommittee of the International Committee of PME some years ago, which are included in the web site. Questions that reviewers might ask in scrutinizing a proposed paper carefully are not unlike those that would be asked in reviewing a manuscript for a research-oriented journal, as follows.

Journals

Whether the reviewing process is double-blind—in which authors and reviewers are each unaware of each others' identities, as for *Journal for Research in Mathematics Education* (JRME), or single-blind—in which authors are unaware of the identities of reviewers, although reviewers know who the authors are, as in *Educational Studies in Mathematics* (EDUC), the basic questions a reviewer might address are the same, and constitute a form of quality control. For JRME, for instance, the list is as follows.

- Does the research deepen our understanding of important issues in mathematics education? Does it have the potential to lead the field in new directions?
- Do the research questions pertain to issues of significant theoretical or practical concern? Are they well-grounded in theory or in prior research?
- Is there an appropriate match between the research question(s) and the methods and analyses employed to answer the question(s)?
- Does the conduct of the study include the effective application of appropriate data collection, analysis, and interpretation techniques?
- Are the claims and conclusions in the manuscript justified in some acceptable way, and do they logically follow from the data or information presented?
- Is the writing lucid, clear, and well-organized?

These questions are useful not only for reviewers, but also for authors to bear in mind as they write research reports or theoretical papers.

THE PROCESS OF REVIEWING

The foundation editor of Mathematical Thinking and Learning (MTL), Dr Lyn D. English, recently devoted a short editorial to "Reviewing Reviewing" (English, 2008). According to English, MTL was experiencing difficulty in securing reviewers for research manuscripts that had been submitted for consideration for publication. She said she recognized the increasing demands placed on university faculty around the world, but added that it was "disheartening when invited reviewers completely ignore our invitations to review and, in turn, cause the review process to become unnecessarily protracted" (p. 110). With English's comments in mind, it will be useful to reflect on issues associated with mathematics education research manuscripts submitted for possible publication in peer-reviewed journals, or conference proceedings, or for chapters in books.

All three authors of this article have served as editors for research journals, research applications, and books in which research articles are included on the basis of perceived quality. All three of us have had cause to reflect on what might reasonably be expected of authors who choose to submit articles for consideration for publication in well-regarded professional outlets. Rather than tell some of the many professional stories that have entered our experience in these services, we believe it will be helpful to comment on what reviewers ought to expect from authors, and also to reflect on what authors ought to expect from reviewers.

What should reviewers expect from authors?

Editors of international "refereed" mathematics education research journals, and those responsible for evaluating submissions for possible inclusion in prestigious conference proceedings (such as those of PME), have the responsibility of selecting appropriate reviewers for submissions they receive. The task of matching appropriate reviewers with submissions can be fraught with difficulty, particularly in recent times, because of the increasing professional demands being placed on mathematics education scholars in most parts of the world. Reviewing a submission, and then writing a report on that submission that will subsequently be read by the author(s), are time-consuming tasks, Furthermore, it needs to be recognized that the quality of the written review of a research manuscript can have implications for both the authors of the manuscript *and* the reviewers. Reviewers often find themselves wondering how other reviewers would assess the submission. It is only natural to do so, because the quality of each reviewer's professional judgment, and to a certain extent, reputation can be at stake. Reviewers know that the editor, who will read the review, is a highly regarded international scholar. They do not want to write reviews that are obviously at variance with what other reviewers are likely to report.

Nuts-and-bolts matters to be taken into account by prospective authors. Given the pressures on reviewers, it is important that authors do not waste reviewers' time by submitting under-prepared manuscripts. *Before* they actually write a research report intended for submission for possible publication, prospective authors should reflect on the significance of their research theme, on the quality and importance of their results, and on the publication outlet that would be most suited to the report that they are about to prepare. Inexperienced researchers should seek advice from experienced researchers on which journal (or other outlet) would be likely to be most suitable for the report that is to be written. Then, having decided on the journal or proceedings to which they will submit their report, authors should make themselves thoroughly acquainted with the style, formatting and referencing expected of them. Once a first draft is prepared it is wise for all authors, including experienced authors, to have it checked for expression, for typographical errors, for coherence, and for consistently so far as formatting and referencing are concerned.

Our experience is that many authors submit their reports to the “wrong” journal or proceedings, and even those who choose the right potential outlet do not follow formatting and referencing styles demanded by editors of that outlet. Strictly speaking, when this happens it would be fair for editors to return submissions, without comment, stating that they cannot be reviewed until the required formatting and referencing procedures have been adopted.

Matters associated with the quality of expression used in a submission can be contentious, for in the international context it is often the case that researchers find themselves having to write in a language that is not their first language. Writers should in this case do their best to get the manuscript checked for accuracy and fluency in language *before* it is submitted. It should not be the task of a reviewer to try to work out what a sentence, or a paragraph, or even a whole paper, is trying to say. However, in the real world, this is not likely to happen, and hence many reviewers and editors are faced with the time-consuming task of trying to work out meaning. This problem is one which is faced by all publishers who wish to accept articles for possible publication from authors whose first language is not the same as the language they have used in their submission. Some believe that this is an equity issue, with serious ramifications for authors, editors, reviewers, and publishers. Others contend, simply, that it is not the task of reviewers to try to work out meaning in manuscripts in which the quality of expression is poor.

The reality is, however, that in many cases, the editor is gentle and generous, and reviewers find themselves distracted not only by poor expression, but also by annoying, and often idiosyncratic formatting and referencing. At the very least, reviewers should expect that there be a one-one correspondence between citations in the text of a paper and the entries in the reference list at the end of the submission. All too often, there are major differences, with texts cited in the main body of the paper not appearing in reference lists, and works included in the reference lists not being cited anywhere in the main body of the text. It is reasonable, too, for reviewers to expect that any figures/diagrams, etc., mentioned in the text are easy to access in the paper that they have been asked to assess. Sometimes, figures are missing, or cannot be accessed, or if they are available, are poorly drawn, or tiny, or overly large.

A submitted manuscript should be such that an expert reviewer *should* be able to read it, immediately, for meaning, without having to be distracted by annoying factors that stop them from concentrating on the design and on the main results and implications of the research described in the submission. Every author intending to submit a paper for publication should, therefore, make sure that it has all of the main expected features of a good research paper, according to the submission guidelines provided.

The submission should be well *structured*, in its formatting and in its logic. In particular:

- It should have a well-written *abstract* that describes the area of the research, its design, its extent, and its main results, in the briefest and clearest possible terms.
- It should have an *introduction* that provides a succinct, but interesting, background to the main issue(s) addressed in the paper.
- The issue(s) should emerge unambiguously from the introduction. *Research questions* might be presented towards the end of the introduction, or they might emerge out of the literature review that follows the introduction. The meanings of all key terms should be clarified, and the meanings of research questions should be absolutely clear.
- Following the introduction, a succinct, scholarly *review of relevant literature* should be provided. The aim here is to make readers aware of the findings of results of reported research that is clearly pertinent to the issues being considered. One of the main weaknesses of literature reviews in papers submitted for international audiences is that, too often, the works of authors from only one nation are reported. For most questions, worthwhile related research will have been conducted and reported in various nations, and it is the responsibility of researchers/authors to be aware of that. That should not be too difficult in this age of the Internet. References to research papers should be to the original papers whenever possible—rather than to Google (or other) summaries of research, or to summaries provided in papers by other authors. If the key research questions were not presented after the introduction they can be presented here—the advantage of presenting them here is that they can be seen to emerge from the literature. The theoretical base for the study is something else that often emerges, naturally, from the review of the literature. Reviewers should be left in no doubt about the potential unique contributions of the research study.
- The *design* or parameters surrounding the study should now be presented, and details given on how that design was implemented. It is important that details of any samples should be clearly presented – otherwise a reviewer may never get to know the basis and form of the data that were analysed. Issues such as the extent to which the design would allow for representative data to be obtained should be attended to, here. In multi-stage studies it is

essential that reviewers be made aware of how it was intended that data would be obtained at the various stages of the study.

- The actual data obtained should then be summarized, and analysed. *Methods of analysis* should be in accord with statements made in the section on design.
- Finally, *implications, and limitations*, of the research should be presented in unambiguous language. In this section, answers to research questions should be summarised as clearly and as succinctly as possible. All claims made should be based on evidence presented from the data. Furthermore, comments and speculations in relation to any wider gestalt that might apply to the research might be made at this stage of the paper: opinions and conjectures should be clearly labelled as such.

What should authors expect from reviewers?

Mathematics educators who conduct research and submit reports of that research for possible inclusion in international "refereed" mathematics education research journals, or in prestigious conference proceedings, or in edited collections, should be able to assume that, provided they have met the stipulated requirements for submission, their submissions will be fairly dealt with by reviewers. Unfortunately, as in any other area of academia, and despite the goodwill of all concerned, it is not always easy to achieve the desired neutrality in the quality assessment process.

Take, for example, the situation that might arise if two research groups differ sharply in relation to which theories are most pertinent, and which type of research are most needed, with respect to a reasonably well defined area of algebra education. For instance, Research Team *A* might believe that the only worthwhile research efforts at this time are those which assume that a "functions" approach to school algebra, including widespread use of graphing calculators, is what is needed, and that the best method of analysis is qualitative research based on a particular theory. Research Team *B,* on the other hand, does not accept these assumptions: its members believe that what is needed most is a combination of qualitative and quantitative research in which the thought processes that students use when solving equations are identified, and the effectiveness of various teaching and learning approaches, on standard algebra tasks, are quantified and compared, using strict statistical procedures involving random sampling and control groups. What should happen if one or more members of Research Team *A* submit a research-based article for possible publication to some publication outlet? Who should be invited to review the submission?

An appropriate answer to this last question may not be easy. Should the editor seek reviews from persons with known empathy for research of the kind preferred by Team *A*? Or should the editor deliberately send the article for review to persons known *not* to favor the kind of research advocated by Team *A*? If the editor happens to know the algebra education research field intimately then he or she might decide to avoid reviewers with known sympathies to research approaches favored by Team *A* (or Team *B*). Often, however, editors do not have detailed knowledge of the established scholarship, and the politics associated with various research areas. In such cases, reviews might be sought from scholars who are not aware of the main issues associated with the reported research. When this happens, reviews of submissions can be less helpful than might reasonably have been expected.

If it be admitted that the scenario described in the last two paragraphs is not unusual, then the question arises: what should authors reasonably expect from reviewers, particularly if individual differences in emphases can be identified within and between different nations, or different areas in our field?

We believe that there are critically important, "essential" criteria that ought to apply in *any* review situation. These criteria more or less define reasonable expectations that *any* author, or set of authors, might have with respect to reviewers.

- *The neutrality criterion.* Every reviewer should, as much as possible, divest himself or herself of any known biases when reviewing a manuscript. The reviewer should concentrate on whether the research questions are clearly stated, whether the literature review is fair and adequate, whether the design was adequate, whether the analyses were well conducted and reported, and whether implications are justified given the research reported.
- *The wider-than-self criterion*. Editors should avoid seeking the assistance of reviewers who are known unduly to draw attention to their own approaches to research and to their own findings and publications. A mature approach to reviewing requires reviewers to be as disinterested as possible, to the point where they do not expect that their own pet results, theories or methodologies ought to be present, or even mentioned, in manuscripts they review. In particular, the wider-than-self criterion implies that reviewers must not allow old grudges or prejudices to influence what they write in a review.
- *The mentoring criterion*. Every reviewer should continually keep in mind that his or her review should assist writers to develop the quality of their research, particularly in relation to the area under consideration. In other words, reviewers should strive to encourage and to help, as well as to assess. Although most reviewers are very busy people, it is particularly helpful if they can draw authors' attention to literature not mentioned in the manuscript being assessed, which is likely to be of interest. Obvious weaknesses in research design should be pointed out, but in gentle rather than in super-critical language. Comments made by reviewers who consciously take due account of their mentoring role are likely to have a profoundly helpful effect, even on writers whose submissions are not accepted for publication. Well thought out comments can assist writers to feel that the review process has been helpful, insofar as they have learned something of importance that will help them prepare better submissions in the future (Hourcade & Anderson, 1998).
- *The divesting of power criterion*. There can be little doubt that the role of "reviewer" is one that carries with it considerable power. What one writes in

a review has the potential to affect what authors think about themselves, what editors think about authors, and what authors think about how they should conduct their future research. Reviewers should not hide behind a cloak of anonymity to indulge in sarcastic remarks. There is no place at all for sarcasm in professional reviews.

- *The absence of overly negative comments criterion*. Persons who submit papers for possible publication should be prepared to accept reasonable criticism from reviewers. Reviewers should think carefully before they make a comment like "the research described in the submitted manuscript does not make any contribution to knowledge". Reviews that draw attention only to what the reviewer regards as "major" weaknesses are discouraging, and unlikely to be helpful. It is wise to remember that at the other end of each manuscript is an author, or group of authors, who have done their best to carry out and write up a research exercise to the point where they believe that their report is good enough to be submitted for external review and possible publication.

TIME WELL SPENT

It has been noted that doing a quality review of a scholarly paper is time-consuming. However, such reviews are appreciated by authors and editors alike. The keynote struck in such reviews may be summed up in one word: *balance*. There is a balance between negative and positive aspects, between critique and encouragement, between the evaluative and the mentoring purposes of such reviews.

References

English, L.D. (2008). Reviewing reviewing. *Mathematical Thinking and Learning, 10,* 101-102.

Gunstone, R. & Leder, G.C. (with B. Crassins & S. Groves) (1992). *Quantitative Methods in Education Research: A Case Study.* Geelong: Deakin University Press.

Hourcade, J.J. & Anderson, H. (1998). Writing for publication, In J.A. Malone, B. Atweh, & J.R. Northfield (Eds.), *Research and Supervision in Mathematics and Science Education* (pp. 277-298). Mahwah, NJ: Lawrence Erlbaum.

NATIONAL PRESENTATION

María Trigueros
Ana Isabel Sacristán
Lourdes Guerrero

RESEARCH IN MATHEMATICS EDUCATION IN MEXICO: ACHIEVEMENTS AND CHALLENGES

María Trigueros — Instituto Tecnológico Autónomo de México
Ana Isabel Sacristán — Centron de Investigación y de Estudios Avanzados
Lourdes Guerrero — Universidad Michoacana de San Nicolás de Hidalgo

Mexico has a long-standing tradition in research in Mathematics Education. It was one of the first countries worldwide to have research groups specifically dedicated in this area. One of the strongest groups originally began, over 30 years ago, in response to the need for curriculum analysis and development of teaching materials, but it quickly diversified into other research areas and spawned other groups both nationally and in other Spanish-speaking countries. Graduate programs, initially research-specific and later of professional development, have proliferated. There have been important publications produced in Mexico, including two renowned research journals. Mexican researchers have also published internationally, and constantly collaborated with research groups and institutions in other countries. Groups from Mexico have made significant contributions for developing research, both theoretical and empirical, in areas such as History and Epistemology, Algebra, Elementary Mathematics, Advanced Mathematical Thinking, and the use of new technologies. Some of the research has had a direct impact on the national educational system. Other areas are in development, such as those of modeling; teacher development; assessment; and gender, access and equity. In the presentation we will give an overview of the most significant research achievements in our country and point out some of the current challenges for future research.

HISTORICAL BACKGROUND: THE ESTABLISHMENT OF THE FIRST RESEARCH GROUPS IN MATHEMATICS EDUCATION

From the need of mathematics textbooks and programs to the concern of mathematics education in general

The history of Mathematics Education research in Mexico goes back to 1968, when the government launched a major educational reform; in response to the needs of this reform, a group of mathematicians at the Center for Advanced Studies and Research (Cinvestav) were asked, in 1970, to develop a new mathematics curriculum, together with mathematics textbooks for primary schools. As Filloy (2006) recalls, although the general attitude in the group was that of “teachers need to know more mathematics”, some researchers began to be concerned about more general educational issues, problems in the teaching of mathematics, as well as other deficiencies in teachers; they were concerned, as well, with what the “New Maths” tendency entailed. They began researching the History of Mathematics and relating it to Curriculum Design, and soon began producing – in parallel with the development of textbooks and teacher materials – the first academic papers derived from their reflections. In 1975, this first group[1], specifically dedicated to the study of Mathematics Education (or "Educational Mathematics", i.e. *Matemática Educativa* – which is the term the group used), became officially established at Cinvestav, with 11 full time researchers.

The initial aims of the group were the following (Hitt, 2001):

> (1) General research on the learning of mathematics and the methods of teaching it. (2) Experimentation, reviews and corrections of the new primary school mandatory textbooks. (3) Study of the real needs of primary-school teachers and development of different types of auxiliary materials. (4) Structuring of a mathematics curriculum for the teacher-training schools. (5) Study of the problems faced by secondary-school teachers, in particular those in public schools, with emphasis in the development of materials suitable for those levels, both for teachers and for students. (6) The structuring of an undergraduate degree focusing on the teaching of mathematics with the aim of training secondary-school teachers specialized in the teaching of mathematics. (7) Designing Master's and doctoral programs with the same aims. (8) The development of popularization materials.

It is interesting to note that one of the main concerns of the group was not only on research, but also on the training of teachers and human resources. (As is elaborated later in the paper, this concern had far reaching consequences for the development of Mathematics Education research groups across the country and in Ibero-America.) In the early 1970s, the plan for both undergraduate and graduate programs in Mathematics Education was innovative since at the time most mathematics teachers did not have specialized training as teachers.

The training of researchers and of teachers by the SME-Cinvestav, and the origins of the expansion to other institutions

The first Master's program in Mathematics Education was launched in September 1975, with a strong content in mathematics as well as on the history and foundations of mathematics. It is worthwhile mentioning that even in those early days, there was interest in the use of the new computational technologies: in particular programmable calculators were used in some of the mathematics courses (Hitt, 1998). This Master's program has been adapted to today's needs: it includes now differentiated specialization areas according to educational levels or specific research interests.

The doctoral program began around 1982, first at a small-scale, and then more formally in 1992, around the time when the SME became a department (DME).

As of 2006, over 425 students had graduated from the Master's program and over 65 from the doctoral program. Many of the graduates of these programs became involved in Mathematics Education research in over 114 institutions in Mexico and in other countries, particularly those in Latin-America (Figueras, 2006).

[1] The *Sección de Matemática Educativa (SME) del Cinvestav*. Two decades later, this "section" - originally part of the Department of Educational Research (DIE) - became a Department (DME), with, at one point up, to 35 full-time researchers.

Also, from the 1970s, the group at Cinvestav established important researcher and teacher training programs with state universities, technological institutes, teacher-training schools and with the National Autonomous University of Mexico (UNAM). From 1977 a semi-open Master's program in Mathematics Education was launched, first in cooperation with the *Universidad Michoacana de San Nicolás de Hidalgo*, then with the Autonomous University of Guerrero in 1978, and more broadly in the summer of 1979 with other state universities. This expanded when, in 1984, the National Program for the Training and Continuous Education of Mathematics Teachers (*Programa Nacional de Formación y Actualización de Profesores de Matemáticas* – PNFAPM) was launched by the Cinvestav group. This program proved that interchange and joint work between researchers and mathematics teachers in a massive way was possible (Hitt, 1998). Institutions from over 16 states across the country participated: UNAM; the Autonomous Universities of Guerrero, Sinaloa, Sonora, Yucatán, Nuevo León, Ciudad Juárez, Nayarit, Estado de Hidalgo, Estado de México, *Benito Juárez de Oaxaca*; the Universities of Guadalajara, Colima*, Juárez del Estado de Durango, Michoacana de San Nicolás de Hidalgo*; National Pedagogical University (UPN); and the Tecnological Institutes of Ciudad Juárez, Morelia, Durango, Ciudad Madero, Nuevo Laredo, la Laguna, and Chihuahua. Not only did the members of the SME-Cinvestav participate in those programs, but international researchers were invited as well (Hart, 2006).

All of these programs provided the seeds of future research groups, as described earlier. Cinvestav also has the Department of Educational Research (DIE); through textbook and curriculum development, a strong collaboration was initially developed between the SME and the DIE. A smaller group of researchers at DIE has continued its own line of research in Mathematics Education contributing particularly in the area of elementary-school level; at DIE new researchers are formed as well.

The PNFAPM also established academic links with international institutions, and many members of the SME participated in doctoral and post-doctoral studies abroad. Simultaneously to the PNFAPM, national and international conferences and meetings were launched so that researchers and teachers could share and discuss their experiences.

In 2005, a parallel 3-year Master's program for in-service teachers was launched at DME-Cinvestav in cooperation with the ministry of education of the state of Mexico. Over 60 teachers are currently enrolled in this program, and researchers are using it as an opportunity to carry out classroom research and study teacher-training issues.

International links and participation

From its inception, the SME-Cinvestav group began studying what was being done internationally, and developed academic links with foreign researchers and institutions. The first links were done with Brousseau and Glaeser, and later with the *Instituts de Recherche sur l'Enseignement des Mathématiques* (IREM) in Bordeaux and in the *Université Louis Pasteur* in Strasbourg, France, as well as the *École des Hautes Études en Sciences Sociales* in Paris. This established an influence of the

French school of *Didactique.* However, as is recounted by Filloy (1981), Hitt (1998) and Pluvinage (2006), the SME-Cinvestav group also took into account theoretical frameworks from other countries such as the USA (e.g. the work of Bruner and Skinner) the Soviet Union (e.g. that of Kruteski), the UK, as well as the work of Piaget. Other academic links took place with the University of London, UK, Cambridge University, UK, and the University of Toronto, Canada. By the 21st century, many other international links had been established with the DME-Cinvestav. In addition to the aforementioned ones, others include those with the Universities of Granada and Valencia in Spain; the *Université Joseph Fourier* in Grenoble, France; the University of Quebec in Montreal (UQAM), Canada; the Universities of Georgia, and of Massachusetts-Dartmouth in the USA; and the Universities of Nottingham and of Bristol, UK.

Of course, since the 1970s the members of the *Matemática Educativa* group also participated actively in international conferences such CIEAEM, PME and ICME. In the 1980s they organized the first Central-American and Caribbean Meeting on the Training of Teachers and Researchers in Math Education (*Reunión Centroamericana y del Caribe sobre Formación de Profesores e Investigadores en Matemática Educativa*) and the Conference of the International Group for the Psychology in Mathematics Education (PME) held in Oaxtepec.

Areas of research of the *Matemática Educativa* groups at Cinvestav.

Despite its origins at the primary school level, in the beginning much of the research of the SME-Cinvestav centered at the upper educational levels, due in part to the experience of the first members of the group (Hitt, 1996). Gradually, the research interests of the group developed, and research expanded to include all levels from pre-school to university.

As mentioned earlier, one of the main areas of study that the group developed was the use of the history and development of mathematical concepts, as a means to understand difficulties in the learning of mathematical ideas. Epistemological analysis, both in terms of phylogenesis as well as ontogenesis, was also used as framework for curriculum design. Later, some researchers began doing educational experimentation and clinical observations.

The group grew in the 1980s and new areas of research emerged and others were consolidated, particularly those concerned with the use of multimedia (e.g audio-visual media) and of computational technologies for mathematics teaching and learning (see Cuevas et al, 2006 for more on the history of the latter area).

From a global perspective, Hitt (2001) identified the following as some of the main research areas of the Cinvestav groups: curriculum analysis and design; educational experimentation with didactic materials for middle-school; epistemological analysis; clinical observations; design of didactical situations and classroom observations; data exploratory analysis; educational experimentation with mathematics teachers and the identification of cognitive obstacles; new teaching methods and the use of technology; problem-solving. In 2008, internal documents of the DME-Cinvestav describe the following as the main research areas of the members of the *Matemática Educativa* group: arithmetic and algebraic thinking; advanced mathematical thinking and the teaching of calculus and analysis; geometrical thinking; the teaching and learning of statistics and probability; the history and epistemology of mathematics; theoretical foundations; the social construction of mathematical thought; cognition; technology-based learning environments; problem-solving; gender studies in Mathematics Education; teacher-training; and assessment in Mathematics Education.

THE EXPANSION OF MATHEMATICS EDUCATION RESEARCH IN MEXICO

Other than Cinvestav, Mathematics Education as an area of research has not received a determined support from most universities in the country; in spite of this situation, some Mexican researchers have made important contributions to the field and are internationally recognized.

From its beginning in 1970, research in mathematics education in Mexico has grown considerably. Even though some centers, like Cinvestav, concentrate a large group of researchers and programs for researchers' education, they were followed soon by a burst of development across many universities throughout the country. Smaller groups exist nowadays in most of the republic's states and in different universities. Also, new Master's and PhD Degree programs in Mathematics Education have appeared in many of them.

As Waldegg (1998) points out it is only in the 1980s when significant progress could be perceived in the field of Mathematics Education research, since by then there were over 16 research groups working regularly across the country; several specialized research journals started to appear; and researchers regularly and actively organize and participate in national and international conferences and associations. Also, in that decade, research topics, methodologies, and theoretical frameworks became diversified as well as specialized.

Educación Matemática: A Mexican research journal

In 1988, the journal *Educación Matemática* was born as the result of a fusion of several existing journals related with Mathematics teaching: *Lecturas de Educación Matemática, Opera Prima, Matemáticas y Enseñanza* y *Boletín Informativo* published by UNAM, Cinvestav, *Sociedad Matemática Mexicana* and *Asociación Nacional de Profesores de Matemáticas*. When this effort was initiated, several colleagues were invited with the goal of uniting efforts around a common objective: creating a journal, written in Spanish, to publish results of Mathematics Education research in Mexico and other countries, in order to make them accessible to the wide community of Spanish-speaking teachers and researchers.

Twenty years have passed since the foundation of this important journal. It can be said that its original purposes have been accomplished. The journal has become a mandatory reference for researchers from Spanish-speaking countries; many researchers in non Spanish-speaking countries find it useful as well. It reflects the growing interest in research in Mathematics Education in Spanish-speaking countries, as well as the wide variety of interests of this research community. Recent issues contain papers authored by researchers from more than twenty different institutions within Mexico, and from 15 countries. Papers include a wide variety of research areas: From some traditional streams of research such as the construction of knowledge, teacher-training, the history of mathematics teaching, the rigorous analysis of experimental designs, and the teaching and learning of specific concepts; to innovative research interests, such as knowledge and learning in adults, special education, or the relationship between speaking and writing in learning mathematics, as well as issues in school culture. All the papers published in this journal, are rigorously reviewed by recognized national and international researchers.

Results from state-of-the-art reports

The growth in the number of researchers has meant a rapid accumulation of interesting and important research results. In 1992, a first effort to write a state of the art report on research in Mathematics Education in Mexico was promoted by COMIE (Mexican Council for Research on Education), a professional society that promotes research on education. Results of this review, coordinated by Waldegg (1992) included the description of 282 published papers that appeared both in international and Mexican journals and doctoral thesis written since the creation of the first program, until 1991. Ten years later, a new state of the art report coordinated by Avila and Mancera (2003) and a group of collaborators, and promoted also by COMIE, showed how interest in Mathematics Education research had grown. This report included the description of 483 publications that appeared as books, book chapters, reviewed articles in international and national journals, and PhD and Master's degrees thesis.

Research has continued to grow very rapidly. Not only new research groups have grown throughout the country, but publication has continued rising very rapidly. Mexican researchers are members of international associations and participate in larger numbers at international conferences, even though this participation implies the need to write and present in languages other that Spanish.

Some important research groups across the country today

As described above, the *Matemática Educativa* group at Cinvestav has played a fundamental role in the promotion of research in Mathematics Education and in the training of researchers and teachers from Mexico and other Spanish-speaking countries. The creation of the Central American and Caribbean Conference on Mathematics Education was the beginning of a series of meetings where Spanish-speaking researchers could interchange their interests and their work. This conference later became the Mathematics Education Latin-American Meeting (RELME) and extended the original links with all the countries in South America.

Given its size and role in the creation of human resources, the groups at Cinvestav capture the attention of researchers around the word who consider that all researchers in Mexico work at that institution; but this is not the case. Even though most universities in Mexico do not support large departments where research is mainly concerned with Mathematics Education, groups within Mathematics departments or even within Engineering, Psychology or Pedagogy departments have developed. Researchers in these groups are contributing to the development of new areas of research and are also playing a relevant role in research on specific traditional topics.

Most of the research performed in the country, takes place at Mexico City institutions. One important group can be found there at the National Pedagogic University (UPN). This university is concerned with teachers' education. It hosts both undergraduate and graduate programs in Mathematics Education and hosts several groups interested in different research areas. Other universities like UNAM, the Autonomous Technological Institute of Mexico (ITAM), the Metropolitan Autonomous University (UAM), and the Ibero-American University (UIA) have smaller research groups, but some of them are very productive. Against what could be expected, although these groups concentrate much research on Advanced Mathematics Education, they are also interested in research at other school levels and on the use of technology in the classroom.

Research at Mexico City has exerted a strong influence on research groups that have formed in other areas of the country. Researchers from those groups started their work as students of research programs in Mexico City, and continued to develop the line of research they had started there. Nowadays they have turned their attention to local problems and their relationship with national and global research topics. Researchers' contributions at different universities - e.g. the Autonomous Universities of Guerrero, Querétaro, Morelos, Aguascalientes, Zacatecas, Coahuila, Baja California, Sonora Yucatán, Chiapas, Tamaulipas, Quintana Roo, Nuevo León, Campeche, *Universidad Michoacana*, *Universidad Veracruzana, Tecnologico de Chihuauha, Escuela Normal de Zacatecas, Universidad Pedagógica de Zacatecas y de San Luis Potosí, Instituto Tecnológico de Estudios Superiores de Monterrey, Universidad de las Américas*, among others – have been growing continuously and are developing local graduate programs focusing mainly on teachers' continuous education; there are also groups where research in mathematics education is developed. Lately some new groups are arising at institutions other than universities such as at the *Instituto Latinoamericano de Investigación Educativa* (ILCE), the *Instituto Nacional de Evaluación Educativa* (INEE), the *Centro de Estudios sobre evaluación* (CEE) and the Ministry of Education (SEP).

AN OVERVIEW OF THE MAIN RESEARCH AREAS

In the following paragraphs we give an overview of the types of research being conducted in Mexico, categorized by school level. Part of the information reported here comes from the state-of-the-art reports, but other information was derived from a survey of papers in journals, both Mexican – including *Matemática Educativa*, *Relime* and *Revista Mexicana de Investigación Educativa* - and international, in proceedings of some conferences and through personal communications with some members of the community. This search was not exhaustive. Our objective is to present a wide panorama of research interests in our country.

Research at the Elementary-school level:

In the past decade, studies at this level focused strongly on learning analysis, experimental developments, studies about teaching practices, learning styles and analysis of teaching resources. The important state-of-the-art study coordinated by Avila et al. (2003) pointed to the lack of inter-institutional research and the need to develop more studies on teachers' knowledge; mathematics learning at preschool level; teachers' training; measurement; the mathematics used in indigenous cultures and its links with school mathematics.

Research conducted for this paper, found that since 2002, research continued to grow and strengthen in the areas of students' learning; the analysis of teaching resources; problem-solving; probability; and proportion. On issues related to this school level, some inter-institutional research projects emerged (and in general, inter-institutional research keeps growing). We notice a wider body of research on teachers, teachers' knowledge and teachers' use of technology in the classroom. Some emergent research streams relate to the impact of the use of technology; national and regional assessment; and mathematics within indigenous cultures and its introduction into school mathematics.

It is important to point out that research on elementary-school Mathematics Education has impacted school practice through the development of curriculum, textbooks and other teaching materials based on research results.

Research at the Middle-school level:

As reported in the study by Avila et al. (2003), research studies concerned with mathematics at this school level changed their focus from learning processes and curricular analysis, to the learning of specific concepts and evaluation. Research was mainly oriented to study the learning of algebra, particularly the concept of variable, the use of technologies and modeling. Some other lines of research also emerged, such as arithmetic learning; the use of technology, which was introduced in the latest curricular reforms; and the teaching and learning of probability. Authors of this study pointed to some important lines of research that were still not developed: mainly studies about teachers; and studies about telesecundaria - the distance education middle-school program based on video lessons developed for schools in remote rural, and some suburban, areas.

Research at this school level grew considerably after 2002. Some research areas, in particular those related to the teaching and learning of algebra; of negative numbers; the use of technology; and problem-solving, have continued to evolve and have had an important impact at international level. Other areas - such as that relating to assessment – are now consolidated areas; and new lines have appeared, in particular, studies focusing on teachers' knowledge and teachers' practices at school, gender studies and the teaching and learning of geometry.

Inter-institutional research has continued growing, still dominated by Cinvestav and the same groups at universities throughout the country, but new institutions have joined with the emergence of important groups working on evaluation.

At this educational level, it is important to emphasize the emergence of two large projects of introduction of technology for the teaching of mathematics, which were designed on the basis of national and international research results: The *Educación Sec 21* project and the Teaching Mathematics with Technology (EMAT) program. Through these projects, activities were produced based on research results; teacher-training workshops were held across the country; and several collaborations between international and national researchers - as well as between institutions such as the Ministry of Education (SEP), Cinvestav, ITAM, ILCE and the Organization of Ibero-American States (OEI) - were established. These programs also had a strong impact around the country, and promoted evaluation studies and research in this area.

Research at the High-school level

According to the report by Avila et al. (2003), during the 1990s and early 21st century, research on high-school mathematics education continued growing steadily. Most studies concentrated on students' learning; on the learning of specific concepts from the curriculum; and on problem-solving. Some new research lines seemed to be receiving more attention, for example, those on the teaching and learning of geometry, particularly those linked to the learning of proof; as well as those on probability and statistics. New strands of research started to develop: there were some studies on mathematical reasoning; attitudes and conceptions; the use of technology; and on teachers' knowledge. Although research grew in that decade, reviewers commented that there was a need of much more studies about teachers, school practices and evaluation.

Contrary to what happens at elementary and middle school levels, the presence of research from the Mathematics Education Department at Cinvestav and the UPN is not as strong. There are many institutions interested in research at this school level, probably because many universities have high-school programs and because it is important to know more about the knowledge students bring to the university; there are, however, less inter-institutional studies.

Development has continued in recent years. Research papers continue to grow steadily at this school level. The learning of algebra and calculus are still the areas where more research is conducted, but some of the lines that were in development before, seem to have consolidated. This is the case of research related to the areas of geometry and problem-solving. Geometry is still dominated by studies that focus on proof, but the emergence of a focus on concepts and on the use of technology can also be observed. Also, there are more studies about classroom culture and some about other general aspects of Mathematics Education. On the other hand, although the number of assessment studies has grown at elementary levels, there are very few at this school level.

Research at the University level

According to the 2003 report by Avila et al., there was a considerable increase in the number and quality of research in the 1990s at the upper educational level, but, by far, research concentrated on the learning of concepts of one-variable calculus. There was some incipient research related to calculus of several variables, to differential equations, to analysis, on probability, statistics and complex variables. There were some studies focusing on teaching resources, particularly on the design and use of teaching software, and some on teaching practices. The authors of the 2003 report pointed out that it was important to focus research on teaching practices and teachers, to widen the spectrum of concepts studied and to conduct more research on the use of technology in the classroom.

Research conducted for this paper, found that since 2002, research at this school level has grown considerably. Research topics continue along the same lines. Some of them, such as research related to the teaching and learning of calculus, and of probability and statistics concepts and their learning have been strengthened, and new areas of interest have appeared. Probably the most important research topics to emerge are that studying the learning of concepts in linear algebra, followed by research on applied mathematics, or the mathematical needs of undergraduate programs other than mathematics. Some research has been conducted on modeling and problem-solving, differential equations and proof. There is still a lack of studies about classroom and teachers' practices and about the use of technology at university level.

Research on Adult Education

Before 2002, there were some small efforts of research focusing on adult education. However, it is important to notice that at that period most of the production of didactical resources was based on research results. More recently there has been a growing interest, mainly from researchers from UPN, to investigate more about adult's knowledge of mathematics, numeracy, ethno-mathematics and the study on mathematics used in indigenous cultures; as well as that of school mathematics in practice. But these efforts are not enough considering the educational problems of this sector of the population, in our country.

Mexican contribution to Mathematics Education knowledge and international participation: a summary

Mexican researchers have made some important contributions to what we know now about the learning and teaching of mathematics. Mexican research started, as pointed out above, with an emphasis on epistemological and historical studies about the development and the learning of different mathematical concepts. That tradition has continued and has been joined with other types of studies using different theoretical approaches, some of them developed by Mexican researchers or where Mexican researchers have contributed to their development.

Some contributions that are widely recognized are research studies on the learning of Algebra. Mexican studies range from pre-algebra and the learning of negative numbers, to studies on the different uses of variable, learning of equations and the use of technology in the teaching of algebra. There are also important contributions in elementary mathematics education, especially in the areas related with the learning of numbers and fractions.

In general, research on the use of technology has had a wide impact. Mexican researchers have participated in several international meetings where issues related to technology and the learning and teaching of mathematics have been discussed. They have developed pedagogical models that have been compared with those developed in other countries and innovative approaches to teach with technology, and to assess the development of such projects.

Research on Advanced Mathematical Thinking is another area where Mexican researchers have made important contributions to the corpus of Mathematics Education knowledge. In particular, studies on the learning of Calculus, on the learning of Linear Algebra and Differential Equations have pioneered these areas of research.

Research on problem-solving has also been an area where Mexican researchers have contributed to Mathematics Education knowledge. More recently, new developments in the area of modeling are also playing an important role.

As mentioned above, Mexican researchers have been present, since the beginning, at international conferences, symposiums and meetings. Each year there is a growing number of Mexican participants at international conferences. Mexico has also organized several important international conferences and meetings celebrated in different regions of the country. The participation of Mexican researchers in the international community has been recognized through nominations of several of them to participate as members of international committees and the fact that several of them have joined efforts with researchers from other countries to do collaborative research.

It is important to note the role of Mexican researchers within the context of the Latin-American and Ibero-American communities. Mexico plays a central role in the organization of meetings of institutions in this context, including the publication of journals such as those mentioned above and the Latin American Research Journal on Mathematics Education (RELIME).

CHALLENGES FOR THE FUTURE

Even though research has been growing and spreading across the country there are many challenges that the Mexican community has to face. One important problem within the community is that in the Mexican education system the habit of writing is not promoted. Many researchers contribute with their involvement in teacher-training; in the development of curriculum and materials; or in adult education, but the results of such actions are not communicated through written reports or papers in national journals. This is made worse by the lack of mastery of the English language. Thus, results of these efforts do not get to a wider audience and it results in a lack of accumulated knowledge that could better guide future research efforts and practices. Although the establishment of the National System of Researchers (SNI) have forced the production of publications, there is still a need for the Mexican Mathematics Education community to face this problem so that results obtained by Mexican researchers are better known to a wider national and international audience.

Another issue is that Mexico does not have a national professional society for research in Mathematics Education. There are thus few opportunities for exchanging results and opinions on different education problems, plan together, evaluate the products of research, or to form groups to dedicate attention to specific problems. Such an association is also needed to organize national research meetings. Nowadays, researchers meet at conferences on Mathematics; on Education; or at those for mathematics teachers, where the focus in not specifically on Mathematics Education research. These are necessary actions needed to help create a sense of identity, to help in the training of new qualified researchers and for creating quality standards. All of these are important characteristics for the further professional development of Mathematics Education as a field of research.

A third concern is that, although research results have been taken into account for the development of new curricula, there is still a need of promoting more opportunities of interaction between the policy-makers and the research community. An important problem related to political decisions is the fact that young researchers have many difficulties in finding jobs and positions within research centers and universities. Young researchers are needed to make innovations, to foster new ideas and impact in the dynamics of the community. However, many talents are lost when young talented people leave the field for positions in the job-market that are not related to Mathematics Education.

A fourth area where research results and researchers can, and should, play an important role is in the education, and continuous training and support, of mathematics teachers; as well as the area of adult education. These actions are fundamental if we want the population to have the mathematical knowledge needed for the country to progress.

Finally, collaboration within the Mathematics Education community can help in solving other problems, such as the need to change the negative views and perceptions that mathematicians have about the field of Mathematics Education; or in strengthening the research of small groups within universities both in Mexico City and in the different states, by means of inter-institutional projects.

Endnote

We would like to thank Eugenio Filloy and the faculty of the Department of Mathematics Education at Cinvestav, as well as Eduardo Mancera, Silvia Alatorre, Marcela Santillán and Alicia Ávila for the information they provided, both through personal communications and through their writings.

References

Ávila, A. & Mancera, E. (Coords.) with Block, D., Carvajal, A., Eudave, D., Aguayo, L., & Camarena, P. (2003). El campo de la Educación Matemática 1993–2001. In A. D. López

& Mota (Coord.), *Saberes Científicos, Humanísticos y Tecnológicos: Procesos de Enseñanza y Aprendizaje. Part I* (Vol. 1, pp. 35-353). Colección: La Investigación en México 1992–2002, No. 7. México: COMIE, SEP, CESU.

