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CONTEXT MATTERS: EQUITY, SUCCESS, AND THE FUTURE OF MATHEMATICS EDUCATION

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This article presents a reflection on my research over the past 10 years, both the theoretical framings that have proven to be useful as well as some specific research findings. After presenting a definition of equity and its four dimensions (access, achievement, identity, power), I unpack a number of successful learning contexts in which I have conducted research and what they have revealed to me about equity. The contexts I explore include: 9 US high schools, 1 successful teacher community, 23 teacher candidates, and the achievement gap. Finally, I conclude with ways in which teaching and learning contexts, especially successful ones, might play a more prominent role in future research.

Contexts have always mattered to me. Perhaps it is because I was raised to believe that communities shape and support individuals into the beings they become. Some contexts bring out the best in me, while others hide my strengths. Considering my worldview, it makes sense that my research would pay particular attention to contexts.

In my research, I do not strive for the empirical findings to be generalizable to all students, or even all US students. My focus has always been to document successful learning environments for students who have been marginalized by society, highlighting the origins of such learning environments--be they personal or institutional. I do so for two main reasons: 1) as an existence proof to those in doubt that these environments and their associated student outcomes can be created and 2) as a means for informing how we might build more such contexts for learning. By marginalization, I mean through processes such as racialization, classism, sexism, and language bias. However, that is not to say that many of the foundational pieces of these successful environments are not applicable in settings where the students are white and/or middle/high income.

Contexts matter for a number of reasons. A focus on context helps remind us that no category of teachers or students (urban students, African American students, Latina/o students, even female, bilingual Latinas born in the US) is homogeneous. In fact, our beliefs, our lived experiences, our knowledge bases, and our agendas all influence how we "perform" in a given setting. All good teachers focus on context. They recognize the fact that among other things, a student's mathematical thinking is grounded in the kind of problem presented, how that student is positioned in the classroom with respect to others (DeAvila, 1988; Forman & Ansell, 2002), the norms of interaction (Seeger, et al., 1998), and the tools available to express one's ideas (Khisty & Viego, 1999; Moschkovich; 2007). For me, a focus on the context of learning also serves as a humanizing tool in mathematics education research. It moves us away from a kind of objectified way of knowing something (e.g., students or the "one" path to equity). And, contrary to what the larger public, many alternative certification programs, and some mathematicians think, mathematics teaching is too complex to be reduced to a list of basic skills or even strategies that can be followed by any college graduate. So, while it is important for mathematics educators to present their research in ways that are accessible to policy makers (Lubienski, in press), giving voice to the contextual factors that enable or constrain learning in a given situation is equally important. Richer descriptions of educational settings and their origins also are more likely to move away from a US-centric perspective and towards a more global reality in reporting mathematics education research.

My work is deeply grounded in socio-cultural theory, drawing on the notion that learning is intricately connected to the contexts in which it occurs (Lave, 1991; Lave & Wenger, 1991; Cobb, 2000; Atweh et al., 2001). We see this most clearly in research that has considered out-of-school vs. in-school mathematics performance (Nunes, Carraher, & Schliemann, 1993; Civil, 2006). Almost at the flip of a switch, highly competent street vendors are unable to complete similar mathematical problems when imported into a "school math" context. Like Franke and Kazemi (2001) who seek to "capture the evolutionary character of teacher learning rather than the more static characteristics (p. 56)," I aim to document the nature of effective teaching and learning contexts, not just their distilled "characteristics."

Most members of the mathematics education research community would agree that equity is a valued goal, maybe even the reason behind their research. However, much less consensus arises when the question is raised: *how do you think we should address equity?* Increase teacher content knowledge, create more multicultural curricula, develop professional learning communities, exert greater control over school policies, partner universities with local schools are just a few of the strategies that might start the list. For the most part, highlighting (successful) contexts is not likely to be an answer. Yet, attending to context is key for equity purposes. In this paper, I will unpack a few contexts in which I have conducted research and what they have revealed to me about equity along four dimensions. Then, I conclude with ways in which teaching and learning contexts, especially successful ones, might play a larger role in our future research. The contexts I will explore include: 9 US high schools, 1 successful teacher community, 23 teacher candidates, and the achievement gap.

Framing Equity

I begin with a definition of equity, partly because it is critical to how we might explore successful contexts and because so many definitions of equity exist. Equity means fairness, not sameness. So, when we look for evidence that we are achieving equity, we should not expect to find that everyone ends up in the same place. In 2002, I argued (Gutierrez, 2002) that at a basic level, equity means "the inability to predict mathematics achievement and participation based solely on student characteristics such as race, class, ethnicity, sex, beliefs, and proficiency in the dominant language" (p. 153). I argued for a focus on the dominant interpretation of this meaning as well as a critical one (something I will discuss later) and how equity could relate to the sustainability of our planet. It was important for me at the time to consider not just learning outcomes as they relate to a schooling context, but also to learning outcomes that relate to life and our relationships around the globe. I would like to elaborate on that definition to include four dimensions: access, achievement, identity, and power. Let me explain.

Access relates to the tangible resources that students have available to them to participate in mathematics. These resources include such things as: quality mathematics teachers, adequate technology and supplies in the classroom, a rigorous curriculum, a classroom environment that invites participation, reasonable class sizes, and supports for learning outside of class hours. The

Access dimension reflects the predominant equity mindset of math educators in the 1980s that students are affected by their "opportunity to learn" and continues today in more nuanced forms (Nasir & Cobb, 2007). However, a focus on access is a necessary but insufficient approach to equity, in part because equal access assumes sameness.

Beyond opportunities to learn, we also care about student outcomes, or what I categorize as *Achievement*. This dimension is measured by tangible results for students at all levels of mathematics. Achievement involves participation in a given class, course taking patterns, standardized test scores, and participation in the math pipeline (e.g., majoring in mathematics in college, having a math-based career), among other things. Moving from mere access to achievement is important when considering that there are serious economic and social consequences for not having enough math credits to graduate from high school, not scoring high enough on a standardized achievement test to gain acceptance to college, or not being able to major in a math-based field that can confer a higher salary and prestige in society. The achievement dimension was most prominent in the late 1980s and early 1990s when a greater emphasis was placed on standardized test scores and continues today into the more narrowly defined "achievement gap," something I will discuss later in this article.

However, because there is a danger of students having to down play some of their personal, cultural, or linguistic capacities in order to participate in the classroom or the math pipeline and because some groups of students historically have experienced greater discrimination in schools, issues of *Identity* have started to play a larger role in equity research in mathematics education (Abreu & Cline, 2007; Martin, 2000; 2007). From my view, students should be able to become better persons in their own eyes, not just in the eyes of others. For most mathematics educators, identity issues might include understanding mathematics as a cultural practice in ways that might further develop the appreciation of one's "roots." Examples of this approach are present in the ethnomathematics program (D'Ambrosio, 2006). But, we cannot stop there, as identity is much more than just one's past. More centrally, the identity dimension concerns itself with a balance between self and others. A window/mirror metaphor is useful here (Gutiérrez, 2006). That is, students need to have opportunities to see themselves in the curriculum (mirror) as well as have a view onto a broader world (window). For example, using mathematics to analyze social justice issues might offer a mirror to students who have been marginalized by society while it provides a window to students who benefit from the status quo. Identity incorporates the question of whether students find mathematics not just "real world" as defined by textbooks or teachers, but also meaningful to their lives. It includes whether students have opportunties to draw upon their cultural and linguistic resources (e.g., other languages and dialects, algorithms from other countries, different frames of reference) when doing mathematics. As such, we need to pay attention to the contexts of schooling and to whose perspectives and practices are "socially valorized" (Abreu & Cline, 2007; Abreu, 1999, Civil, 2006). The goal is not to replace traditional mathematics with a pre-defined "culturally relevant mathematics," but rather to strike a balance between the number of windows and mirrors provided to any given student in his/her math career.

However, even if students have access to quality mathematics, achieve a high standard of academic outcomes as defined by the status quo, and have opportunities to "be themselves and better themselves" while doing mathematics, it is not enough to call it equity if mathematics as a field and/or our relationships on this planet do not change. As such, a final piece of equity involves *Power*. The Power dimension takes up issues of social transformation at many levels. This dimension could be measured in voice in the classroom (e.g., who gets to talk, who decides the curriculum) (Morales, 2007; Zevenbergen, 2000; Adler, 1998), opportunities for students to use math as an analytical tool to critique society (e.g., exploring "risk" in society) (Mukhophadyay & Greer, 2001; Skovsmose & Valero, 2001; Gutstein, 2006), alternative notions of knowledge (D'Ambrosio, 2006), and rethinking the field of mathematics as a more humanistic enterprise (e.g., recognizing that math needs people, not just people need math) (See Gutiérrez, 2002, for a more developed argument).

For the most part, Access and Achievement can be thought of as comprising the dominant axis. By dominant, I mean:

> …mathematics that reflects the status quo in society, that gets valued in high stakes testing and credentialing, that privileges a static formalism, and that is involved in making sense of a world that favors the views and perspectives of a relatively elite group. (Gutiérrez, 2007; p. 39).

These are the components students will need to be able to show mastery in the discipline as it is currently defined and to participate economically in society. This axis, where access is a precursor to achievement, measures how well students can play the game called mathematics.

On the other hand, Identity and Power make up the critical axis. By critical, I mean:

> …mathematics that squarely acknowledges the position of students as members of a society rife with issues of power and domination. Critical mathematics takes students' cultural identities and builds mathematics around them in ways that address social and political issues in society, especially highlighting the perspectives of marginalized groups. This is a mathematics that challenges static formalism, as embedded in a tradition that favors the West. (Gutiérrez, 2007, p. 40).

The critical axis ensures that students' frames of reference and resources are acknowledged in ways that help build critical citizens (Skovsmose & Valero, 2001). In some sense, identity can be seen as a precursor to power. This axis builds upon the idea that mathematics is a human practice that reflects the agendas, priorities, and framings that participants bring to it. As such, a diverse body of people are needed to practice mathematics, not just to build a 21st century workforce, but so that they might participate democratically. Moreover, mathematics needs a diverse body of people so that the field can sustain itself in the most vibrant way possible.

To be clear, all four dimensions are necessary if we are to have true equity. Learning dominant mathematics may be necessary for students to be able to critically analyze the world; while being able to critically analyze the world may provide entrance into dominant mathematics. It is not enough to learn how to play the game; students must also be able to change the game. But, changing the game requires being able to play it well enough to be taken seriously. As researchers concerned with equity, we must keep in mind all four dimensions, even if that means that at times one or two dimensions temporarily shift to the background. A natural tension exists between mastering the dominant frame while learning to vary or challenge that frame. As such, access, achievement, identity, and power are not going to be equally or fully present in any given situation. For example, teachers cannot be expected to address power issues everyday in the classroom in ways that are meaningful to every student. Similarly, when identity or power issues are being brought to the surface, at times the connection to mastering dominant mathematics may take a lower priority. The goal is to attend to and measure all four dimensions over time.

Equity in Teaching and Learning Contexts

Given this broader definition of equity, we might ask ourselves: How do access, achievement, identity, and power play out in different contexts? Which contexts matter? How do they matter for promoting equity? In this section, I will unpack a few contexts in which I have conducted research and argue what they have revealed to me about equity. In each of these contexts, I ask: what is the nature of this context and how does it contribute to our understanding of equity?

Nine US High Schools

I have always believed we learn best from understanding "success" cases. In that vein, I first began my research trajectory with the question: What is the nature of a public school that propels its students to not only take more mathematics than is required by the district, but also to show significant gains in standardized achievement? Steeped in "opportunity to learn" theories, my first cut was to take an institutional/policy analysis, focusing on tracking as it affected students' access (Gutierrez, 1996). I drew upon the Longitudinal Study of American Youth (Miller, Suchner, Hoffer, & Brown, 1992), a data set following students from grades 7 - 12. Using hierarchical linear modeling to capture the effects of students nested within schools, I sorted the 52 schools based upon overall student gains in mathematics, course taking patterns, and differentiation within student outcomes. From the larger data set, I chose nine US high schools that were non-selective and serving a large proportion of Latina/o, African American, and/or working class students. Four of these schools were chosen for their clear student gains and signs of success; four other schools were chosen for negligible signs of success with little or no gains (e.g., less than 50 percent of students at the school reached the second year of Algebra by grade 12); one school was chosen to represent middle-of-the-road schools. My goal was to understand the nature of these schools and their accompanying success (or lack thereof). I supplemented the quantitative student data with teacher questionaires, teacher interviews, and school documents.

Though much research at that time had focused almost exclusively on the practices and outcomes of individual teachers or school wide cultures, I changed the contextual frame to consider teacher community in relation to institutional issues. For me, a single teacher was not the appropriate context for getting at broader notions of equity. And, a school level analysis was likely to minimize the role of subject matter in teachers' everyday work commitments (Gutiérrez, 1998; Stodolsky & Grossman, 1995; Talbert, 1995; Siskin & Little, 1995). I was interested in the four schools where a large proportion of their students were excelling in mathematics and where that distribution was spread out over the entire student body. For me, that had to involve more than one maverick teacher or a silver bullet policy. The math department seemed a useful unit of analysis.

What distinguished the effective math departments from the ineffective ones? Tracking was not the pivotal policy. In fact, two of the four successful schools had tracking policies in place with support structures to push adolescents towards higher-level courses and half of the ineffective schools were de-tracked. The number of formal departmental meetings, years and degrees of staff members, math/science magnet designation, and overall school culture also were not key to distinguishing success. Instead, the effective departments stood out as different from the ineffective ones in four main aspects of their organization and culture. They had a rigorous and common curriculum, commitment to a collective enterprise, commitment to students, and innovative instructional practices.

A rigorous and common curriculum meant there were very few lower level math courses in which students could get lost or bored. In fact, students were offered little choice in the kinds of courses they could take, as streamlined paths led to the most advanced courses, and 3-year minimum requirements for graduation were implemented. Additional courses were created to get students back on track or help them double-up courses in a given year so they did not lose sight of the end goal. In their curricular design and their course requirements, these effective math departments presented to students a culture that taking higher-level math courses was not only expected, but also just the norm.

The second component to these effective math departments was a commitment to a collective enterprise. That is, unlike the norms of privacy found in many schools, teachers in these departments regarded themselves as part of a community of practice (Lave & Wenger, 1991, Wenger, 1999), learning from and with colleagues. One of the first signs of this collective priority was the practice of rotating teachers' courses assignments so that no single teacher owned all of a single category of students (e.g., freshmen, seniors, honors students) or subject matter (e.g., all geometry classes, all algebra classes). Teachers explained that rotating the courses meant that they not only had a chance to get a broader sense of the math curriculum (e.g., reminding themselves of how algebra is the foundation of calculus), but it also allowed for repeat students--ones who were in a teacher's class for more than one year. The impact of these repeat students was that teachers often had to think twice about judging a student as either innately competent or incompetent, as they noticed that some students were just late bloomers, going through family issues, or better at certain topics than others. This course rotation also led to more teachers discussing their work, and sharing lesson plans. While many of the ineffective departments could be described as operating under an "independent contractor" mode, the effective departments relied upon each other for professional development. At times they attended workshops and courses together based upon the subject matter taught, while other times they required individual teachers to report back to the group on things they had attended. These departments could be described as having collective autonomy in the sense that they did not conduct all business as a whole group. Rather, they had a common vision of what they were trying to accomplish and used frequent discussions and activities to address their goals.

The third component was a commitment to all students. More than just a slogan, this commitment came through in teachers' actions. For example, rather than the deficit frames or stereotypes held by members of the ineffective departments, teachers held constructive conceptions of students (e.g., as creative, smart) and held them accountable to high expectations. Partly related to the "repeat" students that teachers mentioned, they held flexible conceptions of the learning process (e.g., that not all learning could be easily measured, that maturity contributed to proficiency). They also shared the responsibility for learning, seeing it as partly their role to motivate students to want to learn.

The fourth component distinguishing the effective math departments from the ineffective ones was innovative instructional practices. In terms of instruction, while I found ineffective and/or traditional teachers in effective departments and successful teachers in ineffective departments, they were exceptions rather than the rule. Overall, while teachers in the effective math departments for the most part continued to lecture, they moved beyond worksheets and practice of basic skills. Moreover, as a group they attempted to make the mathematics relevant to students' lives, partly by offering choices of topics for larger projects. Some such projects included basketball standings, ages of actors/actresses at the time of receiving an Oscar, and African American voter registration. Technology was also more prominent in these effective departments than in the ineffective ones. The majority of teachers used graphing calculators to model concepts and to help students see dynamic patterns or "the bigger picture." Moreover, students were expected to work in groups--partly to attend to the personal need for students to be engaged with peers, but also to encourage reasoning and conjecturing.

Although I have outlined the four components individually here, no single component would be enough to create the success these departments saw. More likely, the effects were synergistic--building off of each other. I termed this departmental culture "Organized for Advancement" (Gutiérrez, 1995; 1996) suggesting it involved a conscious "stance" (Cochran-Smith & Lytle, 1999) on the part of teachers to organize themselves and structure their work in ways that advocated for students and their learning above everything else. That is, it is not the mere presence of these components as resources for teachers that matter, it is also the meanings that emerge for teachers and students as these resources are put into use in local contexts (Adler, 2001).

From an equity standpoint, three of the four dimensions are highlighted: access, achievement, and identity. More specifically, when mathematics departments organize their formal and informal policies, courses, interactions, and supports for students in ways that promote high standards, students not only gain access to quality mathematics, they tend to achieve in ways that relate to both broader participation and test scores. When students are offered the opportunity to choose their own topics for projects, to a certain extent they are invited to express their identities and/or draw upon their cultural resources. What was clear to me at the end of this study was that although I could distill the results of the nine schools into a set of four characteristics that distinguished effective from ineffective, I was only scratching the surface. I needed to explore in greater depth the nature of a single math department, partly to understand the dynamics involved. And, while I was convinced that these OFA math departments were addressing access and achievement, I was skeptical that identity and power issues were sufficiently acknowledged (Gutiérrez, 1999).

A Successful Teacher Community

The focus shifted in my next study to ask not only what was the nature of a successful mathematics department, but also how was this teacher community created and sustained? Again, I continued to search for: what does this community reveal about equity? This math department was situated within a school that served 67 percent Latina/o students, 15 percent African American, and with 98 percent qualifying for free lunch. Their success was measured by: students taking more than the required number of math courses while in high school, large number of students in calculus (30 in 1996; 42 in 1997; 61 in 1998; 80 in 1999), calculus classes reflecting the broader student body (e.g., with respect to race/ethnicity, class, language, and school success), and 80 percent of the calculus students college bound (Gutiérrez, 2003). The following vignette attempts to capture the school context.

> We enter Union High School through the backside of the building and pass through a set of metal detectors and two armed Chicago Police officers standing post. Students (primarily working class and Latina/Latino) no longer enter through the front because it faces a main road that provided access for a shooting in the 1980's. Streams of students with large red identification tags swinging from their necks push past each other to get to their classrooms and to socialize with their friends. Students are ushered through the halls by security staff in red shirts and teachers (mainly white and middle class) who also display identitification tags. A look at school test scores indicates many of the freshmen are several years below grade level in skills, especially mathematics and English. Union is what the media often portrays as the degradation of public schools in the inner city.
>
> We might expect this school to offer an array of low level ("business math," "consumer math") courses, a watered-down curriculum with perhaps one AP calculus where those few student who make it through the public school system are still interested in college and a possible career in math. Instead, we find 3 full calculus classes.
>
> Each teacher has his own personal style. One has a dry sense of humor, cracking jokes with his students and then quickly getting down to business. Another has a soothing voice accompanied by energetic presentations and passion about mathematics. Still another has a relaxed and youthful air to being with students who are close to him in age. In all three classrooms, we see Latina/o students (primarily) with some African Americans and just a few whites all working in groups, communicating and arguing about mathematical concepts and strategies for approaching problems. They alternate between Spanish and English language, between graphing calculators and pencil/paper forms, between time spent at their desks and at the chalkboard or their small-group white boards, between their textbook written by Harvard professors and worksheets made by their own teachers, between understanding mathematics as the "forest" (big picture/concepts) and the "trees" (details/symbols), and learning from examples that incorporate students' and teachers' lives--all with the goal of understanding the meaning of derivative and integral.
>
> In each class, teachers are walking around to groups of students posing provocative questions and/or providing feedback for student work. Mostly, the teachers project a facilitator role, encouraging the students to help each other. Students pick up on this fact and are getting up from their tables to confer with other groups before returning to share the information obtained or to tutor other students when everyone in their group has reached an answer. These classes could not be described as quiet. Rather, they have the "hum" of intellectual activity that would make most teachers proud. And, with forty percent of the school's senior class present in these three calculus classes, who wouldn't be? These classes reflect both some of the goals that NCTM has put forth in the Standards and the formats used in countries where math achievement is high.

Through classroom observations, teacher and student interviews, and an analysis of school documents, the strong role of teacher community came through. In the words of one teacher:

> I think actually individual really good teachers help some kids that wouldn't make it otherwise, but I think the task of a department or of a school is to build up a community, a spirit, a plan that makes it broader than just one individual teacher, you know. And I think that may be the key lesson of what we've done at Union, that it's bigger than one teacher. And the power of a bunch of teachers working together is like greater than, the whole is greater than the sum of its parts.

In fact, only through community were teachers able to support students cognitively and emotionally in ways that advanced them to calculus.

Like the math departments that were Organized for Advancement, this department rotated its course assignments so that no single teacher owned a set of students or topics. The lack of teacher tracking in this successful teacher community was less a result of a school policy and more reflective of the stance that teachers took to create more democracy and opportunities for learning among themselves. Teachers could also be found sharing and discussing curricular materials; communicating and reflecting on students and their teaching; reinforcing to each other that all students can learn calculus; relying upon each other for professional development and support for students. Like the OFA departments I had studied, a key feature of this successful context lay in teachers placing students' needs, not just mathematics, at the center of their work.

While the broader mathematics education community has embraced the idea of "Lesson Study" (Fernandez & Yoshida, 2004; Crockett, 2002) and "Communities of Practice" (Stein et al., 1998; Franke & Kazemi, 2001; Sherin & Han, 2004), it is important for equity purposes to consider whether teacher community should be an end in itself (as a universal model of professionalism and growth) or a means to something larger. In fact, the vision of student empowerment, not just professionalism, drove the norms and practices of this teacher community. In the words of the department chairperson:

> More than anything we provide a vision for kids…having them believe in themselves as a group, having them be able to do math as a group, having them believe they can go to college as a group, and then at a whole 'nother level, um, it's like a political level…Organizing, I mean, I, I mean, at some level my way of teaching tries to organize them to be actors rather than acted upon.

As such, we learn that for equity purposes, the guiding mission of a community of practice may be as important if not more so than its presence.

Upon further exploration, this successful teacher community could not easily be distilled into a set of static characteristics without regard to how the community developed or was sustained through threat. A look into the history of this community of practice showed it was built partly on the biographies of the most veteran teachers (many of whom held identities that were marginalized in society), partly on a university partnership that provided professional development, and partly on strategic recruitment and socialization of new members over a period of 10 years (Gutiérrez & Morales, 2002). When teachers' practices and beliefs were threatened by a new principal who sought to focus staff on basic skills, the community's strong commitment to students and a reform curriculum, coordination of courses, mentoring of new teachers, and joint lesson planning allowed them to continue many of their practices without administrative support or sanctioning. Their community of practice had effectively helped them subvert the system so they could continue to be advocates for students. As such, this study highlighted the importance of not only chronicling the nature of a successful teaching/learning context, but better understanding the origins and trajectory of that context so that we might build others like it (Gutiérrez, 2002a).

Again an important aspect of this math teacher community moved beyond mere access and achievement (in terms of how many students made it to calculus) to include issues of identity and power. Identity issues included language and culture, but in complex ways (Gutierrez, 2003; in press a). Teachers did not rely upon Mayan mathematics, or some pre-scripted contexts for Latina/o students such as tortillas instead of bread. Rather, they developed a deep understanding of their students (e.g., who uses Spanish when and with whom, who prefers graphing calculators to paper-pencil forms, who is a leader in the school, etc.) and used that knowledge to create working groups and an atmosphere where students felt comfortable using Spanish or code-switching (regardless of their English proficiency levels) and negotiating that practice with non-Spanish speakers. Like the window/mirror analogy, teachers wanted to build upon the resources that students already possessed (Moschkovich, 2007), but they also saw the importance of students communicating their arguments in English--a language for which they would be held accountable on standardized tests. This meant sometimes students helped each other present their work. Identity issues also came through in the potlucks that teachers hosted. Students celebrated their mathematical successes with family members and invited speakers, in the midst of home cooked foods.

While more attention has been brought to the kinds of clear-cut curricular interventions that can give power to students, (e.g., using mathematics to explore whether there is discrimination in the ways banks loan money) the issue of power in this teaching context related more to student voice/ownership in the classroom and to an understanding of the ways mathematics and power are related in society. At the time of the study, the calculus students showed outward signs of agency (e.g., developing t-shirts that claimed the calculus space, creating a second "honors" assembly because their efforts had not been acknowledged in the larger school's gathering, creating a body of calculus representatives that provided feedback on teaching to their instructors). However, the true nature of power became more prominent a year later when I had the opportunity to follow 8 of the graduates into their college years at the University of Illinois, the flagship university of the state. Having moved from their neighborhood communities where most of their interactions involved other brown skinned, mainly working class people, the university setting presented a new space where they were often challenged to prove themselves in terms of intellect and their right to be present. Whether it was deficit-oriented professors or white and/or middle class students with negative stereotypes of urban schools, the high school graduates argued that just bringing out "the calculus card" was enough to change the power dynamics. That is, they understood and were able to draw upon the social capital conferred to them by having participated in a calculus program in high school.

23 Teacher Candidates

Having learned the importance of "community" and "stance" in the work of effective mathematics teachers, I shifted my focus to teacher education. More specifically, I studied 23 teacher candidates who remained as a cohort for 2 years as they moved through our certification program. I wanted to know how one might develop in individuals the knowledge and disposition to teach high quality mathematics to urban students. The context of the program in which I work is primarily white, middle class females, strong in mathematics (mainly procedural knowledge) with little exposure to or solidarity with marginalized students. While (re)learning mathematics in ways they were not taught is important (Ball, 1988), the more formidable struggle is to get teacher candidates to recognize that not all students are like them. Part of that challenge lies in getting them to acknowledge and build upon students' frames of mind.

I was frustrated with the limitations of readings and cases studies and was committed to the idea that "learning is becoming" (Wenger, 1999). As such, I was most interested in engaging my pre-service teachers in a community of practice like the successful teacher communities I had studied. I had already spent two years working on a similar project with a local teacher on a professional development grant. While she was committed to the highest levels of professionalism and engaged my pre-service teachers in a kind of community of practice, she did not hold a "stance" on teaching that placed her mainly African American and working class students and their needs first. She received her national board certification during the final year in which we worked together, however, in my eyes she was only minimally successful along the access dimension of equity, and unsuccessful along the other three dimensions. [See Reed & Oppong (2005) for similar results on national board certified teachers]. At best, the pre-service teachers in that project were able to identify beliefs and practices they would not replicate. At worst, our partnership further engrained already held stereotypes of working class students and students of color.

For the new project, I chose a teacher who had won awards for his teaching of calculus at the college level, who chose to teach in an alternative high school serving students who had been unsuccessful in other schools, who put his Latina/o and African American students and their needs first, and who chose to teach an NSF-supported mathematics curriculum. Although only in his first year of teaching, he offered greater opportunities for modeling the kind of equity practice for which I was looking. As such, we engaged in a one-year partnership with him and his students.

The partnership project had several components that attempted to engage pre-service teachers in the kinds of practices that effective teachers of marginalized students do on a regular basis (Gutiérrez, 2004). The university students were asked to: email a high school student on a weekly basis about things other than just math class, do mathematical problems that the high school students were doing, view video of the high school students doing the same problems, think about those math problems from the point of view of the student and the teacher, debrief with the partner teacher the events of the classroom video, prepare lesson plans for the high school classes, and host a fieldtrip to the University of Illinois where the high school students were given a chance to understand college life.

The success of the teaching context with which we partnered lay in a high percentage of students engaged on a daily basis in Interactive Mathematics Program (IMP) activities, focusing on conceptual understanding. Although no standardized achievement data was available, most of the students received solid grades in the two courses with which we partnered: algebra and data analysis/probability. Because the high school students had left their previous schools for reasons of childcare, gang involvement, or lack of support, their commitment to the math classroom here signaled a certain level of achievement. Like many of the math departments I have studied, effective teaching in this context involved a heavy reliance on cooperative learning, emphasis on National Council of Teachers of Mathematics (NCTM) process standards, students working in

Spanish and English, regular use of graphing calculators, student presentations of their work, a rigorous curriculum, and supplemental activities that were relevant to students' lives.

Because I was interested in developing in pre-service teachers their knowledge and disposition to teach quality mathematics, the partnership project aimed to get them to experience mathematics in ways that reflected the goals of the NCTM Principles and Standards (NCTM, 2000). The students had read and discussed the principles and standards, some agreeing more than others that it represented guidelines for a quality mathematics curriculum. Now, they were being given an opportunity to do activities from a real text, to see high school students doing those very activities, to video-conference with their teacher his approach and its consequences, and to decide for themselves whether this was a quality mathematics curriculum.

In the beginning of the project most of the teacher candidates were impressed with the manner in which the Interactive Mathematics Program engaged them in concepts not just procedures and gave them opportunities to connect their mathematical understanding with other topics or the real world. On first pass, and with themselves as the reference frame, the pre-service teachers saw the curriculum as quality mathematics. However, when asked to reflect on the students with which we were partnering and how this curriculum might be appropriate, they were less sure, pointing out that the IMP curriculum assumed a certain level of proficiency in basic skills (something they did not feel the students had) and offered few opportunities to practice the ideas learned. They were also concerned that students could get lost in the heavily reading-based text. Their perspectives failed to fully engage equity issues, reflecting somewhat of a deficit framing on the students of color who were our partners.

When the pre-service teachers had the opportunity to view video of the high school students doing the activity, they were happily surprised to find that the students were engaged in reasoning and problem solving, making conjectures and defending their arguments. At the end of this session, many of them changed their minds and saw the power of a curriculum that focused on concepts and that required students to collect their own data. They saw the manner in which the teacher framed his questions to draw out his students' thinking and fostered student-student interactions. In fact, some argued that this kind of curriculum and teaching was at the heart of addressing equity issues in school because so many inner city kids were usually only asked to memorize procedures or prepare for standardized tests. From our equity lens, they were now able to see issues of access (to a quality curriculum and quality teacher) and some achievement (ability to make conjectures and defend arguments).

As the year progressed, we became more and more familiar with the classrooms with which we partnered and developed a more natural feel to the debriefings of lessons with the teacher. As we gained trust and shared common rituals, the pre-service teachers were able to pose more nuanced questions and our partner teacher was able to be more vulnerable with us. In community, we sought to better understand the students' needs and to best support them to develop mathematical proficiency. At one point in the year, the teacher was recounting a situation that had happened in his class and he wondered outloud if he had done the right thing. He explained that one day in class he was lamenting how the context of the problems in IMP failed to address the lived realities of his students. As the conversation between he and his students pursued, the idea of white textbook writers arose and what might this curriculum look like if the high school students had written it instead. At the time of the conversation he was aiming to show solidarity with students, recognizing that their identities and lived realities were important. However, upon later reflection, he wondered whether bringing up the subject and showing his disappointment with the curriculum would now make it hard for students to want to do the activities in class on Monday. Should he even have said anything to his students?

This question stimulated much discussion among us and raised the issue of whether "high quality curriculum" could be considered in universalitistic terms. Several of the students saw the importance of recognizing the bias in curricular materials with one's students. In the end, the teacher ended up creating a separate project on the probability of "seeing oneself" in a variety of magazines in his data analysis and probability class.

For the first time, my students were starting to see issues of identity and power. They realized that they had not considered whose perspective was privileged in designing curriculum and that teaching involves making these kinds of in-the-moment decisions that can create or break down solidarity and trust with students. More than just having access to an NSF-supported curriculum that challenged them to reason, problem solve, communicate, make connections, and represent mathematics to each other and to their teacher or being able to use Spanish in class, they saw that students also should be given opportunities to see themselves in the curriculum or analzye the world around them. At the end of the project, what was less clear for the pre-service teachers was the extent to which students were being prepared for standardized tests they might encounter in their lives. Their struggle highlights the tensions between dominant and critical mathematics.

What this study revealed to me was the importance of successful contexts, not just for students in public schools, but also for developing teachers. That is, a key feature of the context of learning that differed between the teachers with which we partnered was being able to see real outcomes for students in ways that began to address their identities and power in society, especially with a teacher who held a stance of solidarity with his students. That is, my pre-service teachers were only able to abstract ideas and strategies from things they witnessed or participated in. This feature of success further highlights how all communities of practice are not equal (Gutiérrez, 2005).

I would like to end with some research I have been doing on the achievement gap (Gutiérrez, in press; Lubienski & Gutiérrez, in press) because it helps illuminate how our attention has been diverted from broader equity.

The "Achievement Gap"

"Gap gazing" along with more general "gaps analyses" is an example of the kind of work that is currently embraced in the mathematics education research community as a way to address equity. By gap gazing, I mean research that documents the gap in mathematics achievement between rich and poor students and between primarily brown/black students and white students, while offering little in the way of intervention. In its most simplistic form, this approach points out there is a problem but fails to offer a solution. Even researchers who conduct gaps analyses with the purpose of closing "the gap" fail to recognize that it is the analytic lens itself that is the problem, not just the absence of a proposed solution. Though I see many more problems with using the achievement gap as an analytic lens [See Gutiérrez, in press b], two concerns that are pertinent here are: a) it abstracts data from contexts and b) it ignores the many successful contexts serving marginalized students that have been documented in the literature. Let me explain.

Most of the research conducted on the achievement gap involves large-scale data sets. In these data sets, there is little room for attending to local dynamics, as the purpose is to define generalizable trends. However, by failing to attend to contexts, a "gaps" focus renders policies as "one size fits all," eventhough we know that teaching and learning are not universalistic (Ladson-Billings, 1995a, b). That is, such analyses fail to attend to the meanings that students and teachers ascribe to practices and resources that are at their disposal. In addition, the most significant variables shown to close the achievement gap do so only minimally and do not involve schooling contexts (Lee, 2003; Hedges & Nowell, 1999). Rather, they focus on income or family background, something of which few mathematics educators or researchers have control. Partly because gaps analyses provide little understanding of successful learning contexts beyond a few static variables and because they rely on correlations, they are almost useless in helping us understand either the dynamic relation between these variables or how to develop more such effective learning environments. Moreover, without the larger socio-political frame, achievement gap analyses perpetuate the notion that the problem of low achievement in mathematics is a technical one. That is, if only we knew better how to develop teacher knowledge or teach students of color, we could close the gap.

The fact is, we know quite a bit about what is successful in terms of teaching marginalized students mathematics. For example, we know that effective teachers of diverse students (especially teachers of Latinos and African Americans) come to know their students in meaningful ways (e.g., do not rely upon stereotypes, are able to relate to their students in ways that attend to their mathematical and personal needs, build upon their cultural/linguistic resources), scaffold instruction onto students' previous learning experiences without watering down the curriculum, create classroom environments that have the feel of "family" (including a heavy reliance on group work), believe all students can learn advanced mathematics, and draw upon a deep and profound understanding of mathematics when choosing tasks (NRC, 2004). Programs such as QUASAR, MESA, Project SEED, and to some extent the Algebra Project have had success with students who are not being reached by traditional schooling practices (Hilliard, 2003). What often is lacking is not the knowledge, but the public "will" to support or develop more successful learning contexts such as these.

Perhaps more importantly, an achievement gap focus fails our definition of equity, as it attends only to the dominant mathematics that comprises the access and achievement dimensions. Equity problems among students are complex; no one variable is the lever. Therefore, although the policy arena may pressure us to keep things simple, the designs of our studies and ultimately our solutions must mirror that complexity.

Future Research

My point in highlighting some of the research I have conducted on successful learning environments for marginalized students is not to say that these environments are generalizable to all students. Rather it is to suggest that only in deeper exploration of these environments can we begin to understand the meanings that emerge for teachers and students. Moreover, my emphasis on the potentially dangerous consequences of using an achievement gap focus is not to suggest that everyone must conduct the same research or even that marginalized students do not benefit at times from studies that document inequities. However, in an era of randomized trials and experimental designs, I argue there is a great need to reclaim a space for studies that focus on learning in context, especially if we are committed to a definition of equity that moves beyond mere access and achievement.

More specifically, I encourage the mathematics education community to conduct less research that documents the achievement gap, identifies causes of the achievement gap, and/or focuses on single variables to predict student success. Instead, we need more research on effective/successful teaching and learning environments for black, Latina/o, First Nations, English language learners, and working class students. More rich descriptions of these contexts, including their origins of development are necessary if we are to fully engage a diverse society. Indeed, we need to learn more about and build upon effective models that already exist. From these studies more intervention work is possible. A cursory read of this article could leave one wondering if I am calling for erasure of all large-scale quantitative research or research on populations other than marginalized students. I am not. However, I am challenging us to consider the ways in which the contexts we study come through in our work and how that may relate to our stated goals of equity.

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LEARNERS WITH/IN CONTEXTS THAT LEARN[1]

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In this writing I draw on complexity thinking to interpret and elaborate the notion of 'context.' In particular, I explore implications of a conception of context that includes but exceeds learners—that is, context understood as an adaptive, learning phenomenon. In the process, I consider a range of contexts that are at play in mathematical knowing. The discussion is illustrated through reference to ongoing research into teachers' mathematics knowledge.

1. Learning Contexts

The topic of *context* is not new to the educational research literature. It has been prominent at least since the middle of the 20th century, thanks in large part to the embrace of behaviorism. The emergence of a scientifically rigorous framework, that redefined learning as changes in an individual that are due to events in the individual's setting, ignited an explosion of interest in the notion of *context* and all that it entails.

Of course, behaviorism has long-since been displaced in discussions of learning and teaching. But one of its enduring legacies is a tendency to interpret the word "*context*" in a particular way. *Context* is most often understood in terms of the learner's surroundings (*environment*), rather than in terms of the webs of relationships that include but exceed the learner/agent (*ecology*). The latter sense of *context* is more faithful to the word's Latin root, whose translation is *weave together*.

These two interpretations of context correspond to contrasting conceptions of learners. *Context-as-environment* conjures images of insulated or isolated actors, who occupy but are separate from their situations. *Context-as-ecology* suggests entangled, embedded, and nested agents, whose boundaries are not clearly delineated. Bateson illustrates the point:

> Suppose I am a blind man, and I use a stick. I go tap, tap, tap. Where do *I* start?
>
> Is my mental system bounded at the handle of the stick? Is it bounded by my skin? Does it start halfway up the stick? Does it start at the tip of the stick?
>
> But these are nonsense questions. The stick is a pathway along which transformations of difference are being transmitted. The way to delineate the system is to draw the limiting line in such a way that you do not cut any of these pathways in ways which leave things inexplicable. (1972, p. 459)

Bateson is not just blurring the boundaries of cognitive agents here; he is also implicating the observer in the phenomenon observed. It is the observer who draws "the limiting line" that defines the learner/agent, while at the same time specifies the context. One might thus say the notion of context is multiply vexed, especially when made an explicit object of inquiry. With the observer implicated and the boundaries of learner/agent understood as contingent, it is clear that the context is also an adapting and evolving form. That is, contexts are not merely settings for learners; they are learners themselves.

This expanded the notion of *learner* has been at the core of complexity thinking for the past few decades. Complexity thinking can be defined as the study of systems that learn (Davis, Sumara, & Luce-Kapler, 2008). That is, complex systems (or *learners*) are self-determining forms that adapt their structures to maintain coherence in the face of new experiences and

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changing circumstances. Complex systems arise in the co-activities of learning sub-agents. The hierarchical nesting of sub-agents within agents implies that the delineation of a particular agent/learner from its context is often a matter of arbitrary convention or heuristic convenience.

In this writing I explore some of the implications that this expanded conception of learners with/in contexts holds for mathematics education. My intentions are thus: first, to consider a range of contexts that are at play in mathematical knowing; and second, to discuss the roles that these contexts play in my current research preoccupation—teachers' knowledge of mathematics.

2. My Own Context

In my ongoing study of mathematics-for-teaching, I investigate the devices—examples, images, metaphors, and gestures—that teachers select (often unconsciously) and employ in their efforts to render the subject matter meaningful to themselves and to their students.

Some of the data for the study is collected in meetings in which groups of teachers come together to examine their mathematical knowledge. The meetings are typically organized around familiar topics from the school curriculum. Teachers engage not only in representing existing mathematical knowledge, but also in uncovering and elaborating the understandings on which this knowledge is founded. These groups of teachers can thus be seen as *knowledge-producing collectives*.

For example, in a recent series of meetings, a cross-grade group (K–12) of teachers engaged in unpacking the concept of multiplication (Davis & Simmt, 2006). A question posed to the teachers in the latter part of the series was, "Where does a person's understanding of multiplication come from?" The teachers' response, arrived at collectively after about an hour of group discussion, is presented in Figure 1.

Some of the roots of a personal understanding of multiplication

- *innate capacities*
  - distinction-making ability
  - pattern-noticing ability
  - rudimentary quantity sense

- *elaborations in pre-school*
  - experiences of collecting, ordering, sharing, etc.
  - refinement of object/quantity permanence
  - learning of a counting system
  - preliminary development of number sense
  - preliminary development binary operations (esp. addition)

- *elaborations in the early grades*
  - physical action-based metaphors for binary operations
  - mastery of a symbol system for numbers and operations
  - grouping, repeated addition, & skip-counting

- *elaborations in the middle grades*
  - introduction to range of images and applications
  - *in particular*, shift from discrete to continuous contexts
  - conceptual blending of old and new metaphors, including: grouping processes; sequential folds; many-layers; ratios and rates: making grids and rectangular arrays; areas, volumes, and other dimension-changes; number-line-stretching or –compressing; number-line rotating

- *elaborations in the senior grades*
  - broader range of applications
  - algebraic and graphical representations
  - distinguishing arithmetic, geometric, and exponential growth
  - cutting ties to concrete materials and experiences

Figure 1. Teachers' collective response to "Where does a person's understanding of multiplication come from?"

Many of the aspects of multiplication contained in the above response originated in earlier meetings of the teacher collective. But the organization of these aspects into a chronology, that depicts mathematical understanding as a series of nested elaborations, was new.

To my reading, this chronology, spanning both physiological constitutions and socio-cultural positionings, signals that the teachers were enacting some form of tacit transdisciplinarity. Indeed, the presence of neurological, psychological, sociological, linguistic, and anthropological perspectives in the teachers' response suggests entanglement of mathematical knowing in different complex systems. As discussed earlier, these complex systems are themselves learners-with/in-contexts.

3. Contexts of Mathematical Knowing

So what are the learners-with/in-contexts implicated in school mathematics? In surveying some of them, I use the word *web* to denote the connections between learners and contexts, and to emphasize that any separation of the two phenomena is necessarily artificial.

3.1 – The Endoweb: bodily systems with/in bodies

A recent discovery in neurological research, one that Damasio (2005) identified the most significant discovery since 1980, is that human subsystems are *dependent on* but not *determined by* experience. In describing the learning of bodily systems, Damasio noted:

> … in the nervous system, as much as the immune system, *selection from among diverse elements is more important than instruction* to shaping functional structure. (p. 72, italics added)

This finding challenges the commonsense assumption that the immune system, the nervous system, and other bodily systems operate mechanically. Far from being passively directed by outside stimuli, these systems are active and interactive participants in their worlds. In other words, they are learners that make selections, which in turn contribute to their ongoing self-modifications. An immediate implication is that bodily systems must have diverse elements from which to select.

Recent research in developmental psychology suggests that humans emerge from the womb with a repertoire of many well-defined perceptual preferences and abilities. These include a rudimentary sense of quantity and an ability to distinguish among simple patterns (cf. Gopnik, Meltzoff, & Kuhl, 1999). It is clear that biological bodily systems are already making selections prior to any culturally mediated experiences of mathematical learning.

These findings evoke a conception of knowing in which the mind may be viewed as a coherence-seeking ecosystem, subject to ongoing elaborations that are conditioned by new experiences. Learning is the continual expansion of the range of possibilities for selection and interpretation available to the mind. This conception of mind-as-an-ecosystem is radically different from the "blank slate" of John Locke, the stimulus-response structures of behaviorism, or the brain-as-computer models of cognitivists. While constructivists may not find it too foreign, I suspect that some may be limited by a tendency to treat learner" and "individual" as synonyms. Even though each child comprises multiple learning agents—that is, active, discernment-making, co-implicated systems—the lack of attention in education to instruction on the level of bodily subsystems is conspicuous.

3.2 – The Egoweb: me's with/in us's

The self-versus-society dyad has been at the core of some of the most long-standing debates in education. This dyad is based on the deeply entrenched assumption in western thinking that there exists an impassable division between the categories of *individual* and *collective*. This assumption, however, seems to be untenable. As Donald explains:

> The ultimate irony of human existence is that we are supreme individualists, whose individualism depends almost entirely on culture for its realization. (2001, p. 12).

Self and society arise codependently. The relationship between individuals and collectives is not so much a *me-versus-us* dichotomy as a *me-and-us* complementarity.

It seems curious that schooling, an enterprise that has always been organized around collections of people, places so strong an emphasis on individual learning. Whereas some educators have been concerned with the question of who—individual or collective—should be privileged, there is mounting evidence that greater rewards await educators who promote the emergence of a win-win dynamic between the two.

My work with teachers' research collectives is oriented by this sensibility. I am constantly concerned with the question of how a collection of individual teachers (*me*'s) can cohere into a knowledge-producing collective (*us)*. To be more blunt, I am interested in the ways in which a teacher or a researcher might facilitate the emergence of a collective that is more intelligent than its most intelligent member. I am equally interested in the ways in which participation in knowledge-producing collectives promotes individual possibility. To this end, Elaine Simmt and I (Davis & Simmt, 2003) have used studies of complex systems to argue that there exist specific conditions to which teachers can, and indeed must, attend if they wish to promote vibrant complex interactions in their classrooms.

There is a wealth of current intersecting discourses on collective action. They include: situated learning, activity theory, actor-network theory, and various social constructionisms. All of them characterize the group, either metaphorically or literally, as a learner (e.g., Towers & Martin in these proceedings).

3.3 – The Eduweb: formal education with/in an institutionalized culture

About 15 years ago, I attended a session in which a member of a local Chamber of Commerce presented the results of an informal poll of member business owners. He had asked the question, "What sorts of preparation do you expect of the people you hire?"

Basic mathematical competency was ranked 18th on the list. At the top were qualities such as superior communication skills, ability to work in teams, and competence in gathering information. As a whole, the list pointed to the clear expectation that workers be able to cooperate with others in a collectively intelligent knowledge-producing system. There seems to be a problem here around the pervasive belief that a major purpose of schooling is to prepare the child for the adult world.

Commenting on this sort of disconnect between schooling structures and emergent post-schooling realities, Jenkins et al. note:

> Schools are currently still training autonomous problem-solvers, whereas as students enter the workplace, they are increasingly being asked to work in teams, drawing on different sets of expertise, and collaborating to solve problems.(2007, p. 21)

Of course, there are some signs that schools in general, and mathematics education in particular, are beginning to change. Current emphases on problem solving and group work certainly give reason for hope.

Nevertheless, it is evident that the institution of formal schooling is increasingly falling out of step with the rest of our changing institutionalized culture. In our times, the monolithic industrial-age factories of yesteryear, which served as models for early schooling, have given way to the flexible and adaptable Hewlett Packard, Toyota, and Microsoft. Predicting what children who enter first grade will need to know just 12 years hence is becoming increasingly difficult due to the rapid pace of technical, demographic, and economic change. Yet the topics taught in most schools, especially in mathematics, have remained disturbingly similar for many decades.

One ray of hope that I found in the teachers' response has to do with their enacted sense of the nature of mathematics. Multiplication, as described by the teachers, is contingent and adaptive, and arises from emergent experience. The evolution of mathematical knowing in children, as described by the teachers, is not as a linear progression from partial to complete understanding, but rather a continuous expansion of the space of the possible. These insightful conceptions of mathematical knowledge and mathematical learning are dramatically different from those experienced by the teachers when they themselves studied mathematics in grade school. So even though the topics have remained the same, it is clear that mathematics education has been evolving.

3.4 – The Ethnoweb: western culture in/and a multi-cultural world

The accelerating trend of the last few decades towards globalization has forced western society to come to terms with its ethnocentrism. The proliferation of communication technologies and the expansion of global travel have been accompanied by a growing willingness of western culture to get acquainted with and attend to other cultural traditions. Increasingly, westerners are encountering ways of knowing that arose in different places and in different times, as humans adapted their knowing and technologies to fit with diverse contexts.

The sum total of humanity's systems of knowing constitutes the *ethnoweb*—the collected wisdom of humankind, the multiplicity of ways the world has been imagined, the diversity of understandings of the human race's place in the biosphere. An ethnoweb is defined in large part by its technologies. In this context, the term *technology* encompasses not only artifacts and tools, but also the theories, practices, and methods that enable and constrain possibilities for action.

Language and mathematics are among the most powerful technologies that humans possess, as they equip us with habits, associations, and information that draw from the entire collective memory of humanity. They also open up the possibility of abstraction and abstract reasoning. These technologies underpin the human ability to come together in grander cognitive systems, ones with capacities that vastly surpass the abilities of individuals.

Mathematics education researchers were among the first in education to attend to the multiplicity of ways of knowing offered by the *ethnoweb*. Indeed, ethnomathematics has been a prominent and vibrant area of inquiry for the past 20 years.

Awareness of the role of culture in mathematical knowing is also apparent in the teachers' collective response about multiplication. It would take very little effort to reclassify the response under the heading "Some roots of *the western concept* of multiplication." It is obvious that the teachers made a strong connection between emergence of personal insight and recapitulation of cultural knowledge. Personal knowings of school mathematics, it seems, are all about cultural knowledge.

3.5 – The Ecoweb: humans in/and the more-than-human world

Figure 2. The image from the webpage for this year's conference.

One of the principal triggers for structuring this writing around the multiple contexts implicated in mathematics education was the main image of this year's conference (see fig. 2). When I consider this spectacular scene as *context,* my thoughts are pulled away from active classrooms, problem-posing activities, and artifact-rich experiences. Instead, I begin to contemplate the learner/agent of *humanity* in the context of *the more-than-human* world.[2] In turn, I find myself wondering about the mathematics itself and the role that it plays in this context.

So where is the mathematics? There are two classic poles to the debate. The one sees mathematics *out there*—objective and eternal Truths mined from the universe. The other sees mathematics *in here*—culturally specific and fallible truths that are collectively constructed. Both sides are wanting, as both are founded on a radical separation of agent from context (in this case, humanity from nature), which requires that knowledge be located in either the agent or the context. Complexity thinking, however, suggests a conception of knowledge as the interface between the species and the natural world. Knowledge is the site of *connection* rather than the site of *separation*.

This sense of knowledge is present in the teachers' response. Whether it is seen as a statement of individual understanding, of school curriculum, or of western mathematics, the response exhibits a clear conversational character. It is about participation and embeddedness in the universe. It is about a *conversation* in its etymological sense of *living with*.

Mathematical knowledge is about humans *living with* the more-than-human world. This point is illustrated elegantly in the conference image. When I, a human being equipped with just a preliminary knowledge of fractal geometry and chaotic dynamics, look at this image, my perception is drawn to the self-similarity of structures (trees, mountains, water surface) and to the co-evolution of phenomena (e.g., the smoothed rocks in the foreground echo the interactions of water, ice, rocky mountains in the background). Humanity is always implicated in its more-than-human context, and human mathematical knowing is similarly implicated. It is hardly surprising that the mathematics of fractal geometry and chaotic dynamics are emerging at a time when humanity is becoming more eco-centric and more attuned to evolutionary processes. These mathematics in turn will affect human ways of being in the world, and the world itself. The dance is ongoing.

3.6 – The Webweb

The conversation between humanity and the more-than-human world takes me all the way back to the *endoweb*. Where do our innate abilities to make distinctions, notice patterns, and discriminate among quantities come from? Clearly, these capacities are inscribed in our physical beings. They were learned on the species level over millions of years in the ever-unfolding conversation of humanity in/and the more-than-human world.

This brings up the question of the relationships among the different webs. How are the endo-, ego-, edu-, ethno-, and eco-webs related?

At first, it may seem reasonable to suggest that they are nested like Russian dolls. However, this suggestion is not very useful for making sense of how the *ecoweb* is embodied in the *endoweb*. It seems that the relationship is that of webs unfolding from each other and being enfolded in each other, rather than a simple containment pattern.

Analyzing a living phenomenon within its webs of context expands our understanding of the phenomenon because doing so:

- highlights the *vibrancy* of the phenomenon, often by invoking bodily and/or ecosystemic metaphors (e.g., body of knowledge, student body, the body politic);
- frames dynamics in terms of adaptation and evolution, which are notions rooted in biology, rather than in terms of cause-and-effect, which are notions drawn from physics;
- focuses attention on *fit* (coherence) rather than on *match* (correspondence) in

> accounts of how agents/learners or contexts arise and persist. The implication is that the critical criterion for survival is not *optimality/efficiency* (survival of the fittest) but *adequacy/sufficiency* (if something works, it will probably endure).

In light of these benefits, I'm inclined to ask, "How can mathematics education researchers become more deliberate about the transdisciplinary and transphenomenal character of mathematics knowing?" In the second half of this writing, I will use some of the contexts invoked by the teachers in their analysis of multiplication to begin to work through answers to this question.

4. Mathematics-for-Teaching

The topic of teachers' knowledge of mathematics and its relationship to student understanding is one of the most prominent among mathematics education researchers at the moment. Given my interest in the complexity of knowing, it is not surprising that I too am caught up in this hub of the current *eduweb*.

I begin this section with a discussion of how I have grappled with the phenomenon of mathematics-for-teaching in order to arrive at my current understanding. I continue with a description of a developing assessment tool for teachers' mathematical knowledge. I end by presenting some preliminary results of my research project.

4.1 – Grappling with mathematics-for-teaching

One of the most surprising aspects of my first few years as a middle school mathematics teacher was the degree to which I needed to learn mathematics. I took on my first teaching position with the expectation that my undergraduate mathematics program had prepared me well for the demands of middle school mathematics teaching. But I was wrong. In hindsight, I can report that the mathematics learning that I experienced in these years was in many ways deeper and more intense than that in university. It was also qualitatively very different.

Some of the my most vivid memories from this period revolve around what I might describe as re-membering (i.e., in the original sense of "putting back together"). These were moments in which I became conscious of details and associations of which I had not been conscious before or which I had known and forgotten. The topic of multiplication of integers, for example, occupied me for quite a few evenings, as I kept reworking explanations and examples in anticipation of students' questions and difficulties. I remember being dissatisfied with the very few illustrations and explanations provided in the textbook. I had a strong sense that the pictures of grouped negatives and the accompanying pattern-based justifications did not really address the "why's" of the topic.

I recall the first time that I encountered the image of rotating a number line in an arithmetic book. I experienced a feeling of triumph, almost an elation, when I realized that this simple image could be used readily to explain why "a minus times a minus is a plus" (cf. Mazur, 2003). But I was also frustrated when I recognized that this image was not used to explain multiplication of negative integers when I was a student—even though it was later encountered more explicitly in studies of, for example, the complex plane.

Throughout my years as a schoolteacher, I encountered numerous images, analogies, and metaphors that infuse understandings of mathematical topics. Some were to be found in teacher resource materials, others arose in conversations with colleagues, a few were even presented by students. The continual experience of discovering “new” ways of knowing mathematics shaped some of the core convictions that I now bring to my formal research into teachers’ knowledge of mathematics. They include:

- Mathematics-for-teaching should be studied in the moments when teachers struggle with their understanding of concepts.
- The researcher should pay particular attention to the figurative frames and conceptual associations made.
- There is much to be gained by studying the origins and history of mathematical ideas. Even the most inert mathematical concepts can come to life when the historical reasons that occasioned their emergence and the ways in which they evolved over time are traced.
- Given that pedagogy is by definition a social phenomenon, the researcher must be cognizant of the collective nature of teachers’ knowings.
- The researcher must be attentive to the dynamic nature of knowledge, and be acutely aware that attempts to study mathematics contribute to its transformation.

4.2 – Creating an assessment tool

Let’s return to Damasio’s assertion (§3.1) that selection from diverse elements is more important than instruction to shaping functional structure. Although Damasio was referring to bodily systems, I believe that his assertion holds true for educational systems as well. Hence my work is oriented by the assumption that flexible and intelligent learning is enabled by the presence of diverse interpretive possibilities.

An early goal of this work has been the excavation and explicit representation of the diverse images and metaphors associated with specific school mathematics concepts (See Davis & Simmt, 2006, for a more detailed description of the interactive structures and the data collected.) For example, when the teachers’ collective was presented with the question, “What is multiplication?”, the response included:

- adding repeatedly;
- assembling groups from equal-sized subsets;
- hopping along a number-line;
- making sequences of folds;
- layering layers;
- ratios and rates;
- straight-line functions on the *xy*-plane;
- array-generating;
- area-producing and dimension-changing;
- number-line stretching or compressing;
- number-line rotation.

Very similar lists have been produced when I posed the same question to other groups of pre-service and practicing teachers. I tend to attribute the consistency of response to the stability of the *ethnowebs* and *eduwebs* in which mathematics teaching occurs.

The current goal of the research is to develop an assessment tool that might enable a researcher to determine and augment the range of interpretive possibilities that a teacher has available. I use the images, metaphors, and explanations obtained from the teachers’ collectives to create assessment protocols. These protocols are used subsequently to guide individual teachers' explorations of their own mathematical understandings. For instance, the current protocol of multiplication starts with a focus on multiplication of single digit whole numbers, and then proceeds to multiplication of multi-digit whole numbers, unit fractions, non-unit fractions, decimals, signed integers, linear functions (slope), monomials, binomials, vectors, and matrices. At each stage, the interviewees are asked how they understand the topic, and how they might frame the topic for their students.

I specifically look for moments in which the interviewees admit that they are not comfortable with their own understandings. At these moments, it is typical for interviewees to appeal to rules. The following narrated transcript from an interview with Christine, a Grade 6 teacher with more than 20 years of experience, illustrates this point:

> "2 _ 6 = 12," I write on the paper between us.
>
> "Yep," Christine nods.
>
> "So how would you explain why that's true," I ask, adding, "without reducing it to a rule or procedure?"
>
> "Lots of ways. Two groups of six is twelve, or six groups of two. If you count up by twos, after six moves you get to twelve—which is something you can have the kids do on a number line or by playing a hopscotch-like game. … There's that idea you showed us about multiplying as many-folding: Fold a page in two parts, then fold that in six parts. That'll make twelve parts."
>
> "What about $^{+}2$ _ $^{-}6 = {}^{-}12$?"
>
> "Same thing, really, two groups of negative six is negative twelve. Or moving backward six steps twice sends you back twelve. … I guess the paper thing [i.e., sequences of folds] wouldn't work there."
>
> "Okay, then how would you help someone understand $^{-}2$ _ $^{-}6 = {}^{+}12$?"
>
> "Hmm. Well that one's getting really abstract, isn't it? You can't take negative numbers of groups. I guess you could talk about losing things, but that would just confuse people. It's horrible to say, but I think it's easiest if you just follow the rule for these ones. They're not very meaningful anyway."

When considering multiplication of two negative integers, Christine has reached what I call an *Hmm?! moment*. *Hmms* are those instances when a teacher realizes that something is amiss. They typically occur when an interpretation that has always seemed to work suddenly comes up short. As Christine noted, interpretations of multiplication that are based on grouping, on directed motion, or on folding cannot readily accommodate two negative multiplicands.[3]

Unlike *Aha! moments* that flag sudden clarity, coherence, and ready associations, *Hmm?! Moments* flag unexpected turbidity, inconsistency, and jarring breaks. I believe that *Hmm?! moments* are the ones in which teachers are most willing to expand their range of interpretive possibilities with the aid of the assessment tool.

4.3 – Preliminary Results

I consider my research into teacher's knowledge of mathematics to be still at its preliminary stages. Much more work must be done to unearth figurative frames, to refine questions, and to formalize protocols for interpretation. Even so, some consistent results are emerging. Among them:

- It seems that a majority of practicing teachers, at all levels, tend to regard formal mathematical procedures and rules as well considered and necessary.
  So far, only 3 of the approximately 50 teachers interviewed did not resort to the "follow the rule" escape device when working through the multiplication of negative numbers protocol. When questioned about whichever rule was invoked, interviewees were rarely able to unpack it. Even many of those who were able to find meaning in the mathematics often argued that standard procedures must be taught in the same way they have been handed down for generations. When questioned about this belief, a number of teachers adopted a "survival of the fittest" logical stance, reasoning that tried-and-true procedures are necessarily the best. I would classify this manner of response as an ethnoweb phenomenon; it seems to be part of a grander cultural network of beliefs about the nature of mathematics.
- The middle years of schooling are distinct in that they give rise to an explosion in the number of new metaphors, images, and applications. For the most part, these new figurative frameworks are neither obvious nor made explicit to students.
  None of the teachers interviewed so far had any difficulty with illustrating and interpreting mathematical concepts and processes of the elementary school level. Every teacher seemed to have at least one, and often only one, non-algorithmic interpretation of addition, subtraction, multiplication, and division of whole numbers. The difficulties arose when the interviews progressed to number systems beyond the whole numbers. Both elementary and secondary school teachers were often stumped in their efforts to identify appropriate interpretations of concepts and processes involving common fractions, decimals, signed integers, and algebraic expressions. These topics and concepts are typically introduced in the middle years. So I am led to hypothesize that the frequently noted loss of interest in mathematics that students experience in these years is partly due to the absence of meaningful figurative grounding of the mathematics. I regard this phenomenon to be an eduweb phenomenon. It is within the power of the institution of mathematics education to change it.
- Teachers tend to regard metaphors, applications, and images as "window dressing" for concepts and not as the principal means by which understandings are established.
  Many teachers were strangely timid when discussing how they made sense of ideas for themselves and for their students. Some felt the need to frame their responses with apologetics, such as "I know this isn't really mathematical …" or "This might be wrong, but it helps the kids get it …". The tendency to apologize was present even when the teachers were discussing interpretations that are firmly rooted in formal mathematics, such as the ones included in the multiplication list above.

  There are multiple webs at work here. "Teach a rule" resides in the eduweb, although the ethnoweb likely helps to hold it in place. Conversely, I suspect that paying greater attention to the ecoweb and the endoweb, which highlight the connections to the more-than-human world and to embodied knowing, might help address the issue.
- Understandings improve within group settings.

So far, teachers' responses were always *more varied and nuanced in collective settings. These setting included both focus group interviews and more open-ended research sessions. The impact of collectivity was not merely a cumulative one. The diversity and depth manifest in the collective responses suggested that they were more than just the sums of individual responses. While prompts such as "What is multiplication?" occasioned no more than two or three descriptors from individual respondents, they occasioned remarkably rich and stable responses from collectives. Clearly, this is an egoweb phenomenon, in which the collective us provides a more complex response than does a collection of me's.*

- Understandings improve with teaching experience.
  With few exceptions, in individual settings experienced teachers offered a broader range of interpretations of mathematical topics than did novice teachers. As illustrated in the narrated transcript with Christine, experienced teachers often presented their understandings in an automatic "rapid fire" manner, with little conscious reflection. In contrast, novice teachers, especially at the elementary level, were often more hesitant in their responses and more reluctant to consider multiple interpretations. I am most inclined to associate this observation with the eduweb. As teachers become more deeply entwined in it, they benefit from various opportunities to elaborate their understandings, and to expand the range of their explanations.

5. Closing remarks: Towards participatory mathematics education

I believe that a major contribution of complexity thinking to discussions of teaching and learning is its capacity to render explicit that which has fallen into transparency. It is a sensibility that guided my interpretation of phenomena and experiences described in this writing. For example, being explicit about the boundaries, which observers impose in order to separate knowers from contexts, reveals the shortcomings of many structures of modern schooling. Being explicit about figurative devices in mathematics leads to awareness of the centuries of personal struggle and collective contestation through which new images were proposed, debated, embraced, and thoroughly embodied in cultural knowledge.

Although the narrated transcript of Christine's interview recounts one interview with one teacher, the story it tells extends far beyond the ideas of one person. Christine's interpretations come from somewhere. If one were to study the emergence of multiplication in mathematics, one would find that the webs of association in collective knowledge are as tangled, as muddled, and as fragile as those in individual knowledge (cf., Lakoff & Núñez, 2000; Mazur, 2003). And so the entanglements of ethnoweb, eduweb, egoweb, and endoweb come to the fore.

Self-similar dynamics are at play across all these webs, and limiting one context can disrupt all contexts. Christine's *Hmm* does not flag an error; it does not point to an interpretation that needs to be corrected or replaced. Rather, it announces the need to locate a new image, a novel metaphor, a broader analogy, and to blend it with what is already there. These images, metaphors, and analogies are to be found in multiple contexts. Unfortunately, Christine confines herself to the instrumental approach of appealing to a rule. Prompted by the work or critical and cultural theorists, I am particularly troubled by this move. To my thinking, it not only derails the opportunity for the class to operate as a knowledge-producing collective, but also denies the contexts of the eduweb, ethnoweb, and ecoweb.

As we have seen, if mathematics is understood to reside in the interfaces of humanity with/in the more-than-human world, western societies with/in a multicultural world, formal education with/in an institutionalized culture, and other emergent webs, then the lines between learners and contexts become blurred and vanish. If mathematical knowledge is understood to operate in the border regions that humans create to separate learners from contexts, then it is a mathematics that's embedded, contingent, evolving, and continually enacted.

This view of mathematics has immediate implications for education, and in particular for our understanding of the roles that children and their teachers play in shaping mathematics. Researchers (see Deacon, 1997) have asserted that the most potent shapers of language are not adults but children, who plane off linguistic inconsistencies and readily embrace new vocabularies. I suspect that a similar phenomenon is at work in mathematics classrooms. Children and their teachers play a definitive role in shaping mathematics in their co-selection of images, metaphors, applications, gestures, examples, and exercises. They are participants in the production of mathematics, not merely consumers of or inductees into established knowledge.

When seen in this light, teaching cannot be only about replication and perpetuation of existing possibilities. Rather, teaching is participation in a recursively elaborative process of opening up new spaces of possibility. Complexivist teaching is not about prompting a convergence onto pre-existent truths, but about divergence into new interpretive truths. The emphasis is not only on what is, but also on what might be brought forth.

Mathematics-for-teaching should therefore be regarded as one of the most important domains of mathematical study. Its legitimation as a mathematical domain of inquiry may be the first step towards the timely evolution of mathematics education into a more participatory discipline. As Jenkins et al. explained,

> [a] participatory culture is a culture with relatively low barriers to artistic expression and civic engagement, strong support for creating and sharing one's creations, and some type of informal mentorship whereby what is known by the most experienced is passed along to novices. A participatory culture is also one in which members believe their contributions matter, and feel some degree of social connection with one another (at the least they care what other people think about what they have created). (2007, p. 3)

Participatory education extends far beyond the confines of classrooms. It implicates students, teachers, curriculum developers, mathematicians, and mathematics education researchers in a grand shared social and cultural reality. It is a reality in which the endo-, ego-, edu-, and ethnowebs are all activated.

In closing, complexity thinking points to a mathematics education that is committed to the *connection* of individuals, communities, societies, cultures, and species – all of which are learners with/in contexts that learn. We are all participants.

Endnotes

1. I am indebted to Moshe Renert for his critical commentary on earlier drafts of this writing.
2. The phrase "more-than-human world" is borrowed from Abram (1996).
3. I do not mean to present *Hmms* as a new idea. They are not unlike the "cognitive obstacles" proposed by Brousseau (1983) or the "discrepant events" described by Strike and Posner (1985). They are moments in which what is immediately conscious cannot be fitted with established and embodied associations. They are sequences of experiences that lead to surprising results, but the reason for that surprise might not be immediately available for interrogation.

Christine, for example, knows there is a conflict between her rationalization and the rational process. She knows that she's correct in her explanation at the same time that she knows that her explanation is lacking.

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THE CHALLENGE OF SCALE: DESIGNING SCHOOLS AND DISTRICTS AS LEARNING ORGANIZATIONS FOR INSTRUCTIONAL IMPROVEMENT IN MATHEMATICS[1]

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This paper focuses on research that can inform the improvement of mathematics teaching and learning at scale. We first argue that such research should view mathematics teachers' instructional practices as situated in the institutional settings of the schools and districts in which they work. We then discuss a series of hypotheses about school and district structures that might support teachers' ongoing improvement of their classroom practices. In the latter part of the paper, we outline an analytic approach for documenting the institutional settings of mathematics teaching established in particular schools and districts that can feed back to inform instructional improvement efforts.

The central problem that we address in this paper is how mathematics education research can generate knowledge that can contribute to the ongoing improvement of mathematics teaching and learning *at scale*. Researchers in mathematics education and in the learning sciences have made considerable progress in recent years in understanding the cognitive processes that underlie students' constructions of key mathematical ideas in particular mathematical domains and in developing viable instructional designs for supporting students' development of those ideas. This large and growing body of research on mathematical learning provides a grounding for reform proposals advocated by the National Council of Teachers of Mathematics (NCTM) (Kilpatrick, Martin, & Schifter, 2003). Research has also made significant progress in identifying the forms of knowing that are central to the ambitious forms of instructional practice envisioned by NCTM (1989; 2000), and in developing professional development designs that support teachers' reorganization of their instructional practices (e.g., Hill, Rowan, & Ball, 2005; Kazemi & Franke, 2004). In summarizing current investigations of teacher professional development in mathematics and in other fields, Borko (2004) concluded that this research delineates the characteristics of high quality professional development and "provides evidence that high-quality professional development programs can help teachers deepen their knowledge and transform their teaching" (p. 5). The characteristics of high quality professional development she identified included an explicit focus on knowledge of content and students' reasoning, extensive use of records of practice such as student work and classroom video-recordings, and supports for the development of teacher learning communities. Borko went on to propose a research agenda that focuses squarely on scaling up high-quality professional development that supports teachers' development of the types of instructional practices promoted by NCTM. We concur with Borko's judgment that improving teaching and learning at scale is a pressing research issue. To be sure, additional studies are needed of student learning in particular mathematical domains as well as of teachers' learning that can inform the improvement of professional development designs. However, it seems indisputable that research that addresses issues of how to take mathematics

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reform to scale is underdeveloped in comparison with research on mathematics learning and teaching.

The daunting nature of "the problem of scale" is indicated by the well-documented finding that prior large-scale interventions that have attempted to penetrate the instructional core of classroom teaching and learning have rarely produced lasting changes in either teachers' instructional practices or the organization of schools (Elmore, 2004; Gamoran et al., 2003). Schools typically experience continual external pressure to change, a condition that Hesse (1999) termed policy churn. However, classroom teaching and learning processes have proven to be remarkably stable amidst the flux, with teachers often taking the attitude that "this too shall pass." Cuban (1988) likened the situation to that of an ocean tossed by a storm in which all is calm on the sea floor even as the tempest whips up waves at the surface. Research in educational policy and leadership indicates that the limited impact of prior reform efforts is due in part to reformers' failure to take into account the institutional settings in which teachers develop and refine their instructional practices (cf. Stein, 2004). This general situation appears to be little different in mathematics. As an illustration, early cross-site evaluation of Local Systemic Change projects (LSCs) assessed projects in terms of the extent to which they were developing supportive contexts for instructional improvement and for sustainability (Weiss, Rapp, & Montgomery, 1997, 1998). A synthesis of these early reports concluded that LSCs

> encountered problems in balancing immediate project needs— providing professional development –with long-term goals, for example, engaging principals, parents, partners, and others important for sustaining reforms. Securing stakeholder support requires … staffing, resources, and strategic planning. LSCs were often 'short' in these areas, recognizing the need, but 'dropping the ball,' of necessity, to focus on support for teachers. (Bond, Boyd, & Montgomery, 1999, p. 111)

The early evaluations of the LSCs and the broader policy and leadership literature both emphasize that the improvement of mathematics instruction on a large scale is not merely an issue of developing research-based curricula and designing high-quality teacher professional development programs, though both are critical. Instructional improvement at scale also has to be framed as a problem for schools and districts as educational organizations, where both formal and informal structures and relationships shape the environment in which teachers develop and revise their instructional practices (Blumenfeld, Fishman, Krajcik, Marx, & Soloway, in press; Bryk & Schneider, 2002; Coburn, 2003; McLaughlin & Mitra, 2004; Stein, 2004; Tyack & Tobin, 1995).

The "problem of scale" is especially challenging in the case of current mathematics education reform proposals because they require ongoing learning for most mathematics teachers that involves scrutinizing deep-rooted suppositions and assumptions about students, learning, and mathematics. In this regard, Cohen and Ball (1990) observed,

> changing one's teaching is not like changing one's socks. Teachers construct their practices gradually, out of their experience as students, their professional education, and their previous encounters with policies designed to change their practice. Teaching is less a set of garments that can be changed at will than a way of knowing, of seeing, and of being. (pp. 334-5)

Although a significant proportion of mathematics teachers generally support the standards advocated by NCTM, many are not prepared to develop instructional practices of this type themselves (Cohen, 1990; Elmore, Peterson, & McCarthey, 1996; Grant, Peterson, & Shojgreen-Downer, 1996; Sizer, 1992). Policy research confirms that many teachers' current practices emphasize procedural skills at the expense of conceptual understanding of central mathematical ideas (Cohen, McLaughlin, & Talbert, 1993; Darling-Hammond & McLaughlin, 1995; Porter & Brophy, 1988). It is critical that teachers have reason and motivation to develop both the depth of mathematical knowledge and the detailed understanding of students' mathematical reasoning implicated in the kinds of classroom practices recommended by NCTM. This requires that schools and districts initiate wide-scale professional development and support structures that move far beyond the one-shot workshops that make up the professional development repertoire of most districts.

Researchers who work closely with teachers to support and understand their learning will probably not be surprised by Elmore's (1996) succinct synopsis of the results of educational policy research on large-scale reform: the closer that an instructional innovation gets to the core of what takes place between teachers and students in classrooms, the less likely it is that it will implemented and sustained on a large scale. In accounting for this finding, Elmore noted that reformers typically see themselves as developers of new approaches for supporting students' and teachers' learning, not as institution-changing agents. We will capitalize on Elmore's insight in this paper by emphasizing the importance of coming to view mathematics teachers' instructional practices as situated within the institutional setting of the school and district. This perspective implies that supporting the development of teachers' instructional practices will require changing these settings in fundamental ways.

The institutional setting of mathematics teaching as we conceptualize it encompasses district and school policies for instruction in mathematics. It therefore includes both the adoption of curriculum materials and guidelines for the use of those materials (e.g., pacing guides that specify a timeline for completing instructional units) (Ferrini-Mundy & Floden, 2007; Remillard, 2005; Stein & Kim, 2006). The institutional setting also includes the people to whom teachers are accountable and what they are held accountable for (e.g., expectations for the structure of lessons, the nature of students' engagement, and assessed progress of students' learning) (Cobb & McClain, 2006; Elmore, 2004). In addition, the institutional setting includes social supports that give teachers access to new tools and forms of knowledge (e.g., opportunities to participate in formal professional development activities and in informal professional networks, assistance from a school-based mathematics coach or a principal who is an effective instructional leader) (Bryk & Schneider, 2002; Coburn, 2001; Cohen & Hill, 2000; Horn, 2005; Nelson & Sassi, 2005), as well as incentives for teachers to take advantage of these social supports. The findings of a substantial and growing number of studies document that teachers' instructional practices are partially constituted by the materials and resources that they use in their classroom practice, the institutional constraints that they attempt to satisfy, and the formal and informal sources of assistance on which they draw (Cobb, McClain, Lamberg, & Dean, 2003; Coburn, 2005; Spillane, 2005; Stein & Spillane, 2005). The findings of these studies orient us to take account of the institutional settings in which groups of mathematics teachers work when we formulate designs to support their learning and when we develop explanations of their activity in professional development sessions and in their classrooms. In doing so, we would be begin to act as institution-changing agents who understand that supporting teachers' reorganization of their instructional practices involves, in part, supporting the development of institutional settings that enable their learning.

Historically, the "problem of scale" has been viewed as the exclusive preserve of researchers in educational policy. Our contention is that instructional improvement in mathematics has to be framed as a problem of school and district organizational learning, requiring both a broadening of what counts as worthwhile research in mathematics education and a blurring of the boundaries between policy and leadership researchers on the one hand and mathematics educators on the other. As the evaluations of LSC projects illustrate, the price for failing to attend to the institutional settings in which teachers develop and elaborate their instructional practices can be high. The findings of the LSC project evaluation is not an aberration: current research reveals that the extent to which instructional innovations are continued when funding finishes is generally low even when initiatives are successful (McLaughlin, 2006). In our view, this finding is attributable in part to reformers' untenable assumption that teachers are autonomous agents in their classrooms, uninfluenced by what takes place outside the classroom door. In making this assumption, reformers are, in a very real sense, flying blind with little if any knowledge of how to adjust to the settings in which they are working as they collaborate with teachers to support their learning. In contrast, the empirical finding that teachers' instructional practices are partially constituted by the settings in which they work orients us to anticipate and plan for the school and district support structures that need to be developed to support and sustain teachers' ongoing learning. In taking this latter perspective, we would respond to McLaughlin's (2006) call to identify "system supports necessary for individual learning to continue and deepen" (p. 219).

A Personal Perspective on the Institutional Setting of Mathematics Teaching

Our goal up until this point in the paper has been to clarify the importance of situating teachers' instructional practices in the institutional setting of the school and district when addressing "the problem of scale." We have developed the argument by drawing on the literature in mathematics education and in educational policy and leadership. We now approach the same set of issues from a more personal perspective by describing briefly how the institutional setting of mathematics teaching came to the fore in our own work with mathematics teachers and how this focus led us to our current work that seeks to inform the improvement of mathematics teaching at scale. In doing so, we first outline events that occurred in the course of a teacher development project in which one of us participated some years ago.

The goal of the project was to support a group of elementary school teachers' development of instructional practices that involved building on their students' current mathematical reasoning to achieve a significant mathematical agenda.[2] One year after we began working with the teachers, several members of the newly elected school board challenged the changes that were occurring in how mathematics was being taught and learned even though student scores on the state-mandated test were favorable (Cobb et al., 1991). In a localized version of the “math wars,” a two-year struggle ensued in the district that centered on the issue of who controlled the mathematics curriculum (see Dillon, 1993, for an account of these events). As a consequence of their collaboration with us, the teachers had become able to justify their new instructional practices in terms of the quality of their students’ mathematical learning, and they therefore believed that they were more qualified than the school board members to make decisions about mathematics instruction (Simon, 1993). The teachers, supported by their principals, eventually prevailed and were given the authority to make curricular and instructional decisions in mathematics.

The role of the research team in this sequence of events was largely reactive and involved responding to a series of unanticipated crises that threatened the continuation of the project. We had been flying blind because we naively assumed that the teachers had complete autonomy to decide how they would address state and district mathematics objectives. The lesson that we drew from this experience for future work with teachers was that it would be crucial to understand how what transpired outside teachers' classrooms influenced what occurred within them by taking into account what we came to call the institutional setting of teaching (Cobb & McClain, 2001).

As we prepared for a subsequent teacher development project conducted some years later with middle school teachers,[3] our initial plan was to collaborate with researchers in policy or leadership. We sought to identify colleagues who would analyze the institutional setting in which the participating middle-school teachers worked and share their findings with us so that we could adjust our plans for working with the teachers and with other school and district personnel. We were unable, however, to identify any policy researchers who were willing to engage in design research of this type in which the overall goal is to improve the initial design for supporting learning. Policy researchers typically study the outcomes of specific policies and, more recently, the process by which particular policies are implemented or enacted (see Honig, 2006b, for an excellent collection of papers on policy implementation). The focus tends to be on the effects of established interventions or policies, taking the institutional environment in which those policies are implemented as given. As one eminent policy researcher put it, we were looking for feedback on the institutional setting that would allow us to "mess with the intervention" by continually adjusting our design for supporting the collaborating teachers' learning. In other words, policy researchers typically seek to test the effectiveness of interventions as designed, while we as design researchers wanted to adjust and improve our design for supporting teachers' learning as we identified aspects of the institutional setting that were not conductive to change.

As a consequence of this fundamental difference in methodological commitments, we eventually decided to conduct the analyses of the institutional setting in which the collaborating middle-school teachers worked on our own. To do so, we had to develop an analytic approach that was specific to the teaching and learning of mathematics. We will describe this analytic approach in a subsequent section of this paper. For the present, it suffices to note that the resulting analyses enabled us to be proactive rather than merely reactive. We were able to anticipate potential tensions and conflicts that might have influenced the participating teachers before they escalated into full-scale crises. The resulting analyses also contributed to our understanding of the collaborating teachers' activity in both professional development sessions and their classrooms, thus enabling us to be more effective in supporting their learning. In addition, we came to realize that the interviews and group conversations that we conducted with the teachers in order to understand the institutional setting in which they worked also constituted excellent contexts in which to begin developing a collaborative relationship with them. Although our immediate purpose in these interviews and conversations was to generate data that would inform our initial design for supporting the teachers' learning, we necessarily attempted to understand some of the deep-rooted problems with which the teachers had to cope on a daily basis. This interest in teachers' everyday concerns appeared to be a relatively novel experience for the teachers and seemed to indicate to them that we took their viewpoints seriously. Based on this experience and on similar experiences with teachers in other districts, we now routinely initiate conversations of this type when we first begin working with teachers. Understanding how teachers view their institutional setting has become a critical first step in developing a collaborative relationship with teachers.

Investigating Instructional Improvement at Scale

Our current work capitalizes on the analytic approach for assessing the institutional setting of mathematics teaching that we developed while working with this group of middle-school teachers. One of the primary goals of our current project, which is still in its early stages, is to provide districts with the type of support that one of us originally sought when first attempting to establish cooperative relationships with researchers in educational policy. In other words, we seek to provide district and school-level leaders with the type of information about the institutional setting of mathematics teaching that could improve the support structures for the improvement of teaching. To this end we are collaborating with four large, urban districts that have partnered with the University of Pittsburgh's Institute for Learning (IFL) to formulate and implement comprehensive initiatives for improving the teaching and learning of middle-school mathematics. We will follow 30 middle-school mathematics teachers and approximately 17 instructional leaders in each of the four districts for four years to understand how the districts' instructional improvement initiatives are playing out in practice. In doing so, we will conduct one round of data collection and analysis in each district each year for four years to document: 1) the institutional setting of teaching, including formal and informal leaders' instructional leadership practices, 2) the quality of the professional development activities in which the teachers participate, 3) the teachers' instructional practices and mathematical knowledge for teaching, and 4) student mathematics achievement. The resulting longitudinal data on 120 teachers and approximately 68 school and district leaders in 24 schools in four districts will enable us to test a series of hypotheses that we have developed about school and district support structures that might enhance the effectiveness of mathematics professional development. We will discuss these hypothesized support structures later in the paper.

In addition to formally testing our initial hypotheses, we will "mess with the intervention" by sharing our analyses of each annual round of data with the districts and collaborating with them to identify any adjustments that might make the districts' improvement designs for middle-school mathematics more effective. We will then document the consequences of these adjustments in subsequent rounds of data collection, using both qualitative (e.g., semi-structured interviews) and quantitative instruments (surveys, assessment of teachers' mathematical knowledge for teaching, and coding of the quality of instructional lessons and student work). Clearly, we hope that the feedback we provide to the districts will contribute to their ongoing improvement efforts. In addition, we will attempt to augment our hypotheses in the course of the repeated cycles of analysis and design[4] by identifying additional support structures and by specifying the conditions in which particular support structures are important. In doing so, we seek to address a pressing issue identified by Stein (2004): the proactive design of school and district institutional settings for mathematics teachers' ongoing learning. Work of this type clearly falls at the intersection of research in mathematics education on the one hand and research in policy and leadership on the other, and it reflects the differing though complementary backgrounds of the authors of this paper.

As we engage in this work, we find it essential to be cognizant of the broader federal and state policy environments within which the practices of teachers and instructional leaders in the collaborating districts are located. In this regard, it is worth noting that No Child Left Behind (NCLB) is a relatively radical development in federal education policy. In the great society period of the 1960s initiated during the Johnson administration, federal policies were designed to help special needs students reach basic minimum standards. These policies targeted school personnel and charged them with implementing mandated programs for traditionally underserved groups of students (Honig, 2006a). In contrast, NCLB was designed to enable all students to meet high performance standards.[5] It aims to do so by impacting who teaches and what they teach. In practice, the penalties and incentives associated with NCLB influence teachers' instructional practices, mostly though not universally for the worse (Au, 2007). Whereas great society policies targeted school personnel almost exclusively, NCLB targets state, district, and school policy makers by holding them accountable for ensuring that ever-greater proportions of students surpass a proficiency threshold. States are given incentives to design and enact three central components of the federal policy: content standards, tests aligned with the standards, and mechanisms for holding schools accountable for increasing test scores. The resulting state policies constitute key aspects of the settings within which district and school leaders formulate local policies for mathematics.[6] The resulting local policies as they are actually enacted in schools in turn constitute key aspects of what we have termed the institutional setting of mathematics teaching.

The first question that needs to be addressed when locating the institutional setting of mathematics teaching established in a particular school within the broader policy environment is the quality of state policies developed first in reaction to the standards-based reform movement of the 1990s and later under the auspices of NCLB. As Elmore (2004) notes, most state departments of education lacked the capacity to respond effectively to either the comprehensive requirements of standards-based reform or the assessment and accountability mandates of NCLB. As a consequence, tests in states selected to assess students were frequently poorly aligned with state standards (Porter & Smithson, 2001), and most state tests focused on low-level skills rather than central mathematical ideas (Shepard, 2002). The second question that needs to be addressed is how district and school leaders mediate the state policies, especially in cases where the quality of standards and assessments are open to question. Elmore's (2006) reflections on his experiences of working with numerous schools is not encouraging in this regard. He concludes that most district and school leaders have little if any knowledge of how to respond effectively to state accountability policies. In the schools and districts that he studied, standards-based reform became assessment-driven reform (cf. Resnick & Zurawsky, in press). Elmore goes on to report that, in his judgment, only a minority of schools and districts have developed at least a moderately “worked out” strategy for improvement that has the potential to both motivate and support teachers' improvement of their instructional practices. The four districts in which we are currently working fall within this latter group.

In collaboration with IFL, leaders in the four districts have developed comprehensive designs for improving the quality of mathematics teaching and learning. In doing so, they have framed the challenge of improving the quality of classroom learning environments as a problem of organizational learning in which school and district leaders as well as teachers are supported and held accountable for improving their practices. They appear to be “walking the talk” when it comes to reforming middle school mathematics. In this light, Confrey, Bell, and Carrejo (2001) distinguish between what they term the discourse of high stakes accountability and that of instructional improvement. In moving from talk to action, leaders in the four districts are attempting to respond to the discourse of high stakes accountability by engaging in the discourse of instructional improvement. In doing so, they are drawing on a set of Principles for Learning that IFL derived from cognitive science research. These principles include *cognitive challenge* of student activities; *teacher stance toward students* (e.g., modeling, coaching); *active learning* (e.g., performance-based activities that require students to demonstrate their learning); *student engagement; academic rigor in a thinking curriculum* that emphasizes a commitment to a disciplinary knowledge core, high cognitive demand, and active use of knowledge; and the *use of forms of argumentation* that follow established norms of disciplinary reasoning (Institute for Learning, 2002; Resnick & Hall, 1998).

The initial goal for district leaders and their IFL collaborators is to support school leaders' development of a relatively deep understanding of the Principles of Learning as a means of viewing, assessing, and communicating about the quality of classroom instruction. They then support school leaders as they use the principles to lead the improvement of instruction in their schools. As part of this process, school and district leaders attempt to guide the establishment of nested learning communities that are intended to support the learning of both teachers and instructional leaders. These learning communities include: 1) the classroom; 2) the teacher’s grade level team or content groups; 3) the school, where the principal and the teachers create a learning community focused on pedagogy, curriculum, and effective ways to engage the community; and 4) the district, where the superintendent, key staff, and the principals create a learning community focused on the effective management of instruction within a school. To realize the full power of the nested community organization, IFL personnel work with district leaders to specify district-wide instructional programs in core subjects, including mathematics.

In the remainder of this paper, we focus on two types of conceptual tools that are central to our current work and, we contend, to the improvement of mathematics teaching and learning at scale more generally. The first is a theory of action for designing schools and districts as learning organizations for instructional improvement in mathematics. The second is an analytic approach for documenting the institutional setting of mathematics teaching that can produce analyses that inform the ongoing improvement effort.[7]

Designing for Instructional Improvement in Mathematics

The term *theory of action* was coined by Argyris and Schon (1974, 1978) and is central to most current perspectives on organizational learning. As Supovitz and Weathers (2004) clarify, a theory of action comprises "the logic advanced by advocates of reform to explain how an initiative is supposed to bring about intended results" (p. 1). In our terms, a theory of action establishes the rationale for an improvement design and consists of conjectures about both a trajectory of organizational improvement and the specific means of supporting the envisioned improvement process. The IFL has developed a general theory of action for taking ambitious instructional practices to scale in urban districts by drawing on its experience of partnering with such districts since 1995 (Glennan & Resnick, 2004; Resnick & Glennan, 2002). This general theory of action in turn orients the work of IFL staff as they assist district personnel in developing local theories of action that reflect district priorities, needs, and resources. One of the key characteristics of IFL's theory of action is that it is grounded in cognitive science research on students' learning of central disciplinary ideas. In developing its theory of action, IFL first synthesized the cognitive science research in the form of Principles of Learning, which specify characteristics of classroom learning environments that research indicates support student learning processes. IFL then mapped backwards from the Principles of Learning to develop district design principles. We touched on two of these design principles when we discussed IFL's emphasis on establishing nested learning communities and on specifying instructional programs district-wide. Additional design principles include:

- Focus at every level [within the district] on classroom instruction, including core principles of learning and teaching.
- Use coherent standards, curriculum, assessment, and professional development.
- Two-way accountability in relations between [school and district] staff.
- The view that everyone [in the schools and the district] is a learner.
- Provide continuing professional development, based in the schools and linked to instructional program.
- Make pervasive use of data in making decisions at all levels.
  (Glennan & Resnick, 2004, p. 19)

The backward mapping approach is familiar to most mathematics educators who engage in classroom instructional design and is central to Wiggins and McTighe's (1998) influential book on instructional design. Almost thirty years ago, Elmore (1979-80) argued that education policies should be formulated by mapping backwards from the practices of the practitioners that they are intended to influence. As Stein's (2004) and Cohen et al.'s (2007) recent reiterations of Elmore's argument make clear, backward mapping approaches that take practitioners' practices seriously continue to be the exception rather than the rule when formulating policies both at the school and district levels, as is the case with IFL's work, and at the state and federal levels.

In our current work in the four urban districts, we see local initiatives that stem from IFL's design principles as important for the district-wide improvement of mathematics instruction. However, we think that there are additional aspects of the institutional setting of mathematics teaching that also need to be considered. To this end, we have, formulated a series of hypotheses about school and district support structures that we conjecture will be associated with improvement in middle-school mathematics teachers' instructional practices and student learning. In developing these hypotheses, we assumed that a school or district has adopted a research-based instructional program for middle-school mathematics and that the program is aligned with district standards and assessments. In addition, we assume that mathematics teachers have opportunities to participate in sustained professional development that is organized around the instructional materials they use with students. Our hypotheses therefore focus on potential support structures that fall outside mathematics educators' traditional focus on designing high-quality curricula and teacher professional development. To the extent that the hypotheses prove viable, they specify the types of institutional structures that a school or district organizational design might aim to engender as it attempts to improve the quality of mathematics teaching across the organization.

Teacher Networks and Communities

Consistent with Elmore's (1979-80) admonition, we developed our hypotheses by taking as our starting point the forms of research-based instructional practice advocated by organizations such as NCTM (2000). Teachers who have developed instructional practices of this type attempt to achieve a significant mathematical agenda by building on students' current mathematical reasoning. To this end, they engage students in mathematically challenging tasks, maintain the level of challenge as tasks are enacted in the classroom (Stein & Lane, 1996; Stein, Smith, Henningsen, & Silver, 2000), and support students' efforts to communicate their mathematical thinking in classroom discussions (Cobb, Boufi, McClain, & Whitenack, 1997; Hiebert et al., 1997; Lampert, 2001).[8] These forms of instructional practice are complex, demanding, uncertain, and not reducible to predictable routines (Ball & Cohen, 1999; Lampert, 2001; McClain, 2002; Schifter, 1995; Smith, 1996). The findings of a number of investigations indicate that strong within and between school professional networks can be a crucial resource for teachers as they attempt to develop instructional practices in which they place students' reasoning at the center of their instructional decision making (Cobb & McClain, 2001; Franke & Kazemi, 2001b; Gamoran, Secada, & Marrett, 2000; Kazemi & Franke, 2004; Little, 2002; Stein, Silver, & Smith, 1998). Teachers' participation in professional networks and communities therefore constitutes our first hypothesized support structure.

There is abundant evidence that the mere presence of collegial support is not by itself sufficient: both the focus and the depth of teachers' interactions matter. Clearly, it is important that activities and exchanges in teacher communities and networks focus on issues central to classroom instructional practice (Marks & Louis, 1997). Interactions of greater depth might, for example, involve discerning the mathematical intent of instructional tasks or identifying student reasoning strategies, whereas interactions of less depth might, for example, involve determining how to use instructional materials or mapping the curriculum to district or state standards (Coburn & Russell, in press). We anticipate that communities and networks in which interactions focus to a greater extent on student reasoning and the mathematical potential of instructional activities will be more supportive social contexts for teachers' development of ambitious instructional practices than those in which interactions of limited depth predominate (Franke, Kazemi, Shih, Biagetti, & Battey, in press; Stein et al., 1998).

As a point of clarification, it is worth noting that the term community is frequently used differently in the mathematics education literature that it is used in the policy and leadership literature. Most mathematics educators use the term to refer to groups of mathematics teachers who work together on an ongoing basis to improve their instructional practices.[9] In contrast, policy and leadership researchers frequently use the term to refer to a school-wide community comprised of all teachers and instructional leaders participate irrespective of subject matter area (e.g., Bryk & Schneider, 2002; McLaughlin & Talbert, 2006). In an influential article, Bryk, Camburn, and Schneider (1999) define school-wide professional community as follows:

> Three core practices characterize adult behavior in a school-based professional community: (a) reflective dialogue among teachers about instructional practices and student learning; (b) a deprivatization of practice in which teachers observe each others' practices and joint problem solving is modal; and (c) peer collaboration in which teachers engage in actual shared work… Undergirding these practices are shared norms focused on student learning and collective responsibility for school operations and improvement. (pp. 753-754)[10]

As Bryk and Schneider (2002) and Halverson (2005) make explicit, the emergence of school-wide community involves the development of trust as members of the school staff establish and fulfill obligations while engaging in joint tasks such as creating a school improvement plan or formative assessments to inform instruction. Trust in turn moderates the risks inherent both in undertaking organizational change and in teachers' efforts to reorganize their instructional practices. As a first corollary to our first hypothesis, we therefore anticipate that, other things being equal, professional networks and communities will emerge more readily among mathematics teachers in schools in which the three types of practices listed by Bryk et al. (1999) have already been established. As a second corollary, we anticipate that resulting networks and communities among mathematics teachers will support the development of ambitious instructional practices to a greater extent when school-wide discourse about learning and instruction draws primarily on the broader discourse of reform rather than the discourse of high-stakes accountability.

Access of Teacher Networks and Communities to Key Resources

As our discussion of the potentially facilitative role of school-wide community makes clear, mathematics teacher networks and communities do not emerge in an institutional vacuum. Gamoran et al.'s (2003) analysis reveals that to remain viable, teacher networks and communities need access to resources. The second hypothesized support structure focuses on the potential contribution of two specific types of resources to the emergence and development of teacher networks and communities.

The first resource is time built into the school schedule for mathematics teachers to collaborate. As Gamoran et al. (2003) make clear, time for collaboration is a necessary but not sufficient condition for the emergence of teacher networks and communities. More generally, social interactions among school and district staff are in part a function of institutional structure; although institutional arrangements do not directly determine interactions, they can enable and constrain the social relations that emerge between teacher and instructional leaders (Smylie & Evans, 2006; Spillane, Reiser, & Gomez, 2006).

The second hypothesized resource is access to colleagues who are already relatively accomplished in using the adopted instructional program to support students' mathematical learning. In the absence of this resource, it is difficult to envision how interactions within a teacher network will be of sufficient depth to support teachers' development of ambitious instructional practices. In this regard, Penuel, Frank, and Krause (2006) found that improvement in mathematics teachers' instructional practices was associated with access to mentors, coaches, and colleagues who were already expert in the reform initiative. They explain that accomplished coaches and fellow teachers can share exemplars of instructional practice that are tangible to their less experienced colleagues. In cases where a district has invested in school-based mathematics coaches, Penuel et al.'s findings have implications for both the selection of coaches and for the nature of their professional development. We address these issues when we discuss a subsequent hypothesized support structure, leadership content knowledge. For the present, it suffices to note that in addition to providing ongoing support to individual teachers in their classrooms, coaches can both support the emergence of mathematics teacher networks and serve as a resource for school leaders in their roles as instructional leaders (York-Barr & Duke, 2004). In the absence of school-based coaches, school leaders might support exchanges of sufficient depth in teacher networks by requesting that district-level coaches work with mathematics teachers around instructional issues, and by encouraging the most accomplished mathematic teachers in the school to act as teacher leaders (Stein, 2004). In doing so, the school leaders would facilitate the increasing distribution of instructional leadership in mathematics.

Common Discourse about Mathematics, Learning, and Teaching

A third important support structure emerges from the finding that professional development, collaboration between teachers, and collegiality between teachers and formal school and district leaders are rarely effective unless they are connected to a shared vision of effective instruction that gives them meaning and purpose (Elmore et al., 1996; Newman & Associates, 1996; Rosenholtz, 1985, 1989; Rowan, 1990). We developed our third hypothesis by recasting this finding in three ways. First, we narrowed the focus to a vision for mathematics instruction shared by mathematics teachers and formal or positional instructional leaders. At the school level, formal instructional leaders might include the principal, an assistant principal with formal responsibility for curriculum and instruction, a mathematics department head, and a school-based mathematics coach. Second, we cast the third hypothesis in terms of a common instructional discourse rather than a shared instructional vision. The notion of a shared vision is difficult to operationalize and thus difficult to make tractable for empirical analysis. In contrast, relatively rigorous methods have been developed for analyzing discourse that focus not merely on the words that people use, but on the ways in which they use words and thus the meaning that the words have for them (Sfard, in press). We intend to capitalize on these methods to assess the extent to which mathematics teachers and instructional leaders share a common discourse for mathematics instruction (cf. Hill, 2006). Finally, to make the analysis feasible, we narrowed the scope of the hypothesis still further by focusing on discourse about instructional goals and thus on what it is important for students to know and be able to do mathematically,[11] and on how students' development of these forms of mathematical knowledgeability can be supported. Our third hypothesis is therefore that mathematics teachers' improvement of their instructional practice will be greater in schools in which a common instructional discourse has been established. It is worth reiterating that the notion of a common discourse requires not merely that teachers and instructional leaders use similar terms when they talk about mathematics instruction, but that they use those terms in similar ways. The hypothesized support structure is therefore not only concerned with the terms that teachers and instructional leaders use but with the meaning that those terms come to have for them.

Brokers

People in different positional roles in a district, such as mathematics teacher, principal, and district curriculum specialist, have different charges, engage in different forms of practice, and have different professional affiliations (Spillane et al., 2006). These distinct occupational groups are not, however, homogeneous (Honig, 2006a). Instead, each group is structured by informal networks that emerge as people seek out others with similar viewpoints (Coburn, 2001). The development of a common discourse about mathematics teaching and learning across the formal and informal groups whose agendas relate to mathematics instruction is a non-trivial undertaking. We hypothesize that a common discourse about mathematics will emerge more readily in schools and districts in which the various groups are interconnected by brokers who can bridge between differing agendas for mathematics instruction. Brokers participate at least peripherally in the activities of at least two groups and thus have access to the perspectives and meanings of each group (Wenger, 1998). For example, a principal who participates in professional development with mathematics teachers might be able to act as a broker between principals and mathematics teachers in the district, thereby facilitating the alignment of perspectives on mathematics teaching and learning across these two groups (cf. Wenger, 1998). From the point of view of organizational design, this example points to the importance of developing venues in which members of different formal groups co-participate in activities that relate directly to teaching and instructional leadership in mathematics. A priori, we anticipate that the presence of brokers who can bridge between mathematics teachers and formal school leaders, and between school leaders and key units of the district central office, will be particularly critical in supporting the development of a common instructional discourse. School-based mathematics coaches are likely candidates to fulfill this brokering role, although teachers who assume more informal leadership roles in a school and district might also facilitate this form of boundary spanning (Weatherley & Lipsky, 1977).

Negotiating the Meaning of Key Boundary Objects

The fifth hypothesized support structure also centers on the process of developing a common instructional discourse. Mathematics teachers and instructional leaders use a range of tools as an integral aspect of their practices. Star and Griesemer (1989) call tools that are used by members of two or more groups boundary objects. For example, mathematics teachers and instructional leaders in most schools use state mathematics standards and test scores, thereby constituting them as boundary objects. Tools that are produced within a school or district might also be constituted as boundary objects. For example, the mathematics leaders in one of the districts in which we are working are developing detailed curriculum frameworks for middle-school mathematics teachers to use as well as a simplified version for school leaders. In doing so, the district mathematics leaders are designing tools for others to use that reify their own vision of high quality instruction and instructional leadership. However, as Wenger (1998) emphasizes, a reifying object such as a curriculum framework does not adequately capture the richness of practice precisely because it is frozen into a concrete form such as a text.

> What is important about all these objects is that they are only the tip of an iceberg, which indicates larger contexts of significance realized in human practices. Their character as reifications is not only in their form but also in the processes by which they are integrated into these practices. Properly speaking, the products of reification are not simply concrete,

material objects. Rather, they are reflections of these practices, tokens of vast expanses of human meanings. (Wenger, 1998, p. 61)

Consistent with Coburn and Stein's (2006) recent findings, Wenger's analysis implies that organizational designs for instructional improvement that rely almost exclusively on reifications such as pacing guides and curriculum frameworks will be unlikely to bring about the intended changes in practice. The tool itself cannot fully capture the forms of practice envisioned by its designers. However, organizational designs that attempt to influence people's practices by relying almost exclusively on participation in joint activities have their own limitations, not the least of which is the coordination of activity across an organization as complex as a district. There are therefore good reasons for attending to both reifying objects and co-participation in joint activities when developing organizational designs. The forms of practice that the reifying objects are intended to support become an explicit focus of discussion when the use of reifications is negotiated. It is for this reason that the mathematics leaders in the district in which we are working intend to organize professional development for teachers and school leaders around the curriculum frameworks.

In the context of organizational designs of this type, boundary objects can serve as important focal points for the negotiation of meaning and thus the development of a common instructional discourse. The value of boundary objects in this regard stems from the fact that they are integral to the practices of different groups and are therefore directly relevant to the concerns and interests of the members of the groups. Our fifth hypothesis is therefore that a common discourse about mathematics instruction will emerge more readily in schools and districts in which members of various groups explicitly negotiate the meaning and use of key boundary objects. In speaking of *key* boundary objects, we are referring to tools that are used when developing an agenda for mathematics instruction (e.g., curriculum frameworks) and when making mathematics teaching and learning visible (e.g., district formative assessments), as well as tools that are used while actually teaching.

Accountability Relations between Teachers, School leaders, and District leaders

The sixth hypothesized support structure concerns accountability relations between teachers, school leaders, and district leaders. At the classroom level, instruction that supports students' understanding of central mathematical ideas involves what Kazemi and Stipek (2001) term a high press for conceptual thinking. Kazemi and Stipek clarify that teachers maintain a high conceptual press by 1) holding students accountable for developing explanations that consist of a mathematical argument rather than simply a procedural description, 2) attempting to understand relations among multiple solution strategies, and 3) using errors as opportunities to reconceptualize a problem, explore contradictions in solutions, and pursue alternative strategies. Analogously, we hypothesize that instructional improvement will be greater in schools and districts where the following accountability relations have been established:

- Formal school instructional leaders (i.e., principals, assistant principals, mathematics coaches) hold mathematics teachers accountable for maintaining conceptual press for students and, more generally, for developing ambitious instructional practices.
- In schools that have mathematics coaches, the principal holds the coach accountable for assisting mathematics teachers in developing ambitious instructional practices, and for

assisting the principal in understanding the mathematical intent of the adopted instructional program and the challenges involved in using the program effectively to support students' learning.

- District leaders hold school leaders accountable for assisting coaches and mathematics teachers in improving their practices.

We anticipate that the potential of these accountability relations to support instructional improvement will both depend on and will contribute to the development of a common discourse about mathematics, learning, and teaching. In the absence of a common discourse, different school leaders might well hold teachers accountable to different criteria, some of which are at odds with the intent of the district's instructional improvement effort.

In addition to considering formal accountability relations, we also hypothesize that the development of a collective sense of responsibility for instructional improvement among mathematics teachers will be important. We anticipate that the development of mutual accountability between mathematics teachers will depend on the prior emergence of teacher networks and communities. In addition, we speculate that in schools that do not have mathematics coaches, informal teacher leaders will play a critical role in supporting the development of mutual teacher accountability.

Assistance in Improving Instructional and Instructional Leadership Practices

Elmore (2000; 2004) argues, correctly in our view, that it is unethical to hold people accountable for developing particular forms of practice unless their learning of those practices is adequately supported. We would, for example, question a teacher who holds students accountable for producing mathematical arguments to explain their thinking but does little to support the students' development of mathematical argumentation. In Elmore's terms, the teacher has violated the principle of mutual accountability, wherein leaders are accountable to support the learning of those who they hold accountable. The seventh hypothesized support structure therefore focuses on relations of support and assistance, suggesting that instructional improvement will be greater in school and districts where:

- Formal school instructional leaders (i.e., principals, assistant principals, mathematics coaches) are accountable to teachers for assisting them in understanding the mathematical intent of the curriculum, in maintaining conceptual press for students and, more generally, in developing ambitious instructional practices.
- In schools that have mathematics coaches, principals are accountable to coaches to procure material resources necessary to facilitate high quality mathematics instruction, to assist the coach in gaining authority with mathematics teachers,[12] and to use the coach as a resource to support the principal's development as an instructional leader in mathematics.
- District leaders are accountable to school leaders to provide the material resources needed to facilitate high quality mathematics instruction, and to support school leaders' development as instructional leaders.

In schools where relations of mutual accountability for improving instruction have been established between mathematics teachers, it is important that the teachers are also mutually accountable to assist each other. In such cases, the participation of teachers who are already accomplished in using the adopted instructional program would seem critical (Penuel et al., 2006).

Leadership Content Knowledge

The eighth hypothesized support structure follows directly from our arguments about the importance of a common instructional discourse and concerns the leadership content knowledge of school and district leaders. Leadership content knowledge encompasses leaders' understanding of the mathematical intent of the adopted instructional materials, the challenges that teachers face in using these materials effectively, and the challenges in supporting teachers' reorganization of their instructional practices (Stein & Nelson, 2003). Ball, Bass, Hill, and colleagues have demonstrated convincingly that ambitious instructional practices involve the enactment of a specific type of mathematical knowledge that enables teachers to address effectively the problems, questions, and decisions that arise in the course of teaching (Ball & Bass, 2000; Hill & Ball, 2004; Hill et al., 2005). Analogously, Stein and Nelson (2003) argue that effective school and district instructional leadership in mathematics involves the enactment of a subject-matter-specific type of mathematical knowledge, leadership content knowledge, that enables instructional leaders to address questions that arise in the course of leadership. "Standing at the intersection of subject matter knowledge and the practices that define leadership, this form of knowledge would be the special province of principals, superintendents, and other administrators charged with the improvement of teaching and learning" (Stein & Nelson, 2003, p. 424). Stein and Nelson go on to clarify that adequate leadership content knowledge in mathematics would enable school and district leaders to recognize high-quality mathematics instruction when they see it, support its development, and organize the conditions for continuous learning among school and district staff.

The pioneering work of Nelson and Sassi (2005) not withstanding, the research base on the mathematical knowledge for instructional leadership is extremely thin. As a consequence, it is not surprising to find that leadership programs that focus on supporting ambitious mathematics instruction reflect a range of different assumptions about the importance of leadership content knowledge. For example, the Principles of Learning that ground IFL's work with school and district leaders are not subject-matter specific, but instead specify characteristics of high-quality instruction that hold across core subjects.[13] In contrast, Stein and Nelson (2003) argue that principals should be able to assist mathematics teachers in improving their instructional practices. In their view, this requires that principals have a relatively deep understanding of mathematical knowledge for teaching, of what is known about how to teach mathematics effectively, and of how students learn mathematics, as well as "knowing something about teachers-as-learners and about effective ways of teaching teachers" (p. 416). Stein and Nelson extend this line of reasoning by proposing that district leaders who train principals must know everything that principals need to know both as instructional leaders and as leaders of organizations in which the entire instructional staff seeks to improve. In addition,

> district leaders must also have knowledge of how principals learn (i.e., characteristics of principals as learners -- what pre-conceptions do principals often bring to the learning enterprise? How can these be overcome? How much do principals typically know about facilitating teacher learning? (p. 426)

In the absence of a solid research base, we are forced to follow Stein and Nelson in relying on theoretical speculation to develop an initial, tentative hypothesis about leadership content knowledge. Although we would expect to see greater instructional improvement in schools where school leaders have stronger leadership content knowledge, we suggest that it might be sufficient that this type of expertise is distributed across formal and informal leaders rather than residing exclusively with the principal. In other words, we suggest that the depth of leadership content knowledge that principals require is situational and depends in large measure on the expertise of others in the school. In cases where principals can capitalize on the expertise of an effective school-based mathematics coach, for example, the extent of principals' leadership content knowledge in mathematics might not need to be particularly extensive. Deep understanding of general characteristics of high-quality instruction as captured by IFL's Principles of Learning might suffice, provided principals have an overall understanding of the mathematical intent of the instructional program together with an appreciation that using the program effectively is non-trivial accomplishment that requires support for an extended period of time. We speculate that this limited knowledge might enable principals to collaborate effectively with their coaches who have developed the forms of expertise outlined by Stein and Nelson. In such cases, the requisite expertise as instructional leaders in mathematics and as organizational leaders is distributed across the principal and the coach. If this speculation proves viable, it would be important for principal professional development to attend explicitly to the issue of leveraging subject-matter coaches' expertise effectively.

It might also be sufficient for principals' leadership content knowledge to be relatively limited in the absence of a school-based mathematics coach provided there is a core of mathematics teachers in their schools who are relatively accomplished in using the instructional program effectively. In such cases, a key aspect of instructional leadership would be the ability to capitalize on teacher expertise, particularly supporting the emergence of informal teacher leaders. In the absence of teacher expertise, the role of the principal (or another school leader) in supporting mathematics teachers' ongoing learning is critical. We find this conclusion worrying because the depth of leadership content knowledge that Stein and Nelson suggest that principals need seems overwhelming. Support from district leaders would appear to be essential in such cases, perhaps resulting in the transferal of a small group of accomplished mathematics teachers to the school or the funding of a school-based coach.

The forms of leadership content knowledge that Stein and Nelson suggest that district leaders need appear to be impractical. However, Stein and Nelson's proposal is more attainable if we interpret them as specifying the forms of expertise that district leaders should collectively be able to enact. This interpretation points to the importance of the coordination of activity between district administrative units, particularly those responsible for curriculum and instruction and for instructional leadership. We address this issue when we discuss our ninth hypothesis.

It should be apparent that we see considerable merit in Stein and Nelson's analysis of leadership content knowledge at the school and district levels. Our strategy has been to view leadership content knowledge as distributed across school and district personnel. Stein and Nelson acknowledge this approach when they observe that

> where individual administrators do not have the requisite knowledge for the task at hand they can count on the knowledge of others, if teams or task groups are composed with the

> recognition that such knowledge will be requisite and someone, or some combination of people and supportive materials, will need to have it. (p. 444)

Thus, we expect to see greater instructional improvement in schools where principals and other school leaders make effective use of school and district personnel's collective leadership content knowledge.

Coordination within and between District Administrative Units

When we discussed the hypothesis that a common discourse about mathematics, learning, and teaching is an important support structure, we focused primarily on relations between mathematics teachers and school instructional leaders. The rationale we developed for the hypothesis also applies to relations between district administrative units. We have already indicated that key units include those responsible for curriculum and instruction and for school leadership. In an era in which data based decision making has become almost a mantra (Ikemoto & Marsh, 2007), the unit responsible for assessment and evaluation would also appear critical given the importance of the types of data that are collected to assess school, teacher, and student learning. In addition, depending on the district, the unit responsible for special education might also be influential to the extent that it focuses on how mainstream instruction serves groups of students identified as potentially at-risk. As Spillane et al. (2006) indicate, staff in different administrative units whose work contributes to the district's initiative to improve the quality of mathematics teaching and learning might well understand district-wide initiatives differently. We anticipate that the role of brokers who can bridge the perspectives of the various district administrative units will therefore be important in facilitating the development of a common instructional discourse across the district. We speculate that it might be sufficient for brokers who are not mathematics specialists to enact what might be termed an intermediate level of leadership content knowledge provided there is a collective commitment to the general characteristics of high-quality instruction across units. This intermediate level of leadership content knowledge involves what Spillane (2000) terms a function rather than a form interpretation of the instructional program in mathematics. District leaders who make a form interpretation focus on the surface form of mathematics instruction and view features such as group work, class discussions, use of manipulatives, and real world problems as alternative strategies for achieving traditional instructional goals in mathematics. In contrast, district leaders who make a function interpretation understand, for example, that the function of class discussions is to support students' development of mathematical argumentation, and that the specific ways in which group work, class discussions, manipulatives, and real world problems are enacted may or may not contribute to the achievement of this instructional goal. This intermediate level of leadership content knowledge also encompasses what Spillane (2002) terms socioconstructivist assumptions about mathematics teachers' learning, wherein district leaders understand the social supports for learning that teachers' participation in communities and informal networks can provide. We speculate that with the assistance of brokers, it might be possible for other district leaders who are not mathematics specialists to participate at least peripherally in a common discourse about mathematics instruction. We further speculate that the participation of district leaders from different instructional units in such a discourse will make possible the collective enactment of leadership content knowledge at the district level.

As we indicated when introducing the hypothesized support structures, they do not constitute a theory of action. The hypotheses do not, for the most part, specify actions but instead pull together our conjectures about school and district structures that a theory of action should aim to engender in order to support the improvement of mathematics learning and teaching at scale. In developing the hypotheses by mapping backwards from the classroom, we limited the scope of our conjectures by focusing primarily on the establishment of institutional settings that support school and district staff's ongoing improvement of their practices. This backward mapping process could be extended to develop conjectures that are directly related to the traditional concerns of policy researchers. For example, several of the hypothesized support structures involve conjectures about the role of mathematics coaches and school leaders. These conjectures have implications for district hiring and retention policies. In addition, the hypotheses imply that the allocation of frequently scarce material resources should be weighted towards what Elmore (2006) terms the bottom of the system (see also Gamoran et al., 2003). As the notion of distributed leadership is currently fashionable,[14] it is worth noting that the hypotheses do not treat the distribution of instructional leadership as a necessary good. In the absence of a common discourse about mathematics, learning, and teaching, the distribution of leadership can result in a lack of coordination and alignment (Elmore, 2000). Elmore (2006) observes that effective schools and districts do not merely distribute leadership. They also support people's development of leadership capabilities, in part by structuring settings in which they learn and enact leadership. The hypotheses therefore emphasize that the important outcomes of an initiative to improve the quality of mathematics learning and teaching include "the system capacity developed to sustain, extend, and deepen a successful initiative" (p. 219).

Documenting the Institutional Setting of Mathematics Teaching

The data that we will collect in the course of our four-year collaboration with middle-school teachers and instructional leaders in the four districts will allow us to formally test our hypotheses. In analyzing the instructional practices of 120 participating teachers and the institutional settings in which they work, we expect to see greater instructional improvement in schools and districts where the types of support structures that we have discussed have been established. The hypotheses are also central to the second component of our research in which we will share the results of each annual data analysis with the districts and collaborate with them to identify any adjustments that might make the districts' improvement designs for middle-school mathematics more effective. To accomplish this effectively, we require an analytic approach for documenting the institutional setting of mathematics teaching that can produce analyses that feed back to inform the districts' ongoing improvement efforts.

The analytic approach that we will take makes a fundamental distinction between schools and districts viewed as designed organizations and as lived organizations. A school or district viewed as a designed organization consists of formally designated roles and divisions of labor together with official policies, procedures, routines, management systems, organizational units, and the like. Wenger (1998) uses the term designed organization to indicate that its various elements were designed to carry out specific tasks or to perform particular functions. In contrast, a school or school district viewed as a lived organization comprises the groups within which work is actually accomplished together with the interconnections between them. Brown and

Duguid (1991; 2000) clarify that people frequently adjust prescribed organizational routines and procedures to the exigencies of their circumstances (see also Kawatoko, 2000; Ueno, 2000; Wenger, 1998). In doing so, they often develop collaborative relationships that do not correspond to formally appointed groups, committees, and task forces. Instead, as Coburn's (2001) analysis of teacher groups illustrates, the groups within which work is actually organized are sometimes non-canonical and not officially recognized. These non-canonical groups are important elements of the school or districts viewed as a lived organization.

Given the goals of our research, we find it essential to document the districts in which we are working as both designed organizations and as lived organizations. One of our first steps has been to document the districts as designed organizations by interviewing district leaders about the local theories of action for improving middle-school mathematics that they developed in collaboration with IFL. These local theories of action are designs for the future of the organization that reflect the suppositions and assumptions of district leaders and IFL. In each of the four annual rounds of data collection, we will document any changes in the districts' designs for instructional improvement in mathematics by interviewing district leaders and by collecting artifacts such as organizational charts, policy manuals, and curriculum frameworks.

In analyzing the initial interviews that we conducted to document district leaders' current theories of action, we are attempting to explicate their suppositions and assumptions so that we can frame them as testable conjectures. The process of testing these conjectures requires that we document how the theories of action are playing out in practice, thereby documenting the schools and districts in which we are working as lived organizations in each round of data collection. Methodologically, we will use what Hornby and Symon (1994) and Spillane (2000) refer to as a snowballing strategy and Talbert and McLaughlin (1999) term a bottom-up strategy to identify groups within the schools and districts whose agendas are concerned with the teaching and learning of mathematics. The first step in this process involves conducting audio-recorded semi-structured interviews with the participating 30 middle-school mathematics teachers in each district to identify people within the district who influence how the teachers teach mathematics in some significant way. The issues that we will address in these interviews include the professional development activities in which the teachers have participated, their understanding of the district’s policies for mathematics instruction, the people to whom they are accountable, their informal professional networks, and the official sources of assistance on which they draw. We will supplement these interviews with a survey of a set of related issues. The second step in this bottom-up or snowballing process involves interviewing the formal and informal instructional leaders identified in the teacher interviews as influencing their classroom practice. The purpose of these interviews is to understand formal and informal leaders’ agendas as they relate to mathematics instruction and the means by which they attempt to achieve those agendas. We will supplement this second round of interviews by administering surveys to principals and school-based mathematics coaches. We will then continue this snowballing process by interviewing people identified in the second round of interviews as influencing how mathematics is taught in the district. In terms familiar to policy researchers, this bottom-up methodology focuses squarely on the activity of what Weatherley and Lipsky (1977) term street-level bureaucrats whose roles in interpreting and responding to district efforts to improve mathematics instruction are as important as that of district leaders who designed the improvement initiative. The methodology therefore operationalizes the view that what ultimately matters is how district initiatives are enacted in schools and classrooms (cf. McLaughlin, 2006).

In addition to identifying the groups in which the work of instructional improvement is accomplished and documenting aspects of each group's practices, our analysis of the schools and districts as lived organizations will also involve documenting the interconnections between the groups. To do so, we will focus on three types of interconnections, two of which we introduced when presenting our hypotheses about potential support structures. Interconnections of the first type are constituted by the activities of *brokers* who are at least peripheral members of two or more groups. As we noted, brokers can bridge between the perspectives of different groups, thereby facilitating the alignment of the enterprises of different groups. As our hypotheses indicate, our analysis of brokers will be relatively comprehensive and will seek to clarify whether there are brokers between various groups in the school (e.g., mathematics teachers and school instructional leaders), between school instructional leaders and district leaders, and between key units of the district central office. Interconnections of the second type are constituted by *boundary objects* that members of two or more groups use routinely as integral aspects of their practices. Wenger (1998) emphasizes that reifying objects are relatively transparent carriers of meaning for members of the group in which they are created. In contrast, there is the very real possibility that they will be used differently and come to have different meanings as they are appropriated and used by members of other groups. Our analysis will therefore seek to identify boundary objects and to document whether they are used in compatible ways by members of different groups.

The third type of interconnection is constituted by boundary encounters in which members of two or more groups engage in activities together as a routine part of their respective practices. The hypothesized supports structures involve conjectures about the importance of a range of boundary encounters between members of different groups, including those in which they explicitly negotiate the meaning of boundary objects. In addition to documenting the frequency of boundary encounters between members of different groups, our analysis will focus on the nature of their interactions. A recent finding reported by Coburn and Russell (in press) indicates the importance of pushing for this level of detail. Coburn and Russell studied the implementation of elementary mathematics curricula designed to support ambitious instruction in two school districts. As part of their instructional improvement efforts, both districts hired and provided professional development for a cadre of school-based mathematics coaches. Coburn and Russell found that there were significant differences in the depth of the interactions between the coaches and the professional development facilitators in the two districts. In the first district, interactions focused primarily on issues such as discerning the mathematical intent of instructional tasks and on identifying and building on student reasoning strategies. In the second district, interactions focused primarily on how to use instructional materials and on mapping the curriculum to district or state standards. Coburn and Russell also documented the nature of interactions between coaches and teachers in the two districts. They report that teacher-coach interactions increased in depth to a far greater extent in the first district than in second district. In addition, interactions between teachers when a coach was not present also increased in depth in the first but not the second district. Coburn and Russell trace the contrasting depth of interactions in teacher networks to a corresponding contrast in the depth of the routines of interaction in which the two groups of coaches participated during their professional development. In other words, the contrasting routines of interaction in coach professional development became important features of interactions in teacher networks in the two districts. As our first hypothesis indicates, we anticipate that differences in the depth of interactions in a teacher network influence the extent to which participation in the network supports the teachers' development of ambitious instructional practices.

In our view, Coburn and Russell's analysis represents a significant advance in research on instructional improvement at scale. To this point, policy researchers have tended to frame social networks as conduits for information about instructional and instructional leadership practices. However, research in mathematics education makes it abundantly clear that information about ambitious instructional practices is, by itself, insufficient to support teachers' development of this form of practice. Coburn and Russell's analysis focuses more broadly on interactions across groups as well as within social networks, and emphasizes that these interactions are not limited to the sharing of information but can and often do involve co-participation. Furthermore, their findings demonstrate that the focus and depth of co-participation matters. Their analysis therefore establishes a valuable point of contact between research on policy implementation and research on mathematics teachers' learning. This latter body of work documents that teachers' co-participation in activities of sufficient depth with an accomplished teacher or instructional leader is a critical source of support for teachers' development of ambitious practices (e.g., Borko, 2004; Fennema et al., 1996; Franke & Kazemi, 2001a; Goldsmith & Shifter, 1997; Kazemi & Franke, 2004; Wilson & Berne, 1999). Our decision to focus on boundary encounters when analyzing schools and districts as lived organizations was motivated in large measure by this general finding. In this regard, we anticipate that Coburn and Russell's notion of routines of interaction will prove to be a useful analytic tool as we seek to understand whether the nature of the activities in which school and district staff co-participate in one setting influences how they subsequently interact with others as was the case with the coaches that Coburn and Russell studied.

Providing Feedback to Inform Instructional Improvement

In the approach that we have outlined, the analysis of a school or district as a lived organization involves identifying the groups in which work of instructional improvement is actually accomplished and documenting interconnections between these groups. An analysis of the lived organization therefore focuses on what people actually do and the consequences for teachers' instructional practices and students' mathematical learning. In contrast, an analysis of a school or district as a designed organization involves documenting the school or district theory of action for instructional improvement. This design for the future of the organization specifies organizational units and positional roles as well as organizational procedures and routines, and involves conjectures about how the enactment of the design will result in the improvement in teachers' instructional practices and student learning. An analysis of the designed organization documents both this design and the tools and activities that will be employed to realize the design by influencing people's practices in particular ways.[15] One of the primary goals of our current work is to provide feedback to the four collaborating districts that will enable them to identify any adjustments that might make their improvement designs for middle-school mathematics more effective. In doing so, we will necessarily draw on our analyses of the districts as both designed and lived organizations.

In developing feedback for the districts, we will identify gaps between their designs for instructional improvement and the ways in which their designs are actually playing out in practice by comparing our analyses of each district as a designed organization and as a lived organization. This approach will enable us to differentiate cases in which a district's theory of action is not enacted in practice from cases in which the enactment of the theory of action does not lead to the anticipated improvements in teachers' instructional practices (Supovitz & Weathers, 2004). As an illustration, one of the districts with which we are collaborating is investing scarce resources in mathematics coaches with half-time release from teaching for each middle school. The district's theory of action specifies that the coaches' primary responsibilities are to facilitate teacher collaboration and to support individual teachers' learning by co-teaching with them and by observing their instruction and providing constructive feedback. Suppose that the district's investment in mathematics coaches does not result in a noticeable improvement in teachers' instructional practices. It could be the case that the district's theory of action has not been enacted. For example, the coaches might be tutoring individual students and preparing instructional materials for the mathematics teachers in their schools rather than working with teachers in their classrooms. In attempting to understand why this is occurring, we would initially focus on coaches' and school leaders' understanding of the coaches' role in supporting teachers' improvement of their instructional practices. Alternatively, it could be the case that the coaches are working with teachers in their classrooms, but their efforts to support instructional improvement are not effective. In this case, we would initially seek to understand how, specifically, the coaches are attempting to support teachers' learning and would take account of the process by which the coaches were selected and the quality of the professional development in which they participated.

As this illustration indicates, our goal when giving feedback is not merely to assess whether the district's theory of action is being implemented with fidelity, although our analysis will necessarily address this issue. We also seek to understand why the district's theory of action is playing out in a particular way in practice by taking seriously the perspectives and practices of street-level bureaucrats such as teachers, coaches, and school leaders. As a matter of principle, the development of explanations of this type will necessarily draw on analyses of the district viewed as a designed organization and as a lived organization. This is the case even when a district's theory of action results in the anticipated improvements in teachers' instructional practices. A district's theory of action describes intended practices, routines, and procedures in what Feldman and Pentland (2003) term generalized schematic form. The design cannot fully determine the practice of coaching because the exigencies of circumstances necessarily remain open (de Certeau, 1984; Suchman, 1987). Street-level bureaucrats such as teachers, coaches, and school leaders develop their practices in local contexts created in large measure by the actions of others. Teachers', coaches', and school leaders' actions and interactions are in turn enabled and constrained by designed institutional structures (cf. Smylie & Evans, 2006; Spillane et al., 2006). For example, the design for position of coach can serve as a resource for coaching-as-lived by both orienting coaching practice and by providing a ready-made means of accounting for aspects of coaching practice (Feldman & Pentland, 2003). An analysis of a district as a designed organization foregrounds the district design as a potential resource for action as well as the institutional structures that delimit the possibilities for practice. An analysis of a district as a lived organization foregrounds people's agency as they develop their practices within the context of others' institutionally situated actions.

Discussion

In this paper, we have focused on the question of how mathematics education research might contribute to the improvement of mathematics teaching and learning at scale. We addressed this question by first clarifying the value of viewing mathematics teachers' instructional practices as situated in the institutional settings of the schools and districts in which they work. Against this background, we presented a series of hypotheses about school and district structures that might support teachers' ongoing improvement of their classroom practices. We then went on to outline an analytic approach for documenting the institutional settings of mathematics teaching established in particular schools and districts that can feedback to inform the instructional improvement effort.

The most obvious limitation of the hypotheses and the analytic approach is that they do not attend explicitly to issues of equity in students' access to significant mathematical ideas. At the time of writing, we are revising our hypotheses and data collection instruments in an effort to address this deficiency. In doing so, we are adjusting our hypotheses by mapping backwards from the classroom and are drawing on research on instructional practices that distribute learning opportunities equitably (e.g., Boaler, 2002; Civil, 2007; Enyedy & Mukhopadhyay, 2007; Gutiérrez, 2004; Gutstein, 2007; Lubienski, 2002; Martin, 2000; Milner, 2006; Moschkovich, 2007; Tate, 2005). A priori, we anticipate that we will have to elaborate our hypotheses to take account of at least two institutional structures that have direct implications for the distribution of student learning opportunities.

The first support structure concerns whether and to what extent mathematics classes are tracked (Gamoran, Nystrand, Berends, & LePore, 1997; Oakes, Wells, Jones, & Datnow, 1997). Horn's (2006) analysis of students' trajectories through mathematics courses in two high schools is relevant in this regard. In one school, the mathematics teachers worked collectively for all students' academic success in a rigorous de-tracked common sequence of courses, whereas in the second school the mathematics curriculum was tracked and the mathematics teachers did not believe that all students could succeed academically. Horn's analysis revealed that the official, institutional discourses constituted in the two schools associated particular positions in course sequences with particular types of students. These institutional discourses included assumptions about both what were reasonable goals for students in particular positions in course sequences, and about what counted as mathematical competence for students in these positions. As Horn observed, students' location in a course sequence might influence their understandings of themselves as more or less mathematically competent. In addition to focusing on course sequences as designed, Horn also investigated courses as lived and as experienced by students. In doing so, she demonstrated that students' location in a particular course sequence can have different meanings in different mathematics classes in the same school. This finding indicates that any hypothesis about tracking will have to be nuanced to take account of the nature of teachers' instructional practices in particular classes.

The second, related institutional structure that has implications for equity in learning opportunities concerns the categories of mathematics students that are inherent in teachers' and instructional leaders' practices. Horn's work is again relevant because she analyzed the systems for classifying students that the mathematics teachers had constructed in the same two high schools. Horn's (2007) analysis indicates that the teachers' classification systems were related to their views of mathematics as a school subject. In the high school in which mathematics was tracked, the teachers differentiated between formal and informal methods, and viewed the latter as illegitimate. They took a sequential view of school mathematics and assumed that students had to first master prior topics if they were to make adequate progress. This conception of school mathematics was implicit in the teachers' classification of students as more or less motivated to master mathematical formalisms, and as faster and slower in doing so. The teachers' classification of students in terms of stable levels of motivation and ability grounded the need the teachers perceived for separate mathematics courses for different types of students. In sharp contrast, the mathematics teachers at the second school in which mathematics courses had been de-tracked tended to take a non-sequentialist view of school mathematics and instead viewed it as a web of ideas rather than an accumulation of formal procedures. These teachers rejected the categorization of students as fast or slow because it emphasized task completion at the expense of considering multiple strategies. In addition, the teachers in this school viewed it as their responsibility to support students' engagement both by selecting appropriate tasks and by influencing students' learning agendas. Thus, these teachers addressed the challenge of teaching college preparatory mathematics to all students by focusing primarily on their instructional practices rather than on perceived mismatches between students and the curriculum.

Horn's (2006; 2007) analyses make an important contribution by illustrating that content and the nature of teachers' instructional practices matter when addressing issues subsumed under the heading of tracking. Her work is particularly valuable for our purposes because it ties together institutional structure and classroom practice. We anticipate that we will need to adjust most of our hypotheses as we map backwards from a view of high-quality mathematics instruction in which learning opportunities are distributed equitably.

We conclude this paper by returning to the relation between research in educational policy and leadership and in mathematics education. To this point, researchers in these fields have conducted largely independent lines of work on the improvement of teaching and learning (cf. Engestrom, 1998; Franke et al., 2001). Research in educational policy and leadership tends to focus on the designed structural features of schools and how changes in these structures can result in changes in classroom instructional practices. In contrast, research in mathematics education tends to focus on the role of curriculum and professional development in supporting teachers' improvement of their instructional practices and their views of themselves as learners. In this paper, we have argued that mathematics education research that seeks to contribute to the improvement of teaching and learning at scale will have to transcend this dichotomy by drawing on analyses of schools and districts viewed both as designed organizations and as lived organizations. In the interventionist genre of research that we favor, organizational design is at the service of large-scale instructional improvement in mathematics education. In this genre of research, the attempt to contribute to school and district improvement efforts constitutes the context for the generation of useful knowledge about the relation between the institutional settings in which teachers work, the instructional practices they develop in those settings, and their students' mathematical learning. Research of this type reflects de Corte, Greer, and Verschaffel's (1996) adage that if you want to understand something try to change it, and if you want to change something try to understand it.

Notes

[1] The analysis reported in this chapter was supported by the National Science Foundation under grant No. ESI 0554535. The opinions expressed do not necessarily reflect the views of the Foundation. The hypotheses that we discuss in this paper were developed in collaboration with Sarah Green, Erin Henrick, Chuck Munter, John Murphy, Jana Visnovska, and Qing Zhao. We are grateful to Kara Jackson for her constructive comments on a previous draft of this paper.
[2] This project was conducted by Paul Cobb, Erna Yackel, and Terry Wood and involved a collaboration with approximately 20 second-grade teachers who worked in a rural district.
[3] This project was conducted by Paul Cobb and Kay McClain and involved a collaboration with approximately 25 middle-school mathematics teachers who worked in two urban districts.
[4] In engaging in these repeated cycles of analysis and design, we will, in effect, attempt to conduct a design experiment at the level of the school and district.
[5] It might be argued that, in practice, NCLB focuses on schools and districts to ensure that all students achieve minimum competency levels.
[6] As Cohen, Moffitt, and Goldin (2007) note, NCLB positions state, district, and school leaders as both the targets of policy and as makers of policy.
[7] These two types of conceptual tools serve to ground the two aspects of the design research cycle, namely design and analysis (cf. Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Design-Based Research Collaborative, 2003).
[8] The research base for these broad recommendations is presented in a research companion volume to the National Council of Mathematics' (2000) *Principles and Standards for School Mathematics* edited by Kilpatrick et al. (2003).
[9] An important issue that we gloss over in this paper is that of specifying criteria for differentiating between a *group* of teachers who meet to work on issues of mutual interest and a professional teaching *community*. Researchers who have collaborated with practicing teachers to support their learning make clear that a group of teachers who collaborate with each other in some way do not necessarily constitute a community (Franke et al., in press; Nickerson & Moriarty, 2005; Stein et al., 1998). In this regard, Grossman, Wineburg, and Woolworth (2001) observe that communities are frequently brought into existence by the linguistic fiat of the researcher's pen or keyboard. This phenomenon is evident in both the policy and leadership literature and the mathematics education literature. Discussions of the distinction between a teacher group and a professional teaching community in the true sense of the term can be found in Cobb, McClain, et al. (2003), Dean (2005), Krainer (2003) and Secada and Adajian (1997).
[10] School-wide community as defined by Bryk et al. (1999) is closely related to Elmore's (2000) notion of internal accountability within a school and to the broader construct of social capital (cf. Adler & Kwon, 2002; Smylie & Evans, 2006).

[11] A focus on instructional goals takes us onto the slippery terrain of mathematical values (Hiebert, 1999). It is important to note that values are not a matter of mere subjective whim or taste but are instead subject to justification and debate (Rorty, 1982).
[12] The most robust finding in the general literature on school-based coaches is that the coach's effectiveness depends crucially on the role of the principal in legitimizing the importance of the coach's work (York-Barr & Duke, 2004).
[13] IFL offers districts two programs that are content specific: Content-Focused Coaching and Disciplinary Literacies. Content-Focused Coaching provides professional development for school-based coaches in specific subjects including mathematics. The Disciplinary Literacies program supports teams of district subject matter specialists in designing district-wide instructional programs at the secondary level.
[14] Spillane and colleagues (Spillane, 2005; Spillane, Halverson, & Diamond, 2001, 2004) proposed distributed leadership as an analytic perspective that focuses on how the functions of leadership are accomplished rather than on the characteristics and actions of individual positional leaders. However, as so often happens in education, the basic tenets of this analytic approach have been translated into prescriptions for practitioners' actions. In our view, this is a fundamental category error that, if past experience is any guide, might well lead to pernicious consequence (cf. Cobb, 1994, 2002).
[15] The distinction between schools and districts viewed as a designed and as lived organizations is analogous to that between the intended mathematics curriculum and the enacted mathematics curriculum.

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A COMPARATIVE STUDY OF THE ACHIEVEMENT OF GRADUATES OF NATIONAL SCIENCE FOUNDATION-FUNDED AND COMMERCIALLY DEVELOPED HIGH SCHOOL MATHEMATICS CURRICULA IN THEIR FIRST YEAR OF UNIVERSITY MATHEMATICS WORK

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The selection of K-12 mathematics curricula has become a polarizing issue for schools, teachers, parents, and other educators. This has raised important questions about the long term influence of these curricula. This study examined the impact of participation in either a National Science Foundation-funded or commercially developed mathematics curriculum in high school on the difficulty level of the first university mathematics class a student enrolled in, and the grade they earned in that class. The findings indicated that student participation in a particular high school curriculum was related to their mathematics knowledge in eighth grade, their ACT mathematics score, overall high school GPA, and the number of levels of high school mathematics taken. High school curricula also affected the difficulty of their first university mathematics class but not to their grade in that class. The achievement gaps initially identified in eighth grade persisted throughout high school. The results also provide evidence of differential course-taking in college.

There is widespread agreement among teachers, parents and school administrators, mathematics educators, mathematicians, and others of the importance of providing K-12 students with rigorous instruction that promotes mathematical understanding, but less agreement on how to achieve this goal. A manifestation of the lack of agreement are the "mathematics wars" that have pitted defenders of commercially developed mathematics curricula against those advocating curricula supported by the National Science Foundation. In general, commercially developed (CD) mathematics curricula place a greater stress on traditional algorithms and procedures, while National Science Foundation-funded (NSFF) curricula focus on extended problem solving activity in contextually laden situations and consideration of algebra, geometry, probability, statistics, and topics in discrete mathematics each year. Specifically, NSFF curricula refer to those curricula which were funded from a solicitation of proposals (RFP NSF 91-100) through the National Science Foundation in the early the early 1990's (Senk & Thompson, 2003) that were designed to be aligned with the National Council of Teachers of Mathematics' (NCTM) *Curriculum and Evaluation Standards for School Mathematics* (1989).

In a very real sense two NCTM *Standards* documents (1998, 2000), although widely supported, challenged the hegemony of CD mathematics curricula in U.S. schools. Advocates of CD curricula have argued that teaching mathematics using NSFF curricula will lead to poor mathematics performance and would not adequately prepare students for calculus and other mathematically-oriented courses as taught at the university level. Proponents of NSFF curricula

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.

often point to published reports of ongoing and disappointing mathematics achievement of students exposed to CD curricula, and argue that large scale changes in the way we think about, develop, implement and assess school mathematics programs are needed. For example, results from the Trends in International Mathematics and Science Study (1996) indicated that U.S. eighth grade and twelfth grade students performed below international averages. Regional evidence of problems are reflected in a 1999 report by the Minnesota State College and University System that found only 38% of students entering a postsecondary institution in the State College System, almost all of whom would have participated in a CD curriculum, were considered ready to begin their collegiate mathematics coursework at the level of college algebra. College algebra is typically taught as a senior level course in a high school college-intending mathematics curriculum. The corresponding figure for two-year colleges was 18%. Similar findings have been reported in Wisconsin, Iowa, and North and South Dakota.

These findings challenge the assumption that CD curricula, which have dominated K-12 mathematics instruction in Minnesota and elsewhere in the U.S. for many years, have adequately prepared students for college level mathematics. However, little is known about the college level success, or lack thereof, of students who experienced an NSFF mathematics curriculum in high school.

National Science Foundation-Funded Curricula

Theoretical Models

The 1989 and 2000 NCTM *Standards* documents incorporated components of various cognitively oriented theories about the development of mathematics knowledge and the nature of student learning (Piaget, Dienes, and Lesh and Shulman). There is reason to believe that the fundamental principles that underpin the development of the NSFF curricula would promote achievement for diverse student learners. NSF required that each curriculum proposal (NSF RFP 91-100) document how the planned development project would reflect the principles espoused in the 1989 *Standards* document. Thus these new curricula reflect, for the most part, contemporary cognitive approaches to mathematics teaching, learning and assessment.

Development of National Science Foundation-Supported Mathematics Curricula

Fearing for the future levels of technological literacy in the U.S., and for the future of our ability to lead in this area, NSF funded 13 full mathematics programs (3 elementary, 5 middle grades and 5 at the high school level) at a cost of approximately $100 million dollars beginning in the early 1990's. The NSFF curricula were for the most part problem-centered, focusing on problems embedded in realistic settings that took several class periods to explore and resolve. Extensive teacher support materials and professional development opportunities, also funded by NSF, were provided to school districts adopting these programs.

Attributes of National Science Foundation-Funded Mathematics Curricula

Specific attributes of NSFF curricula include (i) Variable methodological and pedagogical approaches that are designed specifically to sustain student interest, including the use of cooperative approaches that are considered fundamental, (ii) Mathematical content reflective of solutions to real world problems, many relating directly to student interests, (iii) A de-emphasis on abstract algorithmic manipulation unrelated to current investigations, (iv) A wide variety in the topics and mathematical domains considered each year, to continually arouse students' interest and expand their horizons relating the many and myriad ways in which mathematics is applicable to everyday life, (v) Providing ongoing experiences with concrete to abstract sequences, suggesting that mathematics is always “about something” and does not exist in its own cocoon apart from the real world, (vi) Reflecting a belief that the development of meaning should be considered a prime concern when involving students in mathematical activity, and (vii) Learning activities that reflect the ideas of those who have most influenced cognitive oriented theory development and research on human learning, (Piaget, Dienes, Bruner, Shulman etc.).

Evidence of the Performance of Middle School and High School Students in National Science Foundation-Funded Curricula

Research on the achievement of students learning from NSFF curricula in grades K-12 (Senk & Thompson, 2003) indicates that these students typically do as well as or better than CD students on achievement measures that typically consist of classroom problem sets (Cichon & Ellis, 2003; Huntley, et al., 2000; Lott et al., 2003; Schoen & Hirsch, 2003; Webb, 2003), as well as standardized tests such as the Stanford Achievement Test (Abeille & Hurley, 2001; Harwell, et al., 2007; Post, et al., in press), Iowa Test of Educational Development Ability to do Quantitative Thinking (Abeille & Hurley, 2001), Iowa Test of Basic Skills (Schoen & Hirsch, 2003), Scholastic Assessment Test mathematics subtest (Webb, 2003), Comprehensive Test of Basic Skills (Webb, 2003), and the Preliminary Scholastic Aptitude Test (Cichon & Ellis, 2003; Lott et al., 2003). Data across different geographical areas and various curricula show that middle school and high school students studying from NSFF curricula perform similarly on non-standardized measures of mathematics achievement or on the mathematics portions of standardized tests when factors that can affect mathematics performance are taken into account.

Criticism of National Science Foundation-Funded Mathematics Curricula

Despite evidence of success, NSFF curricula have come under intense criticism from web-based sources (e.g., Mathematically Correct at http://www.mathematicallycorrect.com/, New York City HOLD http://www.nychold.com/) and other venues (Hill & Parker, 2006; Klein, 2007; Wu 1997). Much of the criticism claims that NSFF students are poorly prepared to succeed in calculus and other university level mathematics courses. Unfortunately, these claims are based on studies that lacked a distinguishable research design and have not made an attempt to minimize selection and other biases of various kinds (see, e.g., Bachelis, 1998; Hill & Parker, 2006).

In sum, little is known about the impact of participation in an NSFF high school mathematics curriculum on student’s university mathematics performance. This study examines one aspect of this impact, that of examining students’ initial college mathematics experience.

Research Question

The main question addressed by our study was: Do university students who completed at least three levels of high school mathematics in a National Science Foundation-funded curriculum differ from those who have had similar exposure to a commercially developed high school curriculum in the difficulty level of their first university mathematics class, or in the grade they earned in that class when taking into account background factors such as prior mathematics achievement, ethnicity, and gender? A high school level was defined to be the content equivalent of a year-long course taught one hour per day and has been commonly referred to as a Carnegie unit. Four levels of difficulty of college level mathematics courses were identified.

Methodology

The methodology in this study was informed in part by *On Evaluating Curricular Effectiveness: Judging the Quality of K-12 Mathematics Evaluations* (Mathematical Sciences Education Board, 2004).

Research Design

This study employed a quasi-experimental design for cross-sectional (retrospective) data (Pedhazur & Schmelkin, 1991) in which the high school mathematics curriculum (group) a student participated in was the independent variable of most interest. Difficulty of students' first university-level mathematics class and the grade earned in that class served as dependent variables.

Sampling

Our samples consisted of postsecondary students enrolled at a single large public university in a mid-western state in the U.S. during the Fall, 2002 or Fall, 2003 terms. All students had graduated from one of 87 high schools in which they had completed at least 3 levels of mathematics and for which at least 10 students enrolled at the University. The latter requirement helped to ensure reasonable high school-based statistical estimates. The maximum number of students from a single high school in our sample was 77 with a median of 26 students, and these students were assumed to be representative of a college-bound population.

Data

Archival data were collected from high school transcripts (ACT mathematics score, GPA in high school mathematics classes and overall GPA, high school percentile rank, number of levels of high school mathematics completed, gender, ethnicity), the State of Minnesota (Basic Skills Test in mathematics administered in eighth grade), individual school districts (information about the kind of mathematics curriculum a student completed in high school), and the University (mathematics grade and course-taking information). If participation in a particular high school curriculum better prepares students for college mathematics, those students would be expected to initially enroll in more difficult (advanced) mathematics classes and to have better grades. Examining mathematics course descriptions at the University led to the construction of a four-point Likert scale to capture course difficulty (1 = course typically completed in high school level, 2 = difficulty level consistent with courses such as college algebra and pre-calculus, 3 = difficulty level consistent with calculus I, 4 = calculus II or beyond).

Data Analyses

The main data analyses centered on answering the question of whether university students who completed at least three levels of high school mathematics in an NSFF curricula differ from those who have had similar exposure to a CD high school curricula in the difficulty of their first University mathematics class, or, the grade they earned in that class when taking into account background factors. Descriptive analyses were performed to explore patterns in the data, followed by two-level hierarchical linear modeling (HLM) that treated students as nested within high schools. A Type I error rate of α = .05 was used for all significance tests.

Results

A key descriptive result was the appearance of a difference of about one-third of a standard deviation favoring CD students on the Eighth Grade Basic Skills Test in mathematics. CD students also had significantly higher ACT mathematics scores (about one-half a standard deviation). Gender and ethnicity differences on these variables also appeared. A sub-analysis indicated that NSFF students who took calculus at the University (there were several levels of calculus offered) had significantly higher grades when compared with their CD counterparts who also took calculus at the University, with GPA means of 3.26 and 3.0, respectively, on a 0-4.0 point scale. Although the effect was weak, it mirrors the finding by Schoen and Hirsch (2003), who also found that NSFF students had higher grades in a university calculus course.

Hierarchical linear modeling (HLM) was used to examine the effect of high school mathematics curriculum on the difficulty of the first University mathematics class a student enrolled in and the grade they earned in that class. The difficulty of a student's first university mathematics class was analyzed assuming an ordinal scale of measurement and student grades an interval scale.

For the difficulty data, a generalized hierarchical model for ordinal data was fitted to difficulty level that included predictors for high school mathematics curriculum, ACT mathematics score, high school mathematics GPA and overall GPA, percentile rank, high school location, gender, and ethnicity. The key result was the significant slope for curriculum group (.858), meaning that curriculum is related to difficulty. Exponentiating the estimated curriculum slope provides information about the size of the effect. Here, exp(.858) = 2.36 means that, with other predictors held constant, a CD student who is "average" on all predictors is more than twice as likely to enroll in more difficult classes as not, compared to an NSFF average student. Female students who were average on all predictors was 2.69 times more likely to enroll in less difficult classes as not, compared to a male student. Students coming from rural high schools were associated with stronger relationships between ACT mathematics scores and difficulty of a student's first University mathematics class. Difficulty was also related to overall high school GPA and ACT math score in expected ways (i.e., students with higher GPAs or ACT math scores were more likely to enroll in more difficult classes).

For the grade data the key result was that the slope for curriculum group was not significant

(-.026, p = .688), meaning that, with the other predictors held constant, the mathematics curriculum a student, who was "average" on other predictors, participated in while in high school was not related to the grade they earned in their first University mathematics class.

In sum, the HLM results indicated that there was evidence that a student's high school mathematics curriculum was significantly related to the difficulty level of their first University mathematics class but not to the grade they earned in that class, once various background factors were taken into account.

Discussion

The main finding, that NSFF students on average tend to enroll initially in a college mathematics course of lower difficulty than CD students, but tend to earn similar grades in these courses, raises important possibilities and important questions. Perhaps CD high school mathematics curricula simply do a better job of preparing students to begin their post-secondary mathematics than do NSFF curricula. Or perhaps the use of placement tests where the goal is to help students initially enroll in the most appropriate post-secondary mathematics class played a role in the findings for difficulty by assigning NSFF students disproportionately to lower level courses. The placement tests at the University are overwhelmingly oriented to precisely those skills that have often been de-emphasized in the NSFF curricula. NSF curricula intentionally reduce the amount of time and attention paid to standard algorithms and repetitive procedures.

NSFF students who took calculus at the University had significantly higher grades when compared with their CD counterparts who also took calculus, suggests that simple conclusions about any curriculum are likely to be problematic. CD students who completed five levels of high school mathematics, which typically included calculus, 48.9% repeated calculus at the University compared to 36.7% of the NSFF students

An important finding was that significant differences in mathematics achievement favoring CD students appeared in eighth grade, before students participated in any particular high school mathematics curriculum. These differences remained throughout high school, meaning that initial achievement gaps were not erased by participation in these newer curricula. These results also provide evidence of a selection bias and reinforce the need to study the factors leading students to participate in one mathematics curriculum or another. In an ongoing work we include predictors in statistical analyses that help to reduce this bias.

We also want to emphasize that, in addition to the usual limitations of research involving retrospective data linked to research design, sampling, instrumentation, etc., this study did not have access to students' high school teachers or administrators and thus we were unable to ascertain the degree to which teachers of these students were faithful to the author's intentions relating to both content and method. This observation is relevant to both the NSFF and CD groups. Our experience with this issue relates to the Local System Change (LSC) Project at the University of Minnesota (1997-2001). This project provided 130 or more hours of targeted professional development to over 1100 middle grades and secondary mathematics teachers, some of whom undoubtedly taught students in our sample. However, we were unable to link particular high school mathematics teachers to particular students.

For those teachers in the LSC Project receiving 20 hours of mentor on-site assistance, we were satisfied that most of the teachers implemented new NSFF curricula with fidelity to the authors' intent, although variability did of course occur. It is important to emphasize that this study was conducted at a time when teachers in the state were newly involved with NSFF curricula. Effective instruction in any curriculum requires that the teacher know what content students have already been exposed to and ideally where the elaboration of various topics will appear in the future. This long term perspective was not possible for the teachers whose students are represented here.

Implications for Future Curricular Research

These preliminary results should advance the ongoing debate regarding the adequacy of National Science Foundation-funded curricula in helping students prepare for subsequent university level mathematics. Teachers, administrators, counselors, researchers, and the public must engage in ongoing discussions concerning the best option for their students. Preparation for University calculus is probably not the only "raison d'être" in high school mathematics. Curriculum adoption decisions are more complex than that. Our ongoing efforts consider these issues in a longitudinal multi-institutional context over a duration of seven semesters.

Endnote

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CALCULUS STUDENTS' ASSIMILATION OF THE RIEMANN INTEGRAL INTO A PREVIOUSLY ESTABLISHED LIMIT STRUCTURE

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Two teaching experiments were conducted in calculus classes in a large public university. Prior to the experiments, an initial conceptual framework was developed based on a mathematical decomposition of the Riemann sum definition of the definite integral in four layers: product, summation, limit, and function. Several of the layers also include sublayers that illustrate various ways of thinking about each layer. Analysis of the data from the teaching experiments guided modification of the framework to also reflect the cognitive development of students. Data shows that students successfully assimilated the integral structure into a previously constructed limit structure, via approximations. Their struggles were concentrated in areas where the Riemann sum structure departed from the limit structures students had previously encountered.

The purpose of this research was to examine student development of the concepts of Riemann sums and definite integrals. A deep understanding of the structure of a Riemann sum is invaluable to students when encountering real-world applications involving nonintegrable functions, when working with numerical methods for approximating integrals based on the Riemann sum structure, and even when setting up an appropriate definite integral to model a particular situation.

Several previous research studies have focused on issues related to the teaching and learning of the mathematical operations used in defining the Riemann integral: multiplication, rate of change, sequences and series, limits, and functions. Zandieh (2000) developed a theoretical framework for the definition of the derivative based on the mathematical decomposition: difference, quotient, limit, and function. Our decomposition of the definite integral follows Zandieh's model of the derivative.

Although multiplication is a concept that is introduced in elementary school, even adults still have trouble with certain aspects of multiplication (Simon & Blume, 1994; Sowder *et al.*, 1998). Sowder *et al.* found that teachers often could not distinguish between word problems that required an additive solution or a multiplicative solution. Another very common problem is the principle that "multiplication makes bigger" and "division makes smaller" (Sowder et al., 1998). Particularly in Riemann sums, this is not the case. In the Riemann sum, $\sum_{i=1}^{n} f(x_i)\Delta x$, as n (the number of intervals) approaches infinity, delta x (the width of the each interval) approaches zero. Thus, we are multiplying very small quantities, and multiplication in this case certainly does not "make bigger".

Simon and Blume (1994) studied pre-service teachers' understanding of multiplication in the context of area. Although the teachers knew to solve area of rectangle problems by multiplying the two quantities, Simon and Blume found that "their choice of multiplication was often the result of having learned a procedure or formula, rather than of a solid conceptual link between their understandings of multiplication and their understandings of area" (p. 476). When asked why they chose to multiply, the students responded, "because it works" (p. 478), " 'cause that's the way we've been taught" (p. 479) and comparable responses.

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.

Sequences and series also play a key role in the definite integral. Mamona (1987) found that many students do not distinguish between sequences and series and often admit to using the two words interchangeably. This can certainly cause confusion when trying to use sequences and series within the definite integral.

The research on the third layer of the definite integral, limit, is extensive. A few of the most relevant studies are discussed here. Cornu (1991) notes that in the teaching of limits, we usually emphasize the *process* of approaching a limit, instead of emphasizing the *concept* of the limit as a particular value. Since the definite integral is defined to be the limit of Riemann sums, it seems likely that students will struggle to understand the concept. Students will need to understand that the definite integral is the *value* of the limit, and not a *process* of reaching a number. Furthermore, many students believe that a sequence "must not reach its limit" (Davis & Vinner, 1986). In regards to Riemann sums, it was found that students believe that the limit of Riemann sums will never produce the exact value of the definite integral, but will only be an approximation.

Oehrtman (2002, 2007) discusses several metaphors that students use to reason about limits. A few that are likely to appear in student explanations of Riemann sums are discussed here. First, in the collapse metaphor, students envision a change that leads to a collapse in dimension. For Riemann sums, we use rectangles (2-dimensions) to approximate the area under the curve. As the number of rectangles approaches infinity, the width of the rectangle gets smaller. Students often envision the rectangles of 2-dimensions "collapsing" into one dimension where the "rectangle" has no width, only a height. This would eliminate the multiplication layer from the students' understanding and confound the summation layer.

Based on the above results, Oehrtman (2004, in press) outlined a research-based design for an introductory calculus sequence in which limit concepts are introduced through ideas about approximations and error analyses that are intuitively understandable yet reflect the structures of rigorous limit definitions. This approach is intended to provide a consistent treatment of limit concepts throughout an entire calculus sequence in which students can build their understanding of limits of functions, derivatives, sequences and series, Riemann integrals, etc. on a common conceptual structure. The course in which this study was conducted was based on these design principles.

Research specifically on Riemann sums and definite integrals is still quite sparse. Some exceptions are the works of Orton, Thompson, and Silverman. First, Orton (1983) studied student understanding of the definite integral mainly in terms of calculating the area under the graph of a function. Thompson (1994) developed and implemented a module to help students understand Riemann sums in a way that was intended to make the Fundamental Theorem conceptually obvious to students, and Thompson and Silverman (in press) discuss the relationship between area under a curve and other views of the definite integral. They argue that a conceptual understanding of the definite integral requires a stronger understanding of the Fundamental Theorem of Calculus. Similarly, we found that students who could evaluate definite integrals using area under a curve often could not solve similar problems unless they were told which curve to graph (Sealey, 2006). Our research draws on these results and expands to discover how students set up definite integrals using Riemann sums.

Theoretical Perspective

We use Piaget's structuralism (1970, 1975) as the theoretical perspective for both the design of the study and the data analysis. Piaget asserted that conceptual structure is reflectively abstracted from actions and coordinations of actions within a self-regulating feedback system. The focus of actions and coordinations of actions as the source of conceptual structure distinguishes reflective abstraction from ordinary abstraction in which concepts are abstracted directly from objects and their properties. In the design and data analysis of this study we focus on the mental and physical *actions* that students are expected to perform and expect that effective instruction must be designed so that these actions reflect the conceptual structures students are expected to understand. Specifically, students cannot develop a conceptual understanding of definite integrals simply by being exposed to several examples of them (ordinary abstraction). Additionally, computation of definite integrals does not typically require actions that reflect the Riemann integral structure, so we cannot expect conceptual understanding to emerge from this type of activity either. Instead, reflective abstraction of the definite integral as a concept requires students *do* something with Riemann sum structures, be made to coordinate and reverse these actions when possible, and be forced to reconcile difficulties through feedback obtained through these interactions.

When developing a new structure, in this case the structure of the Riemann integral, we focus our analysis on assimilation and accommodation (Piaget, 1970, 1975). Both terms are used to describe ways of incorporating new information into a previously established structure. If the new information does not fit into the previously established structure as-is, the learners must either modify their understanding of the new information (assimilation) or reorganize their previously existing conceptual structure so that the new information will fit (accommodation). In regards to our research, students would be incorporating the structure of the Riemann integral into a previously established approximation framework.

Methods and Subjects

A series of teaching experiments (Simon, 1995) was designed to develop students' understanding of the structure of the Riemann sum definition of the definite integral. A teaching experiment consists of an activity, a hypothetical learning trajectory (HLT), and a revised hypothetical learning trajectory based on results of the activity. In this research, there were three activities with difficult applications involving definite integrals – distance and velocity of a falling object; force and pressure of water on a dam; and the energy and force required to stretch a spring.

Participants in the study were from two calculus classes at a large public university in the United States. The first participants were students enrolled in a calculus workshop that was designed to complement their traditional calculus class, and these students formed the pilot group for data collection and analysis. Approximately half of the students reported that they enrolled in the workshop because they love math and wanted to take an additional math course, while the other half of the students reported that math was very difficult for them, and they enrolled in hopes of getting extra help to pass the calculus class. The rest of the participants were students in a second semester calculus class that was developed for engineering students at the same university.

Both classes were videotaped as they worked in groups throughout the semester to solve novel problems that were designed to develop a conceptual understanding of various mathematics topics. Data also consisted of copies of student written work, both homework assignments and notes taken during class. Throughout the course, students had already been familiarized with an approximation framework for limit structures developed by Oehrtman (2004, 2007). Across several mathematical topics (derivatives, limits of sequences and series, limits of functions, etc.), students worked on activities that required them to answer the five approximation questions listed in table 1. The activities for the Riemann integral used these approximation questions to promote the development of the limit structure within the Riemann integral.

Table 1 – Approximation Questions

1) What is being approximated?
2) What are the approximations?
3) What is the error?
4) What is a bound on the error?
5) How can you find an approximation accurate to within any predetermined bound?

Prior to the experiments, an initial framework was developed based on a mathematical decomposition of the Riemann sum definition of the definite integral in four layers: product, summation, limit, and function. Several of the layers also include sublayers that illustrate various ways of thinking about each layer. Analysis of the data from the teaching experiments guided modification of the framework to also reflect the cognitive development of students. Throughout the teaching experiment, it was expected that students would assimilate Riemann integral ideas into their existing conceptual structure of approximations which, in turn, must accommodate some aspects of the Riemann sum structure where it differs from previous applications of approximation ideas.

Results and Discussion

As expected, students in both groups used the approximation language extensively and correctly while solving all three of the problems. Throughout the course, students were given several approximation tasks, and by this point in the course, the students seemed very comfortable with the concepts. Several examples of students' written work are shown in Table 2 below.

Table 2 – Students' Approximation Language

Jake:	The error is the difference between the unknown value and the approximation
Tim:	The error of the approximation is the amount that the approximation is away from the unknown's true value. Because we don't know the unknown's value we cannot determine the error numerically.
Ben:	The numerical bound on the error is the difference between two approximations where one number is an over estimate and one is an under estimate.
Holly:	I know the general area it [the unknown value] will be in because it's between the last two approximations as they have created my upper and lower bound on the error. I can make the bound on the error smaller by making more accurate approximations. For example the first two approximations are much farther apart than the 6th and 7th approximations, so the bound on the error is smaller.

What was most interesting is that students' struggles were concentrated in areas where the details of considering limits of Riemann sums departed from the limit structures students had engaged in while learning about limits of functions & sequences and the definition of the derivative. For example, the students had the most difficulty developing an initial approximation using the product structure in the Riemann sum $f(x_i)\Delta x$ and the appropriate summation. Table 3 contrasts the structures of the definitions of the derivative and the Riemann integral.

Table 3 - Contrasting structures in the definition of the derivative and Riemann integral

Derivative:	**Riemann Integral:** $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$
Differences and	Rate $f(x_i)$ and difference Δx
Quotient of and	Product of $f(x_i)$ and Δx
	Summation of several (n) products $f(x_i)\Delta x$ as we index through i
Limit as approaches 0 – involves making a single interval (from x to) smaller as we apply the limit	Limit as n approaches infinity – involves taking one interval and breaking it up into n subintervals, each of which gets smaller in width as we apply the limit.

While responding to the five approximation questions in a variety of modeling contexts, students struggled most with finding an initial approximation (question two). We hypothesized that this might be due to the structural differences between the definite integral and previous limit concepts. Such a difference is illustrated in Table 2 where we see that generation of a Riemann sum approximation requires different operations, different objects, and an additional layer than in the case of the derivative. More detailed data analysis around these portions of student activity revealed confusion about the meaning of the rate, its product with the difference Δx, and appropriate summation of multiple products on subintervals. For example, when working on an activity about the force exerted on a dam, the students decided to break the dam into small intervals and approximate the force on each interval, but had great difficulty determining the product that gives the appropriate force. Since force is pressure times area (when the pressure is constant on the entire area), the students decided to pretend that the pressure was constant on small intervals to obtain an approximation. However, instead of multiplying this pressure times the area of the small strip, the students multiplied the pressure on the small strip by the area of the entire dam, resulting in a gross overestimate.

Eventually, after overcoming the difficulties involved in the product, some students still had difficulty constructing the appropriate summation. Instead of including each subinterval once in the summation, these students added the top interval of the dam to the top two intervals of the dam, then added the top three intervals and so on. When comparing solution strategies with each other, one of the students explained his mistake, saying “your way’s gonna be more accurate than mine, cause mine is actually going to be extremely high because it takes this one, added to that one, added to that one. See how it gets doubly large, triply large, so ultimately, this is…too big.”

In contrast, after the students found the initial approximation, they had no difficulty finding both overestimates and underestimates, and all of the students understood that the “exact” answer was bounded by these estimates. They were easily able to explain how they knew if an approximation was an overestimate or an underestimate based on the elements of the particular problem (force, pressure, velocity, distance, etc.). One such example is shown in Table 4 below. Note that in this problem with the force of water on a dam, the pressure increases as the depth increases.

Table 4 – Students’ Explanation of Overestimates

Jake:	Yours is still an overestimate because you’re saying that—the, the—over the whole

Ben: I’m saying this pressure applies to the whole thing, but it’s really only applying to the very bottom of that line
Jake: Yeah
Tom: Right
Jake: to that exact spot right there [draws an imaginary horizontal line at the bottom of the strip]

The students also had little subsequent difficulty finding approximations accurate to within a predetermined accuracy, _, and determining conditions on *n* (equivalently) for specific or arbitrary degrees of accuracy. In the examples used in these activities, the underestimate on one interval was the same as the overestimate on the previous interval. Thus, the difference between the overestimate and the underestimate (the bound on the error) is the same as the difference between the overestimate on the last interval and the underestimate on the first interval. All of the students needed a little help from the instructor to make this connection, but were able to explain it in their own words later as well as use this fact to compute the number of intervals needed to obtain a desired accuracy.

The students were never able to determine the exact value of the definite integral in each of the context problems, although the students knew that such a value existed. In other words, the students’ understanding of the limit in the definite integral more of the process of refining approximations, and not so much in terms of limit as a single object. The students were quite successful viewing the limit as a process, finding approximations to any degree of accuracy. This could be due to the structure of the activities presented to the students. The purpose of the activities was to develop the structure of the Riemann sum, including the product, summation, and limit layers. The students were only asked to find approximations.

Despite the structural differences in the limit layer between the definition of the derivative and the Riemann integral, these results suggest that by the time students attended to the limit layer, they had already assimilated the structure of the Riemann integral into the previously established limit structure via approximations, while accommodating their approximation structure to include the differences between the definite integral and previously encountered mathematical topics. The students had the most difficulty developing an initial approximation. After they understood the various parts of the approximation, it was easy for them to refine their approximation to attend to the limit layer. In both the definite integral and the derivative, a refinement of the approximation is needed to make the approximation more accurate. In the derivative, we take one interval, but make the size of the interval smaller. In contrast, with the definite integral, we have multiple intervals and to make the approximation more accurate, we take more intervals, and the size of each interval becomes smaller.

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CONTEXT AND PRESERVICE TEACHERS' CONCEPTIONS OF PROOF

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In this study I examine preservice teachers' conceptions of mathematical proof in the context of evaluating student work. Although previous studies have examined teacher conceptions of proof, few have specifically focused on the effect the context has on teachers' conceptions. In this study the preservice teachers' conceptions of proof shifted when they moved from describing proof in the abstract to evaluating student work. The importance of establishing the validity of a claim and understanding new mathematics dropped when considering student work, while the finding of a solution took on greater importance. This indicates that pre- and in-service teachers' conceptions of proof in the context of teaching may warrant additional attention by researchers and teacher educators (1).

Reform documents (e.g. NCTM, 2000) highlight the importance of grade-level appropriate reasoning, justification, and proof at all grades, both because it is central to mathematics as a discipline and because it facilitates deeper student understanding of mathematical concepts. While existing research (Knuth, 2002) indicates that preservice teachers often possess naïve conceptions of mathematical proof, there has been little research on how teachers think mathematically about proof in the context of teaching. Additionally, there is a growing body of research emphasizing the value of teacher reflection on practice (Cochran-Smith & Lytle, 1999; Mewborn, 2000). This paper reports on the results of a lesson, ultimately envisioned as part of a larger teacher learning program, designed to provide preservice teachers with opportunities to reflect on mathematical content as well as the practice of teaching. Preservice teachers engaged in solving and providing justification for an open-ended mathematical task, examined hypothetical student work and justifications, and responded to reflection questions on the adequacy of the student work and the role of proof in K-12 mathematics. This paper addresses the following question: How were the preservice teachers' conceptions of mathematical proof connected to the context of evaluating student work?

A number of researchers have investigated both students' and pre- and in-service teachers conceptions of mathematical proof (Bell, 1976; van Dormolen, 1977; Fischbein, 1982; Harel & Sowder, 1998), as well as the role of proof in K-12 mathematics classes (Hanna, 1990). This existing research provides various frameworks and taxonomies for classifying individuals' conceptions of proof (for a particularly thorough example see Harel & Sowder, 1998). Although many of the authors explicitly acknowledge the possibility that individual conceptions of proof may vary with the particular task, there has been relatively little attention paid to the role the context plays in influencing individuals' conceptions of proof. Notable exceptions are Herbst and Brach's (2006) research in which they examine high school students' expectations for situations in which they would be asked to generate proof, and Hoyles' (1997) work which examines the connections between curriculum and student conceptions of proof. Although both of these studies point to the important role context plays in shaping individual conceptions of proof, neither explicitly focuses on the impact specific tasks have on an individual's conception of proof. Additionally, although existing research has focused on the individual's conception of proof there has been relatively little attention paid to how teachers think *mathematically* about proof in the context of teaching. I use the phrase "think mathematically about proof" to differentiate from studies (Knuth, 2002) in which teachers were asked about the role of proof in learning mathematics. Instead, my intent is to focus on how teachers' *mathematical* understanding of proof may be connected to the context of teaching.

Theoretical Framework

Harel and Sowder (1998) define "a person's proof scheme… [as] what constitutes ascertaining and persuading for that person" (p. 244), which is an intentionally actor-oriented approach to the definition of proof. In this paper I also consider proof from the actor's perspective; the aspects of proof analyzed in this paper emerged from the teachers' descriptions of proof, not from an a priori definition of proof. To focus on the teachers' conceptions of proof in the context of teaching, I employed a lesson designed to provide preservice teachers with an opportunity to engage in *mathematics immersion*. Mathematics immersion is a set of problems that are designed to 1) have multiple mathematical connections and solution paths; 2) promote the development of more formal reasoning; and 3) promote reflection on the nature of mathematics problem solving and its relevance to the mathematics of teaching. Thus, the goal of *mathematics immersion* is to provide pre- and in-service teachers with opportunities to authentically engage in and reflect on the mathematics of teaching over an extended period of time. Furthermore, mathematics immersion is intended to reflect Cochran-Smith and Lytle's (1999) notion of "knowledge-*of*-practice," within which "it is assumed that the knowledge teachers need to teach well is generated when teachers treat their own classrooms and schools as sites for intentional investigation" (p. 250). This was accomplished in this lesson by providing preservice teachers with an opportunity to engage in substantive mathematics prior to engaging in practices we hope they will employ as a teacher—namely, reflecting on students' thinking and learning on the same or related tasks. It is hoped that this structure will serve as a model for mathematics immersion in the future and that eventually this lesson will be added to in order to develop a mathematics immersion unit that supports teachers in engaging in and reflecting on the mathematics of teaching over time.

As this study only reports on one lesson and not an extended experience with math immersion it has two important limitations. First, this lesson did not explicitly attempt to connect preservice teachers' reflection to larger social issues, which is a strong theme in the knowledge-*of*-practice. Second, while this lesson did attempt to position preservice teachers as responsible for developing appropriate conceptions of mathematical proof for themselves and their students, eventually teachers would need to explore the relationship between their understanding of proof and the mathematical community's accepted standards.

Methods

Thirty-seven education students at a large Midwestern university participated in this study; henceforth the education students will be referred to as teachers. The participants were enrolled in either an elementary, middle, or secondary mathematics methods course. This lesson was conducted over a period of one or two class periods of the methods courses. The teachers completed some work as homework and during class time they worked in small groups and participated in whole-class discussions. Written work from all participating teachers was collected.

The lesson consisted of four activities. First, teachers were asked to complete an open-ended mathematical task: determining the number of blocks required to build the n^{th} stage of a figure that grew quadratically (see figure 1). Second, teachers were asked to reflect on the definition of proof and its role in K-12 mathematics. Third, teachers examined hypothetical middle school student work on the mathematical task they had completed and were asked to discuss and respond to questions about the quality of the students' work and how they would further support the students' learning. The student work was broken into three parts and the teachers responded to questions after each of these parts. Table 1 provides a brief description of the three pieces of student work. Fourth, teachers reflected on whether the students' justifications were adequate proofs and again reflected on the role of proof in K-12 mathematics.

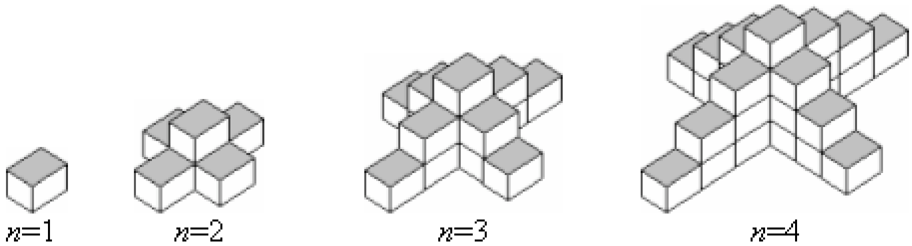

Figure 1. The Mathematical Task

I analyzed the written data from the second activity using an iterative coding process to determine how teachers conceptualized adequate mathematical proof *in the abstract*—that is without the context of student work. I then used these codes, and added to them as necessary, when coding teachers' comments on the adequacy of students' mathematical work during the third and fourth activities. Finally, I made comparisons between the codes teachers used in describing mathematical proof in the abstract and in the context of student work.

Student Work	**Description**
Group A Part I (AI)	Group A used a strategy of removing the "wings" of the structure from the center column to form two rectangles of dimensions n by $n-1$. Although they recognized the width of their rectangles as n in this first part, they did not identify the height as $n-1$ and thus arrive at a different rule for each stage, which they recognize as problematic.
Group A Part II (AII)	The teacher in the scenario asks the students to explain *why* their rectangles are as tall and as wide as they are. After some additional work the students recognize that the height of the rectangles is $n-1$, develop a correct equation, and justify this by saying the wing's height and width are always one less than the height of the center column and when you combine two wings you extend the width by one block.
Group B	Group B creates a table of values for the number of blocks in the n^{th} stage. First they attempt to find a pattern by adding something to n and when this fails they attempt to find a pattern by multiplying n by something. They realize they are multiplying n by consecutive odd numbers and use this to develop a correct equation, which they justify by saying it worked for the ones they tried and it's a pattern.

Table 1. The Student Work

Results

Written data were not collected for all participants on each of the four activities in the lesson as some teachers were absent for part of the lesson or did not hand in written work for each activity. Therefore, unless otherwise stated, results reported as percentages below are out of the number of teachers for which data were collected on the corresponding activity. Recall that much of the data below contrasts how preservice teachers described proof *in the abstract*—that is during the second activity in the lesson prior to looking at student work—with how they described proof while looking at student work, or *in context.*

Adequacy of Student Proofs

The preservice teachers made distinctions between students' attempts at proof. Table 2 shows the number of teachers who indicated that the student work qualified as an adequate proof broken down by the three different groups of student work the teachers were given. The preservice teachers in the elementary methods course did not get to student group B's work.

	Student Work		
	AI (n=32)	AII (n=32)	B (n=20)
Teachers who described student work as adequate proof	0 (0%)	14 (44%)	3 (15%)

Table 2. Adequacy of Students' Proofs

Comparing the Strategies of Student Groups A and B

During the final activity, the preservice teachers were asked to consider the advantages and disadvantages of focusing on the figures, as student group A did, versus focusing on number patterns, as student group B did. The preservice teachers in the elementary methods course did not complete this activity. As table 3 shows, a majority of the teachers did not indicate that either of the strategies was preferable and only a small number of teachers mentioned proof when evaluating the strengths and weaknesses of the two strategies.

n=20	A preferable	B preferable	Neither stronger	Mentioned proof
Number of teachers	3 (15%)	0 (0%)	15 (75%)	2 (10%)

Table 3. Comparing Student Groups A and B

Change from Proof in the Abstract to Proof in Context

Table 4 provides a brief description of the codes reported in this paper that were used when coding teachers' descriptions of adequate proof. Table 5 shows the percentage of teachers whose written work received these codes in the abstract and then in the context of student work. The codes are in descending order of frequency in the abstract.

Code	**Description**
Universal	A proof must *apply* "universally" or to all cases, often within some limiting parameters.
Explain Why	A proof must explain why a solution or method works or is correct.
Establish Validity	A proof must demonstrate the truth or correctness of a statement without any explicit mention that the proof must

	explain *why* the statement is true or correct.
General Argument	A proof must employ an argument that is general in nature; in other words the argument must apply to all cases.
Develop Understanding	Proofs help students understand mathematics.
Is an Equation	A proof is, or must take the form of, an equation or formula.
Answer the Question	A proof must answer the question asked.

Table 4. Descriptions of Codes for Aspects of Mathematical Proof

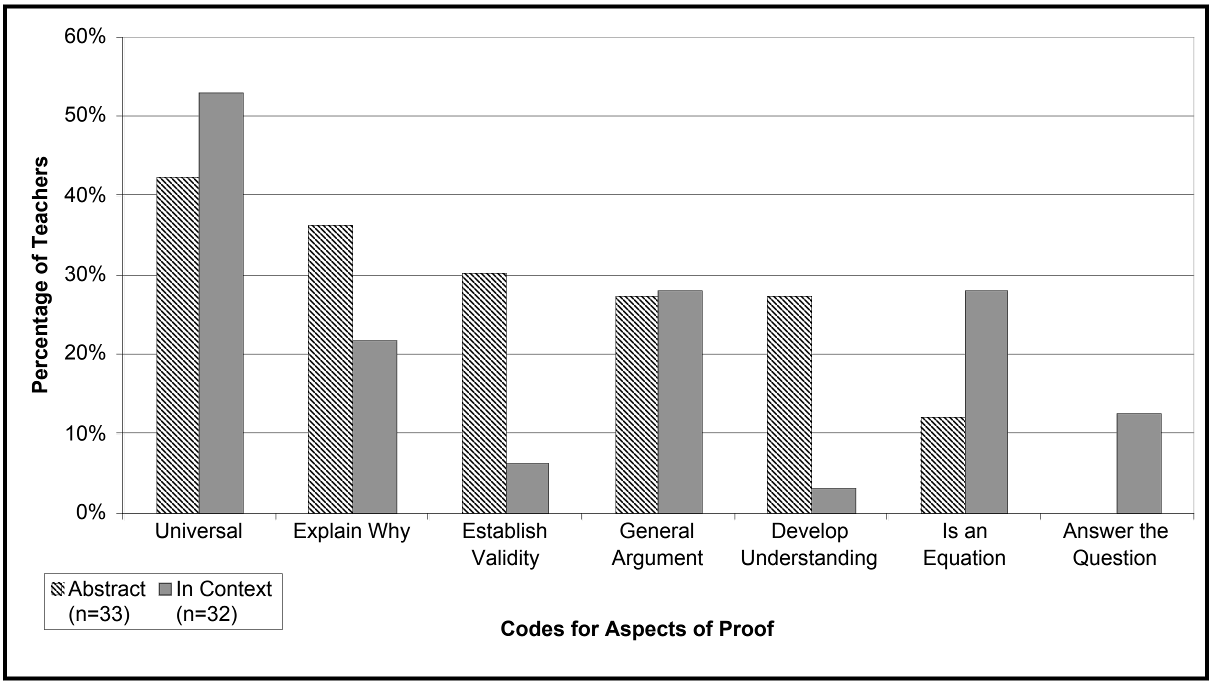

Table 5. Descriptions of Proof as a Percentage of Teachers

Discussion

Establishing Validity and Explaining Why

Establishing the mathematical validity of a statement is an essential aspect of proof (Bell, 1976). However, as Hanna (1990) has pointed out, proofs that explain (why) as opposed to simply prove may be more effective pedagogically. Therefore, it is encouraging to see that both in the abstract and in the context of student work the preservice teachers were concerned with explanations of why with greater frequency than simply establishing validity. While both aspects of proof were mentioned by fewer teachers in the context of student work, establishing validity without an explanation for why was almost nonexistent in the context of student work. Additionally, out of the total number of teachers who mentioned establishing validity (either with or without an explanation of why), the proportion that required an explanation of why rose from 55% in the abstract to 78% when evaluating student work. One reason for this preference for an explanation of why is that the teachers may have viewed the student work as incomplete, but not incorrect, as there were no overt mathematical errors in the student work provided, therefore reducing concern about the validity of the students' work. Another reason for this result may be that the teachers needed to understand the reasoning behind the students' explanations in order to make sense of them.

Developing Understanding

Although mentioned by a relatively low percentage of the preservice teachers, it is encouraging that in the abstract the teachers considered proof to be a tool for developing students' understanding of mathematics, especially in light of Knuth's (2002) work indicating teachers' inattention to this aspect of proof. However, it is troubling that this code all but disappeared in the context of student work. The dramatic drop in this code may indicate that while the preservice teachers consider proof to be an important vehicle for developing understanding, they do not consider it essential to the adequacy of a proof. To the extent that the teachers do not consider understanding essential to a proof's adequacy, they would be in agreement with mathematicians in general; however, it may then be important to provide the teachers with a more explicit focus on the adequacy of mathematical proof *with respect to* students' learning of mathematics so they can continue to focus on understanding in the context of student work.

A Focus on Finding the Answer

The presence of the *universal* and the *general argument* codes both in the abstract and in the context of student work is promising in that they indicate a preference for generality in the student work as opposed to an attention to individual stages. However, from a mathematical perspective, a claim that applies universally but does not employ a general argument to back it up is a conjecture, not a proof. Therefore, it is concerning that in both the abstract and in the context of student work universality received greater attention than the generality of the argument. Moreover, in the context of this lesson, a universal claim is an explicit rule or formula for finding the number of blocks in the n^{th} stage—in short it is the solution to the problem—thus introducing the possibility that the prominence of this code, especially in the context of student work, may in fact indicate a focus on solving the problem. This is supported by the increase in and appearance of the codes *is an equation* and *answer the question* in the context of student work, as in the context of this lesson these codes may be largely about finding the solution to the problem.

The focus on finding the answer may indicate that the preservice teachers see the arrival at the solution as separate from the process of proving. This is further emphasized by juxtaposing the distinctions the teachers made between adequate proofs (table 2) with the comparisons the teachers made between the strategies used by groups A and B (table 3). As seen in table 2, the teachers made strong distinctions between adequate proofs. A much higher percentage of teachers considered student work AII (44%) to be an adequate proof than those who considered student work B to be an adequate proof (15%). However, table 3 indicates that when asked about the advantages and disadvantages of the strategies used by student groups A and B, a large majority of teachers did not privilege either strategy and only a small number of the teachers commented on proof in making this determination.

Conclusion

The results of this study indicate that the preservice teachers' conceptions of proof differed in a variety of ways when they switched from discussing proof in the abstract to examining proof in the context of evaluating student work. While we may be inclined to write these changes off as the teachers not practicing what they preach, I believe that it speaks to deeper theoretical and practical concerns. As discussed earlier, there is a large body of research on individuals' conceptions of proof; however, there has been relatively little attention paid to the way these conceptions may depend on the particular context in which proof is being utilized. The results in this study indicate that this is an area worthy of further investigation as teachers' conceptions of proof *in the context of teaching* may be, and perhaps should be, different from the way they engage proof in other settings. Moreover, mathematics immersion may provide a suitable space within which to investigate this issue.

The results discussed above indicate that while there was some emphasis on proofs that explain why, there was a disconnect between proof and understanding in the context of student work. Furthermore, there was a general shift towards a focus on finding the answer when teachers engaged in evaluating student work. The reason for this shift in teachers' conceptions of proof in the context of a teaching activity may be due to a lack of familiarity with rich examples of how these aspects of proof, and proof itself, can be intimately related to problem solving in the K-12 classroom. This indicates that teachers may require further experiences with mathematics immersion, which provides specific, contextualized examples of students coming to solve and understand mathematical problems through the process of proof. This is particularly important as the NCTM argues that "reasoning and proof cannot simply be taught in a single unit on logic, for example, or by 'doing proofs' in geometry… Reasoning and proof should be a consistent part of students' mathematical experience" (2000, p. 56). If this is to occur teachers will require familiarity with how proof looks in a variety of problems throughout the K-12 curriculum.

Endnote

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EXAMINING FUNCTION UNDERSTANDING FROM CONTEXTS USING CONCEPT MAPS

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This paper describes analysis of function understanding using investigator constructed concept maps. The investigator conducted a design experiment to deliver problems that allowed high school algebra students to expand their understanding of function. The function problems were drawn from motion, cryptology, and sound contexts and increased in difficulty throughout the study. Pre- and post-interview data from the students is used to create concept maps for analysis. The maps help to show trends in learning such as concept acquisition and misconception reduction.

Understanding the concept of function can help students to see connections among a variety of mathematical concepts and serve as an underlying foundation for further study. This can result in an understanding that is an "interconnected web of mathematics" (Hiebert & Carpenter, 1992). For this reason, it is especially important to study how students view and think about the concept of function, and how these interrelated concepts can merge.

However, it is extremely difficult to examine students' conception of function, because of its abstract nature. In addition, there are precious few measures of function understanding, partly because of the diverse nature of the topic. For example, one can test for recall of the definition, but that does little to reveal how a student sees the consistency of function families, or if they believe that a mapping with a finite domain can even be a functional relationship.

This paper reports on a research methodology that not only captures a student's understanding of function at specific points in time, but also helps illuminate the change in functional understanding over an instructional treatment. I constructed concept maps that I used to analyze differences in an individual student's understanding of function, in an attempt to measure change in student thinking about the concept of function from the beginning to the end of an instructional treatment focused on the use of function in motion, cryptology, and sound contexts. I will briefly explain how the concept maps were created, and how these concept maps were used to analyze student understanding of the concept of function.

Theoretical Framework

The theoretical framework for the paper draws from both design experiment methodology, and concept map analysis. In turn, I will address how these areas inform my work.

Design Experiment

Because I was interested in capturing the complexity of learning about the function concept, I used a design experiment methodology for the duration of the treatment and based instruction on articulated design principles (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). These design principles included three aspects: emphasizing the parameters of

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function families, employing computer technology for each problem, and situating all problems given in realistic contexts.

I chose to focus on the parameters of function, for example "a" and "b" of the linear function $f(x) = ax + b$ to promote the more desirable object viewpoint of function (Sfard, 1991). I hypothesized that if students were able to understand these parameters and how they affected the function, they may be able to have a richer understanding of not only these function families, but also how these parameters can affect more abstract functions. I wanted to use computer technology to support students' problem solving within the problems because of the immediate feedback that the technology can afford the students. They were able to use results given to them by their trials to refine their answers and converge on the correct solution. In addition, the careful use of computer technology in instruction has been shown to improve students' notion of function understanding (O'Callaghan, 1998). Lastly, I placed all problems within one of three contexts: motion, cryptology, and sound. These contexts were chosen because they supported a progression of function family complexity. That is, functions of increasing complexity could be chosen for problems as the students worked through the three contexts.

Concept Maps

To depict students' understanding before and after the study, I turned to the idea of creating concept maps based on the students' responses during the pre- and post-interviews. I am using analysis of concept maps in a unique way. Most of the literature related to concept maps has analyzed the maps that students have drawn (cf. Williams, 1998). I am instead using a concept map as an organizer of experimental data for analysis. I am hoping to create a two-dimensional visual representation of a student's concept image of function (Vinner & Dreyfus, 1989). The process and results of this method will be shared in the rest of this paper.

Methods

The interviews discussed in this report are part of a larger study in which students studied the concept of function in three different contexts. Six Algebra II high-school students were given problems on a weekly basis after school. The students worked in pairs on the problems, and were able to interact with the computer and with each other to solve them. These problems allowed students to explore the parameters of the functions used to describe the contexts. The functions within the problems increased in complexity, as did the problem scenarios. Problem choices were guided by a focus on function parameters within the function families with progressive increases in difficulty from simple constant slope and y-intercept in a linear function $f(x) = ax + b$ to a parameter that was a function in itself ("a" in $f(x)$ could be another function, $g(u)$).

For analysis, pre- and post-interview data was entered into a database for each student. Then, a concept map was drawn for items that revealed how students thought about function (Figures 1 and 2). The concept maps in Figures 1 and 2 were created from responses of the following questions: "What is a function?," "What is an example of a function?," "Is a function the same as its graph?," "Can a phenomenon have more than one function to represent it?," and others. Besides this more abstract questioning, the interviews also included problems where the student had to identify if a given graphical representation was a function, as well as problems that required a numerical answer.

I created a “skeleton” concept map that was used for all students and then I expanded upon it to capture each individual’s function understanding. This created a visual representation of the student’s conception of function, with a consistent organization from student to student allowing for pre/post comparison as well as student/student comparison. This “skeleton” consists of clusters based on three items: the definitional aspects of function (definition), the application of function to contextual problems and the worth of the concept in daily living (application), and aspects related to the graphical representation of function (graphical representation). These base items (definition, application, and graphical representation) come from two sources. First, the majority of my own assessment questions were focused on these three aspects of function understanding. Second, the answers that I received in both interviews from all students showed that these three elements were the primary aspects of function that were most salient to the students while working with functions. For example, although other representations were asked about and mentioned by the students during these interviews, the graphical representation elicited the most productive responses from the students. From this cursory analysis, the triad of these items was deemed most important in the students’ conception of function, and therefore was used as a basis for analysis.

The result of this analysis allowed me to create a concept map of understanding that has two noteworthy consequences. First, comparison between and among students is more easily done because of the consistency of the “skeleton” or base map. It is much easier to determine who communicated more unique ideas about a particular aspect of function, such as definitional elements of function. Second, because each secondary idea connects to one of the three core thematic items, the links do not need identifying connecting terms such as “is an example of” or “is defined by”. The need to put these linking thoughts is unnecessary simply by a more explicit concept map structure and design.

I expanded the skeleton map by adding extension ideas where appropriate. These expansions were all done using the student’s responses from certain questions asked during the pre-interview and post-interview. I restricted these extension ideas to only the questions that I asked in both the pre- and post-interviews, even though I asked more questions during the post-interview. In addition, I only included those questions that were asked of every student. These student ideas are coded on each concept map by shape. Items in cloud shapes are those ideas that were communicated to me in the pre-interview only, while those items in squares were communicated during the post-interview only. Items in circles were those ideas that were more persistent. The student mentioned these items in both pre- and post-interviews.

Once I had created the function understanding concept maps for every student, I was able to compare the students’ maps to determine who changed the most during the course of the treatment. I was able to enumerate the change based on the concept maps I created from their responses. I was most interested in change in the definition and graphical representation areas of their concept maps, so I decided to create three ratios: change over all three core ideas, change in definition, and change in graphical representation (see Table 1). I chose to omit a change in application ratio because I was more interested in students that had changed their theoretical conception of function. These ratios were calculated by counting the ideas that changed from pre- to post-interview, and dividing by the total items within that category. For example, student #4 had the highest overall change at 92%, which was the number of total cloud and square shapes in their concept map divided by the total number of extension shapes. So, 92% of the student's map extensions are different from the pre- to the post-interview.

Results and Discussion

From the results shown in Table 1, students 3 and 4 show the most definitional change according to my measure. Analysis of their concept maps in more detail can help to determine the cause of their greater differences.

Measures of Change	Student #					
	1	2	3	4	5	6
Ratio All	0.8	0.4	0.8	**0.92**	0.72	0.61
Ratio Definition	0.2	0.2	**0.33**	0.3	0.18	0.08
Ratio Graphical	0.4	0	0.27	**0.46**	0.36	0.31

Table 1. Ratios of concept map changes

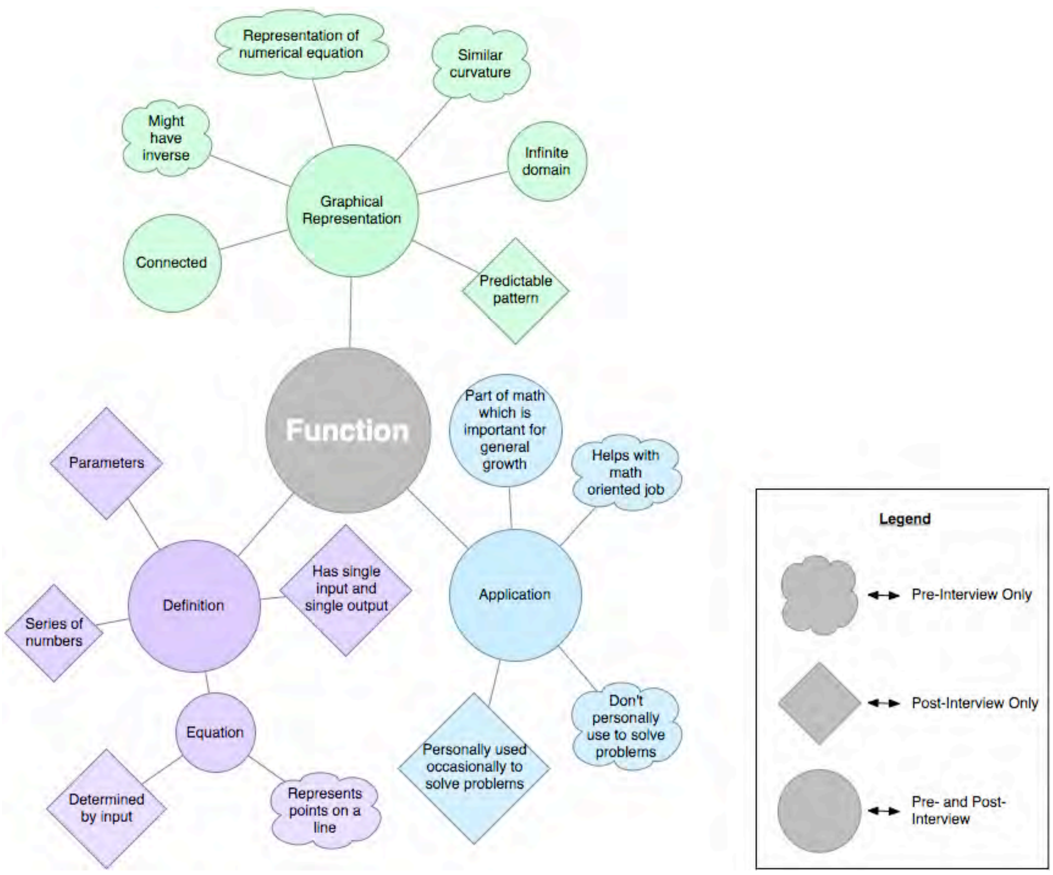

Figure 1. Function concept map of student #3

Student #3 Concept Map Discussion

A number of important conclusions can be gleaned from analysis of these concept maps. There is of course the welcome change in perception of how function is apparent in their daily lives. Student #3 in the pre-interview claimed that he did not use functions to solve problems (see Figure 1). However, in the post-interview this student not only claimed that functions are used occasionally, but that they are probably used, maybe without overtly knowing about it. The nice aspect of this change is that the student used the contexts of the study to explain where functions might be used. This type of application change was consistent among all six students, establishing that the contexts were meaningful for the students.

There are many definition and graphical representation ideas to analyze by looking at the concept map for student #3. First, looking at the definition strand in the lower left corner, the student thought that function was an equation in both interviews. However, the "expansion" or elaboration of that equality was different. In the pre-interview, the student said that the equation represents points on a curve. "I guess it's an equation that represents a line, or an amount of points, a certain amount of points on a certain line."Later, the student said, "It's an equation, but it doesn't have one set answer, it's determined by its input to get a single output."Although the student is still relying on function equaling an equation, the response shows that the student is thinking of the process view of function – or at least starting to. This is supported by the fact that the student thinks that function's equation can be controlled by parameters, thereby giving different answers as the output.

Ideas that were dropped in the post-interview include parts of the student's graphical representation section. Both the fact that a function's graph could have an inverse and the fact that the graph was a representation of a numerical equation was left out of the post-interview. It is true that a graph can help recognize whether a function can have an inverse, but the student did not elaborate this fact. This "inverse" utterance does not appear in the post-interview and could suggest that it was not necessarily a persistent concept for the student. The idea that functions have "similar curvature" shows the student might have dropped a reliance on prototypical functions for identification.

The three items that remained constant in the map are very important to note. As previously mentioned, the student thought that function is an equation, although this was elaborated on in different ways in the two interviews. The other two constants are the misconceptions that the graph of a function is connected, and that function has an infinite domain. Both of these items are understandable, given the content in both standard curricula and the study. However, this might be a limiting factor in both broadening and completing the student's conception of function.

Student #4 Concept Map Discussion

The second map shows a different kind of change over the course of the study (see Figure 2). Although this is one of the two students with the most "change" as defined above, the student did not add many ideas during the semester, but rather dropped a number of items that may be considered either misconceptions or overgeneralizations. Four ideas under the definition category were abandoned. For example, the student dropped the "not random" idea, as well as the idea that functions had specific structures. Also at the end is the idea that there is a steady pattern to a function's graphical representation. This might have resulted from a lack of explicit verbal evidence, as the student still had reservations when presented with graphs of functions with little to no pattern. Further inquiry is needed to confirm that indeed the student has a deeper understanding of these ideas.

For student #4, there was a great moment of documented cognitive dissonance that happened in the post-interview that helped inform the content of the student's concept map of understanding. The student was answering the question of whether or not the graph shown (a scatterplot with three points) was a function. Initially, the student said that it was not a function, because each input was not represented; only three total points were shown on the graph. However, the student then thought about one of the graphs of the encoding function in the cryptography unit, which had 26 discrete points. Here is a transcript of what ensued:

> Student: Because it doesn't have multiple inputs. It can only have one of these three. Wait, but then those codes wouldn't be a function. So then, maybe it is.
> Interviewer: What are you thinking about when you say the codes wouldn't be a function?
> Student: Because you could only put for the inputs 1 through 26.
> Interviewer: Okay, right, and that [contextual relation] was a function.
> Student: Yeah. So, that [interview question] might be.

Through reflection on the activities and problems the student worked with, the student's conception of function was expanded and generalized. The scatterplot shown in the post-interview reminded the student of the encoding functions used in the cryptology lessons. The student is questioning the domain space in a function, and is beginning to realize that discrete domains are acceptable. On the concept map, this is shown as the "might not span x-axis" square.

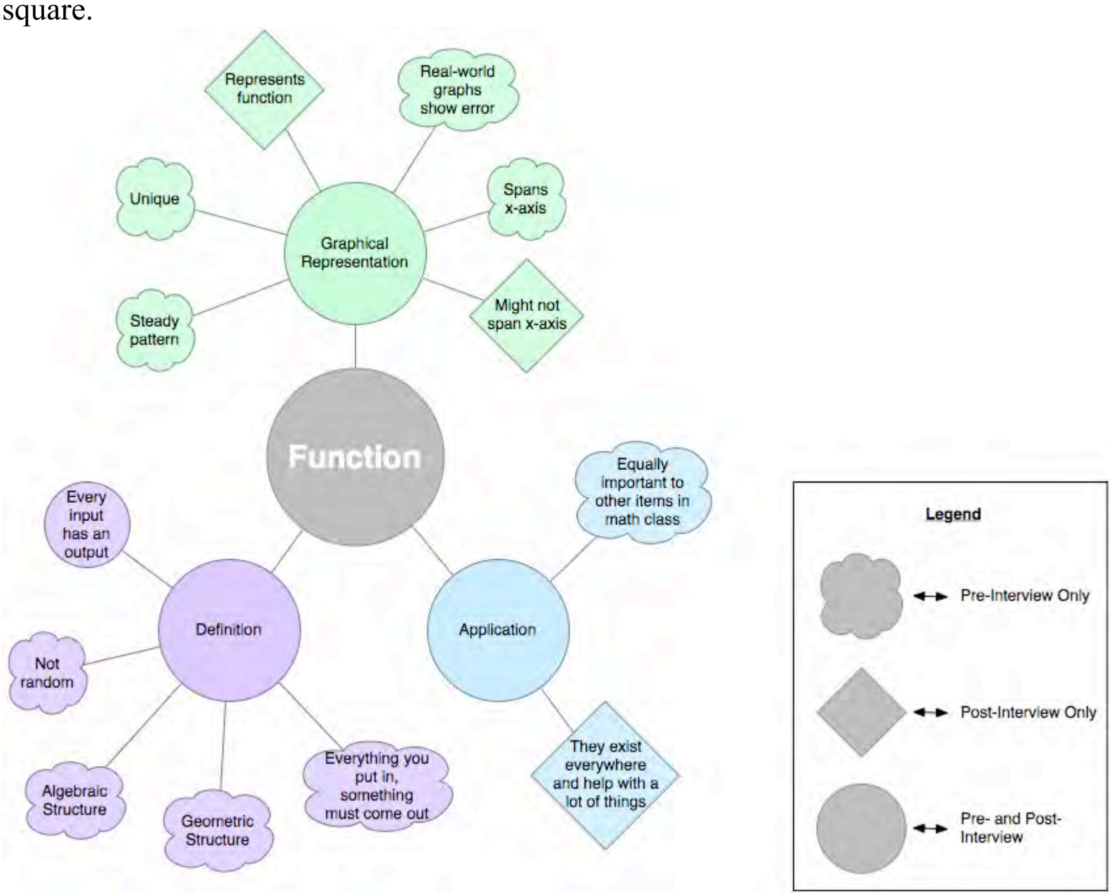

Figure 2. Function concept map of student #4

Conclusion

The concept maps have proven useful for analysis of students' thinking about functional concepts for several reasons. The first is that the non-linear description of their thinking can be displayed graphically. Rather than having a single listing of student understanding, each displayed concept can be linked to multiple items. This can show multiple connections more easily. Concept maps from multiple students can also be compared easily to see trends in understanding. These trends can be shown in students' prior knowledge, gained knowledge, and consistency of knowledge. That is, the concept maps can show not only what the students understood before the treatment, but also if there were any trends among the students after the treatment and if this was consistent among the students in the study. The more consistent that a particular item was among the participants, the more evidence one would have to claim that the treatment contributed toward the development of that knowledge. There are limitations for their use, but they show promise as a research tool.

I will not claim that these maps accurately reflect all of students' ideas about the concept, and I also do not claim that they are accurate isomorphic images of a student's conception of function. (This representational space would not even begin to accurately reflect or represent the complexity of a person's conception.) However, it is helpful for analysis of understanding changes over time. For example, this became evident in looking at the transcribed data from student #4. This student's thinking about a function's domain is in flux, as evidenced by his uncertainty of his answer in the above exchange. The map allows a way to identify this change and record it for further investigation with other data sources.

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VISUALIZATION AND ABSTRACTION: GEOMETRIC REPRESENTATION OF FUNCTIONS OF TWO VARIABLES

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This study uses APOS theory to investigate how students build their understanding of the geometric representation of functions of two variables. A genetic decomposition is proposed. The results of a group of semi-structured student interviews are discussed and the analysis of the interviews leads to a revised genetic decomposition.

Introduction and Research Questions

The notion of a multivariable function is of fundamental importance in advanced mathematics and its applications. Most of the problems in applied mathematics involve this type of functions. Many curricula include a mandatory course on their study, but little is known about students' ideas and difficulties about them. While there are many studies that deal with the general idea of a function, it is necessary to focus on the particularities of multivariable functions to study how students build their understanding about them.

Although there are many published articles that focus on the implementation of new classroom material for teaching multivariable calculus, we could not find research probing student understanding of this notion. This lack of research findings in turn limits our understanding of how students learn the main ideas of the multivariable calculus. This study is part of a project intended to study students understanding of multivariate functions. We focus here on the following research questions: Can students represent and recognize two variable functions in a three dimensional space? Can they visualize surfaces, intersections of surfaces with fundamental planes and projections in a two dimensional plane? Can students relate different representations of these functions?

Theoretical Framework

It is well known that graphs, diagrams, pictures and geometrical shapes or models are a tool for visualization of the abstract concepts in mathematics. By means of these students can build relations between the physical world and the abstract concepts (Several references appear at Hitt, 2002 and Presmeg 2006). The term visualization is used in different meaning between mathematics educators. In this paper the definition used is the one given by Zazkis, Dubinsky and Dautermann (1996), that is, as an act in which an individual establishes a strong connection between an internal construct and something to which access is gained through the senses. For them such a connection can be made in two directions. An act of visualization may consist of any mental construction of objects or processes that an individual associates with objects or events perceived by an external source, or it may consist of the construction, on some external medium such as paper, of objects or events. Consequently, visualization involves a translation from external perceptions or constructions to mental constructions.

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This definition of visualization is used in this report together with APOS theory (Asiala, et al.,1996) to develop a preliminary genetic decomposition to analyze students' ideas and representations of objects and functions in a three dimensional space.

A preliminary genetic decomposition for functions of two variables was developed by the researchers and was reviewed by two independent multivariate calculus teachers. This is a conjecture of how students develop their knowledge of two-variable functions, and their geometric representation:

The objects or schemas that we consider a student may have developed previously to be able to understand the concept of a two variable function are:

- Actions related to the notion of three dimensional space,
- A schema of Cartesian plane which includes the concept of point as an object and representations of relations and other subsets of the plane as processes.
- A schema for real numbers which includes the concept of number as an object, and arithmetic and algebraic transformations as processes.
- The object set, and basic set notation
- A schema for functions of one real variable ( ) which includes the concept of function as process, operations with functions, and coordination of the analytic and geometric representation of functions.

The Cartesian plane, real numbers, and the intuitive notion of space schemas must be coordinated in order to construct the Cartesian space of dimension three, R^3, through the action of assigning a real number to a point in R^2, and the actions of representing the resulting object both as a 3-tuple and as a point in space.

These actions are interiorized into a process that considers all the possible 3-tuples and their representation in space, to construct a process of what can be considered as a three dimensional space R^3.

The space R^3 in turn is coordinated with the schemas for functions of one variable and sets through the process of considering sets of points in R^3 as graphs of functions of one variable and their representations as curves, and also considering other planes or regions in space to construct a schema of subsets of R^3. This schema includes: fundamental planes (meaning vertical and horizontal planes of the form: , , and where *c* is a constant), other subsets, relations and possible domains, and possible sections and contours.

The coordination of the schema of subsets of R^3 with the schema for functions of one variable through the actions of taking a point in the *xy* plane and assigning a number to it can be generalized to consider all the possible assignations in order to construct the notion of a function of two variables as a process. (The emphasis is on the actions of evaluation and finding domain and range of a given function using different representations: table, formula, graphical and verbal). This process is encapsulated so that it is possible that is identified with .

The notion of function of two variables as a process is coordinated to the schema for subsets of R^3 through the action of representing the different points of the function in R^3 to construct graphs of functions of two variables; this action of evaluating the function over a grid on the domain is interiorized so that students are able to work with the process of drawing contours and transversal sections.

Method

An instrument was designed to conduct semi-structured interviews with students and test their understanding of the different components of the genetic decomposition. An initial group of three students was chosen to participate in the first stage of the study. The students were chosen from a group of undergraduate students at a private university who had taken the equivalent of an introductory multivariable calculus course the previous semester. The instructor of the mathematics course they were currently taking chose what he judged to be a good, an average, and a weak student to be interviewed. The interviews lasted for 45-60 minutes, were audio-recorded, and all the students' work on paper was kept as part of the data. The results obtained were independently analyzed by two researchers, and conclusions were negotiated between them. On the basis of the results obtained from these interviews, the researchers concluded that the information on students' constructions about functions of two variables was not enough to fully compare it to the constructions predicted by the genetic decomposition. The instrument was then revised, and complemented with new questions. This new instrument was used in semi-structured interviews 45-60 minutes long with another group of seven students; one of them had also been interviewed with the previous instrument. These students were also selected by the teacher from the same course as the first three, and again chosen so that they represented good, average, and weak students. The interviews were also audio- taped and students' work was kept to be analyzed independently by two researchers, and again, results were then negotiated between them. The purpose of these interviews was to study students' constructions about functions of two variables in analytical and geometrical contexts. In this study we only report results related to the questions of the interview that concern the graphic representation of two variable functions.

Results

Students showed different levels in the construction of the schema for functions of two variables. These different levels are related as well with the development of their visualization capability. We show here some examples of students who have constructed different schemas for this concept as shown by their responses to the interview and the characterization made by the researches of the coordinations predicted by the genetic decomposition that they seem to have constructed.

Some students showed a lack of coordination between their intuitive knowledge of three-dimensional space and the mathematical schemas they use in a multivariable setting. They seem to work at an action level, memorizing some facts and trying to use them to solve the problems, but they don't show evidence of having interiorized these mathematical objects or having constructed relations between them and their intuitive knowledge of space. Rodrigo is such a student. For example, in one of the questions that asked: If you start at point

and move freely in the directions east-west (defined in the problem to be the *x* direction) and up-down (the *z* direction), give an equation that describes the set of all points that may be reached. Rodrigo takes a lot of time thinking and responds:

R-- I'm thinking that we are fixed at y do we? It does have point 3, say here, and I have to describe an equation so that I may move freely on x and on z, so that if we are in three dimensions, it would take … here … we form a plane or not, there it is … negative …ok

I- ok, so you wrote y = x + z + 3

R- 3, what happens is that, according to me, if I write $y = x + z + 3$, then it works here that, that it is x at 3 …

I- let's see, if I give you this green model for a plane, how would you represent the plane you are talking about on the 3D Kit? [The 3D Kit is a physical model of three dimensional space that allows the representation of points, lines, planes, vectors and some surfaces.]

R- what do you mean by how would I represent it?

I- let's see, the set, how will it look? Will it look like a plane? Like a surface?

R- according to me, it looks like a plane… ah ok … let's say that y is fixed at 3, according to me, but now…

Rodrigo wasn't able to model the plane $y = -2$ by correctly positioning a flat rectangular piece of plastic in the 3D Kit. He knew the equation of a plane at an action level, but had not coordinate this action to the geometric representation of the plane. When working with other questions in the interview that involved equations of the form $x = c$, $y = c$, and $z = c$, he recognized them as equations for planes in a 3D space. However he seemed to have memorized this fact, and did not show an understanding of it; he repeated several times things like … *in z, there is a plane …since it is* …, but when asked to describe such situations he was confused.

The analysis of the data shows that the construction of some of the coordinations or interiorizations predicted by the genetic decomposition was not achieved by many of the students. Maria is an example of this group of students. Maria was able to respond quickly and without hesitation to the questions related to the coordination of Cartesian plane, real numbers, and the intuitive notion of space schemas thus demonstrating she had constructed a schema for R^3. She also demonstrated to be able to visualize fundamental planes and to relate them with their analytical representation.

However, she showed difficulties in those questions that asked about the interpretation of the intersection of a given surface in R^3 and a fundamental plane showing that she was not able to visualize the intersection to be able to draw it on a two-dimensional plane. When asked to draw in a two dimensional plane the set of points that satisfy $z = 1$ in the given surface, she started by drawing the graph of plane :

M- in a two dimensional plane? z = 1 … so in z it has a restriction but in, I don't know, the thing is … , well if it were in three … may I draw it first in three?

I- if you want

M- so in x y, … so in x and y it could be any point, isn' it? … as long as it is restricted to … what I have is a plane, would it?

In another problem, she was asked to draw and again, she recognized $y = 3$ as a plane, but she struggled with the intersection curve:

M- we have that with y = 3 … is like, like a cup, and y = 3 … here is y = 3, then it is all 3 [tuples] of x 3 z …

I- how would that look?

M- … something like this, that is the … that is, y can only take 3 as a value?

I- yes

M- only?

I- yes

M- then it would be … a paraboloid … over this, over 3 and then here they are 3 [She draws a solid disc inside the plane $y = 3$.]

I- What is the relationship between this set and the cup you mentioned earlier?

M- well, it is like the tip, no? that is the … the … but no because … yes because x can be positive or negative, but in … that is, it can have any x or any y, any x but in y it can only be 3, like a piece … a slice of the piece

I- here you drew a circle, does it include the points inside the circle?

M- … well in x it has .. it has no point because there is no such … y and z, well neither.

Her inconclusive answer to the last question together with the solid disk she drew confirms that she does not have a clear idea of transversal sections. This same difficulty reappeared in other questions of the interview.

María also showed a lack of coordination of different representations. This lack of coordination can be related with difficulties in visualizing the result of the actions or processes applied on curves that are described by their analytic representation. When Maria was asked to draw or describe the set of points that results when the points on R^2 that satisfy the equation are assigned a height of 3 she responded

M- it is a circle … of radius 1

I- very well

M- then, draw in R^3 *or represent on the Kit the set of points that result if the points in* R^2 *that satisfy the equation are assigned a height z = 3, ah then … we'd be left with … something like this … in the interior* [She first drew the cup-like surface and then quickly fixed her drawing to have the cylinder.]

I- then, that is like a surface?

M- yes, this is a surface because we are in R^3

So her drawing showed that she is able to lift the circle to a height of 3 but she included all the points traced by the movement thus ending up with a cylinder.

These results show that this student has a schema of an intuitive notion of three dimensional space. She can imagine it and imagine points, planes and surfaces in it. She has coordinated schemas of Cartesian plane, real numbers, and the intuitive notion of space to construct the Cartesian space of dimension three (R^3). She has also coordinated her notion of Cartesian space of dimension three with the schemas for certain specific sets, fundamental planes, so she is able to relate these planes with their different representations, geometric, verbal or analytic. However she showed difficulties in the coordination of the schema for fundamental planes and other subsets of R^3, particularly, surfaces and contours. These difficulties were further related to her difficulties to understand the meaning of a two variable function and the possibility to draw graphs for these functions.

There were two students who responded most of the questions related to the graphical representation of two variable functions correctly, but they too showed some difficulties visualizing and representing the intersection of a surface and a fundamental plane. One of them, Rafael struggled with this kind of problem:

*Ra- let's see, draw on a plane of two dimensions the points on the surface that satisfy x = 1, x = 1 would be … here we have x = 1 … x = 1 would simply be this side [*correctly pointing to one side of the boxed surface shown in the given picture*] … x = 1 would simply be all* *this line ... all this line that is on the xy plane ... xy ... in x = 1 ... now ... the convenient plane is ... the plane would be x, xz and it would have a line that goes from here ...*

Rafael drew the line correctly but on an *xz* plane. Again on part c which asked for the section corresponding to $y = 1$:

Ra- Now on part c we have the plane y = 1, which would be this one, now y = 1 would be a plane ... if I consider yz what will be left are these points ... it is a curve ...

Here Rafael draws the intersection curve correctly but on a plane mislabelled as *yz*.

Another source of difficulty he had was the impossibility to visualize the intersection between the plane and the surface. When he was trying to draw the contour corresponding to , he said

Ra- ... I first locate the axes, the x axis, the y axis, this is y zero, ahh now now now now, this y zero, this x zero ... these, the axes and z ... z, ok, consider in a plane of two dimensions points on that satisfy z = 1 ... let's see, in a plane of two dimensions, the points on the surface that satisfy z = 1, z = 1, ...z = 1 .... I am at z minus one, plus one, let's see, here I would try to draw a plane parallel to xy, but finding the points of intersection would be somewhat complicated, If I didn't have, if I didn't have the equation of this, of this surface ...

It seems clear that Rafael knows what he needs to do, he refers to a plane "parallel to the *xy* plane" and also of "finding the points of intersection". However he is unable to visualize and draw the intersection curve.

The same thing happened to Pablo. He responded to all other questions quickly, correctly, and without hesitation. When faced with the need to draw the section of the surface that satisfied $y = 1$ and the contour $z = 1$ he showed some difficulties.

P- ... it would have a minimum when x .... is equal to _, z is equal to -1 .... Then a peak would occur when x is equal to ... z is equal to ... -3 .... Then another peak would occur when x is equal to -1/2 and z is equal to -3 ... the last one would be -1 ... in x equal to -1, z equal to -1 ... a curve ... two minimums and a maximum

Pablo needed to revert to a point by point evaluation strategy and, while he manages to reasonably approximate the general form of the curve, the coordinates he gives are incorrect as he actually describes the section . Then, working with $z = 1$, the point by point strategy did not enable him to obtain the contour $z = 1$ and he gave up.

P- part b says, draw on a two-dimensional plane the points on the surface that satisfy , occurs on a line, here, this is a line and the minimum is the point ... The point -1 in z and y would be -2 and the maximum point for the line would be z equal to 3 and y equal to 2, since it is a line it can be drawn with two points

Pablo had no difficulty with the section corresponding to $x = 1$. He recognized the section was a line before attempting to compute two points on the line.

We can conclude, from the work of these two students that they have coordinated their schema of Cartesian space of dimension three with those for functions of one variable and sets, to construct a schema of subsets of R^3. They seem to be able to solve confidently most of the problems proposed during the interview. The only difficulty they showed is related to the coordination of the representation of a surface in three dimensional space with the representation of its intersection with a specific plane when the surface has a somehow complicated shape. Even when these students knew what they had to do, and could tell that the result would be a curve, they showed difficulties drawing it, probably because they were not able to visualize it.

Discussion and Revised Genetic Decomposition

Results show that most of the interviewed students show evidence of coordinating their intuitive notion of space with those of the Cartesian plane and real numbers, to construct the object R^3; they can locate points in it, and draw a plane parallel to a coordinate plane when a description in terms of natural language is given.

A source of difficulty found was a lack of coordination between some students' intuitive knowledge of 3D space and the mathematical schemas they use in a multivariable setting. Some students seemed to work at an action level, using memorized facts when solving the problems, for example, they knew that equations of the form $x = c$, $y = c$, and $z = c$ are equations of planes, yet they did not show evidence of interiorization of these actions into processes or objects, so they could not relate them to their intuitive knowledge of space.

Few students could coordinate the different representations of fundamental planes and surfaces, in some cases because of problems with visualization. They show difficulties with actions that involve intersections between these objects or projections of those intersections in a two dimensional plane.

It seems that when some students are asked to draw on a two-dimensional plane, they interpret this instruction as drawing on the *xy* plane. It is difficult for them to draw a projection on a plane in R^3, or even think about it. When asked to draw on a two dimensional plane the projection of the intersection between a fundamental plane and a surface, several students insisted on drawing it in the three dimensional space. This was a difficulty that we did not predict in the original genetic decomposition. The fact that it seems to be a common difficulty for most of the students, points out that it needs further study.

It seems that, in some cases, the difficulty that some students have relating their mathematical schemas with common sense understanding of the space that surrounds us has its origin in a language or communication lapse. For example, one difficulty found involved the use of the word "cut" while forming transversal sections of surfaces. For some students the word cut was related to the use of a plane to cut the surface into two parts, not a transversal section. For those students who associated the equation of a surface in R^3 with a solid, the cut resulted in two disconnected solids.

Students were asked to draw a curve on a plane in space, and then to describe it when that set of points was assigned a given height, *z*. Some students lift the curve to the given position, but include in their description all the points traced by the movement, that is a surface instead of a curve. Again, it seems that this difficulty arises because the students assign a different meaning to the action invoked by the word "lift" from that intended. It also can arise from a lack of coordination between their common sense knowledge of three-dimensional space and their mathematical schemas.

It is clear that some of the difficulties that students had when given the graph of a surface and asked to produce graphs of transversal sections and contours, are related to their ability to visualize. It also seems that this ability is independent of the student's understanding of the material, as it could be seen in both the students that did the best and those that did the worst in the interviews. It is important to note that this type of difficulty didn't seem to inhibit the subsequent understanding of basic mathematical constructs, among them that of a function of two variables, even though it may occasionally interfere with the understanding of representations that use level curves or transversal sections.

In the revised genetic decomposition the objects or schemas that we consider a student may have developed previously to understanding the concept are the same as before. The intuitive notion of space now plays a more prominent role since it is now expected that students will be able to coordinate mathematical constructions dealing with subsets of R^3 to their world intuition, through actions of representation and description of those sets physically by means of a model such as the 3D Kit and mathematically. Building the necessary connections between the schema of subsets of R^3 and the intuitive notion of space includes paying more attention to the use of language making the meaning of words explicit by means of performing the actions and reflecting and discussing the results. The revised genetic decomposition also requires that explicit attention be given to fundamental planes, projections and the coordination of these with other subsets of R^3.

Endnote

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CONFRONTING INFINITY VIA PING-PONG BALL CONUNDRUM

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This study examines students' approaches to infinity, before and after instruction, via their engagement with a well-known paradox: the ping-pong ball conundrum. Students' work reveals they perceive infinity as an ongoing process, rather than a completed one, and fail to notice conflicting ideas. We further focus on specific challenging features of this paradox versus other counterintuitive infinity problems.

Imagine you have an infinite set of numbered ping-pong balls and a very large barrel; you are about to embark on an experiment. The experiment will last for exactly 1 minute, no more, no less. Your task is to place the first 10 balls into the barrel and then remove number 1 in 30 seconds. In half of the remaining time, you place balls 11 – 20 into the barrel and remove number 2. Next, in half of the remaining time (and working more and more quickly), place balls 21 – 30 into the barrel and remove number 3. Continue this task ad infinitum. After 60 seconds, at the end of the experiment, how many ping-pong balls are in the barrel?

This ping-pong ball conundrum (Burger and Starbird, 2000) is one of many well-known paradoxes that illustrate the counter-intuitive nature of infinity, which has puzzled mathematicians and philosophers for centuries. From Zeno's paradox to Hilbert's Infinite Hotel, the question of what happens to an infinite iteration once the process is complete has delighted and frustrated minds alike. This study explored university students' responses to the ping-pong ball conundrum, as well as their arguments after instruction.

Theoretical Background

The counterintuitive nature of infinity, as manifested in students' reasoning, is described in prior research (Dreyfus and Tsamir 2004; Fischbein 2001; Fischbein, Tirosh, and Hess, 1979). Tsamir and Tirosh (1996), for instance, explored students' intuitive decisions when comparing cardinalities of two sets. Their conclusions supported Fischbein, Tirosh, and Melamed's (1981) claim that intuitive leaps are necessary to establish meaning about infinity.

Fischbein (2001) observed, when students attempt to form an understanding of abstract concepts, their tacit mental representations in the reasoning process replace the original concepts by more accessible and familiar ones. In Hazzan's (1999) perspective, the use of familiar procedures to make sense of unfamiliar problems is an attempt to reduce the level of abstraction of certain concepts. She suggested that the tendency to apply familiar procedures – such as those of finite set comparison – is indicative of a process conception. Further, Dubinsky, Weller, McDonald, and Brown (2005a,b) – relying on the APOS theory (Dubinsky & McDonald, 2001) – proposed that process and object conceptions of infinity correspond, respectively, to an understanding of potential and actual infinity. In the ping-pong ball conundrum, the process of adding balls ad infinitum corresponds to potential infinity, whereas actual infinity entails a completed infinite process and describes the set of balls as a whole entity. Dubinsky et al. suggested encapsulation of infinity requires "a radical shift in the nature of one's conceptualisation" (2005a, p. 347) and might be quite difficult to achieve.

Extending these ideas, our study uses APOS theory to interpret students' naïve responses, and their attempts to reduce the level of abstraction of properties of infinity as they addressed the ping-pong ball conundrum. Specifically, we address the following questions: (1) What are

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students' ideas of infinity before instruction? (2) Do students' ideas of infinity change following instruction? (3) What specific features of the problem are challenging for students?

Setting and Methodology

Participants in this study were 29 university students; 14 were in the masters' program in mathematics education, and the rest undergraduates with little tertiary-level mathematical training. The study began by presenting participants with the ping-pong ball conundrum as a thought experiment and asking them to record their ideas individually. Group and class discussions ensued, as did instruction on cardinality and infinite sets. The instructional tasks included comparing infinite countable sets using one-to-one correspondence and discussion of the Infinite Hotel Problem. Students then were asked to readdress the original question: At the end of the experiment, how many ping-pong balls are left in the barrel?

Results and Analysis

The mathematical resolution to this paradox lies in the distinction between potential and actual infinity. The process of putting in and taking out ping-pong balls goes on infinitely. However, after the 60 seconds end, so does the process. As such, there are no balls left in barrel at the end of the experiment, since every ball has been taken out at some point.

There was no major difference in the responses from the two groups. Students' initial ideas, in both groups, can be clustered around two main claims:

"The process is impossible since the time interval is halved infinitely many times"

"There are infinitely many balls left in the barrel"

The argument that the rate of in-going balls was greater than that of out-going balls was the popular justification for infinitely many balls at the end of the experiment. This argument was most persuasive both before and after instruction.

While the argument of different rates persisted, students' responses following instruction were more specific with respect to infinity, distinguishing between the infinity of balls taken out and the infinity of balls remaining in the barrel. The following excerpt exemplifies this:

> "You remove a ball infinite number of times so as you go to infinity, you remove all the balls… You could also say you are putting in more (9) balls in than you remove so you eventually add an infinite number of balls. Is one infinity larger than the other?"

In other arguments there was an apparent influence of instruction combined with overwhelming intuitive resistance to the new ideas. For example:

> "There is an infinite number of balls in the barrel, however it is impossible to name a specific ball. As soon as a number is chosen, it is possible to determine the exact time… that ball was removed."

Furthermore, following instruction some students acknowledged a 1-1 correspondence between in-going and out-going balls. Judy wrote:

> "There are still infinitely many balls left in the barrel, because even though there is a one to one correspondence between the sets {1, 2, 3, 4, …} [and] {9, 18, 27, 36, …}, the rate at which you are putting in is more than you are taking out. So even if there are just as many numbers in each set, they will never even out, because the process continues infinitely and you continue to put more in than you take out."

The inherent contradiction in Judy's response, as well as of the 11 other students who presented similar arguments went unnoticed.

Only 3 graduate students suggested that the number of balls in the barrel was 0, but added a comment that pointed to the distinction between what they "learned" and what they "believed". For example, Joe conceded that he could "now entertain the idea that there are no balls in the basket (but [he didn't] like it)."

Students' description of infinity as continuing forever corresponds to potential infinity, and a process conception. The argument for different infinities because of different rates seems to extrapolate common (finite) experiences with rates of change. This is consistent with the observation that students' conceptions of infinity tend to arise by reflecting on their knowledge of finite concepts and extending these familiar properties to the infinite case (Dubinsky et al. 2005a; Dreyfus and Tsamir 2004; Fischbein 2001) and serves an example of "reducing the level of abstraction" (Hazzan, 1999).

Students' resistance to the ideas introduced in instruction appeared surprising in light of prior seemingly successful resolution of other counterintuitive problems, such as the correspondence between natural and even numbers and the resolution of Hilbert's Hotel Infinity paradox. As such, our further analysis attended to the differences in students' response to the ping-pong ball conundrum and their work with other problems involving infinity. Our analysis revealed that while the resolution of ping-pong ball conundrum requires understanding of actually infinity, success with other problems exemplified above does not.

Conclusion

It has been well established that when formal notions are counterintuitive, primary, inaccurate intuitions tend to persist (see among others Fischbein et al., 1979). Moreover, individuals may adapt their formal knowledge in order to maintain consistent intuitions. In the case of infinity, resilient intuitions of potential infinity were prevalent among the participants and persisted following instruction. In the cases discussed here, participants seemed to "construct the state at infinity by mimicking what happens for related finite processes" (Dubinsky et al., 2005b, p. 259). Attending to differences in students' responses to the ping-pong ball conundrum versus other problems, we suggest that not all tasks that involve consideration of cardinal infinity require understanding of actual infinity and that understanding of "infinity small" is more challenging than "infinitely large". This latter observation is of our attention in further research.

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FUNDAMENTAL VECTOR SPACES: WHAT CAN BE LEARNED FROM NON-EXAMPLES

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This study is a contribution to the ongoing research in undergraduate mathematics education, focusing on linear algebra. It explores the students' understanding of the key concepts of linear algebra through the lens of learner-generated examples, identifies some of the difficulties experienced by students, and also isolates some possible obstacles to learning the concepts.

Background and Theoretical Perspective

Linear Algebra has become one of the required undergraduate courses for many disciplines such as computer science, engineering, and economics. As research showed many students leave the course with limited understanding of the subject (Dorier, 2000; Carlson et al, 1997). Part of the difficulty is due to the abstract nature of the subject. Dubinsky (1997) points out that there is a lack of pedagogical strategies that give students a chance to construct their own ideas about concepts in the subject. The pedagogical weakness of linear algebra is that students can pass the course by following a set of procedures. Being faced with non-routine tasks such as those requiring construction of examples can reveal students' weaknesses and misconceptions.

Examples play an important role in mathematics education. Students are usually provided with examples by teachers, but are very rarely faced with example-generation tasks, especially as undergraduates. As research shows (Hazzan & Zazkis, 1999; Watson & Mason, 2004), the construction of examples by students contributes to the development of understanding of the mathematical concepts. Simultaneously, learner-generated examples may highlight difficulties that students experience. This study examines how and in what way example-generation tasks can inform about and influence students' understanding of linear algebra. The APOS theoretical framework was adopted in this study to interpret and analyse students' responses (Asiala et al, 1996).

Method

Participants in this study were 113 students enrolled in Elementary Linear Algebra course. Vector spaces associated with matrices are one of the central concepts of linear algebra and are related to other concepts such as linear transformations. It was observed previously that students commonly confused the span of the columns of a matrix A (namely Col A) with the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$ (namely Nul A). Having been exposed to systems of linear equations, when students see the words 'matrix' and 'span' in the same problem, many of the students may rush into solving the homogeneous system $A\mathbf{x} = \mathbf{0}$ without comprehending what the question asks. The following task was developed to further probe students understanding of these subspaces. It was assigned to be completed in writing after the topics of systems of linear equations, vector and matrix equations, linear dependence and independence in $\mathbf{R}^n$, linear transformations and Invertible Matrix Theorem were covered in the course.

Task: Column space / Null space

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Find an example of a matrix A with real entries for which Nul A and Col A have at least one nonzero vector in common. For this matrix A, find all vectors common to Nul A and Col A. If T is the linear transformation whose standard matrix is A, determine the kernel and range of T.

Every matrix has associated with it two intrinsic and complementary subspaces: the column space and null space. The column space of a matrix A is the set of all linear combinations of the column vectors of A, or the span of the columns of A. The null space of the matrix A is the set of solutions to a homogeneous linear system $A\mathbf{x} = \mathbf{0}$. Both types of subspaces are related to the linear transformation $\mathbf{x}$_$A\mathbf{x}$ in that the column space of A is the range of the linear transformation and the null space of A is the kernel of the linear transformation. In the case where A is a square nxn matrix, the column space and null space of A are both subspaces of $\mathbf{R}^{\mathbf{n}}$, and have the zero vector in common. However, there are cases when these subspaces share nonzero vectors, and so we can ask how large their intersection might be.

The purpose of Task: Column space / Null space was to explore how students treat these special cases, or non-examples. Non-examples are examples which demonstrate the boundaries or necessary conditions of a concept (Watson and Mason, 2004). They can simultaneously be counter-examples to an implicit conjecture. In this task, a matrix A is a non-example of a square matrix with Col A _ Nul $A = \{\mathbf{0}\}$. At the same time, it is a counter-example to the conjecture that for any square matrix A, Col A and Nul A have only the zero vector in common.

Results and Discussions

I first present the summary of students' responses, and then outline how the APOS theoretical framework was used to analyze students' understanding of the fundamental subspaces associated with a matrix. Several types of responses were identified for Null space / Column space task. They are summarized in the Table 1 below.

Table 1: Constructing matrix A with $\mathbf{v}$ in Nul A_Col A, $\mathbf{v} \neq \mathbf{0}$

Type of Response	Example	Frequency
Invertible matrix (necessarily incorrect)	$A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	12%
Singular matrix with Nul A _Col $A = \{\mathbf{0}\}$	$A = \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$	23%
Any other specific singular matrix (necessarily correct)	$A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ or $\begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}$	48%
Semi-general construction	Nul A: $A\mathbf{x} = \mathbf{0}$; Col A: can be any column in A. Pick one column and determine the other column. Let $\mathbf{x}$ equal to first column. $\begin{bmatrix} 2 & a \\ 1 & b \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; $\begin{matrix} a = -4 \\ b = -2 \end{matrix}$ and $A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$.	11%
General construction	$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; $A = \begin{bmatrix} 1 & -1/2 \\ 2 & -1 \end{bmatrix}$.	6%

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Students that constructed an invertible matrix as an example may be operating with an action conception of the vector spaces associated with a matrix A. They need to work with specific numbers and perform calculations explicitly to find the solution set of a homogeneous equation $A\mathbf{x} = \mathbf{0}$, and to compute the column space of A. As a result, computational mistakes lead them to incorrect conclusions.

The process conception of column space and null space may entail realizing that one has to work with a singular matrix since its null space contains a nonzero vector. The action of solving a matrix equation is performed mentally to satisfy one of the requirements of the task. However, an individual has to compute the null space to show that null space and column space share at least one nonzero vector, which doesn't follow from the construction of an example. Students that incorporated the requirement that the null space and column space of a matrix have at least one nonzero vector and limited their potential example space to the set of singular matrices may be working with the concepts of Nul A and Col A as a process. They internalized the procedures for finding these two vector spaces but were still unable to connect them.

An indication that students have encapsulated Nul A and Col A as objects is demonstrated when the students are able to analyze a new situation and recognize how and why to apply the properties of Nul A and Col A. In this task, an individual might recognize that Nul A having at least one nonzero vector implies A is singular. Further, the analysis of Col A having an element in common with Nul A should lead to construction of a matrix such that one of its columns, $\mathbf{a}_i$, also satisfies the matrix equation $A\mathbf{x} = \mathbf{0}$, so that, $A\mathbf{a}_i = \mathbf{0}$. Students that used the general construction methods of Table 1 included all the conditions of Task: Column space / Null space in their example-generation. There is strong evidence that they understand the concepts of Nul A and Col A as object.

Conclusion

Overall, students' examples revealed the connections students make between the different concepts, and students' level of understanding according to the APOS theoretical framework. This study identified that some students' schema of null space doesn't have the connection between the existence of a nontrivial solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ and the singularity of A. Other students internalized the procedures for finding the two vector spaces: Col A and Null A, but were still unable to connect them.

In addition, this study enhances the teaching of linear algebra by developing a set of example-generation tasks that serve not only as an assessment tool but also as an instructional tool. It is hoped that by examining students' learning, the data collected can lead to teaching strategies, which will help students expand their example spaces of mathematical concepts and broaden their concept images/schemas.

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NONSTANDARD MODELS OF ARITHMETIC FOUND IN STUDENT CONCEPTIONS OF INFINITE PROCESSES

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231 undergraduates' written and interview responses to the Tennis Ball Problem (Weller, et al., 2004) suggest that many students appeal to nonstandard models of arithmetic to make sense of infinite iterative processes. This challenges an APOS analysis of this problem, by describing students with fully encapsulated, but alternate, conceptions about infinity that are mathematically (and cognitively) coherent and consistent.

One reason that recent researchers have investigated student conceptions of infinity is because these conceptions have been shown to support and constrain student understandings of important undergraduate mathematical ideas, such as limits, sequences, series, and the real number line (e.g. Sierpinska, 1987). The following study marks another step in establishing a meaningful theory-based account of student reasoning with infinite processes.