Cuevas, C. A., Riestra, J. A., & Zubieta B., G. (2006). Retrospectiva del Área de Microcomputadoras del Departamento de Matemática Educativa. El desarrollo de una línea de investigación y prospectos hacia el futuro. In E. Filloy (Ed.), *Matemática Educativa, Treinta Años: Una Mirada Fugaz, una Mirada Externa y Comprensiva, una Mirada Actual,* (pp. 169-194). Mexico: Santillana/Cinvestav.

Figueras, O. (2006). Preface. In E. Filloy (Ed.), *Matemática Educativa, Treinta Años: Una Mirada Fugaz, una Mirada Externa y Comprensiva, una Mirada Actual,* (pp. 9-16). Mexico: Santillana/Cinvestav.

Filloy, E. (1981). Investigación en Matemática Educativa en México. Un reporte. *Recherches en Didactiques des Mathématiques 2(2),* 233-256.

Filloy, E. (2006). Matemática Educativa: un movimiento cultural y su institucionalización. In E. Filloy (Ed.), *Matemática Educativa, Treinta Años: Una Mirada Fugaz, una Mirada Externa y Comprensiva, una Mirada Actual,* (pp. 19-24). Mexico: Santillana/Cinvestav.

Hart, K. (2006). Working with Cinvestav Mathematics Education Section for over 25 years. In E. Filloy (Ed.), *Matemática Educativa, Treinta Años: Una Mirada Fugaz, una Mirada Externa y Comprensiva, una Mirada Actual,* (pp. 35-42). Mexico: Santillana/Cinvestav.

Hitt E.F. (1996). Preface. In F. Hitt (Ed.), *Investigaciones en Matemática Educativa* (pp. i-iv). México: Grupo Editorial Iberoamérica.

Hitt E.,F. (1998). Matemática Educativa: Investigación y desarrollo 1975-1997. In F. Hitt (Ed.), *Investigaciones en Matemática Educativa II* (pp. 41-65). México: Grupo Editorial Iberoamérica.

Hitt E.F. (2001). Departamento de Matemática Educativa: 25 años de investigación. *Avance y Perspectiva* 20, 17-29.

Pluvinage, F. (2006). Algunas impresiones de mi trayectoria en el DME. In E. Filloy (Ed.), *Matemática Educativa, Treinta Años: Una Mirada Fugaz, una Mirada Externa y Comprensiva, una Mirada Actual,* (pp. 27-33). Mexico: Santillana/Cinvestav.

Waldegg, G. (Coord.) (1995). *Procesos de Enseñanza y Aprendizaje II, Vol. 2*. Colección: La Investigación en los Ochenta, Perspectivas para los Noventa. México: COMIE, Fundación SNTE para la cultura del maestro mexicano.

Waldegg C.G. (1998). La Educación Matemática: ¿Una disciplina científica? *Colección Pedagógica Universitaria 29*, 13-44. Retrieved 5 April, 2008 from http://www.uv.mx /cpue/colped/N_29/la_educación_matemática.htm.

Surveyed Mexican journals:

Educación Matemática. Mexico: Santillana. http://www.santillana.com.mx/educacionmatematica/

Relime - Revista Latinoamericana de Investigación en Matemática Educativa. México: Clame. http://www.clame.org.mx/relime.htm

Revista Mexicana de Investigación Educativa. Mexico: COMIE. http://www.comie. org.mx/v1/revista/portal.php

SHORT ORALS COMMUNICATIONS

A STUDY ABOUT THE SPATIAL ORIENTATION IN THE PLANE, THE LOCALIZATION OF POINTS AS OPPOSED TO THE ARITHMETIC AND GEOMETRIC DESCRIPTION OF THEIR POSITION

Claudia Acuña

Cinvestav

Observing the process of right localization of points allows us to make difference between two types of activities that are supported on geometric considerations (position) and arithmetic (quantization of the position) when the graphic represent rigid movements

The points of reference of the plane are concentrated on the Cartesian axis in that which refers to establishing conditions of sense and direction and the marks of the units upon the axis are the numerical referents. On the other hand the tasks of interpretation of the graphs in terms of their relative positions require the ability to see two objects simultaneously.

We compared the tasks of localization of points in the plane with tasks of description of the arithmetic and geometric aspects related to the localization.

For this work, we have prepared a questionnaire with a sample of 146 high school students between the ages of 15 and 16, under the conditions of their usual mathematics class.

We used a questionnaire with instructions regarding the entry-axis sign-sense and value-displacement relationships, reminding about the names of the axis in regard to the entries of the ordinate pairs and gave a scheme in which we show the initial and final position of a point under a horizontal or vertical translation.

The content of the questionnaire is related to the horizontal-vertical translation of families of points and we ask about 1. The localization of points in the plane; 2. The direction and sense of the translation, and 3. The numerical value of the translation.

The results suggest that the localization tasks are different and easier to develop than those in which one must discover the arithmetical or geometrical changes linked to a rigid transformation such as the translation.

The use of the algorithm of the localization does not induce conscious specialization treatments, not even in those cases where the changes are made in a lineal manner, coinciding with the spatial treatment one makes with the localization of points.

The cognitive functions related to the descriptions of the geometrical and arithmetical aspects of the translated points, seemingly contains factors that are additional to those that are required in the localization of points.

STEPS TO THE CARTESIAN METHOD ON A SPREADSHEET

David Arnau and Luis Puig

Universitat de València

This study is part of an ongoing research[1] into the teaching of the Cartesian Method (CM) to solve verbal arithmetic-algebraic problems. The research is organised by use of the theoretical and methodological framework called Local Theoretical Models (LTM), described in Filloy, Rojano, & Puig (2008).

It is well-known that a spreadsheet environment enhance pupils' ability to name the quantities, to check the relations among them, to generalise from arithmetic, and to extend informal problem solving strategies (Rojano & Sutherland, 1997). The component of teaching of our LTM uses this environment, but does not aim to extend informal strategies. Instead, we have designed a Teaching Model that consists of a version of the steps of the CM adapted to the specific characteristics of spreadsheets, to their power and to their limitations, and to their calculation possibilities and to their system of signs, that we call the Spreadsheet Method (SM).

The study was carried out with 26 students (11-12 year old) from a 2d grade secondary school group. Data have been collected from their paper and pencil work on word problems, prior and after the teaching, and from protocols of their problem solving work by pairs on the spreadsheet, at two stages of the teaching sequence (9 pairs each time). These two stages are characterised by the structure of the problems used: in the first one, the more natural analytical reading of the problem produces an arithmetic network of quantities and relations, whereas in the second one, it produces an algebraic one (see Filloy, Rojano, & Puig, 2008). Through the analysis of data we describe students' performance with regard to their analytical reading of the problem statement, the way they name quantities, the way they construct and represent the relations among quantities, and equalities and equations, in order to show in which sense the SM mediates between an arithmetical and an algebraic way of solving word problems, and the influence of the arithmetical or algebraic structure of the natural analytical reading of the problems they have to solve.

References

Filloy, E., Rojano, T., & Puig, L. (2008). *Educational Algebra. A Theoretical and Empirical Approach*. New York: Springer.

Rojano, T. & Sutherland, R. (1997). Pupil's Strategies and the Cartesian method for Solving Problems: The Role of Spreadsheets. In E. Pekkonen (Ed.), *Proceeding 21th Conference of the Internatrional Group for the Psychology of Mathematics Education* (Vol. IV, pp. 190-197). Lathi, Finland: PME.

[1] This research is supported in part by a grant from the Dirección General de Investigación del Ministerio de Educación y Ciencia, Spain, ref. SEJ2005-06697/EDUC

SCHOOL AND EVERYDAY DISCOURSES IN MATHEMATICAL MODELLING

Jonei Cerqueira Barbosa and Marcelo Leon Caffe de Oliveira

State University of Feira de Santana

In a general form, mathematical modelling is understood as a learning environment in which the students are invited to use mathematics for solving problems from everyday or sciences (Niss, Blum & Galbraith, 2007), so students have to consider the arguments that are related to the situation-problem. The Bernstein's (2000) notions of classification and framing look useful to analyse this point. Classification refers to relations between categories which define what is legitimate in each one. Framing refers to relations within categories by establishing communicative principles. Taking everyday situations to school mathematics doesn't make classification weak, because the school setting put it to work according to its proposal. For instance, Barbosa (2006) distinguishes professional from school modelling. However, what to say about the framing? This question is part of a wider study whose partial results are to be presented here in order to receive comments.

Following the qualitative perspective, a group of students was filmed while solving a modelling task. They were students in a Brazilian countryside's school, and the teacher, named Antonio, was using modelling in his classes by the first time after an in-service course. The data analysis was inspired in grounded theory (Charmaz, 2006). The teacher who was followed shaped the modelling task according the school setting by establishing sequencing and pacing. However, students showed themselves very comfortable to bring up their arguments from the real situation, because it belonged to their known setting. Unexpected discourses were used by the students to approach the situation-problem, so making the framing weaker. These findings allow us to hypothesize the following point: modelling real situations that belong to the student's setting may do the framing weaker between the school and everyday discourses.

References

Barbosa, J.C. (2006). Mathematical Modelling in classroom: A socio-critical and discursive perspective. *Zentralblatt für Didaktik der Mathematik*, *38*(3), 293-301.

Bernstein, B. (2000). *Pedagogy, Symbolic Control and Identity: Theory, Research, Critique* (revised edition). Lanham, Maryland: Rowman & Littlefield Publishers.

Charmaz, K. (2006). *Constructing Grounded Theory: a Practical Guide Through Qualitative Analysis*. Thousand Oaks, CA: Sage.

Niss, M., Blum, W., & Galbraith, P. (2007). Introduction. In W Blum et al. (Eds), *Modelling and Applications in Mathematics Education: The 14th ICMI Study* (pp. 3-32). New York: Springer.

THE USE OF GEOMETRIC CONSTRUCTIONS TO DOCUMENT PRESERVICE MATHEMATICS TEACHERS' GEOMETRIC REASONING

Nermin Bayazit and Elizabeth Jakubowski

Florida State University

In this paper, we consider geometric constructions with compass and straightedge as a tool that helps to document preservice mathematics teachers' geometric reasoning. Owing to the main features of construction problems (accuracy in making conclusions, strict structure, rigorous language and constructivist nature), they can be fruitfully useful to document geometric reasoning. We shall shed new light on construction problems and show how they can be used as indicators of geometric reasoning.

NCTM (2000) stated that geometry is a natural area of mathematics for the development of students' reasoning and justification skills that build across the grades. Geometric constructions appear to have the potential to provide students the opportunity to enrich their visualization and comprehension of geometry, lay a foundation for analysis and apply their creativity (Sanders, 1998). Thornton (1998) discussed that "a construction requires students to make connections between geometric properties and hence bridge between analysis and deduction."

In this presentation, we will discuss how geometric construction problems can be used as a tool for investigating students' knowledge connectedness and geometric reasoning. We will share students' written work on geometric constructions from a junior level geometry course for middle and secondary mathematics teachers. We will analyse the students' work to determine how they made connections with the given information and the desired construction, and if there are any common traits in students' constructions. Moreover, we will look for evidence of internal and external connectedness.

References

National Council of Teachers of Mathematics (2000). *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: The Council.

Sanders, C.V. (1998). Geometric constructions: visualizing and understanding geometry. *Mathematics Teacher, 91(7),* 554-556.

Thornton, S. (1998). Construction geometric reasoning. *Australian Mathematics Teacher, 54(3)*, 6-11.

EXPERTS AND STUDENTS` MISCONCEPTIONS REGARDING CONFIDENCE INTERVALS

Roberto Behar Gutiérrez — Universidad del Valle

Gabriel Yáñez Canal — Universidad Industrial de Santander

We present the results of a research whose purpose was to find out what a sample of experts (statisticians and statistics university professors) and university students understood by confidence intervals. To this goal, a questionnaire was answered by 41 experts and 297 students. The results show that both, students and, experts possess misconceptions regarding confidence intervals. The conception that these intervals contain sample means or single values of the population instead of possible parameter values, and the interpretation of significance levels as a measure of certainty, without any frequency referent, were found to be the most generalized misconceptions**.**

The main results are: (1) roughly a 30% of the experts and half of the students assumed that CI is a sort of truncated range of population values, confirming Fidler's results (2005). The confidence level is assumed by them to be a percentage of population values that are contained in the confidence interval. (2) 32% of experts do not accept that the sample mean is contained in the confidence *interval.* In contrast, the percentage of students that does not accept this fact is 30%. (3) 47% of experts and 48% of students do not interpret in correct form the level of confidence associated at the interval. (4) More than half of the experts and 65% of the students deny that in the long run, if the sample were remade many times, such intervals, in a percentage equal to the confidence level would include the population parameter μ, allowing thus the existence of some intervals that do not include it. (5) 17% of the experts and up to half of the students do not understand that the relation between the interval width and sample size is inverse, namely, increasing the sample decreases the size of the interval. (6) The effect of population variability on the CIs shows the best results: around 90% of the experts and 60% of the students identified that when the variability in the population increases the interval size also increases

References

Cumming, G., Williams, J., & Fidler, F. (2004). Replication and researchers' understanding of confidence intervals and standard error bars. *Understanding Statistics*, *3*, 299 - 311.

Fidler, F. (2005). *From Statistical Significance to Effect Estimation*. Ph.D. Thesis, unpublished. Universidad La Trobe, Melbourne, Australia.

APPLICATION OF FUZZY THEORY TO MEASURE THE UNDERSTANDING OF THE INTEGRAL

Francisco José Boigues, Vicente Estruch, and Ricardo Zalaya

Universidad Politécnica de Valencia

We present a methodology based on the use of metrics Fuzzy and the theoretical model, APOS, for comprehension of a mathematical concept. Our method blends quantitative and qualitative aspects with the aim of determining the level of development in the understanding of the definite integral. We applied this methodology to a sample of students from natural sciences and environmental engineering. At the start of our research we have determinated a cognitive proposal, called genetic decomposition, that a student must build to achieve sufficient understanding of the definite integral. The main schemes included in our proposal are: partition of an interval [*a, b*], Riemann sums for *f(x)* continuous in a real interval [*a, b*] and for a specific partition and finally defining the integral considering a limit of a succession of Riemann sums. Each one of these schemes are comprised several elements and also relationships between them. Afterward, we assign a grade of acquisition to each element, through responses to a questionnaire obtained on interviews with students. Later, these grades were transformed with a Fuzzy metrics $F_d(x,y,t) = \frac{t}{t + d(x,y)}$ in a degree of understanding about development of the definite integral concept.

The results indicate that fuzzy theory is useful to analyse the understanding of a mathematical concept. Additionally allow us get information that helps make more precise some considerations on the thematization of a notion. Also, these results allow us use of this methodology for analysing extensive samples of students.

References

Asiala, M., Brown A., Devries, D.J., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. In J. Kaput, Schoenfeld, & Dubinsky (Eds.). *Research in Collegiate Mathematics Education* (Vol. 2, pp.1-32), Providence, RI: AMS and Washington, DC: MAA.

Cooley, L, Trigueros, M., & Baker, B. (2007). Schema Thematization: a Framework and an Example. *Journal for Research in Mathematics Education,* 38(4), 370-392.

George, A. & Veeramani, P.V. (1994). On some results in fuzzy metric spaces. *Fuzzy Sets and Systems***,** 64, pp. 395-399.

FIRST AND SECOND GRADERS' SPONTANEOUS USE OF PUNCTUATION MARKS WITHIN WRITTEN NUMERALS

Bárbara M. Brizuela and Gabrielle A. Cayton

Tufts University

Recent research has begun to document the role that children ascribe to punctuation marks while they are in the process of learning written numbers and the use they make of these marks while they write numbers. To date, these marks have been neglected and overlooked in mathematics education research. The research questions underlying this study were: How and in what circumstances did children spontaneously use punctuation marks while they were writing numbers? What kinds of unconventional uses of punctuation marks did they generate? The sample for this two-year longitudinal study is 27 first grade (approximate age 6) children and 26 second grade children; 21 of the original 27 first grade children were also in the second grade group. In second grade there were 5 children who had not participated in the study in first grade. Table 1 displays the (conventional and unconventional) production of punctuation marks among the sample of children.

	Total number of numerals written	Total number of uses of punctuation marks	Total number of conventional uses	Total number of unconventional uses
Grade 1	629 (100%)	86 (14%)	76 (12%)	10 (2%)
Grade 2	683 (100%)	151 (22%)	93 (14%)	58 (8%)

Table 1. Number of times punctuation marks were used, both conventionally and unconventionally. Percentages are calculated over the total number of numerals written at each grade level

The following are the types of unconventional uses of punctuation marks made by children in the sample. Percentages are over total number of uses of numerical punctuation:

A. Respects reading of number: 90% of uses punctuation marks in Grade 1 and 82% in Grade 2 continued to respect how the number should be read. B. Respects "batches of digits" rule: none of the unconventional uses of punctuation marks respected the grouping of digits into sets of three. C. Omits zero after comma: 80% of uses of punctuation marks in Grade 1 and 17% of uses in Grade 2 omitted a zero immediately after writing a comma. D. Adds a zero at the end of the number: Only occurred once in first grade (10% of all punctuation marks made in Grade 1). E. Adds zeros within the number: this never occurred with numbers containing punctuation in first grade. 76% of numbers containing punctuation in Grade 2 added a zero within the number. F. Omits or adds a non-zero digit: In Grade 1, 10% of uses of punctuation marks omitted a digit and 0% added a digit; in Grade 2, 10% omitted a digit and 7% added a digit.

CONSTRUCTIVISM AND MATHEMATICS INSTRUCTION: CHALLENGES, MYTHS, MISCONCEPTIONS

Priscilla Brown-Lopez

University of Belize

Durham University

Poor performance in math word problems is prevalent in many countries. This research sought to examine whether the use of a constructivist based instruction can result in improved performance among K-5 students in Belize Central America. The results of the Repeated Measures ANOVA, video recording of students' social interaction and semi-structured interviews revealed that constructing understanding is highly dependent on instructional scaffolding and the socio cultural context of the learner.

ANALYSIS OF THE CONSTRUCTIVIST PARADIGM

Revelations of constructivist mathematical instructions are best understood through open discourse on students' readiness and the effects of prior knowledge on learning. This oral presentation is centred on the proposition that constructing understanding is a highly intricate process, which relies on the experiences of learners and the teachers' ability to guide students to use background experiences to generate new information (Fosnot, 2005). Oral discussion of this stance is linked to research conducted in Belize, Central America to examine whether the use of a constructivist-based instruction will improve performance in math word problems. While the findings are limited to K-5 students in urban and rural Belize, the data suggest that constructing understanding in mathematics requires more than social interaction, authentic resources and prior knowledge thereby posing significant challenges for teachers. Given that constructivism is a "catch phrase" among many educational practitioners, this oral presentation hopes to engage participants in critical analysis of the misconceptions and challenges associated with mathematical instruction in a developing country such as Belize.

Reference

Fosnot, C. T. (2005). *Constructivism: Theory, Perspectives and Practice* (2nd Ed.) Teachers College Press N.Y.

YES, BUT IN THE MATHEMATICAL WAY

Consuelo Campos

Cinvestav

In Mathematical Department of Educative at Cinvestav, Mexico, during 2006 began a program of masters in service focused to 60 professors of basic education. Jointly of this program, a project of evaluation of the master itself is carried out. Among the objectives of the first stage of this project is to understand how it reveals, in first place, the mathematical activity. This paper reports the results obtained and what they suggest. It were observed 48 professors while they taught in its classrooms: two consecutive classes, followed at the end of the second class by one interview. The data were divided in episodes in the sense of Chevallard. For the analysis of the lessons three categories of analysis were considered: 1) presentation of the content and the mathematical tools; 2) use of the mathematical tools y; 3) mathematical justifications. It shows three cases in where the professor reveals some of the most common characteristics have been found: a) the expected mathematical objects were not reached, consequently the mathematical result is not quite clear; b) Limited control of the time and; c) it observes lack of planning in the difficulty identifying exercises that raise serious conflicts; the approach to the mathematical content that promotes limited mathematical activity and confusion as far as the looked or the reached one for thing through activities, the supposition that is a spontaneous transit from an explanation to the application of the concept and, the deficiency as far as its own mathematical knowledge.

The presentation of the mathematical tools implies itself the organization and structures of the mathematical content. A suitable presentation allows identifying through the activities the relevant mathematical aspects, along with its relations and structures. They would be problems, conjectures, results, algorithms. Whereas in the referred cases, remained like exercises: repetitions without context or apparent purpose. The professors have difficulties in the use of the mathematical tools, emphasizes the poor use of the mathematical language: in fact, the lack of precision as far as the definitions, consistency and rigor in the use of the terms, promoting confusion in themselves and their students. In a general way, the justifications from the point of view of the mathematics are weak: the tendency is to handle non-proven results.

But as far as we see in the obtained results we can notice that it is not only the knowledge, cause they knew the topics, but also in the 'non-mathematical way' of the use of this knowledge. We see that to improve the mathematical activity in the classroom is necessary becomes to do it in a 'mathematical way', i.e., recognizing the necessity of the precision in the language, the definitions, the symbolization, the rigor in the justifications, the formalization of the properties, results and algorithms. That is to say, the professor and its lessons must have the characteristics that we adjudged to the mathematics.

TYPES OF SECONDARY STUDENTS' CONJECTURES ON THE CONSECUTIVE NUMBER PROBLEM

Ing-Er Chen — Fooyin University
Fou-Lai Lin — National Taiwan Normal University

This study is to investigate the types of students' conjectures and what is the key mechanism to affect students' successful in conjecturing activity. The source problem was one of the Consecutive Numbers problems: the sum of n consecutive integers is always an even number. On the worksheet, there are four main stages: (a) give examples; (b) observe common properties; (c) formulate conjectures; (d) generate new ideas. 35, 38, 39 students belonging to three classes of grade 7, 8, 9 participated in the study. They were asked to answer the questions and write down their ideas in 40 minutes in the class section.

The results are: there are 62.9%, 84.3% and 87.2% students in grade 7, 8, 9 who were able to propose at least one conjecture. And only 2.8%, 2.6% and 18% students in each grade proposed one more conjectures. Most (63%, 74% and 36% respectively) students gave support examples or counter examples when they were asked to give more conjectures. The conjectures students proposed could be divided into two big categories. The first category is called uncertain conjecture, like "the sum of n consecutive integers is uncertain an even number (it could be even or odd)". The second category is called certain conjecture. These certain conjectures could be divided into three sub-categories and are displayed in Figure 1.

	P'→Q (a)	**P'→Q(b)**	**P'→Q (c)**	**P→Q'**	**P'→Q'**
	The sum of n consecutive even numbers is an even number.	The sum of n consecutive integers is an even number if n=2k, 4k, n≠2.	The sum of n consecutive integers is an even number if the sum of odd numbers is even.	The sum of n consecutive integers is an integer.	The sum of n consecutive integers is not an even number if n=2.
G 7	11.4%	5.7%	5.7%	0	0
G 8	2.6%	5.3%	7.9%	5.3%	0
G 9	10.3%	12.8%	17.9%	0	2.6%
Total	24.3%	23.8%	31.5%	5.3%	2.6%

Figure 1. Types of certain conjectures (modify P, modify Q, and modify P and Q).

In addition, we found that there are 22.9%, 28.9% and 43.6% students in grade 7, 8, 9 using conditional terms to describe the common property. Conditional reasoning is an important mechanism to make conjectures successfully.

IMPLEMENTING MODELING ACTIVITY TO ENHANCE STUDENT'S CONCEPTUAL UNDERSTANDING AND ACTIVE THINKING

Kuan-Jou Chen and Erh-Tsung Chin

National Chunghua University of Education

Taiwanese student's mathematical achievements have been highly ranked in recent international assessments (e.g., TIMSS, 1999, 2003; PISA, 2006). However, behind these positive results, as a mathematics teacher in the first line, what can be noticed is that our students might be good at solving formal mathematical problems, but they still lack the proficiencies of conceptual understanding and active thinking. In this study, we tried to investigate students' algebraic problem solving abilities through implementing a modeling activity whilst they had not learned using symbolic representations to solve problems, in order to enhance their mathematical proficiencies.

This is a case study, adopting modeling theory (Lesh & Zawojewski, 2007) as the theoretical framework, with thirty-four fifth grade pupils in Taiwan. The modeling activity is based on a model-eliciting activity, guiding students to read articles, to prepare questions, to describe questions, and to share strategies to construct their own problem solving strategies and approaches. The contents of the teaching activities include: Discovering the big footprint, measuring animal's footprints, and the relations of footprint size and height and weight. The collected data include: student work sheet responses, classroom teaching video record transcripts, and classroom observation records.

The main results show that, different from the conventional teaching environment, the students were highly motivated in participating in the modelling activity. Through actively discussing and sharing ideas with peers, they successfully constructed their own approaches to solve the algebraic problem. According to these results, it might be reasonable to get a conclusion that implementing modelling activity in the mathematics classroom could effectively enhance student's conceptual understanding and active thinking.

Reference

Lesh, R. & Zawojewski, J. (2007). Problem solving and modelling. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning (*pp.763-804). Greenwich, ct: information age publishing, inc and nctm.

A STUDY ON DEVELOPING AND VALIDATING A QUESTIONNAIRE OF MATHEMATICS TEACHER'S INQUIRY TEACHING COMPETENCY

Erh-Tsung Chin, Chih-Yen Liu, and Cheng-Jung Hsu
National Changhua University of Education

The aim of the study is to develop and validate a questionnaire which can measure high school mathematics teacher's competency of applying inquiry teaching strategy in the teaching activities. The questionnaire might not only help mathematics teachers perceive their own competencies of implementing inquiry-based teaching, but also provide evidence of the growth of teacher's inquiry teaching competency as a research instrument for researchers to conduct relevant research. The development of the questionnaire was started from referring to relevant theories of the nature of mathematics (e.g., Ernest, 1994; Lakatos, 1976; Tymoczko, 1998), followed by reviewing the theories about inquiry-based teaching (e.g., Borasi, 1992), principles and standards of mathematics education (e.g., NCTM, 1998), and the viewpoint from science inquiry (e.g., Trowbridge & Bybee, 1986). Finally the conceptual structure of the questionnaire was induced and the items were designed based on the structure. The questionnaire was administered to 314 high school mathematics teachers through stratified convenience sampling. By applying SPSS 14.0 to conduct statistical analysis, including item analysis, factor analysis and reliability analysis, two scales were yielded which are "teacher's expectation towards inquiry teaching outcome" and "inquiry teaching competency", and 41 items (including three reverse items for consistency checking) are included in the questionnaire. The Cronbach's α coefficient of the whole questionnaire is 0.9604, while the two scales' Cronbach's α coefficients are 0.9097 and 0.9606, respectively. Thus it appears rather high reliability of the internal consistency of the questionnaire. Besides, Pearson's coefficient of correlation of the whole questionnaire is 0.8294, while they are 0.7924 and 0.8057 of the two scales, respectively. It reaches 0.01 significance standard, which indicates the high stability of test and retest of the questionnaire. In addition, the statistical analysis shows that the questionnaire also possesses content validity, expert validity and construct validity.

References

Borasi, R. (1992). *Learning Mmathematics though Inquiry.* Portsmouth, NH: Heinemann.

Ernest, P. (1994). The dialogical nature of mathematics. In P. Ernest (Ed.), *Mathematics, Education, and Philosophy: An International Perspective* (pp. 33-48). London: The Falmer Press.

Lakatos, I. (1976). *Proofs and Refutations*. Cambridge: Cambridge University Press.

NCTM (1998). *Principles and Standards for School Mathematics.* Reston, VA: Author.

Trowbridge, L.W., & Bybee, R.W. (1986). *Becoming a Secondary School Science Teacher.* Columbus, Ohio: Merrill.

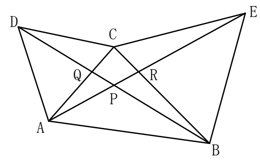
Tymoczko, T. (1998). *New directions in the Philosophy of Mathematics*. N.J.: Princeton University Press.

RETHINKING "DISCOVERY" AS A FUNCTION OF PROOF IN SCHOOL MATHEMATICS

Kimiho Chino
Kokushikan University

Researches have examined the suggestion in teaching and learning for "discovery" based on the analysis of activities of the students (De Villiers, 1998; Miyazaki, 2000). The purpose of this paper is to explore the other potential of "discovery" in lower secondary schools. Targeted students belonged to the ninth grade in Japan. The survey was done from late Sep, 2001 to early Oct, 2001, with the following proof problem with a phrase "as shown in right figure".

Survey Question: As shown in right figure, construct equilateral triangles *ACD* on *AC* and *BCE* on *BC* outside ΔABC, respectively. Connect vertexes *A* and *E*, and *D* and *B*, respectively.



(1) Hanako began writing a proof of $AE = DB$ as follows. Carry on a logical argument to explain why they are congruent.

> Proof: About ΔACE and ΔDCB,
> Because ΔACD is an equilateral triangle, $AC = DC$. …........… (i)
> Because ΔBCE is also an equilateral triangle, $CE = CB$. …… (ii)

(3) Explain the reason why $AE = DB$ regardless of any shape or the size of ΔABC.

Regarding Question (3), responses by students who express a virtually identical valid argument suggest that some students assumed at least the following triangle as ΔABC: a triangle with fixed length of *AC* and *BC* and any size of $\angle ACB$, a triangle with fixed size of $\angle ACB$ and any length of *AC* and *BC*, a triangle which is similar to ΔABC, or any triangle. The character in these students' explanations is to express clearly that (some) equivalent relation(s) referred in the proof did not depend on the length of these sides or the sizes of the angle. Of course, it might be necessary that we carry on detailed researches because the students might have only expressed a part of their ideas. However, it seems reasonable to conclude that each triangle assumed by these students as ΔABC is different. These students have a chance to reflect what "as shown in figure" means with their proof, except for the students who recognize ΔABC is any triangle. It is one of the potentials of "discovery" as a function of his/her proof.

Endnote

This research was funded by Grant-in-Aid for Scientific Research, Nos. 18730538 and 18330187.

References

De Villiers, M. (1998). An alternative approach to proof in dynamic geometry. In R. Lehrer & D. Chazan (Eds.), *Designing Learning Environments for Developing Understanding of Geometry and Space,* (pp. 369-393), London: Lawrence Erlbaum Associates Publishers.

De Villiers, M. (1999). *Rethinking Proof.* Emeryville: Key curriculum press.

Miyazaki, M. (2000). What are essential to apply the discovery function of proof in lower secondary school mathematics? In *Proceedings of 24th Annual Meeting of the International Group for the PME* (Vol.4, pp.1-8). Hiroshima: Hiroshima University.

AN ANALYSIS OF ALGEBRAIC THINKING OF ELEMENTARY SCHOOL STUDENTS

JiYoung Choi — Seoul Daedong Elementary School
JeongSuk Pang — Korea National University of Education

Algebra deals with relationships among quantities, representation of mathematical relationships, and the analysis of change. Traditionally, algebra has been regarded as a subject suited to secondary school students. However, children's extensive experiences with numbers can be the foundation of much of the symbolic and structural emphasis in algebra (NCTM, 2000). All students should learn algebra, and algebraic thinking should be emphasized throughout the elementary grades (Carpenter, Franke, & Levi, 2003). In order to teach algebraic thinking, we need to understand not only the nature of algebraic thinking but also the characteristics of students' thinking.

However, few empirical studies have been conducted with regard to what really constitutes elementary students' algebraic thinking. Given this background, we designed six consecutive lessons in which 4^{th} graders were encouraged to represent their algebraic thinking. The lessons were video-taped and transcribed. Additional data included students' worksheets, informal interviews with focus students and field notes. We investigated how students might recognize patterns in the process of finding the relationships between two quantities, how they might represent a given problem with various mathematical models including algebraic expressions, and how they might perceive the equivalent expressions that were apparently different. Students' errors and cognitive obstacles while they attempted to use algebraic symbols to generalize patterns we also analysed.

This study showed that students recognized patterns through concrete activities with manipulative materials, and employed various mathematical models to represent a given problem situation, accompanying verbal descriptions. When using algebraic expressions, students tended to differentiate the expressions for each operation instead of using a complete but complex expression. Students were able to represent a problem situation with two algebraic expressions but they could not consider the two expressions to be the same. Students had difficulties in using the equal sign and letters for the unknown value while they attempted to generalize a pattern. This study implies how to connect algebraic thinking with students' arithmetic or informal thinking in a meaningful way, and how to teach algebra throughout the elementary grades.

References

Carpenter, T.P., Franke, M.L., & Levi, L. (2003). *Thinking Athematically: Integrating Arithmetic and Algebra in Elementary School*. Portsmouth, NH: Heinemann.

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author.

A FIVE-STEP PROGRAM FOR IMPROVING TEACHER EDUCATION STUDENTS' ALGEBRA CONTENT KNOWLEDGE

M. A. (Ken) Clements and Nerida F. Ellerton

Illinois State University

We summarize our analysis of data from a study investigating cognitive and psychological difficulties and dilemmas experienced by 158 pre-service teacher education students who were taking their last algebra course before they became full-time teachers of middle-school mathematics. All 158 mathematics students had completed full algebra programs in secondary schools in the United States of America, but our initial testing revealed that most of them knew very little of what they had been taught during their school years in the algebra content areas of linear and quadratic equations, linear inequalities, and functions. Furthermore, despite the fact that much of the students' thinking was guided by misconceptions, it was too often the case that they were not aware that their thinking was mathematically faulty.

The main focus of the study was on (a) identifying any misconceptions these students initially held with respect to linear and quadratic equations, inequalities, and functions; and (b) how students' fossilized misconceptions were overcome through a carefully sequenced intervention program. The aim of the study was to make sure that by the conclusion of the program, each and every one of the 158 students had strong mathematics subject knowledge with respect to equations, inequalities, and functions.

The intervention program comprised five steps. First, through a diagnostic testing process, the students were assisted not only to identify fossilised misconceptions that had guided their thinking but also to recognise that they had developed inappropriate levels of confidence in their ability to give correct answers to important algebra tasks. At the second step, the students were individually assisted in the process of replacing inappropriate misconceptions and skills with appropriate conceptions and skills. Then, at the third step, they were required to reflect, metacognitively (in writing), on where they had gone wrong and why. The fourth step involved them in revisiting, from time to time, relevant tasks in order that their confidence and understanding would be consolidated and enhanced. And, finally, by responding to a retention instrument that paralleled the original diagnostic instrument, the students were given the opportunity to demonstrate that their new conceptions were accurate, and their levels of confidence were not misplaced.

Our data analyses revealed that the five-step overall intervention strategy proved to be extremely effective, with all of the prospective teachers not only learning relationally the subject matter knowledge, but also much associated pedagogical content knowledge. As a consequence of coming to understand the content, the students came to know that they really did know what they had learned.

LEARNING TO TEACH MATHEMATICS USING JAPANESE LESSON STUDY: A CASE IN IRELAND

Dolores Corcoran

St Patrick's College

University of Cambridge

This study seeks to examine the development of six Irish pre-service primary teachers as teachers of mathematics during an elective module in the final year of their Bachelor of Education course. The group adopted a protocol for Japanese lesson study (LS) involving collaborative pre-teaching planning sessions, research lessons that were observed and videotaped and post-lesson reflective meetings. Students and lecturer became co-researchers, and a community of practice espousing a 'reform' approach to teaching mathematics was established. There were six student participants and three iterations of the LS cycle. Data for analysis include the video records of six research lessons, reflective journals written by the students after each session, tape recordings of all planning and post-lesson meetings, the student teachers' lesson plans and examples of children's work.

Wenger's (1998) descriptive terms were used to analyse the learning element of the students' participation in the LS community of practice and the dimensions of the Knowledge Quartet (KQ) devised by Rowland, Huckstep and Thwaites (2004) were used to analyse the students' teaching. Thus the KQ as an analytic tool also worked to build community by becoming part of the "shared repertoire of ways of doing things" (Wenger, 1998) within the group.

Three indicators of degrees of participation in a community of practice are offered by Wenger (1998): engagement, imagination and alignment. Each was present in this group and findings are that participation in LS provided an opportunity to build identity and develop mathematics subject knowledge as these student teachers studied their own and each other's teaching.

References

Rowland, T., Huckstep, P., & Thwaites, A. (2004). Reflecting on prospective elementary teachers' mathematics content knowledge: The case of Laura. In M.J. Høines & A.B. Fugelstad (Eds.), *Proceedings of the 28th Conference of International Group for the Psychology of Mathematics Education* (Vol 4, pp. 121 – 128), Bergen, Norway: PME

Wenger, E. (1998). *Communities of Practice*. UK: Cambridge University Press.

A DIDACTIC PROPOSAL FOR INTRODUCING CALCULUS WITH TECHNOLOGY

Carlos A. Cuevas Vallejo
Cinvestav

Magally Martínez Reyes
Universidad Autónoma Estado de México

François Pluvinage
Université Louis Paster
Cinvestav

Some guides, for the differential calculus teaching, result from the study and analysis of history. One guide is that the first differential calculus course must not be constructed over a previous knowledge of formal definitions of the function concept and of the notion of limit, which emerged at the end of their creation.

To visualize a differential calculus course without the use of technology, would unable the professor to take advantage of one of the most important resources he can count on nowadays. We have introduced essentially two kinds of software in a differential calculus course, the first has didactical scenarios which simulate a natural phenomenon. These scenarios, where the student is able to manipulate things, they are of free use on the net. The second kind has tutorial systems which share the teaching job with the professor. For the first part of the course which corresponds to basic concepts of function, dependent and independent variable, parameter and equation, the project of concrete action, named "Project pulleys", was designed. This project has three applets simulating the movement of a pulley loading a specific weight. Additionally we create a whole work environment for the students, including: Directed instructions for the professors with the description of the objects of study, the necessary time prevision and organizational instructions; working instructions for the students, questionnaires, with presupposed spaces, for the students who are conducting the activity. Before the use of the IELM in the institution the failure indexes were of 80%. The traditional pattern of evaluation consists in: exam written 50%, homework 30%, activities works 10% and participation 10%, so that all students that approved the course should have punctuation in each item. Gradually through the last three years this failure index has diminished to 25%, maintaining the traditional pattern of evaluation. Due to the heterogenousity, in previous mathematical knowledge of the student population, an application of a spiral teaching accordingly with the principal of proximal development (Vigotzky) in order to not introduce solved courses. To design and implement modeling activities was, in accordance to the RME scheme (real mathematical instruction), in the form of concrete action projects

WHAT THE GAME OF NIM REVEALED ABOUT CHILDREN'S INTUITIVE UNDERSTANDINGS

Thérèse Dooley

St. Patrick's College, Dublin and University of Cambridge

A diverse range of meanings is attributed to the term intuition (Ben-Zeev & Star, 2001). Sometimes, it is seen as a raw and unrefined form of knowledge; at other times it is regarded as being akin to deep insight. In an effort to simplify this complex field, various classifications of intuitive cognition have been proposed. Of main interest to this paper is the distinction between *primary* and *secondary* intuitions made by Fischbein (1987). Primary intuitions are those that have natural roots. They are evident in the kind of responses that a child gives to a problem-situation that do not involve a quantitative, formal analysis of the situation. Secondary intuitions are those that are acquired through educational experience rather than through natural experience. One of the main characteristics of intuition is its resistance to change. Often primary intuitions are so firmly anchored that they co-exist with more scientifically acceptable ones (Fischbein, 1987). At other times, they influence understanding at an implicit, tacit level.

As part of my doctoral research on insight in primary mathematics, I was involved in teaching a class in which there were nineteen girls and thirteen boys, aged 10 - 11 years. Three of the lessons involved a version of the Nim game. This game is played by pairs of students, who begin with a pile of counters. In the simplest version, each in turn takes either one or two counters, the winner being the last to remove a counter from the pile. Quite early, students learn that they will lose if there are three counters remaining and it is their turn. Therefore 'three' is an unsafe position as are six, nine, twelve counters etc. In this presentation I will describe follow-up interviews that I held with pairs of pupils where it was revealed that, in the context of this game, ideas of fairness and parity (the even/odd pattern) dominated their thinking to the extent that they ignored the winning pattern that was emerging. While the former ideas seem to be 'primary', further focus on the 'threes' multiplication pattern and on mathematical activities that challenge ideas of fairness may be warranted at primary school level.