Theoretical Background

Two theories of student thinking in mathematics education have been widely used to analyze and interpret student answers to questions involving infinite processes—*encapsulation* (e.g. Weller, et al., 2004) and the *basic metaphor of infinity* (Lakoff & Nunez, 2000).

Encapsulation is part of the larger APOS framework (Dubinsky & McDonald, 2001), which treats student thinking about mathematics as a set of progressive abstractions through *actions*, *processes*, *objects*, and *schemas*. Encapsulation, or similarly, *reification* (Sfard, 1991), is the type of abstraction by which students are able to reconceptualize a mathematical process as an object, which could itself be a part of other processes. Through this framework, student difficulties with infinite processes are due to an incomplete encapsulation of the iterative processes into completed totalities.

The Basic Metaphor for Infinity (BMI) is also part of a larger framework, which describes all of mathematical thinking as a sequence of metaphors ultimately grounded in physical embodied experience (Lakoff & Nunez, 2000). The BMI is the metaphor that allows us to project onto an infinite process a final end result.

While encapsulation effectively interprets some student responses about infinite processes, the BMI provides a more plausible account for the different common student answers, especially to the Tennis Ball Problem. However, even the BMI gives an incomplete account of the mathematical structures students create in order to make sense of these "end results" of infinite processes.

Methods

The results reported are part of a larger study about student conceptions of concepts foundational to calculus. 235 university calculus students were given a questionnaire which included an adapted version of the Tennis Ball Problem (Weller, et al., 2004). 225 of these students responded to this item.

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Six additional students were later interviewed informally and briefly in a classroom setting to illuminate some of the response patterns identified. Their responses were written down by the interviewer. The primary purpose of these interviews was to illuminate some of the student questionnaire responses in order to refine a set of interview protocols for a more formal future study. This informal part of the study is included here because some interview remarks provide insight about a reasonable interpretation of the questionnaire responses.

Results and Discussion

The modified Tennis Ball Problem is reproduced in Figure 1. The student written response patterns were categorized into the five categories found in Table 1.

Figure 1—Modified Tennis Ball Problem

"Suppose you put two tennis balls numbered 1 and 2 in Bin A and then move ball 1 to Bin B. Then you put balls 3 and 4 into Bin A and move ball 2 to Bin B. Then you put balls 5 and 6 into Bin A and move ball 3 to Bin B. And so on infinitely. How many balls are in Bin A after you are done? Why?"

Table 1—Written response patterns to the Modified Tennis Ball Problem for 225 undergraduates

Response categories		Number of students		
			n	%
a. 0 balls remaining in Bin A		3	1%	
b. Infinitely many balls or half of the balls	(or both)		131	56%
c. Objected to "after you are done"			39	17%
d. A positive finite number (such as "3")			36	15%
e. No answer or didn't know		16	7%	

Almost no one answered the Tennis Ball problem correctly, that 0 balls remain in Bin A. The theory of encapsulation accounts for why 17% of students explicitly objected to visualizing the infinite process as being completed, and why 15% of students thought that there was some small positive number of balls remaining. These students have failed to encapsulate the infinite process into a completed totality. However, many students explicitly replied that there are infinitely many balls left in the bin. It appears that these students have actually succeeded in encapsulating the infinite process into a totality, but the totality they have encapsulated is incorrect.

The BMI provides a more nuanced explanation: it is not that these students have failed to encapsulate the infinite process, but rather that they are metaphorically extending the wrong property of the infinite process to achieve their final end result. When asked why there are infinitely many balls in Bin A, these students usually explain that after each step of the process there is another ball in Bin A, so this number increases to infinity. But when they are asked to *name* a ball in the bin, they are unable to answer.

These students are creating a metaphorical result of an infinite process by extending the "how many balls" property from the finite case to the infinite case. The correct answer, however, relies on the extension of the "which balls" property from the finite case to the infinite case. This is a profoundly different thing to attend to, and it requires using numbers not to *count*, but to *index* (or name) the balls in the bin. The difficulty in getting students to use numbers to index rather than count is precisely the difficulty in getting students to understand Georg Cantor's theory of sets. That this is a difficult transition is mirrored by the difficulty Cantor had in convincing his peers that there really are different sizes of infinity, but that these sizes are revealed not by trying to count the elements in an infinite set, but by trying to index or name the elements.

There is still one more idea that the APOS account cannot explain, and that is why students are unperturbed by the belief that Bin A contains infinitely many balls, none of which can actually be named. In the interviews, the students were presented with the following argument: there cannot be any balls remaining, because if there were then we should be able to name such a ball k, but we know that ball k was moved out on the kth step of the process. Only one of the interviewed students recanted because of this explanation; several actually countered by saying that there exist numbers so big that they cannot be named, and that the "last half" of infinity is in the bin, where the balls don't have names.

This belief suggests a tacit belief in a nonstandard model of arithmetic. A nonstandard model of arithmetic is, essentially, an alternate number system in which all of the same first-order arithmetic statements hold true that are true in our standard number system, but which is not isomorphic to the standard system. Logicians have created such models (Skolem, 1934/1955), and although these models have the same degree of mathematical power and consistency as our standard models, they look significantly different. For instance, in one such nonstandard model of arithmetic, there exist all of the counting numbers, but there are also a bunch of larger infinite numbers which do not have names and cannot be referred to explicitly in mathematical formulas.

This is not the only instance in which students appear to maintain alternate nonstandard models in mathematics (e.g. nonstandard analysis in Ely, 2006). Understanding student thinking in terms of nonstandard models of our numerical reality explains some student responses currently unaccounted for. Certainly many students are unable to encapsulate infinite processes, or fail to appropriately apply the BMI to such processes. But sometimes students *are* able to do these things, and yet end up with an alternate conception of our number system, one that may actually be as powerful and coherent as the conception we are trying to get them to adopt.

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PROBLEMS OF VISUALIZATION IN CALCULUS

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The view that imagery can hinder mathematical thinking might surprise some mathematics educators who contend that understanding is enriched whenever visual imagery is used. In two contrasting cases, we present the view that dynamic visual images, without support from analytic thinking, can be a hindrance in constructing mathematical meaning – contrary to the "panacea" view of imagery which is sometimes expressed.

Visualization and visual thinking have been fundamental during the past two decades of studies to reform the way calculus is taught, and their roles have been recognized and given further emphasis by reform calculus textbooks. Having stated that students should have adequate understanding of visual representations and the ability to make connections between analytic and graphical representations, Hughes-Hallett (1991) advocates a balance between the graphical, the numerical, and the analytical: "A balance is required because it's seeing the links between various approaches that constitutes understanding" (p.125).

Theoretical Framework

In this paper, our work is framed by Krutetskii (1976) who distinguished among main types of mathematical processing by individuals: analytic, visual (or geometric), and harmonic. A student who has a predominance toward the analytic relies strongly on verbal-logical processing and relies little on visual-pictorial processing. Conversely, a student who has a predominance toward the visual relies strongly on visual-pictorial processing and relies less on verbal-logical processing. Harmonic students rely equally on verbal-logical and visual-pictorial processes. Krutetskii also identifies two subgroups of the harmonic type: pictorial-harmonic and abstract-harmonic. Representatives of these subtypes can depict mathematical relations equally well by visual means. Abstract-harmonic types do not feel the need for using visual schemes while solving problems, whereas pictorial-harmonic types have a tendency to rely on visual means. Our work also is framed by Presmeg's (1986, 2000) study in which five kinds of mathematical imagery, associated with visual thinking, are identified: concrete imagery, memory images of formulae, pattern imagery, kinesthetic imagery, and dynamic imagery. In the literature (e.g., Aspinwall, Shaw, & Presmeg, 1997), several difficulties verifying the limitations of imagery have been documented. Presmeg (1986) observed that visual dynamic imagery was effective and visual students who were able to combine visual imagery and analytic means avoided the drawbacks associated with the use of imagery. Wheatley (1997) confirmed Presmeg's findings that dynamic visual imagery produces high levels of mathematical functioning.

Methods

As a component of a larger line of research aimed at better understanding how students learn elementary calculus viewed through their personal and idiosyncratic representations, we developed cases describing two participants, Jack and Bob. Our work is supported by the view that posing and analyzing rich tasks for students provides windows into their thinking with

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ramifications for curriculum and instruction. As a result of observations of what students say and write, and how they represent mathematical situations, researchers make decisions about appropriate ongoing investigations. Generally, subsequent tasks are designed to clarify or validate early assertions. Here, we illustrate the contrasting thinking processes of Bob, whose representational scheme is predominately pictorial-harmonic, and Jack, whose representational scheme is predominately visual. We conducted weekly clinical interviews with the participants, during which they were presented with derivative graphs of functions and asked to draw possible antiderivative graphs as we sought to gain understanding of their mental processes and representations. First, procedural tasks, presented with equations, were administered to demonstrate the participants' understanding of algorithms, techniques, and formulas for computing integrals. Then we prepared an initial set of graphical tasks to start the interviews, and the remaining tasks were developed based on analyses of these weekly interviews. Each clinical interview lasted about 20 minutes and was video-and audio-taped. The participants were asked to think aloud while they were solving the tasks so that we could analyze their responses and strategies as well as describe and make inferences about their representations.

Results

The procedural tasks were easy for them and in a few minutes, they correctly computed the integrals without any difficulty, demonstrating that they were proficient at computing various integrals. In this study, sixteen graphical tasks were designed to examine the participants' mental images, representations, and strategies to create meaning for the derivative graphs. One graphical task will be discussed in this paper; the others tasks as well as the participants' work can be found in Haciomeroglu's (2007) study. The graphical task in Figure 1 with the following instructions was presented to the participants: *The graph of is shown. Let f(0) = 0 and sketch a possible graph of f(x).* Bob examined the *y* values at *x = 0*, *1*, and, *2* and immediately transformed the derivative graph into the graph in Figure 2 by visualizing tangent lines to the graph. At the same time, relying on analytic means, he calculated the area under the derivative graph to find the *y* values of the antiderivative graph at *x = 1* and *x = 2*. Bob's response suggests that he is demonstrating the characteristics of the pictorial-harmonic type, one of Krutetskii's (1976) subtypes of harmonic students, who starts from visual-pictorial means and relies on graphical representations during a solution. Jack drew the graph in Figure 3 on the basis of his estimates for the slopes of tangent lines. That is, he used the *y* values at *x = 0*, *1*, *2* to visualize the tangent lines and transformed the derivative graph into the graph in Figure 3. Jack's thinking was predominantly visual. He relied on graphic representation and did not translate the derivative graph into another representation. When we asked Jack to find the *y* coordinate of the inflection point, he thought that the graph could be stretched as long as it has the same slopes at *x = 0*, *1*, and *2* and that it was not possible to find

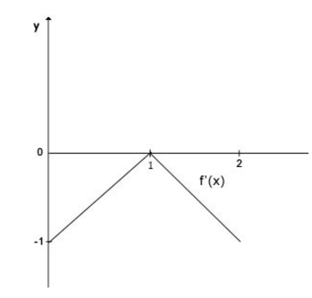

Figure 1.

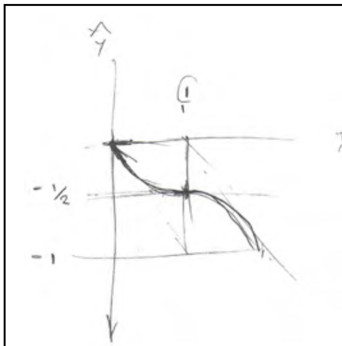

Figure 2.

the inflection point. He said the inflection point was not a fixed point due to vertical stretching. In Figure 4, we considered what he described and stretched his sketch in Figure 3. This graph meets his conditions and illustrates how his visual thinking could be misleading him as his dynamic images of the antiderivative graph hindered his understanding that vertical stretching would change the derivative graph. In the literature (e.g., Presmeg, 1986, 2006), dynamic imagery had been considered to produce high levels of mathematical functioning, but, without the support of analytic thinking, misleading and false dynamic images interfered with Jack's thinking and caused him to think that there were infinitely many inflection points due to vertical stretching.

Discussion

Jack demonstrated the characteristics of visual types who tend to employ only visual modes of thought while solving problems. Bob also demonstrated a strong preference for visual modes of thinking but, when he experienced the need for verbal-logical modes of thinking, he used this component by calculating the area under the derivative graph or estimating the equation of the derivative graph. In clinical interviews, we observed how visual dynamic images, without the support of analytic thinking, controlled Jack's thinking and hindered his understanding. With the help of analytic means, Bob drew precise sketches and avoided some of Jack's difficulties associated with the use of visual dynamic imagery. These difficulties indicate the importance of analytic thinking to support visual thinking while dealing with graphical in the complete understanding of differentiation and integration. The literature suggests that students understand the derivative and test their ideas graphically and analytically (e.g., Hughes-Hallett, 1991). We also believe that understanding of calculus is strongly related to the ability to synthesize visual and analytic thinking, and students' understanding of analytic methods and developing graphic representations of derivative and antiderivative graphs go together as students construct mathematical meaning for the concept of derivative and antiderivative in calculus. Derivatives and antiderivatives of different graphs should be discussed to encourage students to use formal definitions of left- and right-hand derivatives, differentiability, and continuity as well as help them construct appropriate mental images and representations that will facilitate their learning and understanding of calculus.

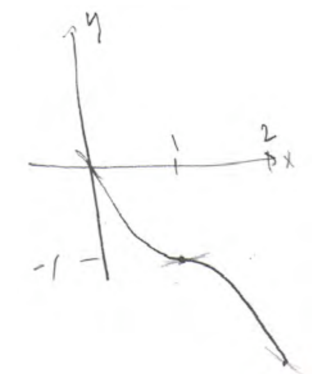

Figure 3.

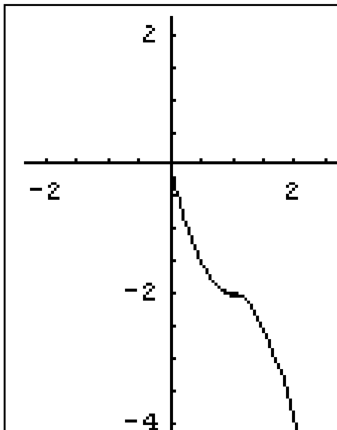

Figure 4.

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THE INTERPLAY AMONG GESTURES, DISCOURSE AND DIAGRAMS IN STUDENTS' GEOMETRICAL REASONING

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This study examines how high school geometry students' multimodal interactions with diagrams (using gestures as well as speech) reveal their reasoning. The study suggests that when information given in a diagram is limited, students complement it with gestural and verbal expressions that involve them in conjecturing. The study suggests graphic representations that can be used to codify gestural interactions in instructional practice.

Research has discussed the gap between the physical properties of geometric diagrams and the conceptual nature of geometric figure (Duval, 1995; Fischbein, 1993; Laborde, 2005; Mariotti, 1995). Herbst (2004) claims that building geometric knowledge requires deductive reasoning with representations of mathematical objects, such as diagrams. From the perspective of learning as participating in a community of practice (Lave & Wenger, 1991), it is expected that learning geometry involves students interactions with diagrams. Studies on gestures in science and mathematics learning suggest that gestures embody students' thinking and spatial reasoning (Cook & Goldin-Meadow, 2006; Nemirovsky & Noble, 1997; Roth & Welzel, 2001). This study identifies how students interact with diagrams in gestural and verbal forms, and how such multimodal interactions may advance their deductive reasoning.

The data in this study is from a corpus of classroom videos in a large project that studies mathematical work in high school geometry. An experimental lesson was developed to provide context for students making conjectures about parallel lines, a topic that had not yet been taught in the class. With a given diagram consisting of sets of intersecting lines (see Figure 1), students were asked to determine the measurements of all angles formed by given intersecting lines, but to actually measure as few of them as possible. We compare the use of gesture and language obtained from video records of that lesson with those of an intact lesson which represented customary teaching practice in that classroom. Video records of intact lessons had been collected on a weekly basis.

Video segments from five experimental classes and an intact lesson are identified where students interact with diagrams in public. We analyzed the transcripts with the video images. We examined the modality in students' discourse. Modality is defined as the uncertain intermediate degrees that lie between polarities (Halliday & Matthiessen, 2004). Modality is used to shows degree in the spectrum between positive and negative statements (of frequency, necessity, obligation, etc.). We take modality in discourse as an indicator of students' judgments about the truth of statements about diagrams. The use of "should," prescribing that diagrams should be in certain way indicates a belief on the necessity of geometrical properties of the diagrams (Herbst, 2004) and suggest students' theoretical reasoning that goes beyond the capture of "spatial-graphical" properties (Laborde, 2005) of the diagrams. Along with analyzing discourse, we "mind read" (McNeill, 1992) gestures, an imagistic form of speakers' utterance, to study students' reasoning and thinking.

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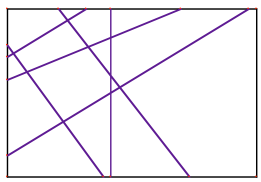	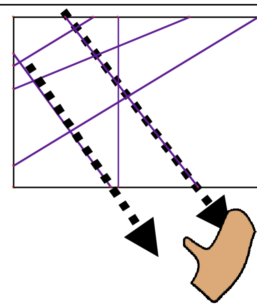	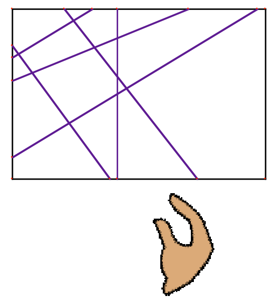	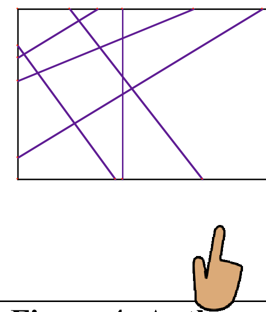
Figure 1: The diagram given on the worksheet	**Figure 2: The open palm indicates the parallelism of the two lines**	**Figure 3: A hypothetical intersecting point**	**Figure 4: Anthony: "There can't be another point down here"**

Results

Our analysis revealed that students' verbal and gestural interactions are different in experimental and intact lessons. In the intact lesson, the diagram, a homework problem in the textbook, provides sufficient information (labels of angles and segments) that suggest students what elements to pay attention to when doing proofs. Thus no additional actions (e.g. labeling) or conjecturing need to be made. Hence, deictic gestures were dominantly used to point to an object on the diagram and to draw the audiences' attention (McNeill, 1992). The student's use of present tense (i.e. is, are) indicates that they are stating the "facts" about the diagram.

However, in the experimental lessons, diagrams only contained representations of the objects mentioned in the problem without any labels or symbols attesting to their assumed relationships. Students had to use gestures to create hypothetical assumptions of diagrams (e.g. if they *would not* be parallel), and justify their conjectures with high degree of certainty (e.g. the two angles *have to* be equal). Gestures were used to represent imaginary diagrams and externalize students' imagistic thoughts of diagrams.

For example, two students, Collin and Anthony conjectured that two given lines should be parallel through proof by contradiction: Collin first stated that the two lines should be parallel by tracing the two lines with his open palm outside the frame (see Figure 2). Then, he assumed that the lines would intersect at a certain point if the statement were false. A hypothetical intersecting point was positioned outside the frame (" if they *would not* be parallel, you get a triangle", see Figure 3). Unlike gesturing with an open palm, Collin narrowed the space between his thumb and index finger, indicating an intersecting point of the two lines. This variation of gestures indicates his differentiating notions between parallelism and intersection.

After making the assumption regarding two lines being intersected, he attempted to get the measurement of the third angle in the virtual triangle formed by the two lines. However, with the angle sum theorem and the known measurements of two other angles with 180 degrees in the virtual triangle, it is impossible to have a third angle. Anthony thus pointed at a spot farther than the one Collin did, (see Figure 4), claiming the impossibility of an intersecting point by two parallel lines (" there *can't* be another point down here").

Conclusions

This study shows the constraints of diagrams can be seen as resource to bring forth students' gestures and verbal expressions in making hypothetical claims about diagrams. Thus gestural and modality expressions are mediation tools to compensate for the limitations of diagrams.

The analysis of gestures highlights the importance of multimodal representations in understanding students' thinking. We identify the contextualized uses of gestures representing geometrical ideas, especially representing those that are not visually available. Further research on gestural uses in contextualized settings will contribute to understanding students' thinking and learning.

In addition, the graphic representations adopted in this study can be utilized as a tool to codify the gestural communication in mathematics classroom. Further developments of graphic gestural expressions that represent the authentic interactions in classrooms can help in capturing the essences of students' interactions, and consequently, in understanding how students make sense of mathematical ideas.

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USES OF LIMIT TOWARD THE ε-δ FORMAL DEFINITION

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The purpose of the study is an attempt to develop a preliminary use of limit toward the ε-δ definition to fill in a gap between students' naïve conceptions and the formal definition. Fifty-five undergraduates were recruited as participants. Among them, four representative students in different levels (in terms of different responses to a problem set) were interviewed.

Most students have extreme difficulties in understanding the limit concept because of its abstract formal definition and the precision of the ε-δ definition. However, ε-δ ideas in the concept of limit are not only essential to calculus but also a conceptual basis in mathematical analysis. Therefore, an approach to limit that is accessible to students and become the foundation for later mathematical development needs to be developed for students to better understand it. This study is an attempt to fill in a gap between students' naïve conceptions about limit and the ε-δ definition.

Theoretical Background

Previous Research on Student Learning on Limit

Various aspects of the learning about limit have been investigated over the last few decades. Most previous researchers have attempted to understand student difficulties with limit by examining three different aspects: the analysis of the *mathematical structure* of limit (Mamona-Downs, 2001; Tall, 1992), *misconceptions and cognitive obstacles* related to limit (Cornu, 1992; Davis & Vinner, 1986; Monaghan, 1991 Przenioslo, 2004; Williams, 2001), and the use of *cognitive theory* (Cottrill et al., 1996; Gray et al., 1999)

The Need for an Additional Approach to Student Thinking about Limit

The different cognitive approaches to student thinking about limit have brought important insights into mathematical learning and teaching (e.g., cognitive structures of limit, students' misconceptions, and how individual students construct the mathematical concept of limit). However, mathematics learning is both symbol- and language-mediated activities. Previous researchers, who explored learning on limit within the cognitive framework, underestimate not only the inherently social nature of thinking, but also the role of discourse and communication in learning. To reveal situated learning difficulties, the need for discourse analysis as a research method is warranted for the methodological necessity of dealing with contextual sensitivity in mathematical learning, such as the use of language, active process of "enculturation into a community of practice" (Cobb, 1994), and continuous changes of learning context.

Design of Study

Research Questions

To investigate student thinking about limit, the research questions for this study include: What are salient characteristics in students' conceptions of limit? Which particular preliminary use of limit do students employ for the ε-δ definition?

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Method

Fifty-five undergraduates recruited from the three different courses in a university of the north-central United States participated in the study: Applied Calculus, Calculus I, and Mathematical Investigation. The Mathematical Investigation course is about mathematics for prospective elementary teachers such as numbers and operations. The task for this study consisted of a 5-task problem set and a follow-up interview questionnaire. The geometric interpretation in the concept of limit causes an obstacle to arriving at the notion of a numerical limit (Davis & Vinner; Monaghan, 1991, Szydlik, 2000). Thus, both arithmetic and geometric contexts were used to investigate students' uses of limit in the problem set. After reviewing the answers to the problem set, four representative students from different levels (in terms of their different responses) were selected for interview to find a preliminary use of limit towards its formal definition.

Some Results from Interviews

There were two noticeable findings from the four interviews. One salient characteristic is that students' uses of limit are context-dependent. Informal negation is employed for a preliminary use toward the formal ε-δ definition of limit. The episodes below were extracted from scripts of the four interviews.

Episode 1: Mike

Mike was a student in applied calculus. He responded that infinity means both the biggest number and the act of going on and on in his questionnaire. The exchange was as follows:

I:	Can you give me an example of the biggest number in your mind?
Mike:	There is no number of infinity as a top. I guess I don't think of it as a value. It's more a concept, I guess.
I:	Instead of value just a concept?
Mike:	Yes.
I:	When you think the biggest number, it means just a concept.
Mike:	Yes.
I:	What is this (the act of going on and on) just like?
Mike:	Then I am relating it to the plot of graphs.
I:	OK, this one is related to graphs. This is a concept
Mike:	Yes.

Mike used both notions of limit, but he employed them in different contexts. The biggest number means just an abstract concept rather than an actual number. The act of going on and on is related to the plot of graphs. His use of limit is context-dependent.

Episode 2: John

John was taking calculus I. John responded $\infty\times(1/\infty) = 1$ for a problem in the problem set which was intended to elicit understandings of discrete limit processes of geometric shapes. The widths of those geometric shapes are decreasing, whereas their heights are increasing at the same rate. Below is part of the dialogue between John and the interviewer.

I:	Did you mean $\infty\times(1/\infty) = 1$?
John:	Yes.
I:	You also answered n approaches ∞ as quickly as 1/n approaches 0. Is that your thinking about $\infty\times(1/\infty)$.
John:	Yes, in the classroom I learned about the rate of growth like…n in the series goes to infinity faster than natural log of n. They both go to infinity but one goes more quickly. So that's what I was thinking here. Since this n right here goes to infinity just as quickly as n right there. So they are cancelled each other…
I:	Can you explain infinite small more?

John: Infinite small is so small that like it doesn't registered any more…
After some big number, you can't get it close enough to see the difference.

In this episode, John tried to justify his reasoning by looking at the difference between limiting processes and the limit ("you can't get it close enough to see the difference").

As to the question of "Is 0.999… equal to 1 or just less than 1?" in the problem-set, Paul answered 0.999… = 1 because there is no positive number that you can add to 0.999… which will make it less than or equal to one. Based on the two responses from John and Paul, the meaning of informal negation is that there is no number that makes a difference between limiting processes and the limit. Informal negation seems to be much easier than the formal definition to introduce infinitesimal. Mathematically, the formal definition of the limit of a sequence is that for $\forall\ \varepsilon > 0$, $\exists\ m$ such that $a_n + \varepsilon > a$, for every $n > m$. Instead of using the formal statement, John and Paul employed its informal negation: for $\forall\ \varepsilon > 0$, there is no m such that $a_n + \varepsilon < a$, for every $n > m$.

Discussion

In the previous example, I have described context-dependence as well as informal negation in uses of limit by examining students' discourse. The first thing to note is that learning transfer is not automatic because students seem to employ different uses of limit in different contexts. Informal negation seems to be used as a tool to fill in the gap between students' naïve uses of limit and the ε-δ definition. What works better for mathematical learning in practice is dependent on whether investigation tools for our research aid us to not only deepen our fundamental understanding, but also provide pragmatic processes of resolving student learning difficulties situated in context. Discourse analysis as a multi-lateral approach is a promising method to reveal the mechanisms of mathematical learning in complex contexts.

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CALCULUS STUDENTS' UNDERSTANDINGS OF THE CONCEPTS OF FUNCTION TRANSFORMATION, FUNCTION COMPOSITION, FUNCTION INVERSE AND THE RELATIONSHIPS AMONG THE THREE CONCEPTS

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The concept of function is fundamental in the learning of mathematics where good understanding equips students with more ways of problem solving. In addition, the function concept is a unifying concept in modern mathematics (Leinhardt, Zaslavsky & Stein, 1990), central to different branches of mathematics (Kleiner, 1989), and essential to related areas of the sciences (Selden & Selden, 1992). A deep understanding of functions and the ability to identify and to switch from one form of a function representation—graphical representation, tabular representation, verbal representation or equation representation—to another are critical and allow students to see relationships, develop a better understanding, broaden and deepen one's understanding, and strengthen one's ability to solve problems (Even, 1998). Further, a strong understanding of the concept of a function is essential for understanding calculus, a critical course for the development of future scientists, engineers and mathematicians (Breidenbach, Dubinsky, Hawks & Nichol, 1992; Carlson, Oehrtman & Thomson, in press). Researchers investigating students' understanding of functions have found that many students, including pre-service teachers, have a limited understanding of functions.

Function ideas are covered heavily in precalculus to prepare students for calculus. Among the function ideas covered in precalculus, are function transformation, function composition and function inversion. These three related concepts are normally taught separately in precalculus and the relationship among them is seldom discussed. Despite this, when students get to calculus, they are expected to not only know and be able to use these concepts but to also be able to apply them in calculus problems and exhibit flexibility in their view of the three concepts. I argue that flexibility among the three concepts and the ability to visualize one of them when working with the other (e.g., a transformation when working with a composition or an inverse) can enable a student to develop rich relationships and the ability to solve problems. While much research has been conducted on students' understanding of functions, little attention has been paid to students' understanding of these three concepts and students' view of how the concepts are related. This study addresses the question: *What are calculus students' understandings of the concepts of function transformation, function composition, function inverse and the relationships among the three concepts?*

This study employs both qualitative and quantitative methods, and is influenced by a constructivist theory of learning and a framework of flexibility (Moschkovich, Schoenfeld, & Arcavi, 1993). There are two sets of participants: college calculus 1 students and high school AP calculus students from institutions in the northeastern United States.

This poster draws on data collected from the participants in a two-step process. In the first phase, a 20-question questionnaire was given to 100 students in the first course in the calculus sequence (including high school students enrolled in AP calculus) gathering data about students' mathematical background, demographic information, as well as their understanding of the concept of function. Based on analysis of these questionnaires for summary statistics, 16 of the 100 respondents were interviewed then in an individual task-based setting (Goldin, 2000). The participants' work (including transcripts of the interview)

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will be analyzed inductively using a combination of grounded theory and a framework of flexibility. Data will be represented and discussed in the poster.

COLLEGE LEVEL STUDENTS' REASONING OF AN OPTIMIZATION PROBLEM: HISTORIC HOTEL MEA (MODEL-ELICITING ACTIVITY)

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Historic Hotel problem (a model-eliciting activity) was implemented at college level. Math major students' and pre-service science and math students' thoughts about an optimization problem revealed through Historic Hotel MEA. The use of calculus and other mathematical concepts appeared in students' ways of thinking is presented in the study. Historic Hotel MEA provided multiple opportunities for formative assessment and made students' thinking of optimization visible.

Model-eliciting activities (MEAs) are based on real-life situations where students, working in small groups, present a mathematical model as a solution to a client's need (Zawojewski and Carmona, 2001). The problem of Historic Hotels is an MEA in which students are asked to develop a mathematical model to maximize profit that can be calculated with a quadratic equation. In the problem, there is a client who inherited a historic hotel and does not have management skills. He wants to determine the rate per room where he was told by the previous owner that all of 80 rooms are occupied when the daily rate is \$60 per room; the rate per room increases by \$1 for every vacant room; and each occupied room has a \$4 cost for service and maintenance. The problem can be solved by using quadratic formulas, 1st and 2nd derivative method, or simply looking at the profit for each value. In this study, college level students are asked to solve this optimization problem to see possible their ways of solution to the problem. Two groups of students were given this non-traditional problem. The first group was a calculus class where the majority of students were engineering major and the second one was an education course required for teaching certification where students were either science or math major. There were 23 groups of 3 in calculus class and 11 groups of 3 in the other.

After the problem was solved in groups, students were asked to present their models to their class-mates. Discussions about different solutions were done after each presentation. Authors and their colleagues observed students and took field notes while students were working on the problem and discussing their results afterwards. This particular type of MEA made students' thinking about optimization as a mathematical concept visible to both their peers and teachers. Formative assessment (Black & William, 1998) cycles that students went through appeared in the problem solving process. Students' final works are analyzed in terms of the function they used, variables and their definitions, graphical representations, and generalizations.

Analysis of solutions of both groups to the problem and observations by authors and their colleagues will be reported in the poster. In addition, math major students' and pre-service teachers' ways of thinking will be discussed. Following questions will also be touched upon in the poster:

- How much calculus knowledge do college level students associate with solving an optimization problem?

- In what ways, did college level students model the maximization of profit in the given context?
- What symbols and tool did students use in solving the problem?
- Were there any differences in the method used between calculus students and students in teacher certification program?

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CONTINUITY AND DISCRETE MATHEMATICS: COUNTING SEATS AT A STADIUM

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Most students have problems to define what continuous or discrete phenomena are. For some students they are the same, for some others continuity can be supposed at any time. We present some activities developed to teach how different those situations are.

Mathematics in context presents an abstract theory in a close-to-the-student way, students then are able to try and experiment with new lines of approach.

The use of Mathematics in Context as a framework allows seamless use of technology such as graphing calculators and computer packages to investigate, verify results, and consolidate the key concepts. This approach is useful as well with regular use of higher technology, or with occasional use. Students with internet access at school, after school, or at home, can freely surf to find out additional information.

Counting Seats

We designed three activities under the Mathematics in Context point of view. Each activity was intended as a continuation of the previous one.

The activity was named "A rock star at the stadium." We supposed that a rock star would present a concert at an Olympic Stadium, that kind of stadium is a rectangle with rounded corners (the "head"). The rows of seats at the head have different number of seats, and this number increases as we go up, for example the first row has 50 seats, the second has 52 seats, the third has 54, etc.

In the first activity the students have to calculate the number of seats at a given row and the total number of seats of the head given the total of rows at the head.

Some answers were just as we expected (using calculus) but there were some solutions that used:

- Geometry, a rectangle and a triangle.
- Geometry, a parallelogram.
- Algebra.

The second activity was intended to explore the use of calculus and to show the errors produced by this approach. Students had to represent the rows of seats as a sum of rectangles and to find a liner function that represents those rectangles. Finally the had to integrate that function and to compare the results of using geometry or algebra and the use of calculus.
As a result of these activities the students realized that integrals are for continuous function and do not "count" discrete objects.

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ATTENDING TO STRUCTURE AND FORM IN ALGEBRA: CHALLENGES IN DESIGNING CAS-CENTERED INSTRUCTION THAT SUPPORTS CONSTRUING PATTERNS AND RELATIONSHIPS AMONG ALGEBRAIC EXPRESSIONS (1)

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We report on a study with adult algebra learners that employed an activity to explore patterns and relationships between expressions that might suggest the sum/difference of cubes identity. Our report describes the activity and discusses aspects of expressions to which students attended. Our results to point to a need for refinements to this activity that might better support re-invention of the identity.

Although it has been shown that technology can be used to support student understanding of calculus (Heid, 1988), there continues to be debate about the role that technology can and should play in algebra learning (NCTM, 1999). We report on a study with adult learners that employed an activity involving a computer algebra system (CAS) to explore patterns and relationships between expressions that might suggest the sum/difference of cubes identity $(a + b)(a^2 - ab + b^2) = a^3 + b^3$. This activity was inspired by the work of Goldenberg (2003), who outlined a similar activity designed to support students in re-inventing the difference of squares identity $(a - b)(a + b) = a^2 - b^2$.

In an effort to understand the sources of their difficulties in deriving the sum/difference of cubes identity, our report explores aspects of expressions to which students attended. We detail the instructional activity and its outcomes, and we suggest refinements to the activity that might better support this population of students in achieving the intended learning goals.

Setting and Participants

Our study was conducted in the context of an intact intermediate-level algebra course offered at a large urban university in the Pacific northwestern United States. The course is designed for students who have not had Algebra II or who require a review of elementary algebra concepts and techniques. The students in this class have a wide variety of dispositions and experiences related to mathematics, a variety of educational backgrounds and interests, and a wide spectrum of ages. The 15 students who participated in our study reflect this diversity.

We initiated our study during the third week of classes, after students had developed some experience in multiplying polynomial expressions and factoring perfect square trinomials and differences of squares. The data we report on are drawn from students' individual written work on the first two parts of the activity. We examined their work for evidence of patterns they might have construed from the sum/difference of cubes expressions generated by the CAS and for connections they drew between those expressions and their factored forms.

Summary Description of the Task

The Cubes activity was designed to support students in deriving the identities for factoring sums and differences of cubes. The activity began by having students explore patterns arising from the multiplication of a given sequence of a binomial and a trinomial whose product results in a sum/difference of cubes. This was followed by having students use those patterns to generate the sum/difference of cubes identity. Specifically, students were presented with factored forms of five sums and differences of cubes, each of which they were asked to multiply out using the EXPAND command on their TI-92 calculator. Students recorded the results of these expansions in the appropriate cells of a table (Figure 1), and they were directed to investigate whether the "indicated multiplication of factors produces interesting results".

Factored form	Expanded form displayed by the calculator
1. $(x + 2)(x^2 - 2x + 4)$	

Figure 1. A portion of a table that students completed in the first part of the activity.

The intent in having students use the CAS to expand the products in this part of the activity, rather than requiring them to do so by hand, was to focus their attention on the form of each final expression, and to reduce the potential of diverting their attention to the mechanics of algebraic manipulations. The next part of the activity aimed to orient students' attention towards patterns in the expanded expressions recorded in their table. To that end, the students were directed to take note of the form of each expanded result produced by the CAS and to "describe how this form relates to that of the corresponding factors".

Analyses and Results

Our analysis of the data unfolded in a sequence of interrelated phases. Following Saldanha and Kieran (2005), we began by listing relationships of form and structure among expressions to which the students might be expected to attend. We then examined students' responses for evidence of explicit or implicit attention to these aspects. On the basis of this first examination we then revised and refined our initial list of aspects to arrive at a comprehensive list of them. The authors then each re-examined the data independently, coding it according to the dimensions in this list and documenting the frequency of their occurrence in the students' responses across task questions. In a final phase, the authors compared their code assignments and resolved their few differences through a process of negotiation that involved re-examining relevant parts of the data whenever necessary. This process converged to a 100% agreement in the coding of the data.

The table below displays a sampling of our results; it describes some of the aspects of the expressions to which we hoped the students would attend and it gives the frequency with which students actually attended to each of them. The aspects are listed in descending order, from the most to the least salient for students, as indicated by our analysis.

Aspect of expressions	**Students who attended to aspect**	
	#	%
Expanded version of each expression is shorter than the factored version	12	80%
The first term of the expanded version is the product of the first term in the binomial and the first term in the trinomial $(\boldsymbol{a} + b)(\boldsymbol{a}^2 - ab + b^2) = \boldsymbol{a}^3 + b^3$	10	67%

The first term of the expanded version is a cube $(a + b)(a^2 - ab + b^2) = \boldsymbol{a^3} + b^3$	8	53%
The second term in the expanded version of the expression is a cube $(a + b)(a^2 - ab + b^2) = a^3 + \boldsymbol{b^3}$	2	13%
The last term of the trinomial is the square of the last term in the binomial $(a + \boldsymbol{b})(a^2 - ab + \boldsymbol{b^2}) = a^3 + b^3$	2	13%
The middle term of the trinomial is the product of the two terms in the binomial (sign withstanding) $(\boldsymbol{a} + \boldsymbol{b})(a^2 - \boldsymbol{ab} + b^2) = a^3 + b^3$	1	7%
The first term of the trinomial is the square of the first term in the binomial $(\boldsymbol{a} + b)(\boldsymbol{a^2} - ab + b^2) = a^3 + b^3$	0	0%

Discussion and Conclusion

The results suggest that, in general, the students did not attend to most of the aspects of the expressions that we deemed important. Due to space constraints, we only discuss a few of the aspects and results in this paper. For instance, of the twelve aspects we targeted, only four were noted by more than 50% of the students. Only one aspect—that the expanded forms of the expressions are shorter than their factored forms—was noted by more than 75% of the students. It is important to note that this modal aspect is arguably the least useful one in terms of helping students generate the intended identity. Moreover, the characteristics apparently noted by the fewest students centered on relationships between the terms in the factored form of the expressions, which are arguably the most useful for generating the identity.

These findings suggest that the expanded versions of the expressions were much more salient for students than were the factors themselves or relationships among the factors. This state of affairs is consistent with, and might therefore be attributed to, students' being inattentive to how the expanded forms of the expressions arose from the factored forms. This hypothesis, in turn, raises questions about whether our activity prompts adequately provoked students to reflect on *why* the expanded forms of the expressions are shorter than their corresponding factored forms. Indeed, since the students did not attend to aspects that could form the basis of a derivation of the sum/difference of cubes identity, our activity sequence stands to benefit from prompts that orient students to reflect on why only certain terms appear in the expanded form of the product of expressions. Changes such as these might better support this population of students in attending to crucial aspects of these expressions and structural relationships among them.

Endnote

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DEVELOPING PRESERVICE TEACHERS' MODELS OF ALGEBRAIC GENERALIZATIONS

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This paper describes elementary preservice teachers' models of connecting algebraic generalizations with a related contextual problem. After a pretest, subjects engaged in model-eliciting activities by finding objects to represent a generalization of the form y=ax+b, and an accompanying survey sheet regarding the intervention (Doerr & Lesh, 2003). Results indicated refinement and expansion of models despite initial difficulty with authentic applications.

There is a pressing need to inform and improve algebraic instruction (Rand Mathematics Study Panel [RAND], 2003). Although research on the teaching of algebra has recently increased (Kieran, 2004, in press), the research base on teachers' knowledge regarding algebraic instruction is still quite limited (Doerr, 2004; Menzel, 2001; RAND, 2003). Calls for reform discuss a variety of expanded views of algebra including a focus on algebra as part of the elementary curriculum and generalizing patterns set in a meaningful context (Kaput, 2000; NCTM, 2000). Research shows that elementary level students are capable of algebraic reasoning (Kaput & Blanton, 2001a), yet many elementary level preservice teachers (Zizkas and Liljedahl, 2002) and classroom teachers (Kaput & Blanton, 2001a, 2001b) may not understand the role of algebraic generalizations nor ways to connect generalizations to an authentic context. We need to study how preservice teachers *can develop* a robust understanding of algebraic generalizations.

To address this goal, we examined preservice teachers' responses to writing a contextualized word problem to accompany a given algebraic generalization of the form, $y = ax + b$. This paper focuses on the preservice teachers' models of algebraic generalizations, and specifically, their ability to connect a given generalization to an authentic context. Using an iterative process supported by our theoretical framework, we conducted a detailed study that addressed the questions: (1) How do preservice elementary teachers develop and refine their interpretations of a given algebraic generalization? (2) How do preservice teachers' attitudes about writing algebraic generalizations change in response to instruction set within a context?

Theoretical Framework

Subject matter knowledge is critical to successful teaching (Ma, 1999; Hill, Schilling, & Ball, 2004). Recent reforms call for increased emphasis on algebraic reasoning at the elementary level, yet most current preservice teachers' preparation in algebra occurred at the secondary level. We know that students can extend a pattern more readily than they can generalize it (Orton & Orton, 1999; Zaskis & Liljedahl, 2002). Classroom teachers participating in staff development projects do successfully enhance algebraic instruction and may successfully "algebrify" the elementary curriculum (Kaput & Blanton, 2001a, 2001b). However, because preservice elementary teachers likely have not had the benefit of such dynamic experiences, it is critical to examine their existing models of algebraic generalizations and explore activities that might perturb and expand their understanding. Hence, we adopted a models and modeling perspective of teacher

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development (Lesh & Doerr, 2003) that provided a framework for constructing tasks and analyzing teachers' understanding as they test, revise, refine, and extend their algebraic thinking during a mathematics methods course. This research focused upon preservice teachers' models of interpretation for connecting algebraic generalizations with an authentic context.

Methodology

This study followed a pretest/intervention/posttest design accompanied by a culminating reflective survey of preservice elementary teachers (N = 58). First, the preservice teachers completed a pretest where they were asked to write an authentic contextual problem for the equation y=2x + 3. A few weeks later, students worked with "algebra rules object boxes" (Rule & Hallagan, 2007), instructional materials consisting of sets of related items that illustrated cases of an algebraic generalization accompanied by two cards for each object-set. Four object-sets and 8 cards were provided in a plastic shoebox to each group of four preservice teachers. The fronts and backs of two cards with algebraic generalizations of the type y=ax + b that accompanied an example object-set (with the object-set shown on the reverse side of one card) are shown in Figure 1.

Figure 1. Cards for the star trim problem. Fronts are to the left; reverse sides to the right.

Star Trim Story Problem

A company makes trims for fancy costumes. They want to create a product line of symmetrical star trims in several sizes. If there is always a center large star surrounded by a symmetrical arrangement of stars in a line, determine the rule for the total number of stars.

Let z be the total number of stars in the trim design.

Let n be the number of stars on one side of the central star

$$z = 2n + 1$$

n = 3

n = 2

n = 1

Stars on trim

Each trim piece has 2n stars plus the middle star to produce 2n + 1.

Preservice teachers first matched the front of a card that presented a story problem to the corresponding object and attempted to define the variables, using "z" and "n". They checked their work by looking at the reverse of the card (shown on the right side of Figure 1). Then they created an algebraic generalization using the two variables and searched for the card with the same equation. Again, they checked their work by reading the back side of the card. After spending time exploring the instructor-made object boxes, subjects were assigned to work in a group to find or make a set of related objects (showing the first three or four iterations of the pattern determined by the equation) that followed an algebraic rule of the form $y = ax + b$. Students also wrote a contextualized problem that supported the object and equation. Then, each group of four preservice teachers shared their algebra rules object box with another group of preservice teachers. This repetitive cycle allowed preservice teachers multiple opportunities to test, revise, and refine their thinking about the generalizations in the object boxes.

After the subjects created their materials, they engaged in a reflective activity. Subjects were asked to: (1) State what you learned about writing algebraic generalizations this semester; (2) Describe your current feelings toward writing generalizations and teaching algebra to elementary students as compared to the first day of the semester; and (3) List pros and cons of making the algebra rules object boxes. Finally, the preservice teachers were given an identical posttest about six weeks later. Consistent with a models and modeling approach, this methodology left a trail of artifacts documenting the development of the preservice teachers' models for generating authentic contexts and generating algebraic generalizations that matched their context.

Results

Table 1 shows the rubric used to score student solutions, example student responses and pretest-posttest scores. Half of the preservice elementary teachers performed poorly on this pretest exercise, indicating their lack of understanding of algebraic generalizations, although we were encouraged that almost 40% showed proficient or basic knowledge. Posttest results indicated marked improvement of almost one score point (posttest mean of 1.82 compared to the pretest mean of 1.05; a calculated t-test statistic indicated the mean score on the pretest was significantly smaller than the posttest mean (P-value < 0.001). On the posttest, the percentage of preservice teachers evidencing proficient or basic performance increased to over 57%, while the percent of those with poor performance decreased to 12.5%.

Preservice teachers generated many interesting authentic examples for $y = ax+b$. Four representative examples are shown in Figure 2. Additional examples are available in Rule and Hallagan (2007). Most of the materials generated by preservice teachers were mathematically correct, although a few errors were found. One group produced the following problem for $z = 4n+3$ that illustrates the two most common errors. It was accompanied with a board showing just three foam shirts, each with four buttons and one extra button attached to the side of each shirt. "A Shirt Story Problem: A mother is buying her son three new button-up shirts. Each shirt has four buttons. The shirts also each come with an extra button. Determine a rule for the number of buttons there are altogether." The first error was that only one case (the case in which three shirts were purchased) was shown. Several other preservice teachers also failed to show multiple cases of their story problems. The second error concerned the correct equation for this situation, which was $z = 5n$ because each shirt had 5 buttons (4+1 extra). If the problem were re-written to say that one extra button was given to the customer when one or more shirts were purchased, the generalization z = 4n+1 would have applied, but only one extra button representing the constant term should have been illustrated.

Determining whether an item in a story is a constant or a multiplied variable is difficult for preservice teachers. On the pretest, many preservice teachers produced story problems for z=2n+3 that indicated similar conceptual problems. “You have two parents. They had a number of biological kids plus three adopted children. How many were there altogether?” This story fits z = 2+n+3. Here is another example with the same mistake: “Two boys were playing basketball and then n more boys decided to play and then three more boys joined them. How many boys were playing altogether?” These types of errors decreased markedly on the posttest, indicating that preservice teachers were learning to differentiate constants from variables as they worked through the activities.

Table 1. Rubric for scoring pretest and posttest problem with example student responses and scores

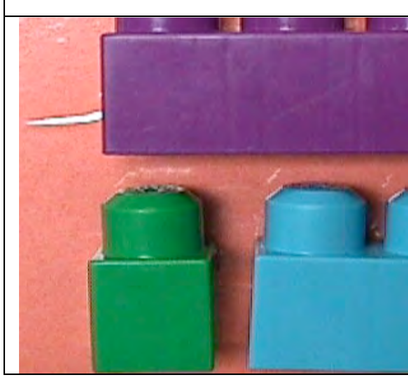
Score	**3**	**2**	**1**	**0**
Performance Level	**Proficient Performance**	**Basic Performance**	**Emerging Knowledge**	**Poor Performance**
Criteria for Judgment	Correct contextual problem using a variable to fit the generalization; Contextual problem is presented by giving several solutions and asking for the generalization or presented as a situation involving a variable.	Correct contextual problem using a variable to fit the generalization, but with poorly stated question; **or** Contextual problem gives values for all variables; it is not a generalization.	Numbers in contextual problem match numbers in given equation, but operations are wrong; **or** Context almost fits, but would make more complex or different equation.	Context" is merely a translation of the numerical expression into English words with or without added objects; no real "context"; **or** Student merely solved for "n"; **or** Incomplete, no attempt.
Example Contextual Problems that were placed in the category	Bicycles have 2 wheels and tricycles have 3 wheels. How many wheels are there in total if there are n bikes and 1 tricycle?	The factory packaged 2 toothbrushes in each box. They had already filled 5 boxes and there were 3 toothbrushes left. How many toothbrushes were there altogether?	2 boys were playing basketball and then n more boys decided to play and then 3 more boys decided to play. How many were there altogether?	Two more than 2 added to 3 is n. Two pennies times n pennies plus three pennies equals what number?
Percent of Pretest Responses	16.0%	23.2%	10.7%	50.0%
Percent of Posttest Responses	37.5%	19.6%	30.4%	12.5%

An analysis of the results of the reflective survey revealed that the algebra rules object box activities had many positive effects, as shown in Table 2. Students learned new techniques to determine algebraic generalizations, better understood the importance of writing algebraic generalizations, and improved in their abilities to generate algebraic generalizations. Preservice teacher comments included, “I learned that algebra isn't really that bad,” and “You should let students discover them not lecture.” In response to question 2, describe your current feelings toward writing generalizations and teaching algebra to elementary students as compared to the first day of the semester, students wrote 139 different comments. Half of these comments indicated a somewhat more favorable feeling towards algebra, and forty-nine percent of these comments indicated a clearly more favorable feeling towards writing generalizations and teaching algebra to elementary students. Students’ responses included such comments as, “they take time and patience to teach and learn. Practicing helps and I'm not so afraid of them anymore,” and

> I'm a lot more comfortable with working and teaching algebraic problems primarily due to the exhaustive practice in and out of class with them. Even at first figuring out how to write an algebraic problem was very difficult but through various methods of practice the concept became clear.

Students’ attitudes were favorable towards teaching generalizations at the conclusion of the project.

Figure 2. Example materials and story problems created by preservice teachers.

	
z=2n+1 Action Figures Problem Jimmy starts out with one firefighter in a collection. Each week, he adds two more when he gets his allowance. If he does this for many weeks, what rule could you use to figure out how many he has total? (Sean O’Hara)	z=3n+2 Free Fruit Problem The grocer at a store is having a special this week: buy fruit in any multiple of three and receive a one-time gift of two pieces of fruit free! Tell a rule to find the total pieces of fruit received for the purchase of different sets of fruit (Megan Green).
	

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z=2n+2 Surface area of Blocks A company makes building blocks that have 4 flat sides, a hollow bottom, and rounded tops. Find a rule for the total area of the flat sides in square units if one small side is one square unit (Joe Wasiluk).	z=3n+2 Flower Garden Problem Maria has a flower garden in her front yard. One day, 2 flowers came up. Every day after that, 3 more flowers came up. How could Maria figure out how many flowers there would be on a certain day in the future? (Sara Faughnan)

Finally, in response to question 3, list pros and cons of making the algebra boxes, students provided many insights. In particular, they recognized the value of these hands-on, concrete materials in developing their confidence and skills with algebra and their importance for the students they will be teaching. Example comments include: “Gave us great practice,” “Helped clarify the concepts,” “Lets students solve problems and assess if they know the material,” “I became more confident in solving algebra problems,” and “Great visual for kids to understand algebra problems.” Many also commented on their enjoyment of the activities and the chance to produce creative materials with such comments as: “Creative and interesting,” and “Fun way for students to learn the material.” They also remarked on the efficacy of group work: “There was not as much stress doing it as a group,” and “Working as a group allowed us to share ideas and strategies.”

Limitations of the algebra rules object box projects mentioned by students included the difficulty of these concepts (“Very complex,” and “Hard to come up with a story problem”), the amount of effort needed to produce the teaching materials (“Spent a lot of time,” and “Takes patience to complete”), and problems with coordinating group members (“Group work not shared equally”). Students also mentioned that some aspects of the project were confusing, making materials cost money, and sometimes the object-sets were hard to construct.

Table 2. Survey results

Question		**Preservice Teacher Responses**	**Percent of All Responses to this Question**
1. State what you learned about writing algebraic generalizations this semester.		Techniques to help write a generalization	55%
		Understanding of the importance of writing generalizations	24%
		Improved in ability to write generalizations	21%
2. Describe your current feelings toward writing generalizations and teaching algebra to elementary students as compared to the first day of the semester.		Now have a somewhat more favorable view of algebra	50%
		Now have a markedly more favorable view of algebra	49%
		Continue to have a negative view of algebra.	1%
3. List pros and cons of making the algebra rules object boxes.	Pros	Helped develop confidence and skills	28%
		Teaches important skills children need	17%
		Able to use creative skills; creative activity	14%
		Liked using manipulatives; activity was concrete	9%
		Enjoyed doing group work	9%
		Fun	9%
		Hands-on activity	8%
		Useful in future classroom	5%
	Cons	Difficult concepts	25%
		Took too much work, too much time	20%
		Problems with group members and a group project	20%

		Confusing	17%
		Expensive	12%
		Hard to make; construction issues	6%

Concluding Remarks

Consistent with prior research of preservice elementary teachers' ability to generalize patterns, these subjects had difficulty in making authentic connections for a simple algebraic expression. This may indicate that this group of subjects does not necessarily have the knowledge to "algebrify" the elementary curriculum. Although our results pose a concern about the ability and readiness of preservice teachers to implement many of the current calls for reform of elementary algebraic instruction, we are hopeful that the *nature of the activities* will help preservice teachers refine and extend their models of algebraic generalization.

We consider the favorable comments made by students in response to the reflective survey to be one of the successes of the project. One of the perceived drawbacks to implementing the algebra boxes was the amount of class time it took to demonstrate the materials, and then have the students try out each others' materials. However, the study was intentionally designed to give the preservice students opportunities to engage in testing, revising and refining their understanding of writing algebraic generalizations. In addition, because attitudes may contribute in part to preservice teachers' willingness to teach algebraic material, we feel it was well worth the effort. As well, many students made favorable remarks about actually creating the algebra boxes. Although some students found the effort confusing, time consuming, and difficult, the pros seem to outweigh the cons, particularly in light of the results of the posttest.

Preservice teachers need extensive experiences in developing their own understanding of the role of algebraic generalization at the elementary level as well as models of generalization that will help them enhance children's understanding in the classroom. This protocol needs to be tested further and a cohort of preservice teachers needs to be identified and followed into the early years of their practice to examine their development in teaching algebraic concepts.

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JUNIOR GRADE STUDENTS' UNDERSTANDING OF LINEAR FUNCTIONS: GROWING PATTERNS LINK SYMBOLIC AND GRAPHIC REPRESENTATIONS

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Twenty-one Grade 6 students were assessed eight months after participating in an experimental lesson sequence in which growing patterns acted as a mediating representation linking symbolic notation to graphic representations of linear functions (y=mx+b). Results indicated students retained an understanding of the link between the coefficient and slope and the constant and y-intercept. In addition, students were able to utilize this understanding to predict how symbolically represented functions would behave when graphed, and to reason about functions that have a negative coefficient/slope.

In this study, I examined the affordances of using geometric growing patterns as a mediating representation to link symbolic notation to graphic representations in order to develop young students' conceptions of linear functions. This study is part of a larger 3-year study in which we have investigated young students' abilities to work constructively with growing patterns as a means of developing an understanding of mathematical structure. The present study reports on the results of a retention test given to Grade 6 students, eight months after they participated in an innovative lessons sequence designed to merge symbolic, geometric, and graphic representations of linear functions.

When considering the domain of linear functions, mathematics educators recommend that students be introduced to various representational forms in order to effectively use these representations as a means of developing conceptions of quantitative relationships (e.g., Janvier, 1987; Moschkovich et al., 1993). It is the interaction among different representations that is necessary for developing an understanding of the connections between symbolic and graphical representations, and an ability to predict changes in the graph of a function that results from transformations of the expression that defines it (e.g. Bloch, 2003). Studies have documented the difficulties that older students encounter when exploring these connections (e.g., Bloch, 2003). For example, when graphing $y=mx+b$, researchers have noted that the connections between m and the slope of the line, and b and the *y-intercept* are not clear (Bardini & Stacey, 2006). Students have difficulty predicting how changes in one parameter will affect the graphic representation, and often conflate m and b, not realizing these properties of lines are independent (Moschkovich, 1996). In addition, studies with older students have shown that there is a propensity to adopt a point-wise approach when considering graphically represented functions, that is, students can plot and read points on a graph but have difficulty thinking of a function in a global way. A point-wise approach can preclude an ability to understand the meaning of slope as representing the rate of change, and results in a relatively simple focus on specific points (Schoenfeld et al. 1993, p. 87). Difficulties outlined in the literature may be due in part to the fact that students are introduced to graphic representation only after they have been taught to think of functions as interpretations of algebraic expressions (Arcavi, 2003) and so the graph is

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considered neither as representing a function, nor as a representation of rate of change, but rather an illustration of the relationship between two algebraic expressions.

In this paper I examine the retained understanding demonstrated by students who participated in a unique instructional sequence that prioritizes visual representations in the forms of growing patterns and graphs, and integrates these with symbolic representations in the form of equations.

Theoretical Framework

The theoretical framework for the three year research project is based on the work of Case and colleagues (e.g., Case & Okamoto, 1996) which has shown that merging numerical and visual representations provides a new set of powerful insights to underpin the early learning of a new mathematics domain. In addition, I looked to the work of researchers who propose that prealgebra learning should centre on the introduction of functions through visual objects (e.g., Noss, Hoyles & Healy, 1997; Yerushalmy & Shternberg, 2001). Our previous studies have shown that growing patterns offer a powerful vehicle for understanding the covariational relations among quantities that underlie mathematical functions (e.g., Moss & Beatty 2006).

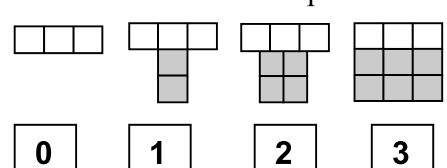
Methodology

Setting and Participants

The data from this study come from a follow-up to the Year 2 of study of Grade 5 students, which was conducted over a 3 month period in 2005/2006 and involved 58 Grade 5 students from 3 classrooms; 1 in an inner-city public school, 1 in a suburban school, and 1 in a university lab school. The follow-up to this study was conducted in 2007 when participants were in Grade 6, and involved 21 students from the laboratory school. All 21 participants had taken part in Year 2 of the study.

Lesson Sequence

During the Year 2 study, students participated in 14 lessons, 5 of which focused specifically on graphic representations of functions. Students initially learned how to build geometric growing patterns based on linear functions of the form $y=mx+b$, with m represented by blocks that increased by a specific number in each successive position of the pattern, and b represented by repeating a constant number of blocks in each position. For example, the pattern below is a representation of the equation $y=2x+3$, where $2x$ is represented by grey blocks equal to 2 times the position number, and +3 is represented by 3 white blocks at every position. The students call this a "times 2 plus 3 rule" because the number of blocks at each position is equal to the position number x2+3 (number of blocks=position number x2+3). The position number of each iteration was explicitly linked to the pattern with the corresponding number of blocks in order to clearly differentiate both variables in the pattern (position number, the independent variable; and number of blocks, the dependent variable). Both components of a composite linear function were delineated by using two different colours of blocks so that students would develop an understanding of the coefficient (the part of the pattern that grows) and the constant (the part of the pattern that stays the same).



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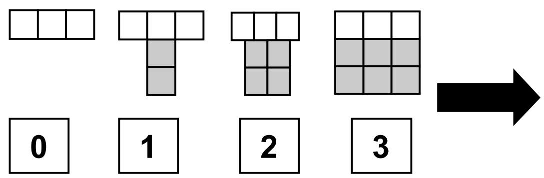
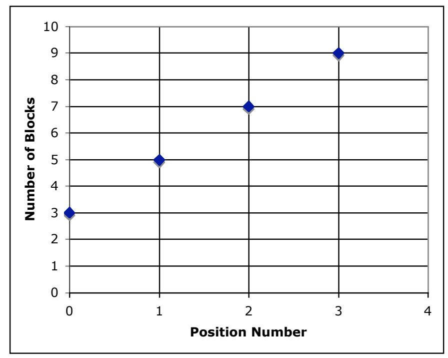
Students then learned to construct linear graphs based on growing patterns. The position numbers of the growing patterns were mapped onto values along the x-axis. The number of blocks was represented by values along the y-axis. The y-axis also represented the "0th position" of a growing pattern - the value of the constant -which was graphed as the y-intercept. When graphing a pattern, students were given functions such as "number of blocks=position number x2+3" and asked to build a growing pattern based on the rule. They were then asked to draw a dot on the graph in order to represent how many blocks were at each position of the pattern they had built. Students were, therefore, graphing "ordered pairs" – position number, number of blocks – without being explicitly asked to do so.





The graphing activities involved students building and graphing the 0th, 1st, 2nd, and 3rd position for three patterns that had different coefficients and the same constant (y=x+1, y=3x+1, y=5x+1). Students were then asked to build three patterns for functions that had similar coefficients and different constants, and predict what the graph would look like (y=3x+1, y=3x+3, y=3x+5). In each case students were asked to compare similarities and differences between the functions, the patterns, and the graphs both within and between the three sets of functions. Thus, students were given the opportunity to explicitly consider how changes in one representation affected the other two representations.

Data Sources

The Grade 6 students were given a pencil and paper Graphing Survey. Items were designed to assess the kinds of misconceptions and difficulties reported in the literature with respect to students' understanding of the links between symbolic and graphic representations of functions. The survey assessed four areas of ability: 1) graph a linear function or discern the function of a given graph; 2) understand the link between m and slope and between b and y-intercept, and determine the graphic outcome of changing m or b in a function; 3) predict functions that would intersect; 4) offer a function for a negative slope (not part of the initial teaching sequence). Students completed the survey during one of their mathematics classes. In addition, six students representing different levels of mathematical achievement (based on teacher rating, report cards, and scores on the standardized Canadian Test of Basic Skills) participated in videotaped individual clinical interviews. All videos were transcribed and coded to assess students' retained levels of understanding.

Results

Results of the Graphing Survey indicated that eight months after instruction, students maintained an understanding of graphic representations of linear function. The table below illustrates the mean scores for each section of the survey for students of high, medium and low achievement levels. Most students were able to identify functions from graphic representations and construct graphs from functions. In addition, most students were able to predict functions that would yield parallel slopes, and slopes that differed in steepness, and could predict changes in slope or *y*-intercept from changing *m* or *b*. Students were also able to predict the intersecting point of two functions. Finally, students were able to reason about functions that would produce negative slopes.

Achievement Level	Discern/Graph a rule (/3)	Slope and *y*-intercept (/13)	Intersecting Lines (/5)	Negative Slope (/2)	Total Mean Score (/23)
High (n=4)	3	12.5	4.5	1.75	21.75
Middle (n=10)	2.9	10.5	4.3	0.9	18.8
Low (n=7)	2.7	7.4	3.2	1.3	14.7

Clinical Interviews

To understand more about students' reasoning I analyzed videotapes of the clinical interviews. In this paper I will report on the reasoning of two students, Ellie, a low-achieving student, and Cassandra, a higher achieving student, both of whom exemplified the kind of thinking found during all six interviews (and in the Graphing Surveys). Ellie's interview was selected to illustrate both the retention of concepts such as the link between the coefficient and the steepness of slope, as well as the ability to make predictions about how different functions behave when graphed. Cassandra's interview illustrates how some students were able to utilize concepts developed during the teaching sequence and extend these to be able to reason about negative slopes, which were unfamiliar to them.

Ellie's Interview – Retention and Prediction

During her interview, Ellie used the term "position number" to refer to values along the *x*-axis, and "0" or the "0^{th} position" to indicate the *y*-axis. She used "times table number" to indicate coefficient, and "addition number" to indicate the constant.

For her first task, Ellie was asked for the function of a given graph (y=5x+3). She looked first at the *y*-intercept, then at the next *x* value (position 1) to determine the coefficient. "*OK this rule is something plus 3 [pointing to the* y*-intercept] because at 0 it starts at 3, and then so 1,2,3,4,5 [counting up from 3 to the next dot on the "position 1 line" (1,8)] so it's 5 – times 5 plus 3 because I counted how many more it would be. If it was without this [pointing to y-intercept] it would just be up to here [pointing to (1,5)] then 1,2,3 [counting up to the dot at (1,8)]. And so it's times 5 plus 3."* Ellie was able to determine the constant by knowing that, just as in pattern building, at the 0 position, or *y*-axis, only the constant is represented. She then checked the difference between the *y*-intercept and the next value (1,8) in order to ascertain how much it was "going up by" – this gave her the coefficient. Ellie demonstrated an understanding that the values of the graphed function would be lower if there was no constant in the rule – the constant "raises" the graphed line up by 3 so that both the coefficient and the constant are responsible for the position of the function on the graph.

Ellie was asked to predict a function that would give a line parallel to y=5x+3. "*A parallel rule for times 5 plus 3 would be times 5 plus 6, or more than 3…any other number other than 3.*

Because times 5 plus 6 [*pointing to the y-intercept of 6*] *would just be there* [*sketching a parallel line*]. *You can't change the times table number because if you do it will be steeper and it won't be parallel."* Ellie began with a partial generalization that in order to get parallel lines for a function, the value of the constant would have to be higher, but then amended this to include any other constant than the one in the given function. Ellie used a point-wise approach when considering the graph, for example, the link between changing the constant in the rule and the resulting change in y-intercept on the graph. However, she also seemed to be able to consider the graph in a more global manner, particularly when she sketched the line that would result from graphing the function $y=5x+6$.

When asked for a function that would give a steeper line, Ellie again responded with a generalized conjecture, this time not supported by any specific examples. Her answer demonstrates an integrated understanding of the how both parameters of a composite function are represented graphically. "*A steeper number than this would be any higher number than 5 would be the times table – the times part. If you have a different times table number then you would get a steeper number – a steeper graph and if you use the same times table number and a different addition number then you would get a parallel graph."*

To predict a function that would intersect $y=5x+3$, Ellie used two different strategies. The first was based on her initial theory that a lower constant and a higher coefficient would result in a line that intersects the given function (this is a viable conjecture given that Ellie only had experience graphing in the first quadrant). *"If you do, let's say times 9 plus 0. So it would start at 0 [pointing to the origin] then there [pointing to (1,9)] then 18 [pointing to (2,18)]. It crosses there [sketching in the line for the function y=9x]*." Once Ellie graphed the two functions she noticed they intersected somewhere between 0 and 1 on the x-axis, and so estimated that they might be intersecting at 0.6 or 0.7. "*It's not exactly in the middle, like 5 in the middle of 10, but it's just over the middle a little bit. It's not exactly 1 or 0 so you have to go in decimals*." The fact that Ellie estimated the intersecting point as a rational number indicates that, rather than viewing the graph as a series of discrete points, Ellie saw the graphed function as continuous.

Ellie was then able to integrating her theory of increasing the coefficient and decreasing the value of the constant with an ability to think of specific values in order to predict a function that would intersect at the "second position", (2,13). *"If you do times 6 plus 1 then you'll go from 1 [drawing a line from y-intercept to (1,7)] to 6 but then you have to go up 1, because it's plus 1 (1,7), and then over to here [extends the line to (2,13)] that's 12 but add 1 [points to the intersection at (2,13)]."*

For the last question, Ellie was able to reason about what it means to have two intersecting lines, based on her experience of building geometric patterns. "*If you built* x*5+3 and* x*6 +1 as patterns they would have the same number of blocks at position 2 because it intersects, so, it has the same number of blocks and… it's just like a graph and blocks are two different ways to lay it out."* Ellie recognized that a linear function underpins both a growing pattern and a graph.

Cassandra – Extending to Negative Integers

During her interview, Cassandra built on her existing knowledge of graphic representations of the form $y=mx+b$ and was able to integrate negative integers into her understanding. The students had only ever worked with positive integers in their functions because it is difficult to represent a negative number when building growing patterns.

In the first instance, when Cassandra was shown the $y=5x+3$ graph and asked for rules that would give parallel lines, she correctly identified that the coefficient would have to be the same, but that you could add or subtract anything to the coefficient and the lines would be parallel. "*Well, there's a couple of different answers, but they would still have to be multiplied by 5. But then it could be, you could add anything with the multiplied by 5. Or subtract anything, and then they will be parallel. Like you could do times 5 minus 4, for example. So that's the subtract*." This was a more general understanding of parallel lines than had been taught in the lesson sequence, during which only constants with positive values had been considered. When asked to show where a negative constant would go on the graph, Cassandra correctly added negative values to the y-axis. This was a very clear indication that Cassandra's ability to consider graphic representations was no longer anchored in concrete pattern building, but had shifted to a more abstract representation so that negative "block" values could be incorporated. When asked about adding negative numbers to the *y*-axis, Cassandra stated "*it can't stop at 0 because there's more than that – so you go below zero so you get into negative.*" This is the first stage of exploring more than one quadrant on the graph.

Cassandra was then shown a graph of $y=-3x+16$. *"It's a rule that gets smaller. I thought maybe it has something to do with what position number it is [pointing to values along the* x-*axis], so I was looking at the 0* [y-*axis*] *and I was thinking 16 minus 0 is 16, and then 16 minus 3 gives you the first position (1,13), and 1x3 is 3 and that's what we took away. So then for the second one [pointing to (2,10)] I thought 16 take away 6 is what it is (2,10), and 2x3 is 6. So the rule I came up with was [writing]* blocks = 16 minus position number x 3.*"* Based on her knowledge that a positive coefficient represents the successive number of blocks added to each position of a pattern, Cassandra reasons that a negative slope shows successive subtraction. However, instead of simply stating her rule recursively as 'minus 3 each time' she is able to link it to the position number, or value of the x-axis, and express it as an explicit rule.

Discussion

The results of the follow-up indicate that students maintained a robust understanding of the links between symbolic and graphic representations of linear functions, in particular, how the parameters of *m* and *b* are represented coupled with an ability to manipulate these parameters. This understanding was grounded in students' previous experience building and graphing growing geometric patterns. In the instructional sequence, the coefficient was linked to the increasing number of blocks at each position in a growing pattern, and students were able to ascertain that the higher the value of the coefficient, the greater the increase in blocks at each position – an initial understanding of rate of growth. This translated to an understanding of the coefficient and steepness of slope, since students could link the increase in coefficient with an increase in blocks at each position, which led to a steeper slope on the graph. Students also developed an understanding of the constant through pattern building by representing the constant at each position number (the part of the pattern that stays the same) and by building the "zeroth" position of a pattern, for which only the blocks representing the constant are shown.

Growing patterns were a mediating representation, linking students' understanding of symbolic to graphic representations. Although students initially worked with patterns, it became evident that they moved from this initial anchoring representation and progressed to using the graphs as the site for problem solving. This was evident in students' abilities not only to interpret and make predictions about how different functions would behave when graphed, but also in some students' ability to extend their understanding to start to include negative integers in composite functions. Nesher (1989) informs us that any systematic learning of a new concept should be done within an environment that is intentionally designed to support a series of transitions, from working with familiar objects (growing patterns) to carrying out mathematical processes using representations of these objects (observing changes in graphic representations of patterns based on changes in the coefficient or constant) to constructing a mathematical object that then can be manipulated (predicting parallel or intersecting functions, integrating an understanding of negative coefficient or constant). Mathematical objects are mental entities abstracted from experience that can be manipulated separately from the experiences that gave rise to them (Kaput, 1989). In our study, the fact that students maintained a strong understanding of graphs over an 8-month period, and could work with these representations without needing to return to concrete pattern building, suggests that they considered graphed functions as mathematical objects.

In addition, it seems that the use of three representations developed students' emergent understanding of the concept of linear functions. Understanding a concept presupposes the ability to recognize that concept in a variety of representations and the ability to handle the concept flexibly within different representational systems (Yerushalmy & Shternberg, 2001). By utilizing the students' understanding of patterns, the connections between symbolic and graphic representations were made as transparent as possible. For these students, the concept of linear function was recognized as the common underpinning of their rules, their patterns and their graphs.

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MATHEMATICS OF COLLEGE ALGEBRA STUDENTS: THE INTERPLAY BETWEEN STUDENTS' SELF-EFFICACY AND FORMAL MATHEMATICAL BELIEFS

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*This paper reports the results of a study of the beliefs of College Algebra students. Subjects (N=195) came from College Algebra classes at two universities in the southern United States. Data sources included a mathematical beliefs survey instrument and on-going individual interviews conducted with 30 of the students. We address two types of mathematical beliefs, **formal mathematical** and **self-efficacy** beliefs, and focus on the inter-connections of these as students solve mathematics problems. Of particular interest is how the students' knowledge of formal mathematical concepts co-exists with their efficacy beliefs in the course of on-going mathematical activity. Drawing from episodes of interviews with one student, the analysis explains the complexity of the students' mathematical beliefs and how these impact the solvers' on-going problem solving actions.*

There is growing consensus that the traditional College Algebra (CA) course is not helping students become quantitatively literate citizens (Hastings, 2006). Based on long-term demographic studies, high D/F/W course rates and the failure of CA to provide students with applicable skills, the *Conference to Improve College Algebra* has called for revamping the CA course (Small, 2002). Other research conducted on CA students surveyed their mathematical beliefs (Frank, 1986), documented their often fragmented conceptual understandings (Carlson, 1997), and examined the effectiveness of instructional strategies (Underwood-Gregg and Yackel, 2000). However, few studies examined how mathematical beliefs influence the ways students interpret and solve mathematics problems. More needs to be known about how beliefs influence students' initiative and efficacy in problem solving.

Purpose and Theory

Our goal is to improve our understanding of the interactions between a student's mathematical beliefs and his/her problem solving actions so that we can develop effective intervention strategies. Drawing from the work of Cooney, Shealy, and Arvold (1998), we view the learner's mathematical beliefs as complex mental structures that aid his/her interpretations in mathematical situations. Muis (2004) noted that previous research employed either a qualitative approach, observing students' problem solving, or a quantitative approach, using students' self-reported survey responses. Studies employing either methodology found significant relationships between students' mathematical beliefs, their engagement with mathematical tasks, and achievement. Muis recommended that further studies employ both qualitative and quantitative analyses in order to develop a more robust understanding of interactions among beliefs, learning, and achievement. Our study included surveys and repeated observations of students solving problems. This is compatible with Vergnaud's (1984) notion that solvers demonstrate their conceptions as "mathematical beliefs-in-action" as they solve problems, and that beliefs serve as conceptual models where solution activity may develop (Vergnaud, 1984, p.7). Often, CA students are viewed as lacking both formal conceptual knowledge and self-efficacy in their actions. We believe this

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is not the case; the mathematical work they do often does make sense to them. The sources of students' conceptual difficulties seem complex and need to be understood.

Subjects and Methods

Subjects (N=195) came from College Algebra classes at two universities in the southern U.S. Subjects completed Yackel's (1984) mathematical beliefs and attitudes survey. We also conducted individual teaching experiments with 30 of the students. Interviews occurred bi-weekly, semester-long and lasted about 40 minutes each. The researchers conducted these interviews by following Cobb and Steffe's (1983) principles of clinical teaching experiments.

Data Collection and Analysis

Data consisted of videotaped interviews, written transcriptions, the researchers' field notes and subjects' written work. Each student's activity was partitioned into episodes that evidenced three critical aspects of the student's solution activity: 1) development of goals, 2) on-going self-monitoring and progress assessment, and 3) problem-solving results. Using Yackel's inventory, we noted episodes where the student's activity appeared to demonstrate mathematical beliefs-in-action. We characterized mathematical beliefs by observing the student's problem solving actions and then drawing inferences about the beliefs on which the actions were based.

Results

The study identified two types of mathematical beliefs that appeared to play important roles in the solutions of the solvers—*formal mathematical* and *self-efficacy* beliefs, and documented the roles of these beliefs in the students' problem solving actions. Formal mathematical beliefs draw from the solver's knowledge of formal concepts such as symbols, functions, and formal algorithms. For example, the ways a student sees fit to manipulate algebraic quantities that involve radical expressions illustrates a formal mathematical belief-in-action; the student has experiences with radicals and has built-up a system of actions that he/she may call upon when problems are faced. This idea is consistent with Schoenfeld's (1985) contention that the mathematical beliefs of students help constitute their mathematical view and play a crucial role in the ways they 'see' the mathematical problems they face. One popular finding about CA students is that they develop idiosyncratic rules from their actions, often based on fragmented understandings (Carlson, 1997). Table 1 summarizes the rules we observed and their connections to formal concepts.

In contrast to *formal mathematical* beliefs, *self-efficacy* beliefs involve the solver's more general sense of their own ability to be successful when problematic situations arise (Bandura, 1997). Students face many choices to proceed as new mathematical situations are met and interpreted. For example, the student may interpret a particular situation and proceed accordingly to invoke a strategy and begin to carry out their solution activity. In making a deliberate decision of how to proceed, the solver usually has some sense of where the proposed action may lead and also a sense of confidence as to the action's potential success. Encountering difficulty, he/she may be able to re-think their original ideas and pursue another course of action. These kind of 'online' decisions draw from the student's beliefs about the nature of mathematical activity and his/her role in it. Self-efficacy beliefs have a major impact on the student's will to remain persistent and focused when unexpected problems suddenly arise in the course of their mathematical activity. Indeed, instructors witness how strong students in advanced mathematics courses effectively monitor their mathematical actions while solving a problem, with a seemingly natural propensity to switch course when they sense that current actions will not achieve success. On the other hand, less competent CA students are usually less likely to begin anew when current ways of operating are unsuccessful. When they are “stuck”, often they quit and require major assistance from a tutor to assist in re-conceptualization of the problem, before beginning a new course. Table 2 shows examples of students demonstrating solution activity within levels of self-efficacy.

Table 1: Mathematical Beliefs Based on Formal Concepts

Belief	***Formal Concept***	***Observed Examples***
Students exhibit misunderstandings of concepts constituted in rule-like action	a) Radical expressions	Errors in simplifying: $\sqrt[\cancel{2}]{\cancel{2}} = 1$ $\frac{2 \pm 2\sqrt{1+e}}{2} = \pm\sqrt{1+e}$
	b) Quadratic equations	Errors in simplifying: $\frac{12 \pm \sqrt{24}}{2} = 6 \pm \sqrt{24}$
Students exhibit fragmented understandings of variables and variable expressions	Polynomials (factoring sum of cubes)	Applying rule for factoring $x^3 + y^3$
Students exhibit fragmented understandings of function	Quadratic functions	Solving for zeros, x_1 and x_2, plotting (x_1, x_2)
Students can often ‘see’ individual pieces but not the unified whole (‘the big picture’)	a) Binomial expansion based on Pascal triangle	Omits the first co-efficient in the expansion, ${}_n C_0$, starts with ${}_n C_1 = 1$
	b) Graphing rational functions	Correctly identifying asymptotes but plotting contradictory points

Our comparisons of the students’ interview and survey results suggest that in CA courses, differentiated instructional approaches may be productive. For example, students exhibiting high efficacy beliefs might benefit from inquiry-based approaches that allow them to focus on the larger problem context from which they can generate hypotheses. Students with low efficacy beliefs might benefit from approaches that are supported by more didactic-based approaches that help them analyze their own actions and provide suggestions for new strategies when the student is unable to hypothesize one. We are not suggesting drilling skills for low efficacy students; instead, instruction that coaches their problem solving strategies.

Table 2: Mathematical Beliefs Based on Self-Efficacy Expectations	
Examples of low self-efficacy: • Student always operating in the 'here and now' – does not reflect on action • Student treats problems as roadblocks; no way to proceed when he/she gets "stuck" • Student does not demonstrate conjectures • Student can only replicate simple results	Examples of high self-efficacy: • Student can distance himself/herself from his actions (reflection) • Student can generate hypotheses about sources of difficulty • Student can test conjectures • Student can self-generate relevant questions • Student can pose novel problems when needed

In the remainder of the paper, we focus on the mathematical activity of a single subject, Brad. We believe that Brad's solution activity illustrates specific examples of beliefs-in-action. Furthermore, the examination of episodes of Brad's mathematical activity helps to illustrate and explain the interplay between his formal mathematical knowledge and his views concerning self-efficacy.

Introducing Brad and Summarizing His Difficulties

Brad was a Business major who had taken College Algebra the previous semester and was repeating the class, a practice common among CA students. In the following paragraphs, we will illustrate and discuss some of the conceptual difficulties demonstrated by Brad during the interviews. We found that Brad's knowledge is typical of College Algebra students; he made common mistakes that many students make as they solve problems. We sought to locate the sources of Brad's conceptual difficulties through our follow-up questions. For example, the errors Brad made simplifying radicals during Interview #1 appeared within his paper-and-pencil work; it was only after much questioning from the interviewer that the probable source of Brad's difficulty was uncovered. The task required Brad to simplify the expression $\mathbf{2\sqrt{50}+12\sqrt{8}}$ and he found an answer, **34**, that did not agree with the answer given in the book, $\mathbf{34\sqrt{2}}$.

Brad: *I've worked it out twice but I didn't get the answer that's in the back of the book. First thing I do is look at radicals and see if I can simplify anything, ... 50 will break down into 2 and 25, which are both perfect squares (sic), so that's what I went ahead and did. And 8 is not a perfect square either, but I know that 2 and 4 are (sic), which are factors of 8, so I went ahead and wrote that down*

$$\mathbf{2\sqrt{50}+12\sqrt{8}=2\sqrt{25*2}+12\sqrt{2*4}}$$

Brad's retrospective reporting of how he tried to solve the problem indicated an overall understanding of the task– he could both re-present and monitor his prior activity in an objective manner. Brad's problem here involved making sense of a discrepancy between his answer and that given in the back of the book.

Brad: *And from here I just go ahead and take the square root of this, **25** , bring the **5** out, which would leave me ... **2·5** and go and bring the **2** out which would be **1**, right ? ... or now it'd just be a **2**, right ?...**(stares in space)**... and then plus **12** then ... that'll just be **1***

and bring out a **4**, *which will be* **2**, *and we multiply by* **2**, *...* **2 x 5** *will be* **10**, *and* **12 x 2** *will be* **24**, *which will leave* **34**, *but that's not what the book got.*

Brad's solution included the incorrect assertion that $\sqrt{2} = 1$. Brad's solution is summarized below (#1-4); the correct solution is indicated within the brackets.

Brad's Solution

1. $2\sqrt{50} + 12\sqrt{8}$
2. $2\sqrt{25*2} + 12\sqrt{2*4}$
3. $2*5*1 + 12*1*2$ $\quad \left[2*5*\sqrt{2} + 12*2*\sqrt{2} \right]$
4. $10 + 24 = 34$ $\quad \left[10\sqrt{2} + 24\sqrt{2} = 34\sqrt{2} \right]$

Brad: *The book got* $34\sqrt{2}$. *I can't figure where* $\sqrt{2}$ *is?*

Interviewer: *It looks awfully close, only thing is the* $\sqrt{2}$ *there in the answer. Why don't you look back at an earlier step and see if there's some place where there could've been a* $\sqrt{2}$ *and maybe it got lost somewhere.*

Brad: ***(reflects)*** *... There and there.* ***(points to*** $\sqrt{25*2}$ ***and*** $12\sqrt{2*4}$ ***)***

Interviewer: *So, what did you do at that point in the process?*

Brad: *I just took the square root of* ***25****, which was* ***5*** *and the square root of* ***2****, ...* ***(long reflection here)*** *... that's* **not perfect**! *..., yes, it's perfect, for some reason, I cannot ...* ***(realizes he has a problem)*** *... no, it's not, is it? You know what I was getting confused with ?* ***(writes*** $\sqrt[2]{2}$ ***)*** *and for some reason I thought I could cancel* ***(cancels 2s in expression*** $\sqrt[\cancel{2}]{\cancel{2}}$ ***)****. Maybe* ***that's*** *where I got lost, that has to be it, because there's no other place.* ***(goes back to his board work and starts with*** $2\sqrt{25*2} + 12\sqrt{2*4}$ ***)*** *Um.* $2*5\sqrt{2}$. *Still gonna keep the* $\sqrt{2}$ *and it's gonna be plus* ***12****, um* $\sqrt{4}$ ***is 2****, still have* $\sqrt{2}$ *there. Then we go ahead and get* $\sqrt{2}$, $5*2\sqrt{2}$ *plus* $12*2\sqrt{2}$, *that stays there.* ***(writes*** $10\sqrt{2} + 24\sqrt{2}$ ***).***

Brad: (**several seconds of reflection**) *I guess this is just like* ***10X*** + ***24X****, the* ***X*** *stays the same and you just go and bring down the radical. Then* ***24*** *and* ***10*** *is* ***34*** ***(writes*** $34\sqrt{2}$ ***)*** *O.K. That's what I was doing. That's the kind of lapse I'll have, that right there. ... for some reason it didn't register with me on the homework.*

While Brad experienced difficulties that related to his knowledge of formal mathematical concepts, he nevertheless demonstrated efficacy knowledge in his actions. For example, we observed that Brad was able to distance himself from his prior activity and objectively review, monitor, and report results to the interviewer. College Algebra students are seldom able to engage in such retrospective analysis of their actions. Brad's actions included a motivation to answer some self-generated questions regarding his overall approach to solving the problem of the discrepancy between his answer and the book's. Unlike many students who merely try to match their work to the answers, Brad developed some working hypotheses, which helped him develop goals of action. He had a sense regarding the source of his error, and with the aid of the interviewer's questions, was able to investigate his prior actions in a new light. He systematically set about to simplify the radicals, never wavering from his belief that his reasoning was sound. Brad's inability to self-diagnose and correct his work with radicals ($\sqrt[\cancel{2}]{\cancel{2}} = 1$), suggests that his misunderstandings about irrational expressions were deep-rooted within his flow of continuous action; it was only with the intervention of the interviewer's questions that Brad became aware of the error and set about to correct it.

One question that usually gets asked about these students is whether or not they have some stable problem solving processes, more general heuristics that they appear to use regardless of the problem. In Brad's case, he did appear to be overly structured throughout the interviews. He usually had a pretty good idea of what he needed to accomplish; however, his fragmented understandings of formal mathematical concepts continued to hamper his actions as he solved problems. In the following paragraphs, we examine Brad's solution activity as he tried to solve a problem that required more sophisticated conceptual knowledge. This will demonstrate how his fragmented formal understandings impacted his conceptual 'big picture' view of things, thus indicating the complexity of his conceptual structures.

Brad's Work Graphing a Quadratic Function

During Interview #4, Brad was asked to graph a quadratic function. This task is among the most difficult for CA students because of the conceptual knowledge involved. Although Brad demonstrated that he had some connected sense of the actions he must perform to solve the problem (i.e., first, find and plot the vertex; next, plot the intercepts, etc.), he experienced difficulty at each juncture, making several errors as he carried out his actions.

Interviewer: *Let's try a graphing problem. How about graphing the following function*: $\mathbf{f(x) = x^2 - 3x + 2}$

Brad: *So first thing I do is have* ***x=-b/2a*** *, so x=minus 3, no, minus minus 3,* ***(writes x= 3/2)*** *so your vertex is* ***x= 3/2*** *and the y is* ***0***. ***[plots the point (0, 3/2) on the y-axis]***. *So you pinpoint your vertex, start plotting points. I took a lot of good notes on this in class so let me look at it.* ***(consults his notes)****... well, maybe not.*

Interviewer: *What about this point here, the vertex. I see you have the vertex there on the y-axis. Do you remember what we called the point on the y-axis?*

Brad: *Yes, the y-intercept, the point that crosses the y-axis. That is where x equals 0, but the vertex is the lowest point on the parabola. 2 is where the graph crosses the y-axis. The vertex is ... the lowest point* ***(looks at his notes)****, now I remember* ***(reads from notes)*** *we take the value of x* ***(points to x=3/2 on the board)*** *and plug it back into the equation to find the y coordinate, so I guess you need to put that 3/2 back into the function.*

Brad then computed the y-coordinate as follows:

Brad's Computations

1. $\mathbf{(3/2)^2 - 3(3/2) + 2}$

2. $\mathbf{9/4 - 3/6 + 2}$

Interviewer: *Well, let's see. Tell me how you got those.*

Brad: *Well I took (3/2) and just squared both 3 and 2 to get 9/4. Then I cross multiplied to get 3/6, which is _ -- can I do that?*

Interviewer: *The 9/4 is fine. Go off over there and show me again how you cross multiplied and got 3/6 or _*

Brad: *Well, let's see, it goes like this.* ***(writes the pair of fractions, and draws arrows) to connect numerators and denominators of fractions)***

$$-\frac{3}{1} \quad \frac{3}{2}$$

Brad: *Wait, I don't think it goes like that. No, you just multiply across, so we get 9/2. I think I remember when we solved those equations a few weeks ago, we did that cross multiplying* $\mathbf{-\frac{3}{1} = \frac{3}{2}}$. Brad re-computed and plotted the correct vertex (3/2, -1/4).

Brad: *Okay, so now I know where it crosses y and the vertex, I think I need to find the places it crosses x, yeah, the x-intercepts. So they must be somewhere around here* ***(points to positive x-axis on graph)****. Okay so now we drop the f(x), the whole thing; then you are finding the x-intercepts* ***(points to f(x)):*** $\mathbf{0 = x^2 - 3x + 2}$

Brad: *Okay, so I think I now solve this and get 2 points for* **x** *(sic). And I think I can make factors* ***(writes the following factored form)*** $\mathbf{0 = (x - 2)(x - 1)}$

Brad: *And I just take the opposites of these points (sic), to get 2 and 1 so we plot* ***(2, 1).*** *No, I think I need to look at my notes again … yeah that's right but I write them like this* ***(writes and plots correct points corresponding to the zeros)*** **(2, 0)** ***and*** **(1, 0)**

After Brad added the two x-intercepts to his graph, the interviewer asked whether there were now enough points to sketch the graph.

Brad: *Well it's a quadratic and most quadratics are parabolas (sic)* ***(points to the vertex, the x and y-intercepts)*** *but we don't know if it zig-zags over here* ***(points to the right of the x-intercept (2, 0))****. You need to go the right beyond this point (2, 0) and plot some more points. But I do know that this point* ***(points to vertex)*** *is the lowest point.*

The interviewer asked Brad if he remembered what kind of function this was, and what were some of its general properties (e.g., line of symmetry, etc.).

Brad: *Wait a second … okay, now I remember that parabolas are, they have a line straight though here* ***(draws vertical line through vertex)****, and they are the same on each side, the line of symmetry. I think we also called these even functions. So, I start here at the vertex and go through these* ***x's*** ***(points to the location of the zeros)*** *and draw the graph... and make it look like it is the same on both sides because of the line through the middle here* ***(points to the line of symmetry)****.*

Brad's solution of the quadratic graphing problem was noteworthy because he demonstrated: (1) a fragmented conceptual understanding of quadratics - he had some connection and coherence about how the different aspects related but could not focus on any one aspect without making mistakes; and (2) many of the same arithmetic operational difficulties that most students have graphing quadratics.

Conclusions

This study contributes to the study of mathematical beliefs in the following ways. First, these examples of formal mathematical beliefs in action help clarify the role of mathematical beliefs as hypothesized by Schoenfeld (1985). The idiosyncratic rules we observed suggest that solvers' informal ways of operating appear to play a prominent role in how they solve problems. Second, our findings about self-efficacy beliefs help clarify the ways that CA students exercise control over their actions. Since most studies of mathematical self-efficacy are conducted by asking students to rate their confidence to solve particular tasks (Pajares, 1996), the results here give specificity to the ways that students actually demonstrate self-efficacy while solving problems.

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NEW PERSPECTIVES ON THE STUDENT-PROFESSOR PROBLEM

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The "student-professor problem" was first reported by Kaput and Clement (1979), who found that many students displayed particular types of errors when constructing equations to describe a linear relationship between two variables. While translating between words and symbols in "student-professor-type" situations are themselves interesting mathematical tasks, Kaput (e.g. Kaput and Sims-Knight, 1983) has suggested that these errors can be most useful as a lens to investigate the underlying mental processes that students use while solving this problem, because students these same processes while working on other math problems. The goal of this study is to investigate the student-professor problem from several new perspectives and, in doing so, describe ways we might help students work more successfully with related algebraic ideas.

Nearly thirty years ago, Kaput and Clement (1979) (and later Clement, Lochhead and Monk (1981)) noted that students had difficulty with the following problem:

> Write an equation using the variables S and P to represent the following statement: 'There are six times as many students as professors at this university.' Use S for the number of students and P for the number of professors. (p. 288)

In multiple surveys of college students, roughly 40-60% solved the problem incorrectly. The most common error made by students was the *reversal error:* $6S=P$. This reversal error was even more common when both variables in the linear relationship have coefficients other than 1 (e.g. when the corresponding equation is $4C=5S$).

This reversal error arises not only when students are constructing an equation based on words, but also when they attempt to construct an equation based on a table of values or a diagram. MacGregor and Stacey (1993) found that similar reversal errors occurred as well in problems with an additive structure and they concluded that students "were not matching the symbols with the words in an item but were expressing features of some underlying cognitive model of a mathematical relationship." (p. 228)

Clement (1982) identified several strategies used by students and suggested that the two essential competencies for correctly solving the problem were recognizing that the letters represented quantities and creating a "hypothetical operation" to make two quantities (such as the number of students and professors) equal. Kaput, Sims-Knight and Clement (1985) suggested that there are two components necessary for success with "student-professor-type" problems: "understanding variables (and the underlying notion of mutual variation of two quantities) and understanding the syntactical features of the algebraic representation of variables." (p. 58) While Rosnick and Clement (1980) noted that a misconception of the equals sign could be associated with incorrect responses, few other researchers have focused on equality misconceptions as a potential source of error.

Not only does the reversal error appear in many situations, but it has also proven difficult to remediate. Clement (1982) found that warning students that they might make an error did little to improve their performance. Rosnick and Clement (1980) tutored students by pointing

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out incorrect features and demonstrating correct solutions but found that most students' conceptual understanding of variable and equation did not change. Several studies (e.g. Philipp, 1992) have shown that replacing *S* and *P* with different letters did not improve students' performance. Similarly, Fisher (1988) found that substituting N_s and N_p (to more actively suggest "number of students" and "number of professors") did not increase performance.

In recent years, increasing attention has been given to students' ideas about the concepts of equality and variables (e.g. Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005)—and to the concept of covariation (e.g. Carlson, Jacobs, Coe, Larsen and Hsu, 2002). It is possible that this recent work could give us a new perspective on the reversal error and the underlying cognitive processes. In addition, while translating between words and symbols is an important mathematical activity, this translation is typically done in the service of solving a problem and not solely for the sake of producing the translation. It is possible that while students are prone to making the reversal error, they would still be able to effectively solve problems involving "student-professor type situations" (e.g. involving a linear relationships between two variable quantities).

Research Questions and Methodology

Much of the previous work on the reversal error has focused on remediation, but less attention has been given to producing detailed descriptions of *successful* student strategies for translating words and diagrams into equations. The first goal of this study is to provide a more detailed account of these strategies and determine if any of these strategies suggest new ways of increasing students' success with translation. This study also seeks to describe students' conceptions of equality and covariation and any correlations between these conceptions and success in "student-professor-type" problems. Finally, this study investigates the different strategies used by students while solving problems in "student-professor-type" situations.

A written survey was given to twenty-seven first-semester calculus students. Four of the items involved "student-professor-type" problems. The first was the standard Student-Professor problem, asking students to translate words to the linear equation $12P=S$. The second was similar but the resulting equation, $11F=2A$, had two coefficients that weren't 1. The third asked students to "create a function" to describe a diagram that represented the relationship $4C=3P$. The fourth gave a situation that could be translated as the equation $5S=3C$ and asked students to predict a value when given a value for one of the variables.

To investigate students' conceptions of the equals sign, they were asked to decide whether several equations were true (such as 5x4=20+3=23 and $a^2+b^2=(a+b)^2$). Their responses were analyzed to determine if they used the equals sign to denote something other than strict equality, or if they viewed the equals sign as denoting an operation rather than a relation (see Knuth, Alibali, McNeil, Weinberg & Stephens, 2005). As part of viewing the equality symbol as operational, a student might describe it as having a "direction" and identify an equation such as $ab+ac=a(b+c)$ as a "reverse distributive property". One item on the survey showed students a bottle being filled with water (shown in Figure 1) and asked them to graph the height versus the volume of the water; this was designed to investigate students' conception of covariation using the framework of Carlson et. al. (2002).

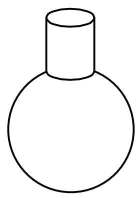

Figure 1

Eighteen students participated in an additional open-ended interview, in which they described their thinking on the written assessment and solved the survey problems again. The

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interviews were transcribed and, along with the written assessments, coded using both previous descriptions from the literature as well as with codes to indicate patterns of reasoning not previously described.

Results

Student Strategies

Students used and described a variety of strategies while working on the translation and prediction tasks. Several students used what appeared to be a static comparison (Clement, 1982) in which they appear to be using the variable as a label instead of a varying quantity. However, MacGregor and Stacey (1993) found that this apparent strategy was a "post hoc explanation of an equation arising from a model formed without conscious intervention" (p. 230). These students may be performing an operation to represent *unequal* groups, such as this student who was working on the student-professor problem:

> Using these variables there are… I used S for students and P for professors. There are twelve times as many students as professors, so if our variables are S and P, if you have just S and P they're the same. So if we have twelve S, multiplying, that would equal twelve times as many students per same number of professors.

A number of students indicated that they were using the equals sign to represent a ratio:

> S: I put students to professors, and then I originally put dots for a ratio, then I wasn't sure if a ratio would be right, so I put equal - I'm not sure if that's the same thing.
> I: Is it the same thing to use the equals sign as the colon?
> S: I guess that's what I wasn't sure of, cause there's twelve times many students as professors at this university… I guess, now thinking about it the equals sign wouldn't work, you'd probably just use the dots for it to say ratio.
> I: What if you did need to write an equation using an equals sign? Could you still write this, or do you think you'd have to write something different?
> S: I probably still could write that, though.

Clement (1982) described the "operative approach" as a frequently successful strategy in which students performed a "hypothetical operation" (p. 21) in which the size of one quantity is changed to be equal to another quantity. In this study, students' strategies could be described in more detail as either *operative reasoning* or *functional reasoning*. In operative reasoning, the student performed hypothetical operations on two quantities:

> I should switch the three and the four, because there are three cows and there are four pigs, but to equalize the numbers, you have to multiply the pigs by three and the cows by four. Thus they both equal twelve.

In functional reasoning, the student performs operations on one of the quantities to transform it into the other quantity. In this strategy, one of the quantities is used as an input to a function and the other is used as an output:

> I took three to four ratio—cause for every four pigs there's three cows, which is point seven five. And so if you're given four pigs, four times point seven five is three, and so you can keep increasing by point seven five.

While working on the value-prediction task, students used different strategies than they used on the translation problems. Several students used a strategy that was similar to functional reasoning but performed the operations *stepwise* instead of all at once. For example, one student divided a group of objects into "test samples" and then counted several of these samples. This corresponded to first dividing the original quantity and then multiplying the result in contrast to multiplying by a fraction or ratio. In this example, the student was working on the problem: "In New York there are three SUVs for every five cars. If there are 165 cars in a parking lot, how many SUVs do you expect there to be?":

> Well because, it says for every five cars… I wanted to… take it out of that one sixty five and once I was done, I was just going to multiply it back out. So... 165 divided by 5 is 33? Yeah. 33, and I just like left that number alone and then there's three for every five, so I did a test sample kind of thing, it's one thing of five cars, so if there's three for every five I just multiplied the 3 by how many test samples there are—which is 33—and I got 99.

Numerous students used *proportional reasoning* to make their prediction by constructing a proportion from the written description and cross-multiplying:

> Okay, there are three SUVs for every five cars, so you can put that into a proportion… So if you set up a proportion and set them equal to each other, so you know like three over five, is SUV—or, is SUVs per car, so you set it equal to x over one sixty five and you can cross multiply and divide through everything and you can find that there's ninety nine SUVs with the hundred sixty five cars.

Student Performance

Roughly half of the students gave correct answers on the translation and function-construction items on the written assessment; roughly one third of the students gave an incorrect answer involving a reversal error. This is in contrast to previous studies in which students had more difficulty translating situations when one of the coefficients was not 1. On the value-prediction item, students performed significantly better, with 70% of student giving correct answers and only 11% supplying an answer that indicated a reversal error. Despite students' success with the value-prediction item, they had difficulty when asked to construct an equation that represented the situation in this item.

While students were more successful predicting values than translating situations into equations, there was no correlation between the strategy a student used to construct an answer on the written assessment and their strategy on the same item during the interview. In addition, there was little correlation between a student producing correct answers or making reversal errors across multiple items. That is, there were few students who consistently produced correct answers or consistently made reversal errors.

Students displayed a flexible conception of the equals sign. They avoided common misconceptions, recognizing that $(a+b)^2$ is not equal to a^2+b^2 and they did not describe $ac+ab=a(b+c)$ as "reverse distribution." However, 20 out of 27 students asserted that the number sentence 5x4=20+3=23 was true and 22 out of 27 asserted that the number sentence $g(x)=x^2=g'(x)=2x$ was true. One student summarized his view of the equals sign as having numerous meanings:

> I'm just not sure about having the equals sign to relate the two. There was probably a better way. The whole thing was that—these two signs are what make the thing equal, and that one just—it would be better to have something to say that one is in relation to this.

While many students described this incorrect interpretation of the equals sign, there was no significant relationship between displaying such a conception and either correctly answering the other questions or a reversal error.

Students were generally successful drawing the graph of height versus volume for a bottle filling with water. In both the written assessment and during the interview, roughly 60% of students drew a correct graph and described their thinking in a way that reflected an understanding of average or instantaneous rate of change (this corresponds to the highest two levels of covariational reasoning using Carlson et. al.'s (2002) framework):

> At first the—I guess the radi—like if you draw this into circles [shades in bottom of bottle] the radius increases from very small to larger and larger, so it will take longer to fill up a point—like if this was a circle here [shades in bottom of bottle] to fill up that circle, then it will… because there's less of a circle to fill there. So it will be quicker at first because it's smaller circles at first, until they start to increase, and then eventually—at whatever location [draws in a horizontal line at the middle of the bottle] it will start to get smaller again.

While most students displayed a relatively sophisticated conception of covariation, there was no significant relationship between their level of covariational reasoning in the interview and either giving a correct answer or a reversal error in the other items. On the written assessment, the only significant relationship with drawing a correct graph was correctly writing a function to describe a diagram.

Analysis and Future Directions

It is somewhat surprising that students' overall success rate—and rate of reversal errors—with the translation and function-creation items was consistent. Previous studies have indicated that the items that used a coefficient other than 1 should be more difficult. More striking is the variability in each student's performance. We might expect certain students to repeatedly make a reversal error or to get all of the items correct, but this was not the case. At the very least, we can say that students' strategies, successes, and errors depend on the problem situation. More importantly, this provides evidence that there may not be a unifying mental mechanism used by each student. Other studies have constructed descriptions of students' mental models in part as a predictor of their performance; since students are not performing consistently, a mental model may not be a useful tool for this prediction. In particular, the lack of relationships between a student's strategy on the written assessment and their strategy during an interview (even while looking at their previous work) suggests that students are either not developing models that they can invoke in multiple situations, or that they have many potential models and the impetus for using a particular one depends on more than the problem situation. There very well may be other important factors such as the student's perceived role as a problem solver and their previous pedagogical experiences, although the current data do not shed light on these issues.

Overall, students were significantly more successful at predicting a value in a "student-professor-type" problem than constructing an equation or formula based on words or a diagram. This suggests that students' strategies depend not only on the problem situation, but also on the specific task they are asked to complete.

In making their predictions, students were able to use various algebraic tools to successfully complete the task despite their translation difficulty. In addition to refining our descriptions of the various mental models students employ when engaged with different tasks, describing students' various algebraic competencies and understandings could shed light on the ways students choose and use their mental models. However, one fundamental algebraic idea—covariation—did not have a strong connection to students' strategies when translating or predicting a value. This could mean that they are using ideas other than those of covariation. It could also mean that the measure of covariation used here hasn't captured the aspect students employed. Alternatively, students may be relying on previous pedagogical experience and memorized algebraic routines to produce their answers (although the data do not provide explicit evidence for this).

Although it didn't affect their performance on the other items, students' flexible interpretation of the equals sign is troubling, although perhaps not completely surprising. Instead of uniquely denoting "sameness," the equals sign seems to be a "Swiss army knife" of symbols, representing a ratio, the co-existence of unequal sets, or an undefined relationship between two objects, ideas, or symbols. One student explicitly described multiple interpretations of the equals sign that led to a reversal error (in the problem, the correct equation was $4C=3P$):

> That one's [4P=3C] the *relationship*, but this one [C=3/4P] is the *equation* on how to get it from number of pigs. If you had this many pigs, then you just multiply it by three fourths and then you'll get the number of cows.

In addition to the "operative" strategy described by Clement (1982), students used several strategies to produce successful solutions. The *functional* strategy is similar to the operative strategy but involves thinking of one variable as an input and performing a sequence of operations on that variable. The sequence of operations could be encapsulated as a rational coefficient. In contrast, the operational strategy involves manipulating both variables to produce equivalent mathematical objects. In the value-prediction item, students used *proportional* and *stepwise* strategies to produce their answers. The proportional strategies involved constructing a proportion based on the situation and cross-multiplying; several students also referred to this as using a ratio. The stepwise strategy involved partitioning a group of objects into several smaller groups and then collecting a number of these new groups.

The proportional and stepwise strategies are particularly notable because they enabled the students to produce computational answers without reversal errors. This suggests a potential pedagogical strategy of developing students' proportional reasoning and partitioning to help them successfully translate between words, diagrams, and equations.

In addition to these potential pedagogical foci, students' usage of functional reasoning—both successful and unsuccessful—underscores the importance of helping them develop flexible conceptions of function. While there was little correlation between students' success with translation problems and their conceptions of equality, their flexible interpretation of the equals sign suggests that more attention be given to developing a robust conception of equality.

As Kaput and Sims-Knight (1983) note, student-professor-type problems are useful as an "acute theoretical lens" (p. 63) through which we can gain insight into students' mental models, understanding of algebraic notation and use of this notation. The results of this preliminary study suggest that there may be previously undocumented mental models that students can successfully use with these kinds of problems. Not only is it useful to describe these models, but students' varying performance suggests that we further need to investigate the interaction with social aspects of problem solving—the aspects beyond the specific problem that cause students to use particular models and interact with the models with particular symbolism and techniques.

The results of this study can be used to inform future studies to develop and refine these new mental models and the further explore the relationship with the idea of equality. By systematically varying the tasks and goals given to students and refining questions to target some of the connections between algebraic ideas, symbolism, and students' solutions, we can create pedagogical strategies to help students develop a more robust understanding of the algebraic concepts.

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PROCESSES OF SYMBOLIZATION DERIVED IN THE USE OF SPREADSHEET

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In this article we report on the procedures high school students used to relate the symbolic language of spreadsheets (Excel) to that used in traditional paper and pencil method. The word algebraic rate problem discussed herein is part of a series of problems implemented in the classroom with the proposal of helping students learn the algebraic language. The students demonstrated progress and difficulties in the understanding of the significance of the algebraic symbols involved.

The incorporation of technology and new theoretical frameworks for describing processes of teaching and learning mathematics has modified the manner in which algebra is perceived in mathematics education. The possibility of relating algebraic language to table of values and graphics with the incorporation of the spreadsheets (Excel) permits the students to work with *algebraic objects* in a dynamic manner different from that encountered in the traditional teaching methodology using paper and pencil.

In this article we report upon the analysis of part of a broader research project which looks into the process in which the students moved from the language of spreadsheet towards the algebraic language used in working with paper and pencil. We describe the form in which a pair of students solved an algebraic *rate* problem which could be modeled on a system of two linear equations with two unknowns. In the description we utilize theoretical elements of Instrumental genesis.

Literature Review

Bednarz & Janvier (1996) defined the characteristics that determine the difference between an arithmetical problem and an algebraic one. The algebraic, or disconnected, problems were characterized by the difficulty presented in establishing direct *bridges* between the known quantities. These must be operated with various unknown quantities at the same time, making it necessary to establish equations in order to solve them. The word *rate* problem reported here is a disconnected in Bednarz & Janvier's (*ibid*) terms.

In the analysis of the work of the students we took the perspective of Instrumental genesis, in particular the triad of technique, task and theory, whose role in teaching and learning of mathematics was discussed by authors such as Artigue (2002); Chevallard (1999), and Lagrange (2000). These authors emphasize the *pragmatic* and *epistemic values* of technique, or the productive potential (efficiency, cost, field of validity), and its contribution to the understanding of the objects involved.

Methodology

The two students who participated in this study were in attendance at a technological high school within the public education sector in Mexico (SEP). They were in their first semester, and they were both fifteen years old.

The work in the classroom, with the students working in pairs, was undertaken as follows: a problem to be solved with the use of spreadsheet was given to each pair of the whole group, the teacher then led group discussions, and finally, preliminary conclusions were arrived at within the group. An interview, quoted from herein, was planned with one of the pairs of students. The task or following word algebraic *rate* problem was solved by the students:

> **PROBLEM 1**. Juan bought 450 notebooks and 300 pens, spending \$6,600. If each notebook cost twice as much as each pen, what was the cost of each notebook and pen? (Martínez, Struck, Palmas y Álvarez, 2001, p. 149.)

This problem can be solved in an algebraic manner by writing and solving a system of equations such as the following.

$$\begin{cases} 450x + 300y = 6600 \\ x = 2y \end{cases}$$

Before giving the students PROBLEM 1 we asked them to establish the equation or system of equations to solve it. In a previous session, the group had solved a problem using spreadsheet and after they had written, directed by the teacher, the system of equations.

Results

The technique and theory the students used to solve the problem could be described in the following manner: after reading the problem the students began to accommodate the information (notebooks, cost, subtotal, pens, cost, subtotal, and total, Figure 1.) into the spreadsheet. This accommodation of the information seemed, up to this point, to be part of a routine for solving problems but the students were simultaneously analyzing the information given in the problem: what they had to look for, or the unknowns (cost of notebooks and cost of pens), the known quantities (number of notebooks and pens), and the total cost. The attempts to understand the problem while accommodating the information could be seen in the hesitation in categorizing the various columns, in organizing the information, and in where to put the data. This accommodation was influenced by a problem previously solved with a different structure, but they perceived it to be similar, which permitted them to start a similar process of solution.

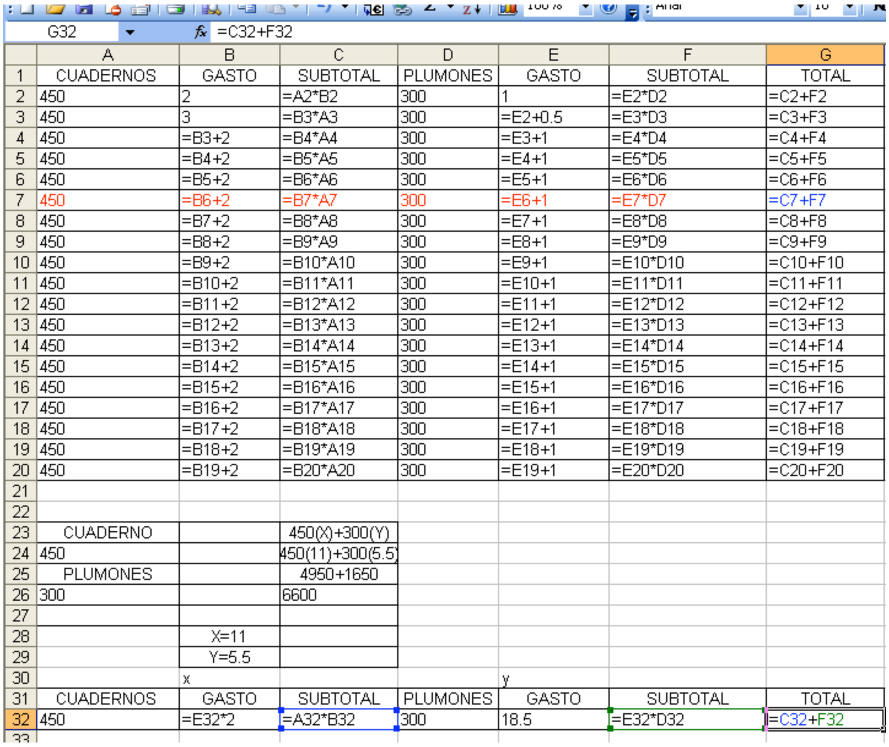
E7 | =E6+1

	A	B	C	D	E	F	G
1	CUADERNOS	GASTO	SUBTOTAL	PLUMONES	GASTO	SUBTOTAL	TOTAL
2	450	2	900	300	1	300	1200
3	450	3	1350	300	1.5	450	1800
4	450	5	2250	300	2.5	750	3000
5	450	7	3150	300	3.5	1050	4200
6	450	9	4050	300	4.5	1350	5400
7	450	11	4950	300	5.5	1650	6600
8	450	13	5850	300	6.5	1950	7800
9	450	15	6750	300	7.5	2250	9000
10	450	17	7650	300	8.5	2550	10200
11	450	19	8550	300	9.5	2850	11400
12	450	21	9450	300	10.5	3150	12600
13	450	23	10350	300	11.5	3450	13800
14	450	25	11250	300	12.5	3750	15000
15	450	27	12150	300	13.5	4050	16200
16	450	29	13050	300	14.5	4350	17400
17	450	31	13950	300	15.5	4650	18600
18	450	33	14850	300	16.5	4950	19800
19	450	35	15750	300	17.5	5250	21000
20	450	37	16650	300	18.5	5550	22200
21							
22							
23	CUADERNO		450(X)+300(Y)				
24	450		450(11)+300(5.5)				
25	PLUMONES		4950+1650				
26	300		6600				
27							
28		X=11					
29		Y=5.5					
30							

Figure 1. Solution procedures of PROBLEM 1.

Thereafter, through trial and error the students began to solve the problem. However, the part of the problem which said “each notebook cost twice as much as each pen” was taken by them to mean the opposite. This meant that when the students considered that the cost of the notebooks was \$10, they determined that the cost of the pens was \$20. This was due to the order in which they had labeled “notebooks and pens” (Figure 1) in the spreadsheet rather than a mistaken interpretation of the statement. After various unsuccessful numeric trials they reread the PROBLEM 1, searching for the relationship between the quantity of notebooks and pens and noticed their error.

Even then, however, they did not establish the relation B=2E o E=B/2, and instead utilized recursive procedures in both columns B (=B3+2) and E (Figure 2). This affected, as will be seen below, the symbolization process on paper and pencil environment.



G32 =C32+F32

	A	B	C	D	E	F	G
1	CUADERNOS	GASTO	SUBTOTAL	PLUMONES	GASTO	SUBTOTAL	TOTAL
2	450	2	=A2*B2	300	1	=E2*D2	=C2+F2
3	450	3	=B3*A3	300	=E2+0.5	=E3*D3	=C3+F3
4	450	=B3+2	=B4*A4	300	=E3+1	=E4*D4	=C4+F4
5	450	=B4+2	=B5*A5	300	=E4+1	=E5*D5	=C5+F5
6	450	=B5+2	=B6*A6	300	=E5+1	=E6*D6	=C6+F6
7	450	=B6+2	=B7*A7	300	=E6+1	=E7*D7	=C7+F7
8	450	=B7+2	=B8*A8	300	=E7+1	=E8*D8	=C8+F8
9	450	=B8+2	=B9*A9	300	=E8+1	=E9*D9	=C9+F9
10	450	=B9+2	=B10*A10	300	=E9+1	=E10*D10	=C10+F10
11	450	=B10+2	=B11*A11	300	=E10+1	=E11*D11	=C11+F11
12	450	=B11+2	=B12*A12	300	=E11+1	=E12*D12	=C12+F12
13	450	=B12+2	=B13*A13	300	=E12+1	=E13*D13	=C13+F13
14	450	=B13+2	=B14*A14	300	=E13+1	=E14*D14	=C14+F14
15	450	=B14+2	=B15*A15	300	=E14+1	=E15*D15	=C15+F15
16	450	=B15+2	=B16*A16	300	=E15+1	=E16*D16	=C16+F16
17	450	=B16+2	=B17*A17	300	=E16+1	=E17*D17	=C17+F17
18	450	=B17+2	=B18*A18	300	=E17+1	=E18*D18	=C18+F18
19	450	=B18+2	=B19*A19	300	=E18+1	=E19*D19	=C19+F19
20	450	=B19+2	=B20*A20	300	=E19+1	=E20*D20	=C20+F20
21							
22							
23	CUADERNO		450(X)+300(Y)				
24	450		450(11)+300(5.5)				
25	PLUMONES		4950+1650				
26	300		6600				
27							
28		X=11					
29		Y=5.5					
30		x			y		
31	CUADERNOS	GASTO	SUBTOTAL	PLUMONES	GASTO	SUBTOTAL	TOTAL
32	450	=E32*2	=A32*B32	300	18.5	=E32*D32	=C32+F32

Figure 2. Values and formulas utilized by the students in approaching PROBLEM 1. With these formulas values shown in Figure 1 were derived. The last column contains information which was added during the interview.

When the students finished solving the problem and found that 11 and 5.5 were the values they were searching for they proceeded to include the algebraic symbolization. However, they had difficulties in determining what x and y were. The meaning of these objects was not clear, and in fact they wrote $11x+5.5y$, and explained to the teacher that x was 450 (the number of notebooks), and that y was 300 (the number of pens); x and y seemed to be considered as labels. During the interaction with the teacher, which was oriented towards the understanding of x and y in their written relation in spreadsheet using it as paper and pencil environment, the students determined that the correct expression was $450x+300y$ (cell C23, Figure 1), wherein x and y appeared to acquire the significance of unknowns or the values to be searched for. They did not write the equation; in other words, they did not make the expression $450x+300y$ equal to 6600. They did not realize that this was a condition which must be satisfied in order to solve the problem, nor did they see the need for writing any other condition.

One of the students of this pair under study began to participate in the group discussion (Excerpt 1) when the teacher was asking the students to establish the other relation, after the group had identified the relation between the cost of the notebooks (x) and the cost of the pens (y), to be $y = \frac{x}{2}$.

Excerpt 1 (Writing the second equation)

L1. Teacher: What other relation besides x and y can we establish? [*Asking the group for the other equation*]

L2. Student 1: [*Starts to speak, but the teacher asked him go to the board*]

L3. Student 1: From here [*points to the equation relation* $y = \frac{x}{2}$ *written on the board*] we can get the results you are asking for because we have that *y* represents, *y* is equal to *x* divided by 2. From there one can get… [*Did not finish the idea and started to explain the other relation for x and y the pair had identified*]. The final result, the number of notebooks times their cost gives us 4950. In the same way we could get the result by multiplying the number of pens by how much each pen cost and then add them together.

L4. Teacher: Write it. How does what you are saying come out in terms of relations?

L5. Student 1: [*Writing on the board*]

$$450x+350y=$$
$$450(11)+350(5.5)=$$
$$6600$$

L6. Teacher: What is $450x+300y$? It is equal to what? [*No one answered*]

The teacher, who was at the back of the classroom, saw that no one responded and asked if the equation was 450*x* plus 300*y* equals to 6600. The students were slow to respond, and the answer some gave was no, including Student 1. The teacher centered the discussion for several minutes on the understanding of the meaning of 450*x* and 300*y*, and then returned to the question of whether 450*x* plus 300*y* equaled 6600 (Excerpt 2).

Excerpt 2 (450*x* plus 300*y* equals 6600?)

L7. Teacher: Does this equal 6600?

L8. Student 2: Yes, because if 450*x* is the cost of the notebooks and 300*y* is the cost of the pens, well these two quantities add up to 6600, in other words, yes.

Student 2 spoke with conviction, but Student 1 in particular was in disagreement, arguing that 6600 could only be arrived at when 11 and 5.5 were used in the calculations. They were discussing for several minutes. The session was drawing to a close, and in conclusion the teacher asked another student to write the relations identified in the problem. The student wrote, with the help of his classmates: $y = \frac{x}{2}$ and $450x+300y=6600$.

Despite the exposition of Student 1 permitted the group to identify the system of linear equations they were searching for, Student 1 seemed to have difficulties with the concept of the equation and solution of an equation. This was brought up during the interview which had been initially designed for the student pair, however, only Student 1 was present.

The interview began with the analysis of a linear equation with two unknowns. The student searched for solutions through trial and error. These were substituted in the equation and the student observed that equality was always reached when the quantities substituted in the equation were the solutions, not when they were arbitrarily selected. Based on this experience the student argued about what an equation was and what a solution or solutions for an equation was.

The teacher returned to the procedure for solving the given problem (figures 1 y 2) and asked the student to remember what they had done previously in the classroom using spreadsheet. The student not only explained the procedure (Excerpt 3) but also wrote an equation (Figure 3).

Excerpt 3 (Solution procedure)

L9. Student 1: We represent *x* as 11 and multiply this by 450, which gives us 4950 [*pointing to the quantities in cells C23, C24 and C25 in spreadsheet, Figure 1*]

L10. Teacher: Oh [*Expressed to demonstrate that attention was being paid to the explanation*]

L11: Student 1: Then we multiply 300 by *y* and this is 300 times 5.5, because *y* equals 5.5.

L12. Teacher: Oh.

L13. Student 1: This gives us 1650. Then from this we have to add the two and we have to come out with the result given here of 6600.

L14: Teacher: Oh. And if asked you to write the equation [*asking it to be written with paper and pencil*]. Okay, to start with, is this an equation or not? [*Pointing to cell C23*]

L15. Student 1: Yes.

L16. Teacher: If I asked you to write it, how would it be?

L17. Student 1: [*Immediately wrote the equation in his paper, substituting the solutions as shown in Figure 3. The manner in which it was written was derived from the previous discussion. Then, in response to the teacher's request, the student explained how it had been done.*]

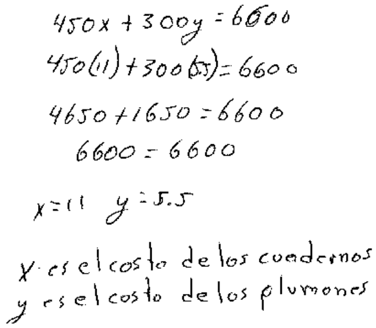


Figure 3. Procedure followed by the student in writing one of the equations on which the PROBLEM 1 was modeled.

The teacher asked the student if this equation was the only one, and as no answer was forthcoming, began to give arbitrary values to *x* to determine the value of *y* and in this manner to show that the identified equation could have many solutions. Student 1 did not seem to understand what the teacher was asking, but was sure that the selection of the values of *x* and *y* could not be arbitrary (Excerpt 4). This was taken up by the teacher in order to emphasize in the symbolization of that other equation in the problem.

Excerpt 4 (How do you symbolize twice something?)

L17. Student 1: Here it has to be double.

L18. Teacher: No, what I want is that *x* to be two [*insisting upon giving arbitrary values*].

L19. Student 1: Ah, okay.

L20. Teacher: Why does it have to be double?

L21. Student 1: Because that indicates that the cost of the pens is… [*was interrupted*].

L22. Teacher: Ah. That means that there is another condition in the problem, that it is not only this.
L23. Student 1: Yes.
L24. Teacher: How do you write this other condition?
L25. Student 1: It would be *y=x_*. [*With this equation the student wanted to express the relation of "each notebook cost twice that of each pen"*].

It was difficult for the student to translate the relation "twice something" of verbal language into symbols. The student practiced with expressions such as $x = y^2$, $y = 2x^2$, $y = x^2$, which were discarded after substituting values into them, until the student identified $y + y = x$ as the other equation being searched for.

Discussion of the Results and Conclusions

In the previous section two processes can be observed in the understanding of meanings: an individual process and a social process. The identification and symbolization of the two equations on which the problem are modeled were the product of the contributions and analysis of the students as a group, along with the teacher. However, in spite of the fact that the interaction of Student 1 permitted the whole group to identify the system of equations, student's knowledge was not sufficient for form a base from which to work and give meaning to the system of identified equations.

In the interview the student went deeper into the meaning of an equation and its solutions, which was important in order to give meaning to the system of equations identified in class. However, other difficulties emerged which had not arisen in the group discussion. In the first part, it was difficult for the student to determine how many equations needed to be written. In the second part, the student had difficulty with the translation of the verbal language into a symbolic language with the relation of "twice something", even when in the group session this relation had previously been symbolized.

The expression $450x+300y$ came out of the work with spreadsheet in the initial process of labeling the columns. For this reason, if the students had related columns B and E through some formula perhaps it would have been easier to generate an algebraic rule to represent the relation of "twice something". However, the technique they used, or the manner in which they had solved the problem, had been based on the use of recursive procedures, over all in columns B and E. These recursive procedures had been selected from the observation of the series of values calculated in these columns. The theory underlying the written recursive formulas seemed to be derived from previous teaching about the search for patterns when there are a series of values.

With the use of this technique the students showed reasoning and routine work. Students analyzing the series of values calculated on columns B and E and they connected them among them using recursive formulas. However, the pragmatic value of this technique, as well as the epistemic value, was not adequate for the modeling of the problem in terms of equations.

In this case, the organization (Figure 1) of the whole information in spreadsheet was exploited in focusing the students to reflect upon the problem, to relate the data in other manners, and to write the system of equations.

However, the transition to algebraic symbolization using spreadsheet environment was not immediate, nor were the processes of the use of algebraic symbols or the elaboration of their meaning. This required the teacher's input and the teacher's becoming involved in both the design of the tasks or problems and in the discussion of the techniques and theory used by the students.

Acknowledgments

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SEVENTH GRADERS' GENERALIZATION STRATEGIES INVOLVING DECREASING LINEAR PATTERNS: COGNITIVE COMPLEXITIES OF TRANSFER

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This paper reports on the results of a teaching experiment with seventh grade students who are members of the same middle school class that is involved in an ongoing longitudinal study that seeks to investigate the development of students' algebraic thinking over three years. We report on the results of eight students who participated in a pre-clinical interview, a teaching experiment over three months, and a post-clinical interview. We report here on the results of two tasks that involve decreasing linear patterns. Students who were successful in finding general forms for increasing linear patterns had difficulty doing so for decreasing patterns. We analyze the cognitive difficulties students exhibited using the theoretical construct of transfer.

In the *21st NCTM Yearbook*, Rosskopf (1953) claims that transfer theories change with developments made in learning theories. For example, faculty psychology prior to the late 1800s claimed that the rigorous work that came with learning formal disciplines and doing hard exercises would successfully facilitate transfer. It was discredited in the early 1900s and replaced with the doctrine of identical elements as a consequence of developments in theories of associationism and connectionism. The introduction of drill and direct practice in the mathematics classroom that consequently emerged from such theories and resulted in the necessity of establishing routine processes in learning was then again replaced with the gestaltist theory in 1912 which emphasizes the need to provide learners with time to mature and to reorganize or reconstruct their experiences. With the gestaltist view came the practice of a developmental approach to learning which saw grouping, reorganizing, structurizing, etc. as fundamental in effecting maximum transfer. Rosskopf (1953) then extracted the following compatible instructional principles: teach for understanding and for concept formation; provide opportunities for students to discover and explore through examples that illustrate the same concept; apply concepts to new problems, and; use acquired principles in examples without the need to memorize the principles. Such practices were the views in the 1950s with a reminder from Rosskopf (1953) that newer perspectives were bound to take shape. He further recommends that "(n)ot only do we need to learn more about what is transferred, but we need to experiment to see how transfer can be facilitated [in larger percentages]" (p. 220).

The current discourse on transfer foregrounds the significant role of the learner as a consequence of developments in research methodology, in particular, advances made in classroom-based design experiments. Traditionally, transfer involves using an existing knowledge acquired in one context in order to handle new or similar information in another context (Anderson, Reder, & Simon, 1996). Lobato's (2003) "actor-driven transfer" basically articulates the primary and central role of the learner in seeing similarity in two different contexts. In other words, similarity is relative to the person and not necessarily the relevant knowledge itself. Thus, epistemological transfer necessitates foregrounding the perception of similarity on the knower. For example, even if knowledge about determining the slope of a

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line and a wheelchair ramp share the same conceptual context, some learners might not see this similarity at all (Lobato & Siebert, 2002).

In this research report, we address the following two research questions:

- Given seventh grade students' existing knowledge of increasing linear patterns, to what extent does such knowledge influence the way in which they deal with decreasing linear patterns? Further, do they construe both increasing and decreasing linear patterns as similar?
- What strategies do they employ in establishing a generalization for a decreasing linear pattern? Are such strategies similar to the ones used to handle increasing linear patterns?

We note that the seventh graders who participated in this study are members of the same middle school class who are involved in an ongoing longitudinal study that seeks to investigate the development of their algebraic thinking over the course of three years that started when they were in sixth grade. In several recently published research reports, we have documented the evolution of their generalization schemes in the case of increasing linear patterns (Becker & Rivera, 2006a, 2006b; Rivera & Becker, 2006). In sixth grade, while the students were successful in developing appropriate classroom mathematical practices in relation to problem situations involving increasing linear patterns, they, however, found decreasing linear patterns very difficult. One possible source of difficulty was an underdeveloped conceptual competence with the integer operations involving negative numbers which prevented them in establishing a direct, closed formula (involving negative slopes). Among the several recommendations we developed last year was to further investigate the students' ability to deal with decreasing patterns in seventh grade. In this research report, we talk about the results of the pre- and the post-clinical interviews of eight students in the class.

Theoretical Framework

In investigating transfer from Lobato's (2003) actor-driven perspective, at least three issues need to be articulated:

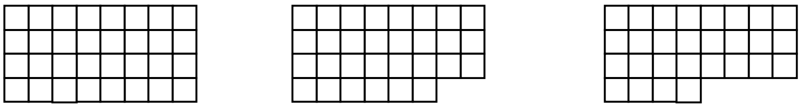
- The need to "scrutinize a given activity for any indication of influence from previous activities" and to "examine how people construe situations as similar" in order to "understand the processes by which individuals generate their own similarities between problems" (Lobato & Siebert, 2002, p. 89);
- The need to assess how transfer demonstrates "the process by which individuals see one situation as being similar to another that they have already thought about" (ibid., p. 90), and:
- The need to reconceptualize the traditional view of transfer as applying perceived or encoded knowledge with the notion that transfer involves constructing relations of similarity which consequently changes the view of transfer situations from being static and unchanging to being "dynamic sites for invention and reorganization" (ibid.).

Decreasing linear patterns take the direct form $y = mx + b$, where $m < 0$. An example is shown in Figure 1. Task situations involving increasing and decreasing linear patterns are structured similarly. Students are usually asked to determine both near (for e.g., item A in the first task in Figure 1) and far generalization (for e.g., items B and C in the first task in Figure 1) tasks and to set up a general formula for any term in the pattern (for e.g., item D in the first task and item A in the second task in Figure 1). Numerical and figural methods that are employed in establishing formulas for increasing linear patterns can also be used in the case of decreasing patterns. Hence, it makes sense to investigate students' ability to generalize decreasing linear patterns from a "transfer perspective."

Methods

A teaching experiment on patterning and generalization, in which decreasing patterns was one among several topics, was conducted in a 7th grade class during Fall 2006 over the course of three months. A pre-clinical videotaped interview was conducted with 10 students to assess their knowledge and understanding of decreasing linear patterns. In the post-clinical videotaped interview, only eight of the 10 students were available for analysis. We note that the 10 students were intensively investigated in the previous year. Figure 1 contains the two tasks on decreasing patterns that the ten students worked through in the pre-interview. Analogous tasks were given in the post-interview but space precludes inclusion here.

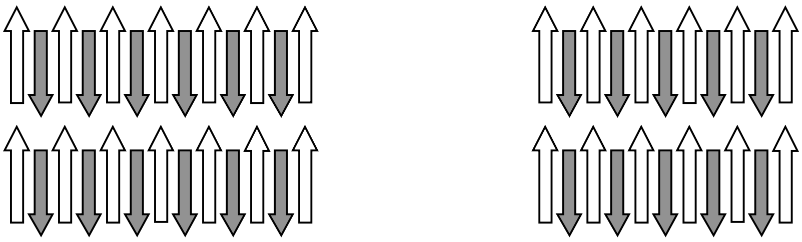
1. Losing Squares Pattern. Take a look at the three different stages in the design below.



Stage 1 Stage 2 Stage 3

A. How many squares are there in stage 1? stage 2? stage 3? **B.** How many squares are there in stage 10? How do you know for sure? **C.** How many squares are there in stage 15? How do you know for sure? **D.** Find a direct formula for the total number of squares in stage n, where n is a positive integer. If you obtained your formula numerically, what might it mean in the context of the above pattern? **E.** How many squares are there in stage 20? What might your answer mean in the context of the given pattern?
F. For what stage number will there be no more squares left? How do you know for sure?

2. Arrow Design Pattern. Take a look at the three different stages in the design below.



Stage 1 Stage 2

A. Find a direct formula for the total number of arrows at any stage n.
If you obtained your formula numerically, what might it mean in the context of the above pattern? **B.** How many arrows are there in the 10th stage? Explain. **C.** For what stage number will there be no more arrows left? How do you know for sure?

Figure 1: Pre-Interview Tasks in Compressed Format

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Results

Pre-interview Results

During the pre-interview, only three of the eight students were able to develop a direct formula for at least one of the tasks; note that all of these students could successfully find direct formulas for increasing linear patterns at the end of sixth grade. One student attempted a recursive formula, using a construction that she was able to explain to develop correct far generalization tasks. For increasing patterns, with a positive number given for the dependent variable, these students regularly could interpret a question such as part F of Task 1 to mean to work backwards and could complete such a task. Given zero as the dependent variable value for a decreasing pattern, none of them approached the task by working backwards. One student used guess and check (what times -2 yields zero); four made a table until they arrived at the zero value; two used mental arithmetic; and one acted it out with squares. For seven students, interpretation of a negative number of squares was a variant of shading the squares or otherwise indicating they were negative; only one said you could not have negative tiles.

Post-interview Results

In the post-interview, six students were able to find a direct formula for at least one decreasing pattern and use that to substitute in to find values for far generalization tasks; this includes the three who were able to do so in the pre-interview. To find the stage number that yields zero squares, again no students worked backward using zero. This time four students started with stage 20 (-6 squares) and worked backwards from there; one student counted off by twos as a way to check his answer for stage 20 (from use of the formula) and along the way found the answer for zero; and one student persisted in guess and check. Now four students said one could not have negative values; two persisted in coloring the squares differently; and two had no response.

Discussion

Given seventh grade students' existing knowledge of increasing linear patterns, to what extent does such knowledge influence the way in which they deal with decreasing linear patterns? Further, do they construe both increasing and decreasing linear patterns as similar? Results of the *pre-interview* indicate that seven of the eight students were unable to use the numerical methods they developed in relation to increasing linear patterns to deal with the two decreasing patterns tasks. While the students tried to apply a familiar numerical method (e.g., finite difference) in obtaining the form $y = mx + b$ in both patterns, what prevented them from obtaining the values could be traced to the negative values of m that consequently prevented them from obtaining the values for b as well. In other words, the combined operation of multiplying by m and adding b was complicated by the fact that the values of m were negative integers. Also, even if the students employed a table or a listing method in dealing with the far generalization tasks (item B in both tasks in Figure 1), most of them could not make sense of the negative values of the dependent terms. Thus, in terms of "transfer," the students had difficulty seeing similarities between increasing and decreasing linear patterns. This difficulty in transfer was again manifested in items C and F in Figure 1 when some of the students who managed to obtain a direct formula for the two patterns did not use their knowledge of "working backwards" in finding an independent term. While results of the *post-interview* indicate some progress in dealing with decreasing patterns, only six of the eight students managed to obtain correct direct formulas. The students at this stage have overcome conceptual difficulties relevant to integer operations involving negative numbers. However, in both pre- and post-interviews, most of the students were unable to justify their generalizations, say, in a visual way, which they could easily do in the case of increasing patterns.

What strategies do they employ in establishing a generalization for a decreasing linear pattern? Are such strategies similar to the ones used to handle increasing linear patterns? In the pre-interview, given that the students had some knowledge of numerical and figural methods in obtaining direct formulas in the case of increasing patterns, seven of the students developed recursive instead of direct formulas. The recursive method was the most frequent method they employed prior to any formal discussion of increasing patterns when they were in the sixth grade. Hence, initial strategies in dealing with unfamiliar tasks were similar in structure. In the post-interview, we note similar strategies as well.

Why is transfer difficult? The middle school students in this study found generalizations involving negative differencing a difficult task to accomplish. In Year 2 of our longitudinal study, we saw that the seventh graders' primary cognitive difficulty with decreasing patterns prior to a teaching experiment was how to handle negative differencing and, especially, how to perform operations involving negative and positive integers in which the rules were not consistent with their existing core arithmetical domain (Rivera & Becker, in press). While we found that they were attempting to "transfer" the existing generalization process they established in the case of increasing linear patterns, they could not, however, make sense of the negative integers and the relevant operations that were used with such types of numbers. For example, Tamara was first asked to establish and justify generalizations for two increasing linear patterns, which she accomplished successfully. Her generalizations were constructive and standard, and she was also able to justify the equivalence of several linear forms with the ones she developed. When Tamara was then asked to obtain a generalization for the Losing Squares Pattern in Figure 1, she immediately saw that every stage after the first involves "minusing 2" squares. She then used multiplication to count the total number of squares at each stage. When she then proceeded to obtain a formula, she was perturbed by the negative value of the common difference and said, "I was trying to think of, just like the last time, I was trying to get a formula. … I was thinking of trying to do with the stage number but I don't get it." The presence of the negative difference, including the necessity of multiplying two differently-signed numbers, partially and significantly hindered her from applying what she knew about constructing general formulas in the case of increasing patterns. In fact, she had to first broaden her knowledge of multiplication to include two factors having opposite signs before she was finally able to state the form $S = -2 \times n + 34$. Further, while she could explain what the numbers m and b meant in the case of increasing patterns, which for her took the constructive form $y = mx + b$, she was unable to justify the forms she established for decreasing linear patterns.

Conclusion: Implications for Teaching Decreasing Patterns

Rosskopf (1953) has echoed the practical need to find ways in which transfer could be facilitated in the greatest number of learners. In this study, we found that it was difficult for students to establish similarity between increasing and decreasing patterns in several aspects. The operations involved in setting up an increasing linear pattern were much more simple compared with those in the case of decreasing linear patterns; negative differencing proved to be too difficult for most students. Further, justifying decreasing patterns seems to require a more sophisticated understanding of the implications of domain and range. One suggestion to facilitate better transfer involving decreasing linear patterns is to use a table method in order to generate an explicit formula, say, from S = 32, 32 - 2(1), 32 - 2(2), ..., until the students are able to see the relation S = 32 - 2(n - 1). Such a formula could then be justified visually (with 32 as the initial number of squares and -2 as the taking away of two squares at each stage, and (n - 1) as representing the fact that the taking away of two squares began with stage 2). We note that the finite difference method that the students conveniently employ in the case of increasing linear patterns will not easily facilitate similarity between increasing and decreasing linear patterns. Applying finite differences in the case of the first task in Figure 1 leads to Tamara's formula, S = -2 x n + 34 which could not be easily justified and which actually confused the students even more. In fact, the students could not explain how to obtain 34 squares when the initial total was 32. They were also confused with the negative coefficient because the pattern decreased beginning with stage 2 and not stage 1.

An unresolved issue that we confronted at the end of year 2 of the study is how to help students apply the working backwards strategy in the case of decreasing patterns. For increasing patterns, they all recognized that, to obtain the independent value in a pattern, given the dependent value, they had to reverse the given operations primarily involving work with whole numbers. For decreasing patterns, students have to deal with negative integers, and the analogy between these and increasing patterns was not apparent to them. How to enable students to recognize relations of similarity leading to effective reorganization and meaningful invention of strategies relevant to decreasing patterns remains a question for further research.

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ATTENDING TO STRUCTURE AND FORM IN ALGEBRA: CHALLENGES IN DESIGNING CAS-CENTERED INSTRUCTION THAT SUPPORTS CONSTRUING PATTERNS AND RELATIONSHIPS AMONG ALGEBRAIC EXPRESSIONS (1)

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We report on a study with adult algebra learners that employed an activity to explore patterns and relationships between expressions that might suggest the sum/difference of cubes identity. Our report describes the activity and discusses aspects of expressions to which students attended. Our results to point to a need for refinements to this activity that might better support re-invention of the identity.

Although it has been shown that technology can be used to support student understanding of calculus (Heid, 1988), there continues to be debate about the role that technology can and should play in algebra learning (NCTM, 1999). We report on a study with adult learners that employed an activity involving a computer algebra system (CAS) to explore patterns and relationships between expressions that might suggest the sum/difference of cubes identity $(a + b)(a^2 – ab + b^2) = a^3 + b^3$. This activity was inspired by the work of Goldenberg (2003), who outlined a similar activity designed to support students in re-inventing the difference of squares identity $(a – b)(a + b) = a^2 – b^2$.

In an effort to understand the sources of their difficulties in deriving the sum/difference of cubes identity, our report explores aspects of expressions to which students attended. We detail the instructional activity and its outcomes, and we suggest refinements to the activity that might better support this population of students in achieving the intended learning goals.

Setting and Participants

Our study was conducted in the context of an intact intermediate-level algebra course offered at a large urban university in the Pacific northwestern United States. The course is designed for students who have not had Algebra II or who require a review of elementary algebra concepts and techniques. The students in this class have a wide variety of dispositions and experiences related to mathematics, a variety of educational backgrounds and interests, and a wide spectrum of ages. The 15 students who participated in our study reflect this diversity.

We initiated our study during the third week of classes, after students had developed some experience in multiplying polynomial expressions and factoring perfect square trinomials and differences of squares. The data we report on are drawn from students' individual written work on the first two parts of the activity. We examined their work for evidence of patterns they might have construed from the sum/difference of cubes expressions generated by the CAS and for connections they drew between those expressions and their factored forms.

Summary Description of the Task

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The Cubes activity was designed to support students in deriving the identities for factoring sums and differences of cubes. The activity began by having students explore patterns arising from the multiplication of a given sequence of a binomial and a trinomial whose product results in a sum/difference of cubes. This was followed by having students use those patterns to generate the sum/difference of cubes identity. Specifically, students were presented with factored forms of five sums and differences of cubes, each of which they were asked to multiply out using the EXPAND command on their TI-92 calculator. Students recorded the results of these expansions in the appropriate cells of a table (Figure 1), and they were directed to investigate whether the "indicated multiplication of factors produces interesting results".

Factored form	Expanded form displayed by the calculator
1. $(x + 2)(x^2 - 2x + 4)$	

Figure 1. A portion of a table that students completed in the first part of the activity.

The intent in having students use the CAS to expand the products in this part of the activity, rather than requiring them to do so by hand, was to focus their attention on the form of each final expression, and to reduce the potential of diverting their attention to the mechanics of algebraic manipulations. The next part of the activity aimed to orient students' attention towards patterns in the expanded expressions recorded in their table. To that end, the students were directed to take note of the form of each expanded result produced by the CAS and to "describe how this form relates to that of the corresponding factors".

Analyses and Results

Our analysis of the data unfolded in a sequence of interrelated phases. Following Saldanha & Kieran (2005), we began by listing relationships of form and structure among expressions to which the students might be expected to attend. We then examined students' responses for evidence of explicit or implicit attention to these aspects. On the basis of this first examination we then revised and refined our initial list of aspects to arrive at a comprehensive list of them. The authors then each re-examined the data independently, coding it according to the dimensions in this list and documenting the frequency of their occurrence in the students' responses across task questions. In a final phase, the authors compared their code assignments and resolved their few differences through a process of negotiation that involved re-examining relevant parts of the data whenever necessary. This process converged to a 100% agreement in the coding of the data.

The table below displays a sampling of our results; it describes some of the aspects of the expressions to which we hoped the students would attend and it gives the frequency with which students actually attended to each of them. The aspects are listed in descending order, from the most to the least salient for students, as indicated by our analysis.

Aspect of expressions	**Students who attended to aspect**	
	#	%
Expanded version of each expression is shorter than the factored version	12	80%

The first term of the expanded version is the product of the first term in the binomial and the first term in the trinomial $(\boldsymbol{a} + b)(\boldsymbol{a}^2 - ab + b^2) = \boldsymbol{a}^3 + b^3$	10	67%
The first term of the expanded version is a cube $(a + b)(a^2 - ab + b^2) = \boldsymbol{a}^3 + b^3$	8	53%
The second term in the expanded version of the expression is a cube $(a + b)(a^2 - ab + b^2) = a^3 + \boldsymbol{b}^3$	2	13%
The last term of the trinomial is the square of the last term in the binomial $(a + \boldsymbol{b})(a^2 - ab + \boldsymbol{b}^2) = a^3 + b^3$	2	13%
The middle term of the trinomial is the product of the two terms in the binomial (sign withstanding) $(\boldsymbol{a} + \boldsymbol{b})(a^2 - \boldsymbol{ab} + b^2) = a^3 + b^3$	1	7%
The first term of the trinomial is the square of the first term in the binomial $(\boldsymbol{a} + b)(\boldsymbol{a}^2 - ab + b^2) = a^3 + b^3$	0	0%

Discussion and Conclusion

The results suggest that, in general, the students did not attend to most of the aspects of the expressions that we deemed important. Due to space constraints, we only discuss a few of the aspects and results in this paper. For instance, of the twelve aspects we targeted, only four were noted by more than 50% of the students. Only one aspect—that the expanded forms of the expressions are shorter than their factored forms—was noted by more than 75% of the students. It is important to note that this modal aspect is arguably the least useful one in terms of helping students generate the intended identity. Moreover, the characteristics apparently noted by the fewest students centered on relationships between the terms in the factored form of the expressions, which are arguably the most useful for generating the identity.

These findings suggest that the expanded versions of the expressions were much more salient for students than were the factors themselves or relationships among the factors. This state of affairs is consistent with, and might therefore be attributed to, students' being inattentive to how the expanded forms of the expressions arose from the factored forms. This hypothesis, in turn, raises questions about whether our activity prompts adequately provoked students to reflect on *why* the expanded forms of the expressions are shorter than their corresponding factored forms. Indeed, since the students did not attend to aspects that could form the basis of a derivation of the sum/difference of cubes identity, our activity sequence stands to benefit from prompts that orient students to reflect on why only certain terms appear in the expanded form of the product of expressions. Changes such as these might better support this population of students in attending to crucial aspects of these expressions and structural relationships among them.

Endnote

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DELINEATING FOUR CONCEPTIONS OF FUNCTION: A CASE OF COMPOSITION AND INVERSE

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Four characterizations of students' conception of function have emerged from the research: (1) Prefunction (2) Action, (3) Process, (4) and Object. Our goal was to further define this framework in terms of student understanding of composition and inverse of functions. From our analysis, we identified levels of sophistication within three of the four conceptions of function. We discuss the implications.

Research has documented the importance of students' progression from procedural to a more dynamic view of function (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & Harel, 1992; Sfard & Linchevski, 1994). Of equal importance is the ability to make connections between the representations. "Understanding a concept in one representation does not necessarily mean that one understands it in another…. Different representations give different insights which allow a better, deeper, more powerful and more complete understanding of a concept." (Even, 1990, p. 524)

Oehrtman, Carlson and Thompson (in press) provide an overview of essential processes involved in knowing and learning the function concept. One focus of their discussion is the notion of action and process conceptions of function (cf. Oehrtman et al, in press; Dubinsky & Harel, 1992). From their investigation, the questions arise: Can we rigidly characterize students' understanding as either process or action? Are there other categories? And, if so, what characterizes them?

Interpretive Framework

The research efforts mentioned above contribute to the knowledge of how to characterize students' understanding of algebra, and more specifically, functions (Breidenbach et al, 1992; Dubinsky & Harel, 1992; Oehrtman et al, in press). Four characterizations of students' conception of function have emerged from the research: (1) Prefunction (2) Action, (3) Process, and (4) Object conceptions (Breidenbach et al, 1992).

Our goal is to further define this interpretive framework in terms of student understanding of composition and inverse of functions. In particular, if the move from one conception to another is transitional marked by oscillations between conceptions, can we categorize the transitional steps? Are there subcategories of these conceptions? In this paper, we investigate students' approach to composition and inverse function tasks to further identify characterizations of students' understanding of function.

Method

We conducted seven, 40 minute, think-aloud interviews. Our protocol was made up of a synthesis of selected tasks from the Pre Calculus Concept Assessment Instrument created by Carlson and her research group, and the tasks we generated to further understand students' thinking about composition and inverse of functions. The participants were all enrolled in a Pre-Calculus course at a large university at the time of the interviews. In each interview session, one researcher took field notes while the other served as an interviewer also taking notes. All interviews were audio taped and reviewed by the researchers. Portions of the audiotapes were transcribed for further analysis. Starting with the four conceptions of

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function defined by Breidenbach et al (1992) the written work of the students, transcripts of the audiotapes and the audiotapes were analyzed in a cyclical process of coding and search for confirming and disconfirming evidence (Strauss & Corbin, 1990) to delineate the categories. Although we identified levels of sophistication for three of the four categories, in this report we will only discuss the levels within the action category.

Results

As a result of this research, we were able to partition three of the four characterizations of students' conception of function with respect to composition and inverse of functions. We identified levels of sophistication within the prefunction, action, and process conceptions defined by Breidenbach, with the goal of delineating the framework in order to have a better understanding of the process of how students come to understand function composition and inverse. Below we delineate three levels of sophistication within the action conception.

The action-0 conception is marked by a flawed conception of mathematical ideas. In particular, students may take actions that are contrary or inconsistent with definitions or concepts accepted by the mathematical community. A student with such a conception often confounds or has an incorrect recollection of previously learned concepts.

Davis (1984) points to evidence of the existence of frames or schemas observable across different people. In our study, we found evidence for a common schema for the word "opposite" to describe the how the inverse of a function relates to the original function algebraically. Task 14 asks "Which of the following best describes the effect of f^{-1}, given f is a one-to-one function and $f(d) = c$?" For all 7 students, algebraically the "opposite" referred to the reciprocal of either the output or the argument of the function. That is, if $f(d) = c$ then $f^{-1}(d) = 1/c$ or $f^{-1}(d) = 1/d$. Student A asserted, "f inverse is one over something." This suggests the subject is perhaps confounding properties of quantities with negative exponents with inverse function notation.

We describe an action-1 conception as superficially correct. A student with this conception is able to take reasonable action based on memorized facts with little to no connections to meaning of the action. Initially in their response to the task, "What does f^{-1}, mean?" 5 students made reference to f^{-1} switching the domain and range or switching the x and the y. Yet when given the function $f(x) = 2x - 1$ and asked for f^{-1} the students' responses revealed that they made little to no connections to meaning. Student C attempted to find the inverse by setting the function equal to zero and solving for x. The interviewer asked Student D "What would the inverse do to the point the point (a, b), if we know (a, b) is a point in f"? Student D asserted that the inverse would not change the point (a, b). Student E computed the inverse by replacing y for $f(x)$, then switching the y and the x in the equation and solving for y. Both students F and G switched the x and y in the function and were unsure what to do next. However on task 14 all suggested either if $f(d) = c$ then $f^{-1}(d) = 1/c$ or $f^{-1}(d) = 1/d$, which contradicts their previous responses. Moreover, in the process of elimination, they all rejected the first multiple-choice item "f^{-1} "undoes" what f does, so $f^{-1}(f(d)) = d$" asserting that "you cannot undo a process." While their initial responses would suggest they have a concept definition (Vinner & Dreyfus, 1989) for inverse of functions, their actions reveal a lack of meaning for a correctly memorized fact.

The action-2 conception we identify as being equivalent to that of Breidenbach's action category, but we note that it is the idea the student has that must be correct, thus arithmetic errors that lead to incorrect solutions are not taken into account. Student responses to tasks coded as action-2 generally displayed procedural knowledge of composition by using a given numerical input to find and output, then using that input to find another output. They could also use tables or expressions to do compositions.

Discussion

Of the 135 tasks coded, 82, or 61% were coded within the action category. The subjects' actions were generally limited to symbol manipulation. Often students who could take no action on a particular task said that they could complete the task if given an explicit equation.

The students displayed compartmentalized notions about inverse. Some could apply well-defined procedures routinely to algebraic expressions. Those whose notion of inverse was the "opposite" or reciprocal did not consider restrictions of the domain of rational functions. They drew "inverse" functions as continuous graphs, reflections of the original graph about the y-axis. Even when faced with the contradictions of their algebraic manipulations, switching the *x* and the *y* and "flipping" the graph about the y-axis they believed the flaw lay in incorrect algebra manipulations instead of conflicting notions.

Moschkovich, Schoenfeld, and Arcavi (1993), Even (1990), and Fridelander and Tabach (2001) point to the importance of the ability to make connections between the representations that define structure in order to develop clear notions about function. A lack of connection between representations of function became a salient deficiency across subjects. Only on lateral and horizontal shifts of the graph did some subjects connect the algebraic representation to the graph and to a lesser extent table.

We do not suggest that our categorizations are exhaustive. That is, our set of data may not have been sufficiently large enough to completely partition the conceptions of function. Nonetheless from the data we collected we were able to further define the analytic framework. Also the intention of our analysis is not to characterize a students' way of thinking as restricted to a particular conception but to describe observable approaches that serve as a lens to understand student thinking. Moreover we argue that perhaps it is the problem type itself that may elicit a particular conception in ones response. In particular, it may not be necessary to express an object or process conception to respond to a task but this does not mean that the subject cannot conceive of the task in that way.

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FROM ARITHMETIC REASONING TO ALGEBRAIC REASONING: PROBLEMS OF REPRESENTATION AND INTERPRETATION OF SYSTEMS OF LINEAR EQUATIONS IN A MIDDLE SCHOOL CLASSROOM

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This study uncovers many problems that 8th grade students encountered when trying to make sense of quantitative situations that could be represented by systems of linear equations in two variables, and graphs of those equations. Mathematics education research from a cognitive perspective over the past 15 years has brought to light the cognitive complexity of school algebra. By coordinating analyses of classroom interactions, student interviews, and teacher interviews concerning those interactions, this study sheds light on some of that complexity and offers an alternative view for helping students and teachers address that complexity: linking quantitative and orthographic reasoning.

Theoretical Framework

Several approaches to the study of algebraic reasoning focus on algebra as "generalized arithmetic," (Carpenter & Levi, 2000; Kaput, 1998) making the assumption that procedures and inscriptions students learned in solving problems in which all values are known would transfer to solving problems in which some values are known and others are unknown, or in which some values could vary. In contrast, we take the position that algebra is much more than generalized arithmetic. Reasoning algebraically requires a shift in goal structure from computing an answer to generating a relation. Such a shift is not easily achieved. It requires a focus on reasoning with quantities rather than on computing with numbers (Blanton & Kaput, 2000; Thompson, 1990). On top of that shift in goal structure and focus is layered a new syntax with its own orthographic rules that are different from the syntax rules of arithmetic (Mikulina, 1991), and graphical representations that need to be interpreted by students, but are not necessarily interpreted as intended (Chazan & Yerulshalmy, 2003). This study builds on our previous work with learner's difficulties with quantitative units when solving word problems (Olive & Ça_layan, 2006) as well as prior research in this domain (Schwartz, 1988; Behr, Harel, Post & Lesh, 1994; Kieran & Sfard, 1999).

Context and Methodology

This study took place in an 8th-grade classroom in a rural middle school in the southeastern United States. The 24 students were between 13 and 14 years old and had been placed in the algebra class based on their success in 7th-grade mathematics. The students were racially and social-economically diverse, with an approximately equal distribution of gender. The first eight class lessons on a unit that focused on graphing and solving systems of linear equations were videotaped using two cameras, one focused on the teacher and the other on the students. Four students were interviewed twice in pairs (a pair of girls and a pair of boys) during the three weeks of the study. The classroom teacher was also interviewed twice during the three weeks. All interviews were videotaped. Excerpts from the classroom videotapes were used during both student and teacher interviews to provoke discussion of the learning that was taking place in the classroom. Excerpts from the videotapes of student interviews were also used in the teacher interviews. This paper focuses on selected problems from Unit 6

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of College Preparatory Mathematics, Algebra 1 (CPM, 2002). Some problems were based on contextual situations involving co-varying quantities that could be modeled by a system of two linear equations, others focused on notational form of linear equations (e.g. y-form) and most involved graphing the linear equations to find a solution to the system as well as using algebraic substitution to find a solution.

Each day the classroom video data from the two cameras were viewed and digitally mixed using a picture-in-picture technology. A written summary of the lesson with time-stamps for video reference was created from the mixed video. These written "lesson graphs" were then used to select excerpts from the classroom video to be used in the student or teacher interviews, and to plan questions and related problems to pose to the interviewees in an effort to understand how the students (and teacher) had interpreted the problems and the classroom discussions that followed from different students' attempts to address the problems. Subsequently, the corpus of classroom video data was reviewed, along with the related lesson graphs to generate possible themes for a more detailed analysis. All student and teacher interviews were transcribed from audio files created from the videotapes of the interviews. A chart of relationships among class lessons, student interviews and teacher interviews was then created. A retrospective analysis, using constant comparison methodology, was then undertaken during which the classroom video, related student interviews and teacher interviews were revisited many times in order to generate a thematic analysis from which the following results emerged.

Results and Conclusions

Issues that arose from our analyses include the following:

- Distinction between an equation in one unknown (e.g. 75=10x) and an equation as a linear function (e.g. y=10x), where both are representing a journey traveled at 10 mph for x hours, and the total trip is 75 miles.
- The algebraic notation "x" standing for both a fixed, unknown quantity and a variable.
- A disconnect between the written equation and the graph representing the situation (e.g. do students understand that all points on the line graph satisfy the equation?).
- Graphs as discreet, point-wise maps of pairs of numbers derived from tables of values.
- Surface association among the numerical values in an equation and coordinates on the graph (e.g. coefficients of x-terms determine the intersection point of two lines).
- Interpretation of "slope" of a line on a graph as iterations of discreet composed units (e.g. "2 up and 1 over") or as "composed unit of differences" (Lobato and Siebert, 2002).
- Solving equations by a series of arithmetical operations (over-generalizing arithmetic operations when using algebraic notation: e.g. -3x=-4x+2 reduces to -4x=-4+2 by taking an x from each side of the equation) and problems with negative variables (Are their *values* always negative?).
- Confusion with the substitution method for solving a system of two linear equations when both equations are equal to the same goal quantity – the solution may not solve either equation (e.g. saving for a bike that costs $300 where the saving rates are different).
- Coordinating quantitative units – especially in situations involving both extensive and intensive units (Olive & Ça_layan, 2006; Thompson, 1990).
- Teacher's selection of equations to solve that help students grasp the mathematics.

One conclusion that emerged from this study is that many students' problems arose from over-generalizing arithmetic procedures that were used successfully in solving situations with known, fixed values, to situations where quantities varied or were unknown (e.g. 5+.25=5.25

but 5+.25x≠5.25x). Another conclusion that we draw from this study is that for many students there is a disconnect between their graphical representations of situations and the algebraic equation representing the same situation. As Chazan and Yerulshalmy (2003) point out, "graphical representations are complex. One cannot simply expect students to be able to read these representations in the ways they are intended" (p. 131).

Overcoming these difficulties requires a shift in students' ways of thinking about quantitative situations, and appropriately designed pedagogical situations along with scaffolding by a sensitive teacher (attuned to students' difficulties) to encourage the shift from thinking about specific values in a situation to coordinating varying quantities. This shift in quantitative reasoning needs to be linked with a necessary expansion of students' orthographic reasoning; they need to learn a new notational syntax that does NOT follow the same rules as the syntax of arithmetic. For example, when we add 50 and 3, we make a contraction of the two numbers and write it as 53, but we cannot do the same for adding 50 to x or adding 50 to 3x or adding 5 to x; but we CAN make a contraction when multiplying 5 by x to get 5x. Our study suggests that linking such algebraic expressions with real quantities and emphasizing the name-unit coordination of those quantities may help students to come to terms with the rules of this new syntax.

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SECONDARY MATHEMATICS TEACHERS' UNDERSTANDING OF EXPONENTIAL GROWTH AND DECAY: THE CASE OF BEN

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This report focuses on a case study of one secondary mathematics teacher and his development of conceptualizing multiplicative behavior in the context of real-world applications involving exponential functions. Data collected through interviews and teaching experiment methods provided evidence for how emphasizing exponentiation as a process can result in an increased ability to describe exponential behavior in robust and powerful ways.

Exponential functions are one of the most important function families taught in high school algebra and expanded upon in college level mathematics courses. This function family provides a model for representing multiplicative growth and decay patterns emerging from our everyday world. It is vital that mathematics teachers provide meaningful opportunities for students to develop robust conceptions of exponential behavior to enhance students' ability to make sense of the world around them.

In this discussion, the researcher presents a description of a secondary mathematics teacher, Ben, and his ways of thinking about exponential behavior. This story focuses on addressing the research questions of this study: (a) What conceptions does Ben hold about the notions of exponential growth and decay? and (b) in what ways does an instructional unit focused on exponential functions and multiplicative reasoning facilitate the development of Ben's understanding of exponential behavior?

Theoretical Framing of the Study

Drawing from the research literature on exponential functions by Confrey and Smith (1994) and Weber (2002), the researcher developed a framework to characterize the notation, language, and reasoning abilities needed to posses a mature understanding of exponential functions. The purpose of this framework was to provide guidance in developing curricular activities on exponential functions, as well as to direct data collection and analysis of this research project. One component of the framework included exponential notation which can oftentimes be a source of difficulty for students due to impoverished understandings of the exponentiation process (Weber, 2002). Findings from Weber's study suggest that students benefit from a more conceptual understanding of exponent operations as opposed to relying on rote memorization.

This report focuses on Ben's development in conceptualizing b^x as representing x factors of b for rational values of x using appropriate notation and language. Interpreting b^x in this manner proved to be a critical conception in transitioning to conceptualizing exponentiation as a process. In this report, the researcher concentrates on this one component of the exponential function framework to provide evidence for how emphasizing exponentiation as a process can result in an increased ability to describe exponential behavior in more robust and powerful ways.

Methods

This case study focused on Ben who was enrolled in a graduate-level mathematics education course, called *Functions: Mathematical Tools for Science*. This exploratory investigation employed task-based interviews and teaching experiment methodology to study teachers' ways of thinking about exponential behavior (Steffe & Thompson, 2000). The researcher conducted two interviews, which were structured around 10 mathematical tasks involving exponential functions, multiplicative reasoning and covariational reasoning. The teaching experiment consisted of six two-hour episodes conducted over the course of four weeks. All interviews and teaching experiment episodes were videotaped. All data was analyzed qualitatively for the purposes of describing in detail teachers' ways of thinking. Qualitative analysis included retrospectively analyzing all videotaped sessions and coding teachers' utterances based on the framework.

Results and Discussion

Ben's ability to describe exponentiation in robust ways seemed to evolve over the course of this study. The interview data, which provided an initial evaluation of Ben's understanding of exponential behavior, suggest that Ben held an image of exponential behavior as a limited recursive process. While Ben was able to think about statically multiplying previous outputs by a factor to obtain the next output value, he was unable to extend this process of recursion for all values of the domain.

In one interview task on comparing a multiplicatively increasing salary (growing by constant percent) with an additively increasing salary (growing by constant amount), Ben was able to articulate the idea of changing rate of change relative to constant rate of change. However when asked to determine the two salary values after 10 years, Ben was unable to use the exponentiation process of repeatedly multiplying the previous value by the given percent for the number of years necessary even though he recognized that "your raises are going to continue to increase as your salary continues to increase because it's dependent on whatever that previous year's salary was." Excerpt 1 illustrates Ben's inability to extend his static image of exponentiation as recursion to more dynamic images of recursion.

Excerpt 1

Ben: You can easily predict what your [additive] salary's going to be in 10 years because that means your salary went up $10,000 [$1000 for 10 years] whereas with the [multiplicative salary] it's a little more difficult to predict where 7% is going to take you in 10 years because it's 7% on top of whatever your previous salary was…you would have to know what year nine was.

This interview passage illuminates Ben's ability to conceptualize the process of repeated addition for linear function situations (lines 1-2) yet he could not apply this notion to the process of exponentiation as repeated multiplication. The evidence gathered in this study revealed how deep-seeded Ben's notion of exponential (constant percentage) growth is within the recursive model of how a future output value is calculated based on the previous output value. The notion of exponential behavior as recursion seems restrictive for Ben in thinking about exponential behavior in terms of an exponentiation model where the growth factor is multiplied by itself by the number of compounding periods (i.e., the number of years passed). Using this dynamic exponentiation as recursion model, the salary after 10 years could be found by taking the initial salary of \$35,000 multiplied by $(1.07)^{10}$ where the growth factor of 1.07 serves as the multiplicative object connecting the initial value and the desired output for a given input value. This model provides the opportunity to think about exponential behavior beyond recursion when previous output values are unknown. Ben did not realize how to jump from an initial salary to the tenth year salary without knowing the intermediate salaries through year nine. This data seems to suggest evidence of Ben's inability to continue the process of repeated multiplication and connect this process with exponentiation. As a result, Ben's model of exponential behavior as recursion restrained him from considering how this process can be continued indefinitely.

During the teaching experiment sessions, Ben was able to make advances in his understanding of exponentiation. Activities for the teaching experiment were explicitly designed to build Ben's ability to use language and notation to represent b^x as x factors of b. Towards the end of the teaching experiment, Ben was able to describe how $(0.635)^x$ could be conceived as having "infinite amounts of the 0.635's you could be multiplying together." Ben followed up by explaining "that's why it's never going to get down to zero cuz there's always going to be a 0.635 you can multiply in there further down the line." During the last episode, Ben was able to describe e^{rt} as "e^r raised to the t power so when you raise to another power you multiply those two powers together it means that I have t amount of e^r so that will give me t amounts of e^r."

While most of Ben's explanations involved an integer amount of factors of the base value, he was able to provide a glimpse into thinking about the effects of multiplying by a full factor. Excerpt 2 illuminates evidence of this shift in Ben's thinking about b^t.

Excerpt 2

Ben: 1 That's how much it's growing by so like each time we add to the t we are
2 increasing by another b so a whole t I mean so like from t is one to t is
3 two I go from having one b to now I'm increasing it by multiplying in
4 another b then another b and another b as t is increasing…I have t
5 amount of b's.

In line 2 of Excerpt 2, Ben was able to articulate how a "whole t" would produce a full factor of b and thus he could continue to multiply by full factors of b for increasing values of t. Ben's ability to think about (a) full factors of b, (b) e^{rt} as $\left(e^r\right)^t$ and representing t amounts of e^r, and (c) $(0.635)^x$ as an infinite amount of 0.635's multiplied together provided powerful opportunities for Ben to conceptualize exponential behavior relative to function situations involving various contexts. The foundation for which dynamic conceptions of exponential function is built upon is first constructed by the ability to view b^x as a multiplicative process of multiplying x factors of b. This fundamental characteristic of exponential behavior can then be extended to exponential functions of the form $f(x) = ab^x$ where f is viewed as the result of the multiplicative process of multiplying the initial value by x factors of b.

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SECONDARY SCHOOL STUDENTS' PREFERENCES AND USES OF MATHEMATICAL REPRESENTATIONS

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Being able to represent the same mathematical concept differently is among many reasons why multiple representations have gained importance in mathematics education. Algebra students' attitudes towards and preferences for representations to solve mathematics problems will be discussed. Results provided will indicate that the type and difficulty of a problem might affect students' preferences along with students' previous knowledge and experiences.

It is suggested that multiple representations provide an environment for students to understand many major and abstract mathematical concepts when moving among different representations of a mathematical idea. The goal of this study is to explore students' preferences towards external mathematical representations in the problem solving process and whether those preferences change when they use a particular representation to solve that mathematical problem. Through the synthesis of various studies on students' preferences for representations, the framework for this study was constructed (such as Cunningham, 2005; Keller and Hirsch, 1998; Senk & Thompson, 2006; Waters, 2004). The framework included internal and external factors affecting students' preferences. Internal factors are primarily related to students' cognitive processes, including preferences, experience and knowledge of the mathematical subject, and beliefs about mathematics. External factors summarize features of the outside environment, which might have an immediate effect on the ways of doing mathematics. The difficulty of the problem and the way that problem is presented could influence preferences or the use of a particular representation. Pedagogical aspects, such as the use of simultaneous multiple representations or weighted use of symbolic representation in instruction, and the use of technology or graphing utilities are examples of external factors.

Methods

Algebra I students in ninth grade (n=25) in a Midwestern high school area were asked to express their preferences to solve several mathematics problems. A representation preference instrument was developed and administered twice in order to investigate students' preferences towards particular representations. The instrument is adapted from Keller and Hirsch (1998). Eight tasks were presented with three representation choices: table, graph, and equation. Two main types of tasks—four contextualized and four non-contextualized—were involved in order to investigate whether the type of question affects students' preferences. Because the type of representation students use to solve a problem could be different from their preferred representation, questions in the first administration did not ask them to solve problems, but only to indicate which representation they would prefer to use. In the second administration, students were asked to solve the same problems and indicate the representation type used. A survey with a four point Likert scale and ten open-ended questions was used in order to obtain information about students' preferences and attitudes towards mathematical representations. Students' responses to open-ended questions were coded and categorized. With the help of a panel of experts, instruments were improved and

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their content validity was obtained. Cronbach's Alpha reliability coefficient for the Representation Preference Instrument was calculated as .61. Data triangulation, member check, and peer debriefing were used to ensure the trustworthiness of the data.

Results and Discussion

Most of the students (96%) agreed that mathematics problems could be solved in various ways using different representations. Although they liked using more than one representation in solving mathematics problems (66%), they agreed it is easier to focus on one representation (72%). Ninety-two percent of the students also agreed that using different representations does not lead to different answers. Students reported that they preferred tables (56%) and equations (40%) to graphs (4%). Graphs (52.4%) and equations (42.9%) were considered more confusing than tables (4.8%). The students' responses to the open-ended questions revealed that personal inclination played an important role in preference for a particular representation. Being uncomfortable with one representation was the reason for some students to prefer a different representation. Previous experience with or knowledge of a representation and knowing how to manage it was another common reason for students to choose one over another.

Cross-tabulation was used to determine whether students' preferences in the first administration and the representation they used to solve the questions in the second administration changed or remained the same. A table was the most preferred and used representation. The dominant arithmetical experience in K-8 education might lead students to prefer tables for numerical representation. For half of the questions, students' preferences and usage of representations was not completely in agreement. It could be concluded that students' intended preferences to solve a particular problem could change when the time actually comes to solve it.

Participants seemed to have certain preferences for specific types of problems. Tables were preferred for the contextual single step problems. Graphs came more into play when more interpretation was involved. Equations were mainly preferred for non-contextual problems. Sometimes students are intentionally taught how to solve a specific type of problem using a certain representation such as detecting slope and y-intercept from an equation without any interpretation.

Students are developing their preferences towards representations where internal factors play important roles. Taking those personal preferences, previous experiences, and knowledge that students bring to the classroom into consideration; teachers could be more beneficial in creating effective teaching and learning environments. Moreover, since external factors might sometimes help and sometimes hinder students in developing their preferences, educators should be careful in designing instruction. Dominant use of a particular representation might restrict students' preferences towards or usage of different representations. Providing multiple representational environments could be beneficial for students to become aware of what kind of information a representation may offer that is different from another representation.

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TOWARDS A THEORY TO LINK MATHEMATICAL TASKS TO STUDENTS' GROWTH OF UNDERSTANDING

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Our focus was to link students' activities when engaged in a number of open-ended tasks to their growth of mathematical understanding. A teaching experiment with pre-algebra students was conducted to enhance understanding of algebra concepts. Basing the analysis of four open-ended tasks on frameworks developed by Pirie and Kieren (1994) and Lesh and Kelly (1994), findings indicate a close relationship between tasks actions required of students and students' actions related to understanding. The findings reported here are a step toward the development of a theory that links mathematical tasks to students' growth of mathematical understanding.

Introduction and Frameworks

Tasks that promote problem-solving and mathematical reasoning are powerful tools of mathematics instruction, having the potential to build meaning, make sense, and connect ideas (Lesh & Kelly, 1994). Lesh and Kelly (1994) state students' understandings are constantly evolving over time and are never complete, suggesting an iterative learning process. They propose that tasks, experiences, and/or prior models, may impede or hasten the growth of this understanding. In describing what makes a mathematical task worthwhile, Smith and Stein (1998) focus on the cognitive demands of tasks on students' thinking and identifying features of the task associated with high and/or low levels. Heid, Hollebrands, Blume, and Piez (2002) acknowledge that their definition of "task" differs from that of Smith and Stein in that they envision tasks as directed by the goals of learning and occurring within a context that uses technology as a tool of instruction. Their instrument identifies the sequence of tasks in which students, working with technology, engage. This paper addresses the role that students' actions play in the growth of understanding fundamental algebraic concepts when engaged in open-ended tasks. Christiansen and Walther (1986) speak to both task and activity as inextricably linked within the instructional context. We define tasks within the role that they played in a teaching experiment taking place during an academic year. Focused on pattern finding, the tasks were open-ended in terms of students' strategies, the representations they chose to describe the patterns, and student outcomes. In this sense, the tasks were non-routine and unlikely to have been previously experienced by students. The initial goals of the teaching experiment tasks were to develop growth in understanding of 1) early algebra concepts including variable, change, co-variation, and 2) representations of mathematical ideas including words, tables, graphs, and symbols (Mojica et al., 2007; Wilson, et al., 2007; Bell & Ives, 2007). Our focus in this paper is to link students' activities when engaged in a number of open-ended tasks to their growth of mathematical understanding.

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The first framework of this study is drawn from the work of Lesh and Kelly (1994) who proposed six student actions that define worthwhile tasks. Definitions of these actions were developed and modified based on data collected in this teaching experiment. Similarly to Lesh and Kelly, we define "model" broadly to include representations of the ideas such as charts, graphs, tables, words, formulas, or concrete materials.

- Constructs model (C): Uses manipulatives or drawings to construct what is given.
- Modifies model (MO): Makes modifications to the given stages - adding/recording data.
- Extends model (ET): Draws/constructs next stage(s) using information from given stages.
- Manipulates model (MA): Uses another representation to record the data (ie. tables, graphs).
- Explains model (EP) : Explains in words features of the task. May include how to 1) construct, verify, or modify the model, or 2) obtain the next stage numbers
- Predicts model (P): Generalizes a rule that uses two variables to predict n^{th} term.

The second framework was selected to determine students' actions related to their growth of understanding (Pirie & Kieren, 1994). This recursive, iterative, and dynamic framework consists of eight levels of student actions. We considered the first five Pirie/Kieren actions in our analysis of the tasks. These terms imply an order in that primitive knowing is the innermost action and formalizing is an outer action. *Folding back* is their term to describe non-linear actions as students move back to "thicken" inner levels of understanding.

- Primitive knowing (PK): Using previous knowledge to engage in the new task
- Image making (IM): Making an image of the task using a variety of concrete actions
- Image having (IH): Taking the visible image and mentally manipulating that image.
- Property noticing (PN): Noticing properties or features related to the task
- Formalizing (F): Generalizing a rule related to the task

Codes that explain the Table 1categories are included in parentheses after each framework term.

Method and Results

The teaching experiment was conducted during three-day, 90-minute sequences in August, November, January, and March with 18 eighth grade students taking pre-algebra. Tasks were selected that required students to work in small groups, often with manipulatives, to find patterns related to perimeter. Two graduate students with five-years of teaching experience guided instruction in both small group and large group sessions. The tasks involved students in looking for patterns to generalize rules from the data they collected. Artifacts and videotapes were sources of data collected during class time, and were coded according to the 6 task actions (Lesh & Kelly, 1994) and 6 understanding actions (Pirie & Kieren, 1994) for each of the four tasks

Table 1 summarizes the analysis of the data for four tasks that required students to generalize rules relating to perimeter. Under each task, the categories of task actions and understanding actions are headings for columns of cells that record, in sequence of occurrence, one observed task action and the corresponding understanding action. The stair sequence task occurred in August, the triangle and square train tasks were undertaken in November, and the hexagon train task in January. Certain task actions appear to be associated with actions of understanding. For example, *constructing the model* is always associated with image making in this analysis. *Modifying the model* tends to be associated with image making, although not exclusively. *Explaining* aspects of the model seem to elicit property noticing from students and usually occurred in response to the teachers' questions. *Extending the model* and *predicting* are associated with generalizing to formalize on the part of students. More variability was observed for actions related to *modifying the model* although additional instances of this action are needed from further analysis of the data. Also, the definition of the term, modify, may not be accurate. Instances of *folding back* are found and movement between inner and outer layers of understanding is evident. These results indicate that there appears to be a relationship between students' tasks actions and their growth of understanding actions.

Stair Sequence Task		`		Square Train Task		Hexagon Train Task	
Task Actions	Undstndg Actions	Task Actions	Undstndg Actions	Task Actions	Undstndg Actions	Task Actions	Undstndg Actions
C	IM	C	IM	MO	IM	MO	IH
MO	IM	MO	IM	MA	F	ET	F
MO	IM	ET	IM	EP	PN	C	IM
EP	IH	EP	PN	C	IM	MO	IM
EP	PN	EP	PN	MO	IM	ET	PN
EP	PN	P	F	MO	IM	EP	PN
EP	PN	P	F	MO	IH	C	IM
P	F	P	F	EP	PN	EP	PN
ET	F	MA	IM	MA	F	P	F
ET	PN	MO	IM	ET	F	MO	PN

Table 1. A comparison of students' task actions with their understanding actions across four pattern finding tasks.

Discussion

The intent of engaging students in open-ended tasks is to enhance their growth of understanding although Lesh and Kelly (1994) suggest that not all tasks are equal in terms of their usefulness for increased understanding. Pirie and Kieren (1994) propose a model that assists our analysis of student actions in relation to this growth of understanding. Their work posits that frequently "folding back" to inner layers is an essential action in the growth of understanding. When teachers begin their instruction by giving students a formula or definition, the students have no supporting structures or inner layers and, according to Pirie and Kieren (1994), the formula or definition becomes "disjoint" from the student's existing knowledge. They hypothesize that further understanding is impossible without the supporting understanding of inner layers. The findings reported here have the potential for evaluating and designing open-ended tasks that encourage students to construct, modify, and explain models so as to build understanding.

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BEGINNING ALGEBRA STUDENTS' NOTIONS OF CHANGE

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For students beginning algebra, the concept of linear functions is central. As students transition from arithmetic, pattern finding activities can act as a bridge to abstraction (Smith, 2003). This poster presents findings from a design experiment that seeks to help students form ideas about covariation and function with pattern finding activities. Mathematical functions can be seen as either a covariation between quantities or a correspondence between values of two quantities. Though students may gain a deeper understanding if they focus on a covariational approach and students usually approach function problems with a covariational view (Borba & Confrey, 1996), much of school curricula present a correspondence approach. Using stasis and change (Smith, 2003) as a frame, our research team designed a sequence of tasks for a class of middle grades students to formulate notions of covariation using patterning activities in efforts to provide conceptual underpinnings for the linear function concept (see Mojica, Lambertus, Wilson, & Berenson, 2007). We conjectured that students would attend to only one variable and generalize recursive patterns and have difficulties transitioning to analyzing change in two variables (Wilson & Stein, 2007). Rather, many students began with a cyclic view of patterns and described it with respect to time, but did not acknowledge this second variable explicitly. The team designed a dialogue to help students focus on linear growth. Also, we used a series of grid activities encouraging students to analyze change in two variables individually and then to coordinate those changes, first with pattern blocks and then with numbers. From these grids, students built tabular representations and described those changes in words. When presented with pattern block activities after experiences with this task, students attended to two variables simultaneously and began to recognize covariation across verbal, symbolic, tabular, graphical, and physical representations. At the end of the study, students were using arguments of involving rates of change to solve problems involving linear functions. The findings suggest that explicit attention to covariation may be a useful strategy in helping students transition from describing change with one variable to the coordination of change across two variables, an essential skill in approaching functions from a covariational perspective.

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LOOKING AT THE STUDENTS' LEARNING OF LINEAR RELATIONSHIPS IN A COMPUTER ENVIRONMENT THROUGH PIAGET'S THEORY

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Manipulating dynamic and linked multiple representations with the help of technology has become one of the promising environments for students to study various algebra concepts. In this research, students studied linear relationships with linked and semi-linked versions of the same multi-representational software. This presentation will share the analysis of students' learning processes in these environments through Piaget's stage independent theory.

Studying the same mathematical idea with different embodiments was among the reasons why the use of multiple external representations has gained more importance in mathematics education. Linked multiple representations are a group of representations in which, upon altering a given representation, every other representation is automatically updated to reflect the same change (Rich, 1995). Semi-linked representations, on the other hand, are defined as those for which the corresponding updates of changes within the representations are available only upon request but are not automatic. Piaget (1952 and 1969) describes the development of knowledge as a process of adaptation and organization. Adaptation occurs when the child interacts with his or her environment. Assimilation is the process whereby the child integrates new information into his or her mental structures, and it involves the interpretation of events in terms of the existing cognitive structure, whereas accommodation refers to changing the cognitive structure. Adaptation is achieved when assimilation and accommodation are brought into equilibrium. Organization is described as the integration of cognitive structures.

Two groups of ninth-grade algebra students (n=20) used VideoPoint computer software but with a different linking property—linked and semi-linked—for an 11-week period to study linear relationships. VideoPoint is a software package that allows one to collect position and time data from digital movies. Data collection methods included mathematics pre-and posttests, follow-up interviews, and clinical computer interviews.

It was concluded that in the linked environment, when a question was asked, students either used the linkage directly to answer the question or they assimilated the new information and went to their previous knowledge to answer the question. When they used the linkage, their explanation for their answer could be based more on the software. However, in the semi-linked environment, students' explanations were based more on the mathematical aspects of the question. When students provided an inappropriate answer to a question and they observed that they were not quite accurate through the linkage or computer feedback, disequilibrium occurred. They needed to go back and interpret this new information through their existing knowledge. If they could not interpret this new information, they needed to accommodate their preexisting knowledge in order to reach equilibrium. Students who trusted their knowledge and answers might not use the linkage at all. On the other hand, some students did not use the linkage when they could have benefited from it.

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REPRESENTATIONS AS A MEANS FOR MATHEMATICAL COMMUNICATION

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Ideas relating to pattern are part of students' informal knowledge of algebra (Lannin, 2006). Pattern finding tasks, especially those which encourage students to make generalizations, promote the development of algebraic reasoning in students. In this year long teaching experiment, we focus on how middle school students communicate their generalizations of patterning tasks with representations.

Students often communicate their ideas about mathematics using representations. Friedlander and Tabach (2001) suggest the use of verbal, numerical, graphical, and algebraic representations can potentially create meaning for students as they develop algebraic concepts. Students also communicate their mathematical understanding using tables, diagrams, gestures, models and metaphors.

The participants in this study are 18 eighth-grade students who are enrolled in a regular mathematics class and are diverse in terms of race and gender. The sources of data used in this study are student artifacts and videotapes of classroom teaching episodes. Analysis includes: creating general description of the videotaped teaching episodes, identifying critical events, and coding videotaped data and student artifacts.

Findings from this study indicate that the participants favored communicating their mathematical ideas about generalizations using the following representations: numeric, both verbal and written words, diagrams, and gestures. None of the students communicated their generalizations about any of the patterns using graphs, even though they had previously graphed linear functions and relations in their regular classroom setting.

Throughout this teaching experiment, students were engaged in patterning tasks in small groups and as a whole class. Representations used for communication in small groups were different than those used during whole class discussions. During small group work, students mainly used words, diagrams, gestures, and numerical representations while generalizing; however, they used words, tables, and algebraic equations and expressions while communicating with peers and instructors during whole class discussions. If teachers want to enhance students' algebraic reasoning through patterning, they need to provide opportunities for students to communicate using a wide variety of representations. Middle school teachers should create learning environments to bridge informal written and verbal representations with more formal systems of representations, like algebraic notation and symbols.

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REPRESENTATIONS USED IN GENERALIZING PATTERNS

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Students use various representations for generalizing patterns and move through levels of complexity (Michael, Elia, Gagatsis, Theoklitou, & Savva, 2006). This study is concerned with external representations, meaning representations that are observed in students' written work and observed group work. Specifically, we examined how students used manipulatives to build patterns, tables, graphs, symbolic notation, numerical notation, and written notation in order to generalize a pattern. This study seeks to answer the following questions: Is there a tie between students' strategies using multiple representations, their justifications and levels of cognitive complexity?

Michael et al. (2006) described three levels of cognitive complexity of understanding relationships in patterns: empirical abstraction of mathematical relations, implicit use of a general rule, and explicit use of a general rule. In the first level, students are continuing to build or create the pattern. In the second level, students predict terms of further positions, although they cannot continue the pattern. Finally, in the third level, students are generalizing the pattern and forming a rule. Lannin (2003) observed a variety of strategies that students use to develop generalizations of patterns: counting, recursion, whole-object, contextual, guess and check, and rate-adjustment.

The participants in the teaching experiment are eighth grade students in a modified year round school. Students have participated in three cycles of a four cycle experiment. The students' activities are video and audio recorded and their written work is collected. By examining and coding video data, the students' work, strategies, and representations can be tracked throughout the lesson. For a pattern building activity in which students are using toothpicks to model the truss structure of a bridge, they generalized early by using a few specific strategies. The students use the model throughout the activity to help them generalize and verify their generalizations. For level one cognitive complexity, continuing the pattern, the students primarily relied on the counting strategy and their model. One of the student's proportional reasoning demonstrated the level two complexity, predicting future terms of the pattern. She used knowledge created by the group earlier in the task, she then predicted future terms (although incorrectly) by using proportions. She demonstrated this technique using a numerical representation. For the level three complexity, the students jumped to finding a formula to describe the pattern. Initially, the students built on the proportional reasoning, later tying a contextual strategy to the problem.

The students' strategies and representations for this particular activity can be followed through the cognitive complexity levels. Further results and analysis will be provided in the presentation.

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THE INFLUENCE OF PROBLEM STATEMENT FORMAT AND PROBLEM SITUATION ON REPRESENTATION PROCESSES

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This paper examines two aspects of the processes employed by college students as they re-present verbal problem statements as algebraic equations. First, the objects of students' initial focus and subsequent knowledge activations are used to describe the schemas engaged during symbolization. Second, the interactions between the problem format and the learners' cognitive activity are examined.

The recognition and subsequent symbolic representation of the relationships embedded in situations are necessary components of applying algebra during problem solving. This study has two purposes related to these components. The first is to develop understanding of the cognitive structures employed by learners as they re-present verbal problem statements as situational and then algebraic experiences. The second is to examine the effects problem statements and situations have on the application of these structures. Algebraic representation is viewed as a constructive process integrating components from and interactions among multiple knowledge areas including understanding of problem statements, problem situations, and algebraic forms. This work builds on studies of symbolization, such as Sims-Knight and Kaput, (1983) and problem solving, such as MacGregor and Stacey, (1998).

Task based interviews are employed to collect data. Four college students are asked to generate linear equations to represent a series of familiar experiences presented in nontraditional formats such as: "If gas costs $3 per gallon, write an equation to relate the total cost T to the number of gallons of gas purchased." The participants' responses are analyzed from two perspectives. First, their initial focus and subsequent knowledge activations are used to describe the schemas employed during symbolization. Second, the interactions between the problem format and the learners' cognitive processes are examined. The variations in how students activate, connect, and utilize these different structures in different situations are compared.

Participants' initial focus is consistently on familiar aspects of the tasks, but those aspects differ based on the learners' experiences. The results indicate there are differences in knowledge application associated with representation effectiveness. Less adept students have firmer expectations concerning the explicit information provided by a problem statement and rely more on that information. They attempt to generate equations without activating experiential knowledge of the situations and subsequently struggle with more abstract problem statements. More adept students may activate situation knowledge and determine a mathematical relation between quantities. They then restructure the situational relation to fit existing algebraic models. The most proficient learners recognize embedded relations in the described situations, activate appropriate algebraic equations, and fit the situation components to the equations. This last group is least affected by statement format.

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HANDLING PROBLEMS: EMBODIED REASONING IN SITUATED MATHEMATICS

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Fifty 4th-17th-grade students participated in individual interviews oriented toward probabilistic intuition. Participants were given a boxful of equal numbers of green and blue marbles, mixed, and a device for scooping 4 ordered marbles and asked to predict the most common sample. Students replied that the outcome with the highest relative frequency would have 2 green and 2 blue marbles. Their verbal reasoning was accompanied by a deictic–metaphoric gesture to the left then right, as if they were separating the colors in the box. Gesture, I argue, bridges direct intuitive grasps of situations to conscious reflection, thus concretizing the prereflective, possibly grounding it in material form such that it emerges as conducive to further elaboration in mimetic symbolic form. Situated mathematical reasoning transpires largely as embodied negotiation among kinesthetic image schemas afforded by available material resources and epistemic forms.

"The soul never thinks without an image" (Aristotle, *On the Soul*, 350 B.C.)
"To see a world in a grain of sand…" (Blake, *Auguries of Innocence*, ~1800)

Objectives

The objective of this study is to contribute to research on mathematical cognition by illuminating implicit processes of embodied reasoning in situated problem solving. I argue that situated mathematical reasoning transpires as an embodied negotiation between material/perceptual affordances of phenomena and evolved cultural–historical cognitive artifacts that include physical utensils, symbolical forms, and figures of speech. To build this argument, I present empirical evidence of students engaging kinesthetic–imagistic reasoning in solving a situated probability problem. I propose that situated mathematical reasoning transpires neither as direct translation (mapping) from phenomena to symbols nor as chains of signification, as some studies of semiotics or anthropology-of-scientific-practice may suggest, but in embodied modalities drawing on both the material and symbolical and operating in complex dynamical feedback iterations, in which these resources reciprocally constrain each other toward achieving structures evaluated as cognitively coherent, locally effective, and socio–mathematically normative. Specifically, I conjecture that material affordances of situated objects may facilitate a gesture-based bridging from unreflective kinesthesia to representational intentionality, a process that is instrumental to individual reasoning stimulated by and conducted in interpersonal discourse practices. I speculate whether the mathematics-education community currently has theoretical, methodological, and pedagogical wherewithal to successfully interpret the nature and mechanism of such commonplace multimodal reasoning so as to draw implications for the design and facilitation of learning environments. Toward this goal, I articulate the nature of special mathematical learning tools—*reflexive artifacts*—that are conducive to drawing on embodied resources such that the resources are coordinated into emergent solution procedures. These tools may currently be rare, yet articulating their nature could be conducive toward

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designing new reflexive artifacts that facilitate students' guided mathematical reasoning. As such, this paper is part of an ongoing project to develop a design framework including principled methodology for implementing constructivist pedagogical philosophy in the form of content-targeted artifacts, activities, and facilitation emphases (see Abrahamson & Wilensky, 2007).

Theoretical Background

Following renewed post-Behaviorist interest in the roles of vision in mathematical reasoning (Arnheim, 1969; Davis, 1993; Goldin, 1987), the mechanisms of imagery, in its broad multimodal sense, have become foci of research on mathematics education (Presmeg, 2006; Schwartz & Heiser, 2006). Yet, whereas design-based researchers have been cognizant of the roles of perception in mathematical learning, attention to the microdynamics of multimodal reasoning has not been articulated or consolidated in the form of a methodology with clear directives for practice (but see Case & Okamoto, 1996, for a neo-Piagetian approach to the design of contexts that support students in drawing on multiple resources and integrating these as a 'central conceptual structure,' e.g., the case of counting, in which speech, perception, and gesture are implicated). It thus appears timely to integrate considerations of multimodal reasoning into design-research methodology including emphases for microgenetic data-analysis of learners' moment-to-moment interactions with artifacts designed to support content learning. Toward outlining such a prospective methodology, this paper closely examines a case study as a means for considering several theoretical models and their attendant methodologies, including the following. McNeill's (1992) foundational taxonomy of gesture types facilitates interpretation of students' multimodal actions as thinking-for-speaking with artifacts, with each micro-moment constituting context for the subsequent embodied-cognition act. Grice's (1989) theory of pragmatics helps us interpret students' gestures as including aspects of ostentation and clarification that respond to the interlocutor's conjectured perspective. Pirie and Kieren (1994) offer a methodology for monitoring a student's personal invention, consolidation, and use of images that ground the meaning of a mathematical concept and later serve as resources for further conceptual differentiation and development. Schwartz and Heiser (2006) draw on their empirical studies of students' situated problem solving to discuss the difficulty of coordinating modalities (e.g., motoric and imaged)—work suggesting the importance of learning environments that facilitate such coordination. Fauconnier and Turner's (2002) *conceptual blend* model illuminates cognitive mechanisms underlying the superimposition of images or percepts, offering a viable extension of standard cognitive-science problem-solving models to include attention to metaphor as image- and not proposition-based. Finally, Hutchins and Palen (1998) delineate a distributed-cognition approach to explain the ubiquitous, quotidian, and inextricable roles instruments play in supporting the coordination of multimodal and multi-person resources in routinized, practice-based, problem solving. These learning-sciences resources have informed the development of a constructivist/socio-constructivist approach to design that treats students' mediated with-tools phenomenology as epigenetic of reinvention and conceptual understanding. In particular, I seek to understand the ongoing construction of mathematics as developing webs of coordinated multimodal resources. Data from design-based research studies constitute useful arenas for investigating tool-based epigenesis of mathematical constructs, particularly due to the designer's nuanced understanding of the artifacts students engage in problem solving—the tools material properties, embedded mechanisms, contexts, and affordances of conceptual emergence.

Data Sources

The data used in this study are drawn from a design-based research project exploring the nature, roles, and mechanisms of intuition in mathematical reasoning and learning. The subject matter content that served as context for this study was probability and basic statistics. The study was conducted in the form of ~75-minute semi-structured clinical interviews in which students engaged individually in problem-solving and construction activities, using a set of innovative learning tools under development. We have interviewed over 50 students, including Grade 4 – 6 students as well as undergraduate and graduate students majoring in mathematics, statistics, or economics programs (for the design rationale, learning tools, and analysis of empirical findings, see Abrahamson, 2007b; Abrahamson & Cendak, 2006). The current study seeks to achieve deep understanding of a particular brief behavior manifested by most of the participant students.

Figure 1. The marble-scooper randomness generator consists of a boxful of hundreds of marbles of two colors and a utensil for scooping out a fixed number of marbles. In the current embodiment, there are equal numbers of green and blue marbles (p = .5), and the scooper accommodates exactly 4 ordered marbles. At the onset of the interview, students are asked, "What will happen when you scoop?" Problem solving and discussion involve several other mixed-media learning tools pertaining to sampling, randomness, distribution, and combinatorics.

Figure 1, above, shows the marble-scooper device developed for our studies. Each concavity in the scooper can hold a single marble. The simultaneous scooping of four marbles out of the hundreds of marbles is arguably commensurate with flipping four coins (or flipping a single coin four times). Thus, the device supports a situated study of the binomial function $(a + b)^4$. There are 16 unique outcomes in operating this stochastic generator (2^4). For equal numbers of green and blue marbles ($p = .5$), the expected outcome distribution in empirical experiments is 1:4:6:4:1, where these five coefficients correspond, respectively, to the cases of selecting exactly 0, 1, 2, 3, or 4 green marbles in any order (the rest would be blue). Thus, the numeral '6' indicates an expected plurality of samples with two green and two blue marbles in any order.

By and large, all our participants predicted the 2-green–2-blue outcome as the most common. When asked to support their prediction, all students initially said either that, "It looks that way" or that they do not know how they know. For example, one applied-mathematics major said, "I don't know the reasoning behind it, but it seems kind of obvious to me." Upon further thought, undergraduate and graduate students articulated this intuition, saying that the most common sample should reflect the green-to-blue ratio in the box. Only upon prompts did these older students apply notions and solution procedures relating to expected value, the law of large numbers, the central limit theory, and the binomial function (the “mathematical reason,” as one student called these). This study focuses on students’ initial, “non-mathematical,” yet accurate judgment. In particular, I examine the microgenesis of seeing and applying symmetry and proportionality in probabilistic reasoning (for a survey of related work, see Jones, Langrall, & Mooney, 2007; in particular, see Tversky & Kahneman, 1974, for classical demonstration of the ‘representativeness heuristic’). A broader objective framing my work is to develop content-targeted learning tools, activities, and facilitation supporting students in acknowledging their non-analytic perceptual intuitions and coordinating these intuitions with standard solution procedures—I view mathematics learning as the self study of perception (Abrahamson, 2007a).

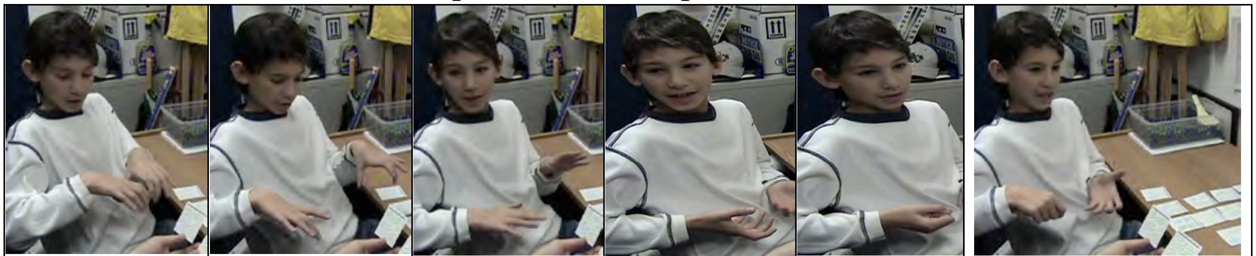
Methods

My research group, Embodied Design Research Laboratory, based in UC Berkeley’s Graduate School of Education (http://edrl.berkeley.edu/), operates primarily in design-based research methodology, in which we investigate mathematical cognition through engaging students in activities with objects of our design (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). Once we have elicited video/audio data, we study student learning using collaborative microgenetic qualitative-analysis methodology (Schoenfeld, Smith, & Arcavi, 1991). Thus, we select and intensely examine short data episodes in attempt to build as complete as possible an understanding of students’ thinking processes. For example—and most pertinent to the current study—, in order to conjecture as to the resources students bring to bear in addressing a situated mathematical problem, we pay attention to students’ gestures as they problem solve (see Alibali, Bassok, Olseth, Syc, & Goldin-Meadow, 1999). Through iterated viewings, comparison, and debate, members of the research team become fluent in the entire data corpus, such that analysis of each participant’s data is contextualized by the complete interview as well as by all other participants’ responses to corresponding items in the interview protocol. Whereas this approach is methodologically incomplete, the potential validity of our insights lies in students’ increased facility with learning tools that are improved iteratively, based on these insights, and in subsequent triangulation with the literature and further data analysis. This paper reports on our analysis of students’ reasoning processes as they initially brought to bear intuitive resources to respond correctly to the marbles problem. More broadly, we are interested in emergent relations between material substance (marbles, scooper, box) and conceptual constructs (e.g., probability, proportion) and the roles embodiment plays in supporting students’ problem-based construction of mathematical content as semiotic coordination of material and epistemic resources. Understanding these complex dynamical processes informs our design of learning environments.

Results, Analysis, and Discussion

While referring verbally to equal proportions of green and blue in the marble box, about 3/4 of the students gestured to one side and then to the other, as though the hundreds of marbles were separated by color to the left and right of the box. In so doing, several students performed a left–right gesture away from the box without any clear referent, some gestured either toward the box itself or to a box they constructed with gestures, and some of the participants indicated—even touched—the middle point of the box immediately prior to gesturing to the "blue half" and the "green half." We will focus on LG, who was typical in manipulating the perceived and imaged.

Sample Data: The 'Equal' Gesture



00:01	00:02	00:03a	00:03b	00:06	00:07
LG (6th grade): If there're equal a*mount*	of *col*ors	on *each*, [hands flap up and down]	*side*,	--	I guess you're less likely [scoops] to get more of one color than the other.

Figure 2. A student engages in multimodal reasoning to support his anticipation of the most frequent outcome in an experiment with the marble-scooper stochastic device (p = .5).

The 4-second pause in LG's utterance suggests that his gestures are integral to the reasoning process, not post hoc. LG's entire sentence is built in the form of an IF–THEN structure: "If…, [then] I guess…" In the IF clause, LG constructs properties of the stochastic generator, whereas in the THEN clause he expresses assumed operative consequences of this construction for the expected empirical outcomes of the projected experiment. Each of these two conjoined clauses is uniquely associated with a proximal seeing of the marble box, yet in the IF clause the marbles are separated, whereas in the THEN clause they are mixed. Thus, in his IF clause, LG considers an event that is complexly related to the distal phenomenon before him. Namely, it is not the case that "there're equal amount of colors *on each side*"—the physical marbles are in fact mixed and not separated. What, then, is the nature of LG's statement? What is he referring to? Why does a casual apprehension of LG's discourse not appear strange to an unreflective interlocutor? Following, I attempt to explain the nature of the gesture—its contexts, mechanisms, and roles.

Data Analysis: The 'Equal' Gesture as a Window Onto Problem-Solving Processes

To begin with, the social situation suggests that there is relevant information to derive from the box so as to respond to the question about the scooper. Hence, the symmetrical shape of the scooper—seen as two concavities on the left and two on the right—foregrounds in the marble box its affordance for bisection. The gesture reveals the body's role in porting these intercontextual constraints back and forth between the material elements of the problem space. In particular, the embodiment of symmetry and balance plays a mediating role between box and scooper. In the process of reasoning through and communicating an idea about the marbles, the student appropriates material properties of available media that include his body and, reflexively, the gesture-constructed marble box. The partitioning gesture toward the constructed marble box is complex deictic–metaphoric. It shows the referents of speech ("amounts of color on each side") in a reconstruction of the box's content, perceived as two en-masse semantic categories.

This mental partitioning organizes the marbles such that they better afford mathematization. Finally, note that the marbles are held in a vessel. This particular vessel, due to prosaic reasons of industrial engineering, production, storage, distribution, and marketing, is structurally simple—a translucent rectangular plastic box. A feature of this box, then, is a straight line, the long side of the box, which faces the student. This long side of the box constitutes an ad hoc measurement tool—a primitive number line—upon which LG tacitly offloads the embodied ratio of the green and blue groups, thus inadvertently concretizing the ratio as part-to-part indexes.

By this interpretation, embedded properties the marbles box (straight frame) constituted a material bridge from a focal artifact (the actual marbles in the rectangular box) reflexively to this artifact's inherent mathematical information in question (green–blue ratio). Thus, pre-verbalized images of symmetry and balance become articulated in bipartite linear form bespeaking proportion. The 'equal' gesture, then, loops from an object and back to it—that is, from the attended categories (color property) of an amorphous object (mixed marbles) and back to the very object (container box), now serendipitously used to ground, elaborate, and communicate embedded aspects of its own properties (half–half). The box of marbles is thus both an originary resource and a vehicle carrying cognitively ergonomic expression of its own properties—it is a *reflexive artifact*, enabling what Noë (2006) describes as using the world to represent itself. Specifically, a reflexive artifact embodies as an affordance an epistemic form for indexing its own properties, thus acting as a cognitive bridge from the phenomenal to the mathematical.

Discussion

Whence did the 'equal' gesture come and what roles did it play? When reasoning and communicating, humans spontaneously leverage their capacity to represent absent objects, or aspects of these objects, within their body space (McNeill, 1992). The epigenesis of a specific gesture is in actual manipulation (Vygotsky, 1978)—the students' 'equal' gesture may be grounded in prior physical actions of sorting and partitioning. Yet it is precisely because the marbles are not readily given to manifesting the 'equal' idea physically, that the participants select an alternative medium, gesture, for conducting and communicating their problem solving. The media-neutral quality of gesture, along with its malleability and portability, make gesture an effective modality for situated problem solving. In particular, gesture marks the body as a buffer for coordinating perceptions of disparate objects that (e)merge as structurally–dynamically homologous. That is, gesture can carry essential structural properties of one element in a situated problem (scooper) to induce it in another element (marbles). Thus, gesture both invests and reveals *epistemic forms* (Collins & Ferguson, 1993), such as *a*:*b*, embedded in situated elements.

In Abrahamson (2004) I suggested that people engaging in situated problem-solving use *embodied spatial articulation*, a type of dynamic visuo–spatial reasoning, to negotiate between body-based mathematical knowledge (kinesthetic schemas) and socially mediated norms of seeing mathematical tools (epistemic forms). That is, people manipulate situations—whether physically or imagistically—so that they can apprehend the situations through familiar schemas that are conducive to determining mathematical properties of the situations (this is the act of modeling, mathematizing). The 'equal' gesture may indicate that the participants were engaging in embodied spatial articulation—they were assimilating the marbles into an embodiment of proportion as a means to anticipate frequency, constructed as expected mode and variance.

Conclusions

Students' mathematical problem solving is not either with or without objects, perceptual or conceptual, situated or symbolic, concrete or abstract. In fact, these pairs of constructs assume an ontology that does not capture the phenomenology of mathematical reasoning. Rather, problem solvers negotiate among embodied schemas afforded by available material and representational resources. In so doing, problem solvers spontaneously conjure and mimetically embody cognitive artifacts that have representational affordances—a symbolic form, a diagram, a figure of speech—as kinesthetic–imagistic substrate supporting the extraction of relevant phenomenal aspects of situated objects. Gesture plays a central role in situated problem solving: gesture bridges from prereflective absorption to reflective attention, from direct intuitive grasps to processes of conscious reasoning and communication. Gesture, a physical action with spatial–dynamic properties, concretizes personal kinesthetic negotiation for inspection, verbalization, and intersubjectivity—gesture grounds embodied quantitative relations for further elaboration.

Reflexive artifacts—e.g., sets of objects affording self-indexing by sorting or tabulation—support embodied mathematical reasoning and therefore merit further design-based research. Of particular interest are cognitive mechanisms governing these perceptual–epistemic negotiations.

Finally, I see a tension between phenomenological and semiotic descriptions of referring-to discourse—a tension that may be hampering productive collaboration within interdisciplinary fields concerned with constructivist pedagogical philosophy, at least with regards to mathematics. This study suggests a need for theoretical perspectives on mathematical cognition that are geared to treat the multimodality of mathematical intuition, reasoning, and practice so as to illuminate issues of design, teaching, and learning. Embodied cognition, informed by phenomenological philosophy, effectively constitutes one such theoretical perspective.

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UNDERSTANDING THE LANGUAGE OF MATHEMATICS IN A STANDARDIZED EXAM

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This study applies an interdisciplinary approach combining linguistics, mathematics, and education to examine test items on a state standardized exam for the eighth grade and the challenges they present for English Language Learners (ELLs). The results of the study suggest that despite being low level cognitive demand, the test items presented challenges for ELLs which included technical vocabulary, complex noun groups, and relational processes (e.g., is, are, have). Implications include a call for collaboration among mathematics teachers and ELL teachers.

Mathematics teachers need to be prepared to work with students from a variety of backgrounds. In fact, a key concern of teacher education programs is preparing mathematics teachers to teach a diverse student body. This concern becomes even more critical as the U.S. population becomes more diverse (Sleeter, 2001). Students who speak a language other than English at home comprise 19.2% of the U.S. population (U.S. Census Bureau, 2005). In the state of Indiana alone, the number of English Language Learners (ELLs) more than doubled between 1995 and 2000 (U.S. Census Bureau, 2000). Between 1993 and 2004, there was a 438% rate of growth in the number of Limited English Proficient children in Indiana (U.S. Census Bureau, 2005). These students face many challenges in coping with the language of the content areas, in particular in mathematics. The achievement gap in mathematics is pronounced for ELLs. Forty-eight percent of students classified as Limited English proficient did not pass the mathematics portion of the eighth grade Indiana Statewide Testing for Educational Progress (ISTEP+) exam.

Educating ELLs is among the equity issues to which the mathematics educational community is attending. The increasing prominence of equity issues in mathematics education is reflected in the designation of equity as the first of six guiding principles for ensuring a high quality mathematics education program by the National Council of Teachers of Mathematics ([NCTM], 2000). The other principles are curriculum, teaching, learning, assessment, and technology. Preparing to address equity issues is critical, and preparation must address both mathematics content and issues around language for ELLs.

The NCTM (2000) identifies communication as one of five process standards to facilitate students' mathematics learning. Among the goals for student learning is that students "use the language of mathematics to express mathematical ideas precisely" (NCTM, 2000, p. 60). Yet, before ELLs can communicate mathematical ideas, they must understand the language that is used to express mathematical knowledge in a variety of instances, such as textbooks, worksheets, and standardized tests. The discourse of mathematics in particular has been identified as problematic for students who have limited experience with the way knowledge is constructed in mathematics classrooms (Lemke, 2003; O'Halloran, 2000). Students' ability to make sense of

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mathematics is impacted by their understanding of the language used in teaching and learning. For example, the word *table* acquires a specialized meaning when used in mathematics – a chart. An ELL must recognize that *table* in the mathematics classroom takes on this alternative meaning than the everyday meaning of table, a piece of furniture. These specialized meanings in mathematics present challenges to ELLs (O'Halloran, 2000) and may hinder their performance in standardized exams.

Research Study

The challenges of learning mathematics go beyond language issues. But clearly the language challenges need to be addressed for students to be successful in mathematics learning. Mathematics has been described as multisemiotic or multimodal, as math symbols, images, diagrams, and language work together to build students' understanding of mathematics (O'Halloran). For ELLs, these language issues present another dimension – access, an equity issue. This equity issue is addressed through our examination of standardized exam items. We explore these questions from an integrated perspective of a mathematics educator and an educational linguist that focuses both on the language and the content with the goal of improving ELLs' mathematics learning. The research questions guiding our study are: (1) What are the language challenges in mathematical tasks on the ISTEP+? (2) What language knowledge is required in order for ELLs to comprehend these tasks? (3) What mathematics knowledge is required in order for ELLs to comprehend these tasks?

We analyze sample mathematics tasks for the eighth grade ISTEP+ exam. The eighth grade level was selected because it is often a gatekeeper to advanced studies in mathematics. Eighth grade is often the grade level at which students are tracked into college preparation courses such as algebra. Thus, students' performance at this grade level and placement into a particular mathematics course has significant impact on their future mathematics course taking, access to science- and mathematics majors and careers, and economic access. Thus, ELLs' performance at this grade level is critical in terms of their access to algebra, and we also know that students' achievement is positively correlated with advanced mathematics course taking (Lee, Croninger, & Smith, 1997).

The items analyzed in this study were released as samples (Indiana Department of Education, 2002). We selected one item from each of the categories of problems included on the eighth grade exam: number sense, computation, algebra and function, geometry, measurement, data analysis and probability, and problem solving. We classified each of the selected items as either (a) what, traditionally, would be referred to as a word problem, (b) a task that required an extended response, or (c) a task that involved understanding a pictorial representation (e.g., graphical). These criteria allow for our analysis to move beyond gauging procedural understanding and include students' meaning making and conceptual understanding.

Theoretical Perspective

This study applies an interdisciplinary approach combining linguistics, mathematics, and education. The linguistic framework that guides this work is systemic functional linguistics (SFL), which has already made some contributions to our understanding of the discourse of mathematics teaching and learning (e.g., Lemke, 2003; O'Halloran, 2000, 2003, 2005; Veel, 1999). This linguistic theory links language structure with social context, enabling us to analyze texts in terms of the linguistic elements that construct meanings (Halliday & Matthiessen, 2004). A functional analysis focuses on grammatical and lexical (vocabulary) features and their realization of particular social contexts and sees language not as a set of rules to be followed but rather as a set of language choices for making meaning (Halliday & Matthiessen, 2004). SFL is a rigorous method of textual analysis and a theoretically coherent instrument used by several researchers in the field of linguistics and education. This approach puts the focus on the content, helping to identify how language works to construct disciplinary knowledge.

We employed the *Task Analysis Framework* (Stein, Smith, Henningsen, & Silver, 2000) to analyze the cognitive demand of the test items. This framework outlines two primary categories - lower level and higher-level demands. The lower level demands category has two subcategories - memorization and procedures without connections, and the two subcategories of the higher level demands category are procedures with connections and doing mathematics. In the Task Analysis Guide, four to six characteristics are outlined for each of the four subcategories and are used to classify a task. A given task may have features in two categories and selecting a category involves some subjectivity. Moreover, tasks may be intended for a given category but implemented at a different level.

Findings

We identified several features across the sample items that may hinder ELLs' understanding of the problem. Our analysis took on two perspectives. We analyzed the items with respect to the mathematics content and language involved in each of the tasks.

Language Requirements

The test items differed in the challenges they presented for ELLs. However, even though these differences were present, all of the test items analyzed fell into in at least one of the following categories. First, technical terminology or the use of a specialized mathematics term characterized several of the items. For example, the use of the term *decrease* requires ELLs to associate the term with subtraction before the student could successfully complete a task. ELLs must engage in multiple levels of comprehension while also having to decipher language subtleties in order to make sense of what is being requested in the test item. Second, the use of complex noun groups was a common feature. We classified a phrase as a complex noun group when it consisted of two or more nouns, each of which must be understood separately before combining those words for a new, single meaning. An example of a complex noun group is *prime factorization* or *the height of a right triangle*. Third, *relational processes* constructed in clauses of *being* and *having* are typical in these mathematical tasks and questions. Such statements are particularly difficult for ELLs, as they require an understanding of language structure before students can accomplish the mathematical task. We present findings that address the overall task and construction of the question that is presented. Following is our discussion of each of these common features providing specific examples for each of the three aforementioned categories of challenges.

The topics addressed in the seven test items included area, percentages, slope, scaling, probability, volume of a three-dimensional object, and principles of a triangle. Below we provide our analysis of the seven test items that we selected. First, we present the task. Second, we provide a linguistic analysis of the task. The analysis of the cognitive demand of the task and identification of the content requirements is described after all the test items are presented. The first item was on number sense:

Test Item 3: The Caspian Sea is the largest lake in the world, with an area of 1.46 x 10^5 square miles. Lake Superior has an area of 3.17 x 10^4 square miles. What is the difference, in square miles, of the areas of the two lakes? Write your answer in scientific notation.

The first two sentences use *relational processes* constructed through the verbs *be* and *have.* In setting up the question, the first sentence *The Caspian Sea is the largest lake in the world, with an area of 1.46x10^5 square miles* uses the relational process *is* to identify that The Caspian Sea is a lake. The preposition *with* is used to show that the Caspian Sea ***has*** 1.46x10^5 miles. This is a difficult sentence construction for ELLs, as they have to make the connection between the preposition *with* showing possession and The Caspian Sea. This sentence construction, a relational process *is* and a preposition with showing possession is highly academic. The second sentence, however, *Lake Superior has an area of 3.17 x 10^4 square miles* is more characteristic of everyday discourse. Here the sentence construction is straightforward and easier for students to understand, as Lake Superior is already identified as a lake in its name and the relational process *has* shows the possessive relationship. There are three features to which students must attend in this question: (a) a Wh- word (i.e., *what*), a common feature of test items, is used; (b) a *relational process* (i.e., *is*) is used; and (c) a complex noun group that identifies the key mathematical focus of the questions is present (i.e., *the difference, in square miles, of the areas of the two lakes*). ELLs have to understand that the words between the commas - *in square miles* – identify square miles as the required measurement unit for the solution. Despite the aforementioned challenges/complexities that ELLs must overcome, the test item is ultimately asking the student to calculate *the difference of the areas of the two lakes.*

The second item was a computation item:

Test Item 4: Last year, the freighter Mariposa carried 20 million tons of cargo. This year, the Mariposa carried 16 million tons of cargo. What is the percent decrease in the amount of cargo carried by the Mariposa from last year to this year? [Choices provided were 20%, 25%, 36%,and 40%)

The question presented in this item has the same pattern as the question discussed above, Wh- word + *is* + complex noun group. The complex noun group is *the percent decrease in the amount of cargo carried by the Mariposa from last year to this year*, and contains the technical term *decrease* requires ELLs to associate the term with subtraction before the student can successfully complete the task. In addition, the word *percent,* which modifies the noun *decrease* is very important in the question, as it carries the meaning needed to determine which operation is required for students to be able to answer the question correctly.

The third item requires students to select the slope of a line when a graph of a line is provided:

Test Item 8: Look at the graph below. (We do not provide the graph here.)

What is the slope of the line in the graph? [Choices provided were: -2, - _ , _, and 2].

This item also follows the same structure as the previous items. The complex noun group *the slope of the line in the graph* is difficult for ELLs in that they are required to understand *slope* – a concept with a specialized mathematical meaning - and work through the complex noun group to be successful on this item. An additional challenge for ELLs is the interaction between graphical representation and the words used to ask the question about the graph.

The geometry item was fourth and required students to calculate the height of a triangle given the dimensions of the three sides. Test Item 12 asked students to "Look at the triangle below" and answer the question "What is the height, in centimeters, of the triangle?" Again, the question structure follows the same format as the other questions. As in test item 3, ELLs have to understand that the words between the commas - *in centimeters* – identify the unit of measurement that they have to use to answer the question, but that what they have to calculate is *the height of the triangle.*

The fifth item was a measurement item:

Test item 16. George and Nancy are designing a house for a school project. They are making a drawing of the house with a scale of _ inch = 1 foot. They want the living room in the house to have a width of 12 feet and a length of 20 feet. What will be the width of the living room, in inches, in the scale drawing?

The question presented in this item follows the same pattern as questions in test items 3, 4, 8, and 12. The only difference is the addition of a prepositional phrase, *in the scale drawing.* This prepositional phrase is especially difficult for ELLs since the noun *scale*, used in the second sentence of the word problem is now being used as an adjective, modifying *drawing*. Therefore, we're not talking about any drawing, but the *scale drawing*. The question, as in test items 3 and 12, identifies the unit of measurement -*in inches* – between commas, and ELLs have to understand that in order to answer the question. Technical vocabulary also occurs with the words *scale*, *width*, and *length*.

The sixth item asked students to calculate the probability of two events:

Test Item 19: Carlos has 120 baseball cards. If he chooses a card at random, the probability is 0.6 that he chooses a pitcher, 0.3 that he chooses an infielder, and 0.1 that he chooses an outfielder. What is the probability that Carlos will choose an infielder's or an outfielder's card?

This test item starts with a statement that uses a relational process, *has*, which establishes a possessive relationship between *Carlos* and *120 baseball cards*. The following sentence is a conditional sentence which starts with the conjunction *If* that determines a condition for the results that show the probability for each card. Each one of the following is a result clause: 1. the probability is 0.6 that he chooses a pitcher, 2. 0.3 that he chooses an infielder, and 3. 0.1 that he chooses an outfielder; yet, they are all combined into a single sentence, with *the probability is* omitted from the second and third result clauses, as this can be presumably understood by the context. This sentence structure can present a challenge for ELLs, given the conjunctive relationship between the conditional (i.e. *If*) clause and each of the result clauses.

The final item was a problem solving item:

Test Item 21: The height of a right triangle is 3 centimeters and the area of the triangle is 6 square centimeters. What is the length, in centimeters, of the hypotenuse of the triangle? [The choices provided were: 5 centimeters, 9 centimeters, 10 centimeters, and 18 centimeters]

This test item also contains relational processes in the first sentence. This sentence contains two clauses linked by the conjunction and, but the clauses follow the same language pattern: a complex noun group + relational process *is* + unit of measurement. The question asked in this test item again follows the same pattern as items 3, 4, 8, 12, and 16: Wh- word + relational process *is* + complex noun group. ELLs need to again understand that the words between the commas - *in centimeters* – identify centimeters as the required unit of measurement for the solution.

The test items selected for this study clearly present challenges for ELLs. These challenges consist of technical vocabulary used with specific mathematical meanings, long and complex noun groups that typically identify what the student is to calculate, and relational processes with verbs of *being* and *having* that build relationships between parts of a sentence and are common features of questions.

Mathematics – Content and Cognitive Demand Requirements

We employed the *Task Analysis Framework* (Stein et al., 2000). Overall, the tasks were classified as low-level cognitive demand. In fact, we classified each of the tasks as *procedures without connections.* Although memorization was a subcomponent of the procedures without connections category, no item was a memorization task. It was not surprising that these items required a low level of cognitive demand as this is a long-standing feature of mathematics standardized exams. There were instances, however, for which a test item had at least one feature of a *procedures with connections task*.

Each of the four categories in the *Task Analysis Framework* had between four and six features. For the seven items examined in this project we identified six features across the tasks. Five of the six features were all of those outlined in the *Task Analysis Framework* for *procedures without connections.* They were (a) algorithmic, (b) limited cognitive ability, (c) lacked connection to concept(s), (d) focused on correct answers, (e) required no explanation, and (f) required more than a limited level of cognitive effort. All of the items except item 4 were algorithmic. All of the problems required limited cognitive ability. In particular, "there is little ambiguity about what needs to be done and how to do it" (p. 16). We note here, however, that this notion of ambiguity differs from ambiguity that an ELL student might encounter with respect to language. Two of the seven items, item 8 and item 16, lacked connection to underlying concepts. Six of the seven items focused on correct items. One item, item 3, required students to show all of their work allowing students to receive partial credit. None of the items required an explanation, and only a few of the items could have classified as requiring more than a limited level of cognitive effort. This final feature was more difficult to judge since it was not always clear what prior instruction a student would have had.

Discussion

The insights gained from this research project highlight the complexity of addressing equity issues in mathematics education, and in particular those issues that concern the learning of mathematics by ELLs. The features we identified not only highlight the complexity of the issues, but our analysis provides insights for the design of assessment instruments. Assessment, especially assessment on standardized examinations is often high stakes; this is especially true for underrepresented students. Understanding how the construction of test items might add an unintended dimension of complexity to a task for ELLs positions teachers, test writers, and policy makers to make more informed decisions and allow for assessment that tells us more about the mathematics that students know and can do.

Specifically, we gained three major insights from this inquiry. The first insight has to do with standardized assessments. We found that the sample test items from the ISTEP+ presented several challenges for ELLs. This was the case for the test items even though they were classified as requiring lower level cognitive demand. As the field of mathematics education advocates teaching and learning that promotes higher level thinking and conceptual understanding, the challenges may become more pronounced as extended response questions are being added to standardized exams. This suggests the continued need to consider both the cognitive demand level of items on standardized exams and the language challenges for ELLs, especially on high stakes exams. As we keep equity issues in mathematics education at the fore, we must be mindful of how these demands might impact ELLs' opportunities to both learn and demonstrate their mathematical understanding.

Secondly, we found a reoccurring pattern in the test items. We believe that understanding the structure of these items can be informative for both mathematics teachers and ELL teachers. These patterns can be understood as resources for preparing ELLs in test taking. We encourage collaboration among these teachers to prepare ELLs for standardized exams.

Finally, our findings demonstrate that the challenges faced by ELLs go beyond vocabulary development. Additionally, ELLs must understand sentence structure and become familiar with language patterns employed in these exams and in mathematics learning, in general. In order for students to be successful on standardized exams, they must be able to determine what is being asked. The language challenges we identified may hinder their understanding of the requirements of the test items, thus impeding their ability to demonstrate their mathematical understanding.

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OPEN-ENDED ASSESSMENT, FORMATIVE FEEDBACK ROUTINES, AND CLASSROOM DISCOURSE: INFLUENCES ON STUDENT LEARNING

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We are in the second year of a three-year study in which we are attempting to help teachers use assessment to increase students' mathematical knowledge. We are using open-ended tasks that are aligned to existing curriculum, are completed after classroom instruction, and are not graded. Students complete these individually and then work in pairs or fours with formative feedback materials (called "hints"). The hints offer explanations of the mathematics inherent in the problems without directly stating the correct answer. Students use their completed tasks and the hints to reach a collective understanding of the mathematics being assessed. In this presentation we will trace student learning by charting changes in the students' mathematical discourse and argumentation across one school year.

Classroom assessment is typically used to indirectly improve student learning by providing feedback to teachers. However, assessment is rarely used for formative purposes with students. Thus, most students see little formative value in *any* form of assessment (Gipps, 1999). While nearly all assessment has some formative intent, these goals are often undermined by more summative goals (Shepard, 1993, 2000).

We are using open-ended tasks with elementary school students in a deliberate manner that emphasizes the formative role of assessment. The open-ended tasks were created to be aligned to the approved state curriculum and the adopted textbook. The tasks are deliberately designed no to directly mimic the tasks in the textbook, however. To eliminate any summative function, the tasks are not graded. Students complete two to three tasks related to one chapter of instruction individually, and the they work in pairs or fours to discuss their solutions. They are provided with "hints" that provide insight into the mathematics inherent in the task but do not directly give away the answer. The goal of the small group discussions is for the group to reach collective agreement on a solution for each task and for individuals to assess the correctness or incorrectness of their own solutions and the thinking that led to those solutions.

This is a multi-faceted, multi-year study, so only one aspect of the study will be reported here. In particular, this presentation will focus on tracing student learning by charting changes in the students' mathematical discourse and argumentation across one school year.

Theoretical Framework

We are drawing on discourse analysis in this work. To analyze students' discourse, we are using the nine discourse types identified by Weaver, Dick, and Rigelman (2005). These nine

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.

discourse types are sequentially organized from least to most cognitively demanding. We are looking at the percentage of "idea turns" that fall in the lower third of Weaver et al. discourse types, the middle third, and the upper third, and we are looking to see if these percentages change across the school year. We are also looking for evidence of Toulmin's argumentation structure of claims, warrants, and data. We are looking to see what counts as evidence for warrants and how warrants are used differently for rebuttals and agreements.

Methods

The overall method of the study is a design-based experiment in which we made refinements to the tasks and hints as we collected and analyzed data. For the portion of the study reported here, we are relying on written student work and videotapes of student dialogue. The data come from students in nine fifth-grade classrooms. Additional data come from one control classroom. Students participated in distal discourse assessments (DDA) at the beginning of the school year. In the DDA, students worked in groups of four to write up solutions to two problems (Which is larger–4/4 or 5/5? Draw a picture and write an explanation for why 2/4 = 6/12.) We have analyzed written student work and videotapes of 27 groups of students. The DDA was repeated, but with different problems, at the end of the school year.

In addition to the pre- and post-instruction DDA measures, we have video of intact groups of students engaging in the post-investigation discussion for each of the 10 textbook chapters for which we have written investigations. We also have each student's individual written work from the investigations. Although we are analyzing all of the investigations, for purposes of this presentation, we have analyzed data that correspond to Chapters 1-3, 4-6, and 7-9 of the text so that we have discourse measures from three different points of the year.

Results

We have found that instructing students to build consensus actually works against high quality discourse. While we initially thought that encouraging consensus would force each student to state an argument and force the group to evaluate the merits of these different arguments, we found that students quickly converged on one answer with minimal discussion when the goal was consensus. This led to a design change in how the goal of the group discussion of the formative feedback was presented. Rather than consensus, which implies that some individuals should abandon their ideas, we began encouraging discussion with a goal of confirming or disconfirming your conclusion.

Second, with regard to Toulmin's (1963) structure of argumentation, we are seeing lots of claims but few warrants in the early year data. Students' primary source of evidence in the pre-DDA and Chapter 1-3 is whether or not the idea advanced by another makes sense to them. Basic mathematical facts and procedures are also relied upon heavily as justification for claims that are made. Neither of these things is surprising; we are interested to see if and how the use of warrants changes over the school year. We are looking for evidence of growth in discourse that leads to co-construction of knowledge and discourse that is oppositional. We are also looking to see if the percentage of discourse turns shifts across the categories of accepting, rejecting, discussing, and ignoring other students' ideas over time.

Discussion

We are attempting to create a classroom environment where students can participate in authentic mathematical discourse (Sfard, 2000) and develop their mathematical expertise. The small group discussions of the students' solutions and the hints are intended to enhance the formative value of assessment and enhance student learning. We have attempted to structure the small group work so that students are encouraged to hold themselves and their peers accountable for mathematical knowledge and discourse. We are hopeful that the collective discussion of the tasks serves scaffold the students' discourse toward what Berieter & Scardamalia (1989) describe as "intentional learning."

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THE DEVELOPMENT OF MATHEMATICS INSTRUCTIONAL APPROACHES

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This qualitative study examines the influences that contribute to sixth-grade teachers' instructional decisions in mathematics. The participating teachers identified similar multiple influences but the nature of and responses to these influences varied. The results of this study suggest the importance of aligning some of the influences on teachers, such as the district, school administrators, and textbooks.

The instructional approaches teachers use in mathematics impact students' attitudes and achievement (e.g., Boaler, 2006). These instructional approaches can vary significantly, even within the same or very similar settings (McLaughlin & Talbert, 2001). Various studies have suggested a host of reasons for these variations (e.g., McLaughlin & Talbert, 2001; Smagorinsky, Cook, & Johnson, 2003). However, few studies have asked teachers to explain their perspectives on the reasoning behind their choices of instructional approaches. Therefore, this empirical study examines two research questions. (1) Why do mathematics teachers teach the way they do? (2) What do they perceive as the influences that helped to shape their instructional approaches?

Theoretical Framework

The theoretical framework for this study is based on Lemke's (1997) view that the direction of an individual's path or trajectory is shaped by the history of events in her or his life and the current context because of the way that past experiences open new potential for participation in current events. For example, a teacher's experiences as a student may help to shape her potential decisions about the ways that she may teach her future students. However, Lemke noted that the shaping that our experiences in various communities provide does not prevent the development of unique perspectives and meanings.

> The ways in which we connect past events and present ones are always partly unique; our meaning systems have a biological ground, a cultural set of historically specific resources, and a socially shaped set of commonalities with others, but they also have a psychological individuality. (pp. 48-49)

In this qualitative study, I demonstrate how individual mathematics teachers' history of past experiences and current context helped to shape their instructional decisions.

Methods and Participants

I gathered data for this narrative inquiry through a series of lesson observations and individual and focus group interviews. The participants in the study, Mary, Kathy, Holli, and Susan, were sixth-grade public school teachers who taught at two elementary schools, Northside and Parkview. These schools, which had high proportions of students of color and low socioeconomic status, were in the same district and adjacent neighborhoods in a small Western city in the United States. At the time the study was conducted, both schools were meeting the Adequate Yearly Progress expectations established by the state.

Evidence

The four teachers in this study each identified an extensive and very similar list of interrelated influences that could be viewed as belonging to two main categories (personal experiences/beliefs and external circumstances) and six subcategories (personal: K-12 experiences, teacher education/professional development, and beliefs; external: school personnel, curriculum and assessment, and student characteristics). However, although they each identified the same types of influences, the nature of these common factors and the teachers' responses to them varied. As a demonstration of these variations, I focus on the interrelationship among five influential factors that contributed to a pivotal aspect of Kathy and Susan's instructional decisions. Both teachers explained that many of their students had low reading levels in comparison to the district-selected mathematics textbook's fairly high reading level. In addressing these two influences of student characteristics and curriculum, three additional influences of their own K-12 experiences, their teacher education and professional development, and their principals all played a part in their instructional decisions.

Kathy

In addition to the influences of student characteristics and the district-selected text curriculum, Kathy's response to the reading level issue was shaped by the three additional influences named above. First, Kathy described her discomfort as a K-12 mathematics student, including a sense of isolation and lack of support. As a result of these experiences, she placed a priority on helping her students feel more comfortable in their studies of mathematics than she had. Second, Kathy's principal expected her to follow the district-selected mathematics text closely unless she provided a justification for making changes. Third, Kathy explained that she had had little teacher education and no professional development related to mathematics instructional approaches. (Her lack of professional development was partially due to her principal's decisions.) Therefore, she relied on the mathematics textbook as a source for learning how to teach mathematics, saying, "If it wasn't for [the text], I wouldn't even know where to begin to teach math." Kathy was understandably uncomfortable making major changes in the curriculum it provided. She commented, "I'd be too afraid I'd mess it up."

Kathy's instructional decisions reflected these five interrelated influences. In most of Kathy's lessons, she followed the district-selected reform-oriented textbook in a scripted manner. She explained that she addressed the conflict between her students' reading levels and that of the text by reading the lesson, questions, and examples aloud to her students and closely supervising their completion of some of the problems with their partners or small groups before assigning homework problems. She identified her interest in making students comfortable as an important reason for reading the text aloud. "I feel this will circumvent any problems…it's all clarification up front."

Susan

In contrast to Kathy, Susan viewed her negative experiences as a K-12 mathematics student as related to a lack of learning about mathematics reasoning and communication. Therefore, she believed it was important to provide her students with opportunities to learn the reasoning and communication she had not. "I had to find a way for them to *understand* math." Both of the principals with whom she had worked had supported her exploration and experimentation with innovative ways to teach mathematics, including providing funds for professional development and additional resources beyond the textbook. Additionally, her current principal evaluated her on meeting the state mathematics standards and on her students' high-stakes test scores, rather than on her use of the textbook.

Susan frequently used small group and whole-class formats to facilitate students' extensive discussion and debate about the meanings of key terms, as well as the associated characteristics, examples, and non-examples of the terms. She followed these lessons with text work, but chose the text topics carefully according to the questions asked most frequently on the state high-stakes tests. "The place where [the curriculum] has to be flexible is the textbook…where it becomes more of a guidebook." Susan explained that the intensive focus on student discussion and debate helped the students to have a strong understanding of the important vocabulary and concepts before they began to work in the text. "It was great in that, by the time [we got to the text], they had a really good picture of what it was in their heads."

Conclusions

In considering how these results contribute to the PMENA goals of understanding the psychological factors of mathematics teaching, the results of this study were consistent with Lemke's (1997) description of the way that individuals are shaped by their history and current context and yet make unique decisions within these influences. Even this abbreviated focus on one aspect of Kathy and Susan's teaching illustrates that each teacher was confronted by similar issues and influenced by similar multiple and intertwined factors. Yet, despite these similarities, Kathy and Susan responded uniquely in ways that were related to the combined effects of their previous experiences and the characteristics of the context in which they taught. In the same way, each of the four teachers in this study demonstrated her own interpretation of the best ways to teach mathematics based on her particular history and context.

These results are revealing in considering the characteristics of successful – and unsuccessful- reform efforts in mathematics. In particular, reform efforts that do not align with many of the other influences experienced by teachers may be overwhelmed by the other factors that contribute to these teachers' instructional decisions (e.g., Smagorinsky et al., 2003). Conversely, these results also highlight reasons why reform efforts that involve alignment of vision and effort across teacher education programs, school districts, school administrators, and groups of teachers are more likely to be successful (e.g., Cobb, McClain, Lamberg, & Dean, 2003) because the teachers do not have to make choices among competing influences.