References

Ben-Zeev, T., & Star, J. (2001). Intuitive mathematics: Theoretical and educational implications. In B. Torff & R.J. Sternberg (Eds.), *Understanding and Teaching the Intuitive Mind: Teacher and Student Learning* (pp. 29-56). Mahwah, NJ: Lawrence Erlbaum.

Fischbein, E. (1987). *Intuition in Science and Mathematics: An Educational Approach*. Dordrecht: D. Reidel Publishing Company.

USING EXAMPLE GENERATION TO EXPLORE STUDENTS' CONCEPT IMAGES OF SEQUENCE PROPERTIES

Antony Edwards and Lara Alcock

Loughborough University

This Short Oral reports on an example generation task given to 101 students in an undergraduate Real Analysis course. We discuss students' responses to three questions concerning monotonicity and boundedness, indicating (i) a high rate of discrepancies between students' concept images and the formal definitions for these concepts and (ii) notable cases in which the given response is not, in fact, an infinite sequence.

It has been shown that students often do not use formal definitions appropriately to judge whether or not mathematical objects belong to certain categories, even when they can correctly state these definitions (Vinner, 1991). Instead they tend to rely on their concept images, which may include spontaneous conceptions derived from the everyday meaning of mathematically precise terminology (Cornu, 1991).

This Short Oral reports on an exploratory pilot study in which 101 students attempted an example generation task involving sequences. We use such a task as a research tool that allows us to investigate students' current concept images for the sequence properties of monotonicity and boundedness. In doing so, we reveal spontaneous conceptions relative to these properties. We also highlight a particular sub-problem: that of students apparently attaching properties to the wrong kind of object.

We discuss a question in more detail where the majority of responses were incorrect - 87% did not combine the definitions *increasing* and *decreasing* to give a sequence satisfying both.

Two further questions are outlined where attempts to generate a sequence that satisfied certain properties apparently led to a failure to control for the requirement that the answer be a sequence. This result echoes that of Dahlberg and Housman (1997), who reported that that some students modify or reinterpret the meaning of a concept if they are unable find examples to satisfy it.

References

Cornu, B. (1991). Limits. In D.O. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 153-166). Dordrecht: Kluwer.

Dahlberg, R.P. & Housman, D.L. (1997). Facilitating learning events through example generation". *Educational Studies in Mathematics*, *33*, 283-299.

Vinner, S. (1991). The role of definitions in teaching and learning mathematics. In D.O. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 65-81). Dordrecht: Kluwer.

SITUATING THE DEVELOPMENT OF STANDARDS-BASED SECONDARY SCHOOL MATHEMATICS TEACHER EDUCATION CURRICULUM IN THE PHILIPPINES

Levi Esteban Elipane and Hiro Ninomiya

Saitama University

In the face of educational reform movements, the responsibilities of various teacher training institutions in providing mathematics teacher education programs would inevitably appear to be more problematic. As *standards* have served as a basis for educational reform brought about by the call of various educational stakeholders for a clearer definition of what students must be able to know and must be able to do, and also by the public's demand for accountability, bodies of research about considering the social dimension in the development and implementation of curriculum have also been gaining much attention in the field. Hence, as the term 'standards' is usually considered universal, and the concept of social dimension is customarily linked to the idea of a *culture* in a certain locality, careful considerations must be performed in order to achieve the balance in the process of developing the curriculum for mathematics teacher education.

This paper is a situational analysis of the standards-based curriculum for secondary school mathematics teacher education in the Philippines. It discusses on the impact of the competency standards set by the Commission on Higher Education of the Philippines for all graduates of every teacher training institution in the archipelago. The focus was narrowed down to mathematics teacher education. Analysis of the standards document was undertaken and the perceptions of various educational stakeholders were pulled together by conducting surveys and interviews based on the standards documents.

Habermas' theory of communicative action, the Situated Learning Theory by Lave, the Notion of Communities of Practice by Lave and Wenger, and D'Ambrosio's work on ethnomathematics serve as sound theoretical backbones for this investigation.

The results of this study deem to inform, and draw implications for, the development of standards-based curriculum for secondary school mathematics teacher education in the Philippines. Issues on the articulation, accessibility, and the perspective that the set of standards take were highlighted.

Indeed, situating the development of standards-based curriculum for secondary school mathematics teacher education in the Philippines entails further investigations. Phenomenological studies, for example, on how each of the educational stakeholders view or understand each of the standards might be undertaken. Furthermore, a deeper look on the cultures and social dimensions, or communities of practice, must also take place.

DEAF ADOLESCENTS COOPERATIVE LEARNING

María del Pilar Fernández-Viader | Mariana Fuentes
University of Barcelona | Autonomous University of Barcelona

We studied the reciprocal correction and solving addition and multiplication operations strategies in two couples of deaf students that communicate in Catalan Sign Language (LSC) (aged 13:06 -15:11 and 12: 07-13:04). The two members of one couple were in the same level of mathematical achievement and the others were in a different level. We observed that the two students who explained the error or the procedure found an adapted way to help his/her companion. We think this is due to the fact they share the Zone of Proximal Development (ZPD), and also because of the common use of a visual language and because they are efficient in using some visual strategies in which they incorporate the digits with different uses and functions.

COOPERATIVE LEARNING

The interaction relationship between peers situates in the philosophy of cooperative learning which establishes that in this relation of giving and receiving help in a reciprocal way, both students benefit of this process. We studied deaf students in tasks of reciprocal correction and solving operations' strategies (Fernández-Viader & Fuentes, 2008).

Method

Students solved operations by couples. Each student posed the operations and problem to the other student, this one solved the operation and then the first one corrected the exercise. Then they interchanged roles. We describe two situations as examples, one in addition and one in multiplication.

Results and conclusions

We think that the students who explained found an adapted way to help his/her companion because of being in the same Zone of Proximal Development (ZPD) and also because they use the same code, so they feel comfortable to justify and explain the procedures. To have a shared language is to enabling curriculum access too. Studying these kinds of strategies is worth for designing teaching strategies and teacher training.

References

Fernández-Viader, M.P. & Fuentes, M. (2008). *Estratègies de Resolució d´Operacions en Adolescents sords*. (Solving operations' strategies in deaf adolescents). ARCE. Research Office, Faculty of Education and Teacher Training, University of Barcelona. (On line).

AN INVESTIGATION OF CLASSROOM PRACTICE WITHIN A PROFESSIONAL DEVELOPMENT STUDY PROGRAMME

Olimpia Figueras, Carolina Guerrero, Juan Carlos Ponce, Rubí Real, and Martha Sánchez
Cinvestav[1]

Patricia Flores
Universidad Pedagógica Nacional

A master's degree study programne (SP) oriented to basic education in-service teachers was set up in 2005. The main purpose of the SP is the transformation of the mathematics class into an Experimental classroom. This setting has three main functions: as a didactical laboratory, as a place to observe mathematics learning processes in Freudenthal's sense (1981) and as a space to reflect about the daily classroom practice.

In a parallel way to the development of the SP, a research agenda was built up. Three studies have been carried out. The first one to characterize the daily classroom practice of the student-teachers is the main theme of the communication.

Sixty student-teachers constitute the first generation of this professional development study programme. Twenty of them were selected to carry out classroom observation for two successive mathematics lessons, both were videotaped. At the end of the session the student-teacher was interviewed in order to inquire about to the mathematical knowledge that he/she considered was constructed.

The didactical triangle framework (Sensevy et. al. quoted in Steinbring, 2005) serves as a means to focus the attention on the relationships among teacher and knowledge, students and knowledge and teachers and students, which constitutes an approach to teachers' beliefs regarding teaching and learning of mathematics, and his/hers expectations of students' performance. In order to analyse the teaching of mathematics processes it was necessary to divide the sessions in episodes, for studying these relationships (see Chevallard, Bosch & Gascón 1997).

Results

From the most important results derived from the first study we can mention the following: 1. Teachers' mathematical knowledge has to be enriched in order that they can propose tasks that promote students' effective mathematical activity in the classroom. 2. Teachers try to go beyond their program, however they need to know what students can achieve and the hindrances they face. 3. Teachers have to gain experience to take their students to proper levels of mathematisation.

References

Chevallard, Y., Bosch, M., & Gascón, J. (1997). *Estudiar Matemáticas. El Eslabón Perdido entre la Enseñanza y el Aprendizaje*. Barcelona: ICE/Horsori.

Freudenthal, H. (1987) 'Major Problems of Mathematics Education'. *Educational Studies in Mathematics* 12, 135-150.

Steinbring, H. (2005). 'Analyzing mathematical teaching-learning situations - the interplay of communicational and epistemological constraints', *Educational Studies in Mathematics,* 59,313-324.

[1] Kathleen Hart, François Pluvinage, Consuelo Campos, Alicia Martínez and David Páez are also part of the research group. Ernesto Sánchez, Adrián de la Rosa and Guadalupe Macías have participated in different stages along with 20 teachers and students that supported and carried out the recollection of the data and made the first report of the classroom observations.

MATHEMATICAL LITERACY IN DEAF ADOLESCENTS

Mariana Fuentes — Autonomous University of Barcelona

María del Pilar Fernández-Viader — University of Barcelona

Solving addition and subtraction strategies of seven deaf adolescents (12:04 to 15:11), non-native signers, are explored. We compared with strategies previously described for hearing and for deaf signing children. Students show an ample repertoire of strategies but produce more quantity of errors than described for hearing children. We think this is a consequence of their delay to access to a well-structured linguistic input that makes difficult the access to curriculum.

DEAF CHILDREN SOLVING OPERATIONS'STRATEGIES

A number of research works have explored deaf children's solving strategies and a few considered the use of sign language in this subject (Secada, 1984; Frostad, 1999, Nunes, 2004). Frostad suggests exploring the relationship between deaf children's delay in mathematics and the kind of solving operations' strategies the students use. We aim to contribute to this research line studying strategies use in a peer-interactive situation context.

Method

Students solved by couples addition and subtraction operations. Each student posed the operation and problem to the other student, this student solved the operation and then the first one corrected the exercise. Then they interchanged roles.

Results and conclusions

Our results agree with Frostad who found that deaf children have an ample repertoire of strategies to solve operations. Students combine vocal counting, and counting using fingers to keep track and as numeral signs. Nevertheless, in subtraction, participants produced a number of errors superior to that expected by age and years of schooling. According to recent studies deaf children do not have an inherent delay in their mathematics ability (Nunes, 2004). We think our participants' delay is due to late access to a well-structured linguistic input that makes difficult the access to curriculum.

References

Frostad, P. (1999). Deaf children's use of cognitive strategies in simple arithmetic problems. *Educational Studies in Mathematics, 40,* 129-153.

Nunes, T. (2004). *Teaching Mathematics to Deaf Children*. London: Whurr.

Secada, W. (1984). *Counting in Sign: The Number String, Accuracy and Use*. Unpublished Ph.D. Thesis, Northwestern University, Evanston, IL, USA.

ANALYSIS OF GRAPHICS IN THE CONTEXT OF PHYSICAL PHENOMENA

Martha Fuentes Márquez — Universidad Pedagógica Nacional
Jorge Peralta Sámano — Univerdad Autónoma Estado de México
Marco Antonio Santillán Vázquez
Universidad Nacional Autónoma de México

We present preliminary results of an ongoing investigation, about some problems faced by high school and college students to build and interpret graphics for physical phenomena.

Graphical representations have provided for the development of mathematics and technology a huge utility. Students in high school and even technical college have difficulties in the construction and interpretation of charts.

THEORETICAL FRAMEWORK

The construction and interpretation of charts and the conjecture of a rule of correspondence can be part of the powers to shape a phenomenon (Lesh and Yoon cited by Confrey, 2007, p. 128) The study of physical phenomena using concepts such as force field, temperature, heat, etc. many of which are built taking as a starting point the intuition and other operating schemes.

OBJECTIVES

Giving account of how students interpret and construct graphs, how operating and understand these phenomena through them. To describe situations in which students build a chart with the intervention of a mediator.

METHODOLOGY

A first example was conducted with 27 college students, who were interviewed using the chart that corresponds to the vertical shooting (***v*** vt *t*), then, we have video recorder the discussion of the behavior of the magnetic force with the distance, interrogating 8 students with ages between 15 and 17 years.

PARTIAL RESULTS

Students find it convenient to use the analogy between gravity and magnetism. With regard to the preparation of charts, perhaps by the continued use of bar charts in Excel, students drew bars instead points to indicate the value of variables without considering whether the **phenomenon is continuous or discrete.**

References

Confrey, J. (2007). *Epistemology and Modeling Overview*. New York: Springer.

THE EMERGENCE OF INTEGERS THROUGH THE SOLUTION AND INVENTION OF ADDITIVE PROBLEMS

Aurora Gallardo and Eduardo Basurto

Cinvestav

The solution of additive problems occupies an important position in mathematical education research. Several classifications have been provided by the likes of Vergnaud (1982) and Nesher (1983). The basis classification used was Bruno and Martinón (1997), which deals with the distinction between a problem's functional structure and its semantic form. The functional structure refers to the type of numerical situations -states, variations and comparisons- while the semantic form refers to the mode of expressing said numerical situations: that is to say, *paying a debt* could be equivalent to subtracting or *reducing part of a debt*, in which case both phrases are known as *equivalent semantic forms*; in other words they are verbal forms that bear the same meaning. These meaning equivalences lay a bridge between mathematical language and natural language, as well as a means of identifying addition and subtraction in the learning of negatives. The equivalent semantic forms can be observed in the following sentences, which are different ways of expressing "Juan had 3 more than Marcos did": Marcos had 3 less than Juan did; Juan had -3 less than Marcos did; Marcos had -3 more than Juan did. These forms are unrealistic in common usage of language, but they are very useful in adittive problem solving. This article reports on research undertaken with 12 and 13-year old students who worked on solving and inventing additive problems for the purpose of extending the numerical domain of natural numbers to integers. The results obtained indicate that a relationship exists among acceptance of negative numbers, semantic forms and the context of problems. For example: the variation of a state is related to the negatives as subtrahend and relative numbers dealing with the monetary and concrete contexts. The combination of successive variations is connected through relative numbers and negative results dealing with the monetary context. The comparisons of variations is related to relative numbers in the temperature context. The variations of variations is related to subtrahends and relative numbers within the context of debts and earnings. The variations of a comparison is connected through relative numbers and the spontaneous assumptions of a state function in the monetary context. We may conlude that a diversification of problem contexts generates new semantic forms.

Bruno, A. & Martinón, A. (1997). *Clasificación Funcional y Semántica de Problemas Aditivos.* Educación Matemática, 9(1), México: Iberoamérica.

Nesher, P. & Greeno, J. (1983). Semantic categories of Word problems. In I.M.A.G. Laboratoire (Ed.), *Proceedings of the V PME* (pp. 63-68.18). France.

Vergnaud, G. (1982). A classification of cognitive tasks and operations of though involved in addition and subtraction problems. In Carpenter, T. Moser (Ed.). *Addition and Subtraction: A Cognitive Perspective* (pp 35-39), Lawrence Erlbaum Associates, Hillsdale. New Jersey.

APPROPRIATION PROCESSES OF CAS: A MULTIDIMENSIONAL STUDY WITH SECONDARY SCHOOL MATHEMATICS TEACHERS

Montserrat García-Campos and Teresa Rojano

Cinvestav

A number of studies have revealed the potential of Computer Algebra Systems (CAS) as a tool for the teaching and learning of algebra. Others indicate that only after acquiring good knowledge of algebraic rules, techniques and methods can it be possible to effectively use it. Here we report on outcomes from a study undertaken with secondary school mathematics teachers, in which we analyze teacher appropriation processes of CAS, in order for them to solve algebraic tasks, as well as the use they make of the tool in teaching algebra within the classroom setting. The theoretical perspective of instrumental genesis is adopted to analyze such processes (Artigue et al, 2001), and an algebra activity classification framework was used in the experimental materials design (Kieran, 2006). Methodology is based on a grid structured around three dimensions (*epistemological, cognitive, and didactic*) (Artigue et al, 2001). A pre–post questionnaire program was applied, with an intermediate CAS workshop for teachers that included individual interviews (stage 1). A subsequent classroom observation was carried out (stage 2). Outcomes from stage 1 show that during the interview, teachers declared that it was possible to use the calculator to teach the usual paper and pencil solving algebraic methods. Nevertheless, in the written post-questionnaire, most of them proposed the use of CAS only as a support or verification tool. This sort of usage is what they applied in the post-questionnaire section in which they were asked to solve algebraic problems. That is, most of them did not use CAS to solve the problems, but used it afterwards to verify that their answers were correct. Outcomes from stage 2, will report on the extent to which this teachers' view impacts their practice in the classroom setting when CAS is incorporated.

References

Artigue, M., Grugeon, B., Assude, T., & Lefant, A. (2001). Teaching and learning algebra: Approaching complexity trough complementary perspectives. In *Proceedings of the 12th ICMI Study Conference* (pp. 9-14). The University of Melbourne Australia.

Kieran, C. (2006). Research on the learning and teaching of school algebra at the Midle secondary, and College Levels. In Lester, F. (Ed.), *Handbook of Research on Mathematics Teaching and Learning.* NCTM.

WHAT VALUES ARE WE TEACHING IN THE MATHEMATICS CLASS ROOM? EXPLORING THE VALUES OF THE IRANIAN MATHEMATICS CURRICULUM

Soheila Gholamazad

Ministry of Education

There is a common belief that mathematics is the most value-free subject in school curriculum (Seah, W. T., Bishop, A. J., FitzSimons, G. E., Clarkson, P. C., 2001), and that mathematics in school should likewise be taught in a value neutral way (Sam, L. C., Ernest, P., 1997). In the last few decades, however, a number of mathematics educators have remarked on the important role of values in the mathematics education (Bishop, 1987; Ernest, 1991). The values teaching and learning, although, inevitably happen in all mathematics classrooms, they appear to be mostly implicit. Therefore, it happens very often that teachers have only limited understanding of what values are being taught and encouraged.

This paper reports on an ongoing research towards exploring the planned values as explicitly and implicitly documented in the Iranian school mathematics curriculum, and to compare them with teachers' understanding of their own intended and implemented values.

The data gathered through examining the k-12 Iranian mathematics curriculum. For organizing the data, we adopted and adjusted the framework that Sam and Ernest (1997) developed for categorizing the values. We, also, observed several mathematics classes at different levels; elementary school, middle school, and secondary school. The results showed the existent gap between intended and implemented values in mathematics curriculum.

References

Bishop, A.J. (1988). *Mathematical Enculturation-a Cultural Perspective on Mathematics Education.* Netherlands: Kluwer Academic Publisher.

Ernest, P. (1989). The impact of beliefs on the teaching mathematics. In P. Ernest (Ed), *Mathematics Teaching: The State of the Art*, London: The Falmer Press.

Seah, W.T., Bishop, A.J., FitzSimons, G.E., & Clarkson, P.C. (2001). *Exploring Issues of Control over Values Teaching in the Mathematics Classroom*. Paper presented at the Annual Conference of the Australian Association for Research in Education, Fremantle, Australia.

Sam, L.C. & Ernest, P. (1997). Values in mathematics education: What is planned and what is espoused? In British Society for research into Learning Mathematics (BSRLM), *Proceeding of the Day Conference* held at university of Nothingham, (pp. 37-44).

THE CONCEPT OF INFINITE SUM. A REVIEW OF TEXTBOOKS

Alejandro S. González-Martín

University of Montréal

One of the new and complex concepts that students encounter fairly early when they follow post-secondary mathematics-oriented courses, is that of numerical series (or just *series*), which can be defined as a sum of infinite terms. Although this concept is complex and contradicts intuition, it has many applications in Physics, Economics, Biology, ... Due to their "mysterious halo", series are usually reduced to their algorithmic aspects, which later produce many misconceptions in understanding the key concept of integral (Bezuidenhout & Olivier, 2000; González-Martín, 2006). Usually little emphasis is placed on the application of series or on the construction of meaning and usually students develop no visual images associated to the concept.

Despite its epistemological complexness, there are not many research results about the teaching and learning of this concept and the different research results we have found do not show any convergence in their approaches. This lack of uniformity may be one of the reasons why there is no impact in the production of textbooks.

Aiming at producing an exhaustive revision of textbooks for the last 15 years in Québec, we have at the moment analysed six texts which have been present in the programs of many post-secondary establishments in Montréal (covering a wide period of years: 1993-2004). Even if these textbooks give a relatively great space to explain content about series (more than 10% generally), the approach used seems to be "traditional" and the register used is almost exclusively the algebraic one, with very few graphical representations. Very few applications of the concept are shown and very little historic reference is used. Moreover, series are usually introduced just as a mathematical object that answers to mathematical needs, so students do not necessarily develop a vision of series out of mathematics or of their applications.

We are aware that the set of textbooks we have chosen is very small to draw general conclusions. However, this set has allowed us to see some tendencies which will better guide our further analyses. Once we have finished these analyses of a significant set of textbooks, we aim at analysing how teachers develop their practises, under the hypothesis that teachers tend to follow the approaches of the textbooks.

References

Bezuidenhout, J. & Olivier, A. (2000). Students' conceptions of the integral. In T. Nakahara & M. Koyama (Eds.), *Proceedings PME24* (Vol. 2, pp. 73-80). Hiroshima, Japan: PME.

González-Martín, A.S. (2006). *La Generalización de la Integral Definida desde las Perspectivas Numérica, Gráfica y Simbólica Utilizando Entornos Informáticos. Problemas de Enseñanza y de Aprendizaje*, Phd Thesis, U. La Laguna (Spain), Isbn: 84-7756-679-8.

THE DEVELOPMENT OF A DIGITAL ASSESSMENT OF EARLY FRACTION LEARNING

Kristy Goodwin

Macquarie University

Screen-based technologies offer a unique pedagogical medium for students' interpretation and creation of mathematical representations to externalize their mental models (Noss, Healy, & Hoyles, 1997). Hence, technological tools provide unique opportunities to assess students' fraction concepts (Clements & Sarama, 2007). However, few studies have examined the affordances of technological tools on the assessment of early mathematics learning. This paper reports the development and implementation of an Early Digital Fraction Assessment (EDFA), as part of a larger study investigating the pedagogical and representational affordances of digital tools on early fraction learning. The sample comprised 40 male students, drawn from Kindergarten and Grade 1 in one Sydney school. Eight case study students, four from each class, were identified for closer analysis. The EDFA consisted of 30 items, assessing the students' ability to describe, recognize and represent fractions and identify corresponding symbol notation. Two levels of the EDFA were administered. The assessment was completed independently on-screen and digitally recorded to capture students' actions and verbal explanations as they completed the tasks. The EDFA was advantageous as it enabled the students to solve open-ended, multimodal tasks many of which assessed concepts beyond curricula expectations. The multimodal nature of the assessment tool allowed students to create and respond to dynamic representations. Hence, the representational capacity of the EDFA allowed the students to create, manipulate and alter their on-screen depictions. As well, it enabled ease of response, particularly for those students who would have found more traditional modes of assessment difficult, given their emerging literacy and fine motor skills. The students' initial fraction representations generally reflected traditional, instructional models emphasizing the vertical partitioning of an area or region (e.g. a circle or square showing halves). Common difficulties included a limited understanding of symbol notation. The dynamic representations enabled by the EDFA elicited representations of common fractions and percentages, with apparent conceptual understanding.

References

Clements, D.H. & Sarama, J. (2007). Effects of a preschool mathematics curriculum: Summative research on the *Building Blocks* project. *Journal for Research in Mathematics Education, 38*(2), 136-162.

Noss, R., Healy, L., & Hoyles, C. (1997). The construction of mathematical meaning: Connecting the visual with the symbolic. *Educational Studies in Mathematics, 33*, 203-233.

STUDENTS' CONCEPTIONS OF TRIGONOMETRIC CONCEPTS

Zahra Gooya and Ali Akbar Rabanifard

Shahid Beheshti University

Misconceptions could be viewed as incomplete or incorrect understanding of concepts by students. These sorts of understandings are potentially capable of misleading or perplexing students in their problem solving attempts and thus, guaranty their failure in this attempt. Even further, because of the connected and intertwined nature of mathematical concepts, these misconceptions might become a serious impediment for students' future mathematical learning. Therefore, it is necessary to conduct various researches to find out the root causes of these misconceptions in order to know why they are formed and how to help students to replace them with correct conceptions.

For this reason, a study was conducted to investigate high school students' conceptions of trigonometric concepts in order to indicate some of the grade 10 students' misconceptions regarding the trigonometric concepts and then, give some suggestions for teaching theses concepts in a way that could prevent the formations of these misconceptions.

The data collected through a set of carefully chosen problems. The analysis of data revealed that high school students have various misconceptions regarding trigonometric concepts, and for this reason, they make many mistakes while solving trigonometric problems.

The study showed that students did not have correct understanding of the concept of Radian. They also considered trigonometric functions as linear functions. Further, they had difficulty estimating the sinus and cosines of angles that were not familiar to them such as 23 degree angle or else. Finally, students did not have reliable conception of trigonometric circle and were not able to use it to solve their trigonometric problems.

Based on these findings, the researchers suggests that to decrease the students' difficulties with trigonometric concepts, we need to revise our teaching and adapt intelligent learning strategies to reduce students' misconceptions regarding trigonometric concepts.

DID I COUNT ALL THE CASES?!

Zahra Gooya and Mani Rezaie

Shahid Beheshti University

Did I count all the cases? It is quite natural for students to frequently ask such question in a mathematics classroom that involves teaching and learning of combinatorics. Questions like "in how many cases, we could color the vertices of a square with two colors" could serve as a useful vehicle to discuss a specific kind of thinking that some researchers have called it "combinatorial thinking"; the kind of reasoning that mathematics teacher could be encountered with, while working with students on all sorts of problems that require some forms of "counting". Thus, researchers are interested to investigate the ways in which the combinatorial thinking is formed and developed in students at all levels from school children to university students.

To do the investigation, an ongoing study has been designed to study the students' combinatorial thinking at elementary and high school as well as first and second year university. The data for the study were collected by presenting author while he acted as mentor for elementary teachers, taking notes and observing them when they were working on counting problems. He also taught a combinatorics course at university. Therefore, he had the vast and divergent experiences in teaching and observing others teaching combinatorics. These experiences revealed that students use extremely personal strategies for counting all the cases before getting any formal instruction in this regard. In this paper, we will present some of our preliminary findings about the nature of students' combinatorial thinking and the ways in which, they were trying to make sure that they indeed had counted all the cases.

IMPROVING THE WAYS OF REASONING IN SIMILARITY IN 14 AND 15 YEARS OLD STUDENTS

Élgar Gualdrón

University of Pamplona

We would like to show some of the findings of a qualitative research, which is part of my doctoral thesis in progress, on improving the mathematical learning of students through a teaching unit designed according to the lines of Charalambos (1991) and the model of geometric reasoning of Van Hiele considering the lines of Gutiérrez and Jaime (1998). The study sample is a group of ninth-grade students (14 and15 years old) of a school in Floridablanca-Colombia. The similarity of flat figures is mostly taught in isolation from homotheties and Thales´ theorem. We believe that by establishing a direct link between similarity and homothety and the theorem of Thales, when teaching, students achieve greater understanding and acquisition of reasoning tools on the subject. Our presentation will describe the process of improving the forms of reasoning for students in the course of development in such unit. We will show excerpts from the actions of a few students with our comments.

Preliminary analyses of the data set (consisting of video tapes, worksheets for the students, some interviews and field notes) show us interesting ways of resolving certain tasks in which participants use a richer language and show a variety of ways in their reasoning.

References

Charalambos, L. (1991). Analyse et réalisation d´une expérience d´enseignement de division. l´homothétie. *Recherches en Didactique des Mathématiques*, *11*(23), 295-324.

Gutiérrez, Á. & Jaime, A. (1998). On the assessment of the Van Hiele levels of knowledge. In reasoning. *Focus on Learning Problems in Mathematics,* 20 (2&3), 27-46.

TRIGONOMETRY LEARNING: OBSTACLES

Hülya Gür
Balikesir University
School of Necatibey Education

In this study, particular types of errors and underlying misconceptions and obstacles that occur in trigonometry lessons are described. One hundred and forty high-school students participated in the study. A diagnostic test that consists of seven trigonometric questions was prepared and carried out. The students' responses to the test were analyzed and categorized. Many obstacles are related to a concept that produces a mathematical object and symbol. For example: sinex is a concept and symbol of trigonometric functions. Many misconceptions are related to process: the ability to use operations. For example: as representing the result of calculation of sin30 and value of sin30. Many misconceptions are related to procept that is, the ability to think of mathematical operations and object. Procept covers both concept and process. For example: sinex is both a function and a value. The study focused on five objectives: What are the errors committed by students in trigonometry? What is a possible categorization of these errors and obstacles? What are the misconceptions and obstacles relating to learning trigonometrically concepts? What are the possible treatments of students' errors, obstacles, and misconceptions? What are the student's answers that help us explore the students' thinking and reflection about learning? The most common errors that the students had in questions were selected. Several problematic areas have been identified such as improper use of equation, order of operations, and value and place of sinex, cosine, misused data, misinterpreted language, logically invalid inference, distorted definition, and technical mechanical errors. The results of this study found that students have some misconceptions and obstacles about trigonometry. One of the two obstacles to effective learning was that trigonometry and other concepts related to it were abstract and non-intuitive because of lecturing. The students had problems with prior and new knowledge about concept, process, and procept in learning trigonometry. The teacher has an important role to play in overcoming it. Teacher's roles are to observe the students, and if they are making mistakes and errors; s/he could discuss and correct them. Not only do students bring their experiences, obstacles, and misconceptions to class, students suggested that repeating a lesson or making it clearer will not help those students who base their reasoning on strongly held misconceptions. The other obstacle is the exam of university entry. This set of results gave an indication that students are prone to common errors even when teachers have adopted different teaching strategies for teaching trigonometry. Students do not come to the classroom as "blank box". Instead, they come with their own ideas and theories constructed from their **everyday experiences**, and they use these theories. Another suggestion of treatment is using **resources, materials, diagrams, and equipments**.

OBSERVING CHILDREN'S INDUCTIVE REASONING PROCESSES WITH VISUAL REPRESENTATIONS FOR MULTIPLICATION

Tony Harries, Priscilla Lopez, Hilary Reid, Patrick Barmby, and Jennifer Suggate

Durham University

We present this work as a short oral presentation as it is a very recently carried out piece of research. In a previous study (Barmby *et al.*, 2008), we put forward a model of understanding of mathematics based on mental representations linked together by reasoning processes. A conclusion that we put forward was that having access to a variety of representations and being able to reason both within and between them contributes to the development of understanding. Therefore, we wished to investigate ways in which we could promote children's reasoning processes as they worked with a variety of representations in multiplication, and whether this led to greater understanding of the operation. Here, our study was informed by the work of Klauer *et al*. (2002) who suggested that "inducing adequate comparison processes in learners would improve the learners' abilities in inductive reasoning". The training program that they used involved comparing visual representations of objects and highlighting similarities and differences. Christou and Papageorgiou (2007) applied these ideas to primary mathematics, confirming Klauer *et al*'s theoretical framework for inductive reasoning. We therefore carried out a classroom study with mixed ability Year 4, 5 and 6 children (ages 8 to 11) in a primary school in the North East of England. The study involved children working in pairs on laptop computers, using a Flash Macromedia program. The program asked children to compare different visual representations for multiplication calculations, identifying similar and different representations. In using this approach, we were able to record all the actions carried out by children on the computer and their discussion, using a recording program called Camtasia. The analysis of the audiovisual data obtained identified whether children were reasoning inductively with the representations and thereby identifying properties of multiplication such as distributivity and commutativity. Based on our findings, we are able to discuss whether an approach based on comparisons of visual representations can be used by teachers in the classroom.

References

Barmby, P., Harries, T., Higgins, S., & Suggate, J. (2008). The array representation and primary children's understanding and reasoning in multiplication. *Educational Studies in Mathematics* (in press).

Christou, C. & Papageorgiou, E. (2007). A framework of mathematics inductive reasoning. *Learning and Instruction*, *17*(1), 55-66.

Klauer, K.J., Willmes, K., & Phye, G.D. (2002). Inducing inductive reasoning: Does it transfer to fluid intelligence? *Contemporary Educational Psuchology*, 27, 1-25.

LEADING LEARNING: IMPLEMENTING THE QUEENSLAND MATHEMATICS SYLLABUS

Ann Heirdsfield — Queensland University of Technology
Janeen Lamb and Gayle Spry — Australian Catholic University

We report on the conduct of a two-year study of how a model of professional development (PD) supported two, year 3 teachers while implementing new content incorporated within the new mathematics syllabus. We explore what supported a professional learning community to develop a sense of agency and leadership for learning and how this was sustained two years on. The aim of the study was to develop teacher content and pedagogical content knowledge to enhance their agency when implementing the syllabus.

The data sources included researcher field notes and interviews with the teachers. The teachers talked about the importance of the collaborative PD and the provision of appropriate literature, websites and suitable materials, and the ongoing access to the researcher as a way to support their growth in content and pedagogy knowledge. They talked about their sense of ownership of the lessons they developed. They also identified preparation of lesson plans as supporting their construction of knowledge. When probed about the reflective discussion at the end of each lesson, the teachers were in agreement that this period of reflection supported their ongoing development.

For the two years following the project one teacher had continued to teach Year 3. When asked to reflect on the PD and how it was structured she commented,

Pam: It really changed my way of thinking…We worked together collaboratively. That made our lessons more successful and we were very honest with each other… Even when [researcher] wasn't there we would actually just sit there and say what does this actually mean? … All the talking helped us to get the language of mental computation to teach it…The readings and websites were good too but I tell you what was great. The concept maps!

There was evidence that the teachers began to lead their own learning. The teachers collaborated, supporting each other's growth in content and pedagogical content knowledge as well as agency. This leading of learning by the teachers themselves is a powerful opportunity to bringing about educational change that has not been realised traditionally (Frost, 2006). It is argued that, for teachers to develop a sense of agency, shared leadership must be possible and this is best reflected within a professional learning community where teachers have the capacity to influence outcomes.

References

Frost, D. (2006). The concept of 'agency' in leadership for learning. *Leading and Managing: Journal of the Australian Council for Educational Leaders, 12*(2), 19-28.

ANALYZING A MULTIPLICATION LESSON IN JAPAN USING A CALMA FRAMEWORK

Kenji Hiraoka and Kaori Yoshida-Miyauchi
Nagasaki University

Hiraoka and Yoshida-Miyauchi (2007) propose a framework for creating or analyzing Japanese lessons from the viewpoint of mathematical activities (Japanese *CALMA* framework) and illustrate it by an example of analyzing a fraction lesson. Hence, the purpose of this study is to refine the *CALMA* framework using another example.

The *CALMA* framework is mainly based on three viewpoints. The first one is a problem-solving-style lesson. Japanese mathematics lessons usually consist of the three stages: "grasping" (*tsukamu*) a problem (introduction), "solving" the problem individually and "developing (*neriageru*)" it collectively (development and turn), and "deepening (*fukameru*) and concluding" the problem (conclusion), according to a process of problem solving (Krulik, 1977; Polya, 1954/1975).

The second is three levels of mathematical richness and structures contained in contexts. Children are expected to have deepened their understanding at the end of a lesson compared the beginning. Therefore, in the *CALMA* framework different levels of mathematics are arranged within a lesson purposefully such as (1) concrete levels, (2) mathematical levels, and then (3) broader levels (cf. Treffers, 1987).

The final viewpoint is five mathematical activities arranged in one lesson: (1) *mathematizing*, (2) *formulating*, (3) *exploring and processing*, (4) *looking back and applying*, (5) *developing, creating, and appreciating*.

In this paper a multiplication lesson in the third grade is examined to refine the *CALMA* framework. The lesson was observed on January 30, 2008 in a small island in Japan. The process, in which children create how to calculate one-digit number times two-digit numbers, is exemplified according to the *CALMA* framework.

Endnote

This research was supported by the Japan Society for the Promotion of Science (Grant-in-Aid for Scientific Research 19530831, recipient: Kenji Hiraoka).

References

Hiraoka, K. & Yoshida-Miyauchi, K. (2007). A framework for creating or analyzing Japanese lessons from the viewpoint of mathematical activities: A fraction lesson. In J. Woo, H, Lew, K, Park, & D. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol.3, pp. 33-40).

Krulik, S. (1977). Problems, problem solving, and strategy games. *Mathematics Teacher, 70*(8), 649-652.

Treffers, A. (1987). *Three Dimensions: A Model of Goal and Theory Description in Mathematics Instruction□The Wiskobas Project*. Dordrecht, The Netherlands: D. Reidel Publishing Company.

VISUALIZATION IN MATHEMATICAL PROBLEM SOLVING: A CASE STUDY WITH ALLISON

Siew Yin Ho

Nanyang Technological University

This paper has been submitted for a Short Oral Communication because the focus is only on a small part of my study on visualization in mathematical problem solving. Furthermore, only one student's data is discussed here.

Renowned mathematician, Paul Hamos, commented on the importance of the ability to visualize (Halmos, 1987, p. 400). Terence Tao, a child who exhibited a formidable mathematical precociousness as reported by Clements (1984), emphasized the importance of visualization in the problem-solving activity of any individual. This short oral presentation focuses on three interviews with a student, Allison, over three years when she was in the fourth grade till the sixth grade. In each grade year, Allison was asked to solve the same set of six related verbal word problems having high degree of visuality. I will be discussing how Allison's method for solving each of the six related verbal word problems changed over the three years, and the implications for teaching. Allison was interviewed on a one-to-one setting. The interview procedure was structured such that Allison was engaged in the highest possible level of intellectual process, thus every opportunity was given for success in each word problem. The audio-recording, the artefacts (Allison's written solutions) and field notes taken during the interview were used to triangulate the data obtained.

As defined by Presmeg (1986, p. 42), a visual method of solution is one which involves visual imagery, with or without diagram, as an essential part of the method of solution, even if reasoning and algebraic methods are also employed. A nonvisual method of solution is one which involves no imagery as an essential part of the method of solution. At Grade Four, Allison solved all the problems using a visual method for each problem. Except for the first problem, the rest of the five problems were novel for her. At Grade Five, she solved the two problems using a nonvisual method as she solved similar problems before. She used a visual method for the rest of the four problems. It is noted that even though two of these four problems were no longer novel to Allison at Grade Five − she, however, used a visual method for solving as she found the problem situation more complex than the first two problems. At Grade Six, Allison 'formalized' her method of solving such problem types and solved five of the six problems using a nonvisual method.

Reference

Clements, M. A. (1984). Terence Tao. *Educational Studies in Mathematics*, *15*(3), 213-238.

Halmos, P. (1987). *I want to be a Mathematician.* NY: Springer-Verlag.

Presmeg, N.C. (1986). Visualization in high school mathematics. *For the Learning of Mathematics*, 6(3), 42-46.

A STUDY OF FIRST GRADERS' PERFORMANCES ON ONE-STEP ADDITION AND SUBTRACTION WORD PROBLEMS

Shiang-Tting Huang — Chung Cheng Elementary School

Kai-Ju Hsieh — National Taichung University

Solving mathematics word problems has been an important part of mathematics lesson for elementary school teachers. At first glance, one-step addition and subtraction word problems are easy to solve. However, it involves many different situations, which make it not as easy as it seems.

The purpose of this study was to investigate first graders' performances on addition and subtraction word problems. The main focuses were to analyse passing rate, error patterns, and causes of misconceptions, with different types of single-step problems.

The participants were 578 first graders in this study. These students were from 19 classes of nine elementary schools of central Taiwan. A total of 20 addition and subtraction word problems were given. These problems were designed based on Fuson's (1992) view, with four categories: change, equalize, combine, and compare. In addition, 26 participants were interviewed in order to understand their problem-solving strategies and mistakes.

The results of this study indicated that students did well in "combine" situations, following by change, equalize and compare situations. Furthermore, the problems with highest and lowest passing rates were the "combine" situation with whole amount unknown, "add to" type of problem (96.02%), and the "compare" situation with referent amount unknown, "fewer than" type of problem (46.37%). After comparing the results with the contents in the most commonly used mathematics textbooks, researchers found that even though the "compare" situation with referent amount unknown, "fewer than" type of problems was commonly seen in the quizzes and/or achievement tests, it has never been introduced in class.

Most common difficulty found during the interview was lack of language skills. Some first grader did not have enough language skills to fully understand word problems, and/or to represent word problems using language of mathematics. As a result First grade teachers might need to spend more time on discussing meanings of every problem and transformations among mathematical representations.