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WHAT ABOUT THE OTHER HALF? MATHEMATICAL THINKING AND THE GED

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This study examines levels of complexity of GED Mathematics Tests and state-level mathematics assessments. Further, this study investigates ways in which GED instructors engage GED students in mathematical reasoning. This study contributes to an understanding of the mathematical experiences of the significant portion of students who dropout of high school and subsequently attempt to attain a GED certificate.

The graduation rate in South Carolina hovers around 50 to 60% (NCES, 2007). Nationally, 20 to almost 50%, depending on the state, of high school freshman are expected to dropout of school before getting a high school diploma (NCES, 2007). Many of these students who dropout of school subsequently seek a GED certificate (NCES, 2004; Rachal & Bingham, 2004). Though a significant and increasing proportion of teenagers are engaging in GED preparation programs (Rachal & Bingham, 2004), scant attention in mathematics education research is focused on the mathematical experiences of GED students. Recent studies investigating the expertise of GED mathematics instructors, the mathematical rigor of the GED, and relationships between the GED mathematics test and typical state-level mathematics tests are lacking. This project brings GED students' mathematical experiences into the fore by considering the following questions:

- What types and levels of mathematical thinking, reasoning, and knowledge do GED and state-level examinations target?
- Is there a difference in GED students' performance on GED test items and items from state-level examinations?
- In what ways do GED mathematics instructors engage GED students in mathematical reasoning?

Theoretical Framework

This study focuses on state-level mathematical examinations in South Carolina and the mathematics portion of the GED. Items from Algebra 1, Algebra 2, and Geometry End of Course (EOC) examinations and items from the GED Mathematics Test will be analyzed for level of mathematical complexity, as outlined by the National Assessment of Educational Progress (2005) framework. There is some research that supports the notion that at least some students who dropout believe that attaining a GED is easier than completing the requirements for getting a high school diploma (Rachal & Bingham, 2004). However, this scholar can find no studies that explore this assumption in the content area of mathematics. We currently know very little about the relative mathematical complexities of state-level examinations and the GED Mathematics Test. We do know, however, that access to quality mathematics education is a social justice issue (Bartell, 2006), and we know that there are patterns in the demographics of students who dropout of school. Ignoring the mathematical education of teenagers who are GED students devalues the experiences and understandings of a population who are disproportionally minority and of low socio-economic status (Chan, Kato, Davenport, & Guven, 2003; Tyler, Murnane, & Willett, 2003; Tyler, Murnane, & Willett, 2002).

Methods

The National Assessment of Educational Progress (NAEP) outlined in 2005 guidelines for analyzing the mathematical complexity of assessment items (Howe, Scheaffer, & Lindquist, 2004). This researcher attained test items from EOC exams in Algebra 1, Algebra 2, and Geometry from South Carolina in addition to items from the GED Mathematics test. These items are scrutinized for their level of mathematical complexity, following the NAEP guidelines, to see whether GED test items differ from state-level mathematics test items in complexity. This analysis addresses the first question of this project. The second question is addressed through recruitment of GED students to participate in completing two mathematics assessments, one of which is comprised of state-level mathematics items, while the other assessment is a GED mathematics test. Scores from students taking these two tests are compared to allow insight into student performance on GED and state-level mathematical assessments. The third question for this study is addressed through observations of tutoring sessions for GED students in mathematics. The researchers visit area GED centers to study the mathematical engagement of GED students.

Results

Based on the little research that exists on GED students and mathematics, it is likely that GED instructors are ill-versed in research-based educational practices and that most GED students are engaged in mathematics through practice and drill (Ward, 2000). What remains to be seen is how the GED Mathematics Test compares to state-level examinations in mathematical complexity and how GED students perform on these assessments. This study, which is being conducted as part of a collaborate creative inquiry project, will ultimately result in understandings of the complexity of the GED Mathematics Test and how this test relates to state-level assessments.

Discussion

Though this study is in process and a full analysis is not yet complete, discussion of the findings will focus on relationships between the results of the study and the three initial research questions. It is expected that further questions will arise during the course of analysis and reflections. In particular, it is expected that further studies will be called for, which explore the motivations and goals of GED students with the aim of transforming these students' engagement with mathematical ideas. For instance, it could be that GED students might be highly receptive to critical mathematical contexts that explore social inequities and challenge power asymmetries, and these critical contexts could be a way of actively engaging GED students in mathematical reasoning. Before studies can be conducted that explore ways of transforming GED students' engagement with mathematical ideas, however, we need to have a much more thorough understanding of what GED students currently experience mathematically.

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GRADING WITHOUT GRADES: THE IMPACT OF PROVIDING FORMATIVE FEEDBACK ON STUDENT ASSIGNMENTS

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Research has shown that providing students with feedback in the form of formative comments instead of summative scores improves not only student learning but also students' attitudes toward the subject (Black, Harrison, Lee, Marshall, & Wiliam, 2004; Butler, 1988). Using this research as a basis, we investigated the impact this type of grading had on first year elementary education undergraduate students. In our program, these students are in cohorts taking the same five courses. In the study, all five of the students' instructors provided students with formative comments only on their assignments. These comments informed students of where they currently stand in terms of their learning and what else they need to do to fully meet the learning goals. No grades or scores were recorded on students' assignments. We hypothesized that with the decreased attention on grades, students would attend more fully to what they are learning. In other words, if students focus on what they are learning, and not what grade they got, they may actually learn more.

The participants completed a pre- and post-survey on their conceptions of grading and assessment at the beginning and the end of the semester. In addition, eight students were interviewed at the beginning and the end of the semester to get a more detailed account of their grading and assessment conceptions. Finally, the eight students who were interviewed will be followed and interviewed during the next two semesters to look at the impact of the study in the students' field placement and student teaching classrooms.

Preliminary results demonstrated that although students' conceptions of grading and assessment were somewhat stable from the beginning of the study to the end, they did expand their views of the purposes of grading and expanded their ideas of the grading process itself. However, the impact grades have on students cannot be ignored—many of the students in the study repeatedly stated that although written comments provide more information and are more motivating than letter grades, since they will eventually get a grade, it is the grade that is ultimately the most important. This finding has important implications for us in the education community, especially as we prepare students to become teachers. These future teachers will be responsible for grading and evaluating future students. Awareness of the power and effect that giving grades has on student learning is therefore of concern not only for these preservice teachers, but in fact everyone in the educational community. Teaching is all about learning; therefore it is important for future teachers to change their understanding of grading from determining "what grade did I get?" to "what did I learn?"

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JUMPSTART – JUNIOR UNDERGRADUATE MATHEMATICS PLACEMENT SURVEY TEST AND REFLECTION TOOL

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The leap from high school into college is broad and daunting for many. Unrealistic expectations and lack of knowledge can make the gap even wider and more inaccessible. Nevada needs a program that can offer foresight to high school students, their parents, their teachers and their counselors about information critical for math course placement during that ever-so-critical first semester or year of college.

Growth within Southern Nevada's System of Higher Education means greater demand for services, classroom space, and course offerings. Unfortunately, remedial programs in mathematics are among those in greater demand. In an effort to minimize the present and future need for remedial mathematics, the Center for Mathematics and Science Education at University of Nevada Las Vegas and the Community College of Southern Nevada have entered into a partnership with the Clark County School District to offer a **J**unior **U**ndergraduate **M**athematics **P**lacement **S**urvey **T**est **A**nd **R**eflection **T**ool.

The JUMP START test, designed for college-intending juniors, provides students and their parents and counselors with information that helps them recognize the benefits of continued mathematics study during the senior year. The exam, resembling a university mathematics placement test, is used to predict college readiness at the time the test is taken.

The primary goal of JUMP START is to significantly reduce the number of freshmen students enrolled in required remedial mathematics courses. This is achieved by using the JUMP START results to advise high school juniors (and their parents and counselors) as to the student's readiness to take college credit mathematics courses. Scoring reports to the students, parents, and counselors suggest anticipated college placement and encourage specific additional mathematics that would likely lift that placement to another level. Students obtaining this information by the end of their junior year are able to take corrective action in the senior year. In this manner, college-intending students are encouraged to "stay on track" in their high school mathematics courses, and high school students are encouraged to take mathematics during their senior year. Both of these effects have been shown to be factors in preventing college students from having to take remedial mathematics courses at the college level. At Ohio State University, where a similar program has been in place since the late 70's, the percentage of first-year university students enrolled in remedial mathematics dropped by 50% over a five-year period, while the enrollment in senior level mathematics classes at the high school level in a participating district increased by 72%.

No attempts are made to compare different teachers. The JUMP START program is designed to be as non-threatening as possible to students and teachers. All participation is voluntary. It is hoped that the program can eventually be offered at no cost since the NSHE will be saving money by reducing mathematics remediation costs (and students and parents will save on tuition costs for courses that do not contribute to their college degree).

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Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.

Each student receives a “personalized” report of their score and the implications of that score. The “freshman” mathematics courses at the self-reported anticipated institutions are outlined with reference to scores that would place a student in individual courses. Based on performance, the student is directed to take an appropriate senior level mathematics course. Parents will also be given a special brochure outlining many of the benefits of continued study of mathematics during the senior year.

A MODEL FOR FRAMING THE COGNITIVE ANALYSIS OF PROGRESSIVE GENERALIZATION IN ELEMENTARY MATHEMATICS

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This theoretical report considers connections in elementary mathematics between a model for progressive generalization adapted from Realistic Mathematics Education, construction of understanding from Cognitively Guided Instruction, and recent recommendations from the National Center for Improving Student Learning and Achievement in Mathematics and Science emphasizing the key mathematical practices of modeling, generalizing, and justifying.

In a recent research-based teacher education and professional development publication from the National Center for Improving Student Learning and Achievement in Mathematics and Science (NCISLA), Carpenter and Romberg (2004) emphasize the role of three key mathematical and scientific practices: *modeling*, *generalizing*, and *justifying*. They assert—

> To fully understand mathematics and science, students must also participate in the practices used by mathematicians and scientists. Typically, these practices go unnoticed and unlearned by most students because they are not made an explicit focus of attention…. Modeling is a practice central to the work of mathematicians and scientists, but traditional instruction has relegated it to the periphery. A commitment to the construction, evaluation, and revision of scientific and mathematical models, however, can engage students in authentic practices that provide coherence to the mathematics and science curriculum. Modeling is related to two other important practices of scientists and mathematicians: generalization and justification. Much of mathematics and science involves making generalizations and constructing arguments to justify them. (p. 3)

Such publications, with video examples of students engaging in these processes, are valuable for teacher education and professional development. Our perspective is that interpretive frameworks for how cognitive processes (such as generalizing) progress are key elements of teachers' knowledge for teaching elementary mathematics.

Theoretical Framework

The process of progressive generalization, as we have conceived and applied it in our research on teachers' and students' conceptions (knowledge and beliefs) (c.f. Smith, 1998; Smith, 2000) is one dimension of the three-dimensional model shown in Figure 1. This model is an elaboration of Freudenthal's Realistic Mathematics Education (RME) and its model of progressive mathematization as described by Gravemeijer (1994). Freudenthal advocated that learning mathematics should consist of the guided reinvention of mathematics, whereby students experience (with guidance) the process by which informal solutions to particular, situated problems become progressively more general and more formal. This process would end in the general axiomatic system of formal mathematics that has traditionally been the starting point for teaching mathematics. The bottom-up approach of RME stands in stark contrast to the top-down strategies that have dominated traditional approaches to mathematics education, each of which have begun with formal, generalized knowledge and procedures.

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.

From the RME perspective, mathematicians start with a problem situated in reality; organize the problem according to mathematical concepts; gradually trim away the reality through generalizing and formalizing, which emphasize the mathematical features of the situation and transform the real problem into a mathematical problem that faithfully represents the real situation; solve the mathematical problem; and make sense of the solution in terms of the original situation. Generalizing consists of "a posteriori constructions of connections rather than premeditated application of general knowledge" (Gravemeijer, 1994, p. 83). Formalizing includes modeling, symbolizing, schematizing, defining, and axiomatizing. This is the end point of the invention of mathematics. Romberg (1998) identified progressive formalization as a curriculum design criteria across long sequences of learning activities, while progressive generalization can occur across much smaller collections of situations as learners make connections among those situations and make generalizations from those connections.

Building on Treffers' (1987) distinction between vertical and horizontal mathematizing, our model separates the elements of the RME model into two distinct processes (progressive generalization and progressive formalization). Consistent with Cobb (1994, 2007), our model also has two layers (labeled *individual* and *social*) to provide for specific attention to and deliberate choices in emphases on (a) individual cognitive conceptions, processes, and contexts and/or (b) social processes and contexts in which the individual functions.

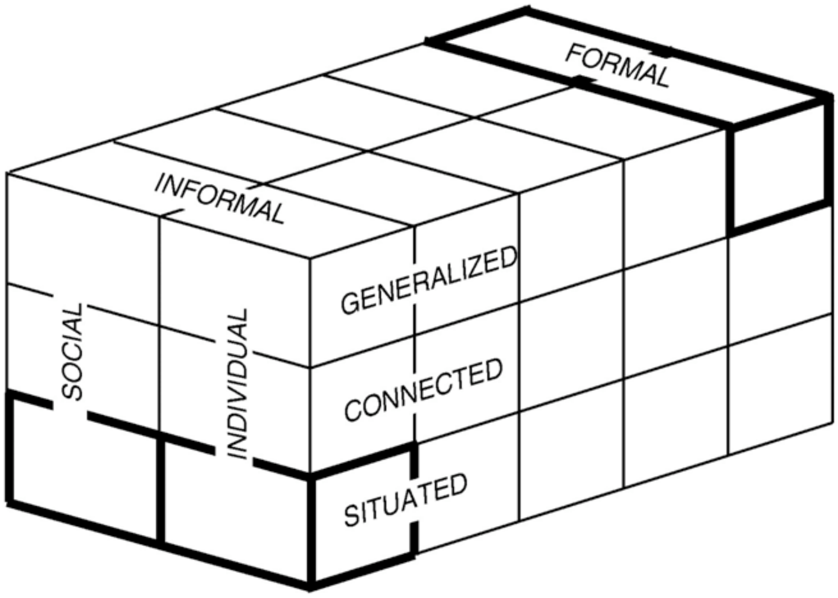

Figure 1. Smith Elaborated RME Model

Three levels of progressive generalization are shown in the vertical dimension of Figure 1 and in greater detail in Figure 2. These three levels are (a) situated conceptions that are linked closely with a problem-solving context or situation; (b) connected conceptions that are connected across contexts; and (c) generalized conceptions that are generalized from, yet linked to, other conceptions and their contexts. In Smith (1998), the first author demonstrates that the process of constructing mathematical generalizations progresses by connecting and abstracting situated conceptions which are originally constructed as context-specific conceptions embedded in particular problem-solving situations. In Smith (2000), the second author recognizes that children's invented algorithms, as described by the Cognitively Guided

Instruction (CGI) Project (c.f. Carpenter, Fennema, Franke, Levi, & Empson, 1999) represent generalized informal strategies for solving a wide variety of different types of story problems in various contexts. These generalizations are learned in the context of solving specific story problems, connecting what is learned in these individual experiences, and generalizing these strategies for later use in solving similar problems. These examples show that generalizations can be expressed using either informal or formal mathematical language.

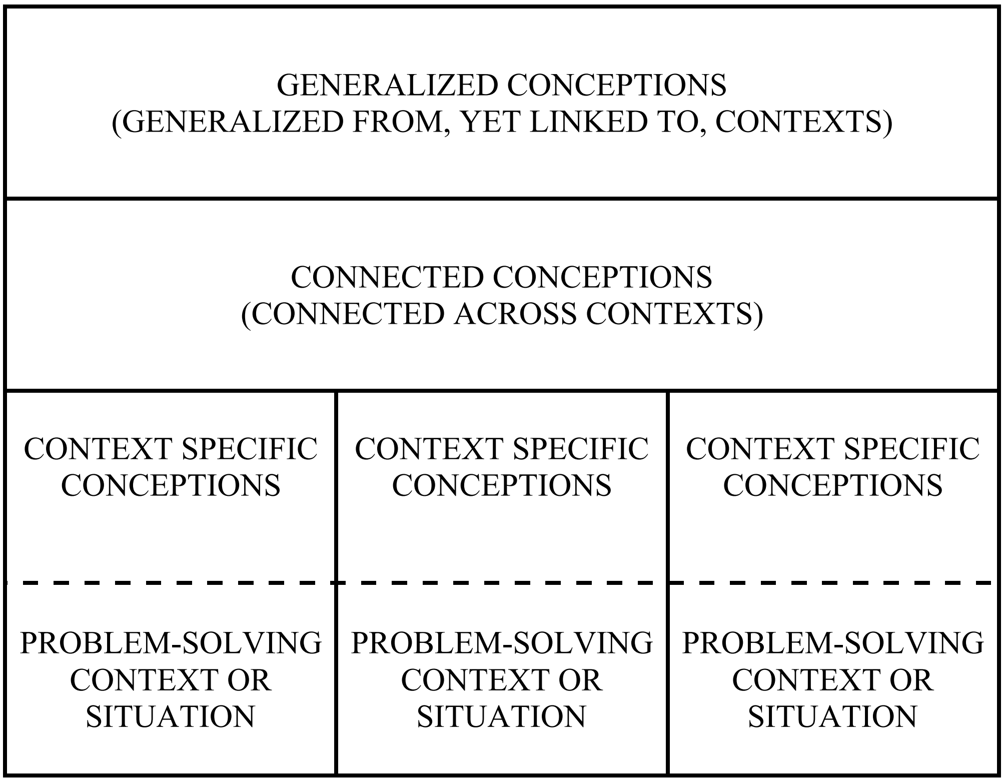

Figure 2. Detail of Progressive Generalization

This theoretical report includes: (1) an example from Smith (1998) detailing the use of the model to describe and explain a teacher's conceptions of even, odd, and super-even numbers; (2) our views of the nature of progressive generalization and its relationship to modeling and justifying; and (3) possible applications of this model for describing students' and teachers' progressive generalizations as authentic cognitive processes central to constructing mathematical understanding.

Using the Model to Analyze a Teacher's Conceptions

This section provides an abbreviated example (Smith, 1998) of the use of this model for analyzing progressive generalization by Sandy, a fifth-grade teacher who engaged during multiple sessions with two other teachers in working through the student activities in the fifth-grade units of a standards-based middle school curriculum. Figure 3 provides a graphical representation of some of the details of the progressive generalization process involving Sandy's conceptions of even, odd, and super-even numbers. As defined in the curriculum materials, a number is super-even if repeated halving eventually results in the number 1. For example, 32 is a super-even number because repeated halving of it results in 16, 8, 4, 2, and 1. This provides an informal definition for powers of 2. The problems to be solved in this example were to determine which of a list of seven numbers are super-even (Problem 7), show if 128 is super-even (P8), and find all super-even numbers less than 1,000 (P9). Table 1 provides selected statements of Sandy's conceptions which are interconnected in a portion of Figure 3.

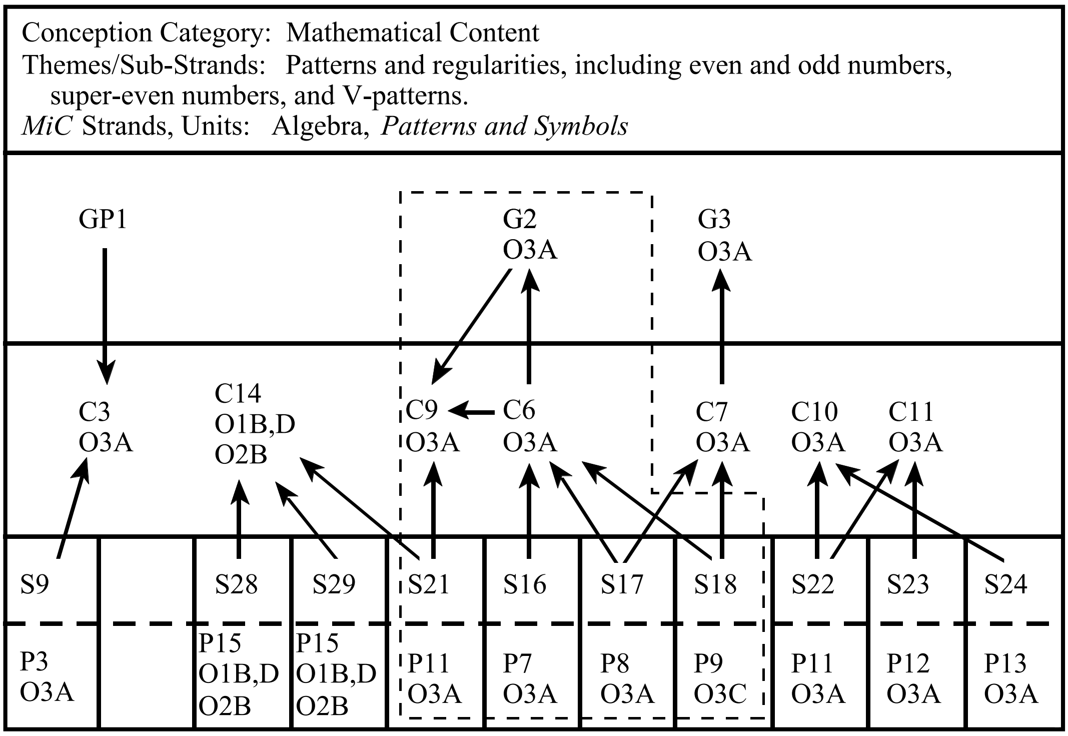

Figure 3. Sandy's Conceptions of Super-Even Numbers

S#	***Statements of Context-Specific Conceptions:***
S16	If you can repeatedly take half of a number without using fractions until you reach one, the number is called "super-even."
S17	In the process of halving a number to determine if it is super-even, once I got to 32 (a known super-even number), I knew it was super-even.
S18	I did a similar problem last year and a few of my students realized they could go up [multiply] instead of down [divide].
C#	***Statements of Connected Conceptions:***
C6	I can double a super-even number to find another, instead of having to test a number for super-evenness by repeatedly dividing.
C7	A large number is super-even if I can divide it by 32 and get another super-even number.
G#	***Statements of Generalized Conceptions:***
G2	The pattern of doubled numbers (1, 2, 4, 8, 16, 32, 64, 128 ...) are super-even numbers.
G3	A large number is super-even if division by any known super-even number results in another recognizable super-even number.

Table 1. Selected Statements During Problems 7-9.

Context-specific conceptions are situated in specific problem contexts, whether of the current problem or previous experiences. They are often stated in words that convey mental images of specific events, results, or experiences. Connected conceptions are often stated as conclusions based on more than one experience or repeated results, but remain connected to the elements of specific contexts. Generalized conceptions are often stated as either informal or formal rules that state or imply that the statement is always true or is useful in many similar contexts. At this level, the mathematical focus dominates the references to specific situations and highlights the general applicability of the models, strategies, or ideas. Careful analysis of a transcript from a problem-solving episode can reveal ideas being brought forward in time to be connected to new experiences and sometimes being generalized to future applications. The application of these generalized conceptions to new problem-solving situations provides additional evidence of their nature.

Progressive Generalization and Its Relationship to Modeling and Justifying

Since 1995, we have been asking preservice teachers enrolled in elementary mathematics teaching methods courses and inservice teachers enrolled in CGI workshops to conduct CGI-style interviews with students in Grades 1-3, posing story problems and analyzing children's understanding of addition and subtraction, multiplication and division, and our base-ten number system. Based on our analyses and grading of hundreds of written reports of these interviews, we have been able to recognize this process of progressive generalization as children explain their strategies for solving specific problems, connect many such experiences together, and begin to generalize their informal strategies and build flexibility in the application of those strategies to new, nonroutine problem situations. We have seen this process of progressive generalization begin with students' use of direct modeling strategies driven by their understanding of the actions and/or relationships expressed in the story problem. As experiences solving story problems continue, these students' collections of generalized strategies eventually include a rich mixture of direct modeling, counting, number fact, and invented algorithm strategies. Direct modeling strategies, by definition, are situated in the specific contexts, quantities, and actions of a story. Invented algorithms are generalizations that children have made from many experiences solving problems. Children apply these invented algorithms in new situations without referring back to specific previous solutions or contexts. Children's invented algorithms are at an informal level—they do not involve the formalized procedures and notations of the standard computational algorithms.

In their research-based recommendations on how to integrate arithmetic and algebraic thinking in elementary mathematics, Carpenter, Franke, and Levi (2003) emphasize that children need to use their understanding of equality and relational thinking to make their informal knowledge of properties of number operations explicit by articulating, refining, and editing conjectures. These conjectures constitute a basic form of informal generalization and mathematical argument that can progress to more formal and rigorous justifications and proofs. Since 2003, we have also been asking some preservice and inservice teachers to conduct student interviews focused on children's understanding of the equal sign and relational thinking. Based on our analysis of these reports, we have been able to recognize connections students make between their developing understanding of properties of base-ten numbers and number operations and their strategies for solving story problems.

In addition, during our six-year longitudinal teaching experiment using the *Investigations in Number, Data, and Space* curriculum materials, we have been able to recognize students' use of connected conceptions in making conjectures and other informal generalizations about the properties of numbers and operations. We have also seen how making these conjectures can be encouraged by teachers in standards-based learning environments through appropriate worthwhile mathematical tasks and orchestration of classroom discourse centered on children's thinking, strategies, conjectures, and informal justifications.

Possible Applications of This Model in Elementary Mathematics

Carpenter et al. (February 2004) summarize how what they have learned from their NCISLA studies can be used by others to develop similar innovative practices in new settings. They explain, "Our conception of teaching for understanding entails teachers forging connections among three bodies of knowledge: (a) the critical concepts, processes and methods of inquiry and argumentation of the content they are teaching; (b) the ways their students' mathematical and scientific thinking develops; and (c) the nature and effects of their teaching practices" (p. 6). We are convinced that the framework for organizing elements of progressive generalization described in this theoretical report provides a succinct and practical model for analyzing and understanding the ways in which the authentic process of generalizing develops from students' experiences solving story problems. As we have had occasion to share this model with teachers, it has provided (a) a rationale for valuing children's invented informal strategies and (b) a model for how solving problems in context and connecting context-specific conceptions through discourse about strategies, thinking, understanding, conjectures, and justifications can lead to valued generalizations that are important learning goals in elementary mathematics.

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ENACTIVE COGNITION AND SPINOZA'S THEORY OF MIND

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The purpose of this paper is to highlight some parallels between enactive cognition and Spinoza's theory of mind. Both frameworks entail a single ontology and a dual epistemology. The two systems can inform and support each other. According to Spinoza, mathematical knowledge develops from common notions, ideas which are universally present in the world. Common notions can be interpreted within the enactive framework as recurrent patterns of enaction. In both frameworks mathematics reflects the deep structure of the world. However, human mathematical knowledge is limited because a human individual, and indeed all of human society, is only an embodied fragment of a large and complex universe. This viewpoint offers an alternative to the constructivist approach and therefore has implications for mathematics education research.

According to the theory of enactive cognition, cognition is the interaction between an organism and the world in which it is embodied (Varela, Thompson, & Rosch, 1991). A major inspiration for enactive cognition is the phenomenology of Merleau-Ponty. Campbell (2001) has proposed a radical enactivism, with Merleau-Ponty's notion of *flesh* as the ontological primitive. This interpretation of enactive cognition has remarkable parallels with Spinoza's theory of mind. In both systems the monist ontology supports two epistemologies. Spinoza can illuminate and inform some aspects of enactive cognition.

In Spinoza's epistemology the mind is "the idea of the body," which is altered by impact with other bodies. *Common notions* are ideas that are present in all interactions between bodies. According to Spinoza, common notions are the foundation of mathematics. From the enactive perspective, common notions can be interpreted as recurrent patterns of enaction.

Spinoza's metaphysics implies a deeply structured world. If it is assumed that there *is* a deep structure to the world, then the objectivity and utility of mathematics is guaranteed. However, humans are only finite, situated fragments of a large and complex universe. For that matter all of human society constitutes only a small part of the whole. Human mathematical understanding is limited by its fragmented, situated engagement with the world.

The approach to Spinoza and enactivism offered herein is not fully compatible with the constructivist perspective. However, mathematics education research may benefit from an interdisciplinary dialogue that offers alternative ideas.

Literature

Enactive cognition is developed in Varela, Thompson, and Rosch (1991). Damasio (1994) provides additional neurobiological support for the theory, and Campbell (2001) proposes a radical enactivism that utilizes Merleau-Ponty's notion of *flesh* as ontological primitive.

The primary source for Spinoza is the *Ethics* (1677/1996). An explication of Spinoza's ideas, particularly as they can be related to mathematics, is given in Hampshire (2005). Parkinson (1954) and Carr (1978) can help to clarify Spinoza's theory of knowledge.

Changeux and Ricoeur (2000) and Damasio (2003) engage Spinoza for the light he can shed on current issues in ethics, philosophy of mind, and neurobiology.

The notion that the universe is deeply structured is a feature of Spinoza's philosophy. Campbell (2002) makes a strong argument for accepting the idea of a deeply structured external reality.

Lakatos (1976) provides an interesting perspective on the limitivist conception of mathematics implied by enactive cognition and Spinoza's theory of mind.

The Embodied Mind

Cartesian dualism rigorously separates the two worlds of mind and body. Various monist formulations, on the other hand, claim that all is mind (the idealists) or all is matter (the materialists). The theory of enactive cognition of Varela, Thompson, and Rosch (1991) exists in a middle ground between the extremes of monism and dualism. The residual dualism of enaction is expressed in the fundamental idea of *double embodiment*. Thus, we are physical beings existing in the world, but also we perceive the world, the world exists in us. According to Merleau-Ponty, whose phenomenology is the philosophical foundation of enactive cognition, "The world is inseparable from the subject, but from a subject which is nothing but a project of the world, and the subject is inseparable from the world, but from a world which the subject itself projects" (1962, pp. x-xi, cited in Varela et al). Varela et al write that "*embodiment* has this double sense: it encompasses both the body as a lived, experiential structure and the body as the context or milieu of cognitive mechanisms" (p. xvi, authors' emphasis).

The embodied point of view implies an organism's active engagement in its cognition rather than its being a passive receptor of perceptual data. The very fact of being embedded in the world means that the organism receives external stimuli that change the internal milieu. A change in the internal milieu in turn will change the way the organism acts, altering the stimuli it receives. Varela et al (1991) describe this in poetic terms as "organism and environment enfold into each other and unfold from one another in the fundamental circularity that is life itself" (p. 217). The circle spins at the flickering pace of phenomenal time. It is impossible to identify which comes first, stimulus or response, and the enactive perspective is to regard organism and the world in which it is embodied as a single, interactive structure. According to Merleau-Ponty, "When the eye and the ear follow an animal in flight, it is impossible to say 'which started first' in the exchange of stimuli and responses" (1963, p. 13, cited in Varela et al).

According to Damasio (1994),

> Perceiving the environment, then, is not just a matter of having the brain receive direct signals from a given stimulus, let alone receiving direct pictures. The organism actively modifies itself so that the interfacing can take place as well as possible. (p. 225)

Damasio's neurobiological approach implies that action is necessary to optimize perception. Varela et al (1991) go further to claim that *no* perception is possible without action. Even the experience of color, for example, depends on active perception. The cycle of action and perception continues unbroken and can only be halted with the cessation of both.

It is mistaken, therefore, to regard the organism as an observer of a changing exterior world. Action must accompany perception, but this action in turn alters the environment. According to Varela et al (1991), "[P]erception is not simply embedded within and constrained by the surrounding world; it also contributes to the enactment of the surrounding world. . . . [T]he organism both initiates and is shaped by the environment" (p. 174).

Cognition is defined by Varela et al (1991) as "Enaction: A history of structural coupling that brings forth the world" (p. 206). The *history of structural coupling* is the dance between organism and the world. They move together in perfect synchrony, neither taking the lead, but both moving to the same melody. The two are more closely intertwined than any lovers. In my understanding of enactive cognition, the boundaries of the organism do not stop at the physical shell of the body, but include organs, blood, and nerves—the body is itself part of the world that it enacts—and therefore cognition arises also in the *body's interaction with itself*. Mind *is* the interaction of organism and the world, including this endogenous activity. Consider the two aspects of embodiment. Firstly, the external manifestation of mind is physical activity. This physical activity includes, but of course is not delimited by, the electrical and chemical activity of the brain. Secondly, the internal manifestation of mind is everyday lived experience. It is this world of lived experience that is brought forth by the history of structural coupling.

Campbell (2001) argues that the residual dualism of double embodiment is "somewhat redolent of more traditional 'interactionist' concerns with mediating between the Cartesian real and ideal worlds than bypassing them altogether" (p. 7). He proposes instead a *radical enactivism* based on Merleau-Ponty's *flesh* as a single ontological primitive encompassing both mind and matter. Accordingly, the "objective 'real' world we are in and the subjective 'ideal' world within us [are] manifestations of the same world" (ibid., p. 3). The "real" and the "ideal" are therefore distinguished in epistemology *not* ontology.

There are remarkable parallels between the theory of enactive cognition and Spinoza's theory of mind. The agreement is even closer for radical enactivism.

Spinoza's Theory of Mind

Spinoza's philosophical system has been discussed recently with regard to ethics, philosophy of mind, and neurobiology (Changeux & Ricoeur, 2000; Damasio, 2003). Spinoza's theory of mind also has implications for the theory of enactive cognition. The parallels between the two may provide a way of understanding the nature of mathematical thinking that could have implications for the philosophy of mathematics education.

Spinoza defines *substance* as "what is in itself and is conceived through itself, that whose concept does not require the concept of another thing, from it must be formed" (E1 D3)[1]: substance is that which is not contingent. Spinoza demonstrates that each substance must be its own cause (E1 P6 C) and the cause of no other substance (E1 P6). Moreover, since these self-caused substances can be conceived, they also exist (E1 P7). This implication follows from Spinoza's first definition: "By cause of itself I understand that whose essence involves existence, or *that whose nature cannot be conceived except as existing*" (E1 D1, emphasis added). There can only be one such substance (E1 P14), which, along with its modes (discussed below), must therefore consist of all that there is. Spinoza calls it God or Nature.

Substance can be known through its *attributes*, which “the intellect perceives of substance, as constituting its essence” (E1 D4). Two attributes of substance are *Thought* and *Extension*: “Thought is an attribute of God, or God is a thinking thing” (E2 P1); “Extension is an attribute of God, or God is an extended thing” (E2 P2). Interestingly, Spinoza argues that although substance necessarily has an infinite number of infinite attributes, Thought and Extension are the only two conceivable by humans.

Thought and Extension are intimately related. A good way to understand this connection is to consider their manifestation on the human level. Firstly, however, it is necessary to determine what kind of thing a human individual is in Spinoza’s system. Spinoza defines *mode* as “the affections of substance, or that which is in another through which it is also conceived” (E1 D5). According to Spinoza, “Particular things are nothing but affections of God’s attributes, or modes by which God’s attributes are expressed in a certain and determinate way” (E1 P25 C). Human individuals as *extended* things are finite modes under the attribute of Extension; human individuals as *thinking* things are finite modes under the attribute of Thought. These two finite modes correspond to body and mind, respectively.

Spinoza defines *idea* as “a concept of the mind which the mind forms because it is a thinking thing” (E2 D3). The fundamental connection between mind and body is expressed as follows: “The object of the idea constituting the human mind is the body, or a certain mode of extension which actually exists, *and nothing else*” (E2 P13, emphasis added). Moreover, “The human mind does not know the human body itself, nor does it know that it exists, except through ideas of affections by which the body is affected” (E2 P19). Changes in the body, in other words, are precipitated by interaction with other finite modes. The mind is simply the procession of ideas of the body as the body itself is changed through these interactions. In addition, “The idea of any mode in which the human body is affected by external bodies must involve the nature of the human body and at the same time the nature of the external body” (E2 P16), meaning that external affects, in other words perceptions, are necessarily integrated and transformed by the human perceiver.

Spinoza’s construction of mind appears to parallel enactive cognition. However, enactive cognition requires an *active* engagement between organism and world, whereas Spinoza implies that ideas can correspond to a *passive* body that is acted on by the world.

A crucial point is that the “order and connection of ideas is the same as the order and connection of things” (E2 P7). In other words, it is better not to think of mind and body as two separate entities, with somehow a causal relationship between them, but rather as *one and the same thing under two different attributes*.

The idea of a single substance in which mind and body are finite modes of the two ways in which this substance can be known, Thought and Extension, recalls Merleau-Ponty’s notion of flesh, discussed above. Extension corresponds to the objective point of view, in which we are embodied in the world; Thought corresponds to the subjective point of view in which the world is embodied in us.

There are, therefore, substantial similarities between the two ways of understanding the relationship between mind and body. The next task is to discuss the ways in which Spinoza and enactive cognition can inform each other concerning the nature of mathematics and mathematical thinking.

A Limitivist Understanding of Mathematics

Spinoza's epistemology has three categories of knowledge: Imagination, Reason, and Intuition. According to Parkinson (1954),

> An imagination is an idea which corresponds to the impact of external things on the body . . . . Spinoza wishes to show that in the process of imagining the mind is passive: that the order of thoughts is not a rational order, imposed from within, but is imposed from the outside. (p. 139)

Imagination can be thought of, therefore, as sensory perception. Concretely, Imagination would correspond to perception of a triangle. However, Spinoza is clear that these sensory perceptions can originate endogenously as well as exogenously. Indeed, mind's-eye perception of a triangle following exposure to the word "triangle" is an act of Imagination (E2, P40, S2).

The second category of knowledge, Reason, follows "from the fact that we have common notions and adequate ideas of the properties of things" (E2, P40, S2). Parkinson (1954, p. 166) argues that Spinoza means Reason to cover general results reached by deductive reasoning, except those forms of deductive reasoning that may be attributed to Intuition, the third category. According to Parkinson (pp. 166-167), there are two aspects of Reason: firstly, universality—the deduction, for example, that the angles of any triangle total two right angles; secondly, inference to the particular—that a certain figure is a triangle implies that its angles total two right angles, because this property is true of all triangles.

The third and highest category of knowledge is Intuition, and "this kind of knowing proceeds from an adequate idea of the formal essence of certain attributes of God [i.e., particular objects] to the adequate knowledge of the essence of things" (E2, P40, S2). Parkinson (1954, pp. 182-185) interprets Intuition to refer also to deductive knowledge, but this time to knowledge concerning particulars rather than universal principles. According to Carr (1978), on the other hand, while inference from universal to particular would be an example of Reason, inference from particular to universal constitutes an example of Intuition. Intuition, in other words, implies a different ordering of our thoughts. In concrete terms, Intuition would be present when a *general* truth is immediately grasped from a *particular* geometrical diagram. Reasoned argument may subsequently lead to proof. In my understanding, mundane recognition that a particular geometrical figure is a triangle would also count as an act of Intuition. I prefer Carr's distinction to Parkinson's, and I would therefore categorize *all* deductive arguments under the heading of Reason.

The geometrical examples have suggested how different forms of mathematical reasoning may be assigned to Spinoza's distinctions. Spinoza's categories of knowledge may also be interpreted tentatively within the enactive framework. Accordingly, recognition that a particular perception is a "triangle," an act of Intuition, activates a specific, habitual manner in which an individual interacts with the world, a recurrent pattern of interaction with the world. A deeper Intuition, in which an individual immediately grasps a general truth, would correspond to formation of a *new*, and potentially habitual, manner of interacting with the world, a new recurrent pattern of interaction. Reason, on the other hand, charts connections between existing recurrent patterns of interaction.

According to Spinoza, "[T]here are certain ideas, or notions, common to all men. For . . . all bodies agree in certain things, which . . . must be perceived adequately, or clearly and distinctly, by all" (E2 P38 C). These *common notions* correspond to those recurrent patterns of interaction that are universal, that are shared by everyone. Moreover, such ideas are necessarily adequate (i.e. true) (E2 P38). According to Hampshire (2005),

> Mathematics in general, and geometry in particular, is the science which Spinoza had chiefly in mind as entirely founded on common notions; the fundamental notions and propositions of geometry and arithmetic impose themselves as self-evidently defining the universal and necessary properties of things. (p. 80)

In other words, the basis of mathematics, the common notions, are those ways of interacting with the world that are shared by all.

While specific mathematical ideas may be true, because they express universal principles, the common notions, the human individual is merely a finite mode and necessarily engages the world in a limited capacity. Therefore, whatever patterns of recurrent of interaction are available to the human individual necessarily entail a constrained understanding of the world as a whole. According to Hampshire (2005),

> Adequate [i.e. true] explanation is only found when we are dealing with an aspect of the material world which is everywhere the same, the same in our own bodies as in all bodies. Then we are concerned only with the laws that prevail throughout Nature and that are systematically related. Along this path we arrive at the order of the intellect and we approach the infinite idea of God or Nature. (pp. 10-11)

It is important to note in this passage that Hampshire refers to *approaching*, rather than *reaching*, the "infinite idea of God or Nature." In other words, the mathematical knowledge that is achievable by humans in their capacities as limited fragments of a very large and complex universe, is necessarily limited. It is worth quoting Hampshire in full on this point:

> The infinite idea of God or Nature must be the true mathematical representation of the deep structure of Nature and of its most general laws, everywhere and at all time in operation. Our mathematical knowledge, as it develops and unfolds, gives us glimpses of this; but the knowledge must always remain fragmentary, because our powers of thought are finite, and because our active thought must always be subject to interruptions. Lastly, the inputs that we receive in interaction with objects in the environment are limited both by our position in the common order of nature and by our sensory equipment and brains, that is, by the limitations of our bodies. Our minds are correspondingly limited. The mind is the active part of the whole person, who is both mind and body. (p. 11)

There is nothing in this discussion of Spinoza's view of mathematics that is inconsistent with enactive cognition. Spinoza, in his "infinite idea of God or Nature," clearly implies a deeply structured world. Is it unreasonable to postulate the same within the enactive framework, and that human mathematical thinking is limited because of the fragmented, situated human engagement with this structure?

Campbell (2002) gives an embodied perspective on the existence of a structured external reality. He argues that the very embodiment of human beings within the noumenal realm in itself assures a correspondence between human ideas and an external reality:

> When one considers that we are conscious, reflective, and free embodied beings constituted of and embedded within the noumenal realm, we are lead to the realization that, subjectively, our autonomous actions can be seen to arise through us from within that very realm. (p. 432)

If the correspondence does exist between human ideas and external reality, then a structured human cognition implies a structured external world. Provided that a world with deep structure is accepted, the objectivity of mathematics follows as a matter of course. The utility of mathematics in modeling aspects of the world is also assured, because the mathematics itself *is* the deep structure of the world: the model is the reality. Extending the metaphor that cognition is the intimate dance between individual and the environment, it can be seen, therefore, that *we all dance to a common melody, a melody that emerges from the groundswell of our being as embodied fragments of a deeply structured universe*.

Human mathematical knowledge will always be limited because humans are only finite, situated fragments of a large and complex universe—in fact, all of human society is only a finite, situated fragment of the whole. However, human mathematical knowledge does increase, and the increasing utility of mathematics implies that the direction of increase is toward greater knowledge of the deep structure of the world. This view is consistent with the method of proofs and refutations of Lakotos (1976), which is a reflection on the way in which human knowledge of the deep structure of the world is enhanced.

Implications for Mathematics Education

The viewpoint discussed herein is constructivist in that recurrent patterns of enaction create *individual* mathematical understanding. To the extent that society is a significant environmental factor for cognition that is characteristically human, mathematical understanding is also a *social* phenomenon. However, the radical constructivism of von Glasersfeld (1995) and the social constructivism of Ernest (1998) are not fully compatible with the views discussed herein because neither requires that mathematical knowledge should reflect a deeply structured external reality.

The existence of a structured external reality as the basis for mathematics indicates that the source of knowledge for the student is the same as that of the teacher. Their mathematical knowledge will be the same because of the shared features of their situated embodiments as human individuals in a deeply structured world—their mathematical knowledge will consist of notions common to both. Just as cognition arises from the interaction between individual and world, mathematical learning, can arise from the interaction between student and teacher, in which the teacher is a significant component of the student's environment. The teacher structures this interaction to facilitate student learning. Mathematical learning results in new recurrent patterns of interaction, intuitive knowledge in Spinoza's sense.

Any academic discipline requires a theoretical framework, a firm foundation to ground its empirical investigations. Mathematics education, in particular, needs to question the nature of mathematics and the nature of education. To my mind, the answers given by mathematics educators should reflect the humanity of education. By means of the ideas presented in this paper I have made some tentative suggestions in this regard. The resulting interdisciplinary approach is consistent with the goals of PME-NA.

Endnotes

1. The quotes from Spinoza's *Ethics* are Curley's translations in Spinoza (1677/1996). According to the usual practice, they are referred to by book (E), proposition (P), definition (D), corollary (C), and scholium (S).

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ELEMENTARY TEACHERS' MATHEMATICS TEXTBOOK TRANSFORMATIONS IN TERMS OF COGNITIVE DEMAND OF PROBLEMS IN FRACTION UNITS

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This study explored elementary teachers' transformations of textbooks into practice in teaching fractions in the context of recent efforts to reform mathematics education. This study focuses on the first chain of teachers' transformations for classroom instructions—that of selecting problems with their textbooks. Factors that support and constrain teachers' textbook transformation approaches are also explored. This study revealed three problem selection patterns in terms of cognitive demand. Three influential factors, textbook cognitive demands, curriculum policy, teachers' view on textbooks were identified.

This study examined teacher's textbook transformations in terms of cognitive demand when teachers select their problems with their textbooks. Research on teachers' textbook use and influential factors has been done over the course of two decades and has provided a substantial number of categories of teachers' textbook use patterns and factors that influence them (e.g., Freeman and Porter, 1989; Kauffman, 2002; Schmidt, Porter, Floden, Freeman, & Schwille, 1987; Stodolsky, 1989; Sosniak and Stodolsky). However, most of the previous studies on textbook use focused on the maximal extent of coverage, such as to what extent teachers use textbooks in planning and teaching school subjects. They do not help us understand how teachers use their textbooks to provide different students' learning opportunities.

Professional Standards for Teaching Mathematics (NCTM, 1991) articulated that students' opportunities for learning are created by, what Stein and Smith (2000) called, cognitive demands of task, "the level and kind of thinking required students to successfully engage with and solve the classroom problems". Although researchers have documented mathematics instructional changes with different frameworks, such as, "new mathematics topics," "variety of manipulatives," etc, to gauge teachers' implementation of reform ideas, opportunities for student learning are not created simply by putting students into groups, by placing manipulatives in front of them, or by handing them a calculator. Rather, it is the level and kind of thinking in which students engage that determines what they will learn (NCTM, 1991).

The purpose of this study is to examine elementary teachers' textbook transformation when selecting problems in terms of cognitive demand and its influential factors. This study focused on the topic of fractions. The detailed research questions are as follows:

1. *What textbook transformation patterns, in terms of cognitive demand, do elementary teachers exhibit in their problem selec*tion?
2. *What kinds of the factors influence teachers' problem selection with textbook in terms of cognitive demand?*

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Conceptual Framework

Cognitive Demands of Problems

Problems here mean a mathematical object in the textbook and in teaching meant for the students to figure out and solve. *Cognitive demands of problems* mean "the kind and level of student thinking required when (students) engage in problems. Two different levels of cognitive demands—high vs. low—identified by Stein and Smith (2002) were used in this study.

Factors Influencing Teachers' Textbook Transformation

To explore factors that influence teacher's textbook use, this study employed three-level factors: *individual-level*, *contextual-level*, and *teachers' opportunity-to-learn factors.* Table 1 shows three levels of factors and the sub-categories this study explored.

Table 1. Factors that influence teachers' transformations

Factor-level	Sub-category
Individual-level factors	
Teacher use of student objectives	
	Teachers' skills and knowledge
	Beliefs about teaching and learning
View about textbooks or textbook use	
Contextual-level factors	
District textbook policy	
Teacher perception of test (e.g., MEAP test, NCLB)	
Teachers' perceptions of the students' mathematics ability	
Teachers' opportunity to learn	
	Professional development opportunities
	Teachers' perceptions about cognitive demands of textbook lessons

Methods

This study originally employed a mixed method design that combines both quantitative (survey) and qualitative approaches (interview & observation) to explore elementary teachers' use patterns. However, due to space limit, this paper addressed only one method, quantitative method.

Participants

A total of 169 teachers participated in this study from second through sixth grade. This is convenient sample. Participants were recruited through Master courses at Mid-Western University and professional development programs in the U.S. Data was collected from 2006 summer to 2007 fall semester.

Data Instrument

The survey was first developed based on the previous studies (e.g., Horizon Research questionnaires (2003) and then the pilot study was conducted. Survey were revised based on their feedback and reformatted the survey to facilitate easier reading. The revised survey is comprised of five parts: (1) background information, (2) teachers' perceptions of the cognitive demand of their textbooks, their problem selection and their questions, (3) individual-level factors, (4) contextual-level factors, (5) teachers' opportunities-to-learn factors (see Table 2). Teachers' textbook transformation patterns were measured using teachers' perception of the kinds and levels of problems presented in their textbook lessons and in their problem selection. They were asked to indicate the frequency of the various types of problems presented in their textbook and those used in teaching. Influence of individual-level factors, contextual level factors, and teachers opportunity to learn factors are measured by asking teachers to how much they would agree or disagree with each of the provided statements.

Data Analysis

After the preliminary statistics using the Statistical Package of the Social Science (SPSS), scatter plots were first used to get sense of the relationship between different variables and textbook transformation patterns were identified based on the average means of scale. Then regression analysis were completed for two purposes: (a) to examine relatively important influences among the three groups of factors and (b) to identify the most influential group of factors among individual-level factors, contextual-level factors, and teachers' opportunity-to-learn factors.

Results

1. *What textbook transformation patterns, in terms of cognitive demand, do elementary teachers exhibit in their problem selection?*

To find the relationship between cognitive demands of textbook problems and selected problems in their teaching, this study used the scatter plot. Figure 1 shows the linear relationship between cognitive demands of textbook problems and those in teaching, indicating that the higher cognitive demand of textbook problems teachers indicates, the higher cognitive demand problems teachers use. From this trend we can draw two different transformation patterns, which are *H-H pattern*, teachers who selected high cognitive demands problems with high cognitive demands textbooks and *L-L pattern*, teachers who selected low cognitive demands problems with low cognitive demands textbooks.

Figure 1 Scatterplot of Textbook Problem Cognitive Demands against Cognitive demand of Problems in Teaching

Figure 1 also shows that there are variations, indicating that there are other possible patterns. To explore precisely textbook transformation patterns, the average rating scale (m=3) were used. For example, if the average composite score of textbook cognitive demands problem is greater than 3, it was considered high cognitive demands textbook;

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otherwise low cognitive demands textbook. Three patterns are identified as shown Figure 1.

2. *What kinds of factors influence teachers' problem selection in terms of cognitive demand?*

This study used a hierarchical regression to explore this question. Based on the literature review, the order of entering three groups of factors was decided. Since individual level factors (e.g., teachers' knowledge) are considering important factors, I first entered individual level factors. In the second step, learning opportunities factors were added. Then contextual level factors were added to create a full model that explains teachers' higher cognitive demands content selection.

The results show that while student objectives among individual-level factors contributed significantly to the prediction of the teachers' problem selection while teachers' knowledge and beliefs did not have direct effect. Second, the addition of factors related to teachers' opportunity to learn highly improved the prediction of teachers' cognitive demands problem selection (24%_68%). Particularly, teachers' perception of textbook cognitive demands had a direct and significant effect, but the effect of student objectives decreased and became insignificant. When contextual level factors were entered, curriculum policy (groups with curriculum policy vs. no policy) was significantly associated with teachers' problem selections, meaning that teachers select slightly high cognitive demands contents when there is curriculum policy. This study has implications to policy makers, curriculum developers, and teacher educators.

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EXPLORING CONNECTED UNDERSTANDING IN CONTEXT

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We discuss connections students made while exploring an open-response task in university honors calculus. As students exercised their agency, they built connections amongst previous tasks, their life experience, graphs of rate and quantity, and conceptually important calculus ideas. Students used metaphor and linguistic invention to justify or explain these connections to others. The conceptually important calculus ideas and connections students built ultimately helped them to build conceptions of the Fundamental Theorem of Calculus

Connected understanding has been an increased focus in recent years (Hahkioniemi, 2006; Martin & Towers, 2006; Megowen & Zandieh, 2005; Walter & Gerson, 2007). The nature of connections that students build, particularly within inquiry-based classrooms, has not been clearly characterized.

Literature Review and Theoretical Framework

Students learn through the exercise of personal agency (Walter & Gerson, 2007). Through sustained mathematical exploration, students activate their agency by making choices and engaging in problem solving activities with a high level of motivation, and a sense of enthusiasm as ideas are built and justified (Cifarelli & Cai, 2005, Zaslavsky, 2005).

To describe connections that students make, we use the idea of conceptual blending (Fauconnier & Turner, 2002) where two mental spaces are connected to form a blended space. Two specific types of connections are metaphor (Presmeg, 1998) and linguistic invention (Brown, 2001; Walter & Johnson, in press) where students use a personal experience to create a parallel story to the mathematical context they are exploring.

Research Methodology and Question

In winter, 2006, the authors conducted a teaching experiment in which 22 students collaboratively explored cognitively important, open-response calculus tasks resulting in multiple solution strategies as well as multiple directions of inquiry. The qualitative data is extracted from three hours of videotape collected within two class periods, in which four students explored the Quabbin Reservoir Task (Hughes-Hallet, 1994). Students were given graphs of inflow and outflow of water in Boston's Quabbin Reservoir as a function of time (see Figure 1) and asked to reason about the quantity of water in the reservoir.

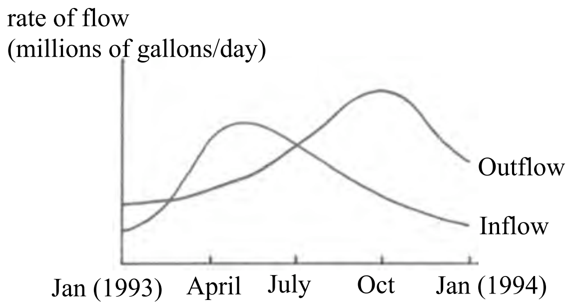

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Figure 1: The Quabbin Reservoir Task (labels added)

In order to explore the question, 'What connections do students make as they explore the Quabbin Reservoir Task?' the videos were transcribed and broken into episodes including cognitively important discussions, questions, refutations, or justifications. Each episode was coded for connections and organized chronologically and in collections of related episodes which were further analyzed with respect to existent theories and emergent theories, resulting in a multilayered analysis supported by strong evidence. The video and transcript were viewed together throughout and theories were triangulated with field notes and student work.

Data and Analysis: An Example from Jay

Jay first drew a graph of the quantity of water vs. time by reasoning about how the rates affected the quantities at individual points. As he drew his graph he made connections between the rate of flow and the change in quantity, while remaining grounded in the context of the reservoir (see Figure 2), building ideas important to understanding the antiderivative.

Jay: If you look here the inflow is kind of the same from right here to right here [*pointing to the maximum inflow at May*] (see Figure 1), but the outflow is going up. And so you know, we'll be saying that this point [quantity] is going, is increasing.

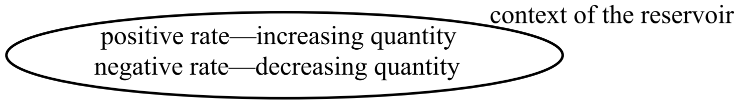

Figure 2: Jay's Blended Space: Rate and Quantity

Next Jay began to compare the quantities of water at the beginning and end of the year, still reasoning about the differences between the rates. Shaun suggested that they could also determine the quantity of water in the reservoir by comparing the areas between the inflow and outflow curves. When later Shaun and Jay disagreed about how much water was in the reservoir at the end of the year, instead of returning to his own reasoning with rates, Jay used areas to justify Shaun's answer and disprove his own (see Figure 1).

Jay: Okay, well look at this. Just look at that section with this section put together and I think you're right because…that little bowl (*a*)…That's all the gain in inflow…And then that bowl (*b*), then that bowl (*c*) are all the outflow, so I think you are right.

In the course of Jay's justification, he used a metaphor of a bowl as a container for the area between the curves. This metaphor was a conceptual blend between Jay's life experience and the quantity of water in the reservoir. Thus, Jay built a three-way connection amongst the area between the inflow and outflow of water, a bowl as a container, and the quantity of water in the reservoir (see Figure 3). These connections were just the beginning of stronger connections Jay later makes as he builds his understanding of area between curves.

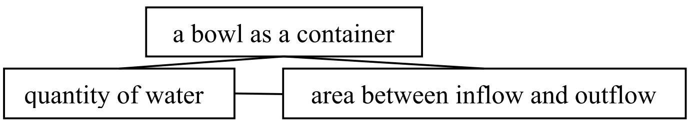

Figure 3: Jay's Bowl Metaphor

Later the group tried to make sense of the units of the area between the curves. Timbre and Jay began to debate whether the units along the horizontal axis (quarters of a year) were appropriate measures of time since the units of the inflow and outflow are millions of gallons per day. Jay suggested that there was no reason to convert the quarters of a year to days. In order to explain his thinking to Timbre, Jay introduced linguistic invention by suggesting that they think about the acceleration of a "dude" jumping out of an Apache.

Jay: Acceleration is the distance over time over another time.…Well in an acceleration such as gravity…you are going at a certain rate, like you're falling, you have a certain rate.…Alright, this dude jumps out of an apache, okay?…His rate is meters per second, but he is going over a period of time. Time is passing while his rate is still a thing over seconds…these passing seconds are not changing whether or not this second, as you know it's a part of a rate, it's different than this passing time. Each notch on this passing time is not effecting whether or not this is meters per second.

Conclusion and Implications

In the larger body of data, students made connections amongst previous tasks, their life experience, graphs of rate and quantity, conceptually important calculus ideas such as: area antiderivative, and concavity. In particular, they connected the quantity of water to both the rate of change and the area independently within the context of the Quabbin Reservoir. Students used metaphor and linguistic invention to justify or explain these connections to others particularly in the face of disagreements or debates. The conceptually important calculus ideas and connections students built while exploring the Quabbin Reservoir Task ultimately helped them to build conceptions of the Fundamental Theorem of Calculus.

When students are allowed to exercise agency in the classroom as they build ideas, they use their rich life experience, and shared classroom experience to justify and build connections between those ideas. Metaphor and linguistic invention can be powerful tools for students to communicate their understandings and in particular connections between ideas to themselves and others. Facilitating an environment where students have the freedom to choose their own connections will allow them to more freely build upon what they already know. Establishing an environment where students explain and justify their thinking will further allow students to build stronger connections through the use of powerful connecting and communicating tools such as metaphor and linguistic invention.

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PROMOTING COMPLEXITY RESEARCH IN MATHEMATICS EDUCATION

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Recently has been an interest in the significance of complexity theories for educational research. Complexity theory allows researchers to experience a shift in thinking to a "middle way" mentality of proscriptive studies. Common principles of researching in this middle way, however, have not yet been articulated. The paper reports on an inquiry about methodological commonalities, the principles that appear to guide complexity-based studies. The inquiry involved content analysis of seven complexity based theses. The commonalities amongst others include: task oriented research, researchers as close participant-observer, multiple research observers and diverse data interpretations. We argue that to the extent that studies framed by complexity theory have common modes of inquiry that are unique, complexity theory is a methodology.

Over the past three decades, the field of mathematics education has witnessed development of new theories of knowing and knowledge such as constructivism, situated cognition, critical theory and post-structuralism. With regard to mathematics education research these new theories inform a proliferation of more descriptive studies among researchers that problematize the assumptions behind prescriptive studies. Recently there has been an interest by several researchers in the significance of complexity theories for educational research. Complexity theory allows mathematics education researchers to experience a shift in thinking from an "either or" mentality between descriptive studies or prescriptive studies to a "middle way" mentality of proscriptive studies that seek to understand and build environments that support learning, teaching and research in complex ways. Canadian mathematics educators, such as Kieren, Simmt and Davis view complexity theory as a theoretically promising, middle way for investigating mathematics educational issues. Common principles of researching in this middle way, however, have not yet been articulated. A central question that we address in this paper is: What might research techniques informed by complexity thinking look like?

Our intent in this paper is to contribute to the new interest in complexity theory. First we offer a brief synopsis of the discourse, complexity theory, and its sub discourse, enactivism. We then report on an inquiry about methodological principles, if any that guide mathematics education researchers who adopt complexity theory. We have dubbed research guided by complexity theory, proscriptive research. The inquiry is based on a content analysis of our own work and the work of other complexity researchers as well as interviews with selected complexity researchers. We seek for the commonalities among the studies with regard to the nature of the research, the role and nature of observers, the form of inquiry structures (including data interpretations and research writing) and the relation between empirical data and theory. We conclude the paper by elaborating on these commonalities.

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Theoretical Framework: Learning as a Complex Process

Davis and Simmt (2006) define complexity science as a science that deals with "self-organizing, self-maintaining, adaptive phenomena—in brief, with systems that learn" (p. 295). Adopting complexity thinking we view individual learners as well as collectives of learners plus other learning systems as complex learning organisms whose study lends itself to research methods that respect this complexity. Complexity research explores how complex adaptive living systems, humans included, learn and live (Capra, 1996). One of the sub discourses of complexity science is enactivism, which is based on the work of Varela and Maturana (1992), and Varela, Thompson and Rosch (1992). Enactivism is a theory that views cognition and learning as an "enactment of a world and a mind on the basis of a history of the variety of actions that a being in the world performs" (Varela, Thompson & Rosch, p. 9). Some Canadian mathematics educators frame their research with enactivism (see for example, Gordon 2001 & Simmt, 2000). Kieren, Davis and Mason (1996) observe:

> At the time when there seem to be conflicting views on mathematical cognition between those which observe it as personally driven and those which observe it as externally driven; between individually based and socially based views; between cognition as fundamentally active or fundamentally receptive; …it is important and perhaps necessary to seek and apply ways of thinking about cognition which are in the middle (p. 9).

In this middle-way mind, body and world; mathematical structures, individual learners and collective of learners; teachers, learning resources and curriculum exist in relation to each other, to the systems that they form (and those that form them) and to their environments. We adopted the complexity theoretical stance for our studies but in doing so we find ourselves enacting a form of inquiry that takes into account the embodied (biological), embedded (socio-cultural) and extended (material-technological) nature of learning systems.

Methods: Analysing Our Three Studies

In the three studies that form the basis of this paper we each drew from enactivism in particular and from complexity theory in general. Each one of us investigated the dynamics that emerge during mathematics learning/teaching activities. For Author 1 (2005) the dynamics were in terms of mathematical attentiveness. Her research involved both classroom-based and extra curricular activities. The participants were secondary school students. It involved two sites—one in Canada and the other in Uganda. For Author 2 (2004) the dynamics were in terms of students' talk and explanations. Her research study was in Namibia and was classroom-based. Yet for Author 3 (2002) the dynamics were about the nature and growth of knowledge of mathematics student-teachers as they undertake teacher education programs. She too had two research sites, one in Tanzania and the other in Canada.

Author 1, Author 2, and Author 3's studies are all qualitative but differ in terms of focus—students learning, classroom practices and teacher learning. The researchers also were African students in Canada. To isolate only the commonalities that pertain to the basis of the studies in complexity theory we analyse the studies alongside four other studies whose researcher did not have an African origin: Glanfield (2004) on teacher understanding, Gordon (2001) on mathematical conversation, Simmt (2000) on mathematical knowing, and Reid (1995) on proofs in mathematics learning. The inquiry method involved content analysis of our three studies and of the other four complexity-based studies. In addition Author 2 carried out semi-structured interviews with all the researchers except Reid. The goal of the inquiry was to ascertain the tacit principles that guided their research, principles that were not drawn from established research methodologies such as ethnography. We sought for the commonalities among the studies with regard to the nature of the research, the role and nature of observers, the form of inquiry structures (including data interpretations and research writing) and the relation between empirical data and theory. This paper particularly focuses on what, in terms of these methodological aspects, appeared common among complexity-based studies.

Results: Commonalities among Complexity-Based Research

Although a number of education researchers at the University of Alberta have drawn from enactivism, and later complexity science to inform their studies, no one has, at least not in detail, explored complexity or enactivism as a research methodology. Perhaps this gap might be explained by the fact that, in theory, neither complexity thinking nor enactivism is a research methodology. Both are mainly thought about as theoretical frameworks. We argue that to the extent that studies framed by complexity theory have common modes of inquiry that are unique, complexity theory is a methodology.

Complexity research has its routes in scientific studies, especially, of brain and other bodily systems. The methodology in the scientific studies is mainly clinical and laboratory based with very few scientists drawing from disciplines in the humanities such as phenomenology and hermeneutics to inform their experimental results. Thus it is interesting that complexity-based, proscriptive studies in mathematics education have quite a lot in common in terms of methodology. These commonalities include: use of task oriented research, researchers taking on the role of close participant-observer, having multiple observers at a site, encouraging diverse interpretations of data, making explicit the conditions of observations, engaging in recursive writing, using organic inquiry structures, and braiding empirical investigations with theory.

Discussion: Guiding Principles of Complexity Research

Put differently, the proposed paper shares methodological insights gained from complexity-based studies. In the proposed paper we illuminate each of the principles with examples from our studies. By doing so we explore how researchers could adopt the complexity or enactivist perspectives to investigate mathematics learning and teaching. We show that complexity studies require flexibility on the side of the researcher. Just like life or adaptation, when proposing a study one may not know, for instance, who the multiple observers are or where the multiple sites and interpretation will be. It is in carrying out the study that a researcher happens to choose directions, explore techniques, seek opportunities and identify subsequent data sources, thus laying down an organic study design. For that reason, instead of thinking in terms of a pre-specified and descriptive research that is an individualistic, static and pre-designed endeavor we look at it as a collective, co-emergent and organic—proscriptive process. Proscriptive studies need only general principles, principles that encompass many potentialities.

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WHY DO STUDENTS HAVE DIFFICULTIES RECOGNIZING THAT A LINEAR EQUATION CONSISTS OF TWO ALGEBRAIC EXPRESSIONS?

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Data from a small empirical study of ninth grade students carried out by the author indicated that although these students could simplify algebraic expressions and solve linear equations with ease, they still had difficulties indicating a relationship between algebraic expressions and linear equations. This report traces some of the developmental roots that aid in explaining why students have difficulties recognizing such a relationship and suggests some approaches to remediation.

Theoretical Framework

The available literature indicates that students face considerable challenges with recognizing and using the structure of expressions and equations (Kieran 1989a). Kieran (1989a) defines linear equations as open number sentences consisting of two expressions which are set equal to one another and names this the *surface structure of an equation*.

Sample of Empirical Work

In a small empirical study of four fifteen-year-old ninth grade students carried out by the author, students' responses to the question "What are the differences between algebraic expressions and linear equations?" were:

Ellen: Equations have an equal sign and variables … something equals something else. Expressions don't have an equal sign.

Tim: Expression is like giving a certain number to a variable, … I think! And then equation is a whole bunch of numbers added together kind of a thing.

Kathy: I think equations don't have variables and expressions do. When you have an expression and you simplify you don't find the exact answer, but you just kind of add like terms together. But when you solve an equation you find the exact answer.

Susan: Expressions don't have the equal sign. They have a variable like x.

As the responses indicate, only Ellen was able to indicate that equations consist of two entities set equal to one another. Yet, Ellen was unable to recognize that these two entities are algebraic expressions. Susan was able to recognize that the equal sign is not a characteristic of algebraic expressions. However, Susan was merely able to recognize the *existence* of the equal sign in equations but not its balancing nature. These responses confirm Kieran's claim.

Questions

This report attempts to address the following questions: 1)Why do students who have had instruction in algebra have difficulties in recognizing the surface structure of equations? 2)What are some possible approaches to remediation of this educational problem?

Developmental Roots and Cognitive Issues

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Various aspects of development can help explain the aforementioned difficulties that students have. First, let us consider issues of memory. Once a new piece of information we learn is stored in our memory, it becomes part of our knowledge base and can be retrieved from memory by *cues* that are highly similar to what is stored in it (Rovee-Collier, 1999). Perhaps when students encounter equations, adequate cues that relate expressions to equations are not provided leading students to think that they are learning something completely new and unrelated to expressions. Second, student difficulties might have roots in *categorization*, in that students might only be able to form *basic-level categories*. These are general categories whose members share overall shapes and characteristic actions (Mervis, 1987). In the case of algebraic expressions and linear equations, perhaps students form one basic-level category for both according to their common characteristics such as numbers, variables, and actions such as combining like terms in simplifying expressions and in simplifying each side of an equation. In addition, students have difficulties in viewing the equal sign as a symbol of balance. This might be because when students were learning arithmetic they viewed the "=" sign as a symbol that *potentiates action* in arriving at the exact answer (Ellis & Tucker, 2000). Once linear equations are introduced, students transfer their knowledge from arithmetic to solving linear equations. Action potentiation caused by the equal sign in the form of finding the value of the unknown is especially apparent in the case when the method shown to students is one which transposes the terms in the equation.

Approaches to Remediation

The available literature suggests that physical manipulatives and metaphors provide scaffolding for abstract ideas (Goldstone & Son, 2005; Boroditsky, 2000). In the case of linear equations balance pans could be used in curricula – as physical manipulatives or metaphors - to represent simple linear equations, emphasizing that each side of the balance beam is an algebraic expression such that when these two expressions are balanced the result is a linear equation. However, the balance pan metaphor can only be used to represent linear equations that have a positive number solution. So, instruction on linear equations could start with equations that have positive number solutions before moving on to complete symbolic manipulations and to linear equations with negative number solutions. In this manner, the process of *concreteness fading* can be used to first introduce students to concrete situations (physical manipulatives) and then to reasoning about these situations explicitly, allowing students to successfully generalize to a more formal understanding of the concepts (Goldstone & Son, 2005).

Another approach to remediation is the use of *priming* tasks. When linear equations are introduced in the classroom, priming tasks can help students retrieve the information stored in their memory that relates to algebraic expressions. Teachers could write an initial exemplar of a linear equation on the board, hide the equal sign and right-hand-side of the equation and ask students what the left-hand-side is and if they have a label for it. The same could then be done with the other side of the equation.

Further research is needed to identify whether these approaches to remediation indeed help in overcoming the educational problem described in this report.

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COGNITIVE IMPACTS OF ATTENTION DEFICIT-HYPERACTIVITY DISORDER (ADHD) ON COLLEGIATE MATHEMATICAL LEARNING

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Many grade K-12 studies have indicated that students with ADHD are more likely to have low academic achievement than non-ADHD students, but little research has been conducted on this subject at the postsecondary level. For years, those who have worked with ADHD college students have generalized results from elementary and secondary students to the college level, without a firm post-secondary research foundation. It is now known that adults with ADHD often experience different symptoms than in childhood (APA, 2000). This new understanding of ADHD experience for adults, coupled with increasing enrollments of students with ADHD in postsecondary settings, means we must increase our understanding of the particular learning needs of the collegiate ADHD student population. Towards that end, my dissertation study investigated the following questions:

Q1. Are the cognitive and affective challenges encountered by college mathematics students with ADHD different from those outlined in published literature for all college students learning mathematics?

Q2. If so, what is the nature of the difference(s)?

I used a purposeful case-sample design with three undergraduate students with ADHD who were enrolled in mathematics courses at two Ph.D. granting universities. I completed constant-comparative analysis of over 30 hours of task-based interviews to create cognitive-affective profiles of students' problem-solving activity. As points of comparison, I created similar profiles for non-ADHD student work both from existing literature and additional task-based interviews with ADHD participants, three non-ADHD college mathematics students, and two college mathematics instructors using the "bottle problem" (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). My profile and theory building was informed by three existing but as-yet-separate theories (Bandura, 1986; Barkley, 1994; Pirie & Kieren, 1994). Barkley's *unified theory of ADHD* is a combination of theory from education and neuropsychology that specifically addresses cognition in those diagnosed with ADHD while Pirie and Kieren's *dynamic model* provided a means for visualizing growth in mathematical understanding. Both were situated in social cognitive theory's interacting personal, behavioral, and environmental factors.

The results of this study describe the patterns of mathematical activity specific to the three college students with ADHD. Additionally, the "bottle problem" interview responses were analyzed, profiled, and compared to similar analyses of responses reported in Carlson et al. (2002). As a result of the analysis and visual profiles (to be displayed on the poster) it became clear that the answer to the first research question was: Yes, there appear to be unique aspects to the kinds of cognitive and affective challenges faced by ADHD college mathematics learners. And, in response to the second research question, the nature of the differences seems to hinge on

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the privileging of visualization and pictorial representations, over symbolic and tabular representations, by college-level ADHD learners.

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LESSON STUDY: THE EFFECTS ON TEACHERS AND STUDENTS IN URBAN SCHOOLS IN THE UNITED STATES

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The focus of this poster session will be to share a professional development model of Lesson Study adapted from education experts such as Lewis (2002), Germain-McCarthy (2001), and Yoshida (1999). This research sought to examine the effects lesson study had on mathematics teachers and students in an urban middle school. In particular what effects lesson study had on teachers' instructional strategies and conceptual understanding as well as students' achievement and conceptual understanding was investigated.

The professional development of teachers is essential to improving the nation's schools, but for most teachers in the United States, professional development means attending a one-day workshop designed to transfer a specific set of ideas, strategies, or materials with little or no follow-up to facilitate classroom implementation (National Research Council, 2002; Smith, 2001). One growing consensus concerning professional development, however, is the opportunity for teachers to engage in collaboration and follow-up discussions with their peers as sustained motivation for instructional change (Guskey, 2000). Lesson study consists of the elements researchers recognize as essential to positively influence instructional practice and students' understanding.

Consequently, 13 middle school mathematics teachers in a large urban school district formed three lesson study groups. Data collection consisted of a pre and post questionnaire, planning and reflection transcripts, observation notes, lesson plans, student's work and assessments. A constant comparative analysis involving the three cases studies was used to analyze the data. The two stages of analysis consisted of both within and across case comparisons. More specifically, the researchers sought to examine the effects lesson study had on middle school mathematics teachers and students in an urban school district and will share findings by addressing the following objectives:

- Describe the design and implementation of lesson study in a large, diverse, urban school district
- Discuss the forms of data collection and analysis
- Share research results related to teachers and students
- Discuss barriers, issues, successes, and future endeavors for lesson studies.

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MATHEMATICAL LEARNING DISABILITIES: AN EXPLORATORY CASE STUDY

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Although it is estimated that between five and eight percent of school-aged children have mathematical learning disabilities (MLD) (Geary, 2004), researchers are still struggling to understand the cognitive deficits that characterize MLD. This field of study has been constrained due to the lack of diagnostic measures. Because no test for MLD exists, the vast majority of studies rely on achievement test scores below a researcher-defined threshold to identify students with MLD (Geary & Hoard, 2005). This has led to an inability to distinguish a low score caused by a disability from a low score due to other factors (e.g. poor teaching). Reliance on achievement measures has resulted in an unintended over classification of particular groups of students (low SES, minority, non-native English speakers) in the MLD group (Hanich, Jordan, Kaplan & Dick 2001). This poster presents an exploratory study aimed at understanding the ways in which the disability manifests itself.

I approach the study of MLD from a Vygotskian perspective, which presumes that a student with a disability "is not simply a child less developed than his peers but is a child who has developed differently" (Vygotsky, Knox, Stevens, Rieber, & Carton, 1993 p. 30). Longitudinal data was collected from weekly tutoring sessions with one student over the course of her 8th, 9th and 10th grade years (approximately 60 sessions total). The student's standardized math test scores place her in the bottom 25th percentile, which is comparable to students in other studies that were classified as having MLD.

A microgenetic analysis of the tutoring sessions indicated that the student solves math problems in non-normative ways. For example, she compensates for her inability to memorize discrete number facts by memorizing lists of multiples. She refers to her "sixes" as "6, 12, 18, 24, 30, 36…". To solve a multiplication problem she counts on her fingers to keep track of the ordinal position as she recites the list. Although this strategy is productive in contexts where she is multiplying two numbers, it did not enable her to efficiently divide. In attempting to solve the problem 2 x __ = 16, she said "You can't do it… cause 2 doesn't go into 16 *evenly*... 5, 10. 6, 12. 7, 14. Oh! 8. Duh.". In this case she was accessing the second element in her "fives", "sixes", "sevens" and "eights" multiples list. Her inability to automatically retrieve number facts constrains her ability to reason about more complex mathematics. This, as well as other non-normative strategies are discussed in detail in the poster presentation. A nuanced understanding of student difficulties is necessary to begin to design diagnostic measures. Only then, can diagnostic measures be created which provide reliable diagnosis and inform remediation.

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SOCIAL COMPARISON THEORY AS A FRAMEWORK FOR MAKING CROSS-NATIONAL RESEARCH PSYCHOLOGICALLY ACCESSIBLE TO US TEACHERS

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Cross-national research has identified many important and compelling differences between US and Japanese students' performance in mathematics. Yet, we believe that the majority of teachers exposed to this literature are left with a message of achievement gaps couched overwhelmingly in terms of US deficits (Moseley & Okamoto, in press; Puchner & Taylor, 2006). It remains an open question as to how to fashion these findings into productive efforts to encourage US teachers to adopt relevant and effective Japanese practices found in the research literature. We argue that this dilemma is the direct result of a lack of useful frameworks for making these findings psychologically accessible to US teachers.

Many US teachers find cross-national research daunting and difficult to interpret in the context of US school practices, not to mention the particular circumstances of their own classrooms. It is clear that research findings have focused intensely on differences between teaching practices in the two nations. We argue that this focus on differences, although based on accurate and compelling results, has not provided a complete comparison. It has inadvertently created a very confusing and intimidating situation for the very teachers that this research was intended to support (Puchner & Taylor, 2006).

In this research we advance a competitive/collaborative social comparison (CCSC) framework for facilitating the implementation of Japanese practices in US classrooms derived from empirical research in social comparison theory (Staple & Koomen, 2005). This framework helps direct our attention to the similarities, as opposed to differences, of teachers in both nations. By applying the CCSC framework, we discuss how to: (a) re-interpret differences, (b) identify similar student learning challenges, and (c) clarify important components of the larger of picture of teacher collaboration. We argue that making these factors prominent produces a more collaborative or integrative mindset, which, in turn, makes the practices of Japanese teachers more accessible to US teachers. Our argument unfolds in two phases, in which we first describe the CCSC framework and its relevance to cross-national research in mathematics, and second we address possible applications of the model that we believe hold promise for making cross-national research more accessible to US teachers.

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BOTH RHYME AND REASON: TOWARD DESIGN THAT GOES BEYOND WHAT MEETS THE EYE

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Drawing on design-based studies where students worked with learning tools for proportionality, probability, and statistics, I appraise whether students had opportunities to construct conceptual understanding of the targeted mathematical content. I conclude that visualizations of perceptually privileged mathematical constructs support effective pedagogical activity only to the extent that they enable students to coordinate perceptual conviction with mathematical operations—intuiting that*, and not* how*, two representations are related constitutes perceptually powerful yet conceptually weak situatedness. In constructivist learning, as in empirical research, regularity apprehended in observed phenomena is measured, expressed, and schematized. Students should articulate or corroborate visual thinking with step-by-step procedures, e.g., synoptic views of multiplicative constructs should include tools for distributed-addition handles.*

"Neither rhyme nor reason can express how much"
(Shakespeare, *As You Like It*, 3:2)

This theory-of-design paper builds on empirical studies of mathematical cognition and cognitive psychology and is inspired by phenomenological and analytic philosophy. I argue that visualizations of perceptually privileged mathematical constructs, e.g., proportionality (Gelman & Williams, 1998), are effective teaching/learning activities only to the extent that they enable students to coordinate intuitive and explicit knowings (Schön, 1981)—apprehending the *that* but not the *how* of quantitative relations embedded in images constitutes perceptually powerful yet conceptually problematic situatedness (e.g., Davis, 1993). Worse, students' false sense of understanding may hamper personal reflection or formative assessment. To demonstrate epistemological tensions and pedagogical promise inherent to *visual thinking* (Arnheim, 1969), I discuss four data episodes pertaining to the domains of proportionality and probability. In each study, participant students can be said to have grounded the mathematical content in the situation only to the extent that they were equipped to articulate their multiplicative judgments additively.

Theoretical Background

A critique proverbially inveighed against traditionalist instruction is that students who demonstrate procedural skill often have not developed deep understanding of the mathematical concept. Yet, could such procedural–conceptual hiatus result from participating in activities that purport to embody constructivist pedagogical philosophy? This paper's point of departure is that some situational contexts intended to help students ground the meaning of a mathematical concept may not do so, even when they appear as though they might. These pedagogical situations have the trappings of powerful learning environments: interactive tools representing

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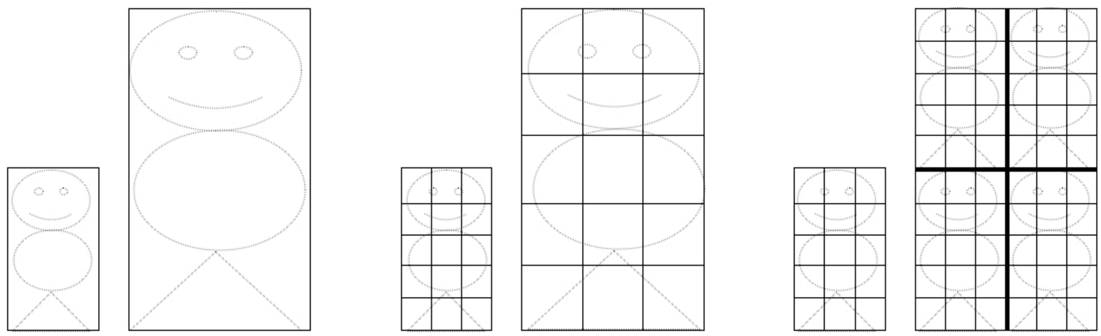
mathematical quantities, symbols, and relations are laid out for manipulation, and rules of situated–symbolic translation are either empirically apparent or provided by a facilitator. Yet, I contend, these rules of translation themselves may not be transparent. Namely, I argue, designers are liable to inadvertently confound perceptual constancies with operatory conservation (Piaget & Inhelder, 1952); psychology with epistemology (Papert, 2000); embodied schemas with mathematical fluency (Abrahamson, 2004). Therefore, sorting out phenomenologically entangled roles of perception and reasoning in mathematical learning is pivotal for constructivist design. Learning tools purporting to make manifest quantitative relations underlying mathematical concepts, I demonstrate, do so only if learners are supported in going beyond what meets the eye.

Rhetorical Case-Study Exposition: Epistemological Nuances of Perceived Equivalence



a. **b.** **c.**

Figure 1. Toward a critique of pictures that purport to ground equivalence: (a) equivalence by perceptual judgment; (b) unit-stretching complementing the phenomenology of geometrical similitude; and (c) grounding geometrical similitude with a fixed unit.

Figure 1a invites the learner to evaluate the geometrical similitude of two rectangles. An intuition of sameness results from the activation of automatic perceptual mechanisms for judging the identity of two percepts (for references, see Abrahamson, 2002). Namely, I contend, *the phenomenology of geometrical similitude is one of identity*—by virtue of judging for geometrical similitude one necessarily considers that the smaller and larger images are the very same object.

Whereas geometrical similitude is first apprehended perceptually, holistically, the multiplicativity of the proportional relations between these objects' corresponding dimensions needs to be learned—whether prescribed or discovered empirically, it is a rule initially dissociated from the phenomenology of perception. That is, the perceptuality per se of geometrical similitude is not multiplicative, additive, or even logarithmic, for that matter—it is a prereflective knowing embedded in our everyday sensory comportment, our being-in-the-world, and is not given to explicitization (Piaget & Inhelder, 1952).

In Figure 1a no tools are provided for a learner to analytically elaborate and verify the intuitive judgment of sameness. Figure 1b presents the larger of the two objects as a zoom-in of the smaller object, as though the two objects are in fact one and the same object as seen from different distances. Figure 1c provides tools for coordinating the perceptual apprehension of similitude with an empirical rule: The two dimensions of the smaller rectangle—height and width—are each multiplied by the same factor so the smaller rectangle fit or become the larger.

The initial apprehension of identity (in Figure 1a) is visually seductive—it subtly carries over as compelling positive affect, through a quantitative lens on geometrical similitude (stretched grids in Figure 1b), to the empirical rule of applying a constant scalar factor to both dimensions (uniform unit grids in Figure 1c), vesting a numerical proportional equivalence (4:3 = 8:6) with truth value. Yet, note that Figure 1c does not elaborate Figure 1b—whereas Figure 1b engages an opaque multiplicative transformation (Kaput & West, 1994), Figure 1c engages the accessible repeated-addition model, an analytic model that deviates from the identity synopsis yet provides semiotic equipment to ground multiplicative constructs as phenomenal–conceptual syntheses.

Examples From Empirical Studies of Design for Diverse Mathematical Content

1. Stretch/Shrink vs. Additive Construction of Proportional Progression

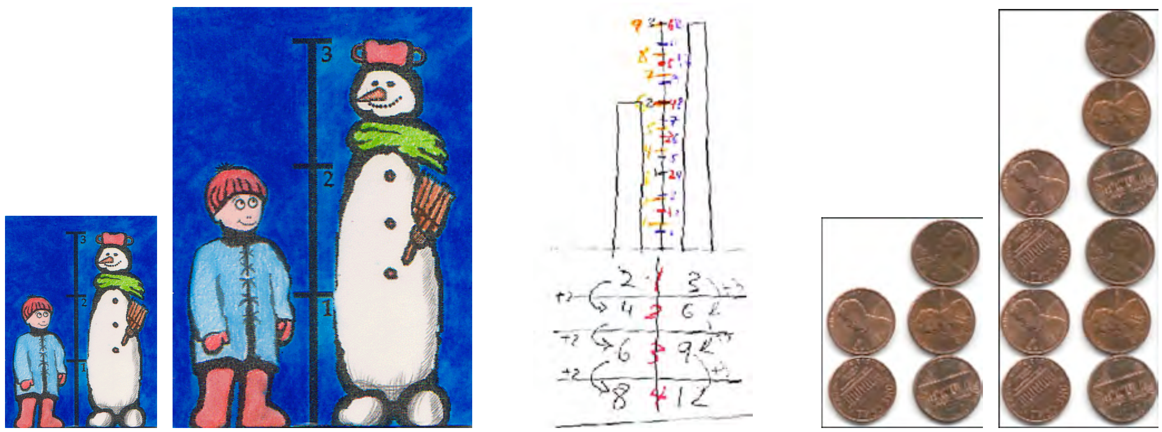


Figure 2. The 'eye-trick' design for proportion: Learning materials and student artifact

In the *eye-trick* design (Abrahamson, 2002), students work with proportionately equivalent card pictures, each displaying a pair of personas, such as Danny & Snowy (see Figure 2). Closing one eye and holding the smaller card nearer to their open eye, students experience an optical illusion as though the cards are identical, due to the similar retinal prints seen through monoscopic vision. Students use a ruler to measure the actual heights of the persons, 2'' & 3'' and 4'' & 6'', then mark these values in a table, under a pair of schematic rectangles (Figure 2). Additional cards, for 6 & 9 and 8 & 12, are judged as similar, measured, and tabulated.

The design's objective is for students to ground proportional equivalence in perceptual identity and thus interpret corresponding measurement pairs as equivalent. Two high-achieving Grade 3 participants determined the arithmetic sequence in each column and iteratively added the constant addend down each (see '+2' and '+3' arrows). Yet, the students did not construct the situation as multiplicative. In fact, they were surprised that differences between the values grew (see numerals 1, 2, 3, & 4 between the columns). Moreover, when subsequently building coin towers (Figure 2), where the iterative rule was "+2 coins here, +3 coins there," neither student perceived any relation to the earlier eye-trick activity, until they had tabulated the coin quantities and recognized the table as identical to the table they had previously built. I concluded that apprehension of induced identity followed by tabulated measurement could be a compelling extension activity yet is problematic as conceptual entry to proportional equivalence.

2. Global Color Density vs. Local Sampling in Statistical Investigation

S.A.M.P.L.E.R., Statistics as Multi-Participant Learning-Environment Resource, is a collaborative activity designed for networked middle-school classrooms (Abrahamson & Wilensky, 2002, 2007). The design introduces students to fundamental statistical constructs, such as sampling. In the first activity, students are asked to estimate the percentage of light-colored squares in an array of thousands of light and dark squares (Figure 3). Typically, students begin by eyeballing the entire array and offering estimates. Yet, to warrant their estimates, students spontaneously sample localized color densities, which they count (e.g., 4 light squares in a 3-by-3 sample of 9 squares), compare, and compile (Figure 3). Thus, the array affords either proportional reasoning or enumeration, and facilitated classroom discussion explores relations between these perspectives so as to support their coordination.

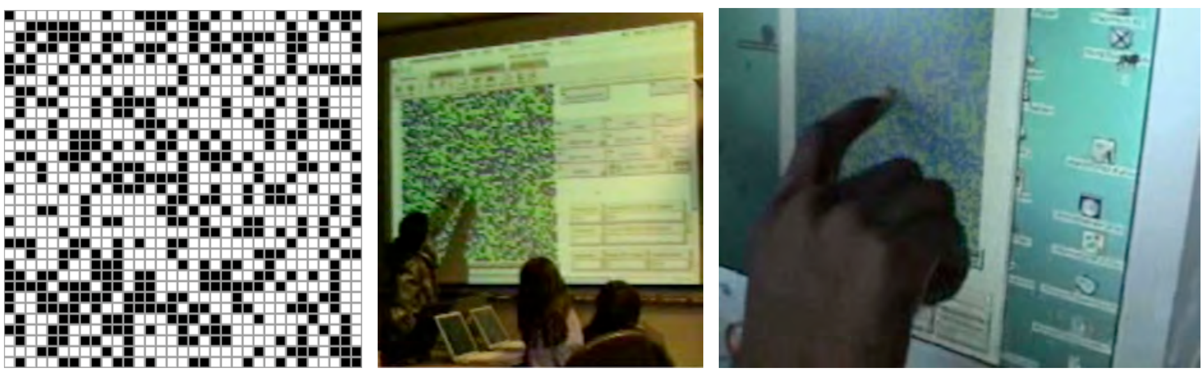


Figure 3. S.A.M.P.L.E.R. activity: coordinating proportional judgment with sampling.

3. Perceptual- vs. Analysis-Based Frequency Expectation

Twenty-eight Grade 4 – 6 students and 25 college students were given a box with equal numbers of mixed green and blue marbles—a few hundred in total—and a utensil for scooping out an ordered sample with exactly 4 marbles (see Figure 4). We asked them to predict what will happen when they scoop. By and large, all students expressed that whereas they cannot know what they will receive on particular trials, their long-run expectation for the greatest relative frequency is of a sample with two green and two blue marbles. When prompted to further warrant their claim, students typically said, “I don’t know the reasoning behind it, but it seems kind of obvious to me” or “I just saw it,” and many articulated that the mode sample should reflect the green-to-blue ratio in the box (Abrahamson, 2007; Abrahamson & Cendak, 2006; see Tversky & Kahneman, 1974, on the ‘representativeness heuristic’). Whereas such synoptic reasoning is accurate, it is delimited in its trajectories as grounds for methodically appropriating normative mathematical strategies, such as calculations pertaining to expected value, the binomial function, or combinatorial analysis. Moreover, such reasoning cannot readily accommodate situations in which the green–blue mix is unequal (such that p is not equal to .5).

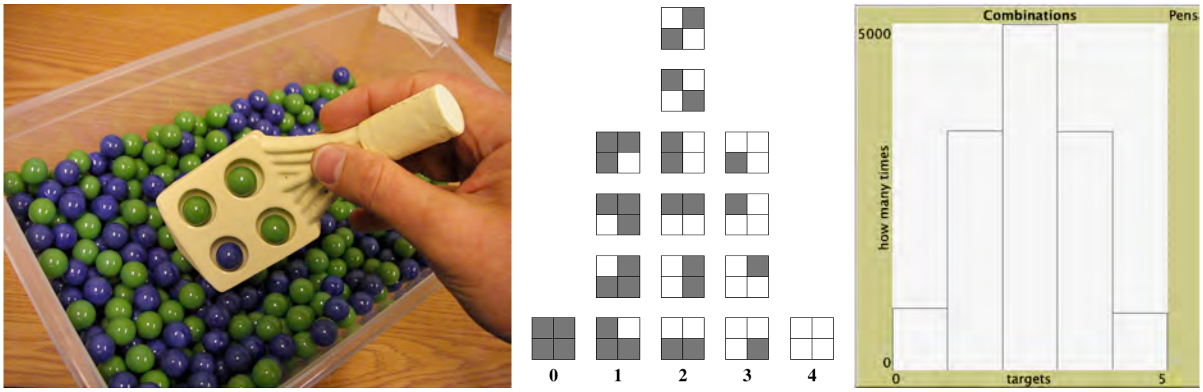


Figure 4. Selected materials used in studies of probabilistic cognition. From left: The *4-block marble scooper* samples from a green/blue mix of marbles; the *combinations tower*—the 4-block's distributed sample space; a computer-based empirical outcome distribution.

We guide students to build the 4-block's sample space and assemble it in accord with the statistic in question, the number of green marbles (Figure 4). Upon beholding this configured set, all the college students and all but one of the Grade 4 – 6 students immediately recognized an analytic warrant for their intuitive assertion that 2-green outcomes would be the most common. Thus, the participants coordinated one intuitive perceptual judgment, which they could not directly articulate, to another perceptual judgment—phenomenologically disparate yet mathematically commensurate—that is more conducive to appropriating normative mathematics.

4. Event- vs. Outcome-Based Expectation for Binomial Distribution

The combinations tower (Figure 4) is designed to facilitate comparison of the marble-scooping sample space (the 16 unique configurations) and distributions of empirical outcomes from numerous simulated trials with this device (Figure 4; note similar shapes). Whereas they appropriated this perceptual–conceptual relation, students' explanations revealed a lacuna in understanding the emergence of distribution as contingent on sample space, probability, and random selection. Specifically, students initially failed to realize that each of the 16 equiprobable outcomes was expected to occur an equal number of times in the experiment and that therefore the 1-4-6-4-1 groups would converge to 1:4:6:4:1 distribution.

Figure 5 shows new interactive features added to the computer-based simulation. Outcomes from experiments with the 4-block device accumulate in their respective columns. Thus, empirical distributions are constructed as "stochastic stretches" of the sample-space distribution and enhanced by re-ordering and color-coding. Middle-school and college students working with these tools were more likely to perceive the empirical outcome distribution as 16 more-or-less equal groups arranged in 5 columns in correspondence with

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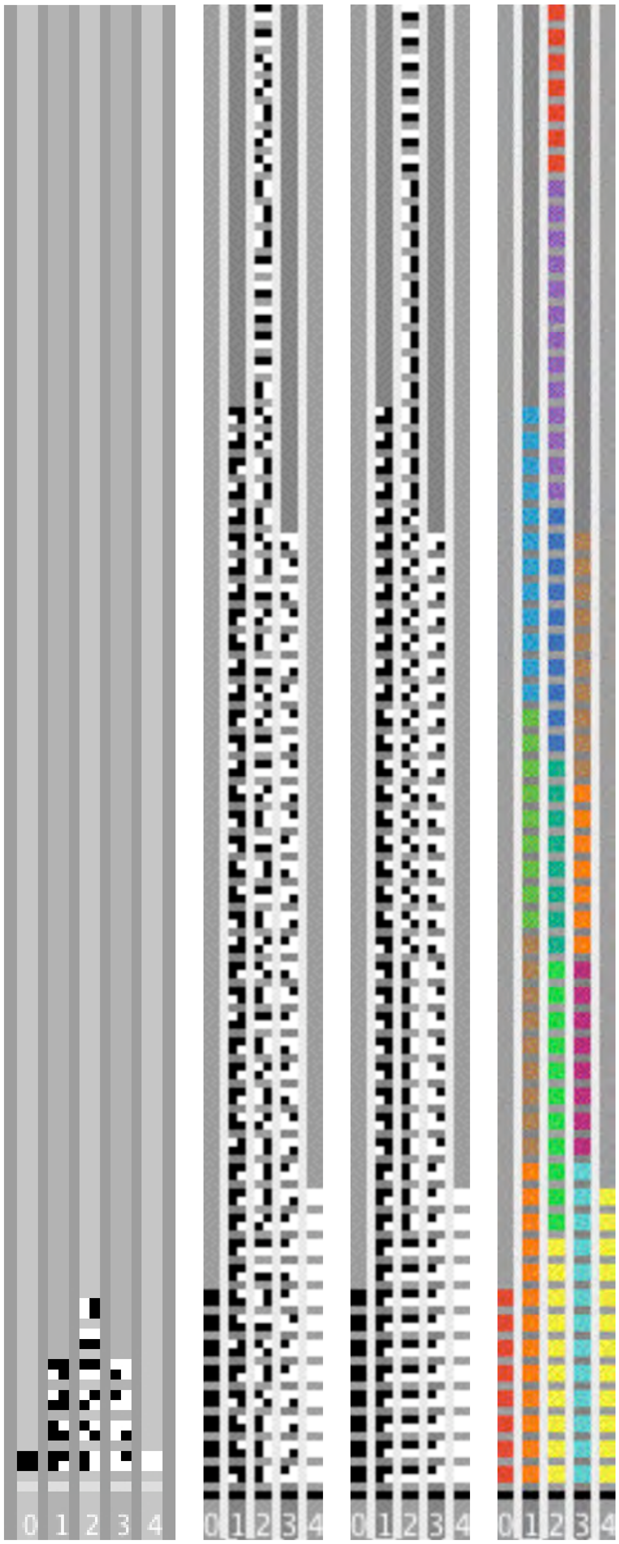
the arbitrary criterion that had designated 4-block events in the sample space by their number of light-colored squares. Such perceptions mark avenues for bridging classicist and frequentist approaches to stochasm.

Figure 5. ***4-Block Stalagmite*****, a computer-based simulation, supports a seeing of empirical distribution as emerging from the sample space. From left: 16 unique possible outcomes in a combinations tower; empirical results from a simulated experiment where random outcomes "drop" into respective columns; a re-ordering of these outcomes by type; and color-coding the 16 groups.**



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Summary and Conclusion

Mathematics, at least K – 12 constructivist curriculum, is an inherently empirical discipline: As in the discipline of physics, mathematical knowledge develops as the methodical articulation of perceptually apprehended regularities governing relations among quantities situated in phenomena under inquiry. As in physics, direct apprehension of phenomenal regularity is limited by available resources: perceptual and para-perceptual mechanisms, memory, computational algorithms, and representational forms. Some regularity is perceptually privileged, such that we apprehend it synoptically with little or no mathematical training. Such seeing lends an affect of knowing, yet the work then becomes to articulate quantitatively this empirical apprehension—to synthesize intuition and mathematics (Schön, 1981). This paper examined four case studies of constructivist design to investigate conditions supporting such tacit–explicit synthesizing. Learners' initial apprehension of the perceptual information was tacit proportional. Guided attention to discrete properties of the situated quantities enabled students to construct an enumerative–additive rule, such that tacit proportional judgment was reformulated as explicit multiplicative, with the additive procedure acting as a buffer from the tacit to the explicit..

Note that the explicit does not directly articulate or translate the tacit (Abrahamson & Cendak, 2007), because these faculties are epistemologically incompatible. Rather, the explicit voices, concretizes, and ultimately enhances the tacit, rendering the somatic semiotic, i.e., in the form of a socio–mathematical artifact bridging the personal prereflective into the disciplinary domain. Curiously, whereas the mathematical disciplinary continuity is from the addition operation to the multiplication operation, the phenomenological trajectory supported by constructivist discovery-based design may traverse from multiplicative intuitive apprehension to additive analytic procedures. Thus, counter to common premises of curricular design, some additive procedures may be grounded in multiplicative intuition.

Broadly, designed embodiment of mathematical concepts can play critical roles in facilitating learners' tacit–explicit negotiation: Learning tools—their inherent perceptual information and measurement tools—should alternately afford analytic handles on embedded quantitative dimensions or tacit faculties for evaluating the veracity of emergent explicit assertions. These designed embodiments act as 'bridging tools' (Abrahamson & Wilensky, 2007)—they mediate tacit–explicit reciprocal negotiation fostering epistemic synthesis. Specifically, learning activities that afford quantitative intuition grounded in proportional judgment should provide tools for articulating this intuition through pertinent additive processes. Otherwise, students' intuitions remain encapsulated, inarticulate, uncoordinated with robust solution procedures—the students sense they understand a concept, but such understanding is likely no more than tenuous.

In sum, students used additive models to ground explicit multiplicative reasoning in tacit proportional apprehension. I therefore implicate the proportional-vs.-enumerative perceptual tension as a contributing factor to pedagogical challenges of rendering design for multiplicative constructs effective. Moreover, it is perhaps the phenomenological incompatibility of holistic and analytic cognitive resources that raise concerns among some mathematics-education researchers of an overly additive articulation of multiplicative constructs (e.g., Confrey, 1995). Yet, I submit, it is precisely in painstaking synthesis of the embodied and calculated, the synoptic and sequential, the privileged and earned, that core conceptual learning transpires. Thus, designers and teachers should create opportunities for students to coordinate holistic multiplicative action models for situated mathematical problems with distributed-addition solution procedures (Fuson, Kalchman, Abrahamson, & Izsák, 2002); to fit all-at-once synoptic sense with step-by-step sensibility; to express how much as synthesis of rhyme and reason.

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CONVERSATIONS ABOUT CONNECTIONS: A SECONDARY MATHEMATICS TEACHER CONSIDERS QUADRATIC FUNCTIONS AND EQUATIONS

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This paper reports the case of Robert, a knowledgeable and experienced teacher, from a larger study in which secondary mathematics teachers were interviewed about their understanding of mathematical connections in a series of progressively more structured individual interviews. A model for categorizing mathematical connections is presented and used as an interpretive tool. Robert values making connections, groups related ideas confidently, but finds articulating specific mathematical connections difficult.

The National Council of Mathematics Teachers' (NCTM) document, *Principles and Standards for School Mathematics* (2000) identifies "mathematical connections" as one of the curriculum standards for all grades K to 12. In this framework, "… mathematics is not a set of isolated topics but rather a web of closely connected ideas" (NCTM, 2000, p. 200). Making the connections is taken to promote students' understanding of new mathematical ideas. While students might make some useful connections spontaneously, the accepted position in the pedagogical literature is that teachers' interventions are necessary if students are to deal with mathematical connections in a systematic and meaningful way (Weinberg, 2001, Thomas & Santiago, 2002).

Theoretical Framework

In the mathematics education literature, a "mathematical connection" is conceptualized in a variety of ways ranging from mappings between equivalent representations (Hines, 2003) to links between mathematical concepts (Zazkis, 2000), to unifying themes that cut across several domains (Coxford, 1995). Just what a mathematical connection **is** often remains implicit both in the research literature and in teachers' professional dialogue.

In this study, I take a mathematical connection to be a true relationship between two mathematical ideas. I draw on Skemp's notion of a person's mathematical knowledge as a set of hierarchical schemata (Skemp, 1987), composed of mathematical concepts and the connections among them. These connections may be "vertical", in which a concept can be thought of as a composite of simpler concepts, or "horizontal", in which a concept can be thought of as a transformation of another. I propose the following as a starting list of specific types of mathematical connections where A and B represent two mathematical ideas:

1. **A is an alternative representation of B**. The alternative is a different category of representation, for example, symbolic/algebraic, graphic/ geometric, pictoral, physical, oral description, written description.
2. **A is equivalent to B**. An equivalent is a different expression within the same form of representation. For example, 3 + 2 is equivalent to 5 (all expressed in symbolic form).
3. **A is similar to B (A intersects B)**. A and B share some features in common.
4. **A is included in (is a component of) B; B includes (contains) A**.
5. **A is a generalization of B; B is a specific instance (example) of A**.
6. **A implies B (and other logical relationships)**.

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7. **A > B; B < A (and other order relationships)**.
8. **A is a procedure (or algorithm) applied to B.**

Methods

This study extends earlier work about teachers' views of mathematical connections (Businskas, 2005) by probing more deeply into teachers' understanding of specific topics in mathematics, to try to make their notions of mathematical connections explicit. The study asks: in considering their knowledge of a mathematics topic, what kinds of mathematical connections can teachers explicitly identify and describe?

Research Setting

Participants in the study from which this case is derived were ten secondary mathematics teachers who volunteered for the study. Teachers were interviewed individually three times over a three-month period in their schools.

Interviews

The first interview was an acclimatizing interview and drew out information about teachers' background and general views about teaching mathematics and the role of connections.

The second interview was a semi-structured interview about a mathematical topic chosen by the teacher. Prepared questions included:

- Please tell me about your own understanding of this topic… What are the important concepts/ideas and procedures that make up your understanding of this topic?
- Please tell me how the ideas and procedures that you've identified are related to each other or related to other topics in mathematics.
- From your point of view as a teacher, what are the most important concepts and procedures that you want your students to learn?

Teachers' responses that included some reference to connections were followed up with questions asking them to elaborate. If teachers did not spontaneously make any references to connections, I asked further probing questions, and sometimes even leading questions in an attempt to get them to voice an opinion. After the interview, I asked teachers to show me their lesson plans/planning notes for their topic, which I examined for any references to connections.

The third interview was a task-based interview in which all teachers dealt with the same topic – quadratic functions and equations. Teachers were given a set of 82 cards (see list below) containing mathematical terms, formulae and graphs related to this topic, gleaned from a selection of high school mathematics textbooks. They were asked to organize the cards in some way that showed the relationships among them. They were instructed "Please group them in a way that will show how they are connected". After completing the task, they were asked to explain their organization, and were constantly pushed to elaborate their statements about connections.

Terms

algebra	associative property	coefficient	commutative property
complete the square	compression	cone	co-ordinates
curve	derive	directrix	discriminant

distributive property	domain	equation	expansion
exponent	expression	factor	factor theorem
focus	FOIL	formula	function
geometry	graph	guess and check	inequality
intercept	inverse	isolate the variable	maximum
methods of solution	minimum	operations	parabola
perfect square	point	properties of roots	quadratic
quadratic formula	radical	range	relation
remainder theorem	root	simplify	solve
square	square root	substitute	symmetry
table of values	translation	trinomial	value

SymbolsFormulae

a, b, c	(p, q)	$b^2 - 4ac$
$p \pm \sqrt{(-q/a)}$	$(-b/2a, (c - b^2/4a))$	$x = p$
$x = -b/2a$	$x= (-b \pm \sqrt{(b^2 - 4ac)})/2a$	$x= 2c/(-b \pm \sqrt{(b^2 - 4ac)}$
$ax^2 + bx + c = 0$	$x^2 - (r_1 + r_2)x + r_1 r_2 = 0$	$a(x-p)^2 + q = o$
$y = ax^2 + bx + c$	$y = a(x-p)^2 + q$	

Coincidentally, one of the teachers, Robert (a pseudonym), chose "quadratic functions" as his topic, thus providing me with three sources of data (2 interviews and his planning notes) about his thinking about this topic. This paper considers his views.

Analysis

All interviews were transcribed.

For the "chosen topic" interview, I read the text to extract the particular statements made by Robert, that dealt directly with the topic of quadratic functions and equations and with connections, or the teaching of these topics. From the extracted statements, I compiled a summary of Robert's description of his understanding of the topic, and a list of specific references to connections. In these, I looked for ideas, that because of their repetition or his emphasis, might be indicators of consistent themes in Robert's thinking.

For the task-based interview, there is also a visual record of the groupings (see photo below). From these data, I compiled another list of explicit connections that he made.

Results and Discussion

Robert is in his ninth year of teaching. Teaching is his first career and he is at his second school. Robert has a Bachelor's degree with a major in Mathematics. He is confident in his knowledge of the provincial curriculum, and is familiar with the NCTM Standards. He also has a Master's degree in education.

In his teaching, Robert tries to promote his students' understanding of the topic but worries that "not all the kids are going to pick up on all the conceptual knowledge, a lot of kids just learn procedural skills".

He chose the topic of quadratic functions and equations because "I think it has a lot of interesting connections, like visual and algebraic connections and lots of connections to real physical problems". Right from the beginning Robert did not separate his own personal understanding of the topic from his understanding of it for teaching. His own content knowledge and his pedagogical content knowledge seem conflated (Shulman, 1986). In fact, he sees himself as teaching his students everything that he knows about the topic: "I think I try to give them everything I know... I try to tell them everything I know about it".

Here's what Robert had to say about quadratic functions and equations in each of the tasks. His comments are summarized and paraphrased, but all mathematical terminology used is his. Even though Robert was asked to talk about his own understanding of the topic, he switched almost immediately to talking about what his students know (or should know).

"Chosen Topic" Interview

Content Summary

Robert's description of the topic was an almost exact match to concepts and procedures summarized in the curriculum guide.

Explicit Connections

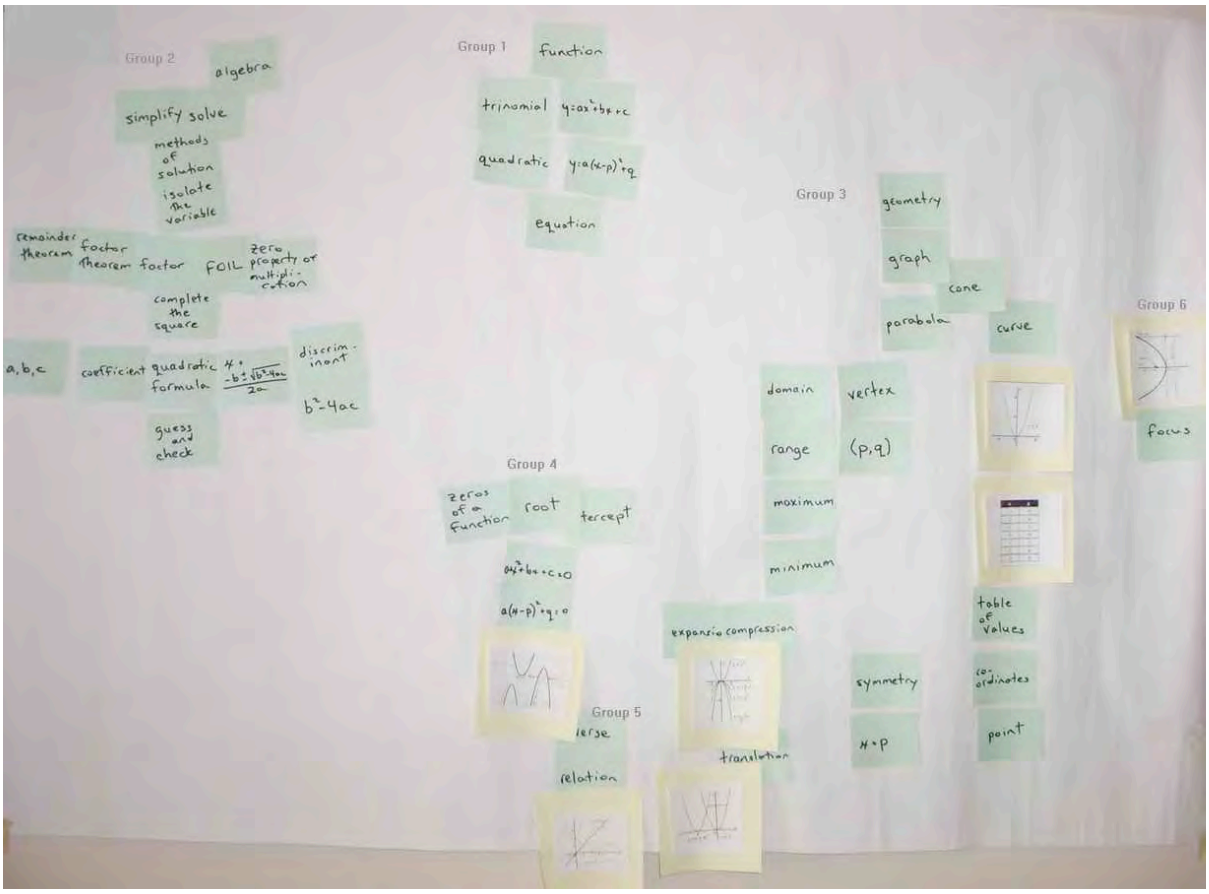
Robert's statements that explicitly referred to mathematical connections are listed below (chronologically) in his own words, and categorized according to the system outlined earlier.

Statement	Type of Connection
If you throw a baseball, it's a quadratic function, a parabola	*Real world [not in the model]* Alternate representation
The connection between obviously an equation and a graph	Alternate representation
The exponent is not going to make it linear	A implies B
Max/min which is the vertex	Equivalent representation
the important relationship is to understand that a graph, the picture view of the equation or the formula, is the same as the algebraic	Alternate representation
I draw the picture, I do the algebra	Alternate representation
the graph just shows them all the different solutions to the equation, all the different pairs of x and y that work in this equation	Alternate representation
these two things are the same things, one's expanded, one's factored [referring to equations]	Equivalent
those with a value up front; typically they have to find zeros that are not going to be whole numbers [referring to coefficient "a" in the equation $ax^2 + bx + c = 0$]	A implies B [non-zero "a" implies that zeros will not be whole numbers]
it's a quadratic function, I'm looking for the maximum point, so I need to find a vertex	A contains B (quad function has max) Equivalent representation
need to understand how to solve a quadratic in order to solve those other ones [i.e. cubic and quartic]	Procedure

Robert repeatedly emphasized that quadratic functions and equations can be alternately represented algebraically and graphically but did not make specific mappings between aspects of the equation and the graph. In this interview, we talked only about mathematical ideas that Robert introduced; the explicit connections that he made seem rather sparse.

Task-Based Interview

After quickly examining all the cards, Robert proceeded to lay them out in groups with little hesitation, completing the task in 15 minutes. He used 57 of the 82 cards. He made six groups (see photo below). The explicit connections that Robert made involving terms in each group are listed in his own words, sometimes slightly paraphrased. Connections among groups are described later.



Group	Explicit connections	Type of connection
1. function…	two different forms of the equation	Equivalent representation
	There's the algebra side and the geometry side	Alternate representation
2. algebra…	factor, complete the square, quadratic formula, guess and check – represents the algebraic skills they need	Procedure
	under factor, you've got the remainder theorem, factor theorem, zero property of multiplication, these are all ways to solve	Procedure
	complete the square is… an algebraic method, but it helps you find the vertex, which makes it easy to graph	Procedure
	the coefficients are a, b, c… you can find the zeros by plugging into the quadratic formula	A is contained in B
3.	a parabola is part of a conic section and it's a	A is contained in B

geometry…	curve	
	vertex with (p,q)	Alternate representation
	table of values give you co-ordinates that are points on the curve	Equivalent representation
	maximum and minimum… they help me to find the range	Alternate representation
	I just had some of the transformations grouped together, expansion, compression… translation	A is an example of B
4. [zeros of a function, root, intercept]	zeros and intercept I put in between the major headings [algebra and geometry] because algebraically that's what you're trying to find, and graphically that's also very easy to see	Alternate representation
5. inverse	inverse which is a relation and an example of it	A is a specific instance of B
6. focus	another example of a conic section	A is an example of B

The connections listed above are within-group connections. Of the explicit connections that Robert offered, only one was offered spontaneously, namely "there's the algebra side and the geometry side". The others were the result of probing, and sometimes repeated probing. Robert was quick and confident in identifying ideas as related but had difficulty articulating what the relationships were.

I also asked Robert "… can you see some extensions where some of the groups that you've identified here might be related to other topics in math?" Again, I report the explicit connections that Robert made in his own words or close paraphrases.

Explicit Connection
Function is not just for parabolas, but for any type of function like linear or cubic, doesn't even have to be a polynomial function.
Remainder theorem, factor theorem… you could use it for solving quadratics but it's mainly for solving polynomials that are like cubic or higher.
Translations, or the transformations are not just moving parabolas… you could move any type of graph.
Domain and range, max and min, are important concepts in anything… calculus and beyond.
Conic sections, there are lots more conic sections we can look at, not just the parabola
Table of values used in any type of graphing.
Symmetry and transformations… not just moving graphs around, but moving objects, symmetrical objects and natural objects

Robert offered all the statements above in response to a single question, without any additional probing. Structurally, all the connections are similar in that they are generalizations. Robert identifies a concept or procedure as one that has a broader scope than the topic of quadratic functions and equations, and then offers examples of other math topics to which the general idea applies. This type of connection is similar to Coxford's themes that run across mathematical topics (Coxford, 1995).

Cards Left Out

When I asked Robert about the cards that he left out, he said "I didn't think that they added anything new to what I had". When asked further to consider them one by one, Robert's reasons fell into two categories:

1. He did not recognize them. Robert left out a third of the cards that showed symbolic expressions or equations. While he repeatedly emphasized the importance of the connections between algebraic and geometric forms of quadratic functions and equations,

he had difficulty making the connection when the algebraic form was the starting point - for example, ($-b/2a$, ($c - b^2/4a$)) are the co-ordinates of the vertex, $x = -b/2a$ is an equation for the axis of symmetry.

2. He didn't think they were important. When pushed further, he recognized some connections but saw them both as far removed and obvious.

Planning

Robert's 15 hand-written pages of planning notes contained definitions, examples, homework assignments, and mostly model solutions for exercises in the text, but only one reference to making a connection - "review completing the square – need it in 2.4 [section 2.4]". He acknowledged later that he does not attend to connections when planning.

Summary

Robert speaks positively about connections in general. His ability to explicitly describe mathematical connections seems related to the "grain size" of the concepts being considered. For example, he was able to list a variety of ideas that are common to quadratic functions and equations and to other topics in mathematics in general. At a finer grain size, dealing with simpler concepts, he found it easy to group them, but quite difficult to describe the relationships among them. At the risk of inferring too much, it seemed that he was blocked by the belief that the specific connections were so obvious, they didn't need to be stated. Explicit connections that he did make were mostly alternate representations, in particular, algebraic and geometrical.

The proposed interpretive model worked well to categorize Robert's responses.

My next step is to analyze the data for the other nine teachers with two ends in view – first, to further refine the interpretive model by using it with a larger data set, and second, to find common themes, which might lay the groundwork for further studies. Robert's case raises two important factors to consider in these further analyses – the grain size of the related ideas and the consistency (or lack of it) between a teacher's general statements about connections and those views in application to specific mathematical topics.

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IMPROVING DIALOGICAL INTERACTION IN CLASSROOMS THROUGH IMPROVING TEACHER PEDAGOGICAL PRACTICES: IMPACT ON STUDENTS' SCORES

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Promoting dialogical interaction is crucial to students' learning mathematics. This study showed that implementing approaches that embed writing and talk into classrooms helps students develop deeper understanding of mathematics concepts. Given that we found statistical difference between control and treatment students' scores, we explored how the teachers change their practices from "traditional" teaching to student-centered practices. Teachers' pedagogical practices in both control and treatment classes were measured using the Reformed Teaching Observation Protocol (RTOP) instrument. The level of teaching in control classes remained the same; whereas, there was a significant improvement in the level of teaching in the treatment (MRA) classes. When students' standardized test scores were compared, the MRA students' scores were significantly higher than that of the control group students.

The central strand of the nature of knowledge is that knowledge is socially constructed within a community through negotiated meaning of experiences, not static or stable, but inconstant (Connolly, 1989; Ernest, 1998). Cobb, Yackel, and Wood (1992) and Ernest (1998) have pointed out that children partly construct their knowledge as a form of collaborative meaning making based on their interaction with others. Seeing mathematics learning as an active problem-solving process, Cobb et al. (1993) stated that social interactions in the classroom can create contradictions and conflicts in children, and in the process of resolving these conflicts, students in turn reorganize their mathematical experiences and mathematical ways of knowing. Similarly, von Glaserfeld (1993) showed at individual level, through dialogical interaction students find inconsistencies in their thoughts and reorganize their conceptual relationships.

Analyzing the role of the teacher in collective argumentation, Yackel (2002) argued that argumentation is crucial to students' learning of mathematical concepts both as a collective and an individual act. Teachers are essential to initiating such an argument, supporting students as they interact, and supplying supports (data, warrant, and backing) that are omitted or left implicit in arguments (Yackel, 2002). That said, it is often hard for teachers to give up old habits in favor of new, student-centered techniques; thus, they need support and guidance (Borko, Davinroy, Bliem, & Cumbo, 2000).

In order to guide teachers, we have developed the mathematics reasoning approach (MRA) (Akkus & Hand, 2005). The intent of the MRA is to provide a framework for teachers to combine different aspects of mathematics teaching and learning such as students' knowledge of mathematics, teachers' knowledge of mathematics (Simon, 1995), negotiation of problem solving methods, and embedding writing into mathematics instruction (Morgan, 1998). Another

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intent of the MRA is to guide students' problem-solving behaviors and enhance students' problem-solving skills via the use of writing (Kenyon, 1989; Morgan, 1998). The MRA is designed to support students' ability to develop a dispositional dialectic for writing in the context of mathematical problem solving that will enhance their mathematical understanding.

There are two templates created for teachers and students (see Figure 1). Using the mathematics reasoning approach requires teachers to engage in a planning phase that requires them to determine the major concepts that they want their students to learn, identify what they (teachers) know and do not know about the topic, and, importantly, align their teaching according to the concept of how students learn. Teachers must create opportunities for students to share their methods, hear alternative methods, and then compare advantages and disadvantages of the methods (Hiebert & Wearne, 2003).

The second component of the mathematics reasoning approach, the student template (Figure 1), consists of a series of questions for students to consider when they are engaged in the problem solving process. The student template resembles Polya's (1945) four-stage problem-solving heuristic (understanding, planning, carrying out the plan, and looking back) or Schoenfeld's (1985) phases of problem solving (analysis, design, exploration, and implementation).The distinct feature of the template from other problem-solving frameworks is it specifically asks students to *compare* their solutions with their peers and to *reflect on* their problem solution after a classroom discussion.

Teacher Template	Student Template
Preparation: - Identify the big ideas of the unit. - Make a concept map that relates sub-concepts to the big ideas. - Consider students' prior knowledge - Consider students' alternative conceptions during the lesson as they connect the prior knowledge to the big ideas **During the unit:** • **Students' knowledge of mathematics** - Give students opportunity to discuss their ideas. - Have students put their ideas on the board for exploration. • **Teacher's knowledge of mathematics** - Use your knowledge to identify students' alternative conceptions. - Guide students to the big ideas identified earlier during the preparation.	• **What is my question (problem)?** - Specify what you are asked (What is (are) the question(s) being asked?). - Outline the information/data given (What information is/are given?). • **What can I claim about the solution?** - Use complete sentences to explain how you will solve the problem. - Tell what procedures you can follow. • **What did I do?** - What steps did I take to solve the problem? - Does my method (procedure) make sense? Why? • **What are my reasons?** - Why did I choose the way I did? - How can I connect my findings to the information given in the problem? - How do I know that my method works? • **What do others say?**

• **Negotiation of ideas** - Create small-group and whole-class discussion. - Encourage students to reflect on each other's ideas. • **Writing** - Have students write about what they have learned in the unit to real audiences (teacher, parents, classmates, lower grades, etc.).	- How do my ideas/solutions compared with others? a. My classmates b. Textbooks/Mathematicians • **Reflection** - How have my ideas changed? - Am I convinced with my solution? Why?

Figure 1. The mathematics reasoning approach teacher and student templates.

Dialogical interaction and writing in the context of problem solving are the two core elements of the MRA. Therefore, this study was guided by the following two research questions:

1. How do the teachers change their pedagogical practices through the use of the mathematics reasoning approach?
2. Is there a difference in students' mathematical performance on a standardized test between the students in the control classes and the students in the treatment classes?

Methods

This study was conducted with three high school algebra teachers, two males (Mike and John) and one female (Amy), who had a total of ten class sections for an Algebra I course, with four sections chosen as control classes and six as treatment classes. The number of students across teachers and group is given in Table 1 below. The teachers participated in the study during the first semester of the school year and were supported during the implementation of the MRA.

Table 1. Distribution of students according to teacher and group.

Teacher	Group	Number of students	Male	Female	High Achv**	Med Achv	Low Achv
Mike	Control	21	11	10	1***	12	3
	Treatment (MRA)	44*	19	25	3	25	7
John	Control	43*	26	17	5	26	8
	Treatment (MRA)	45*	35	10	6	29	6
Amy	Control	24	13	11	1	12	5
	Treatment (MRA)	25*	8	17	2	20	2

* Teachers had two classes in this group.
** Achv (achievement) refers to proficiency level provided by the testing center using students' ITBS scores, relative to each other.
*** The number of students for proficiency level was obtained according to the data reported.

A mixed-research method was used with both quantitative and qualitative data sources. The quantitative data included the Iowa Test of Basic Skills (ITBS) and Iowa Test of Educational Development (ITED) scores that were used to examine performance differences between the control and treatment classes through estimating analysis of covariance (ANCOVA). The qualitative data included observations through both on-site observations and videotape recordings in both MRA and control classrooms and interviews of the teachers that shed light on the level of teacher implementation. This qualitative data enabled the researchers to determine the teachers' level of teaching in both MRA and control classrooms. To document the level of teaching, we adapted an observation protocol called the Reformed Teaching Observation Protocol (RTOP) used to measure "reformed" teaching in mathematics and science classrooms (Sawada, Piburn, Falconer, Turley, Benford, & Bloom, 2000). The instrument consists of 25 items, each rated from 0 (not observed) to 4 (very descriptive). However, we chose 14 items (Chronbach's Alpha was .976) and then categorized them according to the relevancy to each other and reorganized, and four sub-categories were constructed: *Student Voice, Teacher Role, Problem Solving and Reasoning, and Questioning*. The values of Cronbach's alpha for the first three sub-categories were .908, .882, and .952, respectively. There was only one item related to questioning so Cronbach's alpha was not reported. Modified RTOP instrument is given in the appendix.

Analyses

The analyses consisted of a series of passes through various aspects of the data. Regarding qualitative data, the classroom and videotape observations and the field notes were analyzed in terms of the level of teaching (i.e., level of dialogic teaching) in both control and MRA classrooms. The sub-categories of the RTOP were thoroughly analyzed for the teachers' MRA practices. The findings from both types of observations were triangulated with the interviews, email correspondence, and field notes, which provided a general trend of teachers' pedagogical implementation that confirmed our classification of the level of their teaching.

Table 2. Number of observations and RTOP scores provided for teachers.

		Total Number of Observations	Number of On-site RTOP Scores	Number of Videotape RTOP Scores	Number of Final RTOP Scores
Mike	Control	5	5	5	5 (0*)
	MRA	20	11	9	11 (2)
John	Control	5	5	5	5 (0)
	MRA	18	11	8	11 (3)
Amy	Control	5	5	5	5 (0)
	MRA	18	11	7	11 (4)

* Represents the number of final RTOP scores based on the on-site observation.

The percentage of agreement (or the inter-rater reliability) between any pairs of observers for teachers' level of teaching ranged from 90% to 95%. All disparities were then reconciled through discussions resulting in 100% agreement. The final score of the observers was also compared to the lead researcher's score for that particular lesson based on his on-site observations. There was a 90% match between the group and on-site observation scores.

In terms of quantitative data, an analysis of covariance (ANCOVA) was estimated to control for other variables that might impact students' mathematics achievement (Agresti & Finlay, 1997). Students' ITED mathematics scores of the year of the implementation were used as the dependent variable, with teacher and group as the independent variables, and students' ITBS scores from the previous year as the covariate.

Results

Teachers' Pedagogical Practices Improved Across Time

The teachers' levels of implementation were identified using the RTOP instrument. Each lesson observed and/or videotaped was attributed an RTOP score ranging from "0" to "4," with 4 being the highest level of teaching. The mean scores of the observations were assigned to teachers as *final* RTOP scores. Amy and Mike's control teachings were rated .26 and .22, respectively. Their MRA teaching levels were .88 and 1.60, respectively. John's control and MRA teaching levels were 1.40 and 2.10, respectively. Overall, while the teachers' MRA teaching improved, their control teaching level tended to score the same throughout the study. John's routine control teaching level was as high as Mike's MRA teaching and higher than Amy's MRA teaching. Moreover, Mike and Amy occasionally regressed in their implementation of the MRA approach, whereas John continued to improve after the 5th observation. Furthermore, the detailed analysis of the teachers' MRA teachings provides evidence that there was an improvement in the teachers' practices, with "questioning" being the critical component (the right-hand side of Figure 3).

Patterns in the Teachers' Pedagogical Shift to Supporting Dialogic Interaction

We first claim from the complementary qualitative analysis that when these teachers used the MRA, their ability to support dialogic interaction for mathematical problem solving improved across time. Second, we claim that during the process of learning to support dialogic interaction, questioning skill was a forerunner to improving other pedagogical skills important for promoting dialogic interaction, such as giving students voice, promoting student problem solving, and listening attentively to student thinking.

From Figures 3b, 3d, and 3f, it can be seen that all the teachers, at some point, had a high level of questioning yet were unable to incorporate that with problem solving and dialogical interaction. For example, the major area that John needed to work on was his domination of classroom interaction; that is, even though he asked thought-provoking questions, the interaction was only with a particular student. This notion of interaction meant that student voice was lost in his class. Throughout the debriefing sessions, he realized that *"So, you just want me to shut up* *and not tell the answer?"* (Fieldnotes, John, 11/13/2006). John's level of questioning also helped his role during problem solving improve by giving students more opportunities to discover their own methods. He indicated that

> *In math there is always a right answer, and I tell the answer. But now with this project, [I am] trying to step back. …The first time we did, it wasn't as good as it should be. Because they'd never done it before. But they get better at listening. I discovered ways of looking at stories I've never thought of. One time, about a problem, one girl came up an idea, which I never thought of, because I've never given them think time to tell me how they do it.* (Interview, John, 01/11/2006)

Similarly, Mike's classroom observations showed that he often lost track of students' ideas since he focused on his questioning and had a teacher-only-one-student interaction pattern. During the debriefing sessions the lead researcher focused on questioning and the effects of higher-order questioning in other pedagogical areas. Mike worked on his questioning and on not telling students the answer if the correct one did not emerge. To this end, he started to call on different students, which required other students to be engaged in problem solving and listening to their peers, and he often rephrased his questions to make them explicit to the students (e.g., *Well, what I'm trying to ask is …*). Thus, he managed to incorporate his questioning with problem solving later in his implementation. For example, he spent more than half of a lesson trying to come up with different ways of solving an equation. The students were fully engaged, working together to come up with, as they said at the end of the lesson, *"different routes to get to the town if the high-way is closed"* (Videotape, Mike, MRA-No. 10). On the other hand, Mike solved four different problems in 15 minutes in his control class, without any discussion or getting students' ideas. This type of teaching was his regular control teaching throughout the project. This view also came out during the interview, as he defined learning: *"It [learning] goes like hand in hand. I have x amount of knowledge and skills that I can pass on to the students"* (Interview, Mike, 01/17/2005). Even though he acknowledged students' own knowledge and ways of knowing, this idea was consistent neither with his pedagogical practices nor with his definition of the teacher's role in the classroom, as he indicated during the interview: *"I think it [the role of the teacher] depends on what type of learning atmosphere we want for the day. Sometimes, I can be more of a facilitator, and other times, if it is information type, I need to lecture and give it to them directly"* (Interview, Mike, 01/17/2005).

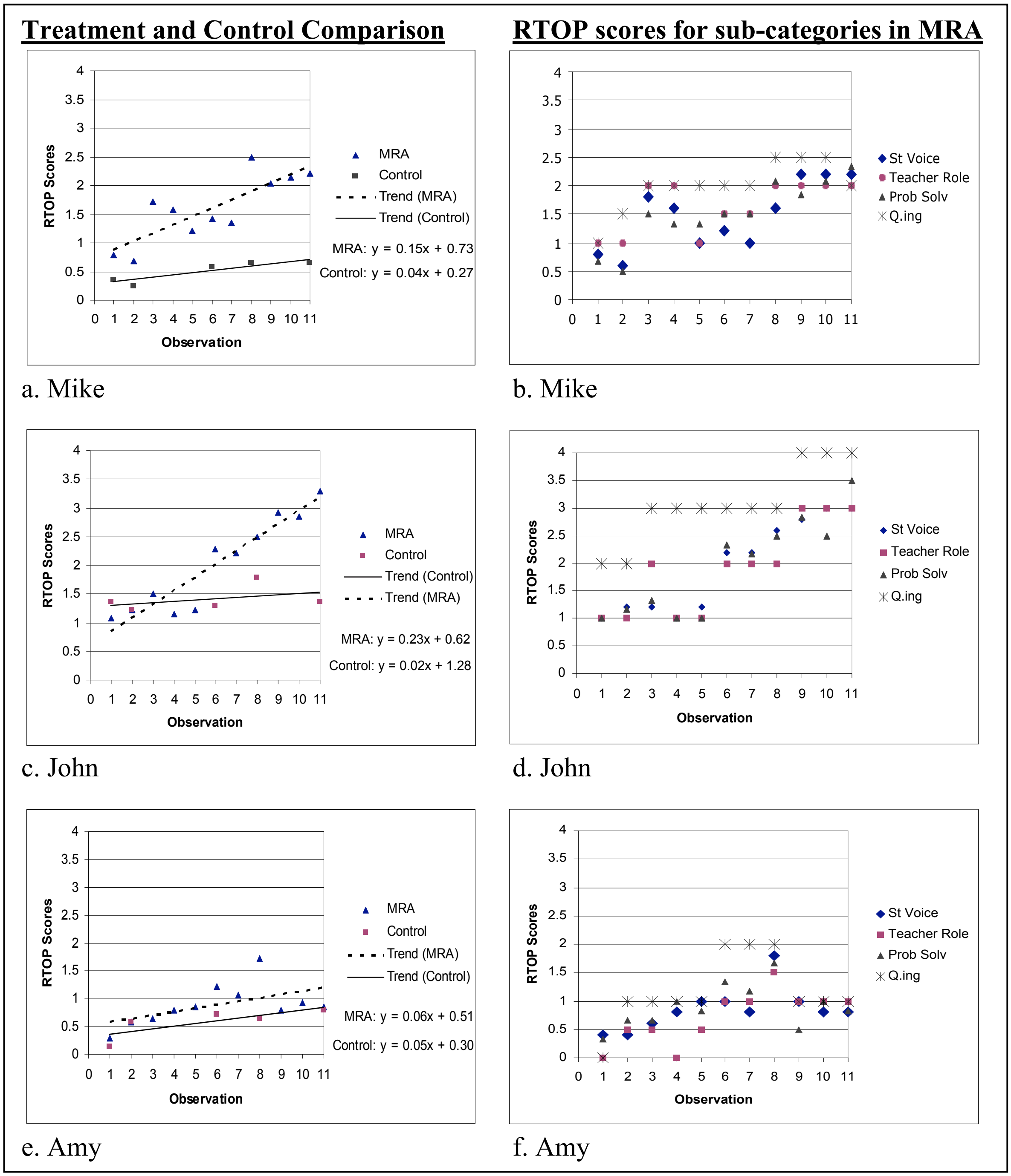

Figure 2. Teachers' levels of teaching and RTOP scores for sub-categories in MRA.

The same perspective on teaching was also held by Amy. Her implementation of the MRA was not as high as Mike or John's. Yet, there were instances where she tried harder to get students involved in discussion, as can be seen in her eighth observation (Figure 3f). From the observation, it appeared as though Amy was struggling with managing classroom discipline. As such she tended to go back to her traditional teaching, where she lectured most of the time. She often indicated during the debriefing session and the interview that *"these students are not used to this kind of learning, so lecturing works better for them"* (Debrief, Amy,2/05/2005).=

Treatment Improved ITED Scores Significantly over Control

Before estimating an ANCOVA model, possible violations of key assumptions were investigated. The overall ANCOVA model was estimated using the ITED mathematics test of the same year as the dependent variable, with the teacher and group as the independent variables, and the ITBS Mathematics Problem Solving and Data Interpretation test of the previous year as the covariate. The model yielded a significant main effect of group in favor of the MRA (F (1, 163) = 5.381, p = .022, η^2 = .032). Moreover, even though there was no significant interaction effect (F (2, 163) = .350, p = .706), the analyses of pairwise comparisons indicated that John's MRA classes (M = 284.554, SD = 18.721) significantly outperformed Mike's control class (M = 273.645, SD = 14.700), t (52) = 2.288, p< .05, Amy's control class (M = 268.555, SD = 14.361), t (54) = 3.518, p< .05, and John's own control classes (M = 276.154, SD = 18.585), t (75) = 1.976, p< .05. Corresponding Cohen's d effect sizes were, respectively, .618 SD, .916 SD, and .450 SD (see Table 5). Even though the difference was not significant, Cohen's d effect size difference between John and Amy's control classes was .437 SD. Moreover, Mike and Amy's MRA classes had higher mean scores than their control classes (significantly higher in Amy's case). Mean Square Error was 345.281, and adjusted R^2 was .316 for this model. Finally, Levene's test of equality of error variance showed non-significant results (F (5, 164) = 1.154, p = .334), which confirms that the error variance of the dependent variable is equal across groups.

Table3. Pairwise comparisons in Cohen's d effect size.

		t (d.f.)	Cohen's d	Magnitude
John (MRA)	Mike (Control)	2.288^* (52)	.618	Medium
	Amy (Control)	3.518^* (54)	.916	Large
	John (Control)	1.976^* (75)	.450	Small
	Mike (MRA)	1.165 (71)	.273	Small
	Amy (MRA)	1.119 (60)	.291	Small
John (Control)	Mike (Control)	.531 (60)	.143	Small
	Amy (Control)	1.686 (55)	.437	Small
Mike (MRA)	Mike (Control)	1.203 (49)	.332	Small
Amy (MRA)	Amy (Control)	2.075^* (40)	.623	Medium

* p < .05

Discussion and Conclusion

This study focuses on examining the changes in *pedagogical* practices when three teachers shift from their traditional teaching to more student-centered practices. The study also looked at the *performance* differences on the Iowa Test of Educational Development (ITED) between the students in the control classes and the students in the treatment classes. The study particularly focused on changes in the teachers' pedagogical practices when using the MRA. The major findings of this study are that implementing a student-oriented approach such as the MRA which includes embedded writing-to-learn strategies does have an impact on student performance and that teachers' adoption of the required pedagogical practices varied as they attempted to move away from their traditional practices.

The complementary qualitative analysis unpacks the evidence we gathered to document our first claim that there was improvement across time in the three teachers' ability to support dialogic interaction for mathematical problem solving. Second, in order to support our description of how teachers shifted their pedagogical practices over the time of the study, we illustrated our evidence that teacher questioning skills were a forerunner to improving other pedagogical skills important for promoting dialogic interaction, such as giving students voice, promoting student problem solving, and listening attentively to students' ideas. Finally, the results from quantitative analysis indicated that students in treatment classes did gain more mathematical understanding compared to the students in control classes.

The three teachers in the study attempted to promote dialogical interaction between students by changing their pedagogical practices where they first focused on changing their questioning strategies which enabled them to improve other pedagogical areas. This change in questioning approach promoted greater negotiation of meaning between individuals through a series of "dialogues." This negotiation process was supported in two ways with the MRA: classroom discussion and individual writing that required the teachers to listen to students' ideas. This study showed that implementing approaches such as the MRA helps students develop deeper understanding of mathematics concepts.

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STUDENTS' LEARNING THROUGH THEIR OWN ACTIVITY: TOWARD A BASIS FOR A SCIENTIFIC APPROACH TO TASK DESIGN AND SEQUENCING (1)

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We offer an account of Erin's re-invention of a common denominator algorithm for division of fractions. This account is a first step in research to better understand how learners develop mathematical concepts through their own activity in the context of a carefully planned sequence of mathematical tasks. Whereas the overall goal of our research is to look across mathematics learning of different concepts at different age levels to better understand how learners learn through their activity with mathematical tasks, we present the analysis of this initial study to demonstrate a phenomenon that is worth studying and a methodology that allows the generation of accounts of learning.

Teaching can be defined as intentional actions to support and foster the learning of one or more learners. Just as a good coach seeks to take full advantage of a player's athletic ability and just as a good mid-wife seeks to take full advantage of a mother's innate ability to birth a child, a good teacher and a good curriculum developer take full advantage of the child's innate ability to learn. However, as mathematics educators are we doing so? Do we have sufficient understanding of how students develop mathematical concepts to be able to create the conditions in which their learning is optimized?

The work discussed in this paper is based on the observation that humans develop concepts through their own mental and physical activity. For example, young children develop a concept of number through their activity. The development of the concept of number, perhaps the greatest single mathematical achievement in a person's life, does not happen in response to the posing of mathematical problems. In fact, even the question "how many?" is meaningless to the child who has no concept of number. Further, there is nothing you can tell or show this child about number that will be meaningful to her. Instead Mom and Dad engage the child in counting, first rote recitation of number names and later coordination of the recitation of each name with touching of the next object. Even the coordination of these two activities can be done with no concept of number. However, the engagement in these coordinated activities eventually leads to an initial concept of number (i.e., a concept of manyness). Are the parents important in orienting attention? Yes. Are the social practices developed important? Yes. Nonetheless, there is something important for mathematics educators to understand about how the child's own activity contributes to this significant learning.

The research described here is part of a larger effort to develop an understanding of the processes by which learners learn through their own activity and engagement with mathematical tasks. The broad goal of this effort is to contribute to a description of these processes that can inform mathematics curriculum development, teaching, and research on mathematics learning and teaching. We envision a scientific basis for task design and sequencing. By *scientific* we mean a number of interrelated things including: insights into learning processes are driven by systematic empirical study of learners' conceptual advances as they engage with designed tasks; that these insights generate viable data-driven hypotheses about learners' advances in terms of their extant conceptual structures, goals, and mental and physical activity; that these hypotheses and consequent models provide a basis for the principled design (creation and refinement) of task sequences intended to leverage learner's own activity in ways that can optimize their learning of particular concepts. *Scientific* also subsumes an empirically-based understanding of the social aspects of instruction (e.g., roles, tool use, norms), however, these aspects are back-grounded in the design of the study presented here and not the focus of our analysis.

Whereas the overall goal of our research is to look across mathematics learning of different concepts at different age levels to better understand how learners learn through their activity with mathematical tasks, we present the analysis of this initial study to demonstrate a phenomenon that is worth studying and a methodology that allows the generation of accounts of learning. Ultimately, we would have a diverse set of accounts from which we can derive empirically-based hypotheses about the mechanisms of learning through activity.

Our paper describes part of a teaching experiment conducted with individual students to create a context for studying learning through learners' own activity and the engagement with tasks. The mathematical domain for the entire study was division of fractions. Our report focuses on one student's re-invention (Freudenthal , 1973), with understanding, of the common denominator algorithm for division of fractions ($\frac{a}{b} \div \frac{c}{b} = a \div c$).

Conceptual Framework

The notion that students learn through their activity is not a new one; it is embodied in Piaget's (2001) construct of *reflective abstraction.* This construct, which suggests that the development of new conceptions derives from activity and reflection on activity, orients our study. Building on Piaget's work, von Glasersfeld (1995) claimed that reflection is an inborn mental ability and tendency of human learners, that it is often not conscious, and that it involves creating records of experience, sorting and comparing records, and identifying patterns in those records. In other words, seeing commonalities in one's experience is a natural and spontaneous activity of the human mind. However, Inhelder, Sinclair, and Bovet (1974) cautioned, "Whatever the degree of regularity [from the observer's perspective, the observations] are always organized by the human learner" (p. 12). Following Piaget and von Glasersfeld, we understand this to mean that the commonalities perceived by the learner are a function of the learner's goals and her current assimilatory schemes.

Methodology

In this study, we were attempting to focus in on one aspect of the learning process, how a learner learns through her own activity as she engages with a carefully sequenced set of mathematical tasks. Our focus was on the student's progress while interacting with the task sequence. For this purpose, we adapted a single-subject teaching experiment. The choice of a singe-subject approach reflected failed attempts to use data collected in whole classes and small groups. The single-subject design minimizes periods of time in which the subjects are watching or listening to another – situations in which the researchers have no indication of the student's mental activity. Such periods in multi-subject data proved to severely compromise our ability to generate compelling accounts of the learning process.

Because of our focus on students learning through their own activity, the researcher's interactions were limited to posing problems that were part of a carefully planned sequence and probing thinking. The researcher's role did not include demonstrating solutions, giving hints, or asking leading questions. Although a social analysis of the situation would reveal important aspects of the norms established in the teaching experiment sessions, our focus dictated that these social features be considered as background and not the primary subject of analysis. One adaptation that was included in the design were frequent assessments of what the student seemed to have learned and of what was still to be learned. These interspersed assessments, which were in addition to the pre and post assessments, were designed to allow for clear delineation of the period of time during which the learning took place.

Three students—all prospective elementary teachers—participated individually in a series of teaching experiment sessions. In the initial sessions students developed a conceptualization of division of fractions as a quotitive division relationship [our words] of quantities. This report describes the session in which they re-invented a common-denominator algorithm for division of fractions and data from the post assessment conducted a week later.

Analysis of the video recording of the teaching experiment sessions, transcriptions of the sessions, and written artifacts of the student's work were aimed at building up an account of the student's learning process. This involved initially identifying relevant data to piece together the *descriptive* story (without inference) of what happened. The data were then reviewed multiple times to find small inferential steps that could be added to the account to begin to *explain* aspects of the story. At every point in the analysis, disconfirming data were considered as well as alternative explanations and challenges from members of the research team. Even if no disconfirming evidence was present, explanations were excluded whenever a sufficiently compelling case could not be made based on the evidence available. The result of this process is an account that can be offered for public scrutiny/critique. The excerpts of the account presented here are from Erin's sessions. Data were similar for all three students.

Results

Erin began this section of the teaching experiment able to create division of fractions word problems. She was comfortable doing the invert-and-multiply algorithm. However, she did not know why this set of actions was appropriate. Our instructional goal was to foster Erin's re-invention of the common-denominator algorithm for division of fractions with understanding through the posing of a sequence of tasks. Our account of Erin's participation in the teaching experiment sessions and her learning process is described in three chronological phases.

Phase 1

We began by asking Erin to draw diagrams using rectangles to solve the problems given. The first problems were word problems whose dividends and divisors had common denominators. *Problem One*: *I have seven-eighths of a gallon of ice cream and I want to give each of my friends a one-eighth-gallon portion. To how many friends can I give ice cream?* Erin drew a rectangle to represent a gallon of ice cream and shaded out 1/8 of a gallon to create a representation of 7/8 of a gallon (the unshaded area). She immediately reported the answer as 7. When questioned, she reported, "Because you have seven eighths, seven of these one eighths, and each of the portions you are giving is one eighth. You have seven to give."

Problem Two: *A scuba diver has two hours worth of air in her tank. If each dive to the bottom of the bay takes three-eighths of an hour, how many dives can she make with the air she has?* In the second problem, the divisor did not divide the dividend evenly. Erin used a similar diagram strategy and reported the whole number part of the quotient and the remainder. The interviewer (I) probed further:

257. I: So, if we are interested in how many dives she can make, including fractions of a dive…
258. E: Five and a third.
259. I: Why do you say five and a third?
260. E: Because she needs three of these [indicates 1/8s], and we have one, so one out of three.

Erin's activity sequence for solving division of fractions problems using diagrams follows. In parentheses, we illustrate the step based on how she solved the third problem involving 11/5÷3/5:

1) Draws the original quantity, the dividend (draws 3 whole rectangles, divides them into fifths, shades out 4/5 in the third rectangle leaving 11/5 to work with).
2) Identifies groups the size of the divisor (circles each 3/5).
3) Counts those groups (counts 3 whole groups).
4) Identifies the remainder as ungrouped part of dividend (r=2/5).
5) Identifies the fractional part of the quotient (fpq) by determining the fraction of the divisor represented by the remainder (2/5 is 2/3 of 3/5).

Note that Erin created her activity sequence from her extant knowledge. Although she was not invited to solve the problem in any way she chose (told to used rectangular diagrams), she was not taught how to create diagrammatic solutions to the word problems.

In the final part of Phase 1, Erin is given a series of division of fractions expressions (e.g., 8/5 divided by 3/5) devoid of problem context. She continues the same activity sequence and explains her work in terms of the size of the groups being counted (e.g., "2 and 2/3 groups of 3/5").

Phase 2

In the second phase, the researcher poses problems for which Erin is asked to do a diagram solution mentally – without actually drawing.

R: I am going to give you another [division of fractions expression], but the numbers are going to be messy enough that you are not going to want to draw it. … can you anticipate what you would get if you would draw it? So, if I say I have 23/25, let me write it, 23/25 ÷ 7/25. If you were to draw it do you know what you would get?

S: Do you want me to tell you how I'd do it, or just the answer?

I: Tell me the answer.

S: I don't know the answer unless I'd do the -- unless I flip them [referring to invert-and-multiply algorithm].

I: Without flipping, can you imagine what you would get if you drew diagrams? Do you want to talk through what you would draw; is that what you are saying? [Erin nods.] OK. Go ahead and do that.

E: I would draw one large rectangle split into 1/25's and I would have 23/25, and then I would section off 7/25- groups. So I could do three groups, and then I'd have two --
R: [Interrupting] How do you know you'd have 3 groups?
E: Because 7 X 3 = 21. … So, I could have 3 groups with that and then there would be 2 left over. So it would be 2. I'd have 3 2/7.

Erin's inability to give an immediate answer and need to do a mental run of the problem are evidence that she had neither observed a numerical pattern nor made a curtailment of her diagram solution. The former could happen with no understanding, just an observation that a particular pattern seems to hold. The latter would be an anticipation of part or all of the solution process, an understanding of the relationships involved that makes going through all of the steps unnecessary.

Erin's mental run was an attempt to capture each step of her diagram solution strategy; she had made no curtailments. Erin was still paying attention to the denominators as she described what she would draw. She spontaneously called on her whole-number multiplication, because she was thinking about how many groups of 7 boxes (that were already created as 25ths) she would have out of the 23 boxes (already created as 25ths). This use of whole-number multiplication is not a reflection of a decision to disregard the denominators, but a function of a shift in focus. One could think of Erin's thought process as being analogous to the use of a pronoun. Once one has established the identity of the antecedent, one can operate with the pronoun without particular attention to the antecedent. Thus, when Erin says "7 parts", it is as if "parts" is a pronoun representing 25ths. Similarly, when Erin was drawing diagrams, she did not need to focus on the size of the parts at this point in her activity sequence, she counted the number of parts to make each group and then counted the number of groups she had formed (steps 2 and 3). The already established diagram embodied the size information. The shift to the mental run problems required that Erin concentrate on the number of available parts and the number of parts per group, the numbers she would need to work with, and to use her whole number multiplication. Multiplication was required because she no longer had countable items in front of her.

For the next problem (compute 7/167 ÷ 2/167 mentally), once again Erin needs to do a mental run of the diagram solution.

Phase 3

In Phase 3, we observed the emergence of the curtailment that was not previously available. The researcher gives Erin 7/103 ÷ 2/103. Although only the denominators have been changed from the previous problem, he does not call Erin's attention to this fact. Erin immediately says, "3 _." Erin thus shows an anticipation that she did not have in the prior problems. As we review further evidence, we invite the reader to recall the distinction we made earlier between an anticipation based on the observation of a number pattern and the anticipation of the results of an understood process (a curtailment).

The researcher asks Erin to explain:
R: You did that awfully quick. How did you know it so quickly?
E: Because I didn't work with those numbers over here. I didn't use 167. [Erin refers to the denominator of the previous problem (probably by mistake), but that does not change the explanation.]
R: How come you didn't use them.
E: I don't know. I didn't need them…

The researcher gives no indication that she has done something that is valid or significant. Rather, he challenges her to defend her claim:

R: What do you mean you don't need it? If you drew it, you need it. Right? … Are you saying you don't need it to draw it either?

E: No, I need it to draw it. Or if, like if I want it to represent 7/167, that wouldn't be the same as like 7.

R: OK.

E: Though, well [reflecting on what she just said], I think it might be, if I'm just drawing --

R: Are you saying that you have a way to draw that without worrying about one hundred sixty seven? Draw me a picture.

E: I might be wrong.

Erin draws one rectangle divided into 7 parts.

R: So what did you draw there?

E: I drew 7 equal parts of a whole.

R: And what does that have to do with this problem? [referring to 7/167÷2/167]

E: … so I have 7 parts, and I want to break, separate that into groups of 2, and I can separate it into one, two, three groups, and I have one, one but I need two to make it, so I have one out of two for the other one too, so it's three and a half.

R: Three and a half what?

E: Three and a half [pause] parts [immediately self corrects], groups.

R: Three and a half groups. And how big are those groups?

E: Two.

R: Two what?

E: Two parts.

R: This is my problem…

E: [Anticipating his objection, Erin breaks in.] Well, I could just say they are two one sixty sevenths if I wanted to.

R: Is that what you'd say?

E: Yeah.

R: So then what did you draw? You started out by drawing 7 parts. What is that?

E: Each of these [points to 1 section in diagram] represents 1/167.…

Whereas Erin can think about the parts as representing a particular size part (e.g., 167), she shows that she can think about the parts as representing *any* size part. When asked whether she would change her diagram for the problem 7/103 ÷ 2/103, she explained "No … I'll just say this [points to one section of her diagram] is 1/103".

In subsequent problems, Erin calculates the answers without doing a mental run of the diagram-drawing strategy. She has now developed a curtailed computational strategy. It is important to note that the curtailment was not the result of observing a pattern in the numbers. The data showing this development indicate that she instead came to see the equivalence of particular problems and could explain the logical necessity of their relationship.

Erin went on to demonstrate that she was able to do problems with improper fractions and mixed numbers (converts to improper fractions). However, when given a problem with unlike denominators, she realized and indicated that she was unable to use her newly developed algorithm, because she did not have common denominators. However, when the researcher asked if she could make common denominators, she did so and solved the problem. Note, for the purposes of this study, we were not interested in whether she spontaneously made common denominators, nor whether she understood the mathematics involved in doing so. It was indicative of her understanding that she did not over generalize to problems with uncommon denominators.

Post-assessment

A week later, Erin was given a post-assessment using an interview format. When confronted with a division of fractions problem (25/17 ÷ 8/17) and told she could not invert and multiply, she used the algorithm she had re-invented during the last session and justified it:

> …I'm really seeing how many … 8/17 pieces are in 25/17 pieces, … I know like what the piece stands for, so I don't have to use the bottom, the denominator … If I had it written out with a picture, I would have like twenty five sections and there were really one seventeenth each of them, and then I would see how many groups of eight I could get from the twenty five … the eights would be the same like, they each would be one seventeenth plus one seventeenth plus one seventeenth. They would all stand for one seventeenth. But it would just be understood.

Later in the interview, Erin was given the following problem:

One student who used this algorithm, went back and looked at two thousand four hundred divided by four hundred and said it explains why the answer of two thousand four hundred divided by four hundred is equal to twenty four divided by four. Can you explain what this person was thinking?

Erin's explanation follows:

> Well, what if you drew these things? You just drew 24 little rectangles…OK. If you had twenty-four of those and each stood for a hundred, you really have 2400. So if you were dividing it into four hundreds, you would only need four of the rectangles, which would equal, each one is equal to one hundred. So that would be four hundred for four of them, whatever you separated those into it. And that's why that would be equal to twenty four divided by four, because if you just had twenty four divided by four with each rectangle standing for one, you would still get the same answer, but they would just stand for different things.

Erin's explanation indicates that she has a deep understanding of the reasoning that she developed in the prior sessions. She learned that there are equivalent division problems that can be thought of as having the same number of units, but different common units. These units can be parts of a whole or can be composite units, multiples of 1. She had developed an understanding that as long as the units are the same for the dividend and the divisor, the problem is equivalent to a problem in which the unit is 1. This was the basis of her reinvention of a common denominator algorithm for division of fractions. Most importantly, Erin did not just know *that* the relationship holds and that the algorithm works, she understood the logical necessity involved and could articulate it. [The importance of this distinction is developed in more detail in Simon (2006).]

Discussion

The account of Erin, on which this report is based, provides an example of a student learning from her engagement with a carefully sequenced set of tasks. Although classroom discussions, collaborative problem solving, and other more elaborate types of classroom interaction are important parts of mathematics instruction, this account focuses attention on a valuable and not sufficiently understood aspect of mathematics learning – a part that could be more scientifically accounted for in lesson design.

The task sequence employed was crucial to Erin's learning. However, our research is not focused on analyzing successful task sequences. The account of Erin is the first step in generating accounts of successful learning through activity with task sequences. The ultimate purpose of which is to understand learning through the learners' own activity at a level that permits the generation of a useful set of design principles for creating successful task sequences.

Endnote

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STUDENTS' UNDERSTANDING AND DIFFICULTIES WITH MULTIPLICATIVE COMMUTATIVITY

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The objective of this study was to analyze children's understanding and difficulties with multiplicative commutativity. In this study, 58 fourth- and fifth-graders were individually interviewed, solved 8 multiplication word problems, and explained one given commutative strategy. Their responses suggest that many children recognize the commutativity if they are directly shown, but do not commute two factors when they solve equal grouping and price problems. Children's explanations for the given strategy and their own strategies indicate that their understanding of the property in contextualized problems is rather limited.

The commutativity of multiplication is one of the critical properties that may facilitate fluent problem solving for arithmetic and algebraic problems. In U.S., the commutative property of multiplication is often taught as a pattern or rule to reduce the multiplication facts to learn. However, students' conceptual understanding of the property in problems with contexts has not been well documented. In this study, I investigate students' understanding and difficulties with the commutativity in equal grouping and price contexts.

Theoretical Framework

Math educators have advocated increased use of problems contextualized in everyday situations in a current math reform movement (Baroody, 1987; Carroll, 1997; NCTM, 2000; Schoenfeld, 1988, 1992). Results from a number of research studies supported the argument that children are more successful solving problems when the problems are presented in everyday situations than problems with numbers and symbols with no context (Baroody, 1987; Carraher, Carraher, & Schliemann, 1987; Saxe, 1988). Children solving contextualized problems were not only more successful coming up with correct answers but also demonstrating more understanding of solution procedures. In a recent National Research Council report (Kilpatrick, Swafford, & Findell, 2001), the authors asserted that one of the ways to help children develop number sense is to provide them with word problems.

Results on students' problem solving of multiplication problems indicated advantages of story problems in facilitating children's understanding of multiplication. For example, Carpenter and his colleagues (Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993) showed that more than 70% of kindergartners in their study solved a multiplication problem correctly using counters after completing 8 months in kindergarten where teachers spent a large portion of their math classes solving word problems. Other researchers also documented mathematically sophisticated strategies that children in grades 3-5 constructed when they were given word problems (Baek, 1998, 2005; Kamii, 1994; Lampert, 1986a, 1986b).

Multiplication problems situated in everyday contexts have been classified to several categories (for review of the problem types see Greer, 1992). In the elementary school, equal grouping and price situations have been frequently used in word problems in order to assist or assess children's understanding of multiplication in curricula or research studies. For

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example, one of the NSF funded curriculum series, *Investigations in Number, Data, and Space* (TERC, 1998), introduces multiplication as *things that come in groups*, using equal grouping problems such as, "2 cartons of eggs. Each carton has 12 eggs." (p. 2). Price problems, such as "A bag of cookies cost $3. How much do you need to buy 5 bags?," are also often used in multiplication lessons. In these problems, the multiplier and multiplicand play different roles, which means that for the cookie problem, for example, 3+3+3+3+3 matches the problem better than 5+5+5. Schwartz (1988) argued that these different roles of the numbers make multiplication and division *referent transforming operations*, distinctively different than addition and subtraction, which he called *referent preserving operations*.

Research studies that documented children's strategies for multiplication word problems often used equal grouping or price problems. Many strategies reported in the studies are based on the distributive and associative properties. For example, one of the students in Baek's study, Laura, solved the problem 153 bags of 37 balloons in each bag, using distributivity and associativity. She knew that 100 groups of 37 is 3700, and 50 groups is 1850 because it is a half of 100 groups. Then she figured out 3 more groups of 37 by adding three 37s. She computed the final answer 5661 by computing 3700+1850+111 (Baek, 2005, p. 245). Results from these studies indicated that word problem contexts not only help children make sense of problem situations, but also provide contexts that facilitate students' partitioning strategies using the distributive and associative properties, and in turn, construct sophisticated computational procedures.

However, other researchers asserted that multiplication problems in equal grouping or price situations hinder students' understanding of the commutativity (Fischbein, Deri, Nello, & Marino, 1985; Nunes & Bryant, 1995; Schiliemann, Araujo, Cassunde, Macedo, & Miceas, 1998). They connected the structure of multiplication to students' understanding of the operation, and discussed their difficulty with the commutativity in multiplication. Fischbein and his colleagues argued that repeated addition is children's intuitive model for multiplication, and different roles that multiplier and multiplicand play hinder students' success in solving multiplication when the multiplier is smaller than 1, whereas the size of the multiplicand does not matter. Fischbein's study has been replicated with many other age group participants and produced the same results (e.g. Bell, Greer, Grimison, & Mangan, 1989; Greer, 1988; Mangan, 1986).

Nunes and Bryant (1995) reported Brazilian students' limited understanding of the commutatativity in equal grouping situations. The students in the study demonstrated more difficulty recognizing the property in equal grouping situations than arrays. The authors, however, concluded that students' overall difficulty with the property regardless of the situations overshadowed impacts of contexts. They results showed that although many children recognized that a×b is the same as b×a, they also thought that the products would be the same if the additive difference between two factors is the same (e.g. 35×17=33×19).

In a study by Schliemann and his colleagues (Schilemann, Araujo, Cassunde, Macedo, & Niceas, 1998), Brazilian street vendor and schooled students solved price multiplication problems. The authors found that street vendors were more limited in use of commutativity. They also found that the use of commutativity is closely tied to a greater success of solving multiplication problems. They concluded that street vendors' experiences with price problems impose limits on the development of their commutative understanding.

In contrast, ter Heege (1985) and Squire, Davies, and Bryant (2004) reported that some success of children in ages of 9-11 years. Ter Heege documented that Dutch fourth graders in the study used the commutativity in solving single-digit multiplication problems. However, it is not clear how students understood the roles of the multiplier and multiplicand since the problems were presented in number sentences without any contexts. In addition, the author did not report how widely the property was used among children. Squire and her colleagues administered a timed multiple-choice assessment using equal grouping, array, and Cartesian product contexts, and showed that many British students correctly identified the commutative problems regardless of the contexts. However, since students were given only 40 seconds for each problem, and they chose an answer for b×a when the product for a×b was given, it is not clear how they understood the property.

In this study, I further investigated children's understanding and difficulty of the commutativity in solving in equal grouping/price multiplication problems. Specifically, I aimed to better understand children's use of the property in problem solving as well as their recognition and explanation the property when they are shown a commutative strategy.

Methods

Thirty-two fourth graders and twenty-six fifth graders in three multi-graded classrooms were individually interviewed. They were asked to solve eight multidigit multiplication word problems and to explain and replicate a commutative strategy. The eight word problems included two 1-digit × 2-digit numbers, four 2-digit × 2-digit numbers, and two 2-digit × 3-digit numbers in equal grouping and price contexts. For the commutative strategy task, the children were told a child in a different class constructed the strategy for the problem, 127 children paying \$6 each for a field trip, and were shown the strategy (Figure 1). Children were asked if the strategy would solve the problem, and if so, why adding the multiplier, instead of the multiplicand makes sense.

$$\begin{array}{rl}
127 & \\
\underline{127} & \\
254 & 2 \\
\underline{254} & 2 \\
508 & 4 \\
\underline{254} & 2 \\
762 & 6
\end{array}$$

Figure 1. The given commutative strategy

During the interviews, children did not have any concrete tools available, but only paper and pencil. The classroom teachers had participated in a research program designed to help teachers focus on children's mathematical thinking and routinely encouraged children to construct meaningful solution procedures for word problems and to discuss meaning, similarities, and differences of their strategies. The teachers did not teach formal algorithms, and discouraged the use of them if children could not explain why they work. Many children in this study were observed to routinely use the algorithm for addition and invented strategies for multiplication. All interviews were audio-taped and children's written work was collected. Children's strategies for solving word problems analyzed by strategy types and use of the commutativity.

Results

Commutativity in Children's Strategies

Among children's 464 solution strategies for the eight word problems, 303 strategies were valid and widely ranged from drawing tallies, to partitioning, and to the formal algorithm. Among the valid strategies, students' understanding of the commutativity was identifiable in 267 strategies, and 36 unidentifiable strategies involved partitioning both factors or the formal algorithm. When children kept the multiplicand intact, and computed the multiplicand the multiplier times, the commutativity in the strategy was classified as identifiable. For example, a fourth grader, Dana, used a doubling strategy for the problem 153 bags with 37 balloons in each bag. In this strategy she kept the multiplicand intact and added two 37s for 2 bags and two 74s for 4 bags, and continued the doubling and additional adding to figure out 153 bags (Figure 2). This strategy clearly shows that Dana did not commuted the multiplier and multiplicand, even though it would have been significantly more efficient if she had computed 37 groups of 153s, instead of 153 groups of 37s.

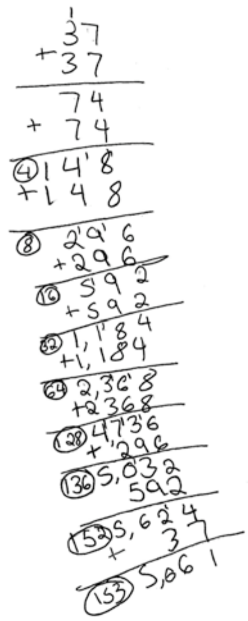

Figure 2. Dana's commativity identifiable strategy

In other hand, Sean, a fourth grader, partitioned both factors for the problem 43 boxes with 24 cards per box, his understanding of commutativity was unidentifiable. Sean explained his strategy that he wrote down 43 boxes on the top and kept 24 cards in his head. He computed 20x40, 20x3, 4x40, and 4x3, and then added all the partial products (Figure 3). Sean's abstract strategy does not allow a reasonable identification of how he understood the roles of the two factors, and his strategy would work the same way if he switches the roles of the factors.

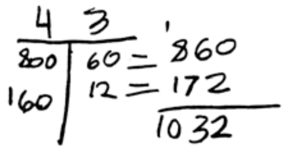

Figure 3. Sean's commutativity unidentifiable strategy

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Among 267 strategies that the commutativity was identifiable, children kept the role of the multiplier and multiplicand as given in the problem context in 207 strategies (77.5%). Students' explanations of their strategies during the interviews revealed more clearly that their choice of not-commuted strategies was not a random choice for many children, but an important part of their strategy in making sense of the solution processes. For example, in solving the problem about 43 boxes with 24 cards in each box, Maureen first started by writing down "43×10", but quickly erased 43, saying "no, there are 24 cards" (Figure 4). Then Maureen changed it to "24×10" for 10 boxes with 24 cards in each, and solved the problem by computing 24×10 for 10 boxes, 240×4 for 40 boxes, and adding 12 for 4×3 and 60 for 20×3 for 3 more boxes.

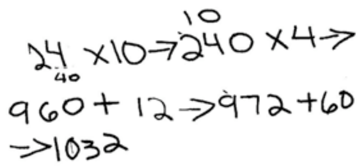

Figure 4. Maureen's strategy, partitioning the multiplier.

There were 60 strategies (22.5%) that children partitioned the multiplicand. Unlike children who partitioned the multiplier, those who partitioned the multiplicand often did not use the problem context in their explanation of the partial products. For example, in solving the problem, 47 children paying $34 each, Laura partitioned 34 into three 10s and 4 (Figure 5). Laura explained her strategy that she calculated 47×10 because 10 is from 34 and she knew "times 10." However, she did not mention dollars or children in her explanation.

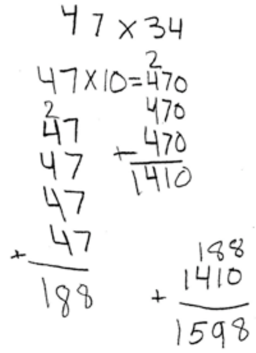

Figure 5. Laura's strategy, partitioning the multiplicand

Commutativity in a Given Strategy

Initially 30 of the 58 children (12 of the 32 fourth graders and 18 of the 26 fifth graders) responded that the given strategy in Figure 1 made sense to them. When the interviewer asked to explain the strategy using the references of children and dollars in the problem, eight of them became confused, and changed to say that the strategy was wrong because the strategy added the number of children, instead of the number of dollars. The twenty-two children who maintained that the strategy works for the problem were asked to explain why adding the multiplier would solve this problem. Six of them stated the property as a rule that multiplication is "reversible", or 127×6 and 6×127 are the same because the numbers are "switched around". Five children tried to explain the property by providing more examples, such as 3×4 = 12 and 4×3 = 12, and 4×6 =24 and 6×4 = 24. The rest eleven children provided more generalizable justifications, including explanations such as 127+127 means $127+$127 when each of the 127 children brings $1 at a time. Although there were variations of clarity and certainty in these 11 children's explanations, these children understood the property as more than a rule.

Conclusion

It is very common to see teachers teaching commutativity in a third grade class as a way to reduce multiplication facts that children need to learn. However, it is not clear how children understand or use the property. Do they know the commutativity as rule or fact? Or do they understand what it means or why it works? If they know or understand the property, do they use the property flexibly in their problem solving?

Ironically, the findings of this study are consistent with previous conflicting studies. First, as Squire and her colleagues (2004) found, many children (30 of 58) recognized the property. However, their justification of the property and use of it in problem solving indicate that their understanding of the property is not deep or not deep enough to take advantage of it. This aspect of the findings is consistent with many other studies at the beginning section that reported students' struggles.

This gap between students' recognition, justification, and use of the commutativity when problems are contextualized in equal grouping or price problems presents us several challenges. Are we satisfied as long as children know a×b=b×a as a rule? Or do children need to understand why it works? Or is a flexible use of the property the true barometer of knowing the commutativity? If we can come to a consensus on the standard, the next challenge will be to develop instructional strategies that will help children understand and use of the commutativity in number sentences and various contextualized problems.

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THE COMPLEXITY OF AFFECT IN AN URBAN MATHEMATICS CLASSROOM

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We outline a major exploratory study of affective interactions around conceptually challenging classroom mathematics in low-income high-minority urban communities. We describe procedures for classroom observations, individual interviews, and analysis with respect to four aspects: flow of mathematical ideas, affective events, social interactions, and teacher interventions. A particular episode exemplifies the interplay of mathematical ideas and key affective events.

This report partially describes an exploratory study, now under way, investigating the development of *powerful affect* around *conceptually challenging mathematics* in urban classrooms in low-income, predominantly minority communities. Considerable research suggests the critical importance of affective issues for children in schools within urban communities (e.g. Anderson, 1999). Here we briefly outline the scope of the broader study and offer an illustrative example describing the interplay between the flow of mathematical ideas and key affective events in a particular classroom episode. Additional data and further interpretations, drawn from a different school participating in this study, are discussed in a related report (Epstein, Schorr, Goldin, Warner, Arias, Sanchez, Dunn, & Cain, 2007).

Theoretical Framework

Our framework is eclectic and interdisciplinary, drawing on urban studies and social psychology, as well as mathematics education and cognitive science. We build on research paying close attention to children's mathematical thinking as they construct representations and justify their solutions to challenging problems (Davis & Maher, 1997; Maher, Martino, & Alston, 1993). But to gain insight about the basis for students' success and their developing confidence in mathematics, we see it as necessary to attend to issues of affect, context, and culture, as well as mathematical activity and interactions (Martin, 2000; Moschkovich, 2002). We see the development of mathematical knowledge in students taking place in a social context of discourse within the classroom community (Cobb & Yackel, 1998), and influenced by the wider, urban community of the children. And we are informed by work on affect in mathematics learning, and beliefs in mathematics education (DeBellis and Goldin, 2006; Goldin, 2000, 2002; Leder, Pehkonen, & Törner, 2002).

Conceptually challenging mathematical activity involves gaining or changing some understanding. Children need to "figure something out," and may experience a sense of impasse. In this domain mathematical meanings are at least as important as procedures, and problems are often nonroutine for students. Classroom activity may involve mathematical discussion and argumentation, exploration, individual students expressing their own ideas and conjectures, "wrong answers" and "blind alleys" as well as fruitful suggestions, and

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questions posed not only by the teacher but by students to each other. While not all teachers may take the approach of encouraging mathematical exploration and discussion, this study focuses on the classrooms of teachers selected because they do so. Such processes of learning and doing mathematics evoke significant affective responses in class. Children (and teachers) may experience a variety of changing *emotional feelings* – e.g., curiosity, puzzlement, bewilderment or confusion, anticipation, frustration, annoyance or anger, fear or threat, defensiveness, suspicion, pleasure, elation, satisfaction, a sense of safety and trust, etc. Over time, students usually develop a range of more stable *attitudes* towards mathematics, coming to see it as something interesting or dull, enjoyable or hateful. They form enduring *beliefs* about mathematics (e.g., what it is, or what it is good for), and about themselves in relation to mathematics (e.g., about their own ability). They develop *values* in relation to mathematics and the doing of mathematics – whether and why mathematics is valued, or what kind of mathematical understanding or performance they will value.

By "powerful positive affect" we mean patterns of affect and behavior that foster children's intimate engagement, interest, concentration, persistence, and mathematical success. The main conjecture underlying the research is that powerful affect in relation to conceptually challenging mathematics is fundamental to developing mathematical ability and essential to present and future mathematical achievement. Our focus is on aspects of urban classroom environments that are likely to influence the development of such powerful affect.

Research Questions and Methods

Research questions for the overall study include: How do teachers contribute qualitatively to creating an emotionally safe classroom environment for urban students to engage in exploring conceptually challenging mathematics? How do teachers contribute to (or impede) the development of powerful mathematical affect in such an environment? What are the affective and cognitive consequences for urban children learning mathematics, including students' social interactions, emotional states, and mathematical learning? How do these develop over the course of the school year?

By an emotionally safe environment, we mean one where mathematical inquiry (including mistakes, false starts, discovery, criticism of each others' ideas, and impasse) does not lead to experiences of fear, pain, humiliation, shame, or domination and submission. We mean an environment where children's experiences include trust, confidence, dignity, and shared respect (with the teacher and with other students) in the doing of mathematics.

Three middle-school classrooms in two urban school districts with large numbers of minority and low SES students were selected for the larger study. In each of the three classrooms, the research team collected data during at least four cycles spanning the school year. Each cycle includes a pre-interview with the teacher and the videotaping of two consecutive lessons. Four "focus students" in each class were followed particularly closely during the year. Immediately after the second lesson in each cycle, the research team viewed the videotapes and selected several evocative segments as a basis for individual follow-up interviews with each focus student and with the teacher. As time permitted, interviews were also carried out with some other students in one of the classes. The methodological intent is to capture the classroom activity and interactions as closely as possible, gaining as much data as possible about the students' affective responses to and feelings about the class sessions, the mathematics, and the social interactions taking place in the classroom.

One senior researcher, one graduate student observer and two videographers with roving cameras following focus students were in the classroom during each lesson. A stationary camera captured an overview of the class. Each interview was conducted by one member of the research team and videotaped by a single camera. Classroom and interview videos were transcribed, and analyzed through four "lenses," with the goal of creating narratives describing: (1) the flow and development of mathematical ideas; (2) key affective events, by which we mean instances where strong feeling or emotion is expressed or inferred, and the immediately subsequent development; (3) social interactions among the students; and (4) significant interventions by the teacher (see also Powell, Francisco, & Maher, 2003).

Observations and Preliminary Inferences

The episode described below occurred in a 7th grade classroom in a public school district designated by the state as one of its most disadvantaged. In this middle school, more than 99% of the students are classified as minority and more than 70% are eligible for free or reduced lunch. The school population over the past decade has changed from being predominantly African-American to include a rapidly growing number of Hispanic students. The data were collected during the first cycle of observations, when school had been in session for less than three weeks, and students were still being assigned to the class. On the first day of the cycle this class included 18 students, 11 girls and 7 boys, assigned by the teacher to work collaboratively in small groups. Our illustrative discussion pertains to the first 26 minutes of this class, augmented by analyses of interviews with three of the students and the teacher after the session. The lesson was based on an investigation from "Variables and Patterns," a unit from *Connected Mathematics 2* (Lappan et al, 2006).

Mr. Pedran [all names are fictitious], the classroom teacher, is a Caucasian male in his early 30s who has been teaching in the school for about five years. Tyanna, a project focus student, is an outspoken African-American girl. Taurean, an African-American boy, is also quite forthcoming in offering his ideas in class. Ryan, another focus student, is a slender soft-spoken boy whose family has emigrated from the Dominican Republic. Denzel, a frequently volatile African-American boy, had been assigned to a Special Education class. He was included in Mr. Pedran's class in response to a special request by his mother. Nammi is a very confident and assertive African-American girl. Shaniyah, Tyanna's partner, is an African-American girl.

The previous day, students had prepared graphs from a four-column table that they had completed working collaboratively in small groups. In the table, column 1 represented the time in hours from 1 to 6. In columns 2 through 4, distance was to be recorded for each hour of time at a particular speed: 50 mph in column 2, 55 mph in column 3, and 60 mph in column 4. Ryan had asked Mr. Pedran the previous day to let him share his graph with the class. After a short discussion clarifying the data to be graphed from the day before, Mr. Pedran asked Ryan to share his graph using a transparency of his original work that Mr. Pedran had prepared (Fig. 1). Mr. Pedran had also made copies of Ryan's graph, and distributed one to each group as Ryan came forward. Note that Ryan had *not* drawn three distinct lines to represent the vehicles traveling at three speeds, but had connected all of the points in the order that he had plotted them, creating what Tyanna refers to below as "zigzag lines." For each time point on the horizontal axis, Ryan had plotted the three points (representing distances for 50, 55 and 60 mph), yielding vertical line segments when he connected them. Ryan had also incorrectly completed the middle column (for 55 mph) in his table; the first three entries were correct, but all the subsequent entries were calculated by adding 5 (instead of 55) to the previous cell (Fig. 2).

The transcripts from which excerpts are reported below have been coded according to broad, preliminary categories: SA (notable student affect); TA (notable teacher affect); SI (notable social interaction); TI (notable teacher intervention). More elaborate coding is being developed within the ongoing analysis. Times are in minutes and seconds from the start of the class period.

Segment 1: Getting started. (2:33) *Mr. Pedran:* So yesterday, you guys remember what we were talking about yesterday? [TI - focusing and connecting with previous lesson] *Tyanna:* Oh, yeah - I know what we was talking about. We was talking about the - um - the graph that we did yesterday about - um - about how like, like the multiples of 55 - and then uh - and uh - about multiples of 55, and 50 and 60 - and you had to put them on the graph in the right way. *Taurean:* We was talking about miles per hour and the distance that you go. *Mr. Pedran:* OK - the distance that you go. Miles per hour and the distance that you go. Anybody else? *Tyanna:* You said that Ryan was going to show us his graph because his graph was the same way as yours. *Mr. Pedran:* Well - it was ... *Tyanna:* Well, not the same way - but it was the right way – [SA - belief that any student's idea, put forth with support from the teacher, must be correct, or "the right way"] (3:55) *Mr. Pedran:* Well, when I show something, it's not about - uh - right or wrong - it's - if you remember the last time we had one of these debates, we still think that our way is the way we want to do things, and others think that theirs is the way. Remember when we had to have a talk about "face"? [TI, TA - underscoring the value of being open to the ideas of others] *Tyanna:* Oh yeah - and then you were like: "Get out of my face!" *Mr. Pedran:* Mm – hmm.

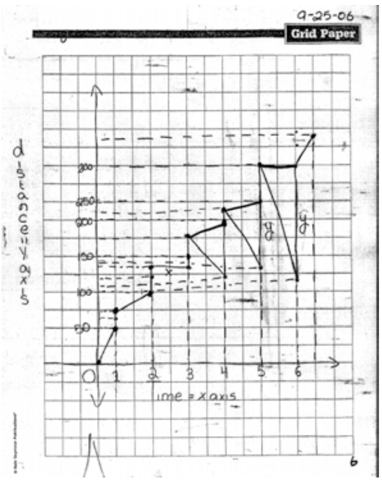
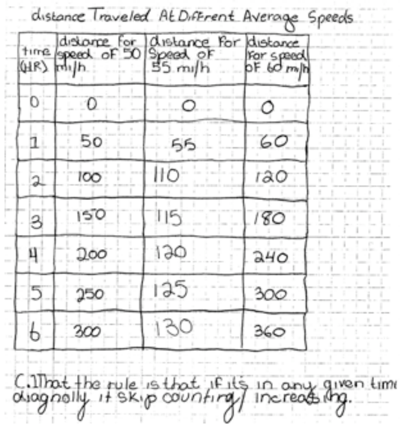


Figure 1: Ryan's graph



distance Traveled At Different Average Speeds

time (HR)	distance for speed of 50 mi/h	distance For Speed of 55 mi/h	distance for speed of 60 m/h
0	0	0	0
1	50	55	60
2	100	110	120
3	150	115	180
4	200	120	240
5	250	125	300
6	300	130	360

C.) That the rule is that if it's in any given time diagonally it skip counting increasing.

Figure 2: Ryan's table

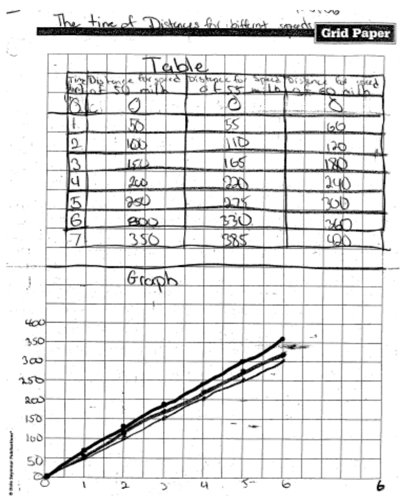


Figure 3: Nammi's table and graph

Ryan stood at his desk, smiling shyly, and moved toward the overhead projector with the completed table from his group in his hand. Denzel, a member of Ryan's group, forcibly took the table from Ryan's hand. When Ryan protested, Denzel indicated that he needed that table, and that Ryan could refer to the completed one that Mr. Pedran had at the overhead. Ryan hesitantly approached the overhead, and began his explanation.

Segment 2: Ryan's presentation and class discussion. (7:42) *Ryan:* (speaking very softly, almost inaudibly, in front of the class) So when I looked at the chart, I put time on the x-axis and distance in the y-axis, and when I saw the, um, distance in the chart, I looked at 1 in the time and 50 in the distance. So I thought 1 goes over there and 50 goes down there and next in the chart - then next after 50 in the chart is 55, so I thought 1 and 55, and after 55 it's 60 and I did the same thing, and it came up to here. (Ryan indicates the points on the overhead to illustrate each pair of coordinates.) [SA - hesitant anticipation and pride] (8:46) *Mr. Pedran:* Ok - so - ok - so what do you guys think? Do you guys agree with the placement of the points? *Nammi:* No. I don't. [SA - confidence] *Mr. Pedran:* Ok - well let me let Tyanna - Tyanna had a question. If anyone has questions anytime, feel free to ask. Go ahead Tyanna! [TI - encouraging open discourse] *Tyanna:* Why you put your lines like "zigzags"? Why not go straight up? [SA - confidence, curiosity, confusion] *Ryan:* These (pointing to the middle of the graph)? *Tyanna:* No - the zigzag lines. *Ryan:* (lowers his voice and responds, but inaudibly) [SA - nervousness and uncertainty] *Tyanna:* I can't hear you - Huh? [SA – tenacity] *Ryan:* I just connected the dots. *Tyanna:* Oh - oh – oh! *Ryan:* It's because this ... [SA – persistence] *Nammi:* My question is - why connect all the dots together and not just put each one in a separate line? *Tyanna:* Yeah - because it's like - all together. It's like - it's like one big graph but it's only - it's only like separate graphs. *Nammi:* It's three separate graphs - it's three separate datas. [SA – confidence] *Mr. Pedran:* So it's three separate sets of data. And so - well I thought - Ryan wanted to share. And when I looked at it, I thought it was an interesting way that he decided to connect the points. So you think it should be something different? Do you want to come up and show us? [TI -mediating student discourse, leaving the question of "right or wrong" open]

Nammi walked forward to join Ryan at the overhead and began to examine his graph, apparently trying to see her representation embedded in his. After attempting unsuccessfully to trace her three lines by going across Ryan's vertical lines as he tried to show her the appropriate points on his graph, she asked Mr. Pedran if she could make a transparency of her own graphic representation. Ryan stood quietly beside Nammi as she constructed a graph on a second overhead beginning with the straight line formed from the data for distances at 50 mph. Ryan tried to indicate that his data also showed this - but she waved him away. Ryan moved back toward his group, Denzel stood up to meet him with a "high-five", the comment "No sweat. Man!" and a big smile of appreciation. Mr. Pedran challenged the rest of the class to consider Ryan's representation as it compared to theirs, and students began talking within their groups. While Nammi was interacting with Ryan and preparing her graph (Fig. 3), Tyanna and her partner, Shaniyah, compared their graph with Ryan's.

Segment 3: Tyanna, Shaniyah, and Ryan. (11:27) *Shaniyah:* Okay - you choose three graphs. I don't get why he connected them. [SI - trying to make sense of Ryan's graph] *Tyanna:* Now - It looks like - now it's like one big graph that's like coordinates of different numbers. If you - if you - it would be better - but this graph here is confusing - because he's got this one - this one. *Mr. Pedran:* Do you think that he's graphed the same information as you? [TI - focusing the students, challenging them to look closely at the two representations] *Shaniyah:* No! *Tyanna:* Because - because we connected ours. Ours looks totally different. He like - he like added his own numbers in - ... and it looks confusing. [SA - puzzlement, confusion and desire to understand] *Mr. Pedran:* Well - that's the way - These lines here - I think are the way he makes sure he lines them up - the dotted lines (from the plotted points to the vertical axis)? These, I'm not sure. Would you - maybe do you want to ask him why he did that? (Tyanna shrugs) Maybe if he's not in front of the class he can explain it to you. Do you want to go talk to him? He's right over there. [TI - giving responsibility for communicating and comparing to the students]

Tyanna joined Ryan, who once again explained that he was connecting the points sequentially from his chart and that the dotted lines were added to help determine the number on the vertical axis. Tyanna indicated her understanding with another: "Oh!" After several minutes of discussion in small groups, Mr. Pedran asked Nammi to present her graph using the overhead that she had produced showing her three lines (Fig. 3). A whole-class discussion followed with apparent consensus that Nammi's graph seemed the more appropriate way to represent the data. Subsequent review of the work of the class, however, indicated that a number of students' graphs were similar to Ryan's.

Segment 4: Mr. Pedran's conclusion. (25:51) *Mr. Pedran:* Let's not - let's not debate what he should have and what he did do. My question is that he chose to do this, and he wanted to share that, and I'm glad he did, because when we look at it, he decided to connect all the dots - and if we look at Nammi's - and a lots of other people do this - Ryan's just - he's - he's just got what Nammi's got in here. What is that? [TI, TA – pulling the discussion together, affirming the value of Ryan's presentation] *Shaniyah:* He just connected the dots. *Tyanna:* But in different forms. (26:24) *Mr. Pedran:* But in a different form. He's got the same - same - I noticed a couple things within here, but if we were just to follow the green line or the blue line ... (Nammi and Mr. Pedran had attempted to overlay the lines representing the three data sets using different colors – and had been perplexed by the points that resulted from Ryan's computations in the middle column.) It's difficult to do at the same time - he's got the same thing - that's why I wanted to point that out. [TA - uncertainty around Ryan's solution and how to connect the two representations]

Retrospective interviews. Ryan's interview tended to confirm the interpretations of his affect noted in the transcripts. When asked how he felt about presenting his solution, he responded: "It made me feel excited, but when I got up there, it made me feel a little bit nervous." His interview indicated that at no time did he recognize any real difference in the two solutions, nor did he recognize that any of his plotted points were incorrect. Rather he stated: "That - uh – this" (his graph) "was the same thing as the graph" (constructed by Nammi) "but she didn't know it." Finally, he said again that explaining to Nammi and Tyanna "made me feel excited again, but I didn't want to tell her. ... Because then - I don't know ... I feel like they'll get embarrassed." Tyanna's retrospective interview, while confirming that she was convinced about her own graph and had been confused by Ryan's, indicated that she thought that both might be correct. In reflecting on her questions for Ryan, Tyanna said, "I be feeling like, 'Oh my God I got this wrong, they're gonna crack on me' - I feel bad when I get things wrong." She went on to say that when the teacher and the class affirm her ideas, "... that make me just feel good!" Nammi was much more convinced that her understanding of the problem was correct and that the three sets of data should be graphed separately. She expressed confusion about Ryan's representation, noting that she thought others in the class were confused as well but were hesitant to say anything. In his post-interview, Mr. Pedran expressed both satisfaction and uncertainty about the class discussion: "I think the kids did a good job of listening to him (Ryan). I think that he (Ryan) did a good job explaining what it was that he did ... and probably he felt good that his work was being shown to the whole class. I liked the way that she (Tyanna) had a question and she asked it, and I still haven't answered the question yet ... We're going to get back to that issue ... why he chose to connect the dots the way he did or if they see anything wrong there ... or if he sees anything wrong there."

Discussion

In this short episode, the rich variety of ideas, beliefs, and expectations underscores both the complexity and the importance of analyzing actual classroom lessons, especially in urban settings. Students are not only expressing mathematical ideas and beliefs about school expectations, but are grappling with accompanying feelings toward each other. Note the two very different responses in Segment 1, where Tyanna remembered the patterns of numbers, multiples of the three speeds, increasing by 5 from 50 to 60, while Taurean recalled that the data had been describing miles and distance. This dichotomy presages the action of the following 20 minutes, as the class considers the two different representations of the data. In studying the curriculum unit, it was noted that nowhere before this investigation had students been asked to represent more than one set of data on the same graph, nor did the directions for the task indicate that the data sets should be graphed as separate lines. The episode reported here raises questions for researchers and practitioners alike concerning how best to address the ensuing uncertainty in the students, and the accompanying cognitive and affective challenges to a teacher committed to a role of facilitator rather than "teller." Note the interaction about why Ryan's representation was selected for consideration. Tyanna appeared to believe that the selection must have been made because it was "right." Mr. Pedran suggested instead that looking at alternative representations, different ways of doing things, might be interesting for the class. He and Tyanna agreed about the potential for strong emotional feelings about a student's solution, and the need to allow each person "emotional space" in comparing mathematical ideas. As we study the narrative of this episode, these and other potential questions and implications about teaching and learning become salient. This will inform our continuing analysis of the complete set of data with respect to the four "lenses" described above.

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UNIVERSITY HONORS CALCULUS STUDENTS NEGOTIATING MEANING FOR CONVENTIONAL VOCABULARY

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University students participating in an inquiry-based introductory honors calculus course draw conclusions about the volume of water in a reservoir based on graphs of the rates of water entering and exiting a reservoir over a period of one year. Through grounded theory research methods, students' introduction and negotiation of meaning for conventional mathematics vocabulary were characterized and analyzed in the context of their participation in mathematical discourse. Students shared and created personal experiences through a process of linguistic invention to describe their personal interpretations of "point of inflection" and "concavity." This study opens further dialogue on Sfard's (2001) notion of learning mathematics as becoming a participant in mathematical discourse.

Conventional mathematical vocabulary is often used in university mathematics classes. However, the appropriation of mathematical vocabulary and notation by learners of mathematics has not been widely studied. Students participating in mathematical discourse develop their own methods of communication as they negotiate relationships between ideas and the words and symbols used to express those ideas (Walter, 2005). This study analyzes negotiation of meaning for the mathematical terms "point of inflection" and "concavity" in mathematical discourse among university honors calculus students.

Theoretical Framework

Psychologists often view human thought and speech as closely related. Vocal speech is that which is audible, while inner speech is human thought that resembles vocal speech (Vygotsky,1986). Developmentally, egocentric vocal speech is considered to be an intermediate state between inner speech and vocal social speech. In egocentric speech, individuals speak without the specific purpose of communicating with others (Piaget, 1997; Vygotsky, 1986). In social speech, on the other hand, the speaker takes on the point of view of the listener (Piaget, 1997). Brown (2001) cites Habermas and Gadamer in describing a dualistic nature to the relationship between humans and language itself. While Gadamer views humans as part of language, Habermas views humans as operating on language. These views on the nature of language suggest a variety of purposes and products for language, including organization of human thought, communication, and participation in society.

We view learning mathematics as becoming a participant in mathematical discourse (Sfard, 2001). Meaningful participation in mathematical discourse includes constant negotiation of meaning (Doerfler, 2000). We interpret the social context in which mathematical learning occurs as analogous to "humans as part of language," and negotiation of meaning as analogous to "humans operating on language" (Brown, 2001). Here, we focus on negotiation of meaning by learners of mathematics and on two processes of social speech through which meaning may be negotiated. The first process may be seen as a learner's comparison of their own usage of words with how other speakers in the discourse use the same words. Learners exercise personal agency as they purposefully choose (Walter, 2005;

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.

Walter & Gerson, 2007) to invite their peers to share their interpretations of conventional mathematical terminology. As a result of engagement in the negotiation of meaning, individual participants evaluate the way in which they believe terminology is or should be used and compare and contrast their own expectations with how vocabulary is used in the present discourse (Blumer, 1969). The second process may be seen as learners' borrowing of words from other contexts to compare newly negotiated meaning with shared experiences in a process of linguistic invention (Walter & Johnson, in press).

Method

This study took place in a university honors calculus classroom at a large private university in the Rocky Mountain Region of the United States. About 20 students met with two professors three mornings a week for two hours to investigate open-ended problems and discus related mathematical concepts. This paper focuses on four university students enrolled in the calculus course. At the time of the course, Daniel was a sophomore majoring in actuarial science who took a statistics course and Advanced Placement calculus courses in high school. Jamie was a senior majoring in geology who had previously taken this a first semester calculus course from another instructor at the same university. Julie was in her first semester at the university and planning to major in mathematics education. Justin was a junior who had not declared a course of study but expressed interest in engineering, mathematics, and mathematics education.

In the seventh week of the fourteen-week course, the participants began to work on the Quabbin Reservoir Task (Figure 1), which the instructors adapted from Connally, et al. (1998). The task shows a graph of inflow and outflow of water in a reservoir over the period of one year. Participants are asked to describe and create a graph of the volume of water in the reservoir during that year.

The Quabbin Reservoir in the western part of Massachusetts provides most of Boston's water. The graph in Figure 6.38 represents the flow of water in and out of the Quabbin Reservoir throughout 1993.

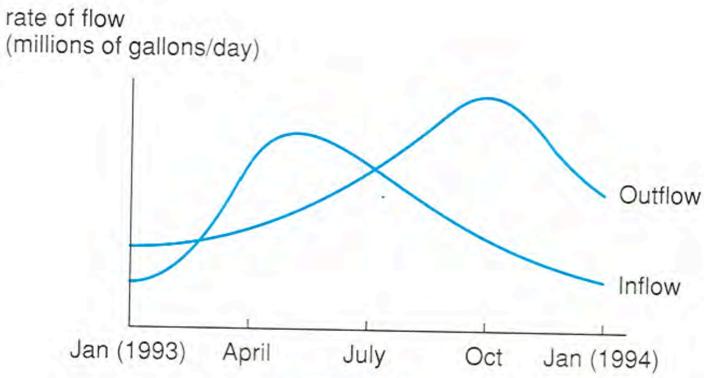

Figure 6.38

a. Sketch a possible graph for the quantity of water in the reservoir, as a function of time.

b. Explain the changes in the quantity of water in the reservoir in terms of the relationships between outflow and inflow during each quarter of the year. How are these changes evident in your graph in part (a)?

Figure 1: Quabbin Reservoir Task

We used grounded theory methodology (Strauss & Corbin, 1998) to conduct a constant comparative analysis of video, transcript, original student work, and researcher field notes relevant to student discourse on the Quabbin Reservoir Task. The codes, which are described below, helped us to identify and delineate categories of language use and events in mathematical discourse. As subcategories developed within individual codes we then identified relationships between different subcategories. Patterns and themes in the process of negotiation of meaning emerged through this continuous review of data.

We began coding the transcript according to the mathematical concepts that were the focus of the discourse. For example, utterances could focus on concepts such as rate, volume, increasing, decreasing, points of inflection, and concavity. The participants used different forms of language, drawn from a variety of contexts, to refer to these concepts. For example, "rate" was described in the context of the reservoir (R) as "the flow of the water." In terms of graph interpretation (G), "rate" was often described as "slope." The participants also referred to "rate" and "volume" as "velocity" and "displacement" respectively, which reflected language that they had used in an earlier task (the Desert Motion Task) about a woman driving through a desert (V) (diSessa, Hammer & Sherin, 1991). Other contexts (O) were also introduced, including "rate" as the speed of a person going down a slippery slide (S), and a context that made reference to the physical sensation of drawing (D).

We then coded language as egocentric or social. Due to the mutual respect and camaraderie among the participants, it was difficult to code any utterance as purely egocentric. There were few utterances captured by the camera that were not acknowledged by the other participants, and therefore any utterances that may have originated as egocentric speech were responded to as social speech. However, within this category of social speech there was a difference in the use of pronouns. The transcript analyzed here is coded in the far right column according to the type of pronouns used as first person (1), second person (2), or third person (3). Line numbers are given for reference purposes in the far left column.

Data and Analysis

To complete the Quabbin Reservoir Task, Daniel created a graph of the volume of water in the reservoir versus time (Figure 2). He then created a graph of net flow of water (Figure 3) by graphing an estimated difference between the inflow and outflow of water on the original graph (Figure 1). Although he did not label the horizontal axis of the graph with points A-G until after the episodes described here, Daniel's labels are shown in Figure 2 and referred to in the transcript to indicate the points and intervals that Daniel points to with his pencil on all three graphs.

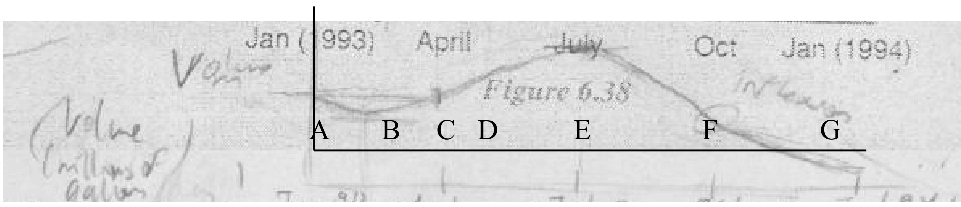

Figure 2: Daniel's Graph of Volume with the Horizontal Axis Labeled

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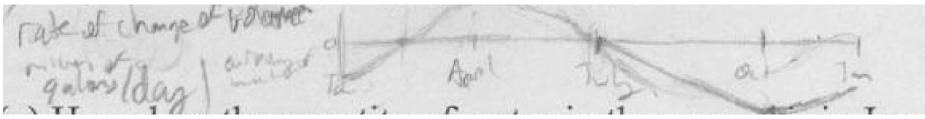

Figure 3: Daniel's Graph of Net Flow

In the transcript that follows, Daniel uses social speech to explain his volume graph to Justin, Jamie and Julie.

1	13:35	Daniel:	Like right here [*volume graph from A to B*] it is going negative until about that point [*indicating point B on the rate graph and the volume graph*].	3 G
2		Julie:	Right.	
3	13:50	Daniel:	So, it has a negative slope. And then it starts going positive up to that point [*point E, July, on the rate graph*]. And so it levels off at zero [*point E, July, on the volume graph*].	3 G
4			Cause the v-, the v- [*pauses*] I don't know what you call that. The velocity of the flow of the water or something? The velocity of this is zero. Which is correct on our velocity chart [*pointing to the rate graph*].	1 V
5			And then it starts going negative again. And then it starts kind of sloping out. And it has, its greatest slope is right here [*point F, October, on the original graph*], so that's its inflection point [*point F, October, on the volume graph*].	3 G

Daniel primarily uses the third person to describe intervals on the graphs. His use of the first person in line 4 may indicate uncertainty about his own language in describing the shape of the volume graph. In line 3, Daniel describes the slope of the graph as negative. However, in line 4 Daniel hesitates, saying that *he* doesn't quite know how to call that which is zero. He uses "the velocity of the flow of the water," which is later shortened to "velocity," as in his use of "velocity chart" to refer to the graph of rate in line 4. As Daniel continues to describe the shape of the graph of volume, he uses the term "inflection point" in line 5. Daniel describes the location of the inflection point as the point with the greatest slope.

Jamie and Julie continue to question Daniel about inflection points, and Daniel develops different methods for explaining what an inflection point is. He begins to refer to maximum velocity quite frequently, reminiscent of language used in the Desert Motion Task (V). Daniel's explanations of inflection points also focus on his ideas about the shape of the graph (G) rather than events in the context of the reservoir. Discussion of the inflection point differs from discussion of other portions of the graph as Daniel repeatedly chooses to use first and second person pronouns.

In the next transcript Julie invites Daniel to again share his meaning for the term "inflection." Daniel hesitates in line 14, and then introduces the context of going down a playground slide in order to share his meaning. He draws curves (Figure 4) he explains the point of inflection to Julie.

13	19:13	Julie:	I don't think I understand the inflection idea.	1

14	19:19	Daniel:	Okay, so if you're, like, here's kind of an, idea, okay. So if you're drawing, a curve. Um, you just. Ah. [*pauses*]	2 D
15			Okay, so the inflection point is where the velocity is the highest, so, like,	3 V
16			if it, if you were like going on a slide and if you're falling down on it, your speed would be increasing, you'd be going like down really really quick-	2 S
17		Julie:	Um hm.	
18	19:48	Daniel:	Then at some point you'd start to level off. And where would your velocity be highest?	2 S V
19	19:53	Julie:	Right there.	
20	19:55	Daniel:	Yeah, like, before you start to slope off.	2 S
21			Yeah, that's the inflection point. Like where the velocity is highest.	3 V
22		Julie:	Okay.	

In line 15, Daniel again refers to maximum velocity to describe the point of inflection. In lines 16 through 20, Daniel introduces a context that involves velocity, or the speed of a person riding down a slide. He describes this person in the second person, inviting Julie to draw upon her own experiences in order to notice a feature of the graph that she may not have noticed before: the point of inflection. To justify having found the correct point, Daniel again returns to his third person reference to maximum velocity in line 22.

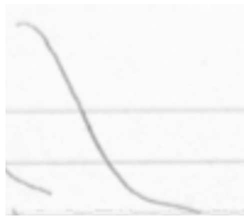

Figure 4

Daniel offers Julie another opportunity to draw on her own experience in order to notice the point of inflection as he is noticing it. Using second person pronouns, he invites Julie to *feel* what it is like to draw a point of inflection. He animatedly narrates the process of drawing in the first person.

23	20:05	Daniel:	Like if you were to draw, like a line, like you can kind of feel the point where you start to curve off.	2 D
24		Julie:	Right.	
25	20:13	Daniel:	Cause you're like drawing and AHHH!	2 D
26		Julie:	[*laughs*]	
27	20:14	Daniel:	Like I'm falling and then whew.	1 S

Daniel has consistently described a point of inflection as the point where the velocity is at a maximum. He has also found other ways to describe a point of inflection to his peers. Although members of the class have used the term "concavity" while describing points of inflection, Daniel has not referred to a point of inflection in terms of concavity. In the following transcript, Julie asks Daniel to clarify the connection between inflection and concavity.

28	20:17	Julie:	And so, when you were talking before the inflection point, did, did the concave, right? Did it start the concave or was that the point of-	2

29	20:30	Justin:	It was the point where they changed, wasn't it?	3 G
30	20:33	Daniel:	I don't remember the concave thing.	1

Any apparent neglect of the term "concavity" on Daniel's part may have to do with his admitting that he doesn't remember anything about it. The other participants at the table now take their turn to help Daniel to notice and describe a feature of the inflection point that he may not have noticed previously.

31	20:36	Justin:	Concave, like, because this-	3
32	20:37	Jamie:	[*as if reciting*] Concave up like a cup, concave down like a frown.	3 O
33	20:41	Justin:	[*laughs*] So concave down, like that, [*drawing a curve on Daniel's curve shaped like a parabola opening down*] that's concave down.	3 G
34		Daniel:	Um hm.	
35	20:44	Justin:	This is concave up [*drawing curve shaped like a parabola opening up, Figure 5*], so they-	3 G
36		Daniel:	Ohh!	
37	20:46	Justin:	-change at that point of inflection.	3 G
38		Daniel:	Ohhh! That's smart.	
39	20:48	Justin:	So that's what you're saying, the point, that's, the velocity's highest at that point of inflection on a displacement graph.	2 V
40		Daniel:	Oh, okay.	
41	20:58	Julie:	So it's when they change.	3 G
42	20:59	Daniel:	Ooh. Concavitivi-ness-us	

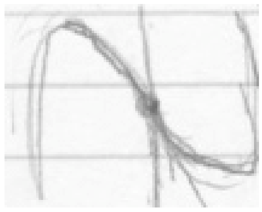

Figure 5

In line 32, Jamie draws on her on experience with her description of concavity. She refers to common objects in her experience, a cup and a frown, to describe the shapes of concave down and concave up through analogy. The tone in which she recites this analogy suggests that she has heard the rhyme before, perhaps during her experience in a previous calculus course.

Justin recognizes and expounds on Jamie's rhyme by drawing parabolic shapes on the graph that Daniel has drawn. Daniel expresses excitement as he notices the concavity of the graph that he has drawn.

Although he has not spoken a great deal in the transcript to this point, Justin reveals that he has been listening to Daniel's explanations of inflection point. He not only modifies Daniel's drawing to show him the relationship between concavity and the inflection point, but he also uses Daniel's words in line 39 to describe an inflection point as having the "highest" velocity, just as Daniel has been describing it.

Daniel is excited to recognize the change in the shape of the graph, and continues to trace the parabolic shapes that Justin drew. He creatively describes the concept of concavity in his use of the term "concavitivi-ness-us." Meanwhile, in her characteristically soft voice, Julie verbalizes the answer to her question in line 28 by describing a point of inflection as "when they [the concavity of an interval on the graph] change."

Discussion

In the data presented here, Daniel was able to describe a point of inflection to his peers without referring to concavity. His descriptions used language that referred to his personal experiences. Daniel developed a personal definition for a point of inflection as a point with maximum velocity. His definition used vocabulary from a previous task, a context within which all the participants had recently participated in much mathematical discourse. Daniel also participated in linguistic invention (Walter & Johnson, in press) to relate the concept of point of inflection to other personal experiences. He used first and second person pronouns to invite the other participants to share his experiences and notice the things that he was noticing. One of these experiences was enacted on the spot, as Daniel invited Julie to draw a curve and "feel" the point of inflection. Physical experiences has been found to affect students' representations of motion (Speiser & Walter, 1996), but this may be a rare example in which the actual creation of a graph becomes the shared experience in motion.

Daniel's experiences in describing his interpretation of a point of inflection may have prepared him to recognize and appreciate Justin's descriptions of concavity. In masterful exhibition of social speech, Justin not only explained how he viewed concavity, but also attempted to take on Daniel's point of view by basing his explanation in Daniel's own words and representations. By consistently inviting one another to share their personal interpretations of "concavity" and "point of inflection," and carefully listening, the participants were able to negotiate not only a common usage for conventional mathematics terminology but also identify relationships between mathematical concepts.

As students negotiate meaning for conventional mathematics vocabulary, they use personal experiences to build mathematical discourse. As demonstrated by the participants in this study, respectful listening, questioning and seeking to understand one another may better enable learners and teachers take on the point of view of others in social speech. We suggest that in order to learn mathematics, a learner must creatively use and negotiate meaning for vocabulary to be an active participant in mathematical discourse. This study extends conversations about how learners negotiate meaning for the purposes of participation in a broader discourse of mathematics.

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CONNECTING EFFECTIVE TEACHING AND STUDENT LEARNING USING THE PIRIE-KIEREN THEORY OF STUDENTS' GROWTH OF UNDERSTANDING

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In order to better understand the connections between effective teaching and student learning, two secondary mathematics teachers' uses of instructional representations are viewed from the perspective of the Pirie-Kieren theory of students' growth of mathematical understanding. Results indicate the need to examine what and how teachers communicate mathematical ideas during instruction and how that affects students' learning of those ideas.

The purpose of this qualitative study is to examine mathematics teachers' uses of instructional representations in order to better understand how theory of effective teaching can be connected to theory of student learning. Teaching is considered effective when teacher knowledge contributes to positive student learning outcomes (Ball, Lubienski, & Mewborn, 2001; Borko & Putnam, 1996; Fennema & Franke, 1992; Shulman, 1986; Wilson, Shulman, & Richert, 1987). Teachers' knowledge has a direct impact on their instructional practices and how effective those practices are for student learning (Fennema & Franke, 1992). However, the relationships among the knowledge components teachers use during practice that contribute to effective teaching, and thus to student learning, are not well understood (Ball, 2000; Grossman, Wilson, & Shulman, 1989; Shulman, 1986; Wilson et al., 1987).

The practice of teaching is a complex web of inter-related procedures in which teachers attempt to convey their knowledge of subject in such a way that students can grow in their understanding of that subject. This study focuses on the teacher practice of using instructional representations. Instructional representations can be described as the words, pictures, graphs, objects, numbers, symbols, and contexts (including examples, metaphors, and analogies) that teachers use during instruction to communicate abstract mathematical ideas to students (Berenson & Nason, 2003). They are the external representations created and/or used by a teacher during instruction. The concept of using representations as a means of communicating abstract mathematical ideas is a prominent focus of research in teacher knowledge and student learning (Borko & Putnam, 1996). The ways teachers use instructional representations connect what teachers know about mathematics and what they know about teaching it (Ball et al., 2001; Berenson & Nason, 2003; Wilson et al., 1987). Furthermore, teachers' uses of instructional representations provide a connection between effective teaching and student learning. This study uses a theoretical framework based on students' growth of understanding in order to better understand how teachers' uses of instructional representations may contribute to effective teaching and thus, to student learning.

Theoretical Framework

The Pirie-Kieren (1994) theory concerns a student's growth of understanding of a particular mathematical concept or idea whereby the processes of learning are dynamic, recursive, and nonlinear. The theory describes embedded layers that consist of eight potential levels in the growth of understanding for a specific student and a given topic or concept. This study considers only the first five activity levels: 1) primitive knowing, 2) image making, 3) image having, 4) property noticing, and 5) formalizing. The first three levels are based on informal and context-dependent levels of understanding and the next two levels are more formal and abstract levels. A key feature of the Pirie-Kieren theory also considered in this study is the activity of folding back, an activity that is vital to growth in understanding (Martin & Pirie, 1998; Martin, 2000; Pirie & Kieren, 1994). Folding back occurs when a student is faced with a problem that is not immediately solvable within the current level of understanding and reverts to inner levels of understanding. The Pirie-Kieren theory is used to examine the processes of mathematical learning, but this study adapts the theory to analyze the uses of instructional representations by two secondary mathematics teachers and how those uses may facilitate students' growth of understanding.

Methods

In this case study of teachers' uses of instructional representations, two secondary geometry teachers, David and Nancy (pseudonyms) were each observed on five occasions during classroom instruction and interviewed twice about their ideas concerning instructional representations, once before the observations and once after all observations were complete. David was an experienced teacher and Nancy was a novice teacher, both teaching honors geometry, at the time of data collection. Their uses of instructional representations during instruction were coded for their use within the activity levels adapted from the Pirie-Kieren theory.

Primitive knowing is defined by Pirie and Kieren as the knowledge a student brings to a setting. In this study, an instructional representation is used within this activity level if it is used to encourage students to review or recall concepts that have been previously taught by the teacher. In the Pirie-Kieren theory, *image making* occurs when a student uses actions to form mental images and in *image having*, the student can use those images without the need for the prior actions. These two activity levels are combined in this study to refer to informal and context-dependent uses of instructional representations. In the next level, *property noticing*, a student can focus on and connect properties of images. Instructional representations are used within property noticing if they are used to prompt students to notice particular properties of a concept and to convey to mathematical meaning. In the level of *formalizing*, a concept takes on a formal mathematical characteristic and a student can enunciate a formal definition or algorithm. In this study, using a formal instructional representation such as an applied algorithm or symbolic formula is considered to be used within the activity level of formalizing.

Results and Discussion

An emergent theme from the analysis of the observational data was the amount of instructional time each teacher used instructional representations within particular activity levels. David spent more time using instructional representations within the activity levels of image making/having and property noticing. David's instructional representations were used most often within the informal activity levels when introducing concepts. He encouraged students to move towards more formal activity levels when reviewing previously taught concepts. Nancy spent more time using instructional representations within the activity levels of property noticing and formalizing. Nancy's instructional representations were most often used within formal activity levels when introducing concepts and when reviewing previously taught concepts, her instructional representations were used within the informal activity levels. David used instructional representations within the formal activity levels when he encouraged students to fold back, while Nancy used informal and context-dependent instructional representations when she encouraged folding back.

Connecting effective teaching to student learning requires examination of what teachers do within the practice of teaching that contribute to students' growth of understanding. According to the National Council of Teacher of Mathematics, "effective mathematics teaching requires a serious commitment to the development of students' understanding of mathematics. Because students learn by connecting new ideas to prior knowledge, teachers must understand what their students already know" (NCTM, 2000, p. 17). The results of this study reveal the importance of examining how teachers communicate their knowledge of mathematics and of how to teach mathematics through their uses of instructional representations and also how they encourage students to fold back with their uses of instructional representations. Furthermore, future research needs to examine students' growth of mathematical understanding based on the instructional representations used by teachers during instruction.

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FACTORS INFLUENCING THE USE OF REPRESENTATIONS IN THE CLASSROOM

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In this paper, we describe a case study focused on the interplay between teacher listening and the use of representations while implementing a reform-oriented curriculum at the 5th grade level. We found that changes in a teacher's use of representations often coincided with student indications of uncertainty or misunderstanding and the teacher's attempt to listen to student thinking.

"The idea of representation is continuous with mathematics itself" (Kaput, 1987, p.25). More specifically, the ability to create meaning using mathematical representations is the essence of doing mathematics and communicating mathematically. Having a set of conventional or taken-as-shared representations allows members of a mathematical community to articulate patterns, relationships, generalizations and, together, to advance the state of mathematical knowledge. Thus, mathematics learners "must progress from idiosyncratic and ad hoc representations of particular problems to conventional, abstract, and general representations that function for a class of problems" (Smith, 2003, p. 264). Teachers, then, must fill the tall order of supporting both the development of individual understandings of mathematics, which may rely on idiosyncratic representations, and developing collective, taken-as-shared conventions and meanings in the classroom (NCTM, 2000). Reform-oriented curricula are designed to support teachers in this effort by including rich tasks that foster discourse around mathematical processes such as representation and generalization.

In this paper, we examine the role of the teacher in guiding both the individual and collective sense-making process using various representational forms. More specifically, the purpose of this study was to investigate the factors that influenced how a teacher used various representational forms while implementing a reform-oriented curriculum.

Theoretical Perspective

We adopt the definition of representation provided by Davis, Young, and McLaughlin (1982): "A representation may be a combination of something written on paper, something existing in the form of physical objects, and a carefully constructed arrangement of ideas in one's mind" (p. 23). This definition includes externally observable images such as physical objects, diagrams, tables, graphs, and equations, yet also acknowledges the role of the agent, or constructor of representations (NCTM, 2000; Monk, 2003). Given the interaction between internal and external representations, and the social context of the classroom, we view student mathematical learning through the lens of the emergent perspective as described by Cobb and Yackel (1996).

Investigating representations in a social setting requires an examination of how teachers and students interact while discussing representations. When students play an active role in discourse, teachers must continually interpret and respond to student ideas. Hence, the ways in which teachers ask questions, listen to students, and respond to students during instruction may influence their decisions about which representations to introduce and when to introduce them.

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Davis (1997) characterized three types of teacher listening: evaluative, interpretive, and hermeneutic—the first two being of interest for this study. When a teacher is engaged in evaluative listening, she is often listening *for* a particular response, rather than listening *to* the students. In this mode of listening, the teacher evaluates student responses against a preconceived standard for a correct response with the aim of leading students toward a particular learning goal. When a teacher is engaged in interpretative listening, she continues to listen *for* specific responses, but also is careful to listen *to* students in an attempt to interpret student thinking. Although, student thinking is valued and incorporated into the lesson, the trajectory is not greatly affected.

Method, Data Sources, and Analysis

A case-study methodology was used to examine how one fifth-grade teacher used representations when teaching. The teacher used a reform-oriented, NSF-funded mathematics curriculum and adhered to a philosophy that promoted student involvement and encouraged sense making. Three consecutive lessons on patterns were videotaped and transcribed.

The data were retrospectively analyzed using a data reduction approach (Miles & Huberman, 1994). Initially, all transcripts were coded in terms of the representation used (visual, tabular, symbolic) and the mode of listening (evaluative, interpretive) employed by the teacher. A protocol was developed to provide criteria for the classification of listening via prompts and cues observed in the video. In addition, special attention was paid to the context surrounding a change in representation or listening mode.

From this initial round of coding, schematics (such as figure 1) were developed to graphically depict the use of representation and listening for each episode of the observation. Researcher comments were added to describe the context surrounding changes in representation and listening. These schematics were then analyzed to identify the factors that appeared to induce a change in representation. A constant-comparative method (Glaser & Strauss, 1967) was applied to test and revise emerging patterns, which are discussed in the following section.

Results and Discussion

By analyzing the interactions among representations, student verbalizations, and teacher actions that indicated her mode of listening, we detected several patterns. Overall, we noticed that the teacher was often engaged in evaluative listening. During periods of evaluative listening, the teacher introduced representations in a very sequential manner. She listened *for* particular responses that propelled the discussion forward toward her goal of moving students from diagrams and physical objects to tables of values to symbolic representations. When the teacher engaged in interpretive listening, the sequential pattern of representations was often disrupted. Catalysts that prompted changes in listening and representation use were varied, but were typically associated with an apparent disagreement between teacher and student about how a representation was interpreted.

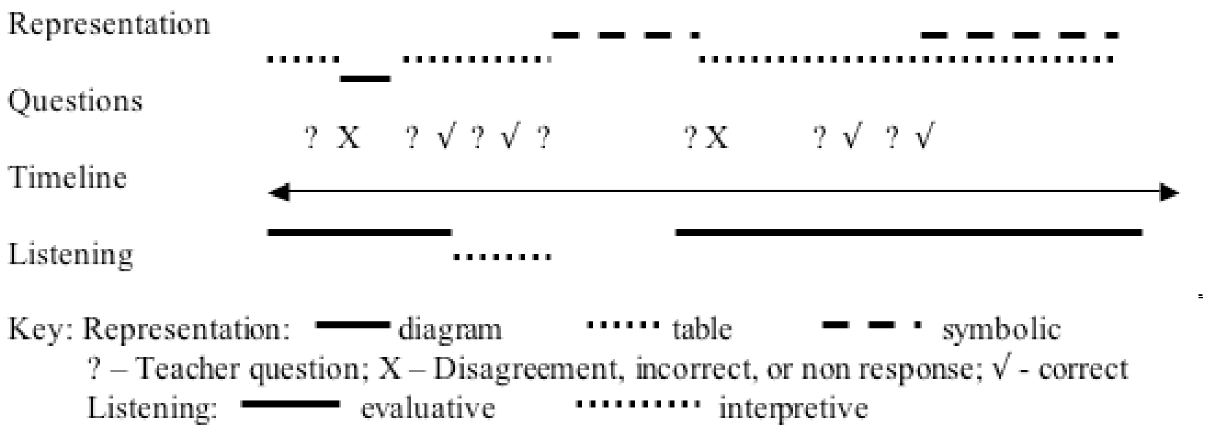

Figure 1. Representation and listening schematic

In the episode depicted in figure 1, the teacher, Emily, discussed a pattern in which three toothpicks were added at each step to extend a string of toothpick-outlined squares. Emily created a table to represent the sequence of steps in the pattern. She led a discussion about values in the table, and eventually led the students to a symbolic generalization.

Figure 1 illustrates the representations (diagram, table, and symbols) that were the focus of discussion during the episode. Because the teacher perceived these representations to be sequential (diagram to table to symbols), correct responses to the teacher's questions seemed to provide the teacher with the green light to continue discussing the current representation or to move up to the next level of representation. On the other hand, a disagreement either led to further discussion of the current representation or precipitated a change to a "lower level" representation.

Likewise, such disagreements and changes in representation corresponded to a change in Emily's mode of listening. After a disagreement, her listening became more interpretive. She offered students the opportunity to explain their thinking, which often required reference to a lower level of representation. Perhaps this downward trend occurred because the diagram, in particular, was a closer representation of actual context and thus facilitated sense-making in the situation.

Conclusions

The complex social environment of the classroom makes a teacher's job of facilitating the negotiation of mathematical norms and practices, while monitoring individual understandings, a difficult task. Listening *to* students can help teachers know when a change of representation may help students build a bridge from their own mental constructs to a shared understanding of how information is encapsulated in conventional representations. Although much previous research has focused on individuals' use of representations, this work considers issues affecting representational choices in the context of the teaching. As a result our work is well aligned with the goals of PME-NA and with the theme of the 2007 Conference, "Exploring Mathematics Education in Context."

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A COMPUTERIZED TIME-SAMPLING TOOL FOR OBSERVING ELEMENTARY MATHEMATICS CLASSROOMS

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Our research focuses on the development of first-grade students' conceptual and procedural knowledge in addition and subtraction. Mathematics educators (e.g., Ginsburg, 1997) have noted that the development of mathematical cognition should ideally be studied in relation to the instruction children receive in school. Our main research objective, therefore, is to describe children's thinking in addition and subtraction as it develops under various forms of mathematics instruction. To this end, we are presently engaged in a longitudinal project comprising two main data collection activities. First, we are interviewing first-grade children on their procedural and conceptual knowledge related to addition and subtraction and associated foundational concepts of place value and part-whole relations. Second, we are collecting detailed observational data of teaching practices in several first-grade classrooms. Our poster will focus on the software tool (called *Domains of Mathematical Teaching*, or DMT) we created to collect the observational data in these classrooms. Inspired by the instrument constructed by Scanlon and Vellutino (1997) to observe early literacy teachers, the DMT is a digitized time-sampling checklist designed to measure various aspects of teaching that have been shown (or hypothesized) to be related to the development of both conceptual and procedural knowledge (e.g., Hiebert & Wearne, 1996). These data will be used to create distinct instructional "profiles" that will be correlated with observed patterns of children's thinking over time.

Baroody (2003) presented a classification scheme for describing different approaches to mathematics teaching. His framework can account for instructional practices that vary from being primarily procedural to those that are conceptual or "investigative" in focus. Baroody's framework was thus a valuable guide for us as we designed the DMT; as such, the tool measures aspects of teaching, such as classroom organization, specific tasks in which the teacher and students are engaged, and the nature of the discourse occurring in the classroom, that can be aligned with the instructional profiles generated by the framework. Specific operationalizations of the components in Baroody's framework were gleaned from pilot data as well as other studies of mathematics teaching, namely TIMSS (Stigler et al., 1999), Hiebert and Wearne (1996), and the Study of Instructional Improvement (see Ball & Rowan, 2004).

Our poster will (a) describe the theory used to design the DMT, (b) address methodological issues with respect to the instrument, including sampling and the operationalization of the theory used to construct the DMT, and (c) present data collected with the DMT to describe the instructional profile of one first-grade teacher as an illustration of the power and versatility afforded by the instrument. The poster will exhibit portions of the DMT itself, which will entail the display of various screenshots illustrating the techniques used to capture different forms of mathematics teaching.

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CONNECTED MULTIPLE REPRESENTATIONS: A POSSIBLE TEACHING PERSPECTIVE?

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This poster presentation will describe preliminary results from a study that focused on the following question: How can instructors foster the use of multiple representations of mathematical ideas among their students as a tool for mathematical reasoning? The study examined the instructor's own beliefs and the translation of these into pedagogical strategies, as well as students' responses to these strategies.

A number of research studies have explored the connections between mathematical problem solving ability and the ability to translate between different representations of a mathematical idea. Most have reported a definite and positive correlation between the two (Gagatsis and Shiakalli, 2004; Janvier, 1987). These studies provide strong motivation to educators to examine the questions: Can teachers purposefully foster the ability to use diverse representations among their students? If so, how? Can students be taught to reason using different representations? It was with the intent of seeking answers to these questions that I embarked upon this study.

The students with whom I worked in this project were undergraduates studying pre-calculus. Traditionally, such college students have had some difficulties with algebra learning. It was this understanding of their background that led me to believe that such students were prime candidates for a study that focused not merely on honing their algebra skills but on consciously developing their abilities to reason through a variety of representations. The large number of studies that have focused on functions and their different representational forms also spurred me in my efforts.

The two main questions that I examined in this study were: What are the strategies that an instructor can employ with her students to foster the students' ability to use diverse representations? How can students be helped to use the diverse representations to reason mathematically? Thus, I was primarily interested in developing students' ability to use the different forms and the connections between them as a means to develop logical and coherent arguments while performing various mathematical tasks.

In this study, I documented the changes I made in my own instructional and assessment strategies. This metacognitive activity enabled me to examine my own attitudes and beliefs, regarding the use of multiple representations. The second part of the research project is an analysis of the documented work that is part of the coursework.

The poster includes:

(a) a rationale for the study
(b) examples that illustrate the instructor's attempts to foster the use of multiple representations by students, and students' responses to these efforts.
(c) preliminary results from the study

In this project, through these activities, I present findings to the pedagogical conundrums of how best to help students succeed in mathematics by offering them a repertoire of representations for different ideas that they study as tools that facilitate mathematical reasoning. Also, I present some strategies that foster connections between different representations.

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DO HUNGRY HIPPOS AND GREEDY ALLIGATORS EAT INTO ALGEBRA PERFORMANCE?

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Mathematical ideas are expressed using symbols that allow us to be brief and succinct in order to express ourselves economically (Pimm, 1995). Young children often have difficulty with numerals and letters that have a single orientation. The mirror-image *greater than* and *less than* symbols are easily reversed and cause confusion when introduced to elementary students. Thus it is popular wisdom for early elementary teachers to introduce the idea of a hungry animal with its mouth wide open for eating the larger number in order to distinguish between the inequalities. Cute. Simplistic. But is it good mathematics?

Pre-algebra and algebra students have difficulty with inequalities. An item analysis of algebraic skills on the Early Mathematics Placement Testing program (Rachlin, 2001) showed that students generally find it more difficult to solve inequalities than equations. It is certainly expedient to teach about hungry hippos and greedy alligators in the early grades, however, there is a question as to whether the conceptual underpinning needed for higher mathematics is served by this expediency. Is it possible for students to be successful if they learn about inequalities in the elementary grades using conventions that are mathematical (rather than zoological) and that use the salient features of the symbols to make sense of the inequality symbols? A preliminary step in finding out whether educators can improve students' understanding of inequalities, is to first explore conceptions of inequality symbols among in-service and pre-service teachers.

In wide-ranging discussions and written reflections with pre-service and in-service teachers ($n = 93$) on distinguishing between the inequality symbols, five common perceptions have emerged. The ideas are listed from most common to least common with explanations and commentaries found in the poster.

1. *Greedy Alligators or Hungry Hippos*: The wide mouth or symbol is directed in the correct direction toward the greater number. Referents are numbers.
2. *Spacing Comparison of the Symbols*: The greater number faces the wider opening in the symbol. The smaller number is closest to the point.
3. *Number Line Similarities*: The arrows at the ends of a number line are related to the directionality of inequality symbols.
4. *The Lazy L*: If a capital L is compressed from the top, it forms a misshapen inequality symbol for *less than*. The L is a mnemonic for *less than*.
5. *Reading from Left to Right*: As you read from left to right, the greater part is first; therefore that sign is greater than.

All students need to make sense of the mathematics they are learning. It is certainly expedient to teach ways to help students distinguish between the confusing inequality symbols. However, the results reported here show adults struggling for meaning when deciphering these symbols. Young children, and older students alike, should be given the chance to construct meaning for these symbols.

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ELEMENTARY FUNCTION IN CONTEXT: AN ELEMENT THAT HELPS IN EVALUATION OF UNDERSTANDING

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This study reports how 70 students working in small groups constructed and interpreted graphs of some elementary functions. The students who were in the second year of high school had received some class room instructions on the subject prior to de task. The study helps to evaluate the level of understanding of the students in polynomial functions of the type: $y = ax^n + b,$ where $a, b \in \mathbf{R}$, $a \neq 0$ and $n = 1, 2 \text{ y } 3$.

Theoretical Framework

Listed below are some of the elements that play important roles in the study; studying and understanding, working in small groups, functions, data representation and its conversion among others. A theoretical posture is taken in relation to the elements, giving brief explanation only on *understanding* in this section. Hieberth and Carpenter (1992) observed that an idea, a procedure or mathematical step is understandable if its mental representation is part of a representational network. According to the authors the the attempt or creation of an external representation by students reveals or speaks volume about the student's frame of reference –their internal or already existing representational network. They also considered that one product of understanding is an increment in transference.

Methodology

(i) A learning environment in line with that of Brandford, Brown and Cocking (1999) was designed. (ii) Class instruction was given to 70 students. They work on the functions especially on their graphical and algebraic representations laying emphasis on the conversion of the data representation. These were carried out strictly from mathematical viewpoint. Attention is paid to the global behavior of the graph. (iii) Instruments to measure students' understanding were applied. These consisting of 5 situations, with a total of 14 questions. The following areas were explored; qualitative and quantitative interpretation of the graphs at a point, in an interval and globally. The contexts of the study are: 1) the water consumption by two families, 2) distance traveled by two vehicles in a given time, 3) the variation or change in area of a rectangle (of a constant perimeter) as the length of one of its sides is varied and 4) the production of three pots of different sizes. The 5rd involved the drawing of graphs of a real life situation which is cost of transport fares of two journeys along a route of at least 12 km using 2 types of vehicles mostly used in Mexico City.

Results

In general terms working in small groups enabled each group to have "relative" success in the tasks. The average mark scored was 70%. Their performance which indicates how pertinent relations were drawn with other stated functions that were not contained in the instruction possibly indicates that the students applied their knowledge with some degree of "flexibility" in the face of "new" situations. In other words they were able to transfer or applied the knowledge earlier acquired. This according to Hiebert and Carpenter (1992) is a product of their understanding.

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MISCONCEPTIONS IN CALCULUS TEXTBOOKS

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A mathematics educator and a mathematician developed a theoretical framework for the analysis of text content potentially supporting the development of mathematical misconceptions in calculus textbooks. Sample texts were then analyzed and the framework refined.

Rightly or wrongly, high school and university textbooks continue to fundamentally influence classroom practice. Some effort has been put into content analysis and exploring the ways in which textbooks are used in classrooms and beyond (for example, Love & Pimm,1996; McCrory, 2006). However, rarely have we taken a really close look at what is in the textbooks, with the focus on *how* the material is presented and what kind of learning may be implied. In many cases, “Unscientific market research is chiefly used to determine content and approach” (Clements, 2007, page 55) in published textbooks.

Evidence suggests (Davis, 2001; Biza et al, 2005) that formed beliefs (situations in which new knowledge is incompatible with the prior knowledge) are very strong in one’s cognitive environment. It is important to identify these and to apply adequate means to overcome them. Textbooks can play a role in strengthening or dispelling students’ incorrect or synthetic models.

Using an initial framework, we examined a number of textbooks commonly used to teach calculus in Ontario, Canada, at the grade 12 and first year university levels, with the aim of examining how they potentially contributed to students' misconceptions. We found material in several textbooks that could support (or at best fail to correct) various such mathematical misconceptions. We then strengthened and broadened our framework based on this work.

Textbooks have the potential to significantly support or inhibit the quality of mathematics teaching and learning if they are used extensively. The fact that (mathematically incorrect) synthetic models can be quite robust means that a serious effort needs to be employed to devise strategies aimed at weakening the beliefs that inhabit them; textbooks are one means to do so. We offer a theoretical framework for identifying and analyzing textbook errors that have the potential to strengthen mathematical misconceptions, and believe it warrants further study.

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STUDENT INTERPRETATIONS OF A GRAPH OF EVERYDAY MOTION: AN ANALYSIS OF HOW JOINT FOCUS OF ATTENTION MEDIATES CONCEPTIONS

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This study analyzes how four pairs of middle school students collaboratively worked on interpreting a piecewise linear distance-time graph depicting everyday motion. Successful completion of the task called for a great deal of flexibility from the students. Questions about distance required students to attend only to vertical displacement, while questions about speed required students to attend to the ratio between changes in distance and time. We analyzed student interpretations of horizontal segments, canonically representing no movement. All pairs consistently interpreted segment "e" located on the *x*-axis as "stopped" or "not moving."[1] In contrast, when interpreting the other horizontal segments in the graph (located at various heights above the *x*-axis), three pairs shifted among contradictory interpretations (see Table 1).

Table 1: *Interpretations of horizontal segments during peer discussions*

	Session 1		Session 2	
Part of Graph	Moving	Not Moving	Moving	Not Moving
a	4	3	2	4
c	4	3	2	4
e	0	4	1	4
g	2	2	2	4

The numbers represent the number of pairs (out of four) that interpreted each segment as moving or not moving at least once during a peer discussion session.

We assume that the process of learning to interpret graphs involves students appropriating (Rogoff, 1990) the meanings and perspectives needed to accomplish this task (Moschkovich, 2004). Our analysis documents how the pairs of students generated and coordinated meanings for inscriptions and utterances, sometimes working from a shared focus of attention, and at other times failing to achieve a shared focus of attention. Changes in students' joint focus of attention may explain their shifting interpretations of horizontal segments in the graph. While responding to some questions, students' joint focus of attention was on the height or length of a particular horizontal segment. In contrast, the pairs who established a joint focus of attention on the slope of the lines consistently interpreted all horizontal segments as stopped.

Previous studies of students' initial conceptions about rate (Thompson & Thompson, 1996; Thompson, 1994; Thompson & Thompson, 1994; Lobato & Siebert, 2002; Lobato & Thanheiser, 2002) help us contextualize these findings and provide evidence of persistent student conceptions in related problem contexts. For example, Lobato et al. (2002) relate this difficulty in viewing rate as a measure to the challenge students face while working with intensive quantities. Our data provide similar findings about students' conceptions while working on the task of interpreting and using intensive quantities (i.e. the slope of a segment) in distance-time graphs.

Analyzing students' conceptions in terms of joint focus of attention can shed light on ongoing questions regarding the origin of student conceptions while interpreting graphs depicting distance-time relationships. Highlighting the role of joint focus of attention also shifts the analysis from individual conceptions to the social dimensions of cognition, and to the critical role of shared meanings and coordinated perspectives in the process of appropriation.

Endnote

1. With the exception of one pair who made one utterance about "e" meaning "moving really slow," but who otherwise consistently interpreted "e" as stopped throughout the remainder of the discussion session.