References

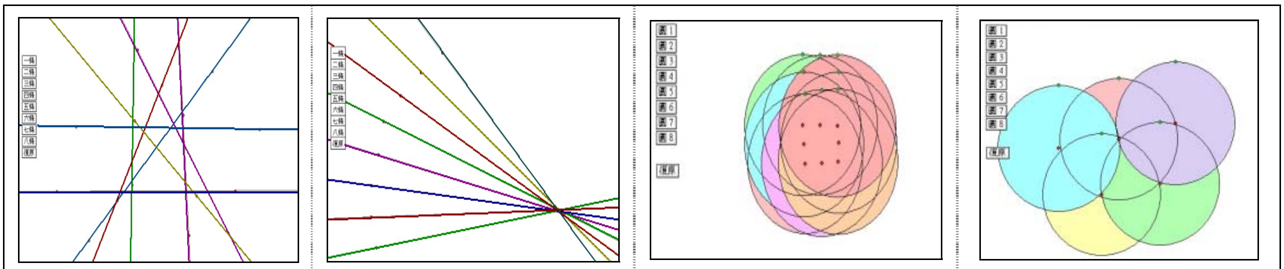
Fuson, K.C. (1992). Research on whole number addition and subtraction. In D. Grouws (Ed.), H*andbook of Research on Mathematics Teaching and Learning* (pp. 243-275). New York: Macmillan.

THE EFFECTS OF A SPATIAL REASONING SCAFFOLD SYSTEM FOR THE ELEMENTARY SCHOOL STUDENTS

Pi-Hsia Hung	Yuan Chen	Kuo-Hung Tseng
National University of Tainan	National Kaohsiung Normal University	Meiho Inssitute of Technology

This study utilizes quasi experimental design to investigate the effects of a spatial reasoning scaffold system for the 6th graders. The growth slopes of spatial reasoning and creative geometric designs are analyzed. Totally, 534 students were included in this research. Three hundred and forty-two students were in the experimental group. Thirty of them are the gifted students. There are 192 students in the control group. A three-wave dynamic assessment design is adopted. The Hierarchical Linear Model (HLM) is applied to analyze the differences of growth slopes and intercepts among three groups. The result shows that the slopes on spatial reasoning ability of general and gifted experimental group are both higher than control group. The variance of slope accounted by groups is 70%. Furthermore, the variance of slope accounted by district is 14%.

The e-learning scaffold system also documents the frequently encountered myth conceptions (shown in the figure included below) the students demonstrated in their learning processes. The teaching implications of these myth conceptions are discussed. The learning system provides students an interactive exploration environment and on-line peer supportive examples. The results suggest abstract geometric concepts can be visualized and internalized at an earlier age, if mind-tool can be effectively implemented.



Crossed Design	Radiation Design	Spiral Design	Concurrent Design
22.4%	25.0 %	52.9 %	36.5 %

Figure 1. Typical myth conceptions in the spatial reasoning scaffold system.

ENGAGING WITH POST-COMPULSORY MATHEMATICS

Paola Iannone and Elena Nardi — University of East Anglia
Liz Bills — University of Oxford

In recent years many studies have focused on why students decide ***not*** to undertake advanced studies in mathematics (Brown, Brown and Bibby, 2008). Here we report on a small-scale project which aimed at investigating reasons why students ***do*** decide to engage with mathematics after the age of 16, investigating reasons for engagement rather than disengagement from the subject. We issued a questionnaire to 120 (96 returns) students in the East of England who were already in their first year of A-level (age 17) trying to ascertain reasons why they had chosen to study mathematics. We found the students to be high achievers at GCSE level (end of the compulsory school in the UK): 86 out of the 96 students achieved top marks in their GCSE examination confirming the hypothesis that mathematics is perceived to be a '*special subject*" which can only be studied at advanced level by the top achievers. This perception emerged as particularly strong among female students. From our analysis we constructed four student types. **Student types 1 and 2** (male and female) are taking mathematics with two other sciences at A-level, are planning to study mathematics at university; they put the emphasis on enjoyment and challenge among the reasons why they have chosen this subject. **Student type 3** is male, is taking a science and a humanity with mathematics and he puts the emphasis on challenge and difficulty as the main reasons for choosing this subject. **Student type 4** is also male, is taking two more sciences for A-levels and he emphasises utility for his choice of subject. Contrary to previous reporting in the literature utility (in terms of increasing employability) featured very low in the students' responses. The findings helped us in designing an outreach day where we invited students from local schools to attend lectures, group work sessions and a panel discussion. Analysis of the student evaluation forms revealed that, amongst the reasons for being apprehensive about studying mathematics at university level, was a strong feeling of insecurity about what studying mathematics at university might 'really be'.

References

Bills L., Cooker M., Huggins R., Iannone P., & Nardi E. (2006). *Promoting Mathematics as a Field of Study* (A UEA Teaching Fellowship Report). ISBN 978-0-9539983-9-8.

Brown M., Brown P., & Bibby T. (2008). "I would rather die": Reasons given by 16-year-olds for not continuing their study of mathematics. *Research in Mathematics Education,* 10(1), 3-18.

ZEN AND ART OF "NERIAGE": FACILITATING CONSENSUS BUILDING IN MATHEMATICAL INQUIRY LESSONS

Noriyuki Inoue

University of San Diego

One danger of integrating inquiry activities into mathematics lessons is that it can easily fall into an "everyone is right" direction, where different strategies are accepted without in-depth discussions on the cogency and efficiency of different approaches and perspectives. To overcome this, Japanese elementary school teachers typically go through a series of trainings on how to help the students examine and co-determine the best mathematical strategy (*neriage* in Japanese). In this *neriage* stage, the teachers encourage students to carefully listen to other students' ideas and discuss the strengths and weaknesses of different problem solving strategies. In this process, the teachers rarely make authoritarian judgments. Rather, the teachers serve as the facilitator of consensus building where the students make judgments on the cogency and efficiency of different strategies without limiting their perspective to the problem solving strategy that they used or simply determining whether their answers are right or wrong. Based on the assumption that this "*neriage*" can be useful in other cultural contexts, a video-based lesson study project was conducted to investigate how US teachers could effectively incorporate consensus building discussions in their mathematical inquiry lessons. Japanese teachers from a Japanese Saturday school (*hosyuko*) served as their advisors for the US teachers.

In the lesson study, a group of U.S. teachers incorporated consensus building discussions in their open-ended inquiry lessons, watched the videos of their lessons and discussed how to better facilitate deep conceptual discussions in the consensus building stage of their lessons. Through the lesson study, the US teachers learned the importance of releasing control to their students so that they could openly discuss and evaluate the strengths and weaknesses of different strategies from multiple angles. Based on this project, this presentation introduces various key points for implementing consensus building (*neriage*) in mathematical inquiry lessons.

References

Shimizu, Y. (1999). Aspects of mathematics teacher education in Japan: Focusing on teachers' roles. *Journal of Mathematics Teacher Education, 2*, 107-116.

INFLUENCE OF DGS ON PLANE GEOMETRY PROBLEM SOLVING STRATEGIES

Núria Iranzo and Josep Maria Fortuny

Universitat Autònoma of Barcelona

This study is part of an ongoing research[1] on the interpretation of students' behaviors when solving plane analytical geometry problems by analyzing relationships among DGS use, paper-and-pencil work and geometrical thinking. Our theoretical framework is based on Rabardel's (2001) instrumental approach to tool use. We seek for relationships between students' thinking and their use of techniques by exploring the influence of certain techniques on the students' resolution strategies.

OVERVIEW OF THE PILOT RESEARCH

The pilot research has been carried out with 11 secondary students that have worked on geometry focusing on a Euclidean approach and problem solving. For the analysis we mainly consider: a) the solving strategies in the written protocols and the GGB[2] files; b) the audio and video-taped interactions with other students; and c) the opinions about the use of GGB collected in a questionnaire. Through the analysis of data we characterize students' learning behaviors and discuss the idea of instrumentation linking the theoretical perspective and the classroom experiments. So far, we have identified different resolution strategies in the GGB environment. We have classified the students into types, considering: 1) their heuristic strategies (related to geometric properties, to the use of measure tools or to both); 2) the influence of GGB (visualization, geometrical concepts); and 3) the obstacles encountered. We still need to better understand the appropriation processes of the software. We also need to better explore the co-emergence of machine and paper-and-pencil techniques in order to promote argumentation abilities in secondary school geometry.

References

Rabardel, P. (2001). Instrument mediated activity in situations. In A. Blandford, J. Vanderdinckt, & P. Gray (Eds.), *People and Computers, Interactions without Frontiers* (pp. 17-30). Berlin, Germany: Springer.

[1] MEC. Development of an e-learning tutorial system to enhance students' solving problem competence. SEJ2005-02535. The 'ongoing research' condition makes us choose the form of a short oral communication.

[2] Geogebra environment www.geogebra.org

HANS FREUDENTHAL'S EXISTENTIAL VIEW OF HUMAN CONDITION AS A BACKGROUND OF HIS DIDACTICS

Shinya Itoh

University of Tsukuba

Most clarifications of Freudenthal's didactics of mathematics have been based on his view of mathematics. The purpose of this paper is to point out Freudenthal's view of human condition as another source of his didactics and to explain the presence of existential themes in Freudenthal's didactics. For that purpose, I extract a statement on the human condition from Freudenthal (1973, 1978) and indicate how some features of his didactics assume that condition.

Freudenthal recognized the necessity of learning physical and mental activities utilizing freedom of choice as a human condition. For example, Freudenthal (1973) argued: "It *is* true, however, that man must learn numerous physical and mental activities which other creatures are gifted with by instinct." In addition, he considered "freedom of choice" as a "characteristically human situation". Moreover, Freudenthal (1978) referred also to "responsibility": "Freedom of choice is freedom for responsibility. Accepting and bearing responsibility must start in a small way." This is an existential theme, dependent upon "the individual's freedom to choose and the responsibility that accompanies that freedom" (Noddings, 1998).

Freudenthal described the freedom of choice of definitions of mathematical concepts in mathematics learning based on the didactic principle of "re-invention". Freudenthal (1973) asked, for example, "How would a student proceed if he is allowed to re-invent geometry?" and discussed different definitions of parallelogram as follows: "Maybe different students choose different fundamental properties. … He [the student] has learned the act of defining rather than having some definition imposed upon him."

Besides "re-invention", "essential features of mathematics education research and practice" proposed by Freudenthal include "mediating mental objects towards concept attainment instead of abstract terminology" and "mediating mathematical structures instead of transmitting the 'structure of mathematics'" (Keitel, 2005). These essential features, as Freudenthal's didactics imply, demand freedom of choice of definitions of concepts and structures, and reject their "cramming" in pre-specified forms.

References

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht: D. Reidel.

Freudenthal, H. (1978). *Weeding and sowing*. Dordrecht: D. Reidel.

Keitel. C. (2005). Encounters with Hans Freudenthal. In H. ter Heege, T. Goris, R. Keijzer, & L. Wesker (Eds.), *Freudenthal 100* (pp. 19-22). Utrecht: Freudenthal Instituut.

Noddings, N. (1998). *Philosophy of Education*. Boulder: Westview Press.

CHILDREN'S INFORMAL KNOWLEDGE OF MULTIPLICATION

Hyungog Jeon and Kyunghwa Lee

Korea National University of Education

Mathematics educators are all too familiar with situations in which students solve problems with informal knowledge and strategies before receiving the formal education. In this study, we have investigated the informal knowledge exhibited by a child during the process of solving a natural number multiplication problem.

According to prior studies (e.g. Carpenter et al., 1999; Kim, 2002), children develop understanding of mathematical concepts from their own experiences and informal knowledge. Furthermore, informal knowledge plays an important role in development of children's mathematical power, which is the application of mathematical knowledge the ability to new or unfamiliar problems. A number of researches indicate that children develop additive reasoning naturally but multiplication is much more complex than addition. Hence, this study investigated the informal knowledge that emerged during the problem solving process of children.

A clinical interview based on problems occurring in the day-to-day lives of children which involved two-digit multiplication was conducted on a third grade student. The interview was repeated four times. As the child solved each problem unaided, the child was requested to explain the problem solving process and the reasons underlying the process.

Findings from these interviews clarify the acquired informal knowledge of the child as exhibited in the process of problem solving to prior knowledge about numbers, various calculation strategies, and operation sense. Particularly the rule on special numbers (0, 1, 10) made calculations more efficient than other prior knowledge about numbers. Among the various calculation strategies employed by the child, it was the modelling-strategy that allowed the child to overcome difficulties when facing situations in which the child did not know the formal calculation method.

Reference

Carpenter et al. (1999). *Children's Mathematics: Cognitively Guided Instruction*. Reston, VA: The National Council of Teachers of Mathematics, Inc.

Kim, J. (2002). *Development of Instructional Unites Connecting Informal and Formal Mathematical Knowledge of Equivalency and Addition*. Doctoral dissertation, Columbia University.

SPEAKING LIKE A SCIENTIST: STUDENT DISCOURSE IN THE MATHEMATICS CLASSROOM AS AN INDICATOR OF AUTHENTIC ACTIVITY

Christine Johnson, Janet G. Walter, and Hope Gerson

Brigham Young University

The situated cognition view of learning suggests that true learning is best exemplified and promoted in the context of "authentic activity" (Brown, Collins, & Duguid, 1989). Brown et al. describe authentic activity as "the ordinary practices of the culture," in which participants act on situations to solve emergent problems and dilemmas (p. 34). We suggest that the comparison of discourse practices of learners of mathematics with the discourse practices of those who use mathematics within other cultural practices may serve as an additional criterion for evaluating the authenticity of the activity of mathematics learning in the schools

In a university honors calculus class, students collaborated to describe the volume of water in a reservoir based on a given graph of the inflow and outflow of water. We used grounded theory methodology to conduct a constant comparative analysis of video, transcript, and original student work. We coded transcript and segments of video to identify and delineate categories of language use and events in mathematical discourse. As subcategories developed within individual codes we then identified relationships between different subcategories. Once we had developed interpretations for linguistic phenomena based on our system of codes, a colleague noticed specific similarities between our results and those of Ochs, Gonzales, and Jacoby (1996).

In their paper describing discourse practices of members of a physics research group, Ochs et al. (1996) found that when using graphical representations to facilitate reasoning about physical phenomena, the physicists would incorporate the conventions of the graphical representations into their gestures. Ochs et al. also noticed a pattern of personal pronominal subjects combined with predicates of motion or change of state, as in the utterance, "When I come down I'm in the domain state" (p. 331). In this presentation, we provide examples of similar results from our current work with learners of mathematics. We suggest that parallels between the discourse of mathematics learners and the authentic discourse of practitioners may be viewed as indicators of authentic mathematical activity.

References

Brown, J.S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher, 18,* 32-41.

Ochs, E., Gonzales, P., & Jacoby, S. (1996). "When I come down I'm in the domain state": grammar and graphic representation in the interpretive activity of physicists. In E. Ochs, E. Schegloff, & S. Thompson (Eds.), *Interaction and Grammar* (pp. 328-369). Cambridge: Cambridge University Press.

CONCEPTIONS ABOUT THE NOTION OF FUNCTION AND THE ROLE OF THE MODE OF ITS REPRESENTATION

Maria Kaldrimidou and Manolis Moroglou — University of Ioannina
Marianna Tzekaki — Aristotle Univ. of Thessaloniki

Research on conceptions about the notion of function is an important issue for mathematics education. Pupils and students have difficulties in conceptualizing the notion of function. The epistemological complexity of the concept (Sierpinska, 1992) and the diversity of the representations used (Hitt, 1998) are the two main factors that influence the understanding and learning of functions. Previous research on students' conceptions in the case of graphical representation of functions reveals three different approaches to conceive a function: the geometrical, the algebraic and the functional (Kaldrimidou & Ikonomou, 1998). In the same context, students used three different procedures to draw a graph: the point-by-point, the step-by-step and the holistic procedure.

In the present study we try to extend this previous research when functions are represented algebraically and numerically. The main research questions were as follows: a) Does the way students conceive a function depends on its representation? and b) Are the procedures used by the students related to their conceptions about the notion of function?

To this purpose a test was administrated to 190 students (17-years old). The test consisted of six tasks. In each task students were asked to give as much information as possible about the involved function. Functions were represented numerically, algebraically and graphically (2 tasks for each). Students' answers were analysed according to the procedure they used and the way they conceived the function involved. The main results of the analysis were: a) Students' conceptions depend on the function's mode of representation, b) In the case of the graphical representation of a function, conceptions and procedures used are related and c) When a function is represented numerically or algebraically, conceptions and procedures are not related.

References

Hitt, F. (1998). Difficulties in the articulation of different representations linked to the concept of function. *The Journal of Mathematical Behavior*, 17(1), 123-134.

Kaldrimidou, M. & Ikonomou, A. (1998). Epistemological and Metacognitive Factors Involved in the Learning of Mathematics: The Case of Graphic Representations of Functions. In H. Steinbring, M. Bartolini-Bussi, & A. Sierpinska (Eds.), *Language and Communication in the Mathematics Classroom* (pp. 271-288). Reston, Virginia: NCTM.

Sierpinska, A. (1992). On understanding the notion of function. In G. Harel & E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy* (pp. 25-58). U.S.A: MAA.

TEACHERS' ADAPTATIONS OF MATHEMATICS CURRICULUM AND STUDENTS' LEARNING OPPORTUNITIES

Gooyeon Kim

University of Missouri-St

I attempt to explore the relationships of mathematics teachers' adaptations of a standards-based mathematics curriculum material and opportunities for students' mathematical learning. Curriculum materials are viewed as a key vehicle in mathematics education. Teachers, in general, are required to follow the curriculum materials. Recent research studies on the relationships between curriculum materials and teachers reveal that teachers' learning occurs through the use of a standards-based elementary curriculum material (Collopy, 2003), Moreover, there are variations in standards-based curriculum materials in terms of teacher learning demand and support for teacher learning (Stein & Kim, in press). In addition, Kim (2007) suggests that teachers use standards-based mathematics curriculum materials differently and in particular, teachers who use *Everyday Mathematics* show a tendency to adapt the curriculum material. For further examination, in this study, I investigate how teachers' adaptations of *Everyday Mathematics* (UCSMP, 2004) influence students' mathematical learning opportunities.

Data were collected in two urban public schools in the US using classroom observations and pre- and post-observation interviews. Twelve lessons were observed with 6 teachers. The data were analyzed by using a qualitative research method. The preliminary findings suggest that the adaptations that the teachers made from the curriculum result in decrease students' learning opportunities for conceptual understanding in learning place value, fractions, and equivalent fractions. This presentation describes the teacher adaptations, students' learning opportunities, and implications.

References

Collopy, R. (2003). Curriculum materials as a professional development tool: How a mathematics textbook affected two teachers' learning. *Elementary School Journal, 103*, 287-311.

Kim, G. (2007). Teachers' implementation of standards-based mathematics curriculum. In J. H. Woo, C. H. Lew, K.S. Park, & D.Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 1, pp. 240). Seoul, South Korea: PME.

Stein, M.K. & Kim, G. (in press). The role of mathematics curriculum materials in large-scale reform. In J. Remillard, G. Lloyd, & B. Herbel-Eisenmann (Eds.), *Teachers' Use of Mathematics Curriculum Materials: Research Perspectives on Relationships between Teachers and Curriculum*.

GENDER IN MATHEMATICS RELATIONSHIP: COUNSELING UNDERPREPARED STUDENTS

Jillian M. Knowles

Endicott College

I piloted a relational counseling approach to mathematics tutoring with 10 volunteers from an introductory statistics class of 13 students, at a small urban state university in Northeastern U.S.A. I saw that affect such as anxiety was symptomatic of deeper relational issues. Extending McLeod's (1992, 1997) interest in classical Freudian approaches in extreme cases, I saw the promise of psychoanalysis's attention to the unconscious and the present effects of the past on *everyone* not just the extreme. As a framework, I used relational theory developed by Mitchell (1988) who integrated three major relational offshoots from Freudian psychoanalysis: self psychology (cf. Kohut, 1977), object relations, and interpersonal psychology. A student's mathematics self is expressed in current patterns of relationship and behavior designed to preserve it.

I found students' levels of mathematics preparation (in relation to college course demands) interacted with sense of mathematics self to yield three categories. Category III students, in focus here, were underprepared mathematically with underdeveloped mathematics selves (Knowles, 2004). Gender affected how I and these students related. Category III students, Karen and Mulder, scored a low Level 2 (of 4) on the *Algebra Test* (Sokolowski, 1997) and had poor number and operation sense. Their math metaphors, transference of past teacher relationships, and my countertransference reactions, showed that both had been mathematics gender-stereotyped by early teachers. Karen was depressed: math was "cloudy," "my worst subject…always…hard for me to understand" even since 1st grade; teachers had expected little of her *because* she was female. I resisted agreeing, and expected more. Mulder "knew" he was capable because of smart male relatives but he developed a "smart but lazy" persona to account for his poor math performance. Teachers treated him as a bright male who put in little effort. I initially agreed, but I had to recognize his vulnerable mathematics self, and support him as he found his own way. We made changes and they did well. All Category III students in the study had similar gender-stereotyped defences of underdeveloped mathematics selves, men: grandiose overconfidence, women: depressed or anxious underconfidence.

References

Knowles, J.M. (2004). Brief relational mathematics counseling as an approach to mathematics academic support of students taking introductory courses (Doctoral Dissertation, Lesley University, Cambridge, MA, 2004). *Dissertation Abstracts International*, UMI Microform 3134513

Mitchell, S. A. (1988). *Relational Concepts in Psychoanalysis: An integration*. Cambridge, MA: Harvard University Press.

WHAT DOES A FIRST GRADER LEARN IN SCHOOL MATHEMATICS?

Misun Kwon — Seryu Elem. School

JeongSuk Pang — Korea National University of Education

The main goals of the mathematics curriculum are to enable students to do mathematics for themselves and to enjoy it. A student's first experience of mathematics at school is very important for subsequent learning and the creation of a positive mathematical disposition (NCTM, 2000). We will investigate the implications of teaching and learning by comparing a first grader with a pre-schooler who has not learned mathematics at school. We will also study the influences of school mathematics on students' ways of thinking.

We interviewed two six year olds 12 times throughout a year. The interviews were videotaped and then transcribed. In the early stages of the interviews the two children had similar test results. The content of the interviews was taken from a standard grade one mathematics text book.

The first grader employed higher level strategies and more number facts than the pre-schooler did. The first grader was concerned about the accuracy of both his answers and the structure of mathematical expressions, such as the equal sign. He provided the correct answer for a typical task, but often made mistake on non-standard questions. For example, when asked to find out patterns in the 100 chart, he easily identified two typical patterns: increasing by units of one and ten. However, he could not identify the "decreasing" pattern when the numbers were displayed from 100 to 10.

The pre-schooler employed lower level strategies and spent a longer time solving the given problems than the first grader did. She also had difficulty using mathematical language. However, she had a lot of informal knowledge and seemed to have more varied solution methods than the first grader did. For instance, when asked to find patterns in the 100 chart she identified both the increasing and decreasing patterns.

The results showed school mathematics can supply students with effective strategies and sophisticated mathematical techniques. However, this does not automatically foster students' own diverse and creative thinking abilities. This study implies that students' own ways of thinking and representation of mathematics should be encouraged along side traditional approaches and algorithm from the early stages of mathematics education.

References

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author.

UNITIZING APPROACH TO DIVISION OF FRACTIONS

Teruni Lamberg

University of Nevada

This study investigates how teachers visualized and solved division of fractions problem using Lamon's (1999) area model of fractions. The findings indicate that teachers had difficulty visualizing certain aspects of the area model when the unit changed during the problem solving process. A unitizing approach for visualizing and understanding division of fractions is presented.

Students and even adult learners struggle with dividing fractions. Even though "invert and multiply" algorithm is commonly taught, students struggle understanding how and why this method works. Therefore, students need to develop mental images of multiplying and dividing fractions so that can understand how and why these procedures work (Cramer, Wyberg & Leavitt, 2008). Lamon (1999) provides a division of fraction area model for teaching fractions. Cramer et al. points out that very little research exists in the area of multiplying and dividing fractions. Therefore, this study investigated how teachers visualized and made sense of division of fractions using Lamon (1999) model. Thirty teachers participated in a summer institute on fractions in a Western State in the U.S. The teachers were asked to solve problems involving division of fraction problems. Data collected included video recordings of the discussions, individual and collective written records and field notes. The data was coded and analysed to examine how teachers visualized division of fraction problems. The analysis revealed that teachers had difficulty visualizing the unit when it was re-unitized during the problem solving process. A new approach to visualizing division of fractions through unitizing emerged. When teachers applied this new approach, they were able to successfully visualize and solve additional problems involving division of fraction. Unitizing as a means of visualizing division of fractions is not mentioned in the research literature. Further research on this approach is needed.

References

Cramer. K, Wyberg, T. Leavitt, S. (2007). Students making sense of models for fraction multiplication and division. In Lamberg, T. & Wiest, L.R. (Eds.), *Proceedings of the 29th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (pp.21-24), Lake Tahoe, NV: University of Nevada, Reno.

Lamon, S. (1999). *Teaching Fractions and Ratios for Understanding: Essential Content Knowledge and Instructional Strategies for Teachers*. Mahwah, NJ: Lawrence Erlbaum Associates.

CHANGE IN PRESERVICE TEACHERS' UNDERSTANDING ON DIVISION WITH ZERO

Kyong-Hee Melody Lee

University of Southern Indiana

Numerous research studies stated that preservice teachers do not have sufficient knowledge about zero and dividing by zero. It is recommended that more efforts are needed to better equip teachers to deal with zero (Wheeler 1983). Preservice teachers' understanding of division by zero is more by memorization rather than conceptualization (Ball 1990), and the problems in dividing by zero are still far from being outdated (Crespo, 2006). Teachers tend to teach the way they were taught. Then, how should this be taught in the elementary mathematics content course? How can mathematics teacher educators assist preservice teachers in improving their conceptual understanding of division with zero? This paper will explore preservice teachers' understanding and change in their understanding on division with zero.

The participants in this study were enrolled in one of the introductory mathematical concepts for teachers course at a state university. Before the research topic was covered, students took the pretest. After the pretest, the class was divided into eight small groups. Two different learning activities were implemented: making story problems approach and relating multiplication with division approach. Three and ½ weeks later, the students took the unit test that was similar to the pretest. At the end of the semester, students will take the final test that will include questions similar to the pretest and the unit test. All of these tests will be analyzed in this study.

References

Ball, D.L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education, 21,* 132-144.

Nicol, C. (2006). Challenging preservice teachers' mathematical understanding: The case of division by zero. *School Science and Mathematics, 102*(2), 84-97.

Wheeler, M.M. (1983). Much ado about nothing: Preservice elementary school teachers' concept of zero. *Journal for Research in Mathematics Education*, *14*, 147-155.

USING TASKS AND PROJECT WORK TO FOSTER MATHEMATICAL LEARNING: EXEMPLARS FROM THE HONG KONG ELEMENTARY CLASSROOM

King Man Leung

University of East Anglia

In many mathematics classrooms today, students still experience mathematical learning as the acquisition of a set of predetermined procedures and skills. Many teachers still perceive their job as transmitting the content of mathematics by demonstrating correct procedures and making sure students practice the skill of using these procedures (Goos, 2004). When students are able to provide a correct answer to a question posed by the teacher, both teacher and students often appear to feel satisfied and students are left with the impression that they have acquired a kind of mathematical thinking that is valued by society. This is one way in which traditional teaching practice is perpetuated.

In Hong Kong, mathematics curriculum reform has aimed to propel teachers towards a paradigm shift from a largely textbook-based, teacher-centred approach to a more interactive and learner-centred approach. The purpose of the study reported here is to examine HK elementary teachers' use of mathematical tasks at this time of reform (Stein & Smith, 1998), particularly in the context of NCTM's (2000) recommendations for integrating inquiry activities into the mathematics curriculum. This doctoral study, now nearing completion, used a case study approach and its data consists of teacher and student interviews, classroom observations, discussions on lesson planning and students' written work. Its focus is on the teachers' experience and reflection as they move towards a student-centred approach, particularly with regard to whether and how the tasks they used, developed students': (i) mathematical skills and understanding, (ii) application of mathematical knowledge in real-life situations, (iii) thinking abilities and positive attitude towards mathematics and (iv) active participation in the lessons. The study makes a strong case for how increasing and enhancing classroom interaction through task-based teaching and project-based learning can help to foster students' mathematical skills and understanding as well as improve attitudes towards the subject.

References

Goos, M. (2004). Learning mathematics in a classroom community of inquiry. *Journal for Research in Mathematics Education*, 35(4), 258-291.

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author.

Stein, M.K. & Smith, M.S. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School*, 3(4), 268-275.

ELEMENTARY TEACHERS' KNOWLEDGE IN MATHEMATICS AND PEDAGOGY FOR TEACHING

Yeping Li and Rongjin Huang
Texas A&M University

Caibin Tang
Hangzhou Education Research Center on Elementary Mathematics

Efforts to facilitate teachers' learning of mathematics for teaching have led to the increased emphasis not only on pre-service teachers' mathematics preparation, but also on in-service teachers' learning through teaching practices. Yet, much remains to be understood about the extent of changes teaching experience and professional learning may contribute to in-service teachers' knowledge growth in mathematics for teaching. As part of a large research study on teachers' knowledge development in mathematics and pedagogy, this paper focuses on a group of Chinese in-service elementary school teachers' knowledge in mathematics and pedagogy for teaching in general, and their knowledge needed for teaching fraction division in particular.

A total of 18 in-service elementary teachers from two different elementary schools in a south-eastern city in Mainland China participated in this study. These two elementary schools were selected to reflect the average quality level in that city. These teacher participants completed the same survey instrument developed in a previous study (Li & Kulm, submitted). The instrument contains two components, with the first as a survey of teachers' beliefs and perceptions in mathematics and pedagogy and the second as a mathematics test that focused on teachers' mathematics knowledge and knowledge needed in teaching fraction division.

The results revealed a gap between these teachers' limited knowledge about the curriculum they teach and their solid knowledge in mathematics and pedagogy for teaching fraction division. In particular, the results from the survey indicated that (1) sampled participating in-service teachers did not show a high confidence in their knowledge about Chinese national curriculum and some specific content topics; (2) some teachers (about 85%) were (very) confident in their knowledge needed for teaching elementary mathematics, while others were not. At the same time, this group's performance on the mathematics test revealed that these teachers have solid knowledge and skills needed for solving typical school mathematics problems, including computations and word problems. Most teachers were also good at explaining how a division-of-fraction procedure works as it was to happen in a classroom. Some of them could even provide two or three different justifications for the same question. The findings suggest that Chinese teachers benefit from teaching practice for the improvement of their mathematics knowledge for teaching but not their knowledge about mathematics curriculum.

Li, Y. & Kulm, G. (submitted). Knowledge in mathematics and pedagogy for teaching. Manuscript submitted to *ZDM-The International Journal on Mathematics Education.*

MENTOR PREPARATION IN SUPPORT OF FUTURE TEACHERS' LEARNING TO TEACH MATHEMATICS

Pi-Jen Lin

National Hsinchu University of Education

The study was to develop a mentoring program with a school-university partnership for enhancing mentors' knowledge. Four pairs of mentor-intern from a school participated in the study. The course with 78 hours to develop mentors' theoretical and professional knowledge was carried out in a half-year internship. The data included pre- and post-test of pedagogy, self-assessment of mentoring, interview, classroom observation. The transfers of mentoring lesson plan and problem posing were two indicators of the effect of the mentoring program.

Mentors need highly complex knowledge and skills beyond in-service teachers required. To develop mentors' such competence, mentor preparation programs take more considerations into account than in-service teacher development program. Thus, there is a need to devote a mentoring program to help mentors become equipped to mentoring future teachers (Wang & Odell, 2002).

The mentoring program with a school-university partnership takes the assumption of the collaborative inquiry model that knowledge and skills are constructed through practice-centered conversation and collaborative inquiry with a group of mentors in the contexts mentoring. Develop mentors' theoretical and professional knowledge integrated into the course containing 78 hours were implemented in 36 hours summer workshop and school-year initiates with 42 hours. In support of learner-oriented teaching and mentoring for the mentors, the activities of mentoring were developed. The process of mentoring consisted of four phases. Classroom observation and lesson plan were the data for measuring how mentors transferred their knowledge and skill into mentoring practices. Each mentor was also conducted individually with a semi-structure interview.

The mentors learned about seven aspects of problems posing and brought them into mentoring practices. The result suggests that the mentors' development with the support of others is more successful than self-development. The mentoring program offered the support by providing individual mentor or FT with others who could give feedback, question, discuss, and challenge. Through the cooperation of university with school, the mentors brought their mentoring practices and the teacher educator created more opportunities of the justification between theory and practice of mentoring.

References

Wang, J. & Odell, S.J. (2002). Mentored learning to teach according to standards-based reform: A critical review. *Review of Educational Research, 72*(3), 481-546.

IS MATHEMATICS LITERACY A BETTER CRITERION THAN MATHEMATICS ACHIEVEMENT FOR THE EQUITY ISSUE INVESTIGATION?

Su-Wei Lin
National Hualien University of Education

Pi-Hsia Hung
National University of Tainan

Equity in education is more than an issue of fairness and distributive justice. Educational inequality and its many consequences are almost never completely random. They usually affect some groups more than others. Indexes of socio-economic status (SES) are widely used in school because of the known relationship of low educational participation and achievement by socio-economic disadvantaged groups (Cabrera & La Nasa, 2000). The criterion variable is also an important element for the equity investigation design. The OECD thematic review of equity in education is primarily concerned with equality of opportunity while recognizing that relative equality of outcomes is often used as an indicator of equality of opportunity. The purpose of this paper is to compare the variances accounted by study programs and SES variables on senior high school student mathematics literacy and mathematics achievement. The sample included is drawn from the Programme for International Student Assessment (PISA 2006). There were 8815 students in the PISA 2006 mathematics assessment. Among these participants, 4078 students also accepted the Taiwan Assessment of Student Achievement on Mathematics (TASA). The purpose of TASA is to investigate the student mathematics achievement. The PISA assessment focuses on real-world problems, moving beyond the kinds of situations and problems typically encountered in school classrooms. The indicators of SES in PISA student questionnaire were all included to construct the SES variable.

The correlation coefficient between mathematics literacy and achievement is around 0.6. Among the SES indicators, book possessed, father's education and classic literature exposure are relatively stronger predictors. The variances accounted by study program are 17% of TASA and 14% of PISA. The variances accounted by SES are 11% of TASA and 14% of PISA. In other words, TASA is more curriculum sensitive and PISA is more SES sensitive. The results suggest that for the equity issue investigation, functional literacy assessment design might be more productive than the conventional achievement one.

A COMPARATIVE STUDY OF PRE-UNIVERSITY MATHEMATICS TEACHERS' USE OF GEOGEBRA IN TAIWAN AND ENGLAND

Yu-Wen Allison Lu

University of Cambridge

In all mathematics curricula, algebra and geometry are two core strands (Atiyah 2001). It is therefore not surprising that Information and Communication Technology (ICT) has specifically targeted these two strands (Sangwin 2007). The most widely used computer applications for the teaching of geometry are Dynamic Geometry Software (DGS), which offers the drag mode and allows use of geometrical images. Computer Algebra System (CAS) programmes are often utilised in the teaching of algebra. Historically, CAS mainly provides algebraic and numerical computations while DGS provide graphical demonstrations. In recent years, a desire of the need to integrate CAS and DGS has become apparent and the recently published software GeoGebra by Hohenwater (2004) explicitly links the two. It provides a bidirectional combination and a closer connection between the visualisation capabilities of CAS and the dynamic changeability of DGS (Hohenwarter and Jones 2007).

This project aims to investigate how innovative mathematical software can support, enhance or even transform mathematics teaching and learning. In particular, GeoGebra has been chosen as the focus of the research not only because it is open source with much freely available support material but also because of its unique capacity to integrate geometry and algebra. This study is a comparative exploration of the use of GeoGebra in Taiwan and England and allows an understanding of how different cultural traditions not only conceptualise mathematics teaching but also how teachers integrate this software in their teaching of mathematics in pre-university courses. Although there is growing evidence of its being used extensively around the globe, systematic enquiry into the classroom use and effectiveness of GeoGebra is limited. Consequently, this study is one of the first rigorous examinations of this potentially liberating software. It offers not only important insights into its uses but also facilitates its transformative potential in mathematics teaching in England, Taiwan and beyond.

References

Atiyah, M. (2001). Mathematics in the 20th Century: Geometry versus algebra, *Mathematics Today*, 37(2), 46-53.

Sangwin, C. (2007). A brief review of GeoGebra: Dynamic mathematics, *MSOR Connections*, 7(2), 36-38.

Hohenwarter, M. & Jones, K. (2007). BSRLM geometry working group: Ways of linking geometry and algebra: the case of GeoGebra. In D. Küchemann (Ed.), *Proceedings of the British Society for Research into Learning Mathematics*, *27*(3), 126-131.

PUPILS' DIFFICULTIES IN UNDERSTANDING THE CONCEPT OF WEIGHT

Yuki Masuda
University of Tsukuba

The concept of weight is one of the key topics in 3rd grade in the area of quantity and measurement in mathematics curriculum of Japan. However, pupils' difficulties in understanding the concept of weight are reported in previous studies. The result of TIMSS2003, for example, showed only 66.3% of 4th grade Japanese pupils correctly answered that the weight of an object did not change depending on its orientation on a scale (Martin et al, 2004).

In this study, the author investigated pupils' difficulties with the concept of weight to identify major factors of them. A set of seven assessment items was developed under four broader categories: (a) the existence of the weight of an object, (b) conservation of weight, (c) sensitivity to weights of familiar objects, and (d) appearances of objects and its weight. Responses from 1,826 pupils to the items were analysed. The subjects consisted of 609 pupils in 1st - 3rd grade (before the teaching of the concept of weight) and 1,217 in 3rd - 6th grade (after the teaching of the concept).

The results revealed that many of Japanese pupils were confused the concept of weight with force, even after they learned the concept. Further, some of 3rd - 6th grade pupils could not judge the weight independently from the appearance of objects. For example, more than two-third of them thought that steam and air didn't have their weights. Also, nearly 20% perceived that a black ball was heavier than a white. Moreover, only about 40% of 6th grade pupils could estimate the weight of some familiar fruits. They were not sensitive enough to the weight of familiar objects.

These results indicated that it was difficult for most pupils to grasp the benchmark of weight. Both the judgment based on the appearance of objects and confusion with other quantities hindered pupils from understanding the existence and conservation of the weight of an object. Three major factors which caused pupils' difficulties in understanding the concept of weight were identified in this study: (i) judgment based on the appearance of objects, (ii) confusion with other quantities, and (iii) insensitivity to weight of objects. Implications of these findings were discussed for curriculum development with a focus on the sequence of the topis to be taught and needed relations of teaching the concept to other areas of the school curriculum.

References

Martin, M.O., et al (2004). *TIMSS 2003 International Science Report Finding from IEA's Trends in International Mathematics and Science Study at the Fourth and Eighth Grades*. TIMSS & PIRLS International Study Center, Boston College.

Masuda, Y. (2006). Pupils' cognition of the concept of gravity: A comparison between the groups of pupils who have learned and who have not. *Journal of Japan Society of Mathematical Education, 88*(10), 2-11. (In Japanese)

NUMERACY TEST ITEM READABILITY DURING TRANSITION FROM PRE-SCHOOL TO SCHOOL

Judith A. Mousley

Deakin University

The Australian Council for Educational Research (ACER) developed the *I Can do Maths* (ACER, 2000) test kit for the assessment of young children:

> to inform teachers and parents about children's development in numeracy in the early years of schooling … [resulting in] descriptive and normative reports of children's performance in number, measurement and space [geometry], and not simply a score, so that planning a teaching program appropriate to an individual child's needs is made easier (Doig & de Lemos, 2000, p. 5).

However, the Level A booklet is often used to assess the readiness of children to proceed from pre-school to school. It uses drawings of objects such as coins, snakes, and three-dimensional shapes. This study explored the effects of giving kindergarten children (aged 4-5 years) the same questions but supplying moveable objects. The question for this research was whether the use of drawings in the *I Can Do Maths* questions, rather than objects that are more familiar and can be manipulated, could have an influence on the test outcomes.

"Original" and "modified" questions were combined to make the 2 equivalent tests administered to 34 four- and five-year-old children. The modified questions were generally answered correctly more frequently than original questions, particularly with money and counting questions.

Their kindergarten teachers had provided a list of 10 children whom they thought had "higher levels of numeracy", and 10 with "lower levels of numeracy". It was found that higher-level children scored well on test questions using either drawings or objects, while lower-level children scored significantly higher with objects than with drawings. When they could handle the manipulatives, the lower achievers showed that they possessed much of the knowledge demanded by the questions. From observation and analysis of specific questions, it seemed clear that the difference was not in the mathematical knowledge being tested but in the way children coped with two-dimensional illustrations compared with the objects provided.