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A SPECIAL KIND OF MISUNDERSTANDING: MATHEMATICAL RELATIONS IN GEOMETRY

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In this paper, we examine elementary pre-service teachers' understanding of the phrase ``is a special kind of'' as in ``a square is a special kind of rectangle.'' Data collected from 109 students indicated that they possess varying interpretations of the phrase. Our findings suggest that difficulty with shape categorization may be related to the meanings students attach to such phrases.

Several studies have suggested that students have difficulty understanding the equal sign (e.g., Knuth, Stephens, McNeil, & Alibali, 2006; McNeil et al., 2006; Baroody & Ginsburg, 1983; Kieran, 1981; McNeil & Alibali, 2005). These studies indicate that students at all equivalence (relational). Such a conception of the equal sign is a hindrance to understanding levels commonly have an *operational* rather than a *relational* understanding; that is, the equal sign indicates the results of a computational process (operational) rather than mathematical important mathematical and scientific ideas (McNeil & Alibali, 2005; Knuth et al., 2006). Herein we present data from a study of pre-service teachers to argue that this lack of relational understanding may be a symptom of a larger issue. Students may have difficulty understanding mathematical relations in general.

In this paper we examine a different mathematical relation; that represented by the phrases “special kind of” and “is” when categorizing geometric shapes. For example, the phrases “a square is a special kind of rectangle” and “a square is a rectangle” indicate a mathematical relation on the set of quadrilaterals. The relation “is” is reflexive (e.g., a square is a square) and transitive (e.g., a square is a rectangle, a rectangle is a parallelogram, thus a square is a parallelogram) though it is not symmetric (e.g., a rectangle is not a square).

More formally, the statement “a square is a rectangle” means that the set of squares is a subset of the set of rectangles. The statement “a square is a special kind of rectangle” means that the set of squares is a proper subset of the set of rectangles. To generalize, we have A *is* B if and only if the set of all A is contained in the set of all B; A *is a special kind of* B if and only if the set of all A is *properly* contained in the set of all B. Thus “is” is reflexive but “is special kind of” is not. For our present purposes, we do not focus on the distinction between “is” and “is a special kind of” though it is quite possible that students interpret these two expressions differently.

Theoretical Framework

Our literature search has not revealed any studies which explicitly explore students’ relational understandings of mathematical relations other than equality. Although much research has been done concerning students’ understanding of shapes it is not of direct relevance here. Any of the many studies which are the descendants of van Hiele’s levels of geometry would provide us with a framework for talking about students’ understanding of geometric shapes (van Hiele, 1985). Other studies have noted students’ use of prototypical images or metonyms to represent whole classes of objects (e.g., Presmeg, 1992; Landau, Smith, & Jones, 1998). For example, a square may not look like a student’s idealized version or prototype of a rhombus; she or he may not acknowledge that squares are special kinds of rhombi.

However, categorizing shapes may be difficult for reasons beyond deficiencies in geometric understanding and protyping. Students may have difficulties with the relations “is” and “is a special kind of”. Our study is thus more aligned with the studies on students' understanding of the equal sign cited above than with those which discuss geometric categorization.

The connection which we draw between the equal sign and the relations of geometric classification is based upon their co-categorization as relations by the mathematics community. We do not intend to argue that student understanding of mathematical topics can be mapped according to the definitional boundaries established by mathematicians. Rather we use this co-categorization as a theoretical and an analytical tool. It allows us to draw comparisons between difficulties with geometric relations and with equality. It adds an extra dimension of analysis for students' experiences with shape classification.

Methods

Identical pre- and post-tests were given to 109 students in a mathematical content course for pre-service elementary teachers at a large public southwestern university in the U.S. The test was about categorization of quadrilaterals and was part of a larger research study about students' argumentation and reasoning patterns. In between the pre- and post-tests was a week of instruction centered around a Geometer's Sketchpad activity in which students explored and described subsets of quadrilaterals. An observation emerged from the pre- and post-test data; students may have difficulty interpreting the phrase “is a special kind of”. The scope of the present analysis is students' misunderstanding of the relation “is a special kind of”. We will not make comparisons between pre- and post-test nor will we examine the role of instruction.

Results

We identified multiple classes of student answers which indicate misunderstanding of the phrase “is a special kind of”; in the remainder of this paper we describe two such classes. The students were all responding to the question, “True or False? A square is a special kind of rhombus. Give a reason for your answer.” In the first class of answers, students indicated recognition of the phrase but answered false because they associated the phrase with a different pair of geometric objects (e.g., a square and a rectangle). They may have seen “special kind of” as an exclusive relation; a square cannot be a special kind of both a rectangle and a rhombus. In the second class of answers, students answered false because they reasoned with an inverted definition of “special kind of”. That is, they suggest that A is a special kind of B $\Leftrightarrow$ $\{B\} \subset \{A\}$ rather than $\{A\} \subset \{B\}$. Readers should note that many students answered false for reasons other than those we will examine (e.g., limited concept image of a rhombus); our focus here is on difficulties students may have with the phrase “is a special kind of” and whether it is appropriate to compare this to misunderstanding of the equal sign.

Examples of Exclusivity

Some students argued that a square could not be a special kind of rhombus because a square is a special kind of something else. Examples of their reasoning include “a square is a special kind of rectangle” (pre-test) and “a square is a special type of parallelogram” (post-test, different student). Both of these statements are, in fact, true. The students here may consider “special kind of” to be an exclusive relation; a square can only be a special kind of one thing. One student who did not believe that a square is a rhombus reasoned: “I may be confused but I thought it was a special kind of parrallelogram [sic]” (post-test).

Another student answered that a square is not a special kind of rhombus because “a rectangle is a special kind of rhombus” (pre-test). A rectangle is indeed a special kind of rhombus only when that rectangle is a square. Putting aside the fact that a rectangle is not a special kind of rhombus, this perhaps indicates that some students are not comfortable with the transitivity of “is a special kind of” as a relation. If a square is special kind of rectangle (assuming the student knew this) and a rectangle were actually a special kind of rhombus then, by transitivity, a square is a special kind of rhombus.

Examples of an Inverted Definition

Several students gave responses which indicated a faulty understanding of “is a special kind of”. They answered that a square could not be a special kind of rhombus because a square has all the features of a rhombus plus a few others. In a sense, they reversed the set containment that is part of the formal definition of “is a special kind of”. Such students argued that a square is not a special kind of rhombus because “a rhombus is a special kind of square”, “A square is not a rombus [sic] but a rombus [sic] is a square... square always has 90° angles”, “A square also has four Right angles”, and “a square has 90° angle and equal sides, a rhombus doesn’t”; each of these responses is from the post test.

Discussion

Students’ incorrect answers to the question “True or False? A square is a special kind of rhombus. Give a reason for your answer.” reveal that many of them may not understand the mathematical relation “is a special kind of”. By focusing on this mathematical relation we are able to identify an obstacle for students other than proficiency with the geometric shapes. Furthermore, relations come with a set of vocabulary which may have potential as an analytical tool.

Students who have difficulty classifying geometric concepts may misunderstand the mathematical relations which frame tasks and instruction. This issue shares features and implications with student understanding of the equal sign. Both “is a special kind of” and the equal sign involve student interpretation of a mathematical relation. Misconceptions about these relations have implications for student learning and understanding. We have not drawn the same distinction between operational and relational understanding that is relevant to the equal sign. Rather we have noted two classes of misconceptions about “is a special kind of”, exclusivity and an inverted definition.

It is unlikely that students make connections between the equal sign and geometric relations such as “is”. However we have connected them by virtue of their co-categorization in formal mathematics as relations. There are perhaps opportunities for educational research and for developing descriptive frameworks in the space between mathematicians’ and students’ ways of thinking.

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GEOMETRY TEACHERS' PERSPECTIVES ON CONVINCING AND PROVING WHEN INSTALLING A THEOREM IN CLASS

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This paper advances understanding of instructional phenomena by focusing on instructional situations. We argue that the decisions in managing a situation require a teacher to respond to norms by negotiating dispositions that might contradict each other. We illustrate this by examining the installing of theorems in high school geometry.

Recent research on mathematics learning has called attention to the nature of the situation that serves as context for that learning (Brousseau, 1997; Lave & Wenger, 1991; Schoenfeld, 1998). In our research, we conceive of classroom life as organized by recurring *instructional situations*: frames that allow teacher and students to exchange the work they do for claims on the stakes of teaching and learning. Decisions and actions made by teacher or students not only result from individual thinking or belief, but also respond to the norms of the instructional situation in which those decisions and actions are made. Depending on the instructional situation, different actions may be normative. For example, in the situation we've called 'doing proofs' in high school geometry classrooms, it is normative for the teacher to state the proposition to be proved in terms of a specific diagram (Herbst & Brach, 2006), but in the situation of 'installing' a new theorem, it is normative for the teacher to state a theorem in terms of abstract concepts (Herbst & Nachlieli, 2007).

In this paper, we advance understanding of instructional phenomena by focusing on the norms of those instructional situations as they relate to two different activities which have traditionally been studied in their cognitive and epistemological dimensions: convincing (or bringing someone to a state of belief) on the truth of a statement (Harel and Sowder, 1998) and proving (or establishing the truth of a statement for a given community of knowers; Balacheff, 1987). We argue that a teacher's management of those activities requires them to respond to norms of the instructional situations where those activities occur. A teacher's response to those norms is constructed by decisions and actions that articulate various dispositions that might contradict each other. We illustrate this point examining further the situation of installing theorems in the high school geometry class and drawing from teachers' responses to an animated representation of the teaching of a theorem about medians.

Teaching in Classroom: Practical Rationality

Classroom instruction relies on a tacit contractual relationship vis-à-vis the knowledge at stake: the teacher teaches that knowledge to students, the students study that knowledge with the help of the teacher, and the teacher attests to the students' learning of that knowledge (Brousseau & Otte, 1991). That contractual relationship is more specific (in terms of who can or has to do what, when, how, and to get what) depending on the particular kind of symbolic goods that are at stake. The solving of an equation in algebra, and the doing of a proof in geometry, for example, are both activities that require students to lay down a reasoned sequence of statements. But they differ in terms of the role of *reasons* in those sequences, and in the extent to which those reasons need to be explicitly laid down alongside the statements. What is at stake in both situations is not just a claim on the final statement in the sequence (the statement of what x equals, or the statement of the conclusion to be proved) but also a claim on knowing the "method" of solving an equation in one case, and a claim on knowing "how to do a proof" on the other (Chazan & Lueke, in press; Herbst & Brach, 2006). With the expression "instructional situation" we name each of the various frames that enable teacher and students to bill stretches of classroom work on account of the objects of knowledge they have contracted about. We model situations as systems of norms that organize those transactions. By "norm" we mean a central tendency around which actions in instances of a situation tend to be distributed. We posit that those (mostly unspoken) norms shape a teacher's and her students' actions: As they participate in an instructional situation, they hold themselves and each other accountable for responding to the presumption that they should abide by those norms.

We are interested in the situation of "installing theorems", namely the system of norms that regulates the work teacher and students need to do in order to be able to take for granted that the class knows a specific theorem. We explore two norms that we hypothesize to be characteristic of the situation "installing theorem": (1) students should come to believe the statement asserted by the theorem is true, and (2) for a statement to be a theorem it has to be provable. We investigate how teachers manage their way about those two norms as they act and make instructional decisions about the teaching of a theorem.

As the discipline of mathematics accrues its knowledge, the capacity to show that a proof exists is the sole grounds on which the mathematical community comes to officially believe the truth of a statement (Lakatos, 1976). But a mathematician's belief on the plausibility of a result often hinges on other means (e.g., Pólya, 1954). Mathematical proof is only one of the strategies that might obtain ascertainment and conviction on the part of individuals (Harel & Sowder, 1998). Our focus is neither on the discipline nor on the individual, but rather on the public work done in the classroom. In the classroom, teachers are accountable to mathematics as they propose that a statement is a theorem and also accountable to students as they expect students to take such statement as true. It is thus reasonable to hypothesize the two norms listed above as having a hold on the way the teacher goes about her work teaching a theorem. But to say these norms regulate that work means not that they determine or dictate what a given teacher does; rather a given teacher constructs original actions in response to, against the backdrop of, those norms. The value of modeling instructional situations as systems of norms is that it allows us to study the resources with which teachers construct those actions.

We propose that to construct original actions in response to those norms, teachers, in particular, make use of a *practical rationality*: a system of *dispositions*, categories of perception and appreciation that allow them to notice and value possible actions (Bourdieu, 1998; Herbst & Chazan, 2006). This paper explores the practical rationality that geometry teachers invest when handling the two norms: that students need to come to believe the statement of a theorem, and that for a statement to be a theorem it needs to be provable.

"Convincing" and Proving

In a previous analysis (Miyakawa & Herbst, 2007), we identified the term "believe" as one used by teachers to describe a desirable state of affairs for students vis-à-vis a true statement. We use the term "convincing" to describe the work a teacher might do to make students "believe" a statement is true. Clearly, this "convincing" could conceivably be done in several ways, ranging from mere appeal to authority ("trust me") on one extreme, to the organization of an adidactical situation of validation (Brousseau, 1997) on the other. One conceivable way of convincing a class could be by engaging the class in proving the statement. Is the engagement of students in proving a statement a viable way for a geometry teacher to convince students of the truth of a statement? We seek an answer to that question based on the rationality that teachers invest when they do their work.

Before proceeding, we clarify the difference between convincing and proving. In both processes, the statement dealt with is the same. However, the end products each process seeks for that statement, proved and believed, are different. Duval (1991) uses the expressions “logical value” and “epistemic value” to describe the different values attributed to a statement. The former expresses the mathematical value of true, false, or unknown. Independent of this, Duval proposes "epistemic value" to represent “the degree of certitude or conviction attached to a statement” by an individual, often expressed with words such as “probably,” “impossibly,” “certainly,” etc. *Convincing* is a process of attributing or modifying an epistemic value to a statement; *proving* is a process of attributing a logical value. Of course, it is conceivable that proving might also help to convince.

The notion of “cognitive unity” has been proposed to gauge the relationship between conjecturing and proving theorems in classroom activity (Garuti et al., 1998). The notion that there exists “cognitive unity” characterizes the case in which there is continuity between the two processes of conjecturing and proving. This continuity is visible, for example, in the use of the same arguments during conjecturing process and proving process (Garuti et al. 1996, p. 113). We use “cognitive unity” to examine the relationship between convincing and proving in terms of the geometric objects students might be asked to work with and how they might be asked to work with them. The data that we use to examine our question on practical rationality consists of teachers’ reactions to a teaching episode where a lack of continuity in cognitive unity is described.

Method

To pursue our interest in the practical rationality of teachers of geometry, seen as a collective, our study gathers data from groups of experienced teachers of high school geometry who confront together representations of teaching that showcase instances of installing a theorem (see Herbst & Chazan, 2006). We use a novel technique to gather data on teachers’ practical rationality. We create stories of classroom interaction and represent those stories as animations of cartoon characters. These characters interact in ways that might or might not be common in American geometry classes. They showcase instruction that straddles the boundaries between what we hypothesize to be normal and what we expect practitioners might consider odd. The representations of teaching are shown at monthly meetings of experienced teachers. In the discussions that ensue participants point to odd or intriguing moments in a story, suggest alternative, possible stories, or bring concurrent stories of their own collection. We focus on discussions of a story called “Intersection of Medians,” and present a result of the analysis on the transcription of the teachers’ discussion.

Analysis of “Intersection of Medians”

“Intersection of medians” deals with a theorem about the centroid of a triangle: the intersection of the medians of a triangle, when joined to the vertices of the triangle, forms three triangles of equal area. The installation of that theorem in the animation is hypothesized to be odd for the following reasons. The teacher draws a diagram on the board showing the intersection of the medians and the three triangles and asks students, “What do you conjecture the theorem will say about those triangles?” Students are not given an opportunity to measure dimensions and calculate areas. Merely looking at the diagram from their seats, they grope from claim to claim until they succeed, creating the impression that they are using the teacher’s verbal responses to prior guesses, “Close but not quite… Anybody else?” as resources to improve a conjecture. Once the students hit at the right conjecture, the teacher affirms it ("They have the same area, that's right") and proceeds to produce a proof by himself, requiring only limited participation from students. Empirical verification of the theorem is prompted *after* proving by the teacher.

We expected that experienced practitioners would react to the sequence of events in that story. Particularly, we expected that the temporal displacement of the empirical activity to after the proof would be denounced as depriving students of a resource for conjecturing. We also expected that the location in time of this empirical activity would be denounced as potentially bringing to question the capacity of the proof to build conviction.

We suppose that the measurement of area in each of the target triangles (ΔAOB, ΔAOC, and ΔBOC) would normally be done to convince students that those areas are equal in spite of looking different. Typically a student would measure the area by choosing bases and drawing corresponding altitudes for each triangle, measuring those, and then calculating each area separately using the area formula. There are two options for the choice of bases: either the three sides AC, AB, and BC, or the three internal segments AO, BO, or CO. The choice of a base automatically determines the altitudes. In contrast, the proof given in the animation (Fig. 1) requires students to consider not only the three target triangles, but also the mid-size triangles (e.g., ΔAXB), and the small triangles (e.g., ΔAXO). And to prove that the areas of two target triangles are equal in this proof, it was not necessary to consider the bases and heights of these triangles at all — just the bases and heights of the mid-size and small triangles. The comparison of areas of the target triangles is done by subtracting the areas of the small triangles from the mid-size triangles, not considering any multiplicative relationship between heights and bases in the target triangles. Thus this story represents a continuity gap between the convincing and proving processes in terms of the objects used (triangles, altitudes, and bases). Whereas the conception of equal area at play in the proof is one that deals with area of figures as quantities, the one used in the convincing activity deals with area of figures as a number produced out of multiplying measures of quantities (see Herbst, 2005).[1]

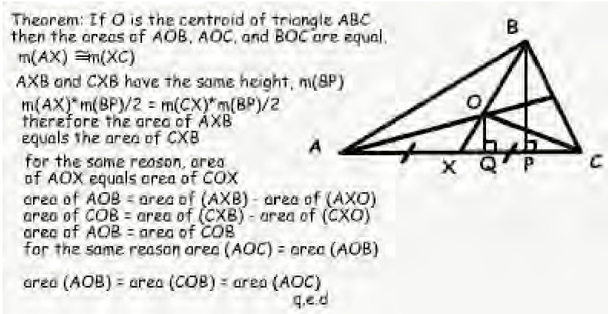

Fig. 1. Proof given in the animation

Results and Discussion

Miyakawa & Herbst (2007) identified the disposition "*it is desirable that students believe the truth of a statement before proving it.*" Participants of study group reacted to the animation by arguing that the conjecturing process did not allow students to "come to believe" the truth of the theorem, and proposed as an alternative that the measurement of areas take place *before* proving. This indicates that teachers value students' thinking to the point that they may spend time on (possibly empirical) work to build conviction of the truth of a claim. We now identify dispositions on the relationship between convincing and proving.

Negative Attitude toward a Gap between Convincing and Proving

Participants identified a gap between measuring and proving. They assumed students would measure the areas of target triangles using the sides of ΔABC as bases.

Ester I'm not sure it [measurement] would give them ideas about proving. I think it might confirm to them that it's, whether it's true or not but I'm not sure, I'm not sure if it would help them get any ideas about how to prove.

Megan Yeah, one drawback though I see, I didn't think of this before, but if he had them do it [measurement] at the beginning, then they'd be looking at those alti—those different altitudes. […] So then you've sort of steered them in the wrong way for the proof. […] When you're doing the proof, you're not really looking at those triangles.

For Megan, the measurement activity constitutes a "drawback," because the objects used in measuring — in particular the altitudes and bases of the three target triangles — are different from the objects used in the proof. This is the gap we mentioned above in the analysis of the animation. She also expresses misgivings ("steer them in the wrong way") that might make her reconsider the measurement activity or the proof itself. The disposition we may identify here is "*it is not desirable for a the teacher to steer students in a wrong way*." This seems to recommend a different course of action than the disposition about conjecturing. On the one hand, since it is desirable that students believe a theorem true before proving it, a measuring activity seems to be advisable. On the other hand, since a teacher should not mislead students, a measuring activity that does not involve the ideas to be used in the proof (or, worse, one that suggests using different ideas) might not be best. These two dispositions suggest a tension in the teaching of this theorem: should the proof be a different one to reinforce what the measurement achieved, or should the measuring activity be left for later to avoid steering students in a direction opposite to that of the proof? And what if the measurement produced areas that were slightly unequal (as happens in the movie)?

Alternatively, a teacher might choose another proof, one that had more cognitive unity with the measuring activity (see endnote 1). This might allow the teacher to avoid the tension between those dispositions. But it might require students to invest different knowledge (e.g., the capacity to look at the same triangle from different points of view, or knowledge of the fact that the centroid divides a median in a 1:2 ratio). We hypothesize that another disposition, such as "*it is desirable that the conceptual complexity of a proof be kept under control,*" is active at the moment of deciding which proof to use, and that might militate against those alternatives. The outcome of such tension might be the decision not to undertake the proof of this theorem at all. Indeed, while teachers acknowledge that every theorem should be provable and could be proved, they also report as a matter of course that not every theorem is actually proved (Miyakawa & Herbst, 2007). The presence of competing dispositions touching on matters of conviction of the truth of statement, cognitive unity, and conceptual complexity, might help explain why some theorems are not proved.

Positive Attitude toward a Gap between Convincing and Proving

We also identified in the same study group session a positive attitude toward the gap between measuring and proving. In particular, Karen liked the proof given in the animation:

Karen That's what's really cool about the proof. Is that you're proving something, you're proving something about these areas without ever finding the base and the height of these other triangles. […] Wait how'd you do that? It's magic.

Later in the same session, she mentioned again the benefit of this particular proof.

Karen […] How does someone come up with this incredible idea to measure, to figure out that these are the same by doing some sort of subtraction deal? And in that way, you are able to highlight what a beautiful proof it is. You know, I like proofs that go around to the back door.

First of all, for Karen, the proof (Fig. 1) is "really cool" and "beautiful." One criterion of beauty visible in Karen's comment is that the proof can be accomplished without using the objects necessary for the measurement ("without ever finding …"). That is to say, she appreciates the discontinuity between measuring and proving ("magic"), contrary to the disposition noted previously which assigns negative value to that gap because it could cause students' difficulty. From Karen's comment we propose the disposition that "*it is desirable for a teacher to show a beautiful proof.*" This disposition supports the use of measurement in the process of convincing, since it helps bring home the point that the proof predicted such result. But Karen did recognize a possible difficulty students might encounter. The following excerpt of transcripts shows that in spite of the difficulty, the gap is worth considering.

Karen It's a roundabout proof. So like that you can't, like if you're stuck on a proof, and you can't figure out how to prove the next step, you back off and go around to other ways to find it. Whereas what happens with the kids is they get stuck and they just, you know, they'll quit or they're, ye--, or they'll yell. But they don't do a whole lot of thinking about how else could I look at this? And we have to keep getting kids to look at how else could I see this? […]

Karen anticipates that students might encounter difficulty and might simply give up on a proof. Despite of this possibility, she allocates value to the process of exploring an argument when students encounter a difficulty in proving: "we have to keep getting kids …" We may identify here another disposition that could be a reason for the former disposition about "beautiful" proof, why it's worth teaching: "*it is desirable for students to bridge the gap between conviction and proof.*" This supports the use of measurement and the proof, and takes into account not only students' difficulty but also the process of overcoming it. It seems that this disposition relates more generally to problem-solving or strategic skills that students need to draw upon when they encounter a difficulty. The need to bridge the gap between what convinces students of the truth of a statement and the proof that such statement is true might be a place in which to develop those skills.

Summary

Different dispositions mentioned in this paper can be summarized as follows (see also Miyakawa & Herbst, 2007). They are activated in response to a representation of teaching in which a gap in the cognitive unity between measuring and proving is identified.

1. *It is desirable that students build conviction of the truth of statement before proving*
2. *It is not desirable for a teacher to steer students in a wrong way*
3. *It is desirable for a teacher to show a beautiful proof*
4. *It is desirable for students to bridge the gap between conviction and proof*

These dispositions sometimes conflict with each other and push teachers to value different actions in teaching. The disposition (2) could push a teacher to avoid proving a particular theorem, so as not to mislead students. On the other hand, (3) and (4) could push a teacher to engage students in appreciating a "beautiful" proof and valuing the work involved in bridging a gap. The proof is indispensable in this case. We may also find that (2) focuses more on students' understanding of the given proof, whereas (3) and (4) focus more on the teaching of specific aspects of proving. We could understand therefore that the matters of conviction of a truth of statement, difficulty caused by the gap of cognitive unity, "beauty" of proof, and problem-solving skill can be taken into account when installing theorems. The participants in our study group allocated various values to these aspects of teaching.

All of these dispositions are not necessarily specific to the situation of installing a theorem even though they bear on the desirability of giving a proof and on the desirability of a particular proof. (1) is apparently specific to the instructional situation of installing theorems. (2) is a more general clause of the didactical contract that makes the teacher responsible to teach true knowledge to learners; it is only activated in this situation by the concomitant identification of a gap between measuring and proving. (3) might help teachers make decisions related to proving not only when installing a theorem but also when choosing problems where students are expected to do the proof. All of them help a teacher navigate the installation of a theorem.

Notes and Acknowledgements

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[1] There are some alternatives for proving this theorem, which might have more continuity in terms of the triangles used between measuring and proving. For example, if the segments AO, BO, and CO are seen as bases of target triangles, any two of three triangles always have the same height and base (e.g., the segment AO as a base of ΔAOB and ΔAOC).

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CONCEPT MAPS, INTERPRETIVE ESSAYS, AND THE DEVELOPMENT OF GEOMETRIC THINKING WITH RESPECT TO AXIOMATIC SYSTEMS

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This study explores the use of concept maps and accompanying essays as a means to foster the development of higher levels of geometric reasoning with respect to finite, neutral, Euclidean, and non-Euclidean geometries and characterizes the journey of preservice secondary teachers toward an integrated understanding of several axiomatic systems.

The combined use of concept maps and interpretive essays can be effectively used to enhance and assess higher levels of geometric reasoning. While the concept map provides a graphical representation of the mathematical connections perceived by the student, the interpretive essay expands upon these relationships and focuses on communicating mathematical ideas in writing. By relying on two avenues of expression, schematic and verbal, students can explicitly communicate aspects of their knowledge of mathematical concepts and/or topics, both strengths and weaknesses, in a more extensive manner. The current study is based on research related to the importance of developing a knowledge-base in which mathematical terms and topics are viewed as an integrated whole rather than as isolated pieces of information (Bishop, Clements, Keitel, Kilpatrick, & Laborde, 1996; Hiebert and Carpenter, 1992; NCTM, 2000); the use of concept maps to enhance and assess knowledge in mathematics (Bolte, 1999; Williams, 2002); the van Hiele levels of geometric thought (Fuys, Geddes, and Tischler, 1988); and studies related to preservice secondary teachers exposure to Euclidean and non-Euclidean geometry (Blair, 2004).

Method of Inquiry and Data Sources

Data for the current study consist of three drafts of a concept map constructed throughout the term and a final concept map and interpretative essay from a survey of geometries course. Qualitative analysis of the data served as the basis for rich descriptions of students who demonstrate different levels of axiomatic reasoning as categorized by the van Hiele levels of geometric thought — from limited facility working deductively within the Euclidean system (beginning Deduction) to ability to compare and contrast different axiomatic systems in a meaningful way (Rigor).

Concept maps were based on 36 instructor-generated terms and evaluated holistically with respect to organization (i.e., depth and quality of links among various terms; clarity and quality of clusters of related terms; quantity of terms used) and accuracy (i.e., evidence of inaccuracies, misconceptions, and omissions). Accompanying essays were evaluated with respect to level of communication (i.e., representation of interpretations and/or understandings of mathematical ideas) and organization (i.e., level of integration of understandings). Trends, patterns, and common errors that emerged from student work form the basis of descriptions of levels of geometric reasoning with respect to different, but related, axiomatic systems.

Results and Conclusions

Results indicated concept maps and accompanying essays provide a valuable means of enhancing and assessing the development of geometric reasoning with respect to various

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axiomatic systems. General descriptors of the van Hiele levels, as noted by Fuys, Geddes, and Tischler (1988), were used to characterize levels of geometric reasoning illustrated by participating students. At the Deduction Level these include establishes interrelationships among networks of theorems, examines effects of changing an initial postulate in a logical sequence, establishes a general principle that unifies several different theorems; at the Rigor Level descriptors include explores how changes in axioms affect the resulting geometry, and establishes theorems in different axiomatic systems, and compares axiomatic systems.

Analysis of student work indicated three levels of depth of understanding and integration of related topics that span the Deduction and Rigor categories. At the most basic level (**Figure 1**), students were able to categorize different geometries according to the parallel property (i.e., no parallel lines, unique parallel lines, multiple parallel lines) that was either stated as part of the axiomatic system or deduced from the axioms. Most students correctly distinguished between finite and infinite geometries, with the exception of Young's geometry. The most common misconception noted was inferring a system contained an infinite number of points because the existence of a specific number of points was not stated in the axioms and the use of non-specific linking words.

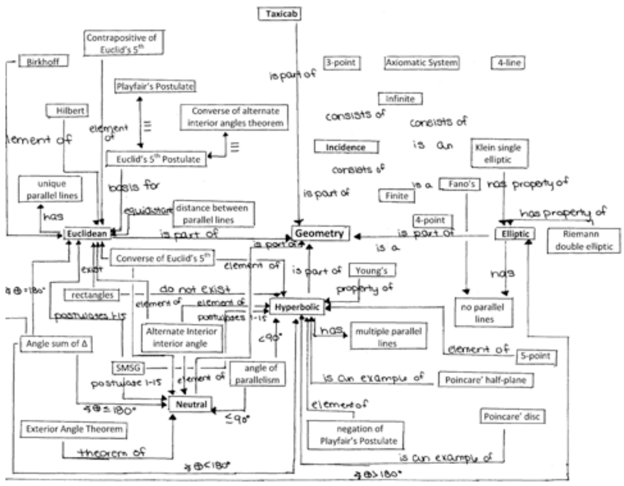

Figure 1. Concept map at basic level

An intermediate level of understanding (**Figure 2**) is indicated by addressing the role of statements equivalent to Euclid's 5th postulate, its negation, and its converse; and distinguishing between neutral geometry and elliptic geometry. The most common misconception at this level involved the converse of Euclid's 5th postulate, which is true in neutral geometry, and the role the Alternate Interior Angle Theorem plays in defining neutral geometry (i.e., it guarantees the existence of parallel lines). These portions of the concept maps indicate the beginning stages of reasoning at the Rigor Level.

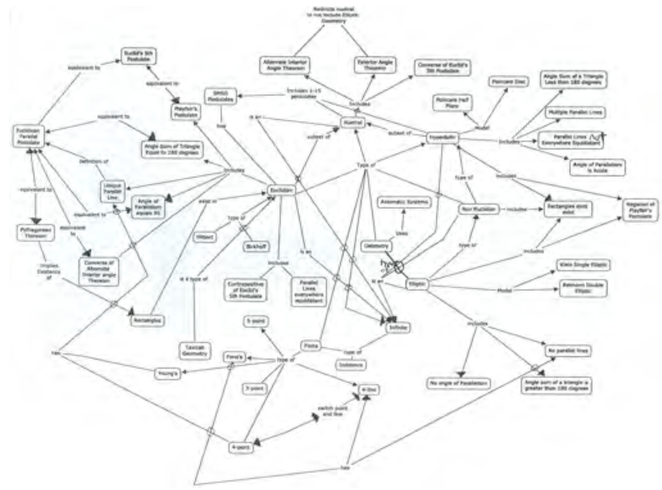

Figure 2. Concept map at an intermediate level

The most integrated level of understanding (**Figure 3**) accurately conveys the relationship between the parallel postulate, the existence of rectangles, the distance between parallel lines, and the sum of the interior angles of a triangle. Although many students correctly linked rectangles and an angle sum of 180° to Euclidean geometry, use of linking works such as “has” or “exist in” rather than explicitly indicating the deduction of these properties from a parallel postulate indicated a more superficial understanding of the relationships.

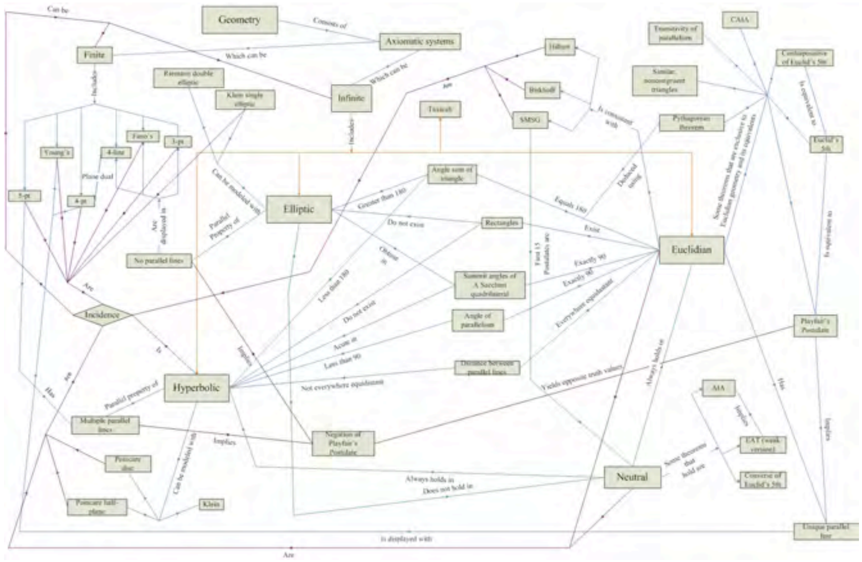

Figure 3. Concept map at an integrated level

In general, it appears the use of descriptive, mathematically meaningful linking words on the concept is one factor that indicates the depth of student thinking. This range and depth of understanding is also supported by the qualitative analysis of the accompanying essays. Comparison of the two drafts completed during the quarter and, at times, a preliminary draft of the final concept map with the final map illustrate a progression from viewing the terms as isolated statements to increasingly integrated defining properties of related axiomatic systems.

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LEARNING TRAJECTORIES: CASE STUDIES OF TWO SECONDARY TEACHERS

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Teachers acknowledge that the ways that students think mathematically influences their instructional practices. Through this study I considered how learning trajectories might provide insights into the ways that teachers understand and make use of students' mathematics in classrooms. Specifically, I examined the ways that two teachers intended student learning to progress in lessons and their reflections of how learning actually progressed. Results indicated that the projected learning trajectory doesn't always coincide with enacted learning trajectory. One reason for these differences was whether the students or the teacher led the direction of the learning trajectories during enactment.

There is a growing body of literature in mathematics education that suggests teachers make instructional changes when they focus on student thinking (Carpenter, Fennema, Peterson, Chiang & Loef, 1989; Lubinski & Jaberg, 1997). In addition, research also indicates that instructional practices that supports and builds on students' thinking will promote students' mathematical understanding (Fennema et al., 1993; Hiebert & Wearne, 1993). Teachers must have an understanding of students' mathematical thinking in order to create learning opportunities that builds on and supports that thinking. Learning trajectories provides a framework for considering the complex ways that teachers make sense of and use their students' mathematical thinking. This research paper will report on a study that examined the learning trajectories that teachers create of students' mathematical thinking through their teaching practice.

Theoretical Framework

In order to support and build on students' mathematical thinking one must consider the relationship between the learning activity and learning process. In other words, when planning lessons one must consider how a student might engage in a mathematics task and what learning might happen because of that engagement. In recent years, mathematics educators have noted this importance of instructional planning when the goal is to build on and support students' current mathematical thinking (Gravemeijer, 2004). Learning trajectories are models that represent children's starting points, the changes that occur due to the mathematical activity, and the interactions that were involved in those changes. In Simon's (1995) discussion about learning trajectories he noted that the "path by which learning might proceed" is hypothetical because "the actual learning trajectory is not knowable in advance. It characterizes an expected tendency" (p. 135). Simon noted that the hypothetical learning trajectory (HLT) includes three components: the intended direction, the learning activities, and the hypothetical learning process. Teachers can refine and modify existing learning trajectories through interactions with children. Steffe (2004) notes, "the construction of learning trajectories of children is one of the most daunting but urgent problems facing mathematics education today" (p. 130). He also mentions that children and a teacher/researcher should co-produce these learning trajectories.

Methods

This research paper details case studies of two high school geometry teachers. These teachers, Judy and Barbara, were selected because they were experienced teachers with advanced college degrees who were willing to discuss their instructional practices and attend working group sessions. Data were collected for both teachers during one semester in one of their semester long geometry classes. Over the course of the semester there were 13–16 classroom observations and six working group sessions. Field notes were taken for each observation. For six of the classroom observations there were 30–60 minute pre-observation and post-observation interviews. The focus of the interviews was the teachers' descriptions of the components of the learning trajectories. When possible, the classes were observed the day before and the day after these six observations. The six lessons for each teacher and all of the working group sessions were videotaped. In addition, all teaching artifacts for the observations and the working group sessions were collected and analyzed.

The complexities of the classroom make it difficult for teachers to focus specifically on students' mathematics. With that in mind, there were six working group sessions that occurred approximately every two weeks. The purpose of the sessions was not to change teachers' instructional practices, but to support the two teachers in their efforts to discuss their students' mathematical thinking. The tasks in these sessions included solving a mathematics tasks, watching videos of students solving tasks, creating tasks to use in their own classrooms, and discussing their student solutions to their developed tasks. In two of the sessions the teachers created or modified a task to use in their own classrooms. The two lessons that included those tasks were part of the six lessons that pre- and post-interviews were conducted.

Data was analyzed based on the learning trajectory framework and using an interpretative stance in order to examine "how individuals experience and interact with their social world, [and] the meaning it has for them" (Merriam & Associates, 2002, p. 4). I began initial analysis during data collection through a research journal. The contents of this journal included my reactions to lessons, ideas about possible answers to the research questions, and themes that were emerging. After data collection I began a retrospective analysis with four stages. During the first stage I coded field notes and interviews. Next, using interview data, field notes, and teaching artifacts I created the teachers' learning trajectories. In order to distinguish between what the teacher planned to happen during the class and what the teacher noted about what actually happened during the class, I created two learning trajectories for each of the six lessons—the projected learning trajectory (PLT) and the enacted learning trajectory (ELT). When necessary, I used video data to fill in missing components of field notes. Third, I used the coding scheme to revisit the data by coding the created learning trajectories. In the process of coding the learning trajectories, I compared the PLTs and ELTs for individual lessons. Finally, I looked for confirming and disconfirming evidence for each of the themes that emerged.

Results

The learning trajectories that I created for both teachers' six lessons were complex due to the factors that influenced their development. I want to acknowledge that I am adapting Simon's (1995) notion of hypothetical learning trajectories to create the teachers' learning trajectories. The learning trajectories were based on the teachers' thoughts about the initial learning goal(s), the students' current mathematical understandings, the learning activities, and the changes that occurred in students' mathematical understandings during the lesson. The learning trajectory began with students' starting points, or their current mathematical understandings. Through some mathematical activity, students progressed through the learning process to conclude at some ending point. There are large learning trajectories, which span multiple concepts, and there are smaller learning trajectories, which cover individual concepts (e.g., a learning trajectory for area of polygons or a learning trajectory for area of regular polygons). In this study, I was interested in the learning trajectory that teachers described for their students during individual lessons, which spanned one to three days and covered anywhere from one to five learning goals. I also want to mention that the learning trajectory can represent learning for one student, a subgroup of students, or a class of students. Figure 1 represents what a learning trajectory might look like. I've chosen to create a simple figure to oversimplify learning trajectories in order to later describe and highlight results. I want to emphasize that learning is not necessarily continuous nor does it fit the curve as shown in the figure.

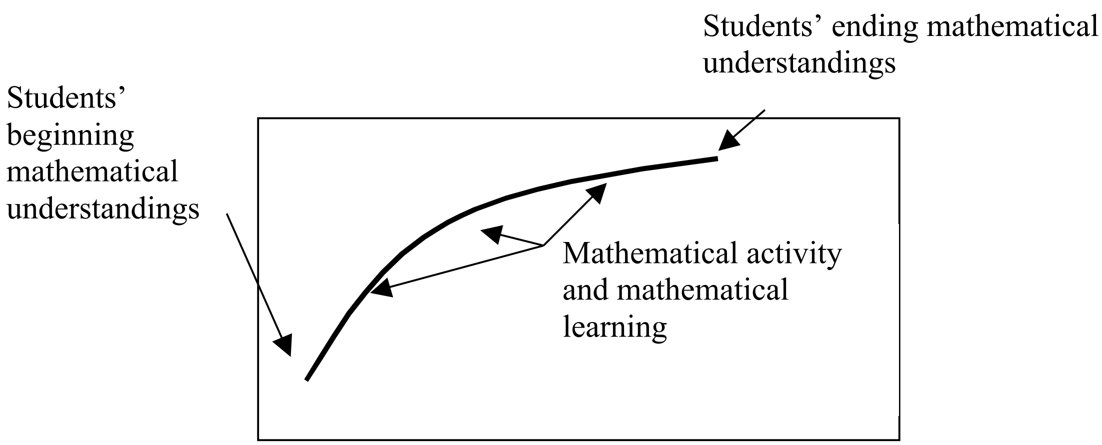

Figure 1. A Learning Trajectory

Let's consider an example from the data. During working session three the teachers chose a task that fit both of their learning goals. The teachers modified the task appropriately for their class and created lessons that incorporated the task. Figure 2 is the task and the diagram that went along with the task. Working session four was spent with the two teachers describing what happened during their classes and analyzing student work.

Given an equilateral triangle ABD and a rectangle EFGC. Suppose point C is allowed to assume different positions on the line segment BD (as C changes positions the points E, F, and G also change accordingly.) Is there a position for point C along line segment BD where the perimeter of the window EFGC is the largest? … area largest?

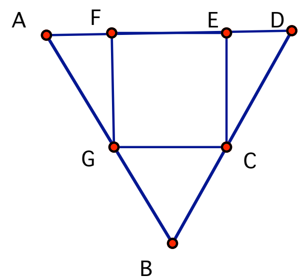

Figure 2. Window Task

This lesson spanned two days in both teachers' classrooms. The goals that the two teachers had for their lessons varied. Barbara's goal for her lesson was for students to engage in a culminating activity that incorporated a lot of concepts from the semester. She expected her stronger students to quickly generalize and use an algebraic representation to support their answers. In addition, she expected some of her students to struggle to begin the task. There was another group of students who she expected would use a measurement strategy, but she thought they would realize that they needed an algebraic solution in order to prove their answer. These different solution approaches signified that Barbara was considering three learning trajectories for her students. She thought her students would think differently about the task and would vary in what they learned from engaging in the task. Judy wanted her students to explore the idea of maximum area and perimeter with some organized investigation. She expected her students to use a measurement strategy and she hoped they would justify their solutions. Judy acknowledged that some of her students would approach the task at a deeper level, but the multiple solutions she described all dealt with the same solution strategy and the same learning progression (just at different paces).

During the first day of the lessons the two teachers' classrooms looked similar with corresponding teacher moves and corresponding student solutions. The two teachers introduced the task in a similar manner. They put a transparency of the Window Task on the overhead projector. The teachers described the task and posed questions to help students begin. Both teachers gave very little guidance about possible solution paths. Then students were asked to work individually for ten minutes to think about possible solutions. After that time the students worked in groups to solve the task. The students solved the task with some variation of a measurement strategy. On the second day there were noticeable differences between the two classes.

Barbara was disappointed that no students approached the task using algebra and that many students solved the task with very little data to support their solutions. On the second day Barbara became more prescriptive in her interactions with students. Barbara created a table of student measurements on the board. She mentioned to students that a measurement solution required a lot of numerical data. Afterwards she made a comment about the difficulty with measurement was that it doesn't prove something was true. Then she asked students if there is a way to represent the changing length of segment CE and students quickly answered by labeling it x. From there the class came up with an equation and graphed that equation on a calculator to see the maximum value for the function.

Judy's class was different. Judy gave students ten minutes to work again in their groups. Then, students explored the task using a teacher-prepared sketch on Geometer Sketchpad (GSP; Jackiw, 1991). During this time Judy walked from group to group to listen to conversations. Following this activity Judy asked particular students to share their solutions. The class looked at a table of values and graphed the values. Judy noticed that many of her students were not connecting the table values to the graphed values, so she decided to change the task to finding the maximum area for a rectangle with a constant perimeter of 24 (the perimeter in the window task varies).

These two teachers made different teacher actions for different reasons during the lesson and, ultimately, progressed through the same task differently. The teachers' enacted learning trajectories were different based on their different learning goals for the learning activity. When comparing the projected and enacted trajectories for each of the teachers' lessons there were some instances where they matched, while in other instances the trajectories were different. In making decisions about the relationship between the two trajectories I examined whether the learning goal, learning activities, and learning process matched or whether they changed during the enacted lesson. I am operating under the assumption that none of the ELTs matched perfectly with the PLTs, so when I claim that the PLT and the ELT match I am claiming that the two learning trajectories matched closely. A change in plans did not automatically suggest a direction change in the learning trajectory or a change in the mathematical activity. For example, on the first day of the lesson on the Window Task Judy felt that there were some students who were not making progress on the task and were not making the necessary connections. On the second day she created a GSP sketch for students to explore. Even though the GSP sketch was not originally planned on the first day I consider the ELT and the PLT to still match. The overall idea of the task did not change; only the method for solving the task was modified. So, instead of students physically measuring the side lengths of the rectangles, the computer program GSP measured the sides. In this portion of the lesson the ELT and the PLT matched closely because the change in method did not change the learning progression nor did it change the overall goal. Now, Judy did make a change in the direction of the lesson partway through the second day's class. Judy created a new task that took the lesson in a different direction than originally intended. This decision was made based on comments that her students made in class that implied they were missing the connection between the numbers that represented the length, width, and area of the rectangle and the graphical representation of those numbers. Notice in Figure 3 that during the beginning of the lesson the PLT and the ELT matched, but along the way the ELT took a different direction.

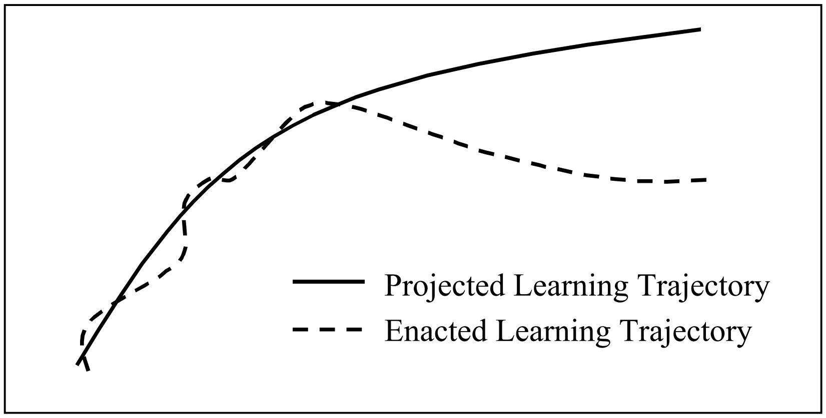

Figure 3. A different direction

When comparing Barbara's PLT and ELT for the lesson on the Window Task it was clear that the students did not progress through the lesson as she planned, which caused the ELT to take a different track than originally planned. Her learning goal for the lesson was not for all of her students to justify their solution with measurement, but to use algebra as a way to prove. Barbara made the decision to bring students back to her original learning goals and learning activity. When she made that decision she became more directive and leading with her comments and questions in order to push students in the direction she wanted them to go. Her actions abruptly changed the ELT so that it matched more closely with what she projected and hoped students would do. Figure 4 illustrates how the enacted learning trajectory started going a different direction than the PLT, but shifted when Barbara intervened.

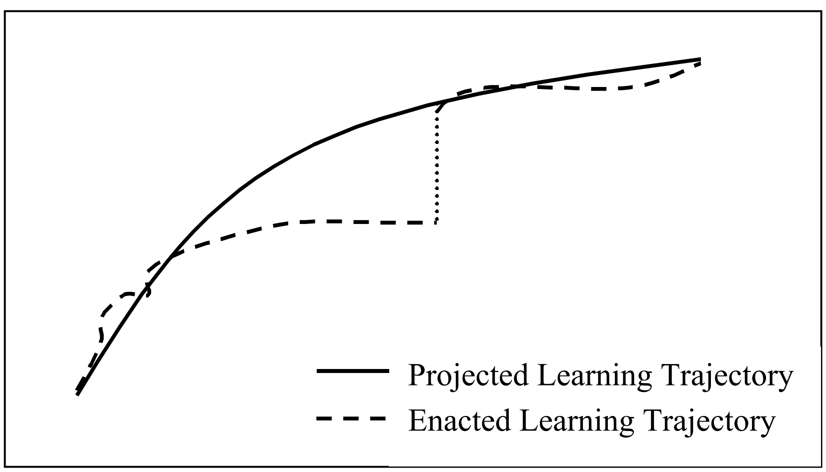

Figure 4. Let's get back on track

There are three findings that I briefly discuss. First, the PLT and ELT did not always match. In Judy's lesson on the Window Task the trajectories didn't match because Judy left the original task to pursue a different task. In Barbara's lesson the trajectories didn't match because students' solved the task in a way that she expected only a subgroup of students to solve. The different solution path led to a different learning progression. From the data there is significant evidence to support the idea that the reasons for why the enacted learning trajectory took the path that it did related to who led the lesson. There are varying degrees to which the ELT was determined by the teacher or by the students. For instance, in Barbara's lesson she led the students back to her projected learning trajectory by being directive. Based on the teachers' intentions and the fact that students were not going in that direction, Barbara intervened. Towards the end of the lesson the ELT matched the PLT. In contrast, in Judy's lesson the students took more of a lead role in the direction of the ELT. Students were left to explore the original task without Judy directing them in particular directions. The students began exploring the relationship between tables and graphs when Judy noticed a gap in knowledge. Based on her students she changed the task. In the example provided both PLT were only partially followed. In other lessons there were instances that the ELT matched the PLT and was led by either students or the teacher. Again, the reasons for the ELT matching the PLT were related to who led the lesson.

Describing students' mathematics was difficult for both teachers. As the study progressed the teachers began to question how they assess their students' mathematical thinking within lessons. Barbara noted, "you know you've asked me that so many times that I'm like I really don't know that I assess that very well" (Interview, p. 43). As one would expect, in instances that the teachers interacted with students doing mathematics the teachers could elaborate on students' mathematics. Barbara noted that during the first day all of her students solved the Window Task with measurement. In her reflection on the first day she was able to describe several student solutions, whereas on the second day she was only able to describe a couple students' mathematics based on who spoke in class. Judy was able to describe, with detail, students' mathematics after both days. She elaborated on the various measurement approaches that she observed during class. For example, one student noted that the perimeter of the window is larger as the point C approached the side of the triangle because the side of the triangle is larger than the height of the triangle. Judy thought this solution was very intuitive and acted on it later in class. Throughout the lesson Judy interacted with students actively engaged in mathematics.

The third finding was teachers created different learning trajectories for different groups of students. Typically, the teacher created two to three groups and used a couple students in each group to describe how all the students in the group were thinking. For example, Barbara thought one group of her students would have trouble solving the window task because she felt like they couldn't "see" the diagram. She thought another group would use measurement and the rest of the students would immediately use algebra. Even though the learning goals and learning activities were similar, Barbara created different hypothetical learning processes for the different groups of students as she felt these students would approach the task differently.

Discussion

This research study presents significant insights into how learning trajectories might be used for teacher development and for research. Specifically, learning trajectories could be used to frame the ways that teachers consider the relationship between learning activities, teacher actions, and students' learning process. Teachers find it difficult to observe and discuss their students' mathematical thinking. This study shows that learning trajectories has the potential to encourage teachers to think about and use their students' mathematical thinking in their instructional planning. Currently, researchers develop learning trajectories through intensive work with children. These learning trajectories are a valuable resource for teachers, but are not yet readily or easily available for teachers' use. Practicing teachers develop their own trajectories based on their years of planning activities and interacting with students. These trajectories are not as detailed or refined as the ones created by researchers, but they can provide powerful insights for teacher education. As researchers continue to develop learning trajectories of childrens' mathematics I think it would also be valuable to examine teachers created learning trajectories.

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LOW ACHIEVING MIDDLE SCHOOL STUDENTS' CONCEPTIONS OF "SAME SHAPE"

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In this study we explore intuitions of "same shape" held by low achieving middle school students through an analysis of their strategies for differentiating, classifying and constructing similar figures. Findings suggest that student's intuitions of "shame shape" are not reliable and vary by context. There is a need to include more diverse figures in the study of similarity.

Despite its usefulness in daily life and critical role in many branches of mathematics, research studies have repeatedly shown that similarity is by far the most challenging context for the study of proportional reasoning. Researchers have claimed that student intuitions and informal experiences can be leveraged to secure a conceptual understanding of this topic. It remains under debate, however, what intuitions exist and how best to draw on them. Among the hypotheses is the idea that students should be formally grounded in classifying distorted and non-distorted figures (Swoboda & Tocki, 2002). This perspective is in sharp contrast to a common assumption made by some middle school math curricula that the idea of same shape is intuitively clear to all students, and can thus be used to define "similar figures" as "figures with the same shape" (Lo, Cox & Mingus, 2006). The purpose of the current study is to uncover the conceptions of "same shape" this particular group of students may have.

Theoretical Framework

We draw from Tall and Vinner's constructs of concept definition and concept image (1981) to help us frame our research questions. Concept definitions are the words used to specify a concept. Concept images include all the mental pictures and associated properties and processes built through experiences over the years. Influenced by an individual's experiences both inside and outside of school, concept images can vary in terms of the degree of richness and connectedness. In our study we explore the concept definitions and concept images of "same shape" held by middle school students. Specifically, the following three research questions guided the design and analysis of the current study:

- What daily experiences provide students with the images of the "same shape"?
- What aspects of an image do students use to determine the similarity?
- What strategies do students use to create a similar copy of a given shape?

Methodology

The study was carried out in an urban Mid-western middle school with a diverse student population in a low-socio-economic neighborhood. Over 80% of the students participate in the free and reduced lunch program. Seven seventh grade students identified as low-achieving on the bases of performance on standardized tests volunteered to be interviewed. Students were first asked about their experiences with "same shape" in daily life. They were then asked to classify four sets of figures that included photographs of their math teacher, L-shapes, M-shapes and a rectangle. Each set contained an original, a 150% dilation, a 200% dilation, and a distorted shape that was created using additive growth.

After the classifying task, students were given selected images from the shape sets and grid paper and asked to draw shapes that are "of the same shape but twice as long on a given side." One additional image was used, a rectangle with a dot inside the upper left corner. These interviews were videotaped and partially transcribed. Student responses were analyzed across individual tasks and then grouped under each research question.

Partial Results

Due to space limitations, we will only discuss the findings for our second research question in this proposal. The findings on the others will be discussed during the presentation.

None of the participants in our study had developed the intention or systematic approach to prove two given figures to be similar. Generally speaking, their determination of the sameness among figures depended on whether they noticed any distorted aspects of the given figures. In the photograph set, those that looked like "real life" were deemed good blow-ups, or "the same shape" as one another. Students refer to specific lengths of portions of the image including the teacher's head or face, using comparative terms such as "skinnier," "fatter," or "smooshed" to describe ways they identified the distortions.

The use of correspondence was the key for students as they differentiated within the remaining shape sets. However, students tended to see a limited number of correspondences and they did not use the identified relationship in a consistent way. Thus it is not surprising that they did not interpret similarity as being transitive in nature. In fact, there were no intuitions that all partitions or parts of a shape need to have the same mathematical relationship as the others. None of the students were able to successfully identify all three similar rectangles from the set, but all could identify at least one pair. Those students who established two distinct subsets of shapes were then given an additional probe to make a cross comparison between the third similar rectangle and the correctly identified pairing. When asked to consider cross-pair comparisons, two out of three students identified one but not both of the rectangles as belonging to both sets.

Discussion

This study has provided some evidence that intuitions and informal experiences with the idea of "same shape" possessed by low achieving seventh grade students my not be compatible to those defined formally in mathematics, thus making the assumption about the intuitions of middle school students problematic. Students in this study have experiences that both enhance and interfere with conceptual understanding of "same shape" and similarity. The findings of this study inform curriculum development as to ways to capitalize on prior constructive experiences, and adjust for potential interfering ones. Our findings also point to the need to include more diverse figures in the study of similarity so that they can see the necessity of having **ALL** corresponding angles be equal **AND** the ratios of **ALL** corresponding sides be equivalent in order to maintain the shape. Further studies are needed to study how this can be achieved among diverse student population.

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PRESERVICE AND INSERVICE SECONDARY MATHEMATICS TEACHERS' VISUALIZATION OF THREE-DIMENSIONAL OBJECTS & THEIR RELATIONSHIPS

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Del Grande (1987, 1990) and Bishop (1980, 1983) have associated geometric understanding with perception of spatial relationships. Recent studies (Gutiérrez & Jaime, 1998; Saads & Davis, 1997) have helped establish a framework for research that considers spatial relationships within the context of 3-dimensional geometry. Gutiérrez (1996) used the work of others (Presmeg, 1986; Yakimanskaya ,1991) to refine definitions for visualization, mental image, external representation, and process. Our research uses the definitions he established.

The National Council of Teachers of Mathematics [NCTM] (2000) has proposed that all students in grades 6-8 should be able to use defining properties to describe and understand the relationships between both 2D and 3D objects, while all students in grades 9-12 should be able to "visualize three-dimensional objects from different perspectives and analyze their cross sections"(p.308) and solve problems involving 3D figures characterized by Cartesian coordinates. In its grades 5-8 *Addenda* series, NCTM suggested using 3D physical models as starting points to develop the concepts of "parallel and perpendicular lines and planes, skew lines, [and] angles in two and three dimensions" (Geddes, 1992, p. 6).

Since the landscape of geometry education in the U.S. includes a definite push towards the inclusion of 3D geometry in the secondary curriculum, we must ask whether U.S. teachers are prepared for this. Gutiérrez (1996) argued that even though "visualization is a basic component in learning and teaching 3-dimensional geometry...[but] there is very limited research activity in this specific area (p.11)."

Research Objectives

The main objective of this research is to begin a series of investigations that shed light on teachers' visualization of basic 3D concepts. How deeply do teachers understand these concepts and their relationships? More specifically, how do they visualize basic relationships and manipulations such as bisections and rotations of fundamental geometric objects in a 3D environment?

Methodology

This study involved 32 preservice and inservice secondary mathematics teachers (22 female and 10 male) at a large research university in the United States. All participants held a bachelor's degree in mathematics or its equivalent, and had successfully completed a geometry course and the calculus sequence including multivariable calculus as undergraduates. The participants were mathematics education graduate students enrolled in two separate pedagogical content knowledge courses. The students participated in the activities described as part of course requirements. They were given the option of having their data used anonymously for research purposes, and all agreed.

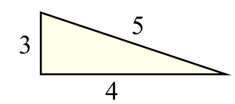
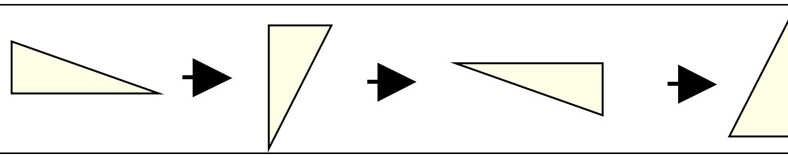
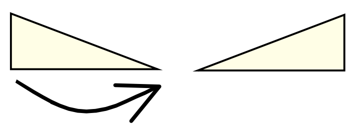
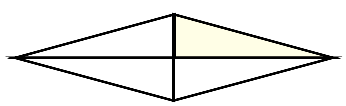
Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.

Eleven items from a written examination were used in this study. The 11 items all represented 3D concepts or 2D concepts in a 3D environment that NCTM has suggested are appropriate for secondary mathematics students. They were designed to assess the teachers' ability to determine relationships between different spatial objects. The examination was developed in light of recommendations suggested by Gutiérrez and Jaime (1998) for paper and pencil assessments in terms of van Hiele levels. After consultations with colleagues and a pilot of the instrument, items were modified as needed. The examination was then administered to the participants at the beginning of each course to determine their abilities to perceive spatial relationships among basic 3D figures. Members of the research team individually analyzed the participants' responses looking for patterns in the data. After the initial, individual analyses were completed, the team verified their findings against each other and the data.

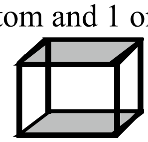
Results

The research findings indicate some teachers persist in working in the plane, even when the situation requires a 3D environment, or they often refer a particular plane.

When given a drawing of a right triangle and asked to rotate it to determine the solid of revolution, 28% of the responses to this typical calculus problem were incorrect. Three of the teachers gave two-dimensional answers as shown below.

***Problem*: If the right triangle shown below is revolved about its 4 inch side, describe, draw and label the lengths of the resulting solid of revolution.**	
Student A Response:	
Student B Response:	
Student C Response:	

Even in a 3D environment, teachers often try to reference familiar 2D objects: horizontal and vertical planes or the *xy*-, *yz*-, and *xz*-planes as is seen in the select items shown in the table below.

Examination Question	Responses of 3 Students (None of whom are Students A, B, C or D)
True or False. Explain. Any 3 distinct points in 3-space lie on a single plane.	False: 2 points could lie on bottom and 1 on top, they are different planes. 
	False: Three points could be in the xz plane or yz plane, or xy plane, not necessarily in the same one.

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	Points could lie on same plane but don't necessarily lie on same line. a (1, 2, 3) b (1, 4, 5) c (0, 6, 9) a, b, and c don't lie on same line because don't all share common x, y, z coordinate.

In another example, Student D responded *a horizontal line* when asked, "What bisects a plane?" The following are the frequencies of the incorrect responses to the open-ended question, "What bisects a 3-space?" *Point* – 4, *Segment* – 1, *Ray* – 2, *Nothing* – 3, *Anything because a space means anything* – 1, *Any geometric figure* – 1, *Forgot definition of space* – 1, *Not sure*/?/*blank*– 6.

Discussion

Internationally, de Villiers (1997) has called for major changes to teacher education programs claiming that even "qualified" secondary mathematics teachers hardly know more geometry than their students. This is supported by Malara's (1998) findings concerning middle school teachers' difficulties with 3D visualization. Teachers seem to understand fundamental geometric concepts in 2D, but profound understanding of these concepts must be questioned when they struggle with them in a 3D environment. The results from this study suggest that teachers have difficulty visualizing geometric objects and manipulations in a 3D environment and would have difficulty teaching the geometric concepts called for at secondary levels.

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WHY CAN'T THIRD GRADERS COUNT THE EDGES OF A CUBE? ANALYZING STUDENT PERFORMANCE ON 3D GEOMETRY TASKS

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How many edges are on a cube? This question was asked of fourth graders as part of the Third International Mathematics and Science Study (TIMSS) and only 35% of US fourth graders correctly answered it (TIMSS 1995,1998). I take this poor performance as a starting point in trying to understand student thinking in three-dimensional geometry. Since the TIMSS report is limited to the percentage of students choosing each of the four responses provided, I continued my investigation by analyzing children's responses on an open-ended written version of the task and examining some students' work in an interview setting. My purpose is to show that Battista and Clements (1996) notion of spatial structuring can be applied to understand students' thinking about various polyhedra.

Theoretical Framework

In trying to understand students' approaches for determining the volume of rectangular prisms Battista and Clements (1996) proposed that meaningful enumeration of cubes in a 3D rectangular array was supported by students' spatial structuring of the array. They considered spatial structuring to be "the mental act of constructing organization or form for an object of set of objects" (p. 282). It involves creating a coherent whole from different parts and recognizing the relationship of the parts to the whole and to each other. They also noted that individuals do not "read off" this structure but have to create it from mental actions. While they applied this construct to children's thinking about arrays of cubes, I found it useful to consider children's thinking about less complex 3D objects.

Method

In addition to the published TIMSS data (TIMSS 1995, 1998), I analyzed assessments given to a group of third-grade students participating in a classroom-based design experiment. The 17 students attended an ethnically diverse public elementary school, GW, in Northern California where 81% of the students are considered economically disadvantaged and 60% are designated English Language Learners. At the beginning of the investigation each of the children took a pencil and paper test which included a constructed response version of the "how many edges on a cube?" task. 13 of the children were also videotaped while participating in a dynamic interview, where I offered students support after they initially attempted problems.

Results

When "how many edges are on a cube?" was asked of students on the TIMSS the children could choose from the responses 6, 8, 12 and 24. The first three choices could be obtained by counting some component part of the cube (faces, vertices, or edges) and the last response arises from multiplying the number of faces (6) by the number of sides on each (4). All of these responses assumed that children would successfully count some part of the cube. The prevalence of the answer 8 in the TIMSS and GW data (see Table 1) suggests that the part of the cube that US students seemed to count were the vertices. Analysis of interview data showed that 3 of the 13 children (23%) were counting the vertices, and they provided the answer of 8. It is worth noting the location of the word "edge" in the task makes it unclear whether the term refers to an edge, a face, or a vertex. When students counted the vertices in the interview, I clarified the meaning the term edge by showing the children a poyhedron that I had constructed and running my finger along some of its edges. None of the 3 children who initially counted vertices were able to successfully revise their answer. While they were able to spatially structure the cube to count the vertices (counting the four on top and then the four on bottom), they had more trouble counting the edges.

Table 1 Answers provided by students

	6	8	12	24	9	Other
TIMSS – US 4th graders	22%	37%	35%	4%	(not an option)	2%
GW Written test	23%	31%	0%	0%	6%	40%
GW Interview (initial answer)	0%	31%	23%	8%	23%	15%

Data from the GW written test suggest that many of the students did not successfully count any parts of the cube. Children offered 4, 5, 7, 9, 10, and 16 as answers. Comparison of written test data and interview data show that only 2 of the 13 children were consistent in their responses, and both of these children were counting the 8 vertices. I inferred from this comparison that most of the students did not have a stable approach for completing the task.

Children's work in the interview showed that 4 of the 13 children (31%) counted only the edges visible in the diagram disregarding the parts that were hidden. They were unable to consider the cube from an alternate perspective and edges seemed to be an uncoordinated set of component parts. These children could be said to be operating on a level similar to Battista and Clements' (1996) "medley of views" level. Two of the children successfully counted all of the edges but did so without imposing a structure on the cube. These two children seem to have constructed a mental model of a cube which included parts not visible in the picture. One noted, "there are two faces on the other side and you have to imagine that the faces are on the paper." Some of the children did engage in spatial structuring of the cube but in only one case did this yield a correct answer. In this case the child noted that there were four edges on the top, four "around the sides" and four on the bottom. Four of the 13 children in the interview imposed structure on the cube after they were asked to show what they were counting. For example one child started counting in a haphazard fashion but when asked to show each edge, he pointed to the 4 vertical edges and then the four edges connecting the front and back to the top and bottom. He neglected to count the other edges and so got an answer of 8.

Discussion

Analysis of student performance on the "how many edges in a cube?" task shows that enumerating this component part of a polyhedron is more complex than simply learning the correct vocabulary (as one might infer from analyzing the TIMSS data for this task). Students need to take into account perspective by recognizing that the drawing only shows part of the cube. In addition, spatially structuring the cube, by decomposing it into constituent parts makes the task more manageable. I posit that spatial structuring develops gradually as students continually face challenging enumeration situations. Battista and Clements (1996) indicated that "perceptual and physical actions …become inputs for the structuring process" (p. 290) so it would seem that curriculum should be developed to allow students the opportunity to manipulate 3D objects so they can begin to engage in this process.

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DISCOVERING THE PYTHAGOREAN THEOREM(S) WITH A GEOBOARD: DEVELOPMENTAL EXPERIENCES OF EIGHTH- AND NINTH-GRADE STUDENTS

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Activities that helped two groups of eighth-grade students (13-14 years) to discovering and stating—in their own terms—the Pythagorean Theorem are described in this poster. Another set of activities on applications of the Pythagorean Theorem was implemented with a third group of ninth-grade students (14-15 years). All activities were at the level of knowledge development (formal definitions, formulae, or rigorous proofs of the mathematical results implied were not used). The designing of these activities was based on a particular version of propositions 12 and 13 in Book II of Euclid's *Elements* (equivalent to our trigonometry Law of Cosines): Students had to construct squares on the sides of obtuse and acute triangles, and to determine whether the area of the largest square so constructed could be completed with the areas of the other two squares. That is, we took as an *attractor* this geometric version of the Law of Cosines.

By constructing shapes with rubber bands on a geoboard and by exploring and manipulating those shapes, students built an understanding of geometrical relationships. Their achievements under this empirical-inductive approach allowed them to develop a network of mathematical relationships in their thinking, setting the stage for more advanced study of results related to those on which the knowledge development experiences here described were based.

Different researchers are discussing the potential of complexity theory as a model to explaining school mathematics learning experiences (e.g., Davies & Simmt, 2003; Jones, 1994). A particular area of the nascent discipline of complexity is the study of complex adaptive systems—cas—(Holland, 1996). A feature of a cas is that agents show coherence under change, they adapt. We can think of each student as well as of a classroom environment as a cas. As learners, we undergo processes of attempts to bring order to ideas which "form a system because they interact in some way, […] we understand or learn when *an attractor* [our emphasis]—in the language of complexity theory—brings order to the collection of ideas" (Jones, 1994, p. 41).

Activities were organized as follows: (a) Introductory activities (2 work sheets, whose purpose was to familiarize students with the construction of polygons on the geoboard and the computation of their areas); (b) Exploration of triangles (4 work sheets, whose purpose was allow students to discover the Pythagorean Theorem by means of constructing squares on the sides of obtuse as well as of acute triangles—and eventually using right triangles—; we identified that students of the three groups also created an understanding of the Converse of the Pythagorean Theorem), and (c) Applications of the Pythagorean Theorem (3 work sheets about determining the area of the square on any of the three sides of a right triangle—given the areas of the squares on the other two sides—and then the lengths of its three sides).

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USING 3D COMPUTER GRAPHICS MULTIMEDIA TO MOTIVATE PRESERVICE ELEMENTARY TEACHERS' LEARNING OF GEOMETRY AND PEDAGOGY

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Preservice elementary teachers interacted with computer graphics demonstrations in conjunction with 2D and 3D manipulative activities to improve understanding and motivation within a geometry-methods course. Subjects described and rated their knowledge of geometry and described their perceived level of preparation to teach it. Modest gains in their geometric knowledge and in their perceived teaching readiness were observed pre-post course.

To improve preservice elementary teachers' geometric experiences, we developed instructional materials in the project, *Using 3D Computer Graphics Intensive Technologies to Encourage Teachers and Students' Involvement in STEM*, supported by a grant from Appalachian State University. One theme was*: Using 3D Computer Graphics Demonstrations to Illustrate How 2D and 3D Geometry is Applied to Create Computer-generated Figures*. The primary challenge in developing materials is that the majority of computer graphics geometry is advanced, yet we must provide information at a PSET level and depict it accurately. It is crucial to create lessons that broaden PSETs' understanding of geometry and help them *connect* what they are *learning* to what they will *teach*. We provided PSETs with experiences in manipulating computer graphics demonstrations followed by 2D and 3D geometric activities with manipulatives, i.e. creating a 2D figure with pattern blocks followed by translating this pattern to a 3D figure with *Polydrons*. We surveyed 87 PSETs in four sections of a geometry methods course (pre/post course) to explore their perceptions of their geometric understanding and readiness to teach. The survey questions were clustered in triads where each PSET was asked: 1) to rate their content knowledge of key geometry concepts; 2) how prepared they felt to teach these concepts; and 3) to define or describe the concept in their own words. Other open-ended questions also asked students to: *List three occupations that use geometry and specify how geometry is used*. The surveys were given pre- and post-course after students had participated in active-learning and technology-enhanced approaches to geometry. Five areas of interest emerged from the pre-course survey. 1) The PSETs generally rated themselves as not understanding geometry topics very well. 2) They tended to define geometry by simply providing lists of words, primarily names of geometric shapes, with little evidence of in-depth understanding of geometric topics. 3) PSETs lacked appropriate geometric terminology. 4) PSETs indicated they felt unprepared to teach geometry to children. 5) PSETs were unable to express how geometry is used professionally. Using the technology-enhanced lessons in class seemed to help PSETs gain perspective on geometry's usefulness. It made them more aware of the importance of devoting adequate instructional time in class and the importance of learning proper geometric vocabulary. Their geometric vocabulary improved and we saw modest conceptual gains in the van Hiele levels evidenced in their answers. Even a dedicated geometry methods course however, provides insufficient time to address the geometric learning needs of PSETs. This study will be extended in Fall 2007 with the addition of evaluation questions from the *Learning Mathematics for Teaching* project.

ANALYZING TEACHER CONTENT KNOWLEDGE OF PROBABILITY: THE MAZE PROBLEM

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Twelve middle school teachers participated in a professional development focused on improving their content and pedagogical content knowledge of probability. A key part of the professional development was the planning, implementation and analysis of a rich probability task, the maze problem. The implementation of these lessons were then videotaped and analyzed for student-teacher interactions about key probabilistic concepts, particularly interactions where misconceptions were identified. These interactions were then written up as scenarios to share with the teachers. The teachers were asked to identify the key probabilistic concepts in each scenario and consider questions they might ask to support and extend the understanding of the students in the scenario. Three of these scenarios are shared and the questions teachers developed are analyzed in relation to current research on student thinking in probability.

Theoretical Perspective and Purpose

Probability is a relatively new topic within the grade school and secondary curricula of the United States (NCTM, 2000). The inclusion of this topic in standards, curriculums, and textbooks has led to new challenges and questions for teachers and teacher educators as they attempt to learn the necessary content and pedagogical content knowledge associated with this topic (Kazima & Adler, 2006). Moreover, Shulman (1986) and more recently Ball, Hill, and Bass (2005) have called upon the mathematics education community to identify and describe specialized teacher knowledge. When teachers are able to not only solve basic probability problems but also identify the multitude of ways students develop and think about probability concepts, they gain a deeper understanding of mathematics, in other words a specialized knowledge of mathematics (Ball, Hill & Bass, 2005).

Probabilistic reasoning is often counter-intuitive for both students and teachers. This counter-intuitive nature of probability has been explored and detailed in numerous studies on student thinking leading to the identification of a variety of common misconceptions such as the outcome approach, representativeness, negative and positive recency effects and the conjunction fallacy (Shaughnessy, 2003; Fischbien & Schnarch, 1997). Understanding how students think probabilistically and handling possible misconceptions students may have is a part of the specialized knowledge of mathematics teachers need. Research on student thinking in probability provided this study with a foundation for identifying student misconceptions and analyzing the ways teachers used their specialized knowledge of mathematics to help students modify and change their misconceptions.

This paper explores the specialized teacher knowledge of probability by focusing on twelve participating teachers' implementation and analysis of one rich task, the maze problem (figure 1). The maze problem was selected because it is a multi-faceted task that does not lend itself to quick algorithmic thinking, but to actually "doing" mathematics (Stein, Smith, Henningsen, & Silver, 2000). The task also provides opportunities to explore numerous topics within initial probabilistic thinking previously identified by research in this area (e.g., Shaugnessy, 2003; Amit

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& Jan, 2006; Benko & Maher, 2006), including topics such as experimental and theoretical probability, equiprobability, compound events, randomness and the law of large numbers. Thus, we asked the question, "What specialized knowledge of mathematics, specifically probability, is evident as teachers analyze real scenarios from their implementations of the maze problem?"

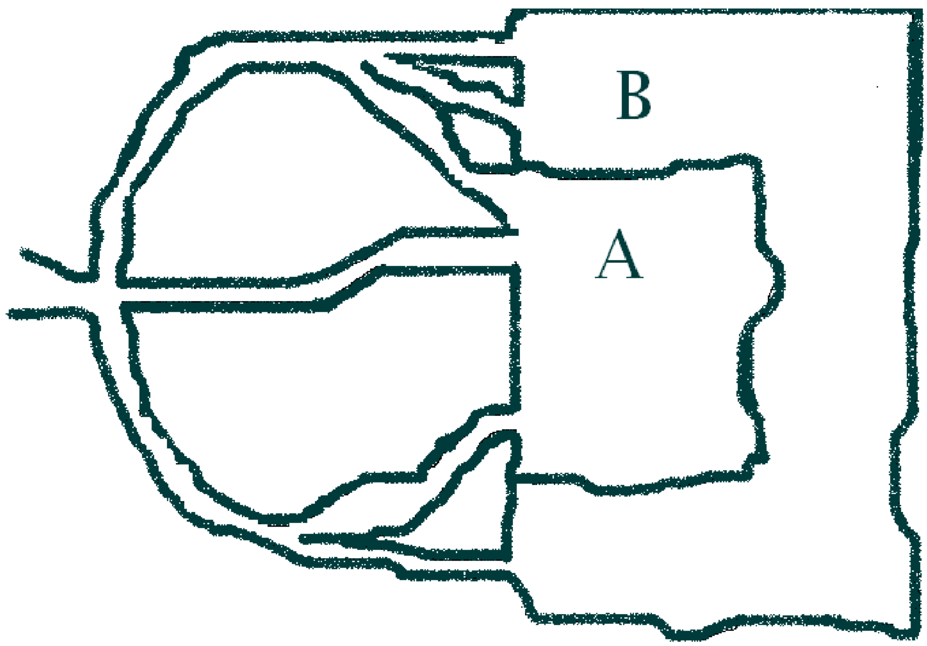

Figure 1. Maze Problem

Methodology

The twelve participating teachers in this study were part of professional development program for middle school mathematics teachers. The program focused on improving the content and pedagogical content knowledge of teachers through summer institutes and follow-up meetings throughout the school year. The summer institutes focused primarily on improving the teachers' content knowledge while the follow-up meetings during the school year were designed to focus primarily on pedagogical content knowledge.

The data collection for this study took place during the school year follow-up meetings. Each teacher was initially asked to determine the experimental and theoretical probability for each room of the maze problem. The teachers were then provided time to plan how they would teach this problem to their students. Fieldnotes were taken during these sessions with the teachers with a focus on key probability ideas that were discussed and any misconceptions the teachers exhibited.

Each of the twelve teachers was videotaped during the implementation of the maze problem and these videos were then analyzed for mathematical topics related to probability. Interactions between teachers and students were coded for topics such as experimental probability, randomness, and law of large numbers. When misconceptions were identified within these interactions, they were written as scenarios that could be shared with the teachers. The scenarios served as an anchor for conservations where we could observe teachers' specialized knowledge of probability. Each scenario details the ideas of the students but does not mention how the teacher interacted with the students. Nine of the scenarios were then chosen to share with the teachers. These scenarios were chosen because each one represented a misconception dealing with a key probability concept the teachers had discussed during their own solving of the maze problem, and are part of the literature on student thinking in probability (e.g., LeCoutre, 1992; Tversky and Kahneman, 1982).

Groups of two or three teachers were then asked to identify the mathematical concepts involved in each scenario. In addition, each group was asked to imagine if they were the teacher in the scenario, what questions or prompts they would ask students to support development of probabilistic reasoning. Each group of teachers was asked to record the above information and share it with the rest of the participating teachers and researchers. The results section below shares three of these nine scenarios, the questions and prompts developed by the groups of teachers, and an analysis of teachers' specialized knowledge in relationship to the questions and prompts they developed.

Results

The maze problem asks students to determine the experimental and theoretical probability of moving through the maze and arriving in the two different rooms, room A and room B. As students develop simulations to determine the experimental probabilities of each path of the maze a variety of topics and corresponding misconceptions can arise.

Scenario 1

The first scenario was initially identified while the teachers worked on developing their own simulations for the maze problem. This scenario was then re-identified in three of our teachers' classrooms during their implementations of the problem.

> *A group of students are setting up an experiment for The Maze. They decide to label the terminating part of each path the numbers 1 through 6 respectively and then use a die to determine the path traveled.*

The mathematical concept the teachers identified for this scenario was independent events. The more apparent misconception identified by LeCoutre (1992) is misunderstanding of equiprobability. Children with an equiprobability bias assume all outcomes of an event are equally likely. In the scenario above the students appear to have an equiprobability bias and they have created an experiment to verify this misconception.

Although independence is not the main mathematical concept of this scenario, the questions the teachers suggested, display pedagogical content knowledge. Their suggestions for working with these students were to ask them the following questions:

- If you walked into the maze, would you see all six paths?
- If you rolled a five how would you know the path to follow?
- Is there an equal chance for each path? Why?

The first two questions appear to be designed to help students understand the maze as a compound rather than simple event. In addition, the last question they came up with suggests the teachers may be aware of the equiprobability misconception. Although the teachers did not identify this mathematical concept their question "Is there an equal chance for each path? Why?" suggests they are trying to help students with this misconception.

Scenario 2

The next scenario deals with issues of sample size. Although the teachers initially struggled with understanding the need for large samples in order to approximate theoretical probabilities (i.e., the law of large numbers) when they solved the problem, most understood the need to run more than one trial. The scenario below occurred in only one classroom because the majority of the teachers required their students to run a certain number of trials rather than leaving this decision to their students.

> *In a classroom, a teacher did not specify the number of trials for each group of students to run. One group did only one trial and determined Room A was most likely. Another group ran twenty trials and determined that Room A and Room B were equally likely.*

The teachers identified three mathematical concepts associated with this scenario: understanding of event, experimental probability and law of large numbers. The teachers were accurate in identifying the law of large numbers and experimental probability as key ideas within this scenario. Their suggestions for working with these students were the following three questions:

- Why just one trial? Is this enough information to make a decision?
- Why stop at 20 trials?
- How can you prove room A and room B are equally likely?

These questions appear to be consistent with the mathematical concepts and might support students developing ideas. For instance, questions such as why only one or why twenty suggest the need for more trials. It is not clear from the teachers' questions if they were aware of the misconception first identified by Konold (1989; 1991) called the outcome approach. Students with an outcome approach to probability believe they need to predict what will happened on the next trial of an experiment rather than what is likely to occur or what will occur most often after numerous trials. Students holding this misconception would not see a need for performing more than one trial. Konold's research suggests students with this misconception would feel comfortable making a decision based on only one trial.

Misconceptions surrounding the effect of sample size are well-established within the literature (e.g., Tversky and Kahneman, 1982) and are notoriously difficult to modify. Although it is not possible to know how students would respond to, "Why stop at 20 trials?" or "How can you prove room A and room B are equally likely?" it is likely that the teachers are underestimating how difficult it is to help students see a need for a large sample size.

Scenario 3

Unlike the other scenarios the final scenario we chose to share in this paper was not initially identified while the teachers solved the maze problem. This scenario is related to the overgeneralization of rational number concepts in determining experimental probabilities.

> *In a classroom, after all trials were run and experimental probabilities were recorded on the board, a teacher asked students to find the experimental probability for the entire class. The results from the class were as follows: 3/20,*

7/20, 8/20, 9/20, 6/20, 5/20. One of the students found that the experimental probability for the class as 38/20.

The original interaction differed slightly from this scenario. The students in the class had all run a different numbers of trials and it was not a student but the teacher that asked the students to "find a common denominator and add the fractions." The students were currently learning about the addition of fractions and the teacher believed this would be an excellent opportunity to practice the concept.

The participating teachers identified the mathematical concept in the following manner, "The student lacks an understanding of why you don't add the fractions. The part and total (whole) both changed." Although the language of fraction, part, and total are problematic because it can lead to the misconception originally exhibited of "adding the fractions" the teachers appear to have identified the main mathematical concept of ratio. The importance of understanding ratio and relative frequency is critical for students in the development of probabilistic thinking (Garfield and Ahlgren, 1988; Fischbein, Pampu, and Manzit, 1970). These and other studies highlight the need for students to be careful of overly generalizing their learning of rational number concepts and procedures.

The teachers' suggestions for these students were the following:

- What does 38/20 mean?
- Is 38 out of 20 more than or less than 100%?
- Is it possible to have a probability greater than 100%?
- Why should we add the top and bottom and not just the top?
- What is the total number of trials for the entire class?
- Was the experiment only performed 20 times for the entire class?

The teachers' first three questions focus on the incorrectness of reporting a probability greater than one. The last three questions focus on trying to help the students understand that as the total number of trials changes it is important to reflect this change in the ratio that is reported. These questions, however, do not directly attack the misconception that combining experimental probabilities is different than adding fractions and the underlying conceptual difficulty.

Conclusions

Each of the scenarios above provide a glimpse of the ways in which teachers must deal with a variety of content and pedagogical issues raised by students when working with an open-ended mathematics task. These issues are particularly difficult for teachers when the content addressed within the task is unfamiliar and often counter-intuitive such as probabilistic reasoning. As exhibited by the final scenario often the misconceptions of students are those also held by teachers and most adults (Fischbein & Schnarch, 1997). In addition, it appears from each of these scenarios that the teachers had a difficult time correctly identifying and naming the main mathematical concepts displayed in each scenario. Due to the importance of language and communication in developing mathematical ideas (Cosogno, Gazzolo, Boero, 2006) this difficulty may lead to continued misconceptions by students. On the other hand the teachers did appear to directly address the main misconceptions of each scenario in their questions for the students, which suggests their pedagogical content knowledge might occasionally out pace their content knowledge. This finding is particularly interesting given the extensive studies documenting teachers limited content knowledge and its potential impact on student understanding (e.g., Ball, Hill & Bass, 2005).

Implications

This project is an attempt to move forward our understanding of the specialized knowledge of mathematics necessary to effectively teach probabilistic thinking. Although a focus on one problem limits the number of topics within probability that might be explored, we argue that this narrow slice allows for a different, but more connected and nuanced view of teacher content and pedagogical content knowledge than has been previously reported in the literature. Clearly some topics associated with the maze problem were more fully explored by our teachers than others because of the individual demands of the age of their students, their state standards, and the affordances and constraints of the maze problem. However, the benefit of providing a common context from which teachers and teacher educators can explore probability we believe outweighs the limitations of focusing on a single task.

Our goals were to help both teachers and teacher educators develop their understanding of how to support the development probabilistic reasoning in students. Since teachers are likely to exhibit some of the same misconceptions as their students, there is a strong need for teachers to think deeply about the mathematics of the task before implementation. Moreover, this experience can provide opportunities to analyze different solution strategies, possible student misconceptions, and possible pedagogical moves.

The strong link between misconceptions of students on the maze problem and the literature on student thinking in probability provided a grounded opportunity to examine teacher thinking and extend the research on specialized knowledge of mathematics in the area of probability.

This study highlights two areas for further research to better understand specialized knowledge of mathematics. First, teachers need support in selecting and using appropriate mathematical language for diagnosing and discussing the mathematical concepts of a task, and in their interactions with students. Second, teachers may benefit from exposure to research on student thinking directly tied to problems of practice, such as the scenarios described in this paper. Research on professional development that incorporates these key features will continue to expand our understanding of the specialized knowledge of mathematics necessary for teaching probability.

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MEANINGS RELATED TO STATISTICAL INTERVALS AND DISTRIBUTIONS AS CONSTRUCTED IN A COMPUTER SIMULATION ENVIRONMENT

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In the present article results of a research with undergraduate students about confidence intervals and their relationship with sampling distributions in a simulation environment are reported. The results show that the students developed correct meanings of concepts like influence of the sample size in sample variability and margin of error in the intervals, but not in the wideness of the intervals. Some difficulties in the interpretation of results of sampling distributions were detected.

A fundamental concept that gives the base for the study of the statistical inference is sampling distributions. A sampling distribution represents the value that a statistic can take (e.g. a proportion) in each random sample in a size given that it is possible to extract them from the same population.

The experience from many teachers and results of some research coincide in that sampling distributions constitute a difficult concept for many students. According to Chance, delMas & Garfield (2004), the difficulty resides in that the sampling distributions require integration and combination of many statistics ideas, such as distribution, sample, population, variability and sampling. Lipson (2002) on his side, attributes the difficulty to the dynamic process of sampling and the diversity of mathematical and symbolic representations of the concept. Other researchers like Moore (1992), associate the difficulties in comprehension of the statistical inference and sampling distributions, to the traditional teaching that has prevailed up to now, -mostly in university level-, where a approach based on theory of probability is adopted, which is considered one of the most complex areas of mathematics.

In the presence of such difficulties, the teaching of sampling distributions and statistical inference by a frequency approach based on computer simulation has been searched. Research works made by Meletiou-Mavrotheris (2004), Lipson (2002) and Cumming and Fidler (2005) suggest that a dynamic environment along with multiple representations may help students to develop an adequate comprehension of these topics. Our research work finds itself in this line.

In the present work we have established the following question, what meanings do students assign to confidence intervals and its relationship with sampling distributions in a computer simulation environment, like the one that Fathom software gives?

Theoretical Framework

In this study, we use the meaning of a mathematical object as stated by Godino and Batanero (1998). In this model, the meaning is conceived as a practical system used by people to solve a field of problems where the concept arises. This system can be observed by means of various elements involved in mathematical activity developed by people in solving problems. There are five elements or meaning components:

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1. *Problem-situations (phenomenological elements)*. It refers to a field of problems or situations where the concept under study arises.
2. *Language (representational elements)*. It is any verbal or written representation used to represent or refer to concepts and features involved in a problem.
3. *Actions (procedural elements)*. These are the procedures or strategies used in solving problems.
4. *Concepts and properties (conceptual elements)*. These are concepts, properties and their relationships with another concepts involved when solving a problem.
5. *Argumentations (validative elements)*. The argumentations or validities used to convince others of the validity of our solutions to the problems or the truth of the properties related to the concepts.

The comprehension of a mathematical object consists of a continuous and progressive process where students acquire and connect different elements involved in the meaning of concept.

Methodology

The study was carried out with a group of 24 students from the Public Politics School of the Autonomous University of Sinaloa (Mexico). The study was conducted when an introductory course on Probability and Statistics was taken by the students. Before of each simulation sessions, the students solved problems by means of probability formulas and tables. Each of these topics was accompanied by a theoretical explanation with computer simulation by the teacher.

Three activities were designed (1.5 hours each one) that involved sampling distributions of a proportion, and they were linked with the confidence intervals. Each of the students had its own computer to solve the activities. At the end of the activities a questionnaire was provided to evaluate its understanding and some students were interviewed. A learning process was designed where the participants' students themselves built a conceptual meaning based on the solved problems. The first two activities emphasized the discovery of sample distribution properties and sample variability mainly. The third activity was centered in exploring relations in sample size, distributions and reliance intervals.

The software Fathom (Finzer, 2002) allowed having three different simultaneous representations, bound to each other, from the same situation (numerical, symbolic and graphical). Such form allowed, by pressing a key or modifying some parameters, the immediate generation of 1000 more samples and to visualize the changes in the representations. At the end of the activity, the students obtained three sampling distributions for sizes of sample (20, 50 and 100) in the same screen, which allowed them to visualize patterns of behavior in the different representations and to establish conjectures about the sampling variability, the form in which the sample size can affect the behavior of the sampling distributions, the dimensions of the confidence interval and the margin of error.

Results and Discussion

In the analysis of meanings constructed by the students we have considered the elements that according to Godino and Batanero´s model (1998) compose the meaning of a mathematical object. When these elements are acquired and connected by the students among a series of activities (in this case computer simulation activities) they inform us about the developed comprehension. We have taken these meanings from student work with class computer sessions, from work forms that were designed for each activity and from some personal interviews.

Phenomenological Elements

The three situations consisted of sampling distributions of a proportion. A representative example of the type of situations contemplated is the following:

A survey made in the Faculty of International Studies and Publics Politics, with a sample random of 20 students, indicates that 6 are smokers, that is to say, 30%.

a) It generates sampling distributions with 1000 samples of size 20, 50 and 100 respectively.

b) It calculates the mean and the standard error of each distribution.

c) It determines a confidence interval of 95% for each distribution and its respective margin of error.

Representational Elements

In the construction of the sampling distributions process, the students made an intensive use of representations (cases tables, charts, formulas and descriptive tables), to explore the relationship between the sample size and the behaviour of distributions and intervals (see figure 1). This contributed to that many students showed an adequate comprehension of the effect of the sample size in sampling variability and the margin of error of intervals, like we will see in the following paragraphs.

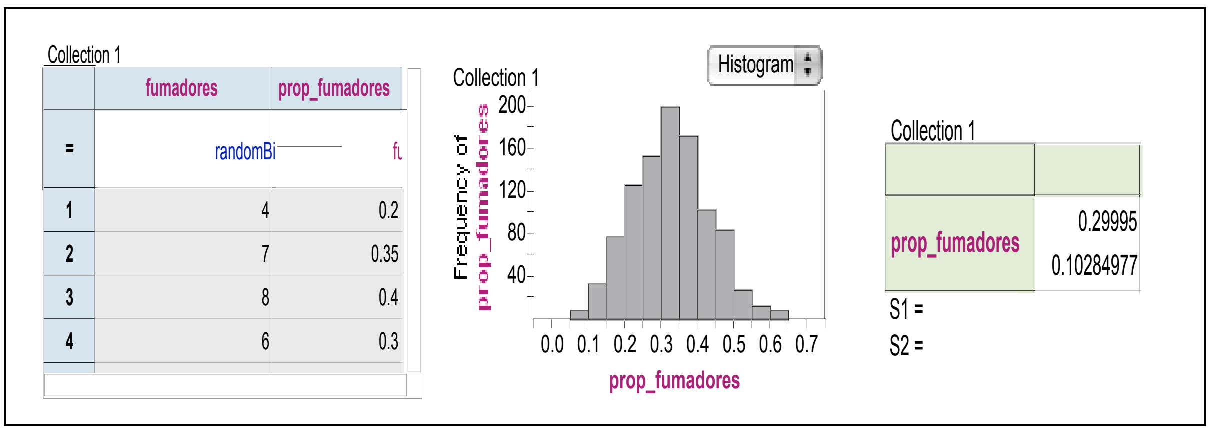

Fig. 1: Some used representations in the sampling distributions exploration.

Procedural Elements

In the simulation process of a sampling distribution, the students were required to identify the parameters of the population (n and p) to simulate the extraction of the samples, subsequently in another column they calculated the proportion of the samples and graphed them (see figure 1), also they constructed a summary table with the most important descriptive measurements.

This part of the process did not imply major difficulties, because of the software flexibility and the experience of the first activity; they were able to continue with the rest with a minimum intervention of the researcher.

Conceptual Elements

The main concept around which the attention of the study was centered was the sampling variability, the effect of the sample size in the sampling distributions and its relationship with the confidence intervals.

In the course of the activities the majority of the students demonstrated an adequate conception of the sampling variability and the expected value of the proportion in one sample. This was due to the activity of the students with the software in which they simulated a repetitive extraction of samples and observed that the results, if they varied from one sample to the other, frequently they corresponded to an interval close to the parameter. For example, once the simulation activities were concluded, in the context of a problem that signalled that in the Mexican population approximately 30% of obese children (binomial population), a student (Karina) was asked to answer what the results of the simulation in the Fathom cases table meant to her. (See figure 2)

Collection 1

	obesos	proporc...	<new>
1	39	0.39	
2	33	0.33	
3	35	0.35	
4	33	0.33	
5	27	0.27	
6	34	0.34	
7	26	0.26	
8	29	0.29	
9	33	0.33	
10	25	0.25	
11	27	0.27	
12	28	0.28	

Fig. 2: Simulation Developer by Karina

R: What does the first number in the table mean?
K: That in the first sample there are 39 obese children, 33 in the second.
R: How many obese children would you expect in a sample of 100?
K: 30
R: But there is a sample of 21.
K: The result is not always the same, they vary.
R: Could you tell me a variation interval?
K: Approximately from 25 to 35.

Another student (Rosario) in the same context answered as follows:

R: Could you explain the meaning of each of the numbers that appear in the table?
Ro: In case 1 there are 20 obese children from the 100 children in the population.
R: What do the 100 numbers represent?
Ro: 100 samples
R: Is it normal that the sample results vary?
Ro: Of course, you do not always get the same.
R: What would be the variation interval?

Ro: From 24% to 34% approximately.

We can see that both students have constructed a correct meaning of sampling variability since they identified that the results vary from one sample to another and they defined an acceptable variation interval around the parameter. This is due to the availability of a chart with a great grouping of samples and their graphic representation. In other words, the representational elements of the software have an important impact in the discovery of properties and the construction of the sampling variability concept.

On the other hand, the sample size influences in sampling distribution properties and its relationship with intervals, the activities were designed so that students simulated sampling distributions for sample sizes of 20, 50 and 100. Simultaneous histograms of distributions were used so that students identified behaviour patterns easily (see figure 3) and the results were captured in a chart to help interpretation.

In the first activity students made inaccurate descriptions of the behaviour of the distributions in relation with the sample size. But, in the following activities, they were asked to focus their attention in the center, the dispersion and the form to be able to discover the properties. Once more the software representations contributed in an important way so that the majority of the students constructed a correct meaning of the effect of sample size in the behaviour of sampling distributions. The interviews and the final survey demonstrate the results. For example in the final interview with a student (Adalberto) answered the following:

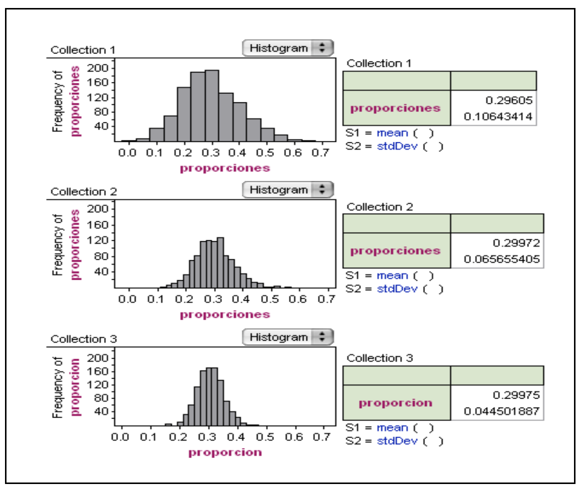

Fig. 3: Sampling distributions for different sample sizes

R: Do you think there is a relationship between the mean of each sample and the mean of the population?

A: Yes because in both the 100 sample size and in the 200 sample size there is no variation and they are near 0.30. That indicates that the 30% data about obese children of the population is important.

R: In the standard deviation? In the sample of 100 we have that it’s worth 0.04 as opposed to the sample of 200 that is 0.03. We can see that the biggest sample has a minor standard deviation.

R: What does that mean?

A: It gives us more confidence because the ranking is smaller.
R: Now construct a distribution with sample size of 500 and compare the three distributions.
A. Practically the mean is the same, it equals to 0.30, or 30%, and dispersions lower, now they are worth 0.02.
R: Which of the three is more precise?
A: The one with a sample size of 500 is more precise because the standard deviation is smaller and the mean is closer, actually is equal to the population proportion.
R. If we continue incrementing the sample size, what do you think will happen with the center of the sampling distribution?
A: It would be the same, it does not vary. It would be equal to 0.30 and the standard deviation will continue to decrease.
R: If we generalize.
A: For diverse sample sizes, we see that the mean is equal (it does not vary a lot) but as the population is bigger the dispersion is smaller.

We observe that Adalberto understands the effect of the sample size in the center and variability of the sampling distributions and he adequately relates it with precision. Karina answered as follows:

R: Construct another sampling distribution of a size of 200 and observe the differences in the center.
K: The mean is the same in both cases, 0.30.
R: Do you think there is a relationship with the 30% of the obese children in the population?
K: Yes
R: Why?
K: Because it's based on the 30% of the population.
R: What happens with the standard deviations?
K: It decreased from 0.04 to 0.03.
R: Now construct a distribution with a sample size of 500.
K: The mean is the same and the standard deviation decreases even more.

About the relationship between sampling distributions and intervals, students were asked to make calculations in a theoretical manner and by simulation. The result of the simulation was very similar and in many cases almost equal to the theoretical results. Thus, the simulation besides of allowing the exploration of the involved concepts and to help understanding the relation among them, can be a tool that provides results sufficiently precise to solve problems. For example, a student obtained the following results in a problem which parameter was p=0.30 and a confidence level of 95%.

Size of sample	Simulation results		Theoretical results	
	Interval	Error	Interval	Error
20	09% - 51%	21%	10% - 50%	20%
50	17% - 43%	13%	17% - 43%	13%
100	21% - 39%	9%	21% - 39%	9%

The fact of calculating the interval limits and the margin of error for the different sample size, and concentrate them in a chart contributed to the students understanding of the effect of sample size in the wideness of the intervals and the margin of error. In a question in the worksheet where they were asked to decide the sample size that gives a best precision (for a given confidence level), all answered 100. Also, they were conscious of the effect of increasing the confidence level in the wideness of the interval. Lets see some answers:

The intervals become wider, but the margin of error increases. (Rosario)

For the 99% they become wider and include all data. (Issel)

They become wider, but the result would not be precise. (Laura)

We can observe that these students have noticed that the confidence level increase the wideness of the interval, but at the same time it makes it more inaccurate and increases the margin of error.

In a final questionnaire, 17 students manage that statistics are variables and 15 of them identify that this variation was in concordance with a distribution form-predictable pattern. While, 21 students indicated that the statistics serve to estimate parameters. Twenty two of the students indicated that the variability of the sampling distribution diminished as the sample size increased and 18 indicated that the average distribution is not altered, and only three of the students identified that the mean of the sampling distribution is equal to the population parameter.

In an item that contained the graph of three sampling distributions, 19 students assigned the correct way of the sample sizes taking care of the variability and 17 correctly identify the average of the population from which the samples come. Respect to the sample size effect in the margin error of a confidence interval, all the students indicated that this diminishes it as increases the sample size, but 15 of them incorrectly indicated that the wide of the interval increases.

Some students had difficulties to interpret the simulation results when these are concentrated in the graph. They saw the sample distribution, when in fact it was a set of samples. This difficulty has been identified in a previous study with another type of students (Sanchez & Inzunsa, 2006).

Conclusions

From the design of the activities that were emphasized in the exploration of sampling variability, the search of relationship between the sample size, the sampling distributions and the confidence intervals, to the dynamic characteristics and the representational elements of the software were all key elements so that a lot of students constructed correct meanings of implicated concepts. Even though various students were not able to develop a correct meaning of some implicated concepts, the results show that there exists a potential in the computer simulation tool in dynamic environments to equip abstract and complex concepts with meaning, like sampling distributions and confidence intervals.

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TEACHERS' PROBABILISTIC THINKING RELATED TO THE REPRESENTATIVENESS HEURISTIC

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This study investigated teachers' understanding of probability related to situations that tend to elicit the use of the representativeness heuristic. Sixteen K-8 teachers' probabilistic thinking was examined before and after being involved in learning to use simulation-based activities designed to address misconceptions related to the representativeness heuristic. Findings from this study suggest that teachers can increase their probability content knowledge and develop more sophisticated understandings while also enhancing their instructional practices for teaching probability through simulation-based activities.

The existence of probabilistic misconceptions has been well documented for the general public (Shaughnessy, 1992; Kahneman, Slovic, & Tversky, 1982). One common misconception in probability occurs when a person assesses the probability of an event based on how representative the event is of its parent population or how closely the process by which it is generated reflects the randomness of the event (Kahneman & Tversky, 1982). In such cases a person is using the *representativeness heuristic* (Kahneman & Tversky, 1982). For example, the judgment that the lottery number 12345 is less likely to occur than the more "random" looking 93827; or the conclusion that after tossing a coin four times and getting HHHH that on the next toss it is more likely to get a Tails (to "balance out" the Heads and reflect the expected 50:50 ratio of Heads to Tails).

Although probabilistic misconceptions have been studied for the general public, very little research has specifically investigated teachers' understandings and misconceptions of probability and their pedagogical understandings related to probability (Stohl, 2005). Further, although the use of probability simulations has been advocated as an important way for students to investigate situations of uncertainty (NCTM, 2000), little research has investigated teachers' use of simulation tools for instruction as well as teachers' understandings gained from the use of simulations (Stohl, 2005).

The purpose of this study is to investigate teachers' understanding of probability related to situations that tend to elicit the use of the representativeness heuristic. This study examines K-8 teachers' probabilistic thinking before and after being involved in learning to use simulation-based activities designed to address misconceptions related to the representativeness heuristic. This study investigates the existence of teachers' misconceptions and the role of simulation-based activities as both a pedagogical tool and a learning tool for teachers. This study addresses the following research questions: (1) Do K-8 teachers' have probabilistic misconceptions related to the representativeness heuristics? (2) Do the probabilistic understandings of teachers change as a result of working with simulation-based activities designed for K-8 classrooms? (3) Are teachers able to retain these new understandings and transfer them to similar tasks?

Methods and Procedures

Sample

This study involved 16 practicing K-8 teachers involved in a K-8 mathematics specialist masters program in the southeastern part of the United States. These teachers were involved in a two-week summer mathematics content course on probability and statistics designed for K-8 teachers.

Procedures

Teachers were given four tasks related to the representativeness heuristic as part of a pre-test for the summer course. At the end of the two-week course a post-test was administered to the teachers that included these same four tasks. Approximately six months later, the same four tasks were administered a third time to the 16 teachers. In addition, at the six-month point, teachers were given four new, but similar, tasks related to the representativeness heuristic. These new tasks were given to test for transfer of knowledge.

The Summer Course

The two-week course focused on both content and pedagogy related to probability and statistics for grades K-8. During the two-week course teachers were involved in learning to use simulation-based activities involving both regular (e.g., coins, dice) and irregular (e.g., tacks, loaded dice) random devices. They also learned to use simulation tools involving the computer (e.g., *Probability Explorer*, [Stohl, 2002]) and the graphing calculator (e.g., *ProbSim* on the TI-84 Plus). Several activities were directly related to the representativeness heuristic and related concepts such as independence and the impact of sample size on variance.

Results and Discussion

The tasks involved in this study were based loosely on those used by Kahneman & Tversky (1982) and Shaughnessy (1981) to elicit misconceptions related to the representativeness heuristic (see Tables 1-4). Each of the tasks are presented in turn along with a summary and discussion of teachers' responses. Each table presents the frequency of responses for the four tasks on the pre-test, post-test (end-of-course), and transfer test (after six months). Finally, the four new tasks also administered on the transfer test are discussed (see Appendix).

For Task 1 (see Table 1), prior to the course, a majority of teachers answered that there would be about 50 heads. Their explanations reflected the idea that the number of heads to tails should coincide with the expected 50:50 ratio. One teacher explained in response to how many heads one would expect to get, "I guess 40 more times. Since there is a 50/50 chance that it will turn up heads each time you toss it, out of 100 it would be about 50 times." This explanation reflects the notion that the sample of tosses needs to balance out to 50/50.

Table 1. Description of Task 1 and the Frequency of Teachers' Responses.

An experiment requires you to flip a penny 100 times and record whether the penny comes up heads or tails. On the first 10 flips the penny comes up heads. After flipping the penny 90 more times, how many heads would you expect to get out of the total 100 flips? Explain.			
	55/45	50/50	Other
Pre	4	11	1
Post	15	1	0
Transfer	11	2	3

After instruction, a majority of the teachers answered that they would expect 55 heads. Their explanations reflected the notion that the second set of tosses was independent from the first 10 tosses and thus the total number of heads would not reflect the 50:50 ratio. One teacher who changed her answer from the pre-test explained, "because each flip is independent you still have a 50/50 chance each time, 50% of the remaining flips would be 45 plus 10 heads already flipped would be 55." After six months, based on their responses, eleven of the teachers seemed to maintain this level of understanding for this task. Three teachers' provided 'other' answers, two of which seemed to reflect a correct intuitive sense for this task, but their explanations were incomplete or non-existent.

For Task 2 (see Table 2), initially, a majority of the teachers correctly explained that either Heads or Tails was equally likely to occur after four successive tosses of a Head, however, there were five teachers that believed that it was more likely for a Tails to occur. For example, one teacher explained, "I would expect it to land on tails simply because the common expectation is that it has landed on heads 4 times, so it is 'time' for it to land on tails." After instruction, 15 out of 16 teachers explained that either Heads or Tails was equally likely. Most of the teachers correctly explained their choices based on the notion of independence of events. For example, one teacher who initially chose Tails explained, "there is still an equally likely chance that the coin will land on either heads or tails. The flipping of the coin is an independent event of the previous 4 trials." Interestingly, several teachers still expressed intuitions leaning toward Tails although they correctly explained why either Heads or Tails was equally likely to occur. After six months, based on responses, teachers overall seemed to maintain a correct understanding of this situation, and based on their explanations seemed to maintain an increased understanding of why their choice was correct.

Table 2. Description of Task 2 and the Frequency of Teachers' Responses.

If a fair coin is tossed, the probability that it will land heads up is 1/2. In four successive tosses the coin lands heads each time. What do you expect to happen when it is tossed a fifth time? Explain your answer.

	Heads	Tails	**Either**	Other
Pre	0	5	11	0
Post	0	0	15	1
Transfer	0	1	14	1

For Task 3 (see Table 3), seven out of 15 teachers initially chose HTTHTH as more likely to occur, most often claiming that this choice reflected the expected 50:50 ratio of Heads to Tails. One teacher explained that HTTHTH "represents 1 head per 2 flips, 1:2 or 50% chance." After instruction, all of the teachers chose "the same chance for each of these sequences," and after six-months 15 out of 16 teachers chose the correct response for this task. However, several teachers seemed to base their correct choice on the idea that each individual flip in the sequence was independent making each of the sequences equally likely, "the coin has a _ chance each time it is flipped to come up heads or tails. It doesn't remember what it did before," instead of treating each sequence as an individual event among a sample space of equally likely events. For example one teacher explained, "out of a possible 2^6 set of possible outcomes, these possibilities are each equally likely." Although the later represents a subtle shift in explanation, it represents a more complete understanding of the situation. Overall, teachers' intuitions about the situation seemed to have shifted away from a dependence on the representativeness heuristic; however, only six teachers' explanations represented this complete understanding on the post-test.

For Task 4 (see Table 4), only eight out of 16 teachers initially chose the small hospital as more likely to have more days when at least 60 percent of the babies born were girls. As one teacher claiming that it makes no difference explained, "it wouldn't make a difference because it is a percent of boys to girls, so it doesn't matter how many of each." However, after instruction, all but one of the teachers chose the 'small hospital.' Moreover, several of the teachers correctly used their understanding of the law of large numbers and its relationship with sample size to describe the differences in variability around the expected value and thus that the number of girls born in the small hospital would be more likely to be at least 60 percent. As one teacher explained, the "law of large numbers states that the bigger your sample size of data the closer you will be to the theoretical probability. You can deviate more from the TP [theoretical probability] in a smaller data size." After six months, 13 of the 16 teachers still chose the 'small hospital' and in most cases were able to correctly explain their answers.

Table 3. Description of Task 3 and the Frequency of Teachers' Responses.

If a fair coin is tossed, the probability that it will land heads up is 1/2. If the coin is tossed 6 times which of the following is more likely to occur? Choose one.

a) `HTTHTH` *b)* `HHHHTH` *c) The same chance for each of these sequences.*
Give a reason for your answer.

	HTTHTH	HHHHTH	**Same**
Pre	7	0	9
Post	0	0	16
Transfer	1	0	15

Table 4. Description of Task 4 and the Frequency of Teachers' Responses.

The chance that a baby will be a girl is about 1/2. Over the course of an entire year, would there be more days when at least 60 percent of the babies born were girls:

a) In a large hospital b) In a small hospital c) Makes no difference.
Give a reason for your answer.

	Large	**Small**	No Difference	Other
Pre	2	8	5	1
Post	1	15	0	0
Transfer	2	13	1	0

At the six-month point four additional tasks were also administered to the teachers (see Appendix). These tasks were created to assess transfer of understanding associated with the representativeness heuristic as reflected in the first four tasks. Transfer Task 1 was intended to assess transfer of understanding associated with the original Task 3. In this new task, instead of working with sequences of coin tosses, the item referred to lottery numbers and asked which of the numbers is more likely to be a winner. For this new task all 16 teachers indicated that the three lottery numbers had the same chance of winning.

Transfer Task 2 was intended to assess transfer of understanding associated with the original Task 4. In this new task, instead of working with births in large and small hospitals, the item referred to drawing different sized samples of balls (10, 100, or 1000) from an urn containing black and white balls in a 50:50 ratio in hopes of getting a sample of balls that is at least 70% black. All of the teachers indicated that a sample of 10 balls would be most likely to result in 70% black.

Transfer Task 3 was intended to assess teachers' understanding related to the notion that when conducting an experiment that the experimental probability approaches the theoretical probability as the sample size gets larger, but that in absolute number the average discrepancy from the theoretical value gets larger. For example, in this task, with the large number of tosses (100,000) the proportion of heads to total tosses would be very likely to be close to 50% (i.e., within 0.51 and 0.49). However, from an absolute point of view the discrepancy between the number of heads and tails could be large in absolute number (e.g., 2000) and still represent a proportion relatively close to the theoretical. All teachers indicated that it would be very likely that the proportion of heads would be between 0.49 and 0.51; however, half of the teachers also indicated that it would be very likely for the number of heads to be within 300 of the number of tails (and one additional teacher who chose 'False' but who's explanation indicated otherwise). These teachers' explanations seem to convey correctly that as the sample size gets larger that the proportion of heads would approach 50%: "these numbers are close to 50% or _ which is what you expect." However, these teachers seemed not to differentiate between the proportion of heads and the absolute number of heads and recognize that it is unlikely for the number of heads out of 100,000 tosses to be close to 50,000 (e.g., within 50,150 and 49,850). Of the eight teachers who indicated that it would not be very likely for the number of heads to be within 300, only three gave clear explanations for their choice, the others gave incorrect or incomplete explanations, or only focused on the proportion being close to 50%. This inconsistency may suggest that some teachers still maintain some notion of "balancing out." However, given the overall success on the other tasks, it is unclear whether this represents a clear dependency on the representativeness heuristic, but instead a misunderstanding of the relative proportions, an overextension of the law of large numbers, or a level of task novelty beyond the understanding necessary for success on the other tasks. Further investigation would be necessary to clarify this inconsistency.

Transfer Task 4 was intended to assess transfer of understanding associated with several concepts related to the representativeness heuristic. In this task participants were asked to choose from two sequences of 150 tosses of a coin the one that was "made up" and not based on an actual experiment. Most of the teachers (13 out of 16) chose James' sequence as the one that was made up explaining that the sequence, for example, "bounced back and forth between H/T very evenly," "never has more than 3 in a row," "seems to follow a pattern," or "looks like he is trying to make it look like 50-50." Two other teachers indicated that they couldn't tell, one indicating that these sequences were both equally likely. Overall, explanations reflected a recognition of the notions of the representativeness heuristic that are often misapplied in situations of uncertainty.

In general, responses to the original four tasks indicate initial dependence on the representativeness heuristic in situations of uncertainty. Results from the tasks after instruction indicate change in teachers' understandings, in that, teachers relied less on the representativeness heuristic and were, to a greater extinct, able to provide better explanations for their answers. After six months these understandings seemed to be maintained and for the most part were transferred to new tasks. However, the novelty of one of the transfer tasks may have been too much for the understandings gained from the work with the simulation activities.

Conclusions

The purpose of this study was to investigate teachers' understanding of probability related to situations that tend to elicit the use of the representativeness heuristic. In particular this study considered the role of simulation-based activities for enhancing both the pedagogical and content knowledge of teachers related to their understanding of probability. Based on the results of this study, some teachers were initially found to have probabilistic misconceptions related to situations involving the representativeness heuristic. However, after instruction involving simulation-based activities, overall, teachers' seemed to be less likely to depend on the representativeness heuristic to make probability assessments. The explanations for their answers were also found to be more sophisticated and reflect understandings of independence and sensitivity to sample size. Further, in general, these understandings were maintained for at least six months, and were successfully applied to new tasks, although these understandings may be limited to more familiar situations and less transferable to the most novel situations. Overall, findings from this study provide promising evidence that teachers can learn probability content and develop more sophisticated understandings while also enhancing their instructional practices for teaching probability through simulation-based activities.

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Appendix

Transfer Tasks and Frequency of Teachers' Responses

Number of teachers choosing a particular response is in parentheses and bolded. The correct response is in italics.

(1) In the State Lottery game of Pick-5 a player picks 5 numbers from 0-9 and the order in which they will come up. Which of the following sequences is more likely to be a winner?

a) 12345 **(0)** b) 66667 **(0)** c) 59372 **(0)** *d) The same chance for each of these sequences.* **(16)**
Give a reason for your answer.

(2) You are selected to participate in a game with the possibility of winning $5,000. In this game you will choose balls with replacement from an urn that contains 50% black balls and 50% white balls. In order to win the $5,000 the sample of balls that you draw must be at least 70% black. You can choose to draw 10, 100, or 1000 balls. Which would you choose?

a) Draw 10 balls **(16)** b) Draw 100 balls **(0)** c) Draw 1000 balls **(0)** d) It makes no difference **(0)**
Give a reason for your answer.

(3) If a fair coin is tossed, the probability that it will land heads up is $\frac{1}{2}$. A fair coin is tossed 100,000 times. Indicate whether the following statements are true or false.
a) It is very likely that the number of heads will be within 300 of the number of tails.
True **(8)** *False* **(8)**
b) It is very likely that the proportion of heads will be between 0.49 and 0.51.
True **(16)** False **(0)**
Explain your answers.

(4) Two students were each asked to toss a penny 150 times. One of the students did the experiment and recorded the results and one student made up their results. The students recorded their tosses in order using **H** for Heads and **T** for Tails. Following are the students' reports. Which student made it up? How can you tell?