References

Australian Council for Educational Research [ACER] (2000). *I Can do Maths: Level A.* Camberwell, Vic: ACER.

de Lemos, M. (2000). *Curriculum and Organisation in the Early Years of Schooling*. Camberwell, Vic: ACER.

RECONCEPTUALISING EARLY MATHEMATICS LEARNING: AN EVALUATION STUDY

Joanne Mulligan — Macquarie University
Lyn English — Queensland University of Technology
Michael Mitchelmore — Macquarie University

There is a growing body of research showing that children's mathematical achievement is closely dependent on their understanding of **pattern and structure**. Recent studies at the Centre for Research in Mathematics and Science Education (CRiMSE) at Macquarie University have shown that pattern and structure is a general underlying characteristic that is common to several mathematical content domains (Mulligan, Mitchelmore & Prescott, 2006). A series of classroom design studies have also shown that young children can learn mathematical concepts very effectively by focusing on crucial features of key mathematical patterns and structures.

A study has been designed to (i) validate a new conceptual framework for mathematics learning based on the development of pattern and structure, and (ii) evaluate the effectiveness of a school-entry mathematics program built on this framework using classroom observations and an interview-based student assessment.

The *Pattern and Structure Mathematics Awareness Program* (PASMAP) to be evaluated focuses on simple repetition patterns, spatial structuring, and the spatial properties of congruence and similarity. Emphasis is also laid on the recognition of similarities and differences and the development of visual memory. The effectiveness of the program will be evaluated in Kindergarten classes from four large primary schools in Brisbane and Sydney, Australia (two in each city). In each school, two of four Kindergarten teachers will trial the integrated PASMAP program; the other two will continue to teach the school's standard program. Narrative profiles of teachers and two target groups of five children within each of the four classes in each school will be compiled as case studies.

The quantitative data will be analysed to find the extent to which (a) children's understanding of pattern and structure is positively correlated with achievement in mathematics, (b) children's achievement in mathematics and numeracy is significantly greater in PASMAP than non-PASMAP children at the end of the experimental year, and (c) any achievement differences are maintained the following year.

References

Mulligan, J.T., Mitchelmore, M.C., & Prescott, A. (2006). Integrating concepts and processes in early mathematics: The Australian Pattern and Structure Mathematics Awareness Project (PASMAP). In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th PME International Conference, 4,* 209-216, Prague: PME.

STUDENT LEARNING PATHS TO MULTI-DIGIT SUBTRACTION: RELATING STUDENTS' METHODS

Aki Murata, Emily Shahan, Laura Bofferding, Yueh Mei Liu, and Jennifer DiBrienza

Stanford University

The study investigated Grades 2 - 4 students' solution methods for multi-digit subtraction problems, examined the gaps between students' uses of invented methods and the current common method (standard algorithm), and discussed meaningful learning paths to develop fluency. We present sample student learning trajectories and related instructional paths based on the findings of the study.

Forty-eight elementary school students in Grades 2 - 4 took a paper test with subtraction problems that differed by grade level, and follow-up interviews were conducted. The data were coded multiple times to identify students' solution methods for each of the test items using the codes from the prior studies (Carpenters, et. al., 1997; Fuson, et. al., 1998). While Grade 2 and 3 students used various counting-based methods and invented methods to solve subtraction problems, by Grade 4, 90% of items were solved by using the common method (algorithm). Students' invented methods were reflective of their thinking process and meaningful, while they required complex multiple steps that needed to be coordinated. The common method simplified the steps, while place value became invisible in the process. Invented methods such as the Decompose-Tens-and Ones (DTO) method and the Begin-With-One-Number (BWON) method took advantage of students' understanding of place value, and the DTO method uses the same steps as the common method. The study suggests that it is possible for students to learn and use multiple methods effectively, and meaningful connections between students' own methods and the common method must be made. The common method may be taught slightly differently to make the mathematics concepts visible (e.g., expanding the multi-digit numbers by places in writing and work with each place at a time). The more explicitly connections are made, the more sensible student learning becomes, and students will extend their understanding to support future learning of mathematics.

References

Carpenter, T.P., Franke, M.L., Jacobs, V.R., Fennema, E., & Empson, S.B. (1998). A longitudinal study of invention and understanding in children's multidigit addition and subtraction. *Journal for Research in Mathematics Education, 29(1),* 3-20.

Fuson, K.C., Wearne, D., Hiebert, J.C., Murray, H.G., Human, P.G., Olivier, A.I., Carpenter, T.P., & Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education, 28(2),* 130-162.

COGNITIVE PROCESSES BEHIND RESPONSES ON ALGEBRA ITEMS IN TIMSS

Margrethe Naalsund

University of Oslo

The aim of the study is to explore thought processes and types of understanding behind Norwegian student responses on selected algebra items in the 2003 TIMSS study, 8th grade, and to what extent the item format plays a role in the students' achievements.

The distinction between two types of mathematical knowledge (e.g. Sfard, 1991: structurally as objects and operationally as processes) will serve as a framework. The study will use a mixed methods design, containing written solutions from approximately 800 students performing two parallel tests on two age levels (8th and 10th grade), together with one-to-one interviews (carried out the same day as the test, in April/May this year).

The TIMSS items applied in this study, are of two different formats (multiple choice and open response), and with two different focuses (letter-symbolic and letter-symbols within the context of word problems). All of the items will be given in two formats (in the two tests) in order to explore to what extent the item format plays a role in student performances within the context of the item (do they reveal the same errors and misconceptions?)

In addition, what are the relations between responses on the particular TIMSS item and responses on the other items in the test, included items emphasising a conceptual understanding of equivalent equations and expressions, and of the algebraic objects involved. (These items will be the same in both tests.) The interplay between transformational skills and comprehension within the context of algebraic symbol language and the students' abilities to see connections between equations/expressions and situational/verbal representations (Kieran, 2007) will be studied, with an emphasis on connections between responses (misconceptions, errors, solution strategies) on TIMSS items. The purpose of the presentation will be to present preliminary results from this research project.

References

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels. In F.K. Lester (Ed.). *Second Handbook of Research on Mathematics Teaching and Learning*. Charlotte, N. C., Information Age.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22,* 1-36.

THE QUADRATIC FORMULA: IS IT A SUCCESSFUL METHOD?

Rosana Nogueira de Lima
Universidade Bandeirante de São Paulo

In this paper, we present an analysis of data collected from three groups of 14-15 year-old students from a public and a private school in the Greater São Paulo area. The students were asked to discuss and analyse the solution, in Figure 1, a non familiar situation for them.

To solve the equation $(x-3)\cdot(x-2)=0$ for real numbers, John answered in a single line that: "$x=3$ or $x=2$" Is his answer correct? Analyse and comment John's answer.

Figure 1. The solution presented to students.

Our analysis consisted in the search for characteristics from the *conceptual embodied world*, the *proceptual symbolic world* and the *formal axiomatic world* (Tall, 2004; Lima, 2007; Lima & Tall, 2008) presented in students' work and the *met-befores* (Tall, 2004; Lima, 2007; Lima & Tall, 2008) they use, analysing how they interfere in the meanings students give to equations and the solving methods they use.

For many students, the quadratic formula seems to be the only valid met-before they know to solve quadratic equations. Eleven students declare that *"John didn't solve the equation"* because he has not used the formula, and other three believe that *"he did use the formula, but he didn't show his work"*.

Presenting a single procedure to solve quadratic equations does not seem to be a very successful approach given the response of these students. The quadratic formula has not proved to be a meaningful method among these students, and it seems to have inhibited them from creating different met-befores to solve quadratics.

The lack of answers discussing algebraic principles shows that formal characteristics of equations and their solving methods may not have been discussed during the learning experience. Such characteristics might prevent them from giving inappropriate embodied meanings to symbols, as in some *procedural embodiments* (Lima & Tall, 2008) and relying on a single met-before for all situations.

References

Lima, R.N. de (2007). *Equações Algébricas no Ensino Médio: Uma Jornada por Diferentes mundos da Matemática*. Thesis. (PhD in Mathematics Education – Pontifícia Universidade Católica de São Paulo), São Paulo.

Lima, R.N. de. & Tall, D.O. (2008). Procedural embodiment and magic in linear equations. *Educational Studies in Mathematics*, *67*(1), 3-18.

Tall, D.O. (2004). Thinking through three worlds of mathematics. In *Proceedings 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 281-288). Bergen, Norway: PME.

APPLICATION OF A METHODOLOGY FOR THE DEVELOPMENT OF INTERACTIVE TECHNOLOGICAL ENVIRONMENTS THAT PROMOTE MATHEMATICS LEARNING

G. Eréndira Núñez and J. Carlos Cortés

Universidad Michoacana

The purpose of this investigation was to obtain information about the learning and teaching phenomena that arouse in a learning environment in which the activities are developed in a computer (Interactive Techonological Environments for Learning Mathematics (ATIAM).

To develop this investigation we applied the methodology so-called ACODESA is related to the Collaborative Learning, the Scientific Debate and the Self-reflection proposed by Hitt (Hitt, 2006), that is based on the activities developed on the computer with the support of the software FUCTIONS AND DERIVATIVES created by Cortes (2002), in which the concept of Derivative is tackled with a numeric approach. The main idea of having the scientific debate at the end of the activities is to construct a concept or to overcome an epistemologic obstacle, where the students have to face didactic situations that promote a cognitive disequilibrium in which the role of the teacher is not to point out the logical contradictions.

Fifteen high-school students participated in the study, with ages between 17 and 18 years old. We carried out eleven sessions, each one of three hours long. We tackled the concept of Derivative with numeric and graphic treatments, using progressions, increases, changing ratios, secant lines and tangent lines.

CONCLUSIONS

The interactions that were generated among the students were numerous. The students learned to express their ideas about. The interactivity among the different actors, the computer and the object of knowledge was strengthened, besides the collaborative learning. The motivation and dynamism in the students was increased in comparison with a traditional class. The experience that takes place in an interactive environment, is much more productive if technology is combined with an appropriate methodological strategy.

References

Cortés C. (2002). *Desarrollo de Software para la Enseñanza del Cálculo Diferencial.* Tesis de Doctorado. Cinvestav, México.

Cortés C. (2002). *Funciones y Derivadas*. Software de apoyo al aprendizaje del cálculo diferencial.

Hitt F. (2006). Utilization de la calculatrice symbolique dans un environnement d'apprentissage coopératif, de débat scientifique et d'auto-réflexion. Versión préliminaire Février 2006.

SOLID GEOMETRY IN THE MEXICAN ELEMENTARY SCHOOL

Francisco Olvera — Universidad Pedagógica Nacional
Gregoria Guillén — Universitat de València
Olimpia Figueras — Cinvestav

The work reported in this communication is part of a research project (Olvera, 2007) structured in a two-stage study. The purpose of one of the stages is to build up an initial Local Theoretical Model (LTM) (in the sense of Filloy, 1999) referred to the teaching and learning of Solid Geometry. The other stage has a double aim: to create a community of practice with in-service primary teachers that enables collabotive work to develop strategies for planning teaching activities to introduce solid geometry in the Mexican elementary school curriculum.

For the building of the initial LTM, the work of researchers related to teaching Solid Geometry and the learning of concepts of this topic were analyzed. Attention was centred on work that could enrich personal knowledge of Geometry and its teaching, and literature linked with mathematical processes or processes of learning mathematical processes (see for example Fielker, 1979 and Guillén, 1997).

The built LTM served as a theoretical framework to analyze documents provided by the Mexican Ministry of Education to teachers and students of the primary school. Among these documents are the national curriculum of primary education and the cost-free textbooks for children. The analyses carried out were focused on: i) a quantitative approach to determine the relevance given in the primary education to Geometry and particularly to Solid Geometry, and ii) a qualitative approach to characterize the teaching model for Solid Geometry.

Arithmetic and Geometry are the two most important topics in the elementary Mexican education, but the latter represents around a half of the former; Solid Geometry is scarcely considered. The teaching model for Solid Geometry imbedded in the curriculum and textbooks does not highlight the rich contexts that the study of the solids provide for learning mathematical processes.

References

Fielker, D.S. (1979). Startegies for teaching geometry to younger children. *Educational Studies in Mathematics*, *10*(1), 85-133.

Filloy, E. & col. (1999). *Aspectos Teóricos del Álgebra Educativa.* Investigaciones en Matemática Educativa. México: Grupo Editorial Iberoamérica

Guillén, G. (1997). *El Modelo de Van Hiele Aplicado a la Geometría de los Sólidos. Observación de Procesos de Aprendizaje.* Non published Doctoral Thesis. Universitat de València, Spain.

Olvera, F. (2007). *La Enseñanza de la Geometría de los Sólidos en la Escuela Primaria.* Non published manuscript (Research doctoral project), Cinvestav, México.

VERBAL AND SYMBOLIC DESCRIPTIONS OF PROPERTIES OF MATHEMATICAL OBJECTS

Mabel Panizza

Universidad de Buenos Aires

This research focuses on students' observations of mathematical objects, on their identification of the characteristic features of these objects and on the descriptions produced thereafter. Considering the abstract nature of mathematical objects, these observations are made on particular semiotic representations. Thus, the features identified are characteristics of (particular) representations of (particular) instances of the object. We were specially interested in the verbal and symbolic descriptions students produce, based on their observations, and on the effects of these descriptions on their knowledge of the mathematical objects.

In our study we found that, sometimes, students may produce an adequate verbal description of a graph or a geometric drawing, without realizing that this description may also stand for other graphs or geometric drawings. Then, a conflict may appear when a student is confronted with a graph (or a geometric drawing) which satisfies his description but is different from the original one. In addition, quite often students think in terms of typical instances of mathematical objects. In these cases, as we showed in Panizza (2006), they identify them as being of a "certain type" (Tversky and Kahneman, 1974), and the identified characters become the "definition" of that type (Duval, 1995), which may also denote other instances, as seen above. In both cases –describing and producing spontaneous definitions– students' meanings remain closely related to the original figures and instances. We developed tasks which can stimulate debate among students so that they can discover the possible coexistence of these different representations –mental, verbal, and symbolic- initially in conflict, and can identify the actual properties of mathematical objects.

In the presentation, we will describe examples of this phenomenon and we will share examples of tasks aimed at supporting students learning about the complex relations between different semiotic descriptions of mathematical properties.

References

Duval, R. (1995). *Sémiosis et Pensée Humaine, Registres Sémiotiques et Apprentissages Intellectuels*. Bern: Peter Lang.

Tversky, A. & Kahneman, D. (1974). Judgement under uncertainty: Heuristics and biases. *Science*, 185, 1124-1130.

Panizza, M. (2006). Substitutions on algebraic statements, based on associations in natural reasoning. In J. Novotná, H. Moraová, M. Krátká, & N. Stehliková (Eds.), *Proceeding 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 312). Prague. Czech Republic: PME.

ROBUST UNDERSTANDING OF VARIATION: AN INTERACTION OF THREE PERSPECTIVES

Susan A. Peters

The Pennsylvania State University

Reporting on the initial phase of an ongoing study of the nature of and influences on conceptions of statistical variation exhibited by secondary mathematics teachers, this presentation will provide a picture of the complexity of a robust understanding of variation as a blend of design, data-centric and modeling perspectives.

Because variation connects to and interrelates with many statistics concepts, a robust understanding of variation is critical for understanding statistics. Research that examines students' reasoning with and about variation reveals that students have many intuitions about variation (e.g., Reading & Shaughnessy, 2004). Current studies tend to focus on students' reasoning about a particular aspect of variation or from a particular perspective (e.g., delMas & Liu, 2005). This study, in part, attempts to complement research on students' reasoning about variation by providing images of reasoning about variation as it is done by teachers as advanced learners of statistics.

This study uses semi-structured interviews with tasks designed to elicit participants' conceptions of variation. Data from a national sample of 16 secondary mathematics teachers who are recognized leaders in statistics education is being analysed using the Structure of the Observed Learning Outcomes Model to guide analysis of participants' conceptions of variations (Biggs & Collis, 1982). Preliminary analysis suggests that flexible and integrated reasoning with and about variation from design, data-centric, and modeling perspectives is indicative of a robust understanding of variation.

This presentation will describe the three perspectives and how the perspectives interact within a robust understanding of variation.

References

Biggs, J.B. & Collis, K.F. (1982). *Evaluating the Quality of Learning: The SOLO Taxonomy (Structure of the Observed Learning Outcome).* New York, NY: Academic Press.

delMas, R. & Liu, Y. (2005). Exploring students' conceptions of the standard deviation. *Statistics Education Research Journal, 4*(1), 55-82.

Reading, C. & Shaughnessy, J.M. (2004). Reasoning about variation. In D. Ben-Zvi & J. Garfield (Eds.), *The Challenge of Developing Statistical Literacy, Reasoning, and Thinking* (pp. 201-226). Dordrecht, The Netherlands: Kluwer Academic Publishers.

TERRITORIALIZATION AND DETERRITORIALIZATION OF ONLINE AND OFFLINE IDENTITIES: THE TRANSFORMATION OF MATHEMATICAL KNOWLEDGE

Maurício Rosa — Lutheran University of Brazil
Marcus Vinicius Maltempi — São Paulo State University at Rio Claro

In previous paper Borba, Malheiros and Santos (2007) showed us that online education courses for mathematics teachers have been the scenario for various studies, some of which have been presented at PME. Also, we developed a course named “Constructing the Concept of Integral through Virtual Role-Playing Game (RPG)”. It was also a useful context to investigate the relations between the construction of online identities and the teaching and learning of Definite Integral. In this manner, the game became a learning and teaching environment of this mathematical concept backed by playful and at distance approach. We used the theoretical support mainly from Turkle’s (1995; 1984) studies and from the Deleuze and Guattari (2005) vision of the construction of concepts. The research paradigm was based in qualitative modality, from textual analysis about the chat in a distance communication plataform (TelEduc).

Thus, this paper highlights one of the possible relations between the construction of online identities in cyberspace through Online RPG and the teaching and learning of the concept of Definite Integral. It is one way to contribute to the Online Mathematics Education. We believe that whenever the construction of online identities is shown in transformation, we can affirm that not only the student but also the teacher can become more “plastic”. There is an evident transformation of the “being” into a “being-with-cyberspace”, which learns and teaches mathematics one with each other, keeping themselves the same. However, the mathematics itself becomes something different. That is, the mathematics is perceived from another perspective through a different vision of the world (cybernetic world).

We have results from our research and we can show that the concept of Definite Integral can be seen from a contextualized scenario more than from “epsilon and delta perspective”. So, each student constructs the mathematical concept from experiences that he/she could not have lived in a conventional classroom. The construction of online identities is in transformation when the concept is been constructed from territorializations and deterritorializations of mathematical ideas, for example, Riemmann Sums, in different immanence plans. That is, under each conceptual character perspective (Deleuze and Guattari, 2005) the Definite Integral concept can be related with an irregular plantation area in a virtual farm and it can be studied from a tractordriver online identity.

References

Borba, M.C., Malheiros, A.P., & Santos, S.C. (2007). Virtual Center for Modeling: a place for exchange among researchers and teachers. In Woo, J.H., Lew, H.C., Park, K.S., & Seo, D.Y. (Eds.), *Proceedings of the 31st PM. Seoul: The Korea Society of Educational Studies in Mathematics, 2007* (Vol. 1, pp 276) CD-ROM.

Deleuze, G.; Guattari, F. (2005) O que é a filosofia?. Translated by Bento Prado Jr. and Alberto Alonso Muñoz. Rio de Janeiro: Editora 34.

Turkle, S. (1995). *Life on the Screen: Identity in the Age of the Internet.* New York, NY: Touchstone Edition.

Turkle, S. (1984). *The Second Self: Computers and the Human Spirit*. New York, NY: Simon & Schuster.

CONVENTION AND INVENTION IN PUPILS' MATHEMATICAL COMMUNICATION

Filip Roubíček

Institute of Mathematics

Communication in the teaching of mathematics is distinguished by using various systems of semiotic representation. For communication in mathematical classes, it is important to acquaint the pupils not only with various forms of mathematical knowledge representation, but also with the rules how to form, interpret, and use them properly (compare Ferrari, 2006). Pupils use *conventional representations* which are proposed to them by teacher and also individual representation means, so called *inventional representations*.

The phenomenon when the pupil uses an expression that does not correspond with the communication context or when the pupil uses such an expression in two different semantic contexts is called the *communication confusion*. *Communication dissonance* is a phenomenon caused by communication confusion that leads to discordance or disagreement between the communicants. These phenomena appear in oral as well as written communication. Pupils acquainting themselves with a certain semiotic system are not usually aware of their mistake, and it is therefore up to the teacher to correct the pupil in using the system. If not doing so, the teacher indicates to the pupil that the pupil's usage of the system is correct which may mean improper acquisition of the semiotic system rules as a consequence.

Observations show that precisely the mathematical terminology and symbolism tends to represent an obstacle for some pupils in understanding mathematics. The problem does not usually consist in terms and symbols themselves but in ways in which they are introduced and used in mathematical classes. If the pupils do not know the rules to create admissible combinations of the symbols, their meaning and usage in various contexts, statements written using the symbols become a formal matter for them in the better case, or a communication and cognitive obstacle in the worst.

Endnote

The paper has been elaborated with support of the Grant Agency of the Academy of Sciences, Grant Project no. KJB700190701, and of the Academy of Sciences of the Czech Republic, Institutional Research Plan no. AV0Z10190503.

References

Ferrari, P. L. (2006). From verbal texts to Symbolic expresions: A Semiotic Approach to Early Algebra. In J. Novotná, M. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education,* (Vol. 3, pp. 73-80). Prague: PME.

DIGITAL TECHNOLOGIES AS A CATALYST FOR CHANGE: SECONDARY SCHOOL MATHEMATICS TEACHERS REFLECT ON THE CHANGES IN THEIR PRACTICE

Ana Isabel Sacristán and Nadia Gil

Center for Research and Advanced Studies

A meaningful incorporation of digital technologies (DT) in education, requires rethinking and changing the teacher's practice and the teaching-learning process; yet these changes are not straightforward (Sacristán et al., 2006). We are concerned with researching the use that in-service mathematics teachers make of DT, the training they require and the changes in their practices that they need to make in order to harness the potential of DT tools. In particular, we are involved in a three-year development and research project – linked to a master's degree program in education for in-service teachers – where participants have been reflecting and documenting the changes in their practice, derived from the incorporation of DT, from various perspectives: (a) The perspective of the teacher and the didactical use of DT. (b) The perspective of the classroom interactions. (c) The possible impact on students and their learning. (d) The technical perspective. (e) The social context. We present data from case studies of four of the participating teachers in this project, all of whom have been teaching for over a decade at the secondary school level.

These case studies illustrate some of the changes in these teachers: overcoming their lack of confidence and reluctance to use DT; their growth in appreciation of the DT tools; realization of the importance of using complementary tools; adapting to new classroom dynamics; developing new assessment methods; etc. We have identified four important factors that help teachers' to change their practice for meaningfully incorporating DT: (i) training; (ii) the actual experience of using DT in his classes; (iii) reflecting on his practice; and (iv) time. The latter seems to be a very important factor; as Goldenberg (2000, p.8) states: "Provide instruction and time for teachers to become creative users of the technology they have."

Endnote

This research was funded in part by the CONACYT Research Grant No. 44632.

References

Goldenberg, E.P. (2000). Thinking (and talking) about technology in math classrooms. [Electronic version]. *Issues in Mathematics Education*. Retrieved 31 January, 2007 from http://www2.edc.org/mcc/PDF/iss_tech.pdf.

Sacristán, A.I., Ursini, S., Trigueros, M., & Gil, N. (2006). Computational Technologies in Mexican Classrooms: The Challenge of Changing a School Culture. In V. Dagiene & R. Mittermeir (Eds.), *Information Technologies at Schools. 2nd International Conference 'Informatics in Secondary Schools: Evolution and Perspectives' Selected Papers* (pp. 95-110). Vilnius: Institute of Mathematics and Informatics.

TARGETING AT EQUITY IN MATHEMATICS EDUCATION: AN INTERVENTION PROJECT IN MULTICULTURAL SCHOOL CONTEXTS

H. Sakonidis, A. Klothou, and A. Nizam

Democritus University of Thrace

Despite important advances in the field of mathematics education, many children, particularly from backgrounds which are traditionally marginalized through school practices, continue to fail in mathematics. This is attributed to these pupils' culture, which differs from that represented in and through school mathematics practices and discourses, thus complicating the learning of the subject matter more than it happens for children whose culture aligns with the latter (e.g., Zevenbergen, 2007). The above suggest that, in order to increase all children's opportunities to succeed in mathematics, social and cultural diversity needs to be acknowledged and exploited as a learning resource. To this direction, the role of classroom communication and teachers' as well as pupil's collaborative activities appear to be crucial (e.g., Gorgorio & Planas, 2002).

The research reported here is situated within the above framework. The data come from a ten years' intervention project, aiming at studying the transformations which the learning and the teaching classroom practices go through, following the introduction of an instructional approach based on a package of innovative educational material that celebrates problem solving strategies and respects cultural diversity. Twenty Greek high schools participated in the study, attended by substantial numbers of Muslim students, who were the main focus of the work carried out. The analysis of the data collected (transcribed teachers' and pupils' interviews, field notes, transcribed lessons, etc.) show a slow but noticeable shift to teaching as well as learning practices which indicate improved levels of trust and understanding among teachers and pupils.

References

Gorgorio, N. & Planas, N. (2002). Teaching mathematics in multilingual classrooms. *Educational Studies in Mathematics*, 47, 7-33.

Zevenbergen, R. & Flavel, S. (2007). Undertaking an archaeological dig in search of pedagogical relay. *The Montana Mathematics Enthusiast,* Monograph 1, 63-74.

TEACHING ADVANCED STUDENTS TO CONSTRUCT PROOFS*

John Selden, Annie Selden, and Kerry McKee
New Mexico State University

We report on the first two iterations of a design experiment to develop a 3-credit course to help advanced undergraduate and graduate students improve their proving abilities. The course was taught in a modified Moore Method way (Mahavier, 1999). There was no book and there were no lectures. We provided notes containing definitions, requests for examples, and theorems to prove. Students presented their proofs in class at the blackboard. Criticism, advice, and often considerable rewriting help were provided. The main data sources were field notes and video recordings of all the classes. The data were analysed between class meetings in an effort to affect students' learning trajectories (Simon, 1995).

A framework for distinguishing students' abilities to write different kinds and aspects of proofs is emerging. For example, we separate proofs into a formal-rhetorical part and a problem-oriented part (Selden & Selden, in press). The formal-rhetorical part is the part that can be written depending only on the formal aspects of definitions and theorems without recourse to their deeper meanings or to problem solving in the sense of Schoenfeld (1985, p. 74). The remaining problem-oriented part does depend on problem solving and a deeper understanding of the concepts. Students' progress in constructing these two parts of proofs seems to develop independently.

In addition, we have noticed two persistent difficulties: (1) "starting in the wrong place," and (2) "reluctance to introduce a fixed, but arbitrary object." The first difficulty refers to starting to prove a theorem by attempting to immediately use the hypotheses, even though it would be more appropriate to first look ahead to what is to be proved. The second difficulty refers to proofs of universally quantified statements, that is, for all (numbers) x P(x). Such proofs often include something like "Let x be a number," meaning x is "fixed, but arbitrary." In an interview, a real analysis student reported having written this aspect of proofs correctly, but only associated doing so with a feeling of appropriateness halfway through the semester. The lack of such feelings of appropriateness may partly explain the persistence of the above two difficulties.

Endnote

*This research was supported by a grant from the Educational Advancement Foundation.

References

Mahavier, W.S. (1999). What is the Moore method? *PRIMUS*, *9*, 339-354.

Schoenfeld, A.H. (1985). *Mathematical Problem Solving*. Orlando, FL: Academic Press.

Selden, J., & Selden, A. (in press). Teaching proving by coordinating aspects of proofs with students' abilities. In M. Blanton, D. Stylianou, & E. Knuth (Eds.), *The Learning and Teaching of Proof Across the Grades.* London: Lawrence Erlbaum Associates/Taylor & Francis.

Simon, M.A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26,* 114-145.

EXAMPLES OF MATHEMATICAL COMPREHENSION AND THE USE OF TASKS IN TEACHING

Armando Sepúlveda, Cynthia Medina, and Diana Itzel Sepúlveda
Universidad Michoacana de San Nicolás de Hidalgo

The following paper analyzes some examples of mathematical understanding shown by high school students when faced with a set of problems or tasks which involve different approaches to solutions, using a teaching method based on problem solving. During the course of the task, the students worked in small groups, presented and defended their ideas and revised their solutions as a result of the feedback and opinions they received during their presentation and discussion. Thus, the students showed different levels of understanding which allowed them to gradually grasp the main ideas related to the solution and, eventually, they solved the tasks. One of the guiding lines of such research is the design of tasks or problems that have specific characteristics (Balanced Assessment Package for the Mathematics Curriculum, 2000): the use of tasks designed for the students to express what they know and to spark their interest in researching what they don't know by means of discussion and exchange of experiences, with a particular method of teaching which combines cooperative work, in small groups and as a whole class, with individual work. Some of the questions which guided our project were: What forms of understanding and methods of solution appear during the processes of problem solving? What is the role of the teacher during the course of the sessions?

The implementation of tasks. As part of an investigation project which is being carried out in Mexico, some of the tasks are being implemented, using the teaching method suggested by Sepúlveda and Santos (2006) which consists of five stages: i) Prior activity; ii) Team work; iii) Team presentation; iv) Group discussion; and v) Individual work. In order to show the kinds of results and analyses which arose from the implementation of these tasks, this paper presents and example of a problem solving task carried out with 24 last year students, from a high school in Morelia, Michoacán, which were, around 17 years of age and which took part in a semester course in problem solving. During one of the two - hour sessions we implemented a task which implied decision making based on the comparison of two sets of data. The objective of this task is to evaluate and promote learning of basic statistical ideas:

Requesting a taxi. "Sara wants to compare two rival taxi companies: Yellow Taxis and Blue Taxis, according to their punctuality, and decide which is the best. She requested the service of each one of the companies on 20 occasions when going to work and registered the time of arrival, before or after the agreed time".

References

Balanced Assessment Package for the Mathematics Curriculum (1999, 2000). *High School Assessment Package* (1 & 2), White Plains, N.Y.: Dale Seymours Publications.

Sepúlveda, A. & Santos, M. (2006). About the development of examples of mathematical comprehension shown by high school students in processes of problem solving. In *Mexican Magazine of Educational Investigation* (31, pp. 1389-1422), México (ISSN 1405-6666).

BOUNDARY OBJECTS AT THE INTERFACE OF COMMUNITIES OF PRACTICE

Dianne Siemon

RMIT University

A short oral is proposed as this presentation is aimed at seeking feedback on the conceptualisation and use of a small number of performance-based tasks in a current research project which is exploring an alternative model of Indigenous teacher education in two remote communities in Arnhem Land, Australia. Referred to as probe tasks, they were originally developed to support pre-service mathematics teacher education at RMIT University. The tasks were subsequently used to identify the mathematics learning needs of remote Indigenous students in Northern Australia as they require relatively low levels of student literacy and were focussed on key number ideas and strategies, an identified area of learning. The teachers involved in this project typically reported that as student responses to the tasks were more readily observed, interpreted, and matched to expected levels of performance, they felt more confident about identifying and responding to student learning needs in a targeted way to positively impact student numeracy learning. This was particularly the case for the Indigenous teacher assistants and secondary-trained teachers with a non-mathematics background (Commonwealth of Australia, 2005). This suggested that the probe tasks and related advice might offer a useful means of building remote Indigenous teachers' pedagogical content knowledge for teaching mathematics.

The probe tasks are currently being used with two groups of Indigenous teacher assistants in an evolving study group environment to prompt discussion of key ideas in first language and English, identify and describe student responses, plan targeted teaching activities, and focus reflective discussions. Three communities of practice are acknowledged for the purposes of the research: the numeracy practices of the local Indigenous community, the practices associated with school mathematics, and the constituted practices of the study group. Anticipated research outcomes include new knowledge about the role of the tasks as boundary objects, that is, objects at the interface of communities of practice around which shared understandings of what is involved in teaching and learning school mathematics can be negotiated. It is this aspect of the research that the presentation will seek to illustrate and clarify.

References:

Commonwealth of Australia (2005). *Supporting Indigenous Students' Achievement in Numeracy – NT Strategic Numeracy Research and Development Project (2003-2004). Final Report.* Canberra: Department of Educaiton, Science and Training

Pawloski, S., Robey, D., & Raven, A. (2000). Supporting shared information systems: Boundary objects, communities, and brokering. In *Proceedings of the 21st International Conference on Information Systems* (pp. 329-338). Brisbane: Association for Information Systems.

ONLINE COLLABORATION IN TEACHER EDUCATION

Jason Silverman and Ellen Clay

Drexel University

It is widely accepted that teachers of mathematics need deep understanding of mathematics, but there is no consensus as to effective or efficient means to help teachers develop it. One viable site for *teacher* mathematical development looks very much like effective *student* learning environments, with students posing questions, developing solutions and questioning and justifying the emergent questions and solutions. We seek to extend this learning environment to an online setting, capitalizing on the benefits of the internet: anytime-anywhere learning with a permanent record of practice. In our online classes, students take part in individual and collective problem solving, synthesis, and reflection in carefully choreographed, online interactional spaces. Our online learning environment has been designed with the explicit goal of positioning teachers to engage in advanced cognitive processes specifically related to mathematics learning and teaching.

The primary methods used to analyze online interactions focus on the participation structure, using discourse and conversation analysis to identify patterns within the interactions: who initiates, the purpose of the initiation, and types of interactions (Teacher[T]-Student[S], S-S, T-S-T, T-S1-S2-S1-S3-S2, etc.). In our work, we note that these linear patterns are not analogous to the settings within which teachers work, where near instantaneous organization and synthesis of multiple student comments, solutions, and viewpoints are important teaching practices. In the proposed talk, we will describe our efforts to code and analyze the online interactions from our classes, which highlight the ways in which the environment captures reform-style mathematical interaction and has the potential for stimulating discussion about how to catalyze such interactions. Further, we will present preliminary analysis of the cognitive activity of those participating in the online interactions using Anderson & Krathwohl's (Anderson et al., 2001) extension of Bloom's taxonomy.

While we believe that this work holds great potential for mathematics teacher development, we are not in the position to disseminate the results of empirical analysis. At this point, we are in the germination and refinement phase. The purpose of this presentation is to share these emerging ideas, to generate feedback and invite future collaboration.

References

Anderson, L., Krathwohl, D., Airasian, P., Cruikshank, K., Mayer, R., Pintrich, P., et al. (Eds.) (2001). *A Taxonomy for Learning, Teaching and Assessing: A Revision of Bloom's Taxonomy of Educational Objectives*, Allyn & Bacon.

GENDER DIFFERENCES AND PISA: AN ICELANDIC STORY

Olof Bjorg Steinthorsdottir — University of North Carolina in Chapel Hill

Bharath Sriraman — The University of Montana

PISA 2003 presented interesting results about students' mathematical achievement in Iceland, where Iceland was the only country that showed significant gender differences in mathematics in favor of girls. These unique results when statistically analyzed, it became evident that the gender differences were only measurable in the rural areas of Iceland. The authors conducted a qualitative study in Iceland in 2007, in which 19 students from rural and urban Iceland who participated in PISA 2003 were interviewed. The purpose of these interviews was to get students to elicit their thoughts on their mathematical experiences, their beliefs about mathematical learning, their thoughts about the PISA results, and their ideas on the reasons behind the unusual PISA 03 results. The data was transcribed, coded and analyzed using techniques from grounded theory in order to build categories and to present feminine and masculine student perspectives on the Icelandic anomaly.

THE MAIN SECTION HEADING STYLE IS PME HEADING 2

Despite the common belief (in many western countries) that the gender differences in mathematical achievement has been liminated, PISA, in addition to the evidence that the presentations at ICME 10 provided, documented statistically significant gender differences in achievement in favor of boys both in the year 2000 , 2003, and 2006. The only one country in PISA 2003 which had statistically significant gender differences in achievement in favor of girls was Iceland.

PISA 2003 presented interesting results about students' mathematical achievement in Iceland, where Iceland was the only country that showed significant gender differences in mathematics in favor of girls. These unique results when statistically analyzed, it became evident that the gender differences were only measurable in the rural areas of Iceland.. The authors conducted a qualitative study in Iceland in 2007, in which 19 students from rural and urban Iceland who participated in PISA 2003 were interviewed in order to investigate these differences and determine factors that contributed to gender differences. The purpose of these interviews was to get students to elicit their thoughts on their mathematical experiences, their beliefs about mathematical learning, their thoughts about the PISA results, and their ideas on the reasons behind the unusual PISA 03 results. The data was transcribed, coded and analyzed using techniques from grounded theory in order to build categories and to present feminine and masculine student perspectives on the Icelandic anomaly. Four general themes emerged from the interviews about why girls did better than boys. They were (1) parental influence and upbringing, (2) peer pressure and the gendered discourse among teenagers, (3) professional ambition, and (4) general human development.

SYNERGISTIC SCAFFOLDING AS A MEANS TO SUPPORT PRESERVICE MATHEMATICS TEACHER LEARNING

Shari L. Stockero — Michigan Technological University

Laura R. Van Zoest — Western Michigan University

This study investigates the role of synergistic scaffolds (Tabak, 2004) in supporting the development of preservice mathematics teachers' knowledge of self-as-teacher by addressing the following research questions: (1) Were there differences in the quality of preservice teachers' initial Mathematics Teaching Autobiographies (MTA) and a revised version completed after scaffolding interventions? (2) What was the relationship between the scaffolds and the revisions made by the preservice teachers? and (3) What were the preservice teachers' perceptions of which scaffolds best supported their learning? Data include MTAs written before and after the introduction of scaffolds, student surveys, and student interviews. Sherin, Reiser and Edelson's (2004) scaffolding analysis framework was used to structure our analysis.

A paired t-test showed that the difference between the two MTA scores was significant ($p < 0.001$). Student surveys and interviews provided evidence that the scaffolds—particularly instructor feedback and a self-assessment rubric—provided focus to the revision process and pushed students to think more deeply than they would have otherwise. The full paper includes detailed analysis of the revisions.

We conclude that carefully designed synergistic scaffolds can support preservice teachers in their exploration of self-as-teacher, and that scaffolds were particularly effective in pushing them to think more deeply about who they are as a teacher and the relationship among their past experiences, current views, and future teaching.

References

Sherin, B., Reiser, B.J., & Edelson, D. (2004). Scaffolding analysis: Extending the scaffolding metaphor to learning artifacts. *Journal of the Learning Sciences, 13*(3), 387-421.

Tabak, I. (2004). Synergy: A complement to emerging patterns of distributed scaffolding. *Journal of the Learning Sciences, 13*(3), 305-335.

USE OF GRAPHS IN CHANGE AND VARIATION MODELING[1]

Liliana Suárez Téllez and Francisco Cordero Osorio

Cinvestav

This work characterizes the undergoing conditions of a learning activity to favour the reconstructions of meanings of mathematical knowledges in a particular school situation. The reference practices and the uses related to certain knowledges can be identified by a) the historical development of mathematical knowledge, b) its immersion in the didactic field and c) the characterization of the students' work. This work emphasizes, in Calculus, the use of graphs to describe the change and variation. This epistemological approach consists of the systemic study of the use of mathematical knowledge in specific learning situations (Buendía & Cordero, 2005). Oresme's work, on the figuration of qualities, provides an explanation of the transformation of the use of mathematical knowledge to deal with change and variation situations (Clagett, 1968). From the debate between functioning and form of the use of geometric figures, this work highlights key elements of the epistemological assumption about the use of graphs in modeling to redefine change and variation. This socioepistemological approach provides a new status for modeling and for the use of graphs, which orients them as generators of knowledge. The socioepistemology of the modeling- use of graphs can be used in situations to work with students and it is integrated by sequences called Modeling Movement Situations (SMM). Our hypothesis is that variation is redefined through the modeling- use of graphs proposal. We also have some evidence about the elements of function of figuration that come out as results from a SMM (Suárez et al, 2005). We also have evidence of different ways of the use of graphs as a result of the characterization of the meanings and procedures that participants use to establish the relationships between the position and speed graphs in a situation of change in different stages such as in the situation design; in which the form is established, the arguments are built and used.

References

Buendia, G. & Cordero, F. (2005). Prediction and the Periodical Aspect as Generators of Knowledge in a Social Practice Framework: A Socioepistemological Study. *Educational Studies in Mathematics,* 58, *3*, 299-333.

Clagett, M. (1968). *Nicole Oresme and the Medieval Geometry of Qualities and Motions*. Madison: University of Wisconsin Press.

Suárez L., Carrillo C., & López J. (2005). Diseño de gráficas a partir de actividades de modelación. *Acta Latinoamericana de Matemática Educativa*, 18, 405-410.

[1] Acknowledge: Supported, in part, by grant 41740-S from Conacyt, Mexico.

VISUAL SUPPORT FOR PROPORTIONAL REASONING: THE DOUBLE NUMBER LINE

K. Subramaniam

Homi Bhabha Centre for Science Education

Several studies have shown that students in the middle and higher grades commonly use an algorithmic approach to solve proportion problems. Conceptual understanding of proportion is facilitated when visual support is provided, at least when either the ratio 'within' or the ratio 'between' the measure spaces is a whole number (Misailidou & Williams, 2003). However, diagrams that show proportional relationships are often restricted to discrete measures, as for example, a picture showing cans to depict the relation '2 cans of yellow paint for every 3 cans of blue'. The passage from whole number based to rational number based multiplicative thinking encounters a significant conceptual barrier (Greer, 1994). In this article we discuss a form of visual support for proportional reasoning – the double number line – that has been proposed but not studied sufficiently. We report two episodes taken from an ongoing study with 11-12 year olds on developing fraction knowledge for reasoning about ratio and proportion.

In the episode reported, students work on a missing value proportion task, where the ratios involved are not whole numbers. Students are able to implement the partition-and-build-up strategy using the double number line representation as a tool for thinking and communicating. This indicates that the intuitive strategies elicited by other pictorial representations are also supported by the continuous representation afforded by the double number line. This adds to other affordances of the model, such as the representation of rational measures, and of visually marking multiplicative transformations both within and across measure spaces (indicated by arrows). A further advantage of the double number line model is that it can represent linear functions that have a non-zero intercept. Such functions form another source of difficulty in reasoning about proportional relationships. A second episode from a different segment illustrates students' constructions about functional relationships using the double number line as a representation.

References

Greer, B. (1994). Extending the meaning of multiplication and division. In G. Harel & J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (61–85), Albany, NY: State University of New York Press.

Misailidou, C. & Williams, J. (2003). Diagnostic assessment of children's proportional reasoning. *Journal of Mathematical Behaviour, 22*, 335-368.

USING MULTILEVEL MODELS TO QUANTIFY QUALITATIVE INSIGHTS FROM DESIGN RESEARCH

Pamela D. Tabor

Southern Cross University

Sloane (2006) proposed uniting "mathematics education research by quantifying qualitative" research findings. Shavelson, Phillips, Towne and Feuer (2003) challenged those utilizing design research to attend to warrants for their research claims, proposing the integration of quasi-experiments within design research. This study explores such an integration.

Design research indicates that the use of base-ten collections materials predisposes children toward the use of 1010 over N10 and that that strategy persists after the tools are no longer available (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997). However, this finding has not been empirically tested. This study used concurrent classroom teaching experiments to test this conjecture while continuing the design iterations to develop instructional sequences that promote facility with mental calculations. The design allowed for the development of different instructional sequences in each classroom. Qualitative analysis indicated that children exposed to collections materials were initially more likely to use 1010 than others. However, this difference appeared to diminish over time. In order to warrant those findings, multilevel models for repeated measures of categorical data were built to model the associations between individual behaviours and the instructional sequence. The research question was: Does exposure to base-ten collections materials permanently predispose children to using 1010 when solving 2-digit addition and subtraction? Results indicate that the collections class was initially significantly more likely to use 1010 ($p < .001$), but this difference did not persist. Details of the model will be presented during the short oral communication. The quantitative analysis reflexively informed the subsequent qualitative analysis.

References

Cobb, P., Gravemeijer, K.P.E., Yackel, E., McClain, K.J., & Whitenack, J.W. (1997). Mathematizing and symbolizing: The emergence of chains of signification in one first-grade classroom. In D. Kirschner & J.A. Whitson (Eds.), *Situated Cognition: Social, Semiotic, and Psychological Perspectives,* (pp. 151-233). Hillsdale, NJ: Erlbaum.

Shavelson, R.J., Phillips, D.C., Towne, L., & Feuer, M.J. (2003). On the science of education design studies. *Educational Researcher, 32*(1), 25-28.

Sloane, F.C. (2006, April 24). *Unifying Mathematics Education Research by Quantifying Qualitative Insight: Keynote Address.* Paper presented at the National Council of Teachers of Mathematics Research Pressession, St. Louis, MO.

ELEMENTARY SCHOOL FIFTH GRADE STUDENTS' THINKING PROCESS IN LINEAR AND QUADRATIC PATTERNS

Dilek Tanışlı and Aynur Özdaş

Anadolu University

A pattern is a systematic configuration of geometric figures, sounds, symbols or actions. Relating patterns in numbers, geometry and measurement helps students understand connections among mathematical topics. Such connections foster the kind of mathematical thinking that serves as a foundation for more abstract ideas studied in later grades. It also help developing algebraic and functional thinking.

The main objective of study being my doctorate thesis is to determine fifth grade students who have different mathematics achievement level (low, mid and high) perception of linear and quadratic figural pattern (find rule of pattern, extend a pattern to the next case and a near case, create a pattern). The study was conducted in a elementary school and participated 12 students. The data of the study was obtained from task-based interviews. And all interviews video-recorded. Two lineer, and two quadratic patterns were asked students in interviews. According to findings of the study in finding rule of linear and quadratic figural patterns students used visual and algebraic approach using recursive, explicit and other strategies. In extending near case in patterns recursive strategies were used mostly, and in extending next case in patterns explicit strategies were used poorly. Most of the students could create linear figural pattern but a few students could create quadratic figural patterns. In addition there is no connection between selection of strategy and student achievement level.

References

Hargreaves, M., Shrrocks-Taylor D., & Threall J. (1998). Children's stratgies with number Pattern. *Educational Studies*, *24(3),* 315-331.

Lan M.H. (2007). The potential of patterning activities to generalization. In *Proceeding of the 31st Conference of the International Group for the Psychology of Mathematics Education.* (Vol. 3, pp. 225-232). Seoul: PME.

Orton, A. & Orton J. (1999). Pattern in the teaching and learning of mathematics. Anthony Orton (Eds.), *Pattern and the Approach to Algebra*. London: Cassell.

Souviney, R.J. (1994). *Learning to Teach Mathematics.* New York: Macmillan Publishing Company.

IMPROVING TEACHERS' TEACHING AUTONOMY BASED ON DEVELOPING TEACHING NORMS

Wen-Huan Tsai

National Hsinchu University of Education

Cobb & McClain (2001) argued that it is not possible to adequately account for individual students' mathematical learning as it occurs in the classroom without also analysing the developing mathematics practice of the classroom. They also argued that it is not possible to adequately account for the process of teachers' development without also analysing the pedagogical community in which they participate. Therefore, the purpose of this study was intended to describe and interpret how normative aspects of teaching (teaching norms) constructed by the professional community and how it affected teachers' teaching practices in their classroom.

The study was based on Cobb & Yackel's (1996) theoretical perspectives of the relations between the psychological constructivist, sociocultural, and emergent perspectives in order to examine both teacher's teaching and students' learning. Through exchange points of view, teachers develop an appreciation for diversity of thought. They become better at seeing another's perspectives, which leads to better pedagogical reasoning on their teaching. In this study, the activities were structured to ensure that knowledge was not only actively developed by teachers but also involved in creating a safe environment for discussing, negotiating, and sharing the meanings of teaching based on what they observed their students learning in their classroom.

The results of this study showed that establishing the teaching norms can foster teacher's teaching autonomy. Teachers with teaching autonomy promoted their students becoming as self-directed learners who were used to ask, inquire, and figure out the answer in their classroom communities. It was found that the process of fostering students' intellectual and social autonomy was consistent with that of enhancing teachers' teaching autonomy. The teaching norms promoted the teachers' teaching autonomy in their teaching practice through the dialogues of the professional community and also developed the learning norms that promoted students' learning autonomy in the classroom communities that are reflexive.

Reference

Cobb, P. & McClain, K. (2001). An approach supporting teachers' leaning in social context. In F.L. Lin & T.J. Cooney (Eds.), *Making Sense of Mathematics Teacher Education*, 207-231. Kluwer Academic Publisher. Printed in the Netherland.

Cobb, P. & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of development research. *Educational Psychologist*. 31(3/4), 175-190.

MATHEMATICAL EDUCATION OF THE TATARS

N. K. Tuktamyshov
Building University

The presentation of the results in the form of a short oral report is determined by its purposes: the introduction of a little-known mathematical history of the Tatars to the international mathematical community; to show the Tatar nation's achievements the in the field of Mathematics since earliest times; to stimulate the mathematical historians' interest in studying Tatar mathematical ideas.

The Tatars are the nation living along the bank of the river Volga since ancient times and their culture is mixed up with the culture of Bulgarian Khanate, Golden Horde, Kazan Khanate, Hun Khaganate. The known manuscripts are "Taftazani" in geometry by Saggetdin Magsud (died in 1389); in arithmetic and algebra by Mukheddin Mukhammad Akhmetshi (15th century); the arithmetic manuals by G. Davletyarov (1898); the geometry manuals by Mukhammed-Zyuya Bakhtiyar (1908) and others. At that time the term of studying was not limited and the classes of Mathematics took about 3-4 hours a week.

MEASURING SPACE, VOLUME, LENGTH, TIME AND MONETARY UNIT

In Bulgaria, people widely used calculation based on the 12 –year animal cycle and solar – lunar calendar. The 12th century mintage was founded on oriental price standards (the basic weight of one myaskal was equal to 4,5 grams). The length unit was measured in a number of different ways: "by ear" ("Ber chakrym" and others) or "by eye" ("a cubit", "a foot", "a finger", etc). People used "kantar" (a large kantar- 40 kg) as the measure of weight and such units as a bucket, a tub, a sack and others as the measure of volume (Berkutov, V.M, 1997).

A SHORT GENESIS OF MATHEMATICAL TERMINOLOGY

Mathematical terminology of 17 -18 centuries was borrowed from Arabic, but since the end of the 19th century the terms have been used in the Tatar literary language. The first terminological dictionary in Tatar containing 1497 terms was issued in 1947. At present, the complete mathematical terminology in Tatar has been substantially developed and present significant importance for all Turkic nations.

CONCLUSION

Many ancient mathematical works have not lost its significance in education until now and they allow: to choose the system of nation –oriented values, to use national psychological peculiarities connected with national historic traditions and patterns of thinking to the benefit of the students. The establishment of mathematical education including higher education in Tatarstan enables to realize the language culture through its style, its associativity, the inferencial and language -bound logic.

References

Berkutov, V.M. (1997). *The Development of Mathematical Education of Bulgar - Tatars*. Kazan, Izdatelstvo "Dom Pechati", 35-50.

PATTERNING IN EARLY CHILDHOOD

Marianna Tzekaki M. Kaldrimidou
Aristotle University of Thessaloniki University of Ioannina

The paper presents a research about patterning abilities of young children (5-6 years old) based on the study and analysis of patterning tasks. In general, current findings in the field of mathematics education promote activities that encourage children to look for repeated forms, structures and relations in different situations from early age, arguing for their significance to the mathematical development (Mulligan et als, 2006). Research about the abilities of young children to recognise, repeat or continue diverse kinds of patterns presents positive results (Fox, 2005), classifying them, however, at different levels or stages of the development of the relative faculties (Michael et al., 2006, Warren, 2005).

Our research was carried out with 75 pre-schoolers, whose patterning abilities were studied before any relevant systematic teaching intervention. The children were examined in patterning tasks selected according to various criteria (familiarity, complexity, variety of material and content) and their results were analyzed on the basis of the context and the specificity of the proposed tasks. The research outcomes confirm previous studies as for the spontaneous ability of young children to continue or complete a given pattern, but indicate, furthermore, a differentiation of this ability in relation to the type and the characteristics of each pattern. This connection allows us to locate elements that could be helpful for the design of appropriate teaching proposals, e.g., with respect to the types of patterning tasks that would allow children to use matching strategies instead of looking for the rule that characterizes the relative pattern.

References

Fox, J. (2005). Child-initiated mathematical patterning in the pre- compulsory years. In H. Chick & J.L. Vincent (Eds.), *Proceedings of 29th PME* (Vol. 2, pp. 313- 320). Melbourne: PME.

Michael, S., Elia, I., Gagatsis, A., Theoklitou, A., & Savva, A. (2006). *Levels of Understanding of Patterns in Multiple Representations.* In Novotna, J., Moraova, H., Kratka, M., & Stehlikova, N. (Eds.), *Proceedings of 30th PME* (Vol. 4, 161-168). Prague: PME.

Mulligan, J., Mitchelmore, M., & Prescott, A. (2006). Integrating concepts and processes in early mathematics: The Australian Pattern and Structure Mathematics Awareness Project (PASMAP). In Novotna, J., Moraova, H., Kratka, M., & Stehlikova, N. (Eds.), *Proceedings of 30th PME* (Vol. 4, 209- 216). Prague: PME.

Warren, E. (2005). Young children's ability to generalize the pattern rule for growing patterns. In H. Chick & J.L. Vincent (Eds.), *Proceedings of the 29th PME* (Vol. 4, 305-312). Melbourne: PME.

MAKING CONNECTIONS WITHIN TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE

Colleen Vale and Alasdair McAndrew

Victoria University

In this presentation we would like to share some of our findings from an on-going project concerning the pedagogical content knowledge (PCK) of junior secondary mathematics teachers who are teaching out of field. A recent government report has confirmed that as many as 50% of junior secondary mathematics teachers in Australian schools do not hold the required tertiary mathematics qualifications (McKenzie, Kos, Walker and Hong, 2008). These findings indicate the crisis in the supply of qualified teachers of mathematics for all levels of secondary mathematics and have significant implications for the mathematical performance and participation rates of Australian students in secondary mathematics.

We are particularly interested in these teachers' mathematical content knowledge and how this knowledge is connected with PCK as defined by Shulman (1987). Chick, Baker, Pham and Cheng (2006) proposed a framework of PCK that groups elements of PCK into three categories. One of the elements in the secondary category, content knowledge in a pedagogical context, is mathematical structure and connections, "evident when the teacher makes connections between concepts and topics, including interdependence of concepts" (Chick et al., 2006, p.299).

Data for this study were gathered during a practice-based professional learning program conducted over the 2007 school year for secondary out of field mathematics teachers. The program included mathematical and professional learning tasks and experienced senior secondary mathematics teachers were mentors. The mathematical content focussed on algebra, functions and calculus. We will discuss the connections that teachers made within mathematics and between mathematics and other elements of PCK that we used to identify elements of the professional learning program that enhanced teacher's PCK of junior and senior secondary mathematics.

References

Chick, H., Baker, M., Pham, T., & Cheng, H. (2006). Aspects of teachers' pedagogical content knowledge for decimals. In J. Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 297-304). Prague: PME.

McKenzie, P., Kos, J., Walker, M., & Hong, J. (2008). *Staff in Australia's Schools*. Canberra: Commonwealth of Australia.

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1-22.

DURABILITY OF PRESERVICE TEACHER LEARNING FROM USING A VIDEO CURRICULUM IN A METHODS COURSE

Laura R. Van Zoest
Western Michigan University

Shari L. Stockero
Michigan Technological University

Practice-based materials are believed to hold great promise for mathematics teacher education (e.g. Barnett, 1998). Not only do they situate learning in a meaningful context, they also provide a means for teachers to closely examine practice, discuss various interpretations of classroom events, and consider outcomes of instructional decisions. Preservice teachers who engaged with the Learning and Teaching Linear Functions [LTLF] video case curriculum (Seago, Mumme & Branca, 2004) in a mathematics methods course increased their level of reflection and their tendency to ground their analyses of teaching in evidence (Stockero, 2006). Furthermore, when compared to a group of their peers who did not engage with a coherent video curriculum, the preservice teachers who engaged with the LTLF videocase curriculum showed a greater tendency to analyze individual student thinking, rather than make ungrounded generalizations about the thinking of students as a group.

While these results are promising, the long-term effects of using the LTLF curriculum remain unknown. The current pilot study extends our understanding of the effects of engaging with the LTLF curriculum by bringing together one year later preservice teachers who had engaged with the materials in their first mathematics methods course. Specifically, we address the question of whether the learning outcomes from using a practice-based professional development video curriculum during preservice teacher education are durable and, if they are, in what ways.

We report on two main forms of data: 1) individual written reflections on a classroom video clip, analysis of student work, and responses to a series of questions related to mathematical and pedagogical issues in the video clip; and 2) a group discussion centered on the video clip that parallels the class discussions in which they had engaged in their university methods course. Analysis centers on comparing the content of the group discussions to that of discussions of the LTLF videocases the beginning teachers had participated in during their initial mathematics methods course to determine whether there is any change in the level of reflection, the focus of the discussions, and the extent to which evidence is used.

This has been submitted as a short oral because the work is in process.

References

Barnett, C. (1998). Mathematics teaching cases as a catalyst for informed strategic inquiry. *Teaching and Teacher Education, 14*(1), 81-93.

Seago, N., Mumme, J., & Branca, N. (2004). *Learning and Teaching Linear Functions: Video Cases for Mathematics Professional Development* (pp. 6-10). Portsmouth, NH: Heinemann.

Stockero, S. L. (2006). The effect of using a video-case curriculum to promote preservice teachers' development of a reflective stance towards mathematics teaching. (Doctoral dissertation, Western Michigan University). DAI, 67 (09A), 3339-3520.

A LONGITUDINAL STUDY OF CURRICULUM IMPLEMENTATION AND GROWTH OF STUDENT UNDERSTANDING USING INTEGRATED MATHEMATICS/SCIENCE MODULES

Philip Wagreich and Howard Goldberg
University of Illinois at Chicago

This presentation will outline the results of a three-year longitudinal study of 4000 students using integrated math/science units developed by the Teaching Integrated Mathematics (TIMS) Project. A major goal of the curriculum is to develop student understanding of proportional reasoning and its applications in the real world. A 38 item "integrated math/science concept inventory" pre-posttest was administered over 3 years. The data showed that there was a very strong linear correlation between grade level and test score and that the intervention produced a significant increase in student understanding. In addition, the effect of SES and other variables was studied. Given the current call in the U.S. for evidence of curricular efficacy, the methodology as well as the results of this study may be of interest.

DESCRIPTION

This presentation will outline the results of a three-year longitudinal study of 13 schools (Kindergarten-8th grade) with a population of over 4000 students, that had implemented a mathematics/science curriculum sequence using units developed by the Teaching Integrated Mathematics Project at the University of Illinois at Chicago. The curriculum sequence focuses on the scientific method and the concept of a variable. A major goal of the curriculum is to develop student understanding of proportional reasoning and its applications in the real world. A 38 item "integrated math/science concept inventory" pretest was administered to students in grades 3-8 at the inception of the project, and a posttest at the end of that academic year and at the end of the next two academic years. A 10-question subset of the test was administered to students in grades 1 and 2. Performance of each student was tracked across the four versions of the test. The data showed that there was a very strong linear correlation between grade level and test score. This enables one to calculate the annual growth in student understanding before the intervention. The study showed that the intervention produced a significant increase in student understanding. In addition, the effect of SES and other variables was studied. Given the current call in the U.S. for evidence of curricular efficacy, the methodology as well as the results of this study may be of interest.

References

Goldberg, H. & Wagreich, P. (1990). A model integrated mathematics science program for the elementary school. *International journal of Educational Research*, *14*(2), 193-214.

Isaacs, A., Wagreich, P., & Gartzman, M. (1997). The Quest for Integration. *American Journal of Education*, *106*(1), 179-206.

Carter, M.A., Beissinger, J., Cirulis, A., Gartzman, M., Kelso, C.R., & Wagreich, P. (2003). Student learning and achievement with math Trailblazers, In S.L. Senk & D.R. Thompson (Eds.), *Standards Oriented School Mathematics Curricula: What does the Research Say About Student Outcomes?* (pp. 45-78). Hilldale, N.J.: Lawrence Erlbaum Associates, Inc.

YOUNG INDIGENSOUS STUDENTS NUMERACY LEARNING: THE ROLE OF ORAL LANGUAGE

Elizabeth Warren and Janelle Young — Australian Catholic University

Eva DeVries — Independent Schools Queensland

This paper reports on a component of a research project, Young Australian Indigenous students Literacy and Numeracy (YAILN), a longitudinal study investigating learning and teaching activities that support Young Indigenous Australian students as they enter formal schooling. The pre and post test results of the School Entry Number Assessment (SENA), an interview conducted with 48 students (average age 4 years and 11 months) indicated that although Indigenous Australian students scored significantly lower on the pre test, intervention focussing on (a)the langue of mathematics, and (b) representations that support mathematical thinking in a play-based context assist these students to begin to bridge the gap in their learning. Of particular importance was using positional language in an oral context and mapping this language onto number concepts.

The use of spoken language in school and the types of interactions teachers utilize can either advantage or disadvantage Indigenous Australian students. Furthermore, the importance of spoken language as the foundation for all learning is often not fully recognized and many young Indigenous Australian children are not able to make a strong start in the early years of schooling as the discourses of the family often do not match that of the school (Cairney, 2003). This mismatch of home and school language has been shown to disadvantage Indigenous students' achievements in literacy and numeracy in the long term (Dickinson, McCabe & Essex, 2006; MCEETYA, 2004).

The research was conducted in 7 preschool classrooms from 5 schools in North Queensland. In Queensland preschool is the first year of formal schooling. Within this region of Australia a considerable number of schools cater for Indigenous students and many from other cultures. The sample consisted of 7 teachers and 125 students (average age 5 years).

The initial results of this research indicate that oral language has a substantive role to play in the development of an understanding of number. It is conjectured that focussing on oral language development in the initial phases of schooling allows all students to begin school on an equal footing, allowing those students with little background in number on school entry the opportunity to 'catch up' with their peers.

THE INTRODUCTION OF EXPLORATORY TALK IN SECOND-LANGUAGE MATHEMATICS CLASSROOMS: A PILOT STUDY

Lyn Webb and Paul Webb

Nelson Mandela Metropolitan University

The South African Department of Education advocates collaborative and constructivist learning, which implies interactive discourse between pupils, and between pupils and their teachers (Setati, 2005). In this paper we draw on a pilot study conducted in the Eastern Cape, where teachers were introduced theoretically to the practice of exploratory talk and then tasked to perform an action research project on introducing discussion in their own mathematics classrooms. The reasons for adopting an exploratory talk approach is that it has been claimed that working in groups and talking with other learners leads to the development of mathematical reasoning (Mercer & Sams, 2006); and because, despite the fact that pupils are often seated in groups in South African classrooms, very little meaningful discourse takes place in these settings (Taylor & Vinjevold, 1999). The research aspect of the project aimed at investigating whether the intervention was successful in terms of enabling teachers to initiate the type of discussion being promoted and, if so, what strategies they used to promote this type of discussion within their context of second-language teaching and learning. The results of the study suggest some successes in terms of teachers initiating exploratory talk and highlighted the fact that these successes were only achieved where code switching to the pupils' main language formed an integral part of the process.

References

Mercer, N. & Sams, C. (2006). Teaching children how to use language to solve maths problems. *Language and Education,* 20(6), 507-528.

Setati, M. (2005). Teaching mathematics in a primary multilingual classroom. *Journal for Research in Mathematics Education*. 36(5) pp. 447 – 466.

Taylor, N. & Vinjevold, P, (1999). *Getting Learning Right.* Report of the President's Education Initiative Research Project. Johannesburg: Joint Education Trust.

TEACHING FOR ABSTRACTION: PERCENTAGES

Paul White — ACU National
Michael Mitchelmore — Macquarie University
Sue Wilson and Ronda Faragher — ACU National

Mitchelmore and White (2000) proposed a theory of learning and teaching by successive abstraction based on the belief that an abstract concept is "the end-product of ... an activity by which we become aware of similarities ... among our experiences" (Skemp, 1986, p. 21). The 4-phase theory of *Teaching for Abstraction* was applied to the teaching of percentages in five Grade 6 classes in regional New South Wales. An 8-lesson unit was devised that fell into three parts: The first three lessons focused on calculation, the next four lessons explored a variety of everyday situations involving percentages, and in the final lesson students constructed their own problems.

The four phases of the theoretical framework were embedded in the unit as follows:

- *Familiarisation*: Students explored with a number of percentage contexts and the calculations that commonly arise in such contexts.
- *Recognition*: Students were guided to compare percentage calculations across different contexts.
- *Reification*: Students were asked to make and explain generalisations.
- *Application*: Students created new problems where percentages were used.

Lesson observations showed a high level of student engagement in discussions. However, some teachers were reluctant to allow exploration and reordered their lessons to model the results of the intended explorations.

The results of pre- and post-interviews and written tests showed that the number of students who could both calculate simple percentages and use them appropriately in context increased substantially as a result of the unit. The improvement in the students' explanations was particularly striking and transcended what could be expected from either memorisation or currency of concepts recently studied. The move away from the inappropriate additive strategies highlighted in the literature was particularly encouraging.

It could be argued that the positive outcomes in this study were simply the result of establishing interactive classrooms. However, we claim that the true cause was the focus on generalizing from familiar contexts which is a feature of our theory. The theoretical model (even if it was not followed rigorously) resulted in new directions for teachers and improved learning for students.

References

Mitchelmore, M.C. & White, P. (2000). Development of angle concepts by progressive abstraction and generalisation. *Educational Studies in Mathematics, 41,* 209-238.

Skemp, R. (1986). *The psychology of learning mathematics* (2nd ed.). Harmondsworth, UK: Penguin.

GEOMETRIC CONJECTURES OF SIXTH AND EIGHTH GRADE STUDENTS

Chao-Jung Wu, National Taiwan Normal University
Miao-Ling Lin, Yong Ping Elementary School
Ju-Chen Chen, National Taiwan Normal University

Conjecturing is the first step of mathematic discovering (Lakatos, 1976) and reasoning (Reid, 2002). The purposes of this study were (1) to contrast 24 6^{th} grade and 27 8^{th} grade students' conjectures and (2) to identify the types of students' conjectures, when they were given geometric conditions and figures.

The participants were sampled from 16 classes of four schools. Because our previous research showed that low achievement pupils produced incomprehensible conjectures, mathematic achievements of all participants ranked in top two-thirds in their classes. Interviews were conducted with the students individually. Following an exercise, students were sequentially given three items and three figures (typical, conjunctive, and extreme) in each item. These items included (1) There is a ΔABC. Point D is the middle of $\overline{AB}$ and E is the middle of $\overline{AC}$. Connecting points D and E forms $\overline{DE}$; (2) Select four different points A, B, C and D sequentially on a circle O. Link $\overline{AC}$, $\overline{AB}$, $\overline{BD}$, and $\overline{CD}$; (3) O is the center of a circle. Draw a diameter BC. Select a point A on this circle and link $\overline{CA}$ and $\overline{AB}$. The participants were required to deliberate any geometric invariance that simultaneously existed according to given conditions.

The **number of conjectures** was a roughly quantitative index and the **correct rate** demonstrated how accurate of students' conjectures. These two indexes derived from two grades were similar. The average number of conjectures of each item was 5.59 and the average correct rate was .78. However, the **relevant rate** which displayed how extensible of conjectures demonstrated some different between 6^{th} and 8^{th}. As for Item 1, 75% of 6^{th} students identified DBCE as a trapezoid was significantly higher than 33% of 8^{th}, but 17% of 6^{th} students directly identifying $\overline{DE} // \overline{BC}$ was not significantly lower than 30% of 8^{th}. This result showed that 6^{th} students preferred to shape identification and 8^{th} students generated more attributes. As for Item 3, 41% of 8^{th} students found that the areas of two triangles with equivalent base and altitude are identical was significantly higher than 17% of 6^{th}. This result demonstrated 8^{th} students proposed more conjectures requiring calculation or inference. Although the participants generated more visual type of conjectures as anticipated, some of them created conjectures with assistant lines. In addition, some participants found the invariance of area proportion, e.g. $\triangle ABC = 4\triangle ADE$ in Item 1 or $\triangle ABC = 2\triangle ACO$ in Item 3.

References

Lakatos, I. (1976). *Proofs and Refutations: The Logic of Mathematical Discovery*. Cambridge: Cambridge University Press.

Reid, D.A. (2002). Conjectures and refutations in grade 5 mathematics. *Journal for Research in Mathematics Education, 33*(1), 5-29.

SHAPING TEACHING ABILITY IN K-8 MATH CLASSROOMS

Zhonghe Wu
National University

Shuhua An
California State University

THE PURPOSES AND RESEARCH QUESTIONS

The goals of this study are to contribute to a better understanding of how to measure teaching ability in mathematics classrooms, to find measurable criteria on teaching ability by identifying the functions, dimensions, and features of teachers' ability for effective teaching, and to examine how different types of teaching abilities result in different learning outcomes.

The following research questions were examined: 1) What are the patterns of K-8 mathematics teachers' teaching? 2) How do these patterns contribute to teaching ability?

METHODOLOGY

Subject. Nine K-8 teachers from six schools in Southern California participated in this study in the academic year 2005-2006.

Procedure. The teachers' classroom teaching was observed and videotaped weekly by two researchers.

Instruments. Data were collected via 119 video lessons, more than ten from each teacher, interviews with teachers, and students' assessments.

Data Analysis: Data analysis was ongoing throughout the period of this study. Interviews, field notes of observations, teachers' and students' reflections were analysed using a qualitative method. The observations and responses from the interviews and reflections will be coded, categorized, and compared for emerging themes.

RESULTS

K-8 math teachers have their teaching patterns, and their teaching patterns are significantly different from one another.

The Features and Dimensions of Teaching Ability

1) Following up on homework, 2) stating learning objectives and orienting students toward the lesson, 3) reviewing prerequisites, 4) presenting new material, 5) guided practice, 6) independent work for practice, 7) assessing performance and providing feedback, 8) differential instruction, 9) time not spent on math instruction

SIGNICANACE OF THE RESEARCH

Although this study confirmed the TIMSS study in teaching patterns, teachers in this study showed differences in their individual style of teaching ability; statistical analysis showed that there are significant differences between teachers in each dimension of teaching ability.

The results of this study indicate that it is the challenge to investigate the measurable criteria for teaching ability, but it provides a new direction to improve classroom teaching. The investigation of the functions, dimensions, and features of teaching ability in this study provides not only concrete and valid instruments to measure teaching ability, but also concrete and practical guidelines for classroom teachers on how to best apply their knowledge to teach math effectively in a balanced way. Furthermore, this study shows that teaching ability can be acquired through daily practice. It is imperative for math teachers to improve their competence in applying knowledge so that it can have a significant effect on student achievement.

VISUALISATION IN MATHEMATICS LEARNING: CANONICAL IMAGES AND SEMIOSIS

Tracy Wylie
Kingsfield School

Laurinda Brown
University of Bristol

In this study, Presmeg (2006) has been followed in using semiotics – the study of the meaning of language, symbols and signs – as a theoretical perspective. A sign can be classified as either iconic, indexical or symbolic.

The term canonical image is used to describe an image that is *economical* in that it gives direct access to the mathematical concept (Breen, 1997). An example of a canonical image is the unit circle image for trigonometry. Another possible definition for a canonical image is an image that affords the flexibility to be used directly in a number of ways with a variety of problems – an image that can be described as iconic, indexical *and* symbolic.

The study was based on six 18-year-old students; five male and one female. The students were video-taped working in pairs on a set of mathematical problems and what was particularly significant was their use of hand gestures.

The data collected showed evidence of "semiotic nodes" (Radford *et al.*), that is, "pieces of the students' semiotic activity where action, gesture, and word work together to achieve knowledge objectification" (p. 56). There is evidence that iconic gesturing (mimicking) and indexical gesturing (pointing to diagram) were being used, which in turn demonstrated the objectification of the mathematical relationships being dealt with. Students were accessing the canonical image for trigonometry to allow them to answer problems on complex numbers and on general trigonometric solutions. This flexibility is illustrated through the different forms of gesturing.

References

Breen, C. (1997). Exploring Imagery in P, M and E. In E. Pehkonen (Ed.), *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 97-104). Finland: PME.

Presmeg, N. (2006). A Semiotic View of the Role of Imagery and Inscriptions in Mathematics Teaching and Learning. In J. Novotnà, H. Moraovà, M. Kràtkà, & N. Stehlikovà (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 19-34). Prague: PME.

Radford, L., Demers, S., Guzmán, J., & Cerulli, M. (2003). Calculators, Graphs, Gestures and the Production of Meaning. In N. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 55-62). Hawaii: PME

A RESEARCH OF IMPLEMENTING THE VALUE-ORIENTED PROBLEM-CENTERED DOUBLE-CYCLES INSTRUCTIONAL MODEL (V-PCDC-IM) IN 8TH GRADE MATHEMATICS CLASSROOM

Shih-Yi Yu and Ching-Kuch Chang
National Chunghua University of Education

Many students felt that mathematics was useless and valueless via traditional lecture teaching and researchers wanted to improve this situation for a long time. This PCDC-IM was created and advanced by Dr. Ching-Kuch Chang on 1995 (Figure 1.). Empirical researches (Chen, 2001; Lin, 2001; Tsai, 2002) showed that PCDC-IM helped students learn mathematics better and also improved students' attitude toward mathematics. In order to enhance students' sanction of mathematical values, we put our intended mathematical values (such as rationalism, openness, practical, preparing, training, multiple, interesting) (Bishop, 1991) (Lin, 1977) into the teaching cycle of PCDC-IM and named it "V-PCDC-IM". The purpose of this study was to implement the V-PCDC-IM in 8^{th} grade mathematics classroom, especially to explore into the teaching process of this model, and the differences of students' sanction of mathematical values. An action research method was mainly adopted in this study, and assist with the questionnaire to investigate the V-PCDC-IM of the two classes of 8^{th} grade. Data collection included teaching journals, classroom observations, field notes, value-oriented teaching material, questionnaires, interviews, and video tapes. Data analysis included content analysis and t-test with the SPSS. We founded that teachers needed to follow some tips to achieve the four components of the teaching cycle of V-PCDC-IM. After implementing V-PCDC-IM, students' sanction of mathematical values will approach to the teacher's intended values. In the items of the questionnaire, the highest sanctions of mathematical values are rationalism, openness, training value, and the lowest sanction of mathematical values is the interesting value. The sanctions of mathematical values between V-PCDC-IM and the traditional instruction are different. The sanctions of V-PCDC-IM are higher than the traditional instruction. The sanctions among intended values, such as openness, progress, practical, interesting, rationalism, training values are all significant.

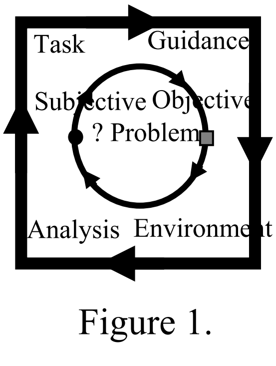


Figure 1.

Reference

Bishop, A. J. (1991). Mathematical values in the teaching process. In A. J. Bishop, A. S. Mellin- Olsen, & J. Van Dormolen (Eds.), *Mathematical Knowledge: Its Growth Through Teaching* (pp. 195-214). Dordrecht, the Nethelands: Kluwer.

Chang, C. K. (1995). Practicality of Implementing Problem-centered Instruction in Taiwan: Summary of Two Project Teachers in Eighth-grade Math Classes. *Chinese Journal of Science Education, 3*(2), 139-165.

POSTER PRESENTATIONS

THE GENETIC DECOMPOSITION OF THE DEFINITE INTEGRAL: A THEORETICAL ELEMENT FOR THE DESIGN OF A TEACHING MODEL USING DERIVE

Francisco José Boigues, Vicente Estruch, and Ricardo Zalaya

Universidad Politécnica de Valencia

We present a work that comprises a significant part of our research on the analysis of the understanding of the definite integral. From a cognitive approach called genetic decomposition, the different stages that students go through during the development of the understanding of the definite integral have been identified.

One of the outcomes of this research is presented here: the implementation of a series of activities using a CAS (DERIVE) aimed at encouraging students to construct the elements of our decomposition proposal. Our general objectives are to promote the understanding of mathematics as a construction of cognitive objects beyond a mere formal development and to bring mathematical research closer to the academic praxis.

Furthermore, there are shown some activities to improve the understanding of the schema of the definite integral.

References

Baker, B., Cooley, L., & Trigueros, M. (2000). A calculus graphing schema. *The Journal for Research in Mathematics Education, 31*(5), 557-578.

Camacho, M. & Depool, R. (2003). Using Derive to understand the concept of definite integral. *International journal for Mathematics Teaching and learning 5,* 1-16.

Czarnocha, B., Loch, S., Prabhu, V., & Vidakovic, D. (2001). The concept of definite integral: coordination of two schemas. In M. van den Heuvel-Penhuizen (Eds.), *Proceedings of the XXV Conference of the P.M.E., (*Vol. 1, pp.-12-17), Utrecht: Freudenthal Institute.

Trigueros, M. (2005). La noción de esquema en la investigación en matemática educativa a nivel superior. *Educación Matemática,* 17(1), 5-31.

MODEL FACILITATED LEARNING: PRESERVICE MATHEMATICS TEACHERS' INITIAL EXPERIENCE WITH HAND HELD GRAPHING CALCULATOR

Lingguo Bu, Lydia Dickey, Elizabeth Jakubowski, Hyewon Kim, J.Michael Spector, and Nermin Tosmur-Bayazit

Florida State University

As the next-generation graphing calculators enter the mathematics classroom in summer 2007, we conducted an exploratory study of prospective mathematics teachers' experience with the graphing calculator guided by theoretical framework of Model Facilitated Learning (MFL). Initial data analysis shows that MFL is particularly useful for mathematics educators and instructional designers to take full advantage of theory, methods, and new affordances of technology in bringing meaningful mathematical experiences to prospective mathematics teachers in technology-supported learning settings.

A next-generation graphing calculator embodies a linked system of dynamic mathematical representations. Taking the stance that new technologies support complex learning and that the learning process of "big ideas" of mathematics is cognitively complex, we have recently turned to the well-established theory of model-based learning and instruction (Milrad et al., 2003; Seel, 2003) for theoretical guidance in an effort to experiment with mathematics instruction using graphing calculators.

In this presentation, we will share sample lessons integrated with the new graphing calculator designed by the instructor of the course as well as the lesson plans proposed by prospective mathematics teachers to highlight their experience with the new technology. Moreover, we will provide examples of snapshots from classroom environment along with student teachers' reflections on the use and integration of this new technology in mathematics teaching and learning.

References

Milrad, M., Spector, M., & Davidsen, P. (2003). Model faciliated learning. In S. Naidu (Ed.), *Learning & Teaching with Technology: Principles and Practice*s, (pp. 13-27). London Kogan Page. Sanders, C.V

Seel, N. M. (2003). Model-centered learning and instruction. *Technology, Instruction, Cognition and Learning, 1*, 59-85.

THE ROLE OF THE CONSERVATION OF AREA IN THE SCHOOL EXPLANATION OF THE CONCEPT OF DEFINITE INTEGRAL

Ma. Guadalupe Cabañas Sánchez and Ricardo Cantoral Uriza

Institute for Research and Advanced Studies of the National Polytechnic Institute

This poster presents the central aspects of didactical proposal related with the definite integral, based on the school explanation of the "area under the curve". We are using the socioepistemological approach to research in Educational Mathematics (Cantoral & Farfán, 2003). From this perspective we ask ourselves what are the uses and contexts of the notion of area prior to their Cauchy definition and of the contexts and procedures in which the definite integral is introduced since Cauchy's work. Thus, the area can be compared, conserved, estimated and measured. The context are characterized as static and dynamic. The contexts and procedures in which the integral is introduced are as follows: Contexts: Conception of function and continuity. Procedures: Conception of primitive function and the distribution of points on the interval of integration where the function is continuous.We are showing in a graphic way the role of conservation of the area in the school explanation of the definite integral. The results reported in Piaget, J., Inhelder, B., Szeminska, A. (1970) and Freudenthal (1983) in relation to the study of area are used in the proposal. Our hypothesis is that before defining integral and the elemental focus of "subdividing and calculating using formulas", a study is required of the notion of area using other activities which can be guided by social practices experienced by students inside and outside the classroom. Activities such as *sharing, comparing and reproducing, measuring, quantifying and conserving* (Cabañas & Cantoral, 2006).

References

Cabañas, G. & Cantoral, R. (2006). La conservación en el estudio de Área. En R. Cantoral, O. Covián, R. Farfán, J. Lezama, & A. Romo (Eds.), *Investigaciones sobre Enseñanza y Aprendizaje de las Matemáticas: Un Reporte Iberoamericano* (pp. 199-226). México DF, México: Diaz de Santos-Comité Latinoamericano de Matemática Educativa A.C.

Cantoral, R., Farfán, R. (2003). Mathematics Education: A vision of its evolution. *Educational Studies in Mathematics,* 53 (3), 255 - 270.

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Holland: D. Riedel Publishing Company.

Piaget, J., Inhelder, B., & Szeminska, A. (1970). *The Child´s Conception of Geometry.* New York; U.S.A.: Basic books, Inc., Publishers.

LEARNING TO ATTEND TO STUDENTS' MATHEMATICAL THINKING: HOW RICH-MEDIA RESOURCES CAN HELP

Chia-Ling Chen, Patricio Herbst, and Vu-Minh Chieu

University of Michigan

The work of teaching mathematics involves attending to students: anticipating what they might do in response to tasks and understanding what they actually do. Teacher education should include opportunities to develop the skill to do that. The practice of lesson planning provides context for it. We investigate how a rich media environment (ThEMaT Composer) based on graphic representation of classroom scenes may engage prospective teachers in anticipating students' responses as they lay out and review a comic-based lesson representation. The poster shows the design of such environment and examples of how prospective secondary mathematics teachers become aware of diverse and timely students' contributions.

OBJECTIVES

This poster shows how prospective teachers use a lesson representation tool to attend to mathematical interactions with students. ThEMaT Composer is a software tool that allows teachers to sketch a lesson in the form of a slide show where slides represent classroom scenes made with cartoon characters. By creating teacher-student dialogues for teacher and student characters in the software, prospective teachers are given the opportunity to virtually implement a lesson. In reviewing the lesson planned they get feedback about the flow of the lesson—feedback that makes them aware of discourse, representation, and diversity issues and that allows them to anticipate finer and more varied responses of students to instructional moves.

It has been argued that the experiences of observing in classroom are not enough to develop the know-how of teaching—virtual settings to practice teaching are needed to scaffold novices' learning to teach. To develop their attention to students' learning and thinking, they need to learn "in" and "from" practice (Ball & Cohen, 1999). Multimedia environments have proved beneficial to develop their insight into the practice of teaching (Lampert & Ball, 1998). ThEMaT Composer can help prospective teachers anticipate events involving teacher-student interaction, raising their capacity to attend and respond to students' input.

References

Ball, D.L. & Cohen, D.K. (1999). Developing practice, developing practitioners. In G. Sykes & L. Darling-Hammond (Eds.), *Teaching as the Learning Profession: Handbook of Policy and Practice* (pp. 3-32). San Francisco: Jossey Bass.

Lampert, M. & Ball, D. (1998). *Teaching, Multimedia, and Mathematics: Investigations of real Practice*. New York: Teachers College.

THE RESEARCH OF YOUNG CHILDREN'S NUMEROSITY DISCRIMINATION IN DIFFERENT ETHNIC DEVELOPING QUANTITATIVE REASONING

Ching-Shu Chen

University of Technology

The Center of Teacher Education

The purpose of this research was to explore the development of quantitative reasoning in young children, and also to analyze the numerosity discrimination of children from different groups. The results will be applied to early childhood mathematical pedagogy. The research subjects came from three kindergartens in elementary schools in Hawaii and Taiwan. Each research group has twenty children. In addition, twenty children were sampled from a private kindergarten in Taiwan. Each school had twenty children who were equal in gender and were an average of 5.5 years old. Research methods used assessment, observation and interview. There were two objective scales to evaluate the children's quantitative reasoning. During the process of the research, the researcher gave opportunities to children who operated objects and named objects, and practiced a small aggregate amount before the formal test. At the same time, the researcher asked the children, without actually counting, but with a quick look with their eyes to estimate two aggregate amounts, then, solved problems. The result showed that the children had a 75% success rate to discriminate numerosity in real objects that fit the developmental model of cognition, but without the cultural difference. Furthermore, when using the scale of half concrete objects (sticker) to test the children, the result revealed that children performed the 2:3 ratios to discriminate numerosity was better than 5:6 ratios. However, Taiwanese children had correct answer scores that were higher than those of Hawaiian children on some problem items. Moreover, to apply the research results to early childhood instruction: 1.To start the concept of quantity learning, then to practice counting to help children have good number concepts. 2. To combine the children's life experiences to give them opportunities of operation, estimation, comparison and prediction so that those children can have a good mathematics foundation.

EXPLORING STUDENTS' SEMANTICS UNDERSTANDING TOWARD LETTERS OF ALGEBRAIC THROUGH QUESTIONING MODEL

Yen-Ting Chen
Chung Hwa University of Medical Technology

Shian Leou
National Kaohsiung Normal University

The study was to explore the conceptual change model of understanding letters' semantics toward algebraic expression in a group of Taiwanese seven graders. After analyzing questionnaire responses of an initial sample of 76 students, 15 students were selected, from three levels of high, medium, and low score, as the final subjects. Comparing the result of questionnaire and the analysis by flow map technique as the role of letters' semantics, the study examined the starting behavior before inquired the 15 students by the guiding model of Questioning. Based on the Growing Model of Mathematic Understanding, a qualitative analysis through a series of Questioning revealed the conceptual change mode of understanding letters' semantics toward algebraic expression about three level students. This study disclosed (1) the different types of question asked by instructor on promoting Progress Understanding and Regress Understanding among three level students (2) the different strategies of solving problem using by three level students (3) the distribution of understanding letters among three level student.

First, the findings showed that different level students hold different perception toward letters' semantics. Secondly, it was also revealed that higher level students engender more frequencies of Progress Understanding and Regress Understanding. Higher level students concentrate on constructing and clarifying the semantics understanding towards algebraic literal symbols; whereas, lower level students focus on constructing the operational skills about algebraic literal symbols. Higher level students can achieve the situation of Formalization, Observation, and Construction; however, lower level students cannot reach the situation of Construction. Third, the examples, from the phase of promoting Progress Understanding, guiding high level and medium level students focus on clarifying the connotation of each formula and comprehending the semantic role of literal symbols; nevertheless, instructor should take more time on lower level students to explain the meaning of examples and the skills of operation. From the phase of promoting Regress Understanding, examples about metacognitive judgment make three level students reach Regress Understanding, but the operational skill about literal symbols also make lower level students reach Regress Understanding. From the strategy of promoting Progress Understanding, three level students utilizing the method of Advanced Question, Pause Question, and Obstructed Question.

THE ISIS PROBLEM: AN INSTRUMENT FOR EXAMINATING FUTURE MATHEMATICS TEACHERS' IDEAS ABOUT PROOF

Dirk De Bock
European University College Brussels
University of Leuven

Brian Greer
Portland State University

The Isis problem, which has a link with the Isis cult of ancient Egypt (Davis and Hersh, 1981), asks: "Find which rectangles with sides of integral length (in some unit) have area and perimeter (numerically) equal, and prove the result." The problem can be initially approached using routine expertise but then requires (for almost all school and college students) adaptive expertise, yet relies on the most rudimentary technical mathematics. It can be extended in numerous ways, for example by asking which triangles with integral sides have the corresponding property (a significantly more difficult problem) or by shifting up dimensionally to ask which cuboids with integer sides have volume and surface area numerically equal. Interesting questions then arise as to which proofs for the original problem are extendible. The problem is notable for the multiplicity and variety of proofs (empirically grounded, algebraic, geometrical) and associated representations. A selection of such proofs provides an instrument for probing students' ideas about proof.

A group of 39 Flemish pre-service mathematics teachers was confronted with the Isis problem. More specifically, we first asked them to solve the problem and to look for more than one solution. Second, we invited them to study five given proofs (factorization, tiles, unit fractions, graph, table) and to rank these proofs from best to worst. The poster will show different self-found proofs in this group of Flemish pre-service mathematics teachers, as well as their rankings of and comments on the five given proofs. The results highlight a preference of many students for algebraic proofs (factorization and unit fractions) as well as their rejection of experimentation.

Because the Isis problem relates two quantities of different dimensionality, it also connects with the considerable body of research showing that students do not understand the basic principle that linear enlargements by factor k result in 2-dimensional quantities, such as area, being enlarged by a factor of k2, and 3-dimensional quantities, such as volume, by a factor of k3 (De Bock, Van Dooren, Janssens, & Verschaffel, 2007), a principle that explains many phenomena in biology and engineering.

References

Davis, P. & Hersh, R. (1981). *The Mathematical Experience*. Boston: Birkhauser.

De Bock, D., Van Dooren, W., Janssens, D., & Verschaffel, L. (2007). *The Illusion of Linearity: From Analysis to Improvement*. New York: Springer.

INEQUATIONS RESOLUTION USING VARIOUS REGISTERS

Vera H. G. De Souza and Tania M. M. Campos

UNIBAN-SP

We present some protocols from a group of 7 Mathematics teachers working on inequations resolution, with activities designed according to a functional graphic approach. In teachers' writing we looked for formal, intuitive and algorithmic aspects, using Fischbein's ideas. Analysis shows that these teachers seem to lack formal aspects and valorize algorithmic ones, as they can graphically treat an inequation but can't relate both treatments, algebraic and graphic, as we hoped, in order to discuss problems connected to inequations algebraic resolution.

We had a question: "In inequations resolution teaching process, can a graphic functional approach bring into light algebraic resolution formal aspects?". To discuss such approach with seven Mathematics teachers, in 2005, we designed a set of activities, to be solved with paper and pencil, asking subjects to compare algebraic and graphic treatments, as in question (1) "Solve algebraically inequation $-2x>0$. Explain your steps", followed by question (2) "Using function $f(x)=-2x$ graph, solve $-2x>0$. Are your answers coherent? Why?". We think we can find formal, intuitive and algorithmic aspects in subjects' mathematical writing on questions like that.

Protocols analysis showed that those teachers were not used to express formal aspects, as we see from an answer to question (1): "$x>0/(-2)$ $x>0$ (It is not true!) $-2x>0$ $2x<0$ $x<0$. It seems to work this way!"

They seemed to valorise algorithmic aspects, because all of them could correctly answer questions like (2), although they have done wrong answers to question (1), as in "$-2x>0$ $(-2x)/(-2)>0/(-2)$ $x>0$". Also, they didn't relate graphic and algebraic treatments, as in answer "$-2x$ is grater than zero when $x<0$! Algebraically $-2x>0$ $x>0/(-2)$ $x>0$. On the contrary??? If x coefficient is negative can't we algebraically solve it?" when teacher tried to compare both resolutions, as asked in question (2), showing that intuitive aspects are even stronger than algorithmic algebraic ones.

It seems to us that lack of formal aspects and emphasis on intuitive ones may explain why we can't positively answer our initial question, at least for this group.

References

D. Raymond. (2000). Basic issues for research in Math. Educ. In T. Nakahara, & M. Koyama (Eds.), *Proceeding 24th Conference of the International Group for the PME* (Vol. 1, pp. 55-69). Hiroshima, JP: PME.

Fischbein, Efraim. (1993). The interaction between the formal, the algorithmic and the intuitive components in a mathematical activity. In R. Biehler et al. (Org.). *Didactics of Mathematics as a scientific Discipline* (pp. 231-240). Dordrecht, HO: Kluwer

CHILDREN´S KNOWLEDGE WITH REGARDS TO ADDITION AND SUBTRACTION IN INFORMAL CONTEXTS

Juan José Díaz, Sergio Arenas Moreno, and Ignacio Martínez Gutiérrez

Universidad Autónoma de Zacatecas

The development of children´s mathematical thinking was analyzed in a variety of situations within the informal context. The sample was 72 children whom where interviewed while attempting to solve addition and subtraction problems during a game. The results indicated an irregular evolutive pattern of informal knowledge.

STUDY

In a previous study it was discovered that the arithmetical knowledge is based on the degree of abstraction related to informal contexts (Bermejo & Diaz, 2007). Kamii & Kato (2005) exponent the importance of arithmetical knowledge through informal activities such as typical children´s games. Our hypothesis assumes that the arithmetical knowledge increases with relation to the technology of the informal context. The sample included the participation of 24 five year olds, 24 six year olds, and 24 seven year olds, all residents of the southern part of Zacatecas, Mx. The material was made up of two addition problems and two subtraction problems that came up during three playtime situations: a game being played in an open area (hop-scotch), a board game (arithmetical oca) and a videogame (Math with Pipo®). The participants where interviewed with regards to the solution of each problem.

The data analysis through an ANOVA 3X2X3X2 with repeated measures in the last two factors revealed that the double Game X Age ($F_{4,132} = 2.46$, $p < .05$) interaction is meaningful. The performance in the addition and subtraction problems increases in the game being played in an open area and in the videogame at 6 years of age although it has a tendency of diminishing at age 7. This competence presents an evolutional pattern during the board game. Therefore, it can be considered that the arithmetical performance is developed with age when problems are solved in a situation similar to the academic context more than in those situations that have a higher level of informality. Informal problems have an evolutional level between the ages of 5 and 6, but they decrease by age 7, which indicates that informal activities stop being interesting for older children due to traditional schooling which implies a lack of the use of educational technology in learning.

References

Kamii, C. & Kato, Y. (2005). Fostering the development of logico-mathematical thinking in a card game at ages 5-6. *Early Education and Development, 16*(3) 367-383.

Bermejo, V. & Díaz, J.J. (2007). The Degree of abstraction in solving addition and subtraction problems. *The Spanish Journal of Psychology, 10*(2), 285-293.

INVESTIGATING BLIND LEARNERS' INTERACTIONS WITH MATHEMATICAL MICROWORLDS

Solange Hassan Ahmad Ali Fernandes — Colégio Nossa Senhora Do Rosário

Lulu Healy — Universidade Bandeirante De São Paulo

Within the domain of mathematics education, the search to understand the potential we have, as human beings, to transform sensations perceived by the sense organs into mathematical knowledge has always been centre stage. Those interested in technology and mathematics education have been particularly active within this debate, challenging the dichotomy abstract-concrete and calling for reconsiderations of the very grounds for cognition: instead of formal operations on abstract symbols, increasingly it is the situated and embodied nature of cognition that is emphasised and under attention. Embodied approaches posit that even the most abstract of symbols have physical grounding and it would seem that the dynamic mathematical representations that digital technologies afford have a role in magnifying the lens onto the ways in which mathematical meanings come about as a result of this grounding process. Yet, perhaps somewhat surprisingly, one set of questions that has not received much attention concerns learners with restricted, or no, access to particular sensory fields. How do these learners engage in the process of building meanings for mathematical objects? What are the groundings by which they make sense of the mathematical activities in which they participate? And how might new mathematical infrastructures be moulded to take these factors into account?

In this poster, we intend to present our attempts to build features designed to support the mathematical activities of students who are blind into mathematical microworlds – accessible and evocative computational worlds, which embed a mathematics that is not only formal but also related to learners' sense of themselves. Many mathematical microworlds exploit computational opportunities to build motion and other visual means of illuminating mathematical structure. Our approach in working with blind students has been to seek alternative media by which to express mathematics in dynamic forms, and especially how sound, coupled usually with tactile explorations, might be employed to model the properties of mathematical objects. We will present two examples of the microworlds we have been working with during projects supported by the Brazilian research foundation FAPESP (Projects 2004/15109-9 and 2005/60655-4). The poster will also focus on how, by considering the particularities of blind students' interactions with the different mediation systems, we might begin to understand better the learning trajectories they follow and the mathematical narratives they construct as they bring to life the computational agents they encounter.

THE INFLUENCE OF THE USE OF GEOGEBRA ON STUDENTS' PRACTICE

Josep Maria Fortuny, Nuria Iranzo, and Markus Hohenwarter

Universitat Autònoma of Barcelona

In this poster presentation we describe an ongoing research study[1] *on the interpretation of students' behaviors when solving plane analytical geometry problems by analyzing relationships between the use of the dynamic geometry system GeoGebra*[2]*, paper-and-pencil work and geometrical thinking. Our theoretical framework is based on Rabardel's (2001) instrumental approach to tool use. In our study we are investigating relationships between students' thinking and their use of techniques by exploring the influence of certain techniques on students' problem-solving strategies. The poster is organized in a three-column structure to present the aim of our research, the theoretical framework, and the analysis of student's work. We conclude with results found so far and planned further research.*

Our pilot research study has been carried out with 11 secondary students that have worked on geometry problems focusing on a Euclidean approach and problem solving. For the analysis we mainly consider: a) solving strategies in the written protocols and the GeoGebra files; b) audio and video-taped interactions with other students; and c) the opinions about the use of GeoGebra collected in a questionnaire. Through the analysis of data we characterize students' learning behaviors and discuss the idea of instrumentation linking the theoretical perspective and the classroom experiments. So far, we have identified different problem-solving strategies in the GeoGebra environment. We have classified students into categories considering: 1) their heuristic strategies (related to geometric properties, to the use of measurement tools or to both); 2) the influence of GeoGebra (visualization, geometrical concepts); and 3) the obstacles encountered. We have found that GeoGebra helps to alleviating technical work and can boost students' geometrical knowledge and understanding.

References

Rabardel, P. (2001). Instrument mediated activity in situations. In A. Blandford, J. Vanderdinckt, & P. Gray (Eds.), *People and Computers, Interactions without Frontiers* (pp. 17-30). Berlin, Germany: Springer.

[1] MEC. Development of an e-learning tutorial system to enhance students' solving problem competence. SEJ2005-02535.

[2] Geogebra environment www.geogebra.org

ANALYSING A TEACHING MODEL OF THE GEOMETRY OF THE SOLIDS IN PRESERVICE TEACHERS' EDUCATION

Edna González and Gregoria Guillén

Universitat de València

The theory of Modelos Teóricos Locales has taken as experimental methodological framework (Filloy, 1999). The aim of this study is to develop an Initial Competence Model that could be a reference for interpreting the Teaching Models presented for the teaching of geometric solids in the Training Plans for teachers. For this, we carried out an work analysis, which we have grouped, such as: i) teachers education, focusing on significant contents for a Training Plan (De Ponte & Chapman, 2006); ii) analysis of mathematical processes and observation of their learning processes (Guillén, 1997; Guillén y Figueras, 2005); iii) Freudenthal Institute (Freudenthal 1973; Treffers 1987). The criteria used were delimited in order to analyze the design and implementation of the Teaching Model. In the analysis, we have identified the contexts used, the components of the mathematical processes that were develop and the different aspects of teaching. In González et.al.(2006) the 6 established categories are described, denominated as: i) About geometry and its teaching. Student and teacher; ii) About geometric contents; iii) How do some of the students learn? What for?; iv) Class Planning; v) Interacting in the class and ... vi) And the language?.

Endnote

Scholarship holder Conacyt. México.

References

Filloy, E. & col. (1999). *Aspectos teóricos del álgebra educativa.* Investigaciones en Matemática Educativa. México, D. F.: G. Ed. Iberoamérica

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht: D. Reidel.

González, E; Guillén, G, & Figueras, O (2006). Estudio exploratorio sobre la puesta en práctica de un modelo de enseñanza para la geometría de los sólidos en Magisterio,. In Bolea, P., González, M.J., & Moreno, M. (Eds.) (2006), *Actas de X Simposio de la Sociedad Española de Investigación en Educación Matemática (SEIEM)* (pp. 195-204). Huesca, España.

Guillén, G. (1999). *El modelo de Van Hiele A a la Geometría de los Sólidos. Observación de Procesos de Aprendizaje*. (Tesis doctoral). Valencia: Universitat de València. Col·lecció: Tesis doctorals en Microfitxes. Valencia: Universitat de València.

Guillén, G. & Figueras, O. (2005). Estudio exploratorio sobre la enseñanza de la geometría en primaria. Curso taller como técnica para la obtención de datos. In A. Maz, B. Gómez, & M. Torralbo (Eds.) (2004). *Investigación en Educación Matemática. Noveno Simposio de la Sociedad Española de Investigación en Educación Matemática (SEIEM)* (pp. 227-234). Córdoba, España.

Ponte, J. & Chapman, O. (2006). Mathematics teachers' knowledge and practice. In Gutiérrez, A. & Boero, P. (Eds.) (2006), *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp. 461-494), Sense Publishers.

Treffers, A. (1987). *Three Dimensions (A Aodel of Goal and Theory Description in Mathematics Instruction - The Wiskobas Project)*. Dordrecht: D. Reidel.

A META-ANALYSIS OF RECENT RESEARCH IN EARLY MATHEMATICS LEARNING AND TECHNOLOGY

Kristy Goodwin, Kate Highfield, Joanne Mulligan, and John Hedberg

Macquarie University

In Australia, the proliferation of technological tools in mathematics classrooms has not been well supported by evidence-based research, particularly in early mathematics learning. This poster reports two stages of document analysis; a review of recent meta-analyses in early mathematics education and technology, and a quantitative analysis of research published in selected mathematics education research journals over the last five years. Selected journals include: Educational Studies in Mathematics; An International Journal; Journal for Research in Mathematics Education; For the Learning of Mathematics; An International Journal of Mathematics Education; Mathematics Education Research Journal and The Journal of Mathematical Behavior.

This analysis revealed four key findings. The first finding is that research on young children's use of technology in early mathematics learning is limited. Of the 512 articles analysed only 1.4% studied children prior to school and 6.6% investigated children in the first three years of school. The second finding revealed is that a limited range of mathematical concepts and domains were investigated. The third key finding was that a limited number (10%, n=51) of articles across all age groups focus on the use technology and of these only 4 articles specifically investigate young children's use of technology in early mathematics. The final key finding is that while there has been a small increase in research published investigating young children's use of technology in early mathematics it would be premature to suggest a growth trend.

These findings have implications for both research development and dissemination in early mathematics education. Broadening the present analysis to include early childhood journals and technology education journals will provide opportunity for a fuller review. The analysis calls for new research agendas and supports current work conducted at the Centre for Research in Mathematics and Science Education (CRiMSE) at Macquarie University. Here, a suite of new studies on young children's early mathematical development and the use of technology, such as programmable toys, dynamic interactive software and interactive whiteboards, are in progress. Further details, including data analysis may be obtained via email (kristygoodwin@mac.com or kate.highfield@aces.mq.edu.au).

DOCUMENTING THE QUALITY OF PROFESSIONAL DEVELOPMENT FOR AMBITIOUS MATHEMATICS TEACHING

Sarah Green, Jana Visnovska, Qing Zhao, and Paul Cobb

Vanderbilt University

We propose a coding scheme for documenting the quality of professional development for mathematics teachers. The instrument reflects theoretical considerations derived from prior professional development studies. It includes four focal rubrics: overall culture of the session, mathematics content, instructional materials, and representations of classroom practice. These rubrics are intended for use alone or in tandem, as appropriate, based on the topic of a particular session.

DIMENSIONS OF HIGH-QUALITY PROFESSIONAL DELVEOPMENT

Our goal in developing this instrument is to document the quality of professional development (PD) that aims, specifically, to support mathematics teachers in improving their instruction in a way that that aligns with ambitious practice (National Council of Teachers of Mathematics, 2000). We draw on the literature indicating the importance of teacher community in supporting teachers' learning. (Cobb & McClain, 2001; Franke & Kazemi, 2001). Our coding scheme explicates dimensions of quality based on concrete indicators of the development of a professional community amongst the participating mathematics teachers.

This poster prominently displays the four rubrics for the critique and consideration of the mathematics education research community. We provide rationales for both topic and content of the rubrics. We also describe adjustments we have made based on preliminary piloting of the instrument, including specific illustrative examples of the observations one might make in a PD session and how the instrument would capture that data. Our goal is to discuss the theoretical soundness of the instrument with colleagues and gather critical feedback for subsequent rounds of revision and field studies.

References

Cobb, P. & McClain, K. (2001). An approach for supporting teachers' learning in social context. In F.L. Lin & T. Cooney (Eds.), *Making Sense of Mathematics Teacher Education.* Dordrecht, The Netherlands: Kluwer.

Franke, M. & Kazemi, E. (2001). Teaching as learning within a community of practice: Characterizing generative growth. In T. Wood, B. C. Nelson, & J. Warfield (Eds.), *Beyond Classical Pedagogy in Elementary Matheamtics: The Facilitative Nature of Teaching* (pp. 47-74). Mahwah, NJ: Erlbaum.

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

STUDENTS' UNDERSTANDING OF WHAT CONSTITUTES A PROOF

Susie Groves and Brian Doig

Deakin University

Children's ideas about justification, explanation and proof, and the classroom conditions and socio-mathematical norms that allow the long-term development of these, has formed the basis of considerable research.

This communication explores Year 6 and 8 students' notions of what constitutes a mathematical proof. The data is derived from written assessments used at the beginning and end of the year with approximately 400 Year 6 and 300 Year 8 students in a large-scale project *Improving Middle Years Mathematics and Science: The role of subject cultures in school and teacher change*[1] (IMYMS). At both Years 6 and 8, two of the items addressed students' notion of proof – an open response item, *Odds and Evens*, common to both levels, and two different multiple choice questions that required students to recognise what constitutes a mathematical proof.

In *Odds and Evens* students were provided with a "domino-type" diagram of the first seven numbers and asked whether it was true that the sum of two odd numbers is even and to explain why. Only a small minority of students (6% and 7% respectively at the beginning and end of Year 6, and 12% and 21% in Year 8) were able to provide a general argument, although a larger number attempted a general argument using single-case key ideas. A large proportion of responses – about 25% in Year 6, and 20% in Year 8 – were incorrect (e.g. two odds make an odd) or un-interpretable.

In the multiple-choice item *Triangle*, Year 6 students were presented with two diagrams to illustrate the statement that Jill tore the corners off a triangle and fitted them together to make a straight line, and were asked whether Jill had proved that the three angles of a triangle always make a straight line. Over 40% of the Year 6 students chose responses that suggest one example is a sufficient proof, while less than 15% showed a clear understanding that a single demonstration is not a proof. The remainder selected that it was not a proof because not all triangles have the same shape. Over 35% of Year 8 students recognised the need for a convincing, logical mathematical explanation to prove Goldbach's conjecture. Nevertheless, over 60% believed that it was enough to show it true for at least 1000 randomly chosen numbers or as many as possible, or to find one number for which it was not true.

If we accept that proof is a critical aspect of mathematics, it is essential that teachers develop their students' understanding of what constitutes proof, but this will clearly require support for teachers to develop appropriate tasks for children.

[1] Funded by the Australian Research Council and the Victorian Department of Education and Training.

PROBLEM SOLVING IN INTERACTIVE ENVIRONMENTS EMPIRICAL RESEARCH ON LEARNING HEURISTIC STRATEGIES THROUGH DYNAMIC GEOMETRY SOFTWARE

Reinhold Haug
Paedagogische Hochschule Freiburg

A growing body of learning material is not only made up of text, but also of dynamic pictures. Furthermore, in computerized learning environments, learning material is not only comprised of written text and static pictures, but also of written (sometimes spoken) text and dynamic pictures. Text and pictures are frequently combined in order to improve students' learning. However, educational and psychological research indicates that many students have no strategies at hand to successfully process dynamic pictures and to appropriately relate text and dynamic pictures. While the conceptualization of learning strategies has a long tradition in research on learning from text, only little research is available with respect to strategies for learning from dynamic pictures (constructions) in geometry software. In my poster I propose a conceptual model for developing heuristic strategies through dynamic geometry software. On the basis of this model, specific heuristic strategies for problem solving are put forward.

In the model of multimedia learning (Mayer, 2001) as well as in the integrated model of text and picture comprehension (Schnotz & Bannert, 2003), four fundamental kinds of cognitive processes are assumed to be relevant to learning from combinations of texts and dynamic pictures: Selecting information / organizing information / transforming information / integrating information.

Based on this situation, on the one hand the poster presentation shows the research questions. On the other hand it explains the heuristic strategies of problem solving with dynamic geometry software. And finally the research design and the test results of a pre- and post-test from the empirical research shows, how successful the strategies can be used in the class room. The final summary attempt to structure the different strategies into process classes:

Selection & Organisation:

- Exploring and discovering constructions and important coherences.
- Using learning-diaries to document functional dependability and further results.

Transformation & Integration:

- Checking and developing conjectures.
- Recognising invariants as special qualities of construction.
- Using auxiliary lines for construction.
- Using learning-diaries and reflection prompts to document heuristic working methods.

References

Mayer, R.E. (2001). *Multimedia Learning*. NY: Cambridge University Press.

Schnotz, W. & Bannert, M. (2003). Construction and interference in learning from multiple representations. *Learning and Instruction, 13,* 141-156.

ROLES OF VISUALIZATION IN MATHEMATICAL PROBLEM SOLVING

Siew Yin Ho

Nanyang Technological University

It was observed that research on visualization did not demonstrate a clear relationship between visualization and success in mathematical problem solving (Hegarty & Kozhevnikov, 1999). Stylianou and Silver (2004) suggested looking into the roles that visualization play in different types of problems. More recently, Presmeg (2006) proposed a list of significant research questions for visualization research; one of which was on what aspects of the use of visualization are effective in mathematical problem solving.

This study focuses on the roles that visualization play in mathematical problem solving. Fifty Primary Five (aged 10.25 to 11 years old) and Primary Six (aged 11.25 to 12 years old) students from five Primary schools were asked to solve a set of six related verbal word problems having high degree of visuality. Each student was interviewed in a one-to-one setting, and asked to write down their solutions on paper. They were also asked to explain their written solutions. The interview procedure was structured such that each student was engaged in the highest possible level of intellectual process, thus every opportunity was given for success in each word problem. The interviews were all audio-recorded. The audio-recordings, the artifacts (the students' written solutions) and field notes taken during the interview were used to triangulate the data obtained.

In the poster, findings of the study will be illustrated with examples of artefacts and vignettes from the relevant interviews. Seven roles of visualization were found in the study. They are: To understand the problem, To allow opportunities to work with a simpler version of the problem, To see connections with a related problem, As a tool to check the solution, To cater to individual learning styles, As a substitute for computation, and To transform a situation into mathematical forms.

References

Hegarty, M. & Kozhevnikov, M. (1999). Types of visual-spatial representations and mathematical problem solving. *Journal of Educational Psychology*, *91*(4), 684-689.

Presmeg, N.C. (2006). Research on visualization in learning and teaching mathematics. In A. Gutierrrez & P. Boero. (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp. 205-235). NY: Sense publishers.

Stylianou, D.A. & Silver, E.A. (2004). The role of visual representations in advanced mathematical problem solving: An examination of expert-novice similarities and differences. *Mathematical Thinking and Learning*, 6(4), 353-387.

THE RELATIONSHIP BETWEEN COGNITIVE LOADING AND ITEM DIFFICULTY FOR THE NUMURICAL OPERATON ITEMS

Chia-Wei Hsiao and Pi-Hsia Hung

National University of Tainan

Large scale assessment routinely release part of the sample items to communicate the assessment theme. Difficulty parameter of each release item is also included in the release documentation. To translate the statistic information into teaching practice adjustment, teachers usually need some professional supports. In this study, a cognitive loading perspective is adopted to interpret the item difficulty parameter. The items of numerical operation of the Southern Taiwan Assessment of Student Achievement on Mathematics (STASA-MAT) were used for the preliminary analysis. The results suggest that cognitive loading components can predict around 45% of the difficulty variance. The implications of these results for math teachers are discussed.

CONTENTS OF THE POSTER

Knowledge about cognitive and processing operations in a model of item difficulty prediction allows test developers to develop teaching strategies that target specific cognitive and processing characteristics (Dimitrov & Raykov, 2003). The purpose of this study is to interpret the difficulty parameter from a cognitive loading perspective. The 2005 to 2007 tests of the STASA-MAT for the 2nd to 4th graders were used for the preliminary analysis. The content of number and computation is chosen for analysis. A 3 cognitive components coding schema (information loading, concepts included, & division operation) was used to predict the difficulty parameter. The results suggest that the three components can predict around 45% difficulty variance. The number of concepts needed for successfully problem solving is the most important predictor. The preliminary results suggest that cognitive loading analysis can be very promising for both test construction and supplemental teaching design.

PARTICULAR VISUAL CHARACTERISTICS

	Information	Concepts	Division
1、There are 256 5th graders of Happy Elementary School. Thirty-two of them are riding bicycle home. Another 144 are taken by parents. The rest of them are walking home. How many students are walking home?	3	1	0
2、Mom makes 522 cookies. Twenty-seven of them are broken. The rest of the cookies are packed. Each bag will have 9 cookies. How many bags are there?	3	2	1

SPREADSHEET INTEGRATED IN ELEMENTARY SCHOOL ALGEBRA LEARNING

Pi-Hsia Hung
National University of Tainan

Chien-Hsun Tseng
National Kaohsiung Normal University

Chun-Yu Chen
Mei-Ho Institute of Technology

The spreadsheet is adopted as an exploration tool to mediate students' algebra collaborative learning of elementary school. Over a two-year period, the students' progress of a 5th grade class was monitor to evaluate the effect of integrated model. A comprehensive framework of technology integration model for elementary school mathematics class was proposed in this paper.

CONTENTS OF THE POSTER

Zbiek, Heid, Blume and Dick (2007) suggested that technology can play a critical role in mathematics education. The technology can be a mind constructing mediator among student, teacher, content and activity. This perspective echoes to Jonassen and Strobel's (2006) view of 'Learning with technology'. This poster presents a validated model of technology integration in meaningful learning. Basing upon Zbiek's (2007) mediation perspective (Fig 1) and Jonassen, Peck and Wilson's (1999) meaningful learning perspective (Fig2), the technology tool is applied in the algebra learning activities of elementary school. The explicit evaluation, Effective Trail, is based on Ajzen's (2002) 'Theory of Planned Behaviour' (TPB), to replace the TACT (Target, Action, Context and Time) indices. The empirical framework of spreadsheets integrated learning which combines effective trail is proposed as Figure 3.

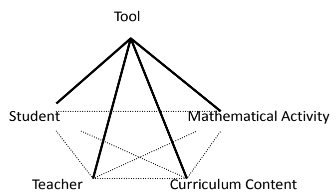


Figure 1. Mediation relationships among technology, student, teacher, activity, and content.

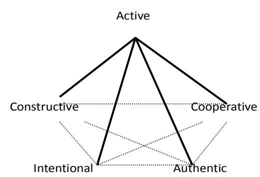


Figure 2. Characteristic of meaningful learning.

PARTICULAR VISUAL CHARACTERISTICS

This research proposes a technology integrated mathematics learning model basing upon current learning theories. The spreadsheet simulation tools were applied for students to explore the concept 'equal relationship' of algebra. The integration model (Fig3) is proposed friendly by the help of multimedia designs. The model is preliminarily verified by students' learning progress and assignments presented. Generally speaking, the single group design suggested that TPB theory is supported.

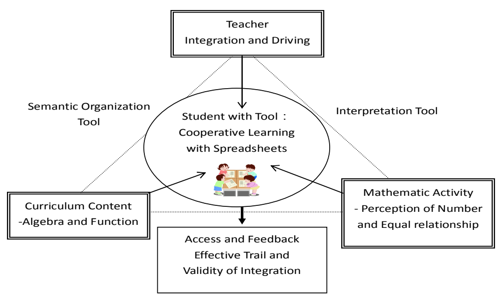


Figure 3. Mediation among Spreadsheets, student, teacher, and Algebra learning.

Reference

Ajzen, I. (2002). *Constructing a TpB Questionnaire: Conceptual and Methodological Considerations*. http://www-unix.oit.umass.edu/~aizen/pdf/tpb.measurement.

Jonassen, D.H. & Strobel, J. (2006). Modeling for meaningful learning. In *Engaged Learning with Emerging Technologies* (pp. 1-29). Netherlands: Springer.

Zbiek, R.M., Heid, M.K., Blume G.W., & Dick, T. (2007). *Research on Technology in Mathematics Education: A Perspective of Constructs*

STUDYING MATHEMATICAL THINKING IN AN ONLINE ENVIRONMENT: STUDENTS' VOICE

Zekeriya Karadag and Douglas McDougall

University of Toronto

This paper documents evidence for revealing hidden potential of students' voice emerged while analyzing students' thinking processes. In a study designed to explore students' mathematical thinking in an online environment, we performed a two-step study each having a significantly different method for data collection. Our aim during the research was to explore, understand, and document what opportunities and challenges exist in online environments. In order to fully explore, we followed the procedure suggested by Charmaz (2006) and collected data for documenting every detail. As a result of this detailed documentation, analysis, and comparison of the data, we focus on 'student's voice' as one of the themes that emerged during analysis stage.

For the first stage of the study, three students were asked to solve mathematical problems and provide their solutions using either paper-and-pencil or drawing software. After we analyzed their solutions, the students were interviewed via email to clarify their methods of thinking, to understand their approach, and to justify our interpretation. In the second stage of the study, one student was recruited and taught how to use Geogebra® (Dynamic mathematics software) and Wink® (Screen casting software). By employing Wink, we could collect data tracking each half second of student's work and analyze more closely by employing "frame analysis method". The frame analysis method is a method to allow capturing students' work done in computer environment and helps to analysing this work by focusing on each frame – a predefined moment of the recording – of the work.

Researchers giving more emphasis on students' voice and listening to them may provide new learning opportunities for students particularly if they are unable to identify their own errors. In addition, we have found that students are quite confident while integrating technology in their work. Scanning their paper-and-pencil work, using online drawing tool to draw their graphs and exploring new software such as Geogebra® and Wink® are the new experiences for them in solving mathematics problem, and they showed improvement in their ability to communicate their solutions. Moreover, having more tools in solving mathematics problems seems to promote their mathematical thinking.

References

Charmaz, K. (2006). *Constructing Grounded Theory: A Practical Guide Through Qualitative Analysis*. London: Sage Publications, Ltd.

CONTENT-RELATED AND GLOBAL CONVICTIONS OF MATHEMATICS TEACHERS AS CONTEXT FACTORS FOR MODELLING COMPETENCY DEVELOPMENT

Sebastian Kuntze and Kristina Reiss

University of Munich

Professional knowledge of mathematics teachers is considered to be an important context factor for instructional practice and for the development of mathematical literacy of the students. Especially instruction- and content-related convictions or beliefs as rather prescriptive components of professional knowledge have been the subject of research identifying characteristics of such convictions and domains of beliefs as well as relationships between these components of professional knowledge (Lin & Cooney, 2001; Törner, 2002; Kuntze & Reiss, 2005). However, studies examining instruction- and content-related beliefs of mathematics teachers and their possible impact on learning outcomes by quantitative methods and, more particularly, by multilevel analysis, are rare. Accordingly, our study aims at investigating convictions of mathematics teachers as context factors for competency development. As the study focuses on modelling competency development of secondary students in the domains of statistics and area measurement, we devote special attention to content-related convictions of the teachers in these domains. The research questions focus on relationships between such content-related and more global convictions (1), on empirical links with variables of the learners (2) as well as on comparisons with prospective teachers (3). The sample consists of more than 80 participating secondary in-service mathematics teachers and more than 2000 students taught by them.

The poster presents the design of this study in detail. Based on an in-depth presentation of the theoretical background, central research questions, information about research instruments, and information about the sample are given. Moreover, first results of a study with more than 200 prospective teachers are discussed.

References

Lin, F.-L. & Cooney, T.J. (Eds.). (2001). *Making Sense of Mathematics Teacher Education*. Dordrecht: Kluwer.

Kuntze, S. & Reiss, K. (2005). Situation-specific and generalized components of professional knowledge of mathematics teachers. In H. L. Chick & J.L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education (PME)* (Vol. 3, pp. 225-232). Melbourne: University.

Törner, G. (2002). Mathematical Beliefs – A Search for a Common Ground: Some Theoretical Considerations on Structuring Beliefs, some Research Questions, and some Phenomenological Observations. In G. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A Hidden Variable in Mathematics Education?* (pp. 73–94). Dordrecht: Kluwer.

A COMPARISON STUDY OF A TEACHER'S REFLECTION

NaYoung Kwon and Chandra Hawley Orrill

University of Georgia

Based on Cohen and Ball's instructional triangle (1999), we sought to understand how one middle school mathematics teacher made sense of student understanding in her classroom. In this study, we compared two case studies with the same teacher to understand teacher reflection. Our purpose was to understand whether the teacher's approach to reflection changed over time and, if so, in what ways did it change.

The two case studies were conducted as part of a larger research effort, the NSF-funded CoSTAR project. Data collected for each case study included daily lesson videotapes during an entire unit of instruction, student interviews in which pairs of students were asked about particular problems or classroom situations, and teacher interviews using the lessons and student interview videotapes. For the purpose of this study, we considered data of Ms. Moseley collected in spring 2003 and fall 2003. We built from our prior analysis of the Fall data (Kwon & Orrill, 2007) by comparing it to the Spring data.

For our analysis, we considered only those comments in which the teacher reflected on her students in her interviews. Using a modified version of Wallach and Even's (2005) categories, we coded instances of *assess*, *describe*, *interpret*, *justify*, and *extend*. Our initial findings showed an increased use of *extend* instances between the spring and fall case study as well as the increase in the use of *justify* in both cases. More *extend* instances indicated that the teacher was focusing more on her own practice as it influenced student understanding. Based on the trends seen across these two case studies with this teacher, we suggest that the interview process, which required the teacher to talk about her students' understandings, supported this teacher in understanding students' thinking and connecting it to her own teaching practice. We will show the final results in tables to compare two case studies. These understandings are critical for the use of reflection for teachers' professional development.

References

Cohen, D.K. & Ball, D.L. (1999). *Instruction, Capacity, and Improvement* (CPRE Research Report Series No. RR-43): Consortium for Policy Research in Education.

Kwon, N. & Orrill, C.H. (2007). Understanding a teacher's reflection: A case study of a middle school mathematics teacher. *School Science and Mathematics, 107*(6), 246-257.

Wallach, T. & Even, R. (2005). Hearing students: The complexity of understanding what they are saying, showing, and doing. *Journal of Mathematics Teacher Education, 8*, 393-417.

INTERNATIONAL GEOGEBRA INSTITUTE: NURTURING A COMMUNITY OF RESEARCHERS AND TEACHER EDUCATORS

Zsolt Lavicza — University of Cambridge
Markus Hohenwarter — Florida Atlantic University
Erhan Selcuk Haciomeroglu — University of Central Florida

Research indicates that despite the established benefits of using technology in mathematics education, the process of embedding technology in classrooms is complex and teachers need more than just being provided with software if the promise of technology is to be more fully realized. GeoGebra is free open-source software that is a versatile tool for visualizing mathematical concepts from elementary through university level as well as linking their algebraic, graphic, and numeric representations. Without any promotion and organized training, GeoGebra has been discovered and is now being used by tens of thousands of enthusiastic teachers and researchers around the world. In the past three years, an extensive self-supporting online community was formed by users of GeoGebra. They share interactive teaching materials on the GeoGebraWiki and support fellow users through an online forum. Volunteers from this community have also translated GeoGebra to 36 languages offering great opportunities to use the software in local languages and in multicultural environments.

A growing body of research suggests that, for the majority of teachers, solely providing software is insufficient and that training and collegial support enhances teachers' willingness to integrate technology into their teaching and develop successful technology-assisted teaching practices. In this regard, we describe our aim of establishing an International GeoGebra Institute (IGI) to provide training and support for teachers and to pursue research projects related to dynamic mathematics software. While our current plan is that IGI will be first established at Florida Atlantic University in the USA, our goal is to collaborate with colleagues and to set up other institutes in various locations. In our presentation, we will outline the ideas and plans for IGI and seek feedback from colleagues. We have chosen the format of a poster presentation in order to inform the mathematics education community about IGI and its starting activities and research projects at universities in the USA and several countries in Europe. Furthermore, we are looking for colleagues who would be interested in collaborating in IGI-related projects.

PROMOTING READING IN MATHEMATICS TO STRENGTHEN STUDENT'S KNOWLEDGE AND COGNITIVE SKILLS

King Man Leung
University of East Anglia

Man Wai Lui
The University of Hong Kong

A well-known and essential learning strategy for life-long learning is 'learning through reading'. Promoting a reading culture among students is therefore one of the key tasks in the curriculum reform with the aim to strengthen students' learning capabilities, especially in the subject of Mathematics. According to the Progress in International Reading Literacy Study (PIRLS) results (2007), Hong Kong students ranked the second among the forty-five countries/regions all over the world. As compared to the result published in 2001, Hong Kong students have a significant improvement. In Mathematics, reading helps students develop thinking skills, enrich knowledge, enhance language proficiency and broaden life experience; and promotion of reading to learn in schools is one of the effective teaching strategies in learning mathematics. While language teachers focus more on the teaching of reading strategies and skills, mathematics teachers should encourage students to apply the relevant skills, and broaden their knowledge and exposure through reading materials in subject matters. Reading across the mathematics curriculum needs to be strengthened and a whole school approach should be adopted to share good practices and nurture a reading culture within the school (CDC, 2002). Students should be encouraged, as early as possible, to make full use of the school libraries to read a wide variety of materials (e.g. mathematics story books & history in mathematics), apart from textbooks or reference books for achieving different learning targets and hence lifelong, independent learning.

This poster presentation reveals a study conducted in different elementary schools over the last two years and shares teachers' experience in effective use of reading strategies to strengthen students' mathematical knowledge and foster their generic skills such as self-study, reading, communication and problem-solving skills. Data was collected in the form of audio-taped interviews with teachers and students, video-taped classroom observations, field notes, documents and students' annotated work. To encourage student's learning through reading in mathematics classrooms, teaching strategies such as group presentation, reading competition, annual book fair as a platform for promoting reading, student's reflective journals and book reports will be reviewed.

References

Baer, J., Baldi, S., Ayotte, K., & Green, P. (2007). *The Reading Literacy of U.S. Fourth-Grade Students in an International Context: Results from the 2001 and 2006 Progress in International Reading Literacy Study (PIRLS) (NCES 2008–017)*. National Center for Education Statistics, Institute of Education Sciences, U.S. Department of Education. Washington, DC.

Curriculum Development Council (2002). *Mathematics Education, Key Learning Area Curriculum Guide (Primary 1 – Secondary 3)*, Hong Kong: The Printing Department.

THE DEVELOPMENT OF COMPUTERIZED NUMBER SENSE ASSESSMENT SYSTEM

Su-Wei Lin — National Hualien University of Education

Pi-Hsia Hung — National University of Tainan

The purpose of this study was to develop a computerized number sense assessment system (CNSA) to investigate the issues of elementary students' number sense. Number sense is the ability of awareness and reasoning for the relationships between the numbers, which were embedded in the context or situation. This study adopted the suggestion of Greenes, Schulman, and Spungin (1993), designed Fit the Facts problem to assess students' number-sense skills by focusing on relationships among numerical data. Fig. 1 presents a sample item. The job for the students is to read the story, note relationships among the data, and use the mouse click and drag the numbers to fill in the blanks so that the story makes sense, both mathematically and contextually.

Do you know that each of your hands has __A__ bones? That means that you have a total of __B__ hand bones. Your foot has __C__ fewer bones than your hand, or __D__ bones. So there are total of __E__ bones of your hands and feet. Your body has a total of __F__ bones. The proportion of total hand and food bones to body bones is __G__.

27
1
$\frac{1}{2}$
206
54
106

Figure 1. The sample item of CNSA.

Around 558 fifth and sixth grade students were stratified sample as norm group for CNSA. Three difficulty levels were developed to assess the students' number sense. The study estimated that there were around 14.5% students who were unable to demonstrate the basic number sense and there around 17% students could perform number sense sensitively. The results showed that the three difficulty levels of CNSA could discriminate different ability levels groups successfully. It suggested that there were discriminative power in CNSA for distinguishing different number sense degrees. The correlation coefficients between CNSA and the Mathematic Computerized Adaptive Ability Test and the Computerized Estimation Test were around .47s for the norm group. In order to having a deep and better understanding the relationship of number sense and general mathematic ability, CNSA was a workable and productive assessment approach.

MEASURING FIDELITY OF IMPLEMENTATION OF THE MATH RECOVERY TUTORING PROGRAM

Charles Munter and Sarah Green

Vanderbilt University

This poster pertains to an evaluation study of Math Recovery, a 12-15 week, pull-out tutoring program for low-achieving 1st-graders. Because of its constructivist approach, MR tutoring is not easily evaluated, as much of the tutors' expertise (or lack thereof) is evident only in the interactions with children and the tutors' responses to those interactions. Developing a measure of fidelity of implementation in this case is not as simple as monitoring adherence to a script, but penetrates to the core of what effective mathematics instruction is and how we might know when it is happening. Additionally, the wealth of video data in this study offers unique opportunities to inquire about the reliability of both coding schemes and sampling plans that might be used in this and future evaluations.

DESCRIPTION & RATIONALE

One crucial component in evaluating an intervention's effectiveness is successful implementation of the intervention. But one must be able to determine the extent of such success – particularly in a randomized design, where it is necessary to control for other factors likely to influence the outcome of interest in order to obtain a valid estimate of the program's effects. Determining the extent to which Math Recovery is enacted as intended requires an explication of what 'good' MR tutoring is and a systematic method for evaluating tutors' practices against that ideal. Math Recovery's instructional and learning frameworks, findings of best practices from the general tutoring literature, and an iterative process of designing a coding scheme based on video recordings of MR tutoring sessions all contribute to the ongoing construction of this method and instrument for measuring fidelity.

In this poster presentation of our current instrument and other important graphical elements such as the MR frameworks and the study's sampling plan, we comment on the process of developing a method for describing and reliably assessing the complex practice of delivering mathematics instruction that is attuned to a child's current understanding and needs. We will simultaneously examine our instrument in its current stage and make reference to the kinds of tools and settings used in the Math Recovery program. We aim to engage our audience in conversations about both the ongoing conceptual development and technical development of this instrument.

Beyond its contribution to methods of quantitative program evaluation and issues of fidelity of implementation, our work has possible implications for how we might describe and account for the complexities of the types of mathematics instruction valued among mathematics educators.

DESCRIBING STUDENTS' COMPETENCE FOR WORKING ON WORD PROBLEMS

Guri A. Nortvedt

University of Oslo

School mathematics traditionally calls for students to solve word problems. Much research on word problems has aimed at investigating expert and novice behaviour (Lesh & Zawojewski, 2007). While research often aims at identifying success factors, efforts of teaching expert strategies to novices have not proved successful. One reason for this is that novices lack the mathematical knowledge necessary to understand and use the expert strategies (ibid.). Other approaches are called for.

The model of domain learning (MDL) is a stage theory describing competence within academic domains as consisting of three components: domain knowledge, strategy repertoire and interest (Alexander, 2003). Mathematical competence would then consist of mathematical knowledge, strategies for doing and communicating mathematics as well as interest for doing and learning mathematics. Unlike other theories, novices are not contrasted by experts. Instead research aim at describing the interplay between the three components at three stages as well as the journey towards competence or proficiency (ibid.).

In a research project I am investigating students' competence for working on multistep word problems through different approaches:

- a protocol analysis of students' think aloud protocols while reading and solving a collection of multistep word problems
- an analysis of how reading comprehension is correlated to solving word problems investigating solving patters for different groups of students on national tests in literacy and numeracy
- an analysis of scaffolding talks between researcher and students working on multistep word problems

The poster will present a framework building on the principles for MDL for describing levels in students' competence for working on word problems, illustrated with analysis of data on one student from the three different approaches.

References

Alexander, P. (2003). The development of expertise: The journey from acclimation to proficiency. *Educational Researcher, 32(8),* 10 - 14.

Lesh, R. & Zawojewski, J. (2007). Problem solving and modelling. In F.K. Jr. Lester, (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning.* (Vol 2, pp. 763 - 805), Charlotte, NC: infoagepub.com: NCTM.

EFFECTS OF A CONTINUOUS QUANTITY CONTEXT ON STUDENTS' UNDERSTANDING OF =

Claire Okazaki, Fay Zenigami, and Melfried Olson

University of Hawai`i

This poster presents research on how grade 1 students in the *Measure Up* (*MU*) curriculum research and development project understand the equal and unequal signs. *MU* adapts Russian research (Davydov, 1975a, 1975b; Minskaya, 1975) using a generalized continuous measurement context to teach mathematics to elementary children. In grade 1, *MU* students compare continuous quantities and perform actions which make equal relationships unequal (or vice versa). Experiences with physical models provide the bases for students' understanding and use of the symbols =, ≠, < or > to represent the comparisons without the need to count. An earlier study conducted during the initial stages of curriculum adaptation from the Russian research suggests that *MU* students develop an understanding of equivalence by grade 2 (Whitman & Okazaki, 2003). The curriculum has since gone through several iterations of revisions, prompting an extension of the study. The following research questions have been identified: 1) What do first graders perceive "equals" to mean at the beginning of the school year, at mid-year and at the end of the school year? 2) In a curriculum based on continuous measurement models, for what contexts is the equal sign used as an indicator of the relation between two equal quantities? For what contexts is the equal sign used as an operator? 3) What effect does an elementary curriculum using a continuous measurement context have on students' understanding of the equals sign?

The poster will feature photographs of students engaged in *MU* tasks designed to develop their understanding of equality and inequality. Work samples will show how students represent comparative relationships with mathematical symbols. Data from 15 to 20 first grade student interviews conducted before they began work in *MU*, at mid-year and at the conclusion of the current academic year, will be shared.

References

Davydov, V.V. (1975a, 1975b). Logical and psychological problems of elementary mathematics an academic subject. In L.P. Steffe (Ed.), *Children's Capacity for Learning Mathematics* (Vol. 7, p. 55-107). Chicago: University of Chicago.

Whitman, N. & Okazaki, C. (July 2003) What "=" means. In N.A., Pateman, B.J. Dougherty, & J.T. Zilliox (Eds.), *Proceedings of 2003 Joint Meeting of Psychology of Mathematics Education and Psychology of Mathematics Education – North America* (Vol. 1, p. 262). Honolulu, HI: Curriculum Research & Development Group, College of Education, University of Hawai`i.

SCAFFOLDING COMPETENCY ACQUISITION: MAPS MOSSAIC

Svetlana Polushkina, Regina Bruder, Bastian Benz, and Bernhard Schmitz

Technische Universität Darmstadt

The present study is carried out within an ongoing interdisciplinary doctoral research project MAPS MOSSAIC (“MAthematical Problem Solving and MOdelling: Scaffolding Self-Regulated Acquisition of Interdisciplinary Competencies“) aiming at conceptualization, implementation and evaluation of an interactive and adaptive mathematical learning environment for fostering the development of mathematical and learning competencies in the students from the seventh grade on. Special focus is laid on promoting the quality of learning processes and outcomes by integrating cognitive and metacognitive adaptive learner support into the software.

The design of the learning environment relies on the concept of problem-based mathematics education in combination with self-regulation (Komorek et al., 2007) and complies with the German national educational standards and the quality criteria for designing learning software (Bruder et al., 2004). The conceptualisation of the learning process and the adopted notion of scaffolding (Benz et al., 2007) allow for systematic discovery of learning weaknesses to generate adequate learning support.

In the present study, several cognitive and metacognitive scaffolds are tested with respect to their influence on the learning processes and outcomes on a sample of seventh grade students working with the learning software on a mathematical task. The results indicate the usefulness of scaffolding the learning process through cognitive and metacognitive guidance for the performance on the mathematical tasks and the application of strategies of self-regulated learning. The poster illustrates the design of the learning software and shows the results of the present study.

References

Benz, B.F., Polushkina, S., Schmitz, B., & Bruder, R. (2007). Developing learning software for the self-regulated learning of mathematics. In M.B. Nunes & M. McPherson (Ed.), *IADIS Multi Conference on Computer Science and Information Systems. IADIS International Conference e-Learning* (pp. 200-204). IADIS Press.

Bruder, R., Offenbartl, S., Osswald, K., & Sauer, S. (2004). Third party certification of computer-based learning environments. *Proceedings of the EISTA '04 International Conference on Education and Information Systems: Technologies and Applications* (pp. 21-25). Orlando, Florida, USA.

Komorek, E., Bruder, R., Collet, C., & Schmitz (2007). Contents and results of an intervention in math lessons in secondary level I with a teaching concept to support mathematic problem solving and self-regulative competencies. In Prenzel, M. (Ed.) (in prep.). *Studies on the Educational Quality of Schools. The Final Report on the DFG Priority Programme.* Münster: Waxmann.

THE PERCENTAGE LOCKER

Iris Rosenthal and Bat-Sheva Ilany

Beit-Berl Academic College

Educators agree that intuition plays an essential role while learning mathematics. Furthermore, researchers agree that in order to learn mathematics the didactic method should lead the learners into intuitive and analytic understanding (Tirosh, Barash, Zamir and Klein, 2000). According to Fischbein, Tirosh and Barash (1998), every math activity includes self-evident intuitive knowledge; algorithmic knowledge; and formal knowledge. In addition, mathematical activity requires knowledge of mathematics education that facilitates, according to Fischbein et al. (ibid.), an ability to make a connection between the three components mentioned above, and creates a fundamental base for learning.

In years of experience with teaching percentage to 6^{th} and 7^{th} grades and student-teachers, we identified obstacles in the way in which students grasp this subject. Students' errors and misunderstandings led us to search for a new method for teaching percentage.

In our model - "The Percentage Locker" - we developed an innovative approach to this subject. We constructed a square table with 100 boxes (10x10 cm each) on top, and a set of rectangle cards in different sizes. These tools allowed students to start working intuitively with different numbers as integers (200, 100, 150 etc.). This was followed by formal activities: finding the percent, finding the amount and finding the integer. Our approach prevents the common misunderstanding in which students tend to see the number 100 as an ultimate integer. This model was tried out with student-teachers and with 6^{th} grade students with substantial success.

In the poster, we will present a model of The Percentage Locker, a set of activities that we developed for this approach, and a description of the students' learning process.

References

Fischbein, A., Tirosh, D., & Barash A. (1998). Intuitive knowledge and logical knowledge as components of mathematical activity. *Teaching Mathematics Guide*, 22, 12-28.

Tirosh, D., Barash A., Zamir P., & Klein R. (2000), *Psychologists Aspects in Mathematical Teaching, Mofet Institute.* Tel – Aviv (in Hebrew).

MOTIVATION'S FACTORS OF CAREER'S CHOICE OF MATHEMATICS' TEACHER

Natalia Sgreccia
Rosario National University
National Council of Scientific and Technical Researches

Marta Massa
Rosario National University

This presentation is inscribed in the frame of a qualitative research where we dectect some motivation's factors of career's choice from an interview to thirteen Mathematics' teachers. These factors are identified according to teaching or to Mathematics. We consider that this study contributes with empirical data to the approach's tools to Mathematics' culture of each teacher.

INTRODUCTION

This presentation is inscribed in the frame of a qualitative research which studies the "Geometry of the teacher" who teach to 12-15 aged students. The research was realized in two parts: Part 1 "Geometry of the teacher from his saying", where experts (Sample 1) and Mathematics' teachers (Sample 2) were interviewed; Part 2 "Geometry of the teacher from his practice", where Mathematics' classrooms were observed.

We used five analysis' dimensions for the Part 1's indictment's results and we asignated three variables for each dimension. This presentation is concentrated on the answers of the Sample 2 to the Variable 3 "Motivation's factors of career's choice" of the Dimension 1 "The teacher as professional of Mathematics' Education".

BRIEF DESCRIPTION OF THE SAMPLE 2

The Sample 2 is composed of 13 teachers, 12 of them have the title of Mathematics' teacher and 1 is Statistics' Licenciate and schoolteacher. Almost of them started to work as teacher immediately they finished their degree studies. The work's antiquity is distributed in a homogeneous manner from 0 to 30 years. Everyone, except one interviewee, has worked in this education's level.

VARIABLE 3 "MOTIVATION'S FACTORS OF CAREER'S CHOICE"

(Between parenthesis we show the quantity of the answers)

Through the obtained information, we observe the emphasis put on the pleasure of teaching, which was appearing from playing games in the chilhood (1), from his passage of elementary school (2), from reward or feedback of learners' understanding (7). Also we detect humane links more generals, like contribution to the person's formation (1), from the solidarity (1) or because a teaching's familiar's tradition (3).

Refers to Mathematics, the answers ponder sensations which oscillate from stillness to defiance and that promote liking for it. One of the interviewees mentions Mathematics' teachers' influence on his professional choice, who showed him order in a demonstration.

FINAL REFLECTION

We consider that the motivation's factors of career's choice of Mathematics' teacher form part of the approach's tools to Mathematics' culture of each teacher, which constitute his Mathematics' discourse's foundations in the classroom. As a result of this, becomes interesting to identify, classify and analyse how the different social practices have intervened in the configuration of that culture.

Y THE PROBLEM GENERATED BY THE CONSTANT USE OF “PROTOTYPE EXAMPLES” (SHAPES IN STANDARD POSITION) IN THE LEARNING PROCESS AND COMPREHENSION OF GEOMETRIC DEFINITIONS

Cruz Evelia Sosa Carrillo, Brenda Alejandra Jiménez Robledo,
and Ellina Beliaeva Longuinenco
Tecnológico de Monterrey

This paper shows the results of a research with students from high school. The research was carried out in a geometry and trigonometry course. The main objective of this research is to make a reflection on the differences between the concept definition and the concept image that students have about a geometric object. As well, the analysis of the possible implications that those differences could generate in the students understanding of mathematical concepts is part of this paper. Moreover, the constant use of prototypes in the learning process could provoke that the relevant attributes of a definition are changed by the students’ mental images. The mathematical knowledge always is communicated through rigorous and formal expressions that use definitions, theorems, axioms and inferences based on those theorems. In the same way as Ouvrier (2006), and Vinner (1991), we consider relevant to study the problem of the knowledge definition, because Mathematics is a theoretical knowledge in which the definition process is relevant. In order to obtain helpful information for the geometric definition problem we performed a research with 16 high school students aged between 15 and 17 years. They worked with the test of table 1. Some of the answers are in the table 2.

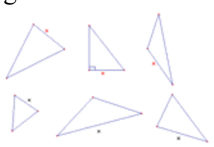
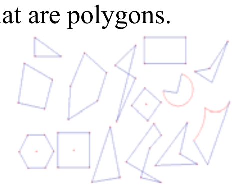
1) Draw the altitude on side X of the following triangles.  2) How do you define an altitude of a triangle?	3) Given the following set of shapes, highlight those that are polygons.  4) What is a polygon?

Table 1. Test

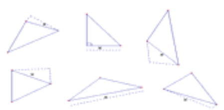
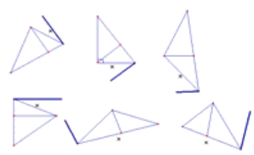
 The Students drew the heights on a horizontal vertical form. The concept image is a prototype	 The students drew the heights to the middle point of the base. The concept image is according to isosceles triangle.

Table 2. Results

According to the results of this research, we believe that knowing a concept definition do not guarantee its understanding, by that, we can not understand a definition and apply its knowledge in a correct way if we only memorize it. As a result, we think that only the constant application in different situations, that are design for the correct learning of the definition, see (Godino, Batanero and Font, 2006).

References

Godino, J.D., Batanero, C., & Font, V. (2006). *Un Enfoque Ontosemiótico del Conocimiento y la Instrucción Matemática.* Departamento de Didáctica de la Matemática. Universidad de Granada.

Ouvrier-Buffet C. (2006). Exploring mathematical definition construction processes. *Educational Studies in Mathematics*. 259-282.

Vinner, S. (1991). The Role of Definitions in teaching and learning of Mathematics. In D. Tall (Ed.), *Advanced Mathematical Thinking*, 65-81.

GRAMMATICAL PARALLELISM AND MATHEMATICAL INVESTIGATION

Susan Staats

University of Minnesota

Grammatical parallelism is a discourse structure that allows speakers to highlight similarities and differences among their ideas in a concise manner (Jakobson, 1960). Because grammatical parallelism is inherently organizational, it is a language resource for students who learn mathematics in a communicatively-oriented classroom. Grammatical parallelism can be represented in transcripts through indented lines, with grammatically similar phrases arranged into columns (Hymes, 1981, shown below); or with boxes around parallel forms that are connected with line segments (Tannen, 1989).

S1: So we gotta try to get
all the like terms on one side and
all the other terms on the other,
so we can plus 2w on both sides which gives you -2w + 4 = 3.
Then you gotta minus 4 and then you get -2w = -1 and
then you got, so you can divide -2 which gives you w is equal to,
I don't know if it's negative ½ or
positive ½.

This poster will compare Hymes' and Tannen's transcription methods for samples of students' algebra discussions, and it will highlight examples of mathematical thinking that are expressed using grammatical parallelism. The poster format is ideal for this presentation because it affords viewers time to study alternative transcription methods and to identify the mathematical thinking that is dependent on grammatical parallelism. Grammatical parallelism allows students to process ideas as they become sociomathematical norms in their classroom (Cobb et al, 2001).

References

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *Journal of the Learning Sciences, 10*(1&2), 113-163.

Hymes, D. (1981). Oral performance and measured verse. In D. Hymes (Ed.), *"In vain I tried to tell you:" Essays in Native American Ethnopoetics* (pp. 309-341). Philadelphia: University of Pennsylvania.

Jakobson, R. (1960). Closing statement: Linguistics and poetics. In T. Sebeok (Ed.), *Style in Language* (pp. 350-377). Cambridge, Massachusetts: MIT.

Tannen, D. (1989). *Talking Yoices: Repetition, Dialogue, and Imagery in Conversational Discourse.* Cambridge: Cambridge University Press.

THE USE OF ILLUSTRATED BOOKS IN MATHEMATICAL PROBLEM-POSING FOR K-2 CHILDREN

Hsin-jung S. Sung and Shuk-kwan S. Leung — Jia-Huang Chen

National Sun Yat-sen University — Kun Shan University

The backdrop of this study is from *How to Solve it*, (Polya, 1945), taking the "mathematics in the making" as an objective in promoting mathematical thinking of young children. In particular, we analyzed children's thinking when they read (or were read) picture books and posed mathematics problems. Three sources of picture books materials were developed, piloted (Chang, 2006; Huang, 2006, van den Heuvel Panhuizen, submitted) and used in this study on problem posing: *Being the fifth, Counting Cats, Colorful Ice-cream.* The potential mathematics related concepts are three: Ordinal Numbers, Cardinal Numbers, and Multiples. Stories were read by child (or by an adult to each child) and after reading the adult asked, "Guess what problem I am going to ask you?" Children guesses were audio-taped. If children were quiet, the adults would asked, "I did the same to your classmates, guess what problem you friend guessed?" Later, they were asked to record the problem in writing or drawings in the form of a diary (Leung & Wu, 2000). Results indicated that with a careful choice of illustrated books, children's responses indicated that they had great imagination and thinking mathematics concepts that were related to these stories. In guessing, children were motivated to ask questions about the stories and then made up problems, although the problems they gave were idiosyncratic and immature. Children were able to use words and drawing to present own mathematics problems. Instructional implications in the integration of mathematics and language learning were considered, especially for K-2 children in-class or at home.

References

Chang, T. (2006). *The Effects of Using Illustrated Books in Young Children's Learning on Arithmetic and Geometric Concepts*. Unpublished master thesis, Institute of Education, National Sun Yat-sen University.

Huang, T. (2006). *The Effects of Using Pictures in Second-Grade Elementary School Children's Learning of Mathematics*. Unpublished master thesis, Institute of Education, National Sun Yat-sen University.

Leung, S.S. & Wu, R. (2000). Sharing problem posing at home through diary writing: Case of grade one. *The Australian Mathematics Primary Classroom*, 5(1), 28-32.

Thatcher, D.H. (2001). Reading in the math class: Selecting and using picture books for Math investigations. *Young Children*, 56 (4), 20-26.

Van den Heuvel Panhuizen (submitted). The PALM project pilot and its results with children in the Netherlands.

AN ANALYSIS ON THE STUDENTS' MATHEMATICAL REASONING STRATEGIES BY THE PRE-SERVICE TEACHERS TEACHING DESIGNING IN KINDERGARTEN CLASSROOM

Juei-Hsin Wang
National Chiayi University

Yen-Ting Chen
Chung Hwa University of Medical Technology

This study discusses kindergarten student's math reasoning strategies by the pre-service teachers' teaching designing. It is based on the spirits of action research to understanding the pre-service teachers' teaching design, teaching processes, inflections which integrate games with different teaching designing in kindergarten classroom, and the students' change in concepts and the feelings about learning before and after teaching. The study gathered data to explore the process of student's reasoning, and mathematical reasoning practices. It is inclusive of teaching activities by planning, and the activities put into practice practically.

First, the researchers discuss mathematical reasoning strategies by the "relying on known to infer unknown" and "exploring mathematics relation".

Second, the researchers understand student's reasoning. The teaching procedure of this study is mainly based on the model of integration of play and teaching. Let the students get the concepts which are required in the play by discussion and sharing; or after the play, begin discussing the questions which happens during the games, and then continue the games and firm the concepts.

Third, the results of mathematical reasoning practices included reasoning knowledge, and using tools in the legitimate ways of reasoning. Students can share different mathematical reasoning strategies, retrospect on their own interpreting patterns and represent new solutions on the games.

Finally, this study gained some insight about mathematical reasoning strategies and design mathematical reasoning course in kindergarten classroom.

A SNAPSHOT OF BELIEFS AND PRACTICES OF A PRE-SERVICE TEACHER

Lyn Webb and Paul Webb

Nelson Mandela Metropolitan University

For the last decade research on teachers' beliefs has made a distinction between mathematics teachers' professed and attributed beliefs (practice) and studies have either found some or no correlation between the two. South African teachers exhibit a variety of levels of mathematical knowledge and knowledge of pedagogy; but many appear to have difficulty in changing their teaching practice towards methods of engaging learners in a learner-centred approach. This difficulty suggests that it would be profitable to know more about the apparently complex relationships and interactions between teachers' beliefs and practices, as well as the effect that changing classroom contexts and activities may have on their practice.

The rationale for using a pre-service teacher was that, although many beginning teachers hold the belief that mathematics is a fixed set of rules and procedures and that learning occurs through solving problems in a step-wise fashion, they are generally not resistant to change and can more easily articulate their thinking in terms of the theory of what they are learning at the time than more experienced teachers who are somewhat removed from their academic experiences (Phillip, Ambrose, Lamb, Sowder, Schappelle, Sowder, Thanheiser, Chauvot, 2007).

In this paper I investigate the beliefs and practices of a novice teacher using questionnaires and graphical representations as well as classroom observations and interviews in order to focus on the explanations of possible disjuncture between beliefs and practice rather than the differences (Speer, 2005). I conclude that, in the same vein as Skott (2001), inconsistency between beliefs and practices may be an observer's perspective that is not necessarily shared by the teacher, and that the view that there is a possible disjuncture does not do justice to the complexity of the practitioner's tasks nor to the rapidly changing contexts and situations that may occur within a single lesson.

References

Philipp, R., Ambrose, R., Lamb, L., Sowder, J., Schappelle, P., Sowder, L., Thanheiser, E., & Chauvot, J. (2007). Effects of early field experiences on the mathematical content knowledge and beliefs of prospective elementary school teachers: an experimental study. *Jounal for Research in Mathematics Education, 38*(5), 438-475.

Skott, J. (2001). The emerging practices of a novice teacher: the roles of his school mathematics images. *Journal of Mathematics Teaching Education, 4*(1), 3-28.

Speer, N. (2005). Issues of method and theory in the study of mathematics teachers' professed and attributed beliefs. *Educational Studies in Mathematics, 58*(3), 361-391.

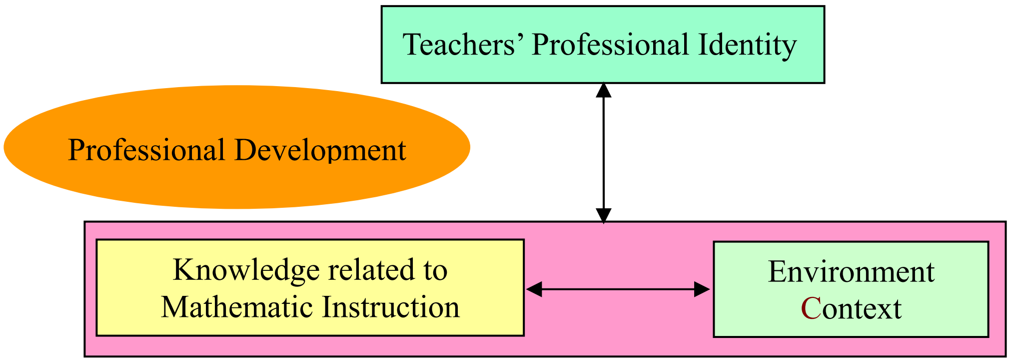
THE DEVELOPMENT OF AN ELEMENTARY TEACHER'S PROFESSIONAL IDENTITY IN MATHEMATICS TEACHING

Shih-Chan Wen — An-ho Elementary School

Yuh-Chyn Leu — National Taipei University of Education

This study is to investigate an elementary mathematics teacher's development in professional identity. Teachers were viewed "as practice" in the past but now they should be viewed "as persons", meaning that each teacher is a unique individual who has his/her own life experiences, identities and different values/beliefs on the issues of education and instruction (Goodson & Walker, 1991).

Teachers' professional identity is the concept of how teachers view themselves as being teachers. The development of teachers' professional identity is the process of how they formulate their profession content through their interactions with significant others (Ex. their colleague teachers) right at their teaching practice site (i.e. schools) and how they "realize and find" themselves from their experiences and reflections (Chou, 2003). The framework of the study is shown as the following figure.



The research method is case-study and the data were collected through classroom observations and in-depth interviews. The results are the content of the case-study teacher's professional identity on the following questions (Newman, 2000): (1) "What kind of elementary mathematics teacher am I?"; (2) "What are my beliefs about teaching elementary mathematics?"; (3) "What do I intend to achieve with my students in my mathematics class?".

References

Chou, S.Q. (2003). *Study on Elementary Teachers' Professional Identity in Curriculum Design.* Taipei: Shi-Ta Books.

Goodson, I. & Walker, R. (1991). *Biography, Identity, and Schooling: Episodes in Educational Research.* London: The Falmer Press.

Newman, C.S. (2000). Seeds of professional development in pre-service teachers: A study of their dreams and goals. *International Journal of Educational Research*, 33(2), 125-217.

A CASE STUDY OF YOUNG CHILDREN NUMERICAL CONCEPT DEVELOPMENT WITH THE PICTURE BOOK

S. C. Wu — National Chiayi University
Y. L. Chang* and W. J. Chen — University MingDao University

Numerical concept is the foundation of influencing young children's mathematics development. Applying Vygotsky's philosophy to children's mathematics learning by using the picture book, the mathematical concept would be embraced in the context within the picture book naturally. Therefore, the mathematical concept would be comprehended easily and indirectly through the intermediary of both written language and pictures, which could be considered as the sticker of children's mathematical concept. Thus, this study aimed to explore the development of five-year-old children's numerical concept by using the picture books that implying the numerical concept. Based on the related literature and the pilot study, a gradational model of the guiding strategy for reading the picture book with the participants was chosen purposefully. First, open-ended questions were proposed for the initial experience. The guiding questions were then provided to observe the realistic performance of their numerical concepts. The final step emphasized children's self-organization and -description for reexamining their developmental status. Participant observations were employed for the data collection. Participants were three five-year-old kids purposefully selected from three families with various socio-economic statuses. Four picture books were selected accordingly and examined by five experts for content validity (*the content of these picture books and the correspondent numerical concepts will be included in the poster and extra handouts, as well as the research design and findings in detail*).

The findings included: 1. "One-to-one correspondence" concept was well-developed. 2. They all exhibited the ability of "counting words". Sometimes they could discover that the interval from 1 to 10 was "1" by reading silently. 3. They could ascertain the order of the given numbers, i.e. they recognized it was the "second" while counting to "2" (starting from 1). 4. They could deal with the addition within 10 with the assistance of fingers. 5. One of them could deal with the multiplication concept and its operations (i.e. 3×2). 6. Two of them who had higher socio-economic status could read to 100 and possessed the principle of the cardinal number (i.e. knowing the total number after counting or remembering the numbers that had been counted) since their parents spent more time accompanying them in learning mathematics. However, the other one (with lower socio-economic status) could only read to 50 because of lacking the interactions with his parents (working mostly). Also, the development of young children's numerical concept was influenced by their previous learning experience, family background, and verbal ability. It further suggested that the construction of their numerical concept should be established on the playful learning environment and their interests in order to promote the learning efficiency.

Endnotes

* Indicates the correspondence author.

RE-INVESTIGATING CHARACTERISTICS OF MATHEMATICAL CONJECTURING

Kai-Lin Yang	Fou-Lai Lin	Jun-De Wu
National Changhua University of Education	National Taiwan Normal University	National Changhua University of Education

The mechanisms and mental processes of mathematical conjecturing have been concerned for educational purposes (e.g. Koedinger, 1998). While school geometric proof is concerned, it is found that activities for mathematical conjecturing are not well designed in present textbooks. Could it be said that brainstorm activities or open-ended problems in textbooks are activities for mathematical conjecturing? It might be; nonetheless, multiple phases of mathematical conjecturing, like phases of science inquiry, are not embraced so that how these activities are arranged depends on how much teachers understand and value them (Thompson, 1985).

The importance of this study is to enhance the understanding of how a mathematician developed her mathematical conjecturing. Sinclair's self-report was the data of this study. She had used her self-report to manifest the aesthetics of mathematical discovery (Sinclair, 2002). Based on the models of Boero (1999) and Chen and Lin (1998), we aim to scrutinize the mathematician's conjecturing and how the movement between private perception and public presentation is actively taken.

The mechanisms for conjecturing consist of appreciating mathematical rigor, knowing the weak validity of experiments, evaluating beautiful generality, enhancing self-efficacy, and associating useful properties. Like scientific experimentation, Sinclair also did experiments of discovering co-variance or invariance. Unlike scientific experimentation, the statuses of co-variance or invariance, premise or conclusion, were discriminated and validated by mathematical proof.

References

Boero, P. (1999). Argumentation and mathematical proof: A complex, productive, unavoidable relationship in mathematics and mathematics education. *International Newsletter on the Teaching and Learning of Mathematical Proof*, 7/8.

Chen, I.E. & Lin, F.L. (1998). A Thinking Model of Mathematics Conjecturing. *Chinese Journal of Science Education, 6*(2), 191-218.

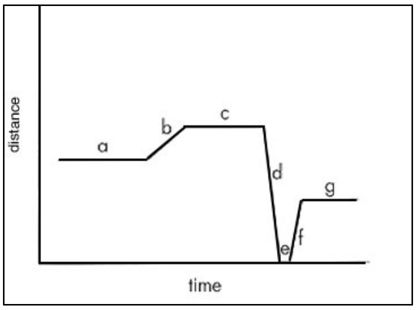
Koedinger, K.R. (1998). Conjecturing and argumentation in high-school geometry students. In R. Lehrer & D. Chazan (Eds.), *Designing Learning Environments for Developing Understanding of Geometry and Space,* (pp. 319-348). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Sinclair, N. (2002). The kissing triangles: The aesthetics of mathematical discovery. *International Journal of Computers for Mathematics Learning, 7*(1), 45-63.

REASONING ABOUT A GRAPH OF MOTION AND A STORY: HOW MULTIPLE RESOURCES MEDIATE INTERPRETATIONS OF HORIZONTAL SEGMENTS

William Zahner, Judit N. Moschkovich, and Tamara Ball
University of California, Santa Cruz

This study builds on previous work on student interpretations of graphs (Bell & Janvier, 1981; Leinhardt, Zaslavsky, & Stein, 1990). We use a sociocultural perspective on mathematical reasoning to describe how four pairs of eighth-grade students interpreted horizontal segments on a distance versus time graph (see Figure 1) using a story about a bicycle trip. While students shifted between two interpretations (*moving* and *not moving*) of the three horizontal segments above the x-axis (segments a, c, and g), pairs consistently interpreted segment e, located on the x-axis, as representing the biker *not moving* (with one exception).



1) This graph shows the distance a biker went during a bike trip. Tell the story of this bike trip. What happened during the trip?
2-3) When is the biker making the most (least) progress or covering the most (least) distance? How do you know?
4) When does the biker stop? How do you know?
5-6) When is the biker going at a slow (fast) and steady speed? How do you know?

Figure 1: The graph and some of the questions students answered. Designed by J. Moschkovich using *Investigations* (TERC, 4th grade, Graphs) and questions from *Connected Mathematics Project*.

The analysis draws on recommendations made by Smith, diSessa, & Roschelle (1993) for analysing student conceptions as valid and context-dependent rather than as misconceptions. Following these recommendations, we examine how students shifted among alternative interpretations of the horizontal segments depending on the affordances and constraints of the mediational means and describe how the location of segments on the graph and the order of the written questions in the problem mediated student interpretations.

References

Bell, A. & Janvier, C. (1981). The interpretation of graphs representing situations. *For the Learning of Mathematics, 2*(1), 34-42.

Leinhardt, G., Zaslavsky, O., & Stein, M.K. (1990). Functions, graphs and graphing: Tasks learning and teaching. *Review of Educational Research, 60*(1), 1-64.

Smith, A., DiSessa, A., & Roschelle, J. (1993). Misconceptions reconceived. *The Journal of the Learning Sciences, 3*(2), 115-163.

AUTHOR INDEX
VOLUME 1

F

G

H

I

J

K