James: **HTHTHTTHHTTHHTHTHTHHHTHHTTHTHTHHTTTHHTHTHTTHHTH**
THHHTTTHTHHTTHTHTHHTTHTHTTHHTTTHTHTHHTTHHTTHTHT
HHTTTHTHTHTHTHHTTHTHHTTHHTTTHTHTHTHTHHTTHTHHTTH
HHTTHTHHH
Adam: **HTHTTTTHTHHTHTHTHTTTHTTHHTHTTHTHTTTHTHHHTHHHHHT**
HHTTHTHHHHHTTTTTHHHTTHHTHHTHTTHTHHTTTTHTTHHTHTH
TTHTTTTTHHHHHHHHHTHHTHTTHHTTTHHHHHTHHTTTTHHTHT
HTTHHHHHT

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James **(13)** Adam **(1)** You can't tell **(2)**

UNDERSTANDING ARITHMETIC MEAN

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Arithmetic mean is a concept which, although simple in form, is quite complicated to fully understand. The purpose of this exploratory study was to examine how experts understood arithmetic mean. Analysis of task-based interviews led to an articulation of two distinct conceptions of arithmetic mean: understanding the algorithm for arithmetic mean and understanding arithmetic mean as a mathematical point of balance. The experts did not have a readily-available description of how these conceptions were interrelated. Further analysis of a follow-up telephone interview led to a hypothesized connection between the conceptions through the use of leveling-off. This hypothesis lays a foundation for future studies of how children may develop a fuller and more connected understanding of arithmetic mean.

The arithmetic mean is a foundational concept for the development of basic data analysis and statistical understandings. Although quite simple from the standpoint of the algorithm, it is a quite complex concept to understand (Mokros & Russell, 1992). Research has been conducted to examine either how children might best come to understand arithmetic mean or what they currently understand (Brousseau, Brousseau, & Warfield, 2002; Cai, 1998; Cai & Moyer, 1995; Cobb, 1999; Cortina, 2001, 2002; Zawojewski & Shaughnessy, 2000). Descriptions of what is needed to understand arithmetic mean deeply have been limited to non-research based reflection. The research study presented here was conducted to examine how experts understand arithmetic mean. The descriptions of their understandings of arithmetic mean are used to hypothesize how educators may help students construct a deeper understanding of arithmetic mean.

Theoretical Framework

The current study was basically constructivist in nature, building upon the major premise that knowledge is not "passively received through the senses or by way of communication, but must be actively built up by the cognizing subject" (Von Glaserfeld, 1995, p. 18). This building up occurs in the learner through a process of assimilation and accommodation of understandings as described by Piaget (Singer & Revenson, 1996). That is, students learn by reflecting on what they have done. They then either organize what they have reflected upon by fitting it into an existing schema (assimilation) or by reorganizing their schema (accommodation) to take the new reflection into account. To develop instructional interventions that assist students in "building up" their understanding of a concept, it is useful to have a sense of what the students currently understand and a description of what the goal understanding might be. In this way, thoughtful instruction can be developed that allows students to reflect on the instruction in such a way that they build up from the current understanding to the goal understanding.

Because the study is based on constructivism, the articulation of the experts' understandings are my description, as a researcher, of what I perceive to be the critical concepts, or relationships between concepts, of others' constructed understanding of arithmetic mean. A goal of the articulation is that it be useful for mathematics education. The

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descriptions of understanding are used to hypothesize ways students may construct understanding of arithmetic mean.

Methodology

The study was exploratory in nature. To develop an articulation of understandings of arithmetic mean, task-based interviews were used to probe the nature of the understandings of several individuals. These individuals were chosen because of their likelihood of possessing sophisticated understandings of arithmetic mean. The methodology was designed to provide information to explore the various ways the experts were thinking about arithmetic mean. An initial interview was developed based upon possible understandings of arithmetic mean that were identified from existing research. After conducting the initial interview, the tasks were adjusted to enhance the information about the nature of experts' understanding of arithmetic mean. For example, in the initial interview the expert was given the following question (titled the "Class Average Problem"):

> *In a certain class there are more than 20 and fewer than 40 students. On a recent test the average passing mark was 75. The average failing mark was 48 and the class average was 66. The teacher then raised every grade 5 points. As a result the average passing mark became 77 1/2 and the average failing mark became 45. If 65 is the established minimum for passing, how many students had their grades changed from failing to passing?*

The expert ended up using variable replacement and algebraic techniques to solve the problem. This particular way of solving hid possible understandings of arithmetic mean, so they task was changed. Experts in the rest of the interview were asked the following problem and were also asked not to use algebra or variable replacements in their solution.

> *A class of students took a test. The class average on the test was 68. The average grade of the students who passed was 80 and the average grade of the students who failed was 64. What percentage of the class passed?*

The interview tasks were designed to help answer questions raised through the review of pertinent literature. The tasks used were designed to be open-ended in order to encourage the person being interviewed to answer in ways that may not have been predicted by the researcher. Three experts were interviewed initially. An analysis of the responses in the interview was then conducted and the results were used to hypothesize and articulate what might be the understandings of arithmetic mean expressed by the experts. The interview questions were then studied. Wording that was considered unclear by the researcher was adjusted and additional questions to probe newly hypothesized understandings were added. A subsequent interview was then created and conducted with four new participants. The analysis of this second phase of interviews was used to continue to hypothesize and articulate understandings. Two very different ways of understanding arithmetic mean were articulated. These understandings appeared to be disjoint in the experts' ways of thinking. This led to a focus in the study on the possible connections of the two different understandings. A telephone interview was then conducted with all previous participants in order to clarify understandings related to this focus. The analysis of this new interview added to the overall articulation of understandings of arithmetic mean.

Results

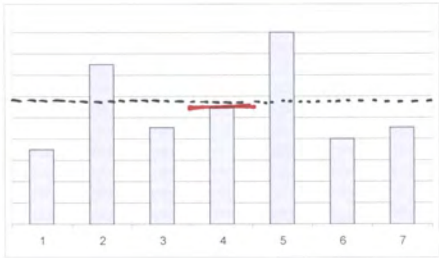
Two distinct conceptions of arithmetic mean appeared to be used by the participants. The first conception is that of understanding the algorithm for arithmetic mean. The second conception is that of understanding arithmetic mean as a point of balance in the data set. Both conceptions have been referred to in previous literature from researchers' reflections on personal understanding. The presented articulations here use research as a basis for the descriptions and attempt to be more specific.

The experts who were interviewed all understood the algorithm for the arithmetic mean. When working with data sets they examined the total accumulation and then shared this accumulation equally with each data point. The experts demonstrated an understanding of the division that was being done as partitive division. When the given representation (i.e. individual data point each represented by a bar in a bar graph) allowed, they would share the accumulation without finding the total accumulation first. The experts also understood that the result of the fair sharing was a per-one ratio; the amount any one data point would be if all accumulation were equally shared. For example, when asked how arithmetic mean was different from the median, one expert described a situation where one person had a lot of money in his pocket and nine people had very little. This was not a good situation for the arithmetic mean because the average money would be "mostly that one person's money." The expert understood that the total accumulation was shared equally and was supposed to represent what the accumulation would be per one person.

Additionally, the experts viewed the arithmetic mean as a ratio-as-measure (Simon & Blume, 1994). That is, the arithmetic mean could be used to replace an entire data set with just one number. The arithmetic mean algorithm (add everything up and divide by how many points you have) results in a per-one ratio. However, this one ratio can represent the entire data set. In the Class Average Problem (presented in the methodology section), one expert used the given average to represent the entire set of students that were in the classification "passing" and the other given average for those students in the classification "failing". He did not view it as a list of the same number repeated a certain number of times (as other expert did) but rather the one number (ratio-as-measure) represented the entire group. The experts did not view the arithmetic mean as simply a number result of a particular formula; rather they understood the implications of the algorithm as the result of partitive division and as a ratio.

The second way that experts understood the arithmetic mean was as a mathematical point of balance. The experts in the study used the word "balance" to describe the work that they were doing with the arithmetic mean. The activity that was most often related to the use of this term was that of equalizing the deviations from a proposed arithmetic mean. When the participants were attempting to find that point of balance, they would propose an arithmetic mean, and then check to see if the deviations on either side of the proposed arithmetic mean were either equivalent (absolute deviations) or cancelled out one another (signed deviations). One problem given was that of a bar graph that depicted building heights. The experts were asked to find the average building height.

Building Height Problem



One expert drew the dashed line to propose a mean and then began to check to see how accurate his estimate was. He added up the units of bar above the proposed mean and compared these to the units of space to the top of the bars below the proposed mean. His goal was to have these amounts be the same. This activity gave him the arithmetic mean.

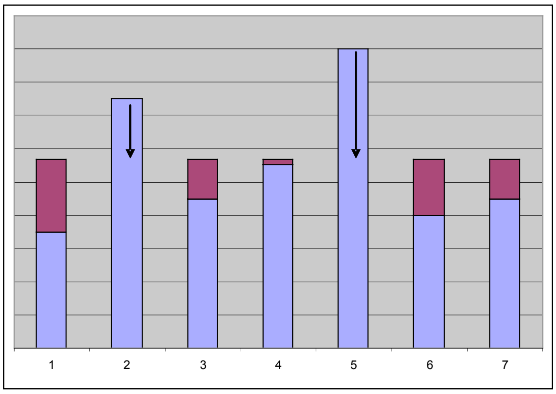
In another task, another expert represented the deviations from his proposed mean numerically and then "cancelled out" positive and negative deviations. A hallmark of all of the experts' understanding of balance was that balance was not some notion based just on physical experience, but rather was a mathematically precise notion of equivalent deviations. The activity of equalizing deviations was how the expert found the mathematical point of balance.

The study not only highlighted the two articulated conceptions of arithmetic mean but also highlighted the point that the experts did not have an overt way to discuss how these two understandings were related other than that they were both the arithmetic mean. The follow-up telephone interview included the question, "Could you explain how you understand that the arithmetic mean defined by the standard algorithm and the mean as a mathematical point of balance are the same thing?" The experts did not have a way (beyond algebra) to show how both concepts were the same even though they could move seamlessly between the understandings. The experts did, however, have a way to think of arithmetic mean that connected individually with each conception. The experts used leveling-off as a way to think about the arithmetic mean when working with both conceptions.

Discussion

The data led me to hypothesize a possible way to relate these understandings through the use of leveling-off. Leveling-off can be used with a bar graph representation of *individual* data points. Each bar's height represents the value of one data point. The bar graph of individual points is not a typical representation for data points in a data set. When using leveling-off, the total value of all the bars is never changed. Instead, taller bars have pieces "taken off" and shorter bars have these pieces added to their bar. The activity of moving pieces is continued until the bars are all the same height.

Leveling-Off Illustrated



The experts understood the algorithm for arithmetic mean as a result of partitive division. The data values were accumulated and then shared fairly with each data point. This was equated to leveling-off by suggesting that the fair sharing could be done by simply moving pieces of a bar graph (or numerical amounts) until every data point was the same value. Thus, when leveling-off was completed, the height of one bar was equated with how much value there was per-one data point.

In other tasks, the experts used leveling-off to explain how they were thinking about arithmetic mean as a mathematical point of balance. The activity of leveling-off allowed the experts to "visualize" the deviations from a proposed mean. In order to obtain balance, and thus find the arithmetic mean, the pieces over and the pieces under had to be equivalent. The experts implied that leveling-off would find a point of balance because any amount over the mean exactly matched an (absolute) amount under the mean. In this sense, the leveling-off was the same as numerically cancelling deviations. When leveling-off to find a point of balance, the experts focused on the deviations from the mean and their equivalence.

Because the experts seemed to connect the notion of leveling-off with both conceptions, it was reasonable to consider the notion as a clue to one aspect of how the two conceptions might be connected. As the experts completed tasks, they moved seamlessly between the two conceptions using the conception with which it was easiest to work in the given task or representation. How did they develop this implicitly connected understanding that allowed them to work flexibly with both conceptions of arithmetic mean? The following is a hypothesis of how a connected understanding may develop in a child. It is generated from the data which suggest that leveling-off is one possible connecting strategy.

A child's earliest experiences with the concept of arithmetic mean may be with two data points. In finding the arithmetic mean, the child may use a variety of ideas. They likely have experiences that include finding the midpoint between the two data points. This might be done physically, such as on a dot plot or number line, or it may be done numerically by adding the two values and using division to share the total amount. The child may use a leveling-off strategy such as increasing one number and decreasing the other number by the same amount. Or, the child may think about the midpoint as equidistant from both values and use a primitive notion of balance. This assortment of experiences might contribute, for some learners, to a connection among notions of leveling-off, partitive division, and balance.

When the data sets become larger and more demanding, the connection among these ideas may become less obvious and therefore more difficult to make explicit. The data in this study suggest that the notion of leveling-off may (perhaps not explicitly) continue to serve as a connecting idea. An early understanding of arithmetic mean is to conceive of the arithmetic mean as a fair sharing of total accumulation. This conception appears to initially involve a specific order: First find the total accumulations and then share that accumulation equally among all data points. This is an activity (often described by the operation of division) that requires a total amount. However, this can develop into understanding the activity of partitive division without first finding the total accumulation. For example, finding the arithmetic mean in a concrete situation such as a set of Unifix towers of different heights can be accomplished through leveling-off. Activities of this type may help the child develop a connection between the strategy of leveling-off and partitive division. Both of these strategies are then also connected to arithmetic mean.

As understanding of the arithmetic mean continues to develop, I hypothesize that the focus of the child may move from the total accumulation of the data set being equally shared to the amounts (or deviations) that data points are from an estimated mean being shared to level-off the data. It is possible to develop either the cancelling of positive and negative deviations or the equalizing of absolute deviations from the activity of leveling-off. Typically, when a data set is given, the arithmetic mean is unknown (such as in the Building Height Problem). Many children will use trial and error of moving pieces of bars to find an equal height. However, in order to use leveling-off in a more efficient manner, an arithmetic mean may be proposed. It is reasonable to assume a child will often have an incorrect estimated arithmetic mean. When the child levels-off the data the result is that there will be extra deviation in one direction or the other. This provides an opportunity for the child to focus on the deviations. As corrections to the estimate are made, the need for equivalent deviations becomes more apparent. Thus, leveling-off is connected to the need for equivalent deviations.

Previous research (Inhelder & Piaget, 1958; Siegler, 1976) suggests that children take much time to develop a conception of balance. It is possible that the conception of physical balance as torque must be developed independently from conceptions of arithmetic mean. Once a conception of arithmetic mean as a point where the deviations are equivalent has been developed, the conception can then be connected with an equivalent conception used in physical balance. It is unclear how the mathematical and physical connection is made. However, I hypothesize that the use of representations for data sets that highlight the physical nature of balance (such as the dot plot) may encourage the connection between the physical and mathematical conceptions. Further research may fill out the proposed progression for learning of arithmetic mean and determine its efficacy.

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PRESERVICE TEACHERS' KNOWLEDGE OF RANDOMNESS AND TEACHING PROBABILITY

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In the past decade the teaching and learning of probability and statistics has been receiving increased attention by the mathematical community. Empirical research details the complexities and difficulties of understanding probability by students at grade levels varying from elementary up to the college level (Fischbein & Schnarch, 1997; Jones, Langrall, Thornton, & Mogill, 1999; Lee, Rider, & Tarr, 2006). Additionally, research has been conducted regarding teachers' understanding of certain probability concepts and how to teach probability (Begg & Edwards, 1999; Mojica, 2006; Watson, 2001). However, little research has been done specifically on preservice mathematics teachers understanding of probability and the teaching of probability.

As curriculum reform requires more attention be given to the learning of probability and statistics, research on preservice teachers' pedagogical knowledge is needed (National Council of Teachers of Mathematics, 2000). This study will advance the current research by giving teacher educators a better understanding of preservice mathematics teachers' knowledge of randomness and probability. With this knowledge, teacher educators can improve the teaching and learning of probability. The results in this paper are taken from a larger study into the relationship between preservice teachers' beliefs, content knowledge, and pedagogical content knowledge of probability. The focus of this paper is on preservice teachers' beliefs about randomness and how those beliefs relate to their perspective of the meaning of probability.

Theoretical Frameworks

One of the fundamental aspects of probability is the notion of randomness and chance. The work done by Batanero and colleagues (Batanero, Green, & Serrano, 1998; Batanero & Serrano, 1999) offers a way to describe students' understandings of randomness. The word 'random' is used in everyday language as well as in mathematics classrooms. The definition of randomness is not clear and this ambiguity can increase the possibility of students having difficulty with this idea. Random is usually used as an adjective such as random number, random experiment, random variable; definitions of these tend to concentrate on the object being described as random rather than a definition of random itself (Batanero et al., 1998).

In addition to the definition being unclear, the meaning of randomness varies depending on a person's perspective of the meaning of probability. There are multiple perspectives that one can have for the meaning of probability; these perspectives fall into two categories: objective and subjective. Within the objective category there are the classical and the frequentist perspectives of probability. From a classical view of probability, one understands the meaning of probability as a ratio of favored outcomes over total outcomes. This view is limited to finite number of outcomes and thus someone with this perspective would believe randomness is tied to equiprobability. "In the classical conception of probability we would say that an object (or an event) is a random member of a given class if there is the same probability for [it as there is for] any other member of its class" (Batanero et al., 1998, p. 115). Someone with a frequentist perspective considers probabilities to be assigned based on the long run behavior of random outcomes. Within this perspective the equiprobability principle need not apply and a preconceived theoretical probability may not be known. Batanero et al. state, “here, we might consider an object as a random member of a class if we could select it through a method providing a given *a priori* relative frequency in the long run to each member of this class” (p. 115). This view of randomness is also an objective property of an event, yet it is based on relative frequencies.

Another view of probability is that it is subjective, meaning that the probability one assigns to an event is subjective to that individuals' beliefs. This view requires an understanding of randomness that is also subjective. What may be random to one person may not be considered random to another. This view is applicable when we know something that may affect our judgment of the randomness of an event.

Methodology

This study is a qualitative case study of 5 preservice teachers. The subjects are juniors and seniors in a mathematics education major with a focus on high school. The participants are purposefully sampled from a 400-level mathematics methods course on teaching with technology. Multiple data sources used for this study include interviews, classroom observations, and documents. Each student participates in an initial interview, a focus group, and a final interview. The interviews are task based and focus on the nature of the preservice teachers’ understanding of probability: their beliefs, content knowledge, and pedagogical content knowledge. Data collection and concurrent analysis is occurring during the Spring 2007 semester.

Results

In the initial interview participants were asked to explain what random means to them. The following table shows their responses and their possible perspective of probability.

Participant & perspective	**“What does random mean to you?”**
Brad [frequentist]	Um it just means, wow, usually use random in the definition… Um, it just means its **messed- jumbled up, uh there’s no real order to it at all** its just, I don’t really know what else to say, just there's no order to it, it just happens, [pause] randomly [laughs].
Yasmin [frequentist]	Random means **not coming in any particular order,** not having a sequence or something you can calculate- like pi.
Jeff [frequentist and classical]	Well if you think about it … **blindly picking something out of a pile**. It is hard to put into words as far as random goes, **not repetitive** … I don’t know, its hard to put into words.
Pam [subjective]	Uh, I guess, in terms of probability when I think of random I think of a **random sample** so it would just be so if you were to have a completely um you know like **not hand picked group of people but just kind of like uh really open sample** and you were to just randomly pick someone from it.
Sam [classical]	Events that happen by chance there's **not any predetermined outcome**.

Definitions of random included – no order, not repetitive, not hand-picked, equal chance, unknown outcome, can’t be calculated. Three of the five subjects displayed a frequentist perspective; their definitions of random referred to the order of outcomes being non-repetitive.

One subject, Pam, did not define random but instead defined what a random sample means. Later in the interview Pam said, "probability is all personalized, I don't know if I'd actually be that great at teaching it because I think about it as more of a choice: you're given information then you do what you want with it." This comment suggests she has more of a subjective perspective. Both Sam and Jeff indicate a classical perspective in their responses that random is equal chance and not predetermined. Three of the five subjects had difficulty defining the meaning of random.

Discussion

These findings support the claims made by Batanero et. al. (1998) that students' have difficulty understanding randomness. In addition, one's understanding of randomness is related to one's perspective of probability. These findings indicate a need for more emphasis on the concept of randomness in teacher education to improve the teaching of probability. Through the use of pedagogically oriented tasks teacher educators can increase awareness of these different perspectives of probability.

Teacher educators can use the tasks from these interviews to stimulate discussion within mathematics education classes. The use of these tasks could possibly raise further questions into the nature of beliefs of randomness and direct improved instruction in this area. Questions for further research include: 1) how does ones' understanding of randomness influence pedagogical decisions/activities? 2) what experiences in teacher education could foster a deeper understanding of randomness? and 3) how does a teachers' understanding of randomness effect students' understanding?

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SAMPLE SPACE REARRANGEMENT (SSR):
THE EXAMPLE OF SWITCHES AND RUNS

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This study continues research in probability education by altering a "classical" problem, referred to as the sequence task. In this task, students are presented with sequences of heads and tails, derived from flipping a fair coin, and asked to consider their chances of occurrence. A new iteration of the task –that maintains the ratio of heads to tails in all of the sequences– provides insight into students' perceptions of randomness. Students' responses indicate their reliance on the representativeness heuristic, in that they attend to sequences by their representative features, rather than by considering independent events. The study presents an unconventional view of the sample space that helps situate students' ideas within conventional probability.

The Sequence Task

Psychologists Tversky and Kahneman (1974) found that students, when considering tosses of a coin, determined the sequence HTHTTH to be more likely than HHHTTT, because the latter sequence did not appear random. Furthermore, they found HTHTTH more likely than HHHHTH, because the latter sequence was not representative of the unbiased nature of a fair coin (i.e., the ratio of the number of heads to the number of tails was not close enough to one). The caveat is that, normatively, all three sequences have the same chances of occurring. A number of researchers in mathematics education (e.g., Shaughnessy, 1977; Cox & Mouw, 1992; Konold, 1989) have continued Tversky and Kahneman's work and as such have furthered research involving the sequence task.

As Shaughnessy (1992) reflects: "There was no attempt made [by Tversky and Kahneman] to probe the thinking of any of these subjects" (p. 473). Shaughnessy's (1977) work brought two new elements to the sequence task. First, "[t]he subjects were asked to *supply a reason* for each of their responses. In this way it was possible to gain some insight into the thinking process of the subjects as they answered the questions" (p. 308). Second his tasks gave students the option of choosing "about the same chance" (p. 309) as one of the forced response items. The results echoed the conclusions of earlier work by Tversky and Kahneman. The sequence BGGBGB was considered more likely to occur than the sequences BBBGGG and BBBBGB. However, with the new "supply a reason" element to the task, Shaughnessy was able to confirm that subjects did find the sequence BBBGGG not representative of randomness, nor was the sequence BBBBGB representative of randomness, because it did not have a representative ratio of boys to girls: all of which had previously been inferred from the number of subjects choosing particular forced response options.

Konold's (1993) research built upon the two elements introduced to the sequence task by Shaughnessy. Konold found that when students were asked to determine which of the sequences were most likely to occur from flipping a fair coin five times, they often chose that all of the sequences were equally likely to occur. However, when students' were asked which of the sequences were the least likely to occur, they picked out a particular sequence. In fact, of the sequence options presented, HTHTH was deemed to be the least likely, because it was

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.

too representative. In other words, it did not reflect the random nature of the task; its archetypical appearance demoted its likelihood.

Researchers have found that humans are poor judges of randomness and, moreover, have determined the evaluation of randomness to be an important component of the sequence task. Batanero and Serrano (1999) found that students' ability to determine what was considered random, in results from flipping a fair coin, was derived from two variables. The proportion of the number of heads to the number of tails in the sequence and "the lengths of the runs and, consequently, the proportion of alterations" (p. 560). Reflecting these findings, Cox and Mouw (1992) state: "The sampling form [of representativeness] is seen both when a small sample is assumed to adequately represent some phenomenon in a population [e.g., 1:1 ratio for coin flips], and also when a sample is expected to appear random" (p. 164). Falk (1981) gave students two long sequences and asked which of the sequences appeared more random. The results show that sequences in which more switches occurred appeared more random. Furthermore, sequences in which a long run occurred appeared to be less random. In other words, randomness was perceived via frequent switches and thus short runs.

Sample Space Rearrangement

As presented, a number of issues have been addressed with subsequent iterations of the sequence task; however, one element remains the same. In each instantiation of previous research (by mathematics educators) presented, subjects were offered sequences with a different population ratio in each of their choices. For example, one option would be HTHTTH (3H:3T), while the second option would be HHHHHT (5H:1T). Cox and Mouw (1992) found disruption of one aspect of the representativeness heuristic, such as the appearance of randomness, did not exclude the population ratio being used a clue. Also, Shaughnessy (1977) found that "many students picked 'the same chance', but gave as a reason 'because each outcome has 3 boys and 3 girls'" (p. 310). While previous research has addressed sequences with a disparate number of heads and tails, this report examines responses from students when all of the choices presented contain the same ratio of heads to tails. As such, the task was constructed based upon an alternative view of the sequences: the view of switches and runs. A "traditional examination" of the sample space, when a fair coin is tossed five times, takes into account the numbers of heads and number of tails. However, rearranging the sample space (see Table 1) shows that the sequences can be organized by the number of switches, denoted S (a change from head to tail or tail to head), and, consequently, the longest run, denoted LR (the number of times the same result occurs).

0S & 5LR	1S & 3LR	1S & 4LR	2S & 2LR	2S & 3LR	3S & 2LR	4S & 1LR
HHHHH	**HHHTT**	HHHHT	HHTHH	HHHTH	HHTHT	**HTHTH**
TTTTT	TTTHH	TTTTH	TTHTT	TTTHT	TTHTH	THTHT
	TTHHH	THHHH	**HHTTH**	HTHHH	HTHHT	
	HHTTT	HTTTT	TTHHT	THTTT	THTTH	
			HTTHH	**THHHT**	**THHTH**	
			THHTT	HTTTH	HTTHT	
					THTHH	
					HTHTT	
P(0S5LR) =2/32	P(1S3LR) =4/32	P(1S4LR) =4/32	P(2S2LR) =6/32	P(2S3LR) =6/32	P(3S2LR) =8/32	P(4S1LR) 2/32

Table 1: Sample Space organized according to Switches and Longest Runs

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.

Results and Analysis

Students were presented with the following:
Which of the sequences is least likely to result from flipping a fair coin five times:(A) HHHTT (B)HHTTH (C)THHHT (D)THHTH (E)HTHTH (F) All sequences are equally likely to occur. Provide reasoning for your response…

Although approximately sixty percent (30/49) of the students –14 to 15 year old high school students, from Vancouver, BC, Canada– incorrectly chose one of the sequences to be the least likely to occur, there was approximately an even split between those who chose *(A): HHHTT* (14 students) and those who chose *(E): HTHTH* (16 students). Justifications for why sequences *(A)* and *(E)* were not representative (of randomness) fell into two categories. Students who chose *(A)* said that the perfect alternation of heads and tails was not reflective of a random process, while students who chose *(E)* said that a run of length three was not indicative of a random process because it was too long. Given the ratio of heads to tails was maintained, determining randomness relied upon switches and, subsequently, longest runs.

This report suggests that heuristic reasoning (more specifically representativeness) is not incongruent with probabilistic reasoning. Illustration of this point draws upon the probabilities associated with the rearrangement of the sample space. For *(A)* there is a 4/32 chance that a sequence of five flips of a coin will have one switch with a longest run of three. For choice *(E)* there is a 2/32 chance that a sequence will have four switches and a longest run of one. From this alternative perspective of the sample space, organized according to switches and runs, all other sequences presented in the task have higher probabilities than sequences *(A)* and *(E)* (e.g., a sequence with three switches and a longest run of two has an 8/32 chance of occurring). As such, it can be inferred from the data that the students are unconventionally, but naturally, looking for features in the sequences which are least likely to occur. From this perspective of a rearranged sample space, it can be argued that the students were correctly choosing which of the sequences are least likely to occur.

Conclusion

An alternative view of the traditional sample space for five flips of a coin via switches and runs, not the ratio of heads to tails, suggests a probabilistic innateness associated with use of the representativeness when determining, heuristically, the randomness of a sequence. Given that "we need to know more about how students do learn to reason probabilistically" (Maher et. al., 1998, p.82), sample space rearrangement may become one such approach.

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TEACHERS' USE OF PROBABILITY EXPERIMENTS AND SIMULATIONS

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With increased curricular attention to probability and access to technologies, teachers are encouraged to use an empirical introduction to probability, as a limit of the stabilized relative frequency, through experiencing repeated trials of the same event, with concrete materials or computer simulations (Batanero, Henry, & Parzysz, 2005). Most teachers have little experience with conducting probability experiments and use of simulation tools and may have difficulty implementing this empirical approach to probability (Stohl, 2005). Specifically, we investigated:

- What is the role of experiments and simulations in middle school probability lessons?
- How do middle school teachers use data generated from probability experiments?

Framework

We make a distinction between a probability experiment and a probability simulation. In a probability experiment, one examines a random phenomena of interest by carrying out specific actions (trials) and observing outcomes of repeated trials. For example, if one is interested in a game that requires the use of two 6-sided dice and we use two physical 6-sided dice to repeatedly play this game, we are conducting an experiment. We define a simulation as a special type of experiment in which a random generating device, either physical (dice, coins, marbles in bag) or computer-based, is used to model random phenomena and perform the needed actions (trials) repeatedly. Thus, if we were interested in the same game that uses two 6-sided dice, but created a model of the dice using two different spinners, then we are simulating the original context through a different physical device. We could also simulate the 2-dice game by using a technology tool. The framework for our study, adapted from Watkins (1981) and Travers (1981), is grounded in two important aspects: 1) considerations of major components in an experiment, including use of a simulation, and 2) decisions a teacher makes during a lesson. Table 1 describes the major components of a probability experiment and the corresponding teacher tasks that may provide insight into the role an experiment or simulation plays in a lesson, with a particular focus on how teachers use empirical data.

Table 1. Components of Framework

Steps in a Probability Experiment	**A teacher should be able to:**
1. Understand the problem context to identify the random event(s) of interest.	• Choose problem contexts amenable to experiments or simulations. • Define the random event in the problem context.
2. Identify the possible outcomes for the random event.	• Construct the sample space. • Understand probabilities of each event. • Decide if students should compute theoretical probabilities *a priori*.
3. Select an appropriate random generating device	• Choose appropriate device(s) that maintain the mathematical characteristics of problem. Either o Use identical device as stated in problem context to carry out *experiment* or

	o Use different device to *simulate* problem.
4. Determine what a trial and sample will consist of.	• Define a trial (a single occurrence) and a sample (a collection of trials).
5. Repeat a number of trials to form samples, possibly repeat sampling.	• Decide on number of repetitions or allow students to make this decision. • Decide whether to structure data collection process and organization of collected data or to allow students this choice. • Understand role of independence or dependence in repeated trials/samples. • Understand the law of large numbers.
6. Analyze results to compute empirical-based probability of random event(s).	• Decide how to use collected data, including any public displays of tables, charts, etc. • Decide how to help students make sense of empirical results. • Understand how to interpret results in relation to sample size and theoretical probabilities (if applicable).
7. Use empirical probabilities to make decision about original problem context	• Interpret empirical results in terms of the original problem context.

Methods

Participants and Data Sources

Teachers in this study were enrolled in a graduate-level course focused on Data Analysis and Probability. One pedagogical objective of the course was increasing teachers' understanding of how to conduct experiments and simulations with concrete materials and technology tools. One course requirement was for teachers to plan, teach, and reflect on a lesson. Of the 29 teachers in the course, nine taught a lesson using an experimental approach to probability. The data include a 15-minute lesson episode and written reflection.

Analysis

Following Powell et al. (2003), the videotapes were viewed several times and described. Verbatim transcripts of the lessons were created, including reproduction of representations created by the teacher, either on the board or overhead projector. Initial descriptions and researcher impressions of each classroom episode were made. For each of the nine teachers, the components of the framework were used to describe the steps in the probability experiments in each lesson and to describe what was known about each of the teacher tasks.

Results and Discussion

Of the nine teachers, five had students conducting experiments in a game context with devices such as dice, coins, spinners, and chips in a bag, while four teachers had students simulate a problem context. Table 2 contains a brief overview of each lesson, including claims about teachers' learning goals. Teachers' were often focusing students on using empirical data to make different types of comparisons. Comparisons were often dependent on students' computing either empirical or theoretical probability. Thus, computation was a major theme. Lessons also illustrated teacher-controlled experiments, where students had little control over collecting data (e.g., sample size), organizing or displaying results. Only two teachers had a learning goal for students to form an argument about the problem context, with one teacher giving students control in data collection and choice of sample size. Teachers instructed students to conduct experiments with small sample sizes, with little repeated sampling. In seven lessons, students conducted 40 or less trials. Only three of the teachers (Whitney, Frank, Helen) displayed data publicly in such a way that students could attend to variability across samples of the same size, while one teacher (Pam) had students verbally share their different results.

Table 2. Summary of Lessons

Teacher	**Grade**	**Experiment or Simulation?**	**Tools used**	**Main Learning Goals for Students**
Kathy	6	Experiment drawing chips	Chips in bag	Predict and then **compare** to data, **Compare** data across samples, **Compare** theoretical to empirical, Provide a context to *compute* probabilities
Pam	6	Simulate gumball machine	Chips in cup	Predict and then **compare** to data
Wanda	6	Simulate die rolls to test fairness	GC-ProbSim	Importance of sample size, Collect data to ***form argument***,
Frank	7	Experiment rolling die	2 dice	**Compare** theoretical to empirical, Provide a context to *compute* probabilities
Whitney	7	Simulate basketball shots	spinner	**Compare** data across samples, Provide a context to *compute* probabilities
Megan	8	Simulate basketball shots	2 different spinners	Provide a context to *compute* probabilities
Helen	8	Experiment die game to evaluate fairness	Sums of 2 dice	Predict and then **compare** to data, Provide a context to *compute* probabilities
Marsha	8	Groups had different experiments	Coins, dice, spinners,etc	Collect data to ***form argument***, Provide a context to *compute* probabilities
Morgan	8	Experiment of Montana Red Dog card game	Deck of cards	Predict and then **compare** to data, Provide a context to *compute* probabilities

All contexts had a known distribution and were amenable to computing a theoretical probability. Teachers demonstrated an overwhelming tendency to privilege the theoretical probability as the best or most reliable estimator of probability, often using empirical differences to justify this preference. Although these teachers are using experiments, their approach does not foster a conception of probability as a limit of a stabilized relative frequency after many trials.

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A FRAMEWORK TO DESCRIBE THE SOLUTION PROCESS FOR RELATED RATES PROBLEMS IN CALCULUS

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Three mathematicians were observed solving related rates problems. This data was analyzed to develop a framework for solving related rates problems. It was found that the mathematicians engaged in a series of phases to generate pieces of their solution. These phases were identified as: draw and label a diagram, construct a meaningful functional relationship, relate the rates, solve for the unknown rate, and check the answer. To complete each phase and construct a piece of the solution, the mathematicians built and refined a mental model of the problem situation. The framework captured how the solution to the problem emerged from the mathematicians' thinking as they responded to related rates problems.

Background

Little research has been published on the solution process for related rates problems in first semester calculus. Martin (2000) conducted a study investigating students' difficulties with geometric related rates problems. In attempting to understand students' difficulties with these problems, Martin broke down the procedure for solving them into seven steps and then classified these steps as either conceptual or procedural as follows:

1. Sketch the situation and label (Conceptual)
2. Summarize the problem and identify given and requested information (Conceptual)
3. Identify the relevant geometric equation (Procedural)
4. Implicitly differentiate the geometric equation (Procedural)
5. Substitute specific values and solve (Procedural)
6. Interpret and report results (Conceptual)
7. Solve an auxiliary problem, e.g. solve a similar triangles problem before being able to use the volume of a cone formula to relate the variables (Varies)

In her study, Martin found that the problems that appeared to be the easiest for students were the ones that required only the selection of the appropriate geometric formula, differentiation, substitution, and algebraic manipulation. The most difficult questions were those that required Step 7, solving an auxiliary problem. She also indicated that while conceptual steps are more difficult for students than the procedural ones, students' poor performance was linked to difficulties with both procedural and conceptual understandings.

The research to date suggests that students have a procedural approach to solving related rates problems (Clark, Cordero, Cottrill, Czarnocha, Devries, St. John, Tolias, & Vidakovic, 1997; Martin, 1996, 2000; White & Mitchelmore, 1996). It has also been reported that students' difficulties appear to stem from their misconceptions about variable, function, and derivative – particularly the chain rule (Clark et al., 1997; Engelke, 2004; White & Mitchelmore, 1996). The ability to engage in covariational reasoning allows a problem solver to construct a mental model of the problem situation that may be manipulated to understand how the system works (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Saldanha & Thompson, 1998; Simon, 1996). The purpose of this study was to develop a framework to describe how a mental model for a related rates problem developed during the solution process.

To describe how a mental model emerged during the solution process, Carlson and Bloom's (2005) multidimensional problem solving framework provided an initial structure with which to examine the data. Their framework describes four phases of the problem solving process: orienting, planning, executing, and checking. They observed that in each phase the problem solver accessed resources and heuristics. Using these ideas, it was my goal to identify the content knowledge and heuristics that were used when solving a related rates problem. It was also desired to describe how content knowledge and heuristics informed the development of the problem solver's mental model of the problem situation.

The Study

The purpose of this study was to generate a framework to describe the solution process for related rates problems in calculus; the resulting framework was subsequently used to analyze the data from a teaching experiment on related rates problems. To study the role of the mental model when solving related rates problems, three mathematicians were asked to solve three related rates problems in a think aloud problem solving session which lasted about 45 minutes. Each mathematician was given a sheet of paper with the three problems printed on it and asked to work through the problem aloud and to attempt to verbalize everything they were thinking. The interviewer asked clarifying questions about their statements when additional information was needed. Each session was videotaped and transcribed for analysis. Discourse analysis was used to analyze the mathematicians' solution processes. Video recordings captured everything the mathematicians said, wrote , and gestured. Gesture became particularly important to be able to identify how the problem solver used and referenced information he had written as part of the solution process. Everything that had been written down was initially coded as an artifact. However, some pieces of written information were defined to be solution artifacts. A solution artifact is more than a piece of information that has been written down as part of the solution process. It is a written piece of information that is referenced or used by the problem solver to continue the solution process; it is the creation of an additional resource. Particular attention was given to how they talked about what they were imagining and how they talked about the diagrams they drew and modified throughout the solution process. Because it is not possible to see what an individual is imagining in his mind, these are the data that allow one to best conjecture how the mind is interpreting and modeling the problem situation.

Results

One of the related rates problems the mathematicians were asked to solve required the problem solver to create a powerful mental model as the geometric figure in the problem situation does not have a commonly known volume formula. The problem was stated as follows: Coffee is poured at a uniform rate of 20 cm^3/sec into a cup whose inside is shaped like a truncated cone. If the upper and lower radii of the cup are 4 cm and 2 cm, respectively, and the height of the cup is 6 cm, how fast will the coffee level be rising when the coffee is halfway up the cup?

One mathematician's solution will be examined to illustrate the framework for the solution process. After reading the question, each of the mathematicians drew a diagram of the problem situation. This diagram was then usually labeled with variables and constants. However, before drawing the diagram, there is evidence that the mental model may have already been revised one or more times. Adam began (transcript lines have been numbered to facilitate the discussion):

Adam:
1. All right, well this is my kind of problem. Coffee is poured at a uniform rate of 20 cubic cm per second into a cup whose inside is shaped like a truncated cone.
2. Oh, this must be one of those horrible, uh, um hotel things, right? They have the little plastic deals with the little conical cups and that always signals bad coffee. So, cause they always make it much too weak, you know. So there you are in this windowless ballroom, drinking out of this flimsy [hand motion]…yeah, I know what you're talking about. All right, a truncated cone.
3. All right, upper and lower radii of the cup are 4 and 2.
4. Um, this makes a difference. I don't know what the upper and lower radii means.
5. Do you mean is the cup 4 inches wide? so it's a shallow cup? or a tall narrow cup?

INT: The cone has actually been truncated so the bottom doesn't actually exist.

Adam: Oh, oh, oh, I see. I'm still thinking about those horrible hotel cones. All right, so it's truncated like this. All right, gotcha, like that. [artifact-diagram]

Observe in Line 2 above that Adam's initial interpretation of the problem situation is of a whole cone and is related to his experience with coffee cups at conference hotels. In Lines 3 and 4, the phrase upper and lower radii appear to be in discordance with his current mental image. After the interviewer explained the truncated aspect of the problem, Adam adjusted his mental image and drew an accurate diagram of the problem situation. Thus, we have observed Adam's mental model develop. In addition to the verbal statements that Adam made, he drew a diagram. Adam's diagram of the problem situation was the first solution artifact that he created. This diagram was further modified as Adam completed the first phase of the problem solving framework: draw a diagram.

Adam continued the solution process by identifying that he needed to find the volume of the coffee cup, and more specifically the volume of a cone:

Adam:
1. Ok, so now, we're going to pour coffee into this thing, and we want to know how fast the coffee is rising when the coffee is halfway up. So, this means we have to figure out first of all what the volume of this thing is. So what's the volume? The volume is, um, is like one half pi *r* squared *h*, or something like that [artifact-equation]
2. cause this is where you could give them the whole deal. There's a factor of 2 pi about the, you know. Oh, there's a factor of 2 pi here cause you do the little shells, and you rotate it around, right? Yeah. [hand motion] I don't remember it, Nicole. pi *r* squared *h*? Does that sound right? I will go on that assumption here. You know, I'm terrible at memorizing formulas. Let's see. Is one half the base times the height, and you rotate it around. No, that's not right. That would be like a cylinder, a cylinder. [artifact-diagram]

As Adam considered pouring coffee into the cup, he transitioned to the next phase of the problem solving process: construct a functional relationship. He restated that he wanted to find how fast the coffee is rising when the coffee is half way up the cup which caused him to identify that he needed to know what the volume is. (Line 1) The volume of a cone formula was not readily accessible as part of Adam's content knowledge. In reconstructing the volume of a cone formula, Adam accessed his knowledge of integral calculus and the shell method. (Line 2)

Adam then spent a considerable amount of time trying to reconstruct the volume of a cone equation. After Adam decided on the formula for the volume of a cone, he proceeded to find the volume of the actual coffee cup by computing the difference between the volume of the entire cone and the volume of the piece of the cone that does not exist. In previous problems, Adam was observed making similar calculations that appear to allow him to better understand what is happening in the problem situation.

After completing these calculations, Adam reconsidered how he should be thinking about the problem situation.

Adam:

1. All right now, ok. So now actually what I need to think about here is, um, should I think about height from the, um, the imaginary tip here? Or should I, should I start thinking about height from here on up? And I think about height from the bottom of the imaginary tip and hope that I made the correct adjustment when I get over here
2. Um, cause all you want to know is the rate of the change in the coffee. So whether I measure from here or from down here, um, that doesn't make a whole lot of difference. So, that's important, ok.
3. Let's see. Um, yep, ok. Now in this case, since we have a cone, the volume depends upon the height. So in this particular case then, the radius is, um, so the radius… We're going down here, what? 2 cm in the space of 6 cm. So it's changing at the rate of um one third cm per cm, so the slope is a third. So that means that this is one half times pi times one third *h* [artifact-volume equation]
4. Ahh, I'm looking back to worry about whether to measure from my imaginary tip, or whether I'm going to start measuring from um the actual bottom. Um, what do I want to do here? This is much, boy this is, there must be a slicker way of doing this in the book cause it would be too hard for the students to be talking about, well, measuring from the tip or here or whatever. Um, so what should I think about? I'm going to go with the imaginary tip. So this is one third *h,* zero is here, 6 is here, and 12 is here. [artifact-modified diagram]

Adam was concerned about how he should measure the height of the cone. (Line 1) This lead him to the revelation, "cause all you want to know is the rate of the change in the coffee. So whether I measure from here or from down here, um, that doesn't make a whole lot of difference. So that's important, ok" and decided that he would measure from the imaginary tip of the cone. (Lines 2, 4)

In the middle of deciding how he should measure the height, Adam also noted that the volume depends on the height. (Line 3) To relate the volume and height, he observed that, "so the radius we're going down here what 2 cm in the space of 6 cm so it's changing at the rate of um one third cm per cm so the slope is a third so that means that this is one half times pi times one third *h*" and modified his volume equation so that it was in terms of height. He constructed a linear relationship between the radius and the height of the cone and used composition to express the volume in terms of height. Thus, it would appear that his content knowledge of functions was accessed to allow him to relate the volume and the height of the cone. Adam also commented that an example in the book would likely be helpful and provide a "slicker" way to solve the problem. (Line 4) However, this resource was not available to him in this session.

He continued:

Adam:

1. So this is what I would do in my office, and then I would think of a slicker way of doing it. Then it would all look nice for the students. That's why we always look like geniuses in front of the black board, cause, ok.
2. Um, so do I have to worry about my offset here? Maybe not, ahh. I'm worrying about what this offset means here. I'd have to offset the bottom, ehh. I'd have to worry about, see what I'd have to worry about is this, uhh, maybe. Let's see. I have to remember to subtract 12 pi from everything. I don't know. What would I do?
3. If I were to think out loud about this, I would say, ok my guess is it doesn't matter. See, because if you had a, if you had a really truly conical coffee cup, not one that was lopped off here, you could just as well imagine. So instead of worrying about, so what I would tell the students is, I would say, well instead of worrying about this kind of cup, um, think about this kind of cup. Just imagine that you already filled it up with coffee to here, um, because who cares about the total volume. All you want to know is how fast the coffee is rising when you're at this volume. So you're, it doesn't really [hand motion] So it's the same answer. Getting the students to see that is perhaps one of the things that distinguish the expert like me from them. …
4. And um, all right. So now, um, in the cup, all right. So this, for that reason I'm just going to look at this long cup. And so now, I'm going to measure height from the tip of the cup. So this is *h,* and here is *h* equal to zero, and here is going to be *h* equal to 12. [artifact-labeled diagram] All right. So halfway up the cup then is going to be, I'm going to worry about the point where *h* equals 9, because that's the point where the height in my imaginary cup corresponds to halfway up the actual cup. [artifact-modified labeled diagram]

As Adam continued to struggle with how to measure the height of the cone, he appeared to have a revelation about how the problem situation worked. (Line 2, 3) He then stated how he would it explain it to students, "well instead of worrying about this kind of cup um think about this kind of cup just imagine that you already filled it up with coffee to here um because who cares about the total volume all you want to know is how fast the coffee is rising when you're at this volume." (Line 3) He realized that it did not matter if the cone had been truncated. Adam's mental model appears to have undergone several changes as he developed his formula for the volume of the cone and what each variable meant in terms of the problem situation. It would seem that each time he played out the situation in his mind he gained a little insight until he finally understood how the situation worked. It is at this point that Adam completed his step of construct a functional relationship between the variables. An important part of completing this step was that he understood exactly what the relationship between the variables was. (Line 3) Thus, his algebraic equation had meaning in the context of the problem.

After simplifying his volume equation to $V = \frac{1}{18}\pi h^3$, Adam continued to his next step of relate the rates:

Adam: And, um, we're interested in *dh dt.* So *dv dt* is *dv dh* times *dh dt.* [artifact-general chain rule form equation $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$] Um, and that means that *dv dt* is 20 cm per second cubed, cm per second, and *dv dh,* uhh is what it's going to be, pi over 9 times *h* squared times *dh dt.* [artifact-differentiated equation $20 = \frac{\pi}{9} h^2 \cdot \frac{dh}{dt}$]

Adam identified $\frac{dh}{dt}$ as the rate he wanted to find and then expressed a relationship between the rates using the chain rule. He did not operate on the volume equation he constructed. Rather, he expressed a relationship between the rates using the chain rule and then associated each rate with a numeric value or a function.

An interesting aspect of Adam's solution is that even though he expressed the volume equation in terms of height, he did not operate on the function he constructed to relate the rates. Instead, when he carried out the relate the rates step, he wrote out the chain rule as $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$. He then proceeded to associate each rate, such as $\frac{dV}{dh}$, with either the appropriate numeric value or function. He did this when solving the previous problems, too. Neither of the other two mathematicians related the rates in this manner on any problem they solved. Rather, they operated on the algebraic equation they constructed.

The completion of Adam's solution came when he substituted in the known values into his differentiated equation and performed appropriate algebraic manipulations to obtain a numerical answer to the problem. Thus, he completed the fourth phase: solve for the unknown rate. After obtaining an answer to the problem, he performed a unit analysis to check his answer for reasonability and complete the fifth phase of check the answer for reasonability.

Five phases were identified in the related rates problem solving process: draw a diagram, construct a functional relationship, relate the rates, solve for the unknown rate, and check the answer for reasonability. In each of the first three phases, the problem solver's mental model appears to be continually evolving as the result of interpreting the problem situation and accessing content knowledge. To begin the draw a diagram phase, the problem solver accessed his knowledge of geometry to imagine what a cone looks like and draw a representation on the paper. Content knowledge of variable was then accessed to refine the diagram by labeling the quantities that were imagined to change with variables thus creating the first solution artifact, a labeled diagram. While transitioning to the phase of construct a functional relationship, the problem solver imagines the problem situation changing over time to decide which variables should be related. Geometric content knowledge is then accessed yet again to facilitate the creation of a functional relationship, the second significant solution artifact. The problem solver then imagined the problem situation changing over time and accessed his content knowledge of rate of change to facilitate completion of the third phase of relate the rates and create another solution artifact, a differentiated function. In the fourth phase of solve for the unknown rate, the problem solver relies almost exclusively on his content knowledge of algebra to substitute in the known quantities and perform appropriate algebraic manipulations to determine the unknown rate, the final solution artifact. Finally, the problem solver checks his answer for reasonability by imagining the problem situation again and performing a unit analysis.

Conclusions

The mathematicians had a richly connected understanding of the concepts of geometry, variable, function, and derivative. This allowed them to easily access information and relate it to the problem situation. This abundant source of content knowledge also facilitated their ability to build a mental model that accurately characterized the problem situation. For example, the mathematicians could easily identify the formula for the volume of a cone and then think about it as a function of two variables that change in tandem. The mathematician could then note that one of the variables could be eliminated by assessing relevant knowledge of geometry and applying function composition. This new function that was created could then be thought about as a function of time. They identified time as the independent variable stated in the given rate, thus making a connection between their function knowledge and understanding of derivative. Consequently, they were able to relate the rates and successfully solve the problem.

This framework focuses on the relationships that exist between the columns of resources and heuristics in Carlson and Bloom's (2005) multidimensional problem solving framework. The problem solver's resources and heuristics are likely supplemented with a mental model of the problem situation. Resources are considered to be one's mathematical content knowledge, heuristics are strategies an individual chooses to apply to the problem situation, and a mental model is a representation in one's mind of the problem situation. The resources, heuristics, and mental model reside in the mind of the individual, and as an individual accesses this knowledge and applies it to the problem situation, solution artifacts are generated.

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AFFECTS OF FOCUSED MATHEMATICAL PROBLEM SOLVING EXPERIENCES ON FIRST-SEMESTER CALCULUS STUDENTS' MATHEMATICAL PROBLEM SOLVING PERFORMANCE

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This report explores developing problem solving skills of first-semester calculus students (n=202) at a midsize (25,000 students) university in the southwest with one-fourth of the student body enrolled at the graduate level. The study was designed to determine base-line problem solving performance among the first-semester calculus students and then measure any change in problem solving performance over the course a fifteen-week semester. Also, an experimental group (n=73) worked 1.5 hours per week in addition to regular class time on challenging group problem solving activities. Findings suggest that students moved from an emergent or developing performance to a proficient performance over the course of the semester at each level of sophistication studied.

Incorporating learning activities that enhance students' ability to become effective mathematical problem solvers is complex in that it requires a framework for understanding problem solving and mechanisms for studying the affects of instruction or learning activities that attempt to improve or enhance students' progress in mathematical problem solving. Lester (1994) indicates that the problem solving literature from 1970 to 1994 suggests that problem solving performance is a function of several independent factors such as knowledge, control, and beliefs.

The interaction of problem solving and competency in using mathematical knowledge cannot be overlooked. Hiebert and Leferve (1986) claim that competency in mathematics involves knowing how concepts, symbols, and procedures relate. This underscores that both conceptual and procedural knowledge are needed to be a successful mathematical problem solver. Studies (White & Michelmore, 1996; Carlson, 1998; Baker, et.al., 2000) find that calculus students have limited conceptual understanding and fare poorly on non-standard mathematics problems.

The work of White and Michelmore (1996) in studying students' conceptual knowledge in calculus and Epperson (2004) motivated the design of the problem solving tasks created to examine problem solving performance among the subjects of this report. The test characterizes a goal of having students move away from applying only procedural knowledge (procedures learned by cues) and move toward applying conceptual knowledge (grasping the relationships between mathematical objects in context). Recent work by Carlson and Bloom (2005) suggests that expert problem solvers possess well-connected conceptual knowledge which "appeared to be an essential attribute for effective decision making and execution throughout the problem-solving process (p. 45)." The tests used in this study focus on snapshots of the artifacts of the heuristics and monitoring problem solving attributes in the executing and checking phases of Carlson and Bloom's (2005) *Multidimensional Problem-Solving Framework.*

Method

The study was conducted at a midsize (25,000 students) university in the southwest with one-fourth of the student body enrolled at the graduate level. A group of 202 first-semester calculus students participated in the study. These students were enrolled in three large lecture classes (class size 60). Each lecture class met for 5 hours per week (3 hours regular lecture format and 2 hours recitation). Each class was divided into two recitation sections—each recitation enrolling less than 40 students. One hour per week of all recitation sections focused on more in-depth calculus problem solving tasks. Also, one of the lecture classes (the experimental group) worked for an additional 1.5 hours per week on challenging group problem solving activities. The experimental group lecture class incorporated attendance and participation in these supplemental 1.5 hour sessions into the overall course grade. Students from the other lecture classes did not attend the additional sessions and no such equivalent was included in their overall course grade.

Data was collected systematically throughout the study in the form of student feedback forms (on group work and study habits), student class work, performance on departmental exams, and a problem solving test that was administered at four different intervals during the 15-week semester. Standardized test scores (such as SAT and placement exam scores) and other demographic and background variables were collected from student records. Incomparability of the time intervals in testing caused by semester time constraints that affected only one of the three sections required reduction of the data set to 135 participants instead of the original 202 for analysis. The remaining group analyzed (n=135) consisted of 64.9% males and 35.1% females. SAT quantitative scores were available for only 55% of the group; however, it is important to note that 38.2% of these scores were in the 500-590 range, 50% in the 600-690 range, and 11.8% were greater than or equal to 700. White students comprised 47.2% of the group and African American, Hispanic, and Asian students comprised 13.4%, 9.4%, and 22.8% of the group, respectively. More than three-fourths of the participants were science and engineering majors.

As a group, the regular problem solving recitation sessions consisted of some short lectures, small group and whole class discussions, student and group presentations and constant feedback and reflection as students worked on carefully selected in-depth mathematical tasks; this model is characterized by Santos-Trigo (1998) and Schoenfeld (Schoenfeld, 1991, 1994) as indicative of a successful problem solving classroom setting. The extra experimental group sessions focused on carefully selected tasks with a more narrow mathematical focus designed to enhance students' procedural and conceptual knowledge on calculus topics as they were studied in the course. The experimental group sessions included student presentations and board work as well as the structure of the regular problem solving recitation sessions. Those students in the experimental group who missed the experimental group sessions were required to complete the work missed in order to receive full participation credit for the missed session.

One of the authors designed a series of tasks that had been developed into four items each to delineate between various levels of sophistication in problem solving. Four tasks were the focus of the test items for this study. The mathematics needed to solve the tasks did not go beyond a typical high school pre-calculus course. Each task was structured in four versions so that the manipulation (procedural knowledge) required to solve each version was essentially the same. However, the difference in the versions (items) was that at each successive level the depth of conceptual understanding of the mathematics and the sophistication of the modeling and generalization decreased. Thus, Item 1 was the most challenging version of the task, while item 4 was the version that required almost no translation into mathematics but simply a straightforward manipulation because almost all of the translation to mathematics had already been completed.

In total, there were sixteen different items created from the original four distinct tasks. A descriptive name was assigned to each task based upon the context given: Projectile Task, Bottle Task, Car Task, and the Inequality Task. The Projectile Task was adapted from prior work by one of the authors (Epperson, 2004). The Car Task is adapted from Carlson's (1998) functionality test and is given in Figure 1. The Bottle Task and the Inequality Task were also adapted from Carlson's functionality test.

As seen in Figure 1, Car Task Item 4 only required a solution for part (b). This represented a basic interpretation from function values shown on the graph. Car Task Item 3 not only asked for the seemingly straightforward interpretation in Item 4, but also required participants to demonstrate their understanding of rate with the inclusion of part (c). Car Task Item 2 layered part (d) onto the parts included on Item 3. Participants with calculus knowledge could present an argument using integrals, but participants without prior knowledge or accessible knowledge of integrals could employ a sense-making approach and observe that car A had been traveling faster than car B during the entire time interval. As noted Carlson's (1998) work, this question focuses on the misconception that higher acceleration means "catching up" and on students' abilities to interpret correct information from a graph. The most complex item, Car Task Item 1, presented all four parts. Part (a) delineates participants' abilities to monitor their progress and interpret graphical information correctly. Although use of calculus was not necessary to answer any of the questions, useful application of a calculus knowledge assists in all but the easiest version of the Car Task.

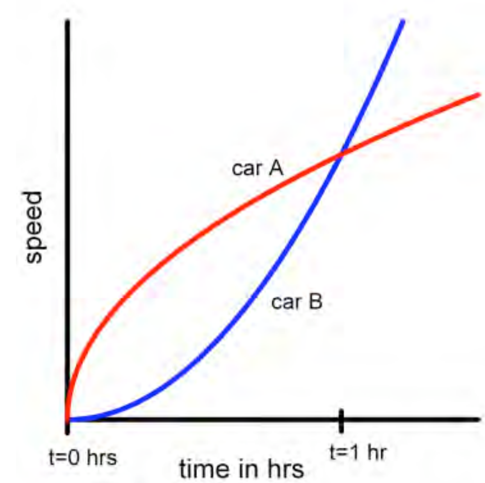
The given graph represents speed vs. time for two cars. (Assume the cars start from the same position and are traveling in the same direction.)

(a) State the relationship between the position of car A and car B at t =1 hr.: Explain.

(b) State the relationship between the speed of car A and car B at t =1 hr.: Explain.

(c) State the relationship between the acceleration of car A and car B at t =1 hr.: Explain.

(d) What is the relative position of the two cars during the time interval between t = 0.75 hr and t =1 hr.? (i.e. is one car pulling away from the other?) Explain.



Item 1: Included parts (a)-(d).
Item 2: Included parts (b)-(d).
Item 3: Included parts (b)-(c).
Item 4: Included part (b) only.

Figure 1. Four Levels of the Car Task

Participants were tested on four occasions in four-to-five week intervals beginning the second week of the semester. The participants were randomly divided into four parallel groups of 4. Four tests were constructed (test A, test B, test C, and test D) and each test included four items: one version of each of the four tasks. Each version of each task occurred on one and only one test and each test had only one question in each version. As in White and Michelmore (1996), a cyclic scheme was used to administer the tests to each of the four groups over the four data collections.

Note in Table 1 that Test A consisted of Item 1 from the Projectile Task, Item 2 from the Bottle Task, Item 3 from the Inequality Task, and Item 4 from the Car Task. Also, note that the items on the Inequality Task labeled Item 3A, Item 2B, Item 1C, and Item 4D denote that the most difficult (with respect to problem solving sophistication needed for solution) version of the Inequality task is on Test C. Thus, the label Item 1C is given to the highest level Inequality problem. A similar scheme is used in enumerating the other items. Each item has a unique identifier that indicates the test on which is occurs and its problem solving level.

Test A	Item 1A: Projectile Task	Item 2A: Bottle Task	Item 3A: Inequality Task	Item 4A: Car Task
Test B	Item 1B: Bottle Task	Item 2B: Inequality Task	Item 3B: Car Task	Item 4B: Projectile Task
Test C	Item 1C: Inequality Task	Item 2C: Car Task	Item 3C: Projectile Task	Item 4C: Bottle Task
Test D	Item 1D: Car Task	Item 2D: Projectile Task	Item 3D: Bottle Task	Item 4D: Inequality Task

Table 1. Problem solving test composition per group.

Participants were randomly placed into four different groups: A, B, C, and D. Their group name corresponded to the test they took first. In the successive administrations of the exam, the following cyclic scheme was used: (A,B,C,D) → (B, C, D, A)→(C, D, A, B)→(D, A, B, C).

For the quantitative analysis of the problem solving tests, a three-fold rubric informed by the work of Carlson et. al. (2002) and Epperson (2004) was developed for each question so that students received three scores for each response for (1) understanding, (2) strategy and accuracy (STAC), and (3) conclusions and justification (CNJU). The tests were scored independently by the two researchers and then compared and discussed to validate the final assessment of performance.

The scores associated with each category of the rubric ranged from zero to five. For each category, a score of zero indicates no attempt at solving the task. Scores of one, two, three, four, or five in each category indicate emergent, developing, proficient, masterful, and sophisticated demonstrations in each category, respectively. Scores were tabulated on frequency charts. For initial data analysis, the scores were grouped into dichotomous sets. The scores 0-2 were grouped in each category to indicate underdeveloped understanding, ineffective strategies and poor accuracy, or poor justification and unsupported conclusions, respectively. Scores higher than 3 were also grouped and indicate expanded understanding, useful strategies and adequate accuracy, or sound justification and conclusions, respectively.

Several statistical analyses used model-smoothed estimates to examine the students' STAC and CNJU score respectively on the 0-5 rubric scale. The STAC and CNJU scores focused primarily on aspects of problem solving involving strategies, heuristics, and reasoning and thus could be measured over time and difficulty level without regard to the specific task used. Only the STAC scores analysis will be included in this report. The data was then modeled using the parameters of testing time and difficulty level with continuous adjustment for quantitative SAT (SAT-Q) scores. To determine the main effects of overall calculus problem-solving-focused instruction (CPSFI), the data was modeled combining the experimental and control groups.

It should be noted that there are plausibly more weak students taking the first two tests who eventually withdrew from the courses before taking the final two tests. The analysis takes this into account by excluding any students who only participated in the first and/or second testing period. In this way, the analysis uses roughly the same set of students at each test time.

A separate analysis examined possible experimental treatment main effects by comparing the control group to the set of students in the experimental group who satisfactorily completed 75 percent or more of the additional group sessions.

Qualitative analysis of the data included categorizing the various problem solving techniques used and responses given on each task. Each test was coded according to these classifications and then organized according to test time, item number, and experimental or control group. Frequency data and descriptive statistics were calculated to detect what proportion of students used the various strategies on each item and how this changed over time. Further analysis, seeks to characterize—using final course grades—those problem solving strategies that strongly or weakly correlated with the top performing students in the classes.

Results

The statistical analysis of the CPSFI main effect reveals that there was a significant rise in the students' strategy and accuracy scores over the course of the semester. SAT-Q score had a significant effect on problem solving performance, but its effect was uniform across the test times and difficulty levels. The analysis indicates that in comparing two students with a 100 point difference in SAT-Q scores their problem solving would differ by .38 on the rubric scale. In Figure 2, the graphs represent the modeled performance of a student with an SAT-Q score of 600 at the four different problem difficulty levels. Note that the modeled performance of a student with 500 SAT-Q could be indicated by a vertical shift of these graphs by -0.38.

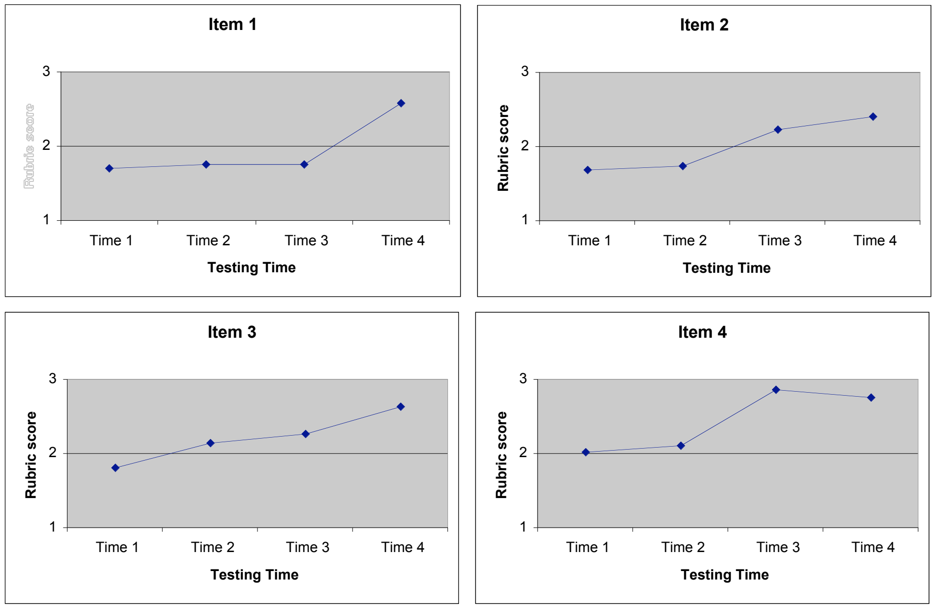


Figure 2. Modeled STAC Performance for Students with SAT-Q score of 600

In a statistical test of main CPSFI effects on STAC scores, the difference between the modeled score at time 4 and at time 1 were estimated and in every case the difference was significant at level .05 as seen in Table 2. The difference between the modeled score at time 4 and time 1 was also calculated for students who earned a course grade of A or B. The differences were only significant at level .05 for items 3 and 4 as seen in Table 2.

Problem version	Estimate of (time 4 – time 1)	p value	A-B student estimates	p-value
Item 1 (most difficult)	.74	.0106	.47	.1983
Item 2 (second difficulty)	.79	.0063	.18	.6350
Item 3 (third difficulty)	.72	.0118	1.05	.0044
Item 4 (least difficult)	.88	.0024	.89	.0170

Table 2. Main Effect of Calculus Problem-solving-focused Instruction on STAC scores

Analysis of the main effect of the experimental treatment revealed no conclusive trends.

Conclusions

The results of the statistical analysis only strongly supported the presence of one of the main effects that were considered in this study, which is the CPSFI main effect. Findings indicate that the base-line problem solving abilities of an incoming calculus student with 600 SAT-Q is at or below *developing* (rubric score 2) for each difficulty level, only reaching *developing* on the simplest item. Even the modeled performance for a student with 700 SAT-Q shows base-line performance above *developing*, but not close to the *proficient* level. The main effect ranged from 0.72 to 0.88 with the greatest effect being on the simplest item. This indicates that upon completion of the course, a student with 600 SAT-Q is closer to the *proficient* level (3) than the *developing* level (2) on items 1, 3, and 4.

Considering that the test is somewhat sensitive to calculus instruction, it should be expected that student problem solving performance would improve with increased exposure to calculus topics. This may indicate that the main effect came more from the expansion of the students' skill knowledge rather than the expansion of their depth of insight and mathematical reasoning. However, their success at using calculus skills effectively to solve problems at a proficient level is certainly a goal of such courses.

Findings suggest that students earning an A or B in the calculus class benefited less from the calculus instruction on the more difficult items 1 and 2 than on simpler items 3 and 4. This may be due to the more procedural nature of the item 3 and item 4 tasks. The A-B students' baseline performance was higher on the more difficult items and lack of significant improvement may be due to a ceiling effect since improvement from the calculus treatment seems to be linked to an increase in procedural rather than conceptual knowledge. This is despite the design of the calculus course to incorporate problem-solving activities in recitation sessions to increase students' conceptual knowledge and use of various problem solving strategies.

The overall indication of the analysis is that there was not a sizeable enough experimental treatment main effect to be detected in the present study. This would indicate that 1.5 hours per week of extra problem solving is not sufficient to boost problem solving performance beyond what is already done by a problem-solving-based calculus treatment as was employed during this study.

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IMPROVISATIONAL ETIQUETTE AND THE GROWTH OF COLLECTIVE MATHEMATICAL UNDERSTANDING (1)

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In this paper we characterise the growth of collective mathematical understanding as an improvisational process. Drawing on elements of improvisational theory, in particular the notion of etiquette, to analyse extracts of data gathered in an apprenticeship training classroom, we demonstrate how this theoretical framework can illuminate and inform collaborative group processes in a mathematics classroom.

Collective mathematical understanding is a phenomenon that has received increasing attention in recent years from a number of different perspectives (e.g., Bowers & Nickerson, 2001; Cobb, 1999; Davis & Simmt, 2003). In referring to collective mathematical understanding we point to the kinds of shared learning and understanding we may see occurring when a group of learners, of any size, work together on a piece of mathematics. We observe collective understanding as a phenomenon that is bound up in the social context of the learning environment, and as such can not be described merely by attending to the actions of individual learners. Elsewhere, we have demonstrated how a collective perspective on mathematical understanding can more fully explain its growth (Martin, Towers & Pirie, 2006; Towers & Martin, 2006). In those works we have employed and extended the theoretical work of Becker (2000), Sawyer (2000, 2001, 2003), and Berliner (1994, 1997), and characterised the growth of collective mathematical understanding as a creative and emergent improvisational process. In further developing our framework, we have documented the ways in which learners' *improvisational coactions* (Towers & Martin, 2006) lead to the growth of collective understanding. Here, we extend that conceptual analysis to a yet finer-grain to explore a key feature of the improvisational process—the etiquette of improvisation.

Improvisational Theory and Improvisational Etiquette

Improvisational theorists have proposed particular characteristics of the improvisational process. There are many more than can be fully articulated here, but those which we suggest are particularly significant in the study of the growth of collective mathematical understanding include the notions of (a) the potential pathways of action, (b) the emergence of collective structure, (c) striking a groove, (d) performance etiquette, (e) listening to group mind, and (f) pursuing the *better* idea. In this paper we seek to elaborate and illustrate one of these key theoretical constructs—improvisational (performance) etiquette—and discuss how this can be employed in considering and describing the growth of collective mathematical understanding.

Becker (2000) talks of improvisational etiquette as requiring that, in a group improvisational performance, "everyone pay close attention to the other players and be prepared to alter what they are doing in response to tiny cues that suggest a new direction that might be interesting to take" (p. 172). It is a process involving "attentiveness, care, and willingness to give ground and take direction from each other" (p. 173) and as such it "calls attention to the granting of equal

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status to everyone's ideas and intuitions" (p. 173). The improvisers "agree, implicitly and collectively, to give priority to what, in their collective judgment, works" (p. 174), and that might lead to something better. In improvisation, when one person does something that is obviously better (in the view of the collective) then "everyone else drops their own ideas and immediately joins in working on that better idea" (p. 175). In mathematics, better is likely to be defined as a mathematical idea, meriting the attention of the group, which appears to advance them towards a solution to the problem.

Methods & Data Sources: The Case of Andy, Joe and Mike

This paper draws on a case study of a small group of apprentice ironworkers as they engage with mathematics in the early stages of their training at a local institute of technology in a large Canadian city. The group and their instructor were observed and video-recorded over a number of sessions, and their in-class work collected for subsequent analysis. Data analysis, using the framework of improvisational theory, was conducted following the method proposed by Powell, Francisco, and Maher (2003), a multi-step process which involves repeated viewing and re-viewing of the video and supporting data to develop valid and coherent coding, analyses, and narrative storylines. One such storyline, focusing on the key improvisational construct explored in this paper, is presented briefly here. In the session discussed in this paper, Joe, Andy, and Mike have been posed the task of establishing the size of choker sling required to lift an assembled structure of four large iron beams into an upright position, and later of determining where the crane should be positioned to accomplish this. The size of choker is something that is dependent on the total weight of the structure to be lifted, and it is this weight that the group first has to calculate. The apprentices have a set of eight technical plans, showing different elevations and views of the framework for the building. The plans contain all the information necessary to assemble the framework, but the form in which this is presented requires considerable extraction from the various diagrams. What makes the task more complex is that fact that on the set of plans, not all the measurements and specifications are in imperial units. Some of the eight drawings are solely labelled in metric whilst others actually mix imperial and metric units. We join the apprentices at the start of the task where they are looking at a number of the technical plans, locating the relevant beams on these and determining their specifications in order to calculate their weights. On the drawings the beams are labelled with two pieces of information: their depth and their weight per metre or foot, depending on whether metric or imperial measures are being used. For example, in this task one of the beams is W ten by twenty-one, indicating a depth of ten inches, and a weight of twenty-one pounds per foot. On a plan this would simply written as 'W10 x 21'.

Extract 1

Mike:	It's a W ten by twenty-one. It's right here (*pointing to a drawing of a beam on one of the plans where the specifications are in imperial units*).
Joe:	It's not what it says here partner (*pointing to the same beam on a different plan where metric units are used*).
Mike:	Yeah, I know. Maybe that's f……g metric or something? I don't know.
Andy:	W two fifty by thirty-three? (*reading from the same page as Joe*)
Mike:	There's no way that's two fifty inches deep. It's way easier if you
Joe:	That's mills (*He means millimetres*). Yeah.

Mike: It's way easier if we do it with inches. It's a W ten by twenty-one.

In the above extract, the apprentices begin by working with the drawn elevations, extracting the mathematical information contained in them that will be necessary for the calculation of the weight of the beam. They understand how to read the plans and they are able to locate the correct beam and its specification from a complex diagram. However, Mike and Joe are using two different diagrams; one is a plan view (the structure from above), the other a cross-sectional elevation of the structure. Because the first of these gives a metric specification and the second an imperial, they become concerned about which units they are working with. Mike has found the beam specification in imperial units, whilst Joe has it in metric. Mike, still thinking in imperial, appears troubled by the figure of two-hundred and fifty ("no way that's two fifty inches deep"), and although he already has a suspicion that Joe is invoking metric units *("Maybe that's f......g metric or something?")*, he is not confident, or happy, to just assume this *is* the case, and suggests they adopt inches as their working measure, believing that for the group this will be 'way easier'. He now proceeds to try and find the total length of the beam from the diagram and in the next extract we see that almost immediately attention turns from the chosen imperial measures to the alternative metric ones.

Extract 2

Joe: Okay. See? It's right there

Mike: I'm just trying to find out how long it is (*pause*). All I need is one measurement. Maybe it's three thousand forty eight? (*reading a length from the plan whose measures are metric*).

Andy: For what?

(*Pause and mumbling from all three apprentices*. *Joe and Mike are looking at one of the plans*.)

Andy: That's what our centimetres will be, three thousand forty eight.

Mike: You think so? Oh yeah, it will. It will be, three thousand forty eight. We know how long it is.

Andy: I already got that (*laughs*).

Mike: Well. That's what we were after.

In the above episode, the apprentices return to the diagrams and are looking for the stated length of the beam. Though the group had initially, as suggested by Mike in Extract 1, pursued a solution using imperial measures, their collective attention now focuses on working in metric units. They locate the correct dimension, three thousand and forty-eight, though they think this is centimetres rather than millimetres (2). There is a slight sense of them not being sure about this measurement, but they end the episode confident that they have the correct piece of information to take forward to a subsequent calculation. They now attempt to use the figures they have abstracted from the plans to calculate the weight of the beam.

Extract 3

Joe: So twenty-one times three forty-eight (*working on calculator*). You already got this down, don't you? (*talking to Andy*) equals? Is that what you got? (*Asking Andy*).

Andy: Yeah (*he sounds uncertain*)

Joe:	In mills?
Andy:	In mills. That's f…..g
Joe:	sixty-four (*He has actually calculated 3048 x 21*)
Andy:	sixty-four thousand. So it would be sixty-four.
Joe:	That's not right.
Andy:	sixty-four?
Joe:	kilograms?
Andy:	Kilograms. Yeah. (*pause*). That can't be right, though. We did something wrong. Those beams ain't a hundred pounds.

In the above extract, having found the required specifications Joe now begins to calculate the weight of the beam. He knows the calculation to perform and obtains the correct answer for that calculation. On seeing that the answer is sixty-four thousand, the group collectively decide this must mean a weight of sixty-four kilograms, a more likely weight for a beam than sixty-four thousand kilograms. However, the mistake here is that they have incorrectly chosen to multiply a metric length by the pounds per foot specification, twenty-one, which has given the wrong answer. At the end of the episode Andy invokes his workplace mathematics understanding to question the group's solution, or at least their units of measurement and therefore the magnitude of the answer. He offers a challenge to the group, pointing out that a beam of the type under consideration "ain't a hundred pounds."

In the next episode, we see how the group responds to the challenge—not by abandoning their solution but by accommodating Andy's challenge and adapting their strategy.

Extract 4

Mike:	Well, you have to do it the metric way. You can't times anything by forty point eight. (*Note that there is no indication on the videotapes of where Mike obtains this figure. He may still be thinking about the three thousand and forty-eight*)
Joe:	(*Pause*). No, I'm not. I'm just timesing it by the weight per kilogram which is twenty-one.
Mike:	Oh yeah.
Andy:	Is that per metre?
Mike:	No, that's per foot.
Andy:	Yeah, we're doing metres though.
Mike:	We got to change something around before we do that.
Andy:	Because if we're changing that. No…
Mike:	Yeah, yeah, yeah. This is per foot.
Andy:	This one that you had over here (*indicating the other plan*) the three fifty and the two fifty times thirty-three, that would be the millimetres.
Mike:	Yeah.
Joe:	Yeah, you're right.
Mike:	That can't be two fifty by thirty-three.
Andy:	That's just the…
Mike:	Oh yeah, thirty-three kilograms per?
Andy:	Per?

Mike: Per metre. (*They both nod*). Okay, yeah. We're happening now. Do it like this. Times it by thirty-three.
Andy: thirty-three kilograms
Mike: thirty-three kilograms per metre.

At the beginning of the above episode, although all three apprentices recognise now that the calculation is incorrect, no one participant is instantly able to determine what to do instead. However, Mike and Andy draw upon their understandings of the mathematical relations involved to realise that they "need to change something around." Andy realises that they don't actually need to convert from imperial to metric, but instead could simply use the metric specification offered on the other diagram, and Mike and Joe agree with this. Mike knows this specification is in kilograms, but is not sure 'per what'. Again, it should be noted that no units are printed on the diagram, so the beam is simply labelled as W250x33. Mike is clear about the mathematics to use here, now that he realises they are working with metric measurements. Once he confirms with Andy that the specification is given in kilograms per metre he knows quickly that all that is required is to multiply the length of the beam by the weight per metre, though it has taken him a moment to be sure of this and he seems to need the confirmation of the group (offered with nods of agreement) before he pursues this solution method.

Analysis and Interpretation

As Becker (2000) notes, the etiquette of collective improvisation is subtle.

> Everyone understands that at every moment everyone (or almost everyone) involved in the improvisation is offering suggestions as to what might be done next, in the form of tentative moves, slight variations that go in one way rather than some of the other possible ways. (p.172)

We see this kind of nuanced coaction (3) play out in the apprentices' collaboration, especially at the beginning of their task where each apprentice offers (sometimes tentative) possibilities for action—which plans to consult, the weights of which beams to calculate, suggestions to use both imperial and metric measures, etc. They locate relevant plans and agree initially to work with imperial specifications. From here on, the contributions of each apprentice continue to build on these collective decisions.

As the collaboration progresses, we see the apprentices listening closely to one another, collectively choosing which pathway to pursue, and demonstrating a willingness to abandon personal motivations (and their own preferred strategies or approaches) and to defer to the "group mind" (Sawyer, 2003). As Becker (2000) notes, as improvisers listen closely to one another "some of [their] suggestions begin to converge and others, less congruent with the developing direction, fall by the wayside" (p. 172). In our data extracts, we see the initial approach of using imperial measures fall by the wayside as the group mind coalesces on the idea of metric measures, and this strategy, though not the one initially favoured, emerges as the better approach and leads the group towards a solution. This is possible because of the willingness of the three apprentices to listen to different ideas, to be receptive to the mathematical value of these, and to collectively decide whether to take them on board. This kind of co-operation we suggest is etiquette in action.

In his analysis of improvisational etiquette, Becker (2000) goes further to reveal how a group works together "when there is no audience immediately present and instead the improvisers are trying to solve a problem for its own sake or their own sake, because it is there to do and they have agreed to devote themselves collectively to doing it" (p. 174). In such situations, he suggests that rather than simply recognising "a hierarchy of ability already...known" the improvisers "ignore the past, ignore reputations, ignore everything but the contribution people make to the collective effort....The rule in these situations is to treat everyone's contribution as 'potentially better' than all the others" (Becker, 2000, p. 175). So, for example, although in Extract 1 Mike favours the use of imperial measurements, in Extract 2 we see Mike and Andy's contributions to the conversation coalesce around the potentially better idea of invoking metric measurement. Here we can see that it is not that a particular person is being afforded special status in the collaborative effort (indeed, the talk shifts fluidly between all three apprentices and all three contribute important ideas to the collective), but rather that particular *mathematical ideas* are afforded special attention—those being the ideas that the group, collectively, determine have the potential to advance them towards a solution to the problem.

Discussion

Becker (2000) suggests that that the kind of improvisational action we have described occurs where the group have "shared a past…which...suggests their long-term participation in an organized world in which the kind of activity they are improvising is common, probably a professional or quasi-professional world where ties of occupation bring people together in joint projects" (p. 175). This description is particularly true of the apprentices discussed here; they are participants in the world of ironworking with its particular structures, conventions and practices. Problem solving (and the associated practices seen here, such as interpreting complex diagrams and plans) is a vital part of their working life, and, with the high degree of unpredictability that exists within the workplace, is potentially an improvisational process. The need to collectively improvise, to be able both to offer ideas and to build on those offered by others, is particularly important in the context of the workplace, where wrong decisions or incorrect calculations have real costs and dangers. In our brief data extracts here, we do see the apprentices setting aside personal histories and reputations and genuinely coacting on the ideas of others based solely on the perceived mathematical value of what they offer, rather than on who they are. Becker (2000) acknowledges that "it is an interesting question-to which I have no answer-as to when people become unselfish…but it is a common observation that, even in settings not likely to produce such a commitment…people do act that way" (p. 175). Like Becker (2000) we do not yet know why people choose to act this way, but we would suggest that environments such as the workplace training classroom (and potentially the school mathematics classroom) offer the possibility of being such a context, in which participants have a set of common mathematical experiences and expectations that can provide the basis for unselfish, collective mathematical action, driven by the desire to solve a problem rather than to advance one's personal standing or reputation. Our data so far, and as illustrated by the extracts in this paper, suggest that, as Becker (2000) states, "the occasions on which this kind of more radical improvisatory experimentation occurs need not be rare or require people of extraordinary talent or psychological abilities (e.g., tolerance for looseness)" (p. 175). The idea of working together is something that teachers of mathematics are increasingly asked to encourage in their classrooms, yet there is a scarcity of detailed models of how these collaborative processes might enable problem solving and/or the growth of mathematical understanding. We suggest that the notion of improvisational etiquette is one key element of such a model, and that attending to the characteristics of improvisational etiquette (in the context of the emerging mathematics) such as "attentiveness, care, and willingness to give ground and take direction from each other" (Becker, 2000, p. 173) within the classroom, may offer one way for teachers to see how group work supports the growth of collective mathematical understanding.

Borrowing from Becker (2000), we also note that one could identify a whole range of kinds of (mathematics education) situations that vary between two poles—those that work on the basis of an etiquette that recognises and maintains a formal ideology of (mathematical) status (and hence render mathematics inaccessible to many) and those whose etiquette requires recognition of differentials in the contribution made to the collective effort (and hence encourage mathematics learners to participate in generating solutions to problems by recognising and valuing the potential worth of every mathematical contribution to the collective action). "These differences are embedded in and supported by differences in larger social organizations, which create the conditions making one or the other possibility more likely" (Becker, 2000, p. 176) and hence we suggest that research attention might fruitfully focus not only on the classroom structures that support improvisational etiquette but also on the larger administrative and societal structures in which individual classrooms and their actors are embedded.

Endnotes

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2. 3,048 millimetres is equivalent to ten feet in imperial units, and has resulted from the conversion of this imperial length to metric before adding to the diagram.

3. Their coaction is evident in these brief extracts of transcript but more clearly observed in the longer (and richer) video record.

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ON PROBLEM SOLVING PROCESSES DEVELOPED WITHIN A COMMUNITY

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The focus of this study is to investigate resources and strategies that high school teachers exhibited while working on problems related to change and variation. The results indicated that during their problem solving processes, they transit from incoherent and limited approaches to more robust ways to solve the tasks as a result of discussing their ideas within a community.

The analysis of the process involved in learning mathematics is complex: What types of tasks and instructional activities help students develop forms of reasoning that are consistent with mathematics practice? How can students construct or develop mathematical knowledge? To what extent does the use of computational tools benefit students' problem-solving approaches? In order to answer these questions, it is necessary to address issues related to the nature of mathematical knowledge, learning scenarios, and the tasks that make it possible for students to understand and use mathematical concepts in problem-solving instruction. In terms of developing teaching activities, it is necessary to take into consideration a very important aspect which is the inquiry of real life situations. This inquiry will allow students to handle data and build models or designs for further analysis so that they are motivated to use mathematics and provide meanings to the concepts and their use (NCTM; 2000; NRC, 2000). In addition, the learning situation in the classroom should become a learning community in which students get used to develop, display, and assess ideas related to problem-solving tasks, both individually and in groups so that they develop criteria to validate those ideas collectively (Boaler, 2000, Manouchehri, 2003).

It is well known that problem-solving tasks are important for students to develop or increase their mathematical knowledge. They help students practice and value processes such as posing questions, seeking for mathematical relations, using different forms of representations, using several arguments to support conjectures and communicating results (Schoenfeld, 1985). Such activities are important to transform the classroom activities into a learning community.

This transformation is not a linear and sporadic process; it depends on the instructor initiative and the ways to structure and organize the learning activities, for example, the instructor should select the tasks and the strategic instruction in terms of the type of knowledge and abilities sought, and the activities that can be developed by the students in this learning situation (Manouchehri, 2003). Thus, the research questions in this report are: How can a group of students become a learning community? What influence does a learning community have on encouraging students to constantly reflect on the approaches used to perform problem-solving tasks? What ways of mathematical reasoning do students develop and use in this learning community?

Conceptual Framework

The term "learning community" is used to refer to a classroom environment where students are interested in achieving a shared goal (in this case, comprehending mathematics ideas or solving mathematics problems) as they get engaged in collaborative work. Thus, students infer, share, discuss, and explain the arguments that support their assumptions. In

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this context, students, in a collaborative way, find criteria to explore and assess their own conjectures. The instructor organizes and guides the activities so that students work individually, in small groups, and within the whole group. Furthermore, the teacher must provide activities to help students analyze situations using different resources and representations; stimulate the comparison of their results; promote individual and collective work; and encourage the need of learning, increasing, and redefining new mathematical concepts and topics (NCTM, 2000).

In this type of environments, students not only learn, explore, and examine concepts, methods, and mathematical processes in classroom activities, but also learn habits that are consistence with mathematical practice. Learning mathematics cannot be separated from this interactive participation that takes place in the classroom throughout the learning period because both of them have influence on each other (Boaler, 2000).

Problem-solving tasks based on real contexts require asking and answering questions arisen from the different perspectives embedded in the situations; therefore, there is not necessarily a unique answer like those required in most math problems. In such approach, there may be several answers, according to the conditions and assumptions made. In order to answer these problems, it is necessary that students represent and explore different approaches to create and redefine previous inferences, changing from their initial incoherent and unorganized inferences to better organized and structured ones. It is useful to observe and trace each student's assumptions to be aware of them and be able to analyze the episodes they go through to understand the situation, as well as the procedures they follow in order to answer the question or questions asked. This will give us information that will allow us to identify certain aspects of the process of the student's cognitive development (Lesh y Doerr, 2003).

A mathematical concept may have different meanings in different contexts and the ways in which it is used may be different, even if the same definition is used. During their academic studies in mathematics, teachers develop some meanings of concepts that should be seen as a part of the broad group of meanings of concepts. When students are solving problems, in a non-mathematical context, which requires the use of their previous knowledge and computational tools or other resources, they need to listen to other students and discuss their ideas openly in order to reflect, improve, and restructure the associated meaning with the concepts involved. This perspective is different from the traditional way of learning. (Biehler, 2005)

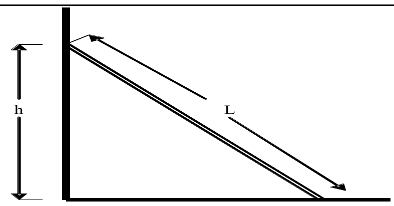
Procedures and Participants

The observations were made during a course which purpose was to develop students' problem-solving skills and improve their comprehension of mathematical concepts, processes, and their ways of reasoning. This study was conducted with seven students of a master program in mathematical education who were in-service teachers at high school level. All of them have a bachelor degree in science in areas like mathematics (three), engineering (two), and business (one). There were sixteen sessions, each one of three hours weekly. The contexts of the problems included situations related to population, credits, concentration (of substances), movement, volume calculations, and substance mixture preparations. While working on the tasks, the participants were allowed to use computers, calculators, or any other resources. In all sessions, students worked individually, in pairs, and then each pair presented their results to the class. Students recorded their approaches on the computer or in their notebook, and they also made reports; all of the evidences were collected at the end of every session. The presentations and group discussions were audio taped.

Results and Discussion

The information used was taken from the reports made by the students while working with two of the activities; the first one was assigned in the first part of the course and the other one in the last part of the course. The first activity did not have only one solution and could have been approached from different perspectives by using several mathematical concepts and procedures. The second one required students to use concepts such as successions and recursive procedures. The information was analyzed in order to identify characteristic aspects of the episodes followed by the participants when solving each problem: understanding the problem, devising a plan, carrying out the plan, and looking back throughout the different phases of classroom work, focusing to the mathematical concepts and process, and in the representation used.

Problem A. *A ladder which has a length of L meters is leaning on a wall and the top edge is at a distance of h meters from the floor. All of a sudden, the ladder begins to fall in such a way that the top edge moves downwards at a constant speed of v m/sec. Describe the speed at which the other edge of the ladder moves.*



The reports of the individual work indicated that the initial comprehension of the problem was similar among students: to determine the speed at which the bottom edge of the ladder moved. Three participants first claimed that it moved at the same speed as the top edge. However, only two of these participants explored to support the claim that their first assumption had been wrong (see Fig. 1 and Table 1).

Al lado sin identificar le llamo p, entonces $L^2 = h^2 + p^2$.
El valor de L es:
El valor de p es:
Si la escalera se desliza, lo hará una distancia, entonces si la v=d/t

La velocidad del punto inferior es

$p ? \sqrt{L^2 ? h^2}$

Analizamos en primer lugar el desplazamiento del extremo inferior, dando valores al largo de la escalera (L), a la altura (h) y al piso donde está apoyada por su parte inferior la escalera (p). Obteniendo la siguiente tabla:

L	h	p	Desplazamiento
10	8	6	1.141428429
10	7	7.14142843	0.858571571
10	6	8	0.660254038
10	5	8.66025404	0.504897352
10	4	9.16515139	0.374240624
10	3	9.53939201	0.258566957
10	2	9.79795897	0.1519154
10	1	9.94987437	0.050125629
10	0	10	

$V ? \frac{\sqrt{h^2 ? p^2}}{t} ? \sqrt{L^2 ? h^2}$

Supuse que la parte superior de la escalera se movía a una velocidad de 1 m/s. De la tabla se puede ver que si h recorre 1 m en un segundo, p recorre 1.14 m en el mismo tiempo. Los extremos no tienen la misma velocidad .

Fig.1 Excerpt of student G's report after individual work.

Sabemos que la rapidez es la derivada de la posición con respecto del tiempo. Si nuestro sistema de referencia es como en la figura (la pared vertical es el eje *y*, y el suelo es el eje *x*), entonces de la figura (Triangulo rectángulo) tenemos la siguiente relación:

$y^2 + x^2 = L^2$ (1)

Como la escalara (extremo superior) baja, entonces de acuerdo a nuestro sistema de referencia es una rapidez en sentido negativo. Entonces derivando implícitamente (1) respecto de t, tenemos:

$2y\frac{dy}{dt} + 2x\frac{dx}{dt} = 0$ el dato que se nos proporciona es $\frac{dy}{dt} = -v$, es decir la rapidez con la que cae (2)

El otro extremo de la escalera se mueve en el sentido positivo del eje x (de acuerdo al sistema de referencia), la rapidez con que se mueve es dx/dt, despejando de (2), tenemos:

$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$, (3) como la rapidez es constante, entonces cuando y = h, podemos calcular x de (1):

$x = \sqrt{L^2 - y^2}$, sustituyendo en (3) tenemos:

$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt} = \frac{vy}{\sqrt{L^2 - y^2}}$, esta es la expresión para la rapidez con que se mueve el extremo inferior de la escalera.

Fig. 2 Excerpt of student F's report after individual work

The initial assumptions shown by five of the seven students were based on the use of the Pythagoras theorem, in which the right triangle is formed by the ladder, wall, and floor (fig.

1). Within this group, one of them considered this relation as a situation of implicit functions and used calculus to determine the speed of the movement of the bottom edge (fig. 2); four students used this relation to observe how the bottom point moved by using a numerical approach in Excel (fig. 1), within these four, one of them used functions and derivatives on one variable to get the expression that would provide the speed as a function of time. Among all of the participants, only one used the concepts related to the movement of the angle and kinetic and potential energy in his first assumption. It is possible to see that at this stage, most students reached a level of comprehension of the problem: they identified the unknown quantity and the relations between important quantities, the changing process among each one, and at the same time they made suppositions about these changes that leaded them to make assumptions related to the aspects that characterize the answer to the question.

According to the reports handed in, the sharing of ideas among students in pair work led them to change the type of strategies used to solve their problem. Their comprehension of the problem was focused on describing how the position of the bottom edge of the ladder changed. One pair of students used the terms velocity and speed as synonyms, not in the same sense as in physics, but as the quotient of dividing distance by time. The strategies used to reach the answer were divided in three types: one using numerical approximations and the other two using approximations based on functions and calculus, one of which led the students to use simple differential equations.

Student	Considerations	
	Individually	Pairs
J	Both edges move at the same speed.	Use Pythagoras theorem, assign values and explore using procedures in Excel, use speed as the quotient of distance divided by time.
G	Both edges move at the same speed. Explores and rejects the assumption. Then, continues exploring.	
I	Both edges move at the same speed, reflects on the situation, rejects the assumption and finds a relation.	Use Pythagoras theorem, and implicit functions and derivatives. Speed is considered the derivative of position. Algebraic representation.
F	Establishes implicit functional relations and uses implicit derivatives to obtain the speed.	
C	Both edges move at the same speed.	Use Pythagoras theorem, and functions and derivatives as one variable. Speed is conceived as the derivative of the position. Use algebraic representation without explicitly establishing time as an independent variable.
B	Establishes functional relations and uses the derivative to obtain the speed.	
D	Uses concepts of the movement of the angle and the law of conservation of energy.	

Table 1. Approaches to the problem made by students during individual and pair work.

During group presentations, these last two assumptions were considered, and participants determined that both led them to the decision that analogue expressions were used for the expression which would result in the position and speed as a function of time of the bottom edge of the ladder.

Con estas ideas, englobamos los resultados para plantear la solución final:
Supusimos que en el tiempo t= 0, la escalera no estaba vertical, sino que tenía una altura del piso a su extremo superior, que denotamos por A, de 8 unidades, la escalera media 10 unidades y la distancia del extremo inferior (B) a la pared es de 6 unidades por el teorema de Pitágoras.
$D_A(t)$ = Vt que es igual a la distancia recorrida por la escalera cuando se va desplazando. $\sqrt{L^2 ? (A(t))^2}$
A(t) =A_0 – Vt que es la posición del punto A en el eje y en el tiempo t.
Entonces B(t) = es la posición del punto B en el eje x en el tiempo t.
B_0 = y $d_B(t)$ = B(t) –B_0. Con estos resultados podemos sacar la expresión general, que nos indica la posición en que se encuentra el extremo inferior de la escalera en el tiempo t ,la cual sería:

.

Ahora para sacar la velocidad instantánea a la que se desplaza ese extremo inferior, hay que encontrar la derivada de esa expresión con respecto a t, que es la variable que se mueve. La derivada es la siguiente:

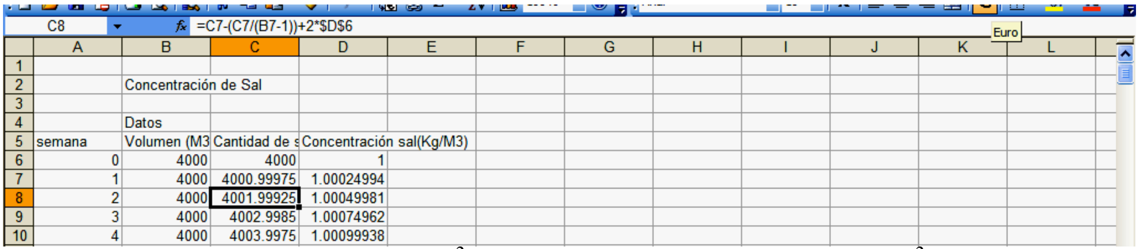
en donde si sustituimos diferentes valores de t, nos dará la rapidez a la que se mueve el extremo inferior de la escalera. Nótese que efectivamente la velocidad no es constante, cuando la rapidez si es constante en el otro extremo.

Fig. 3 Determination of A(t) y B(t) carried out by Student B

At this stage, the instructor asked some questions concerning the different assumptions, such as: Do they describe how the bottom edge of the ladder moves? How are they different? This guided students to reflect about what they had done. There were some observations about the "particular nature of the numerical approaches, and also about the general nature of functional ones. Therefore, the instructos asked if the numerical approximation could be used and how. Student B explained how he used the numerical approximation, which helped him to understand the way in which the magnitudes involved were related. Throughout this process of reflection, the necessity of using both the adequate algebraic representation and the convergence of the two assumptions based on functional expressions became evident. The students realized that they had to be careful with the symbols and definition of variables, relations, and conditions of the problem (see Figure 3).

Problem B. *Initially, a pond has n units of volume of natural water, with a c uniform concentration of salt. Due to the sun, one unit of volume of water evaporates every week. Besides, one unit of volume of the rest is taken out and both are replaced with two units of natural water so that the concentration of salt does not increase. Describe the concentration of salt in the pond.*

The first approaches to understand the problem, which were made individually, are characterized for being numeric. Students assigned values to the initial volume of water and the initial concentration (or quantity) of salt in the pond, and used procedures in Excel to determine the concentration of salt at the end of every week without giving any extra information (see Figure 4). Their main interest was to determine the quantity of salt as a strategy to get the salt concentration, which determines the quotient of the quantity of salt and volume of the water. On the whole, students identified a recursive process, such as the one carried out by the student G (see Fig. 4), which helped them find the amount of salt in terms of what it had been the previous week C7, volume B7, and the initial concentration D6. Furthermore, they were even able to obtain the concentration of salt at the end of the following weeks by repeating the procedure.



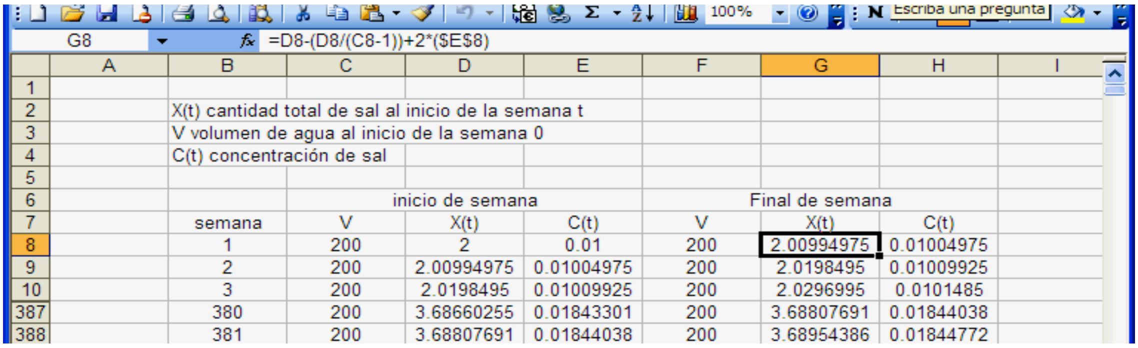
C8 =C7-(C7/(B7-1))+2*D6

	A	B	C	D
1				
2		Concentración de Sal		
3				
4		Datos		
5	semana	Volumen (M3	Cantidad de s	Concentración sal(Kg/M3)
6	0	4000	4000	1
7	1	4000	4000.99975	1.00024994
8	2	4000	4001.99925	1.00049981
9	3	4000	4002.9985	1.00074962
10	4	4000	4003.9975	1.00099938

Figure 4.Student G assigned 400 m^3 of V and a concentration of $1kg/m^3$ (Individual report)

At this phase, two students found it difficult to understand and apply the concept of "concentration" when they were representing the process of variation in the concentration of salt in the pond. Pair work helped students improve the representation of the process in Excel by making visible the relevant amounts at the beginning and end of the week, and also, by posing questions such as: What should I obtain? What is the amount of concentration of salt per week? How do I get these sums? We observed that when students work in pairs, they are compelled to explain and clarify the process used to find the answers (see Figure 5).

The presentations for the whole group made it possible to assess what had previously been done in pairs, in terms of answering other participants' questions and doubts. This allowed them to identify structural aspects of the procedure and also consider criteria to assess the relation between the initial information and the results obtained.



G8 =D8-(D8/(C8-1))+2*(E8)

	A	B	C	D	E	F	G	H
1								
2		X(t) cantidad total de sal al inicio de la semana t						
3		V volumen de agua al inicio de la semana 0						
4		C(t) concentración de sal						
5								
6				inicio de semana			Final de semana	
7		semana	V	X(t)	C(t)	V	X(t)	C(t)
8		1	200	2	0.01	200	2.00994975	0.01004975
9		2	200	2.00994975	0.01004975	200	2.0198495	0.01009925
10		3	200	2.0198495	0.01009925	200	2.0296995	0.0101485
387		380	200	3.68660255	0.01843301	200	3.68807691	0.01844038
388		381	200	3.68807691	0.01844038	200	3.68954386	0.01844772

Figure 5. Modified table carried out by pair of Students C and G.

This activity led students to seek an algebraic representation of the process. The contributions that came up within the group were analyzed in pairs. Each student wrote a new report individually, then they discussed it with their partner, and each pair made another presentation for the whole group. In these presentations, all participants accepted the expression and propositions made by one of the pairs of students (Students G and C). The pair highlighted the recursive nature of the process followed, and used this detail, even though it was used in an incorrect way, to obtain a simple algebraic expression, which guided them to establish that the process in week 2 was a repetition, with the only difference that the initial amount of salt is X_1. The same process was repeated for the following weeks (see Fig. 6).

Students used this expression to determine the time in which the concentration would reach a value that is equal to the double of the initial amount of concentration. The previously statement would be represented in the following equation: $2\frac{X_0}{n} = \frac{X_0(n^2-2)^k}{n^{k+1}(n-1)^k}$, using a numeric procedure in Excel.

In the group discussion, nobody doubted about the results obtained, and they did not consider the procedure performed at the first phase, in which a slow increase in the concentration was visible. The instructor took part in the activity to help students identify this difference and guide them to reflect about what they had done, the model concepts, and the representations of a phenomena or situation. The students worked on the problem again in pairs. Some of them were able to obtain the symbolic representation of the process and then informed the group of this result. During the discussion, it became evident that different approaches, if they are correct, should lead to the same result.

Ya que al iniciar la primera semana hay $X_0 = nc$ cantidad de sal en el acuario.
Al final de cada una de las siguientes semanas habrá:

$$X_1 = X_0 - \frac{X_0}{n-1} + 2c = (\frac{n-2}{n-1})X_0 + 2\frac{X_0}{n} = X_0 \frac{(n^2-2)}{n(n-1)}$$

si consideramos que al inicio de la semana 2 hay X_1 de sal :

$$X_2 = X_1 \frac{(n^2-2)}{n(n-1)} = X_0 \frac{(n^2-2)^2}{n^2(n-1)^2} \quad ,...y\ tambien \quad X_k = X_0 \frac{(n^2-2)^k}{n^k(n-1)^k}$$

$$la\ concentración\ en\ la\ semana\ k\ es : C(k) = X_0 \frac{(n^2-2)^k}{n^{k+1}(n-1)^k}$$

Figure 6. Wrong recursive representation, made by student G, which is initially accepted by the class.

Reflection

While students were searching the answers to the activities, they developed attitudes and concepts they had not shown previously; for instance, they established criteria to value the answers given to the questions asked, and searched for more general answers instead of particular ones. Students also valued some procedures more than others. During pair work and group work, students who worked with numerical approaches accepted other types of approaches, especially if “advanced” mathematical concepts and theories were used (for example, calculus). Even though the numerical approach in Excel (not using paper and pencil) could have provided them with the necessary information to answer the questions asked in the problem as well as required to describe the situation. The fact that they had favored the approaches that implied “advanced” mathematical concepts was possibly due to the students’ limited knowledge of the tool or it may also be because of beliefs developed during their previous academic training (Schoenfeld, 1985). The discussion concerning a situation that had multiple answers allowed students to pay attention to mathematical concepts such as precision, exactitude, approximation, function, and others, of which students proved to have a limited understanding associated to algorithmic activities.

It is important to reflect on relevant problem solving behavior that the participants exhibited during the development of the sessions. The initial approaches to the problem shown by the participants, in general, were incoherent and seemed to aim at representing the problem algebraically. However, examining those first attempts led them to recognize that it was crucial to visualize and comprehend relevant data and particular relationships associated with the task. In this context, the participants recognized that there may be various ways to approach the problem and even those with certain limitations were important to be discussed for meaning and mathematical properties embedded in the solution process. The structure and development of the sessions seemed to be a fundamental ingredient to generate a learning community that values and accepts that learning takes place within an inquiry environment that reflect principles that are consistent with the practice and development of the discipline. In particular, the participants recognized that initial incoherent attempts to solve the problems can be transformed into robust approaches when the learning environment values and promotes the active participation of the learners.

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ROLES OF REPRESENTATIONS IN THE U.S. AND JAPANESE CURRICULA: LEARNING IN AND OUTSIDE OF CONTEXTS

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The paper examined the uses of visual representations in Japanese and U.S. elementary mathematics textbook series and investigated how they support student learning differently. U.S. series used representations to narrow student thinking to help them solve problems in the manner that have already been learned; Japanese series used representations to create space for students to analyze contexts using mathematics. The purpose of the contextual problems in U.S. textbooks is for application of the previously learned concepts and methods, while Japanese textbooks used them for mathematical analysis and explorations.

With recent attention to international studies, we are increasingly becoming aware that mathematics is not taught in the same way in different countries (e.g., Third International Mathematics and Science Study, Programme for International Student Assessment). This paper will examine how differently visual representations are used in U.S. and Japanese textbook series and how they support student learning of mathematics. Role of representations in contextual problems (story problems) and non-contextual problems (number problems) in the two series are analyzed, and the different expectations and goals of mathematics teaching and learning suggested by the uses of the representations are discussed.

Perspectives

The Third International Mathematics and Science Study (National Center of Education Statistics, 2003) compared different aspects of mathematics education in different cultures to examine the reasons for achievement differences (e.g., video study of teaching), and one of the core studies examined mathematics curricula (Schmidt, et. al., 1997). The study found that, although some common mathematics topics are taught, the organization and the presentation of these topics varied across countries. Small and frugally bound textbooks used in Asian countries were compared with large and expensive U.S. textbooks that cover varied goals in the "*mile wide and inch deep"* curricula across the fifty states (Schmidt, et. all, 1997).

Japan is one of the high-scoring countries in the international studies mentioned above. The Japanese curriculum focuses on a few core topics, and there is little repetition and re-teaching of these topics. Concepts are typically introduced as an extension of students' prior learning to make the connections among their learning experiences stronger. Moreover, a long time is spent on each major topic to create successful learning by all students. These curricular approaches minimize the need for re-teaching. Topic presentation is carefully thought out with common visual representations to connect core ideas across topics and across grades: students' mathematics experiences are centered around supportive representations and situations to help students build meanings (see other studies for discussion, such as Mayer, Sims, and Tajika, 1995). For instance, the National Center for Education Statistics (2003) found, in comparing videos of teaching as a part of TIMSS, the Japanese instruction made two to four times more use of visual representations than did other countries.

National Council of Teachers of Mathematics (NCTM, 2000) defines representation as the "act of capturing a mathematical concept or relationships in some form and to the form itself (p.67)". Representations are an essential part of and an effective tool for learning and doing mathematics, and it is important for students to understand mathematical concepts and be able to use multiple representations to show their understanding of the concepts. This paper will examine how these representations are used in the U. S. and Japanese textbook series and their roles in student learning of mathematics.

Methods

One U.S. and one Japanese elementary mathematics textbook series are analyzed for the study. *Scott Foresman Addison Wesley Mathematics* (Pearson, 2004) and *Study with Your Friends: Mathematics* (Gakkkotosho, 2005) are selected because they are one of the most used textbooks in each country (Horizon Research, 2002; Japanese Ministry of Education 2006). All Grades 1 – 6 instructional units were analyzed for their uses of representations. At first, the units were marked for their use of different representations. From there, the five most frequent visual representations across the two textbook series across all grades were identified (namely, pictorial representation (1), tape diagrams, number lines, ten frames, and base-ten blocks), and the units in the two series that used them were further analyzed for their uses in terms of contextual problems (story problems) and non-contextual problems (numerical problems). The units in the U.S. and Japanese textbooks that had similar instructional goals (for addition and subtraction) were then pulled out and contrasted for the structural differences for lessons. These topics are selected because they are the key topics in lower-elementary grade levels when students use visual representations for the first time for mathematical concepts, and it is considered important to understand the early mediating role of the representations. Teachers' instructional manuals were also examined for additional information in terms of instruction as well as homework assignments.

Results and Discussion

For both textbook series, pictorial representations (actual pictures of objects) are the primary representation for the early part of their learning (Grade 1 and early Grade 2); ten frames are used as students work with numbers less than 10; and base-ten block representations are used for multi-digit number operations. While many units in both textbook series use visual representations, their uses differ in an important way. The U.S. series uses representations to accompany more problems that are not contextual. For example, pictures of flowers accompany solutions to the numerical problems such as 3 + 5. In contrast, the Japanese series use representations mainly for contextual (story) problems. Table 1 shows the percents of the visual representation uses in both textbook series for contextual and non-contextual (numerical) problems.

Table 1. Percentages of addition and subtraction units in Grade 1 – 3 that used visual representations for contextual and non-contextual problem solving in two textbook series

Types of Representations	Scott Foresman Addison Wesley Mathematics		Study with Your Friends Mathematics	
	Contextual	Non-context	Contextual	Non-context
Pictorial	50%	65%	88%	6%
Tape diagram .	15%	5%	53%	0
Number line	10%	50%	0	18%
Ten frame	0	35%	12%	6%
Base-ten blocks and cubes	15%	75%	53%	18%
Average percentage of representation use	*18%*	*46%*	*41%*	*10%*

Note. Only the units in each textbook series that concerned addition and subtraction concepts were analyzed (e.g., geometry and measurement chapters were not analyzed). There were 20 such units in Scott Foresman Addison Wesley Mathematics, and 17 units in Study with Your Friends Mathematics for their Grades 1 – 3 volumes.

In average (of five frequent representations), the U.S. series uses the representations 18% of the time for contextual problems and 46% of the time for non-contextual problems. In contrast, the Japanese series uses the representations 41% of the time for contextual problems, and only 10% of the time for non-contextual problems.

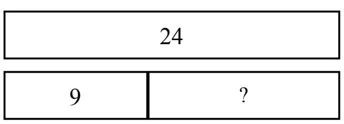
Zooming in further, Figure 1 contrasts typical guiding steps and questions used for problem-solving using tape diagrams in the two textbook series (for contextual problems).

Figure 1. Examples of guiding steps and questions used for problem solving with tape diagrams in two textbook series (as shown in student books)

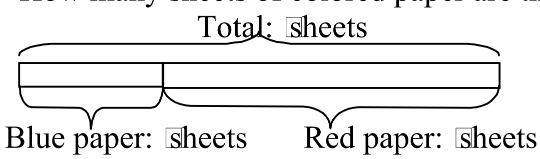
<table>
<tr>
<td><u>Scott Foresman Addison Wesley Mathematics</u>
Grade 3, Unit 2 (early Grade 3); Addition and subtraction number sense</td>
<td><u>Study with Your Friends Mathematics</u>
Grade 2, Unit 7 (mid Grade 2); Addition and subtraction (1)</td>
</tr>
<tr>
<td>
Problem: Mitch bought a cap and a pennant. How much did he spend altogether?

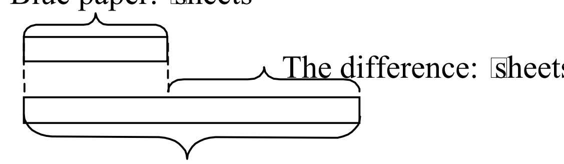
Eagles Baseball Souvenirs
Cap $12
T-shirt $18
Pennant $4
Poster $3

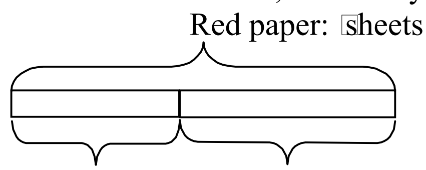
Guiding steps:
1. Identify the main idea.
2. Draw a picture to show the main idea*.
3. Describe the two parts in this problem.
4. Write a number sentence for the problem.

Note. * in teachers’ manuals, the sample picture for this steps is shown as below:


</td>
<td>
Problem: There are 38 blue sheets of paper and 63 red sheets.

Guiding questions:
1. How many sheets of colored paper are there?


2. Which one is how many more?


3. She uses 25 red sheets, how many will be left?


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The Japanese example was taken from the Grade 2 unit on addition and subtraction; the U.S. example was taken from the Grade 3 unit on addition and subtraction number sense. These two units were chosen because they introduced the use of tape diagrams for the first time for students, and thus believed to provide necessary scaffolding steps for using the diagrams for problem solving. In the U.S. example, the problem was given at the beginning (*Mitch bought a cap and a pennant. How much did he spend altogether?*), the necessary information followed (a table summarizing the prices of the items), and students are then guided by a set of questions (in student books) to solve the original problem. For the Japanese example, the context is given at the beginning (*There are 38 blue sheets of paper and 63 red sheets*), and questions follow to examine the given situations that accompany different tape diagram representations. The questions guided students’ attention to different aspects of the addition and subtraction relationships in the given context and how the tape diagram could show the relationships. The U.S. approaches utilize the representation to solve a problem, while the Japanese approach focuses on analysis of the context using representations.

U.S. and Japanese lesson structures are also different in the two textbook series. A U.S. lesson starts with a simple problem (often non-contextual) that shows a step-by-step solution of the problem. Following the example, several problems are given that require students to take the same steps for solution. Representations guide the problem-solving at this point, but they are typically shown in an identical manner from one problem to the next so students may focus on the patterns with the set of problems and finding answers. There may be one or two story problems embedded in the lesson, while the focus of the lesson is clearly on learning how to follow the given steps and finding answers to a set of similar problems.

The Japanese lesson structure is different. A lesson begins with one problem, and for an entire lesson (that can range from one to several class periods), students take time to represent, explain, solve, and discuss the problem and their approaches to solving the problem. Problems in Japanese textbooks are often "worked out" and relevant representations support the entire problem-solving process (Mayer, Sims, and Tajika, 1995). Representations are used for students to analyze and understand the problem and generate a space for instructional conversation. Each instructional unit includes a set of non-contextual problems, but they are not the main focus of the learning.

The U.S. and Japanese textbook series took different approaches in incorporating representations to support students' learning: U.S. series used representations to narrow student thinking to help them solve problems in the manner students previously learned with numerals; Japanese series used representations to create space for students to analyze contexts using mathematics. The purpose of the contextual problems in U.S. curricula is for application of the previously learned concepts, while in Japanese curricula, it is for analysis of mathematics in situations. In the application approach, students enter the problem situations knowing that they are to use what they have just learned to solve them, thus quickly move to make the contextual problems into numerical ones (Boaler, 1994; Brown, Collins, and Deguid, 1989). In the analysis approach, while students know that mathematical ideas will follow contextual problems, space is allowed to think of the presented situations without a known goal that students engage in the situation in more-meaningful ways.

Conclusions

Comparative studies are often useful in highlighting the underlying assumptions of cultural practices. Different instructional approaches are often reflections of different beliefs on how students should learn mathematics, and the different uses of representations are examples of such. If the goal of instruction is to support the development of competent mathematics students who can follow steps and learn to solve numerical problems quickly and accurately, the U.S. approach to focus on simple steps and solving multiple problems of the same kind will make sense. If we believe that mathematics instruction should support students' deeper conceptual understanding of how mathematics works in contexts (Kilpatrick, et. al., 2001; NCTM, 2000), Japanese approach will work better.

We need to question ourselves of the purposes when we use contextual problems in mathematics classrooms. If contexts are there to simply practice mathematical concepts students have already learned, students won't investigate how mathematics works in the contexts carefully because of the shared instructional priority on numerical problem solving. If we want to use contexts meaningfully in mathematics instruction, we will need to find new ways to present problems so that students will focus on the investigation and analysis of mathematical ideas in the contexts and not on practicing already-learned concepts and methods. The Japanese approach presents an example on how contextual problems may help create instructional space where students can analyze mathematics and communicate their ideas that is guided by representation uses.

Contextual problems do not always need to happen at the end of the chapters, and there are different ways to utilize the contexts to support student learning. As we are now faced with the challenge to support *all* students learning mathematics who come to the classrooms with different values and experiences, we need to rethink the way we value the differences (for their interpretation of the contexts in instructional conversation) and how to make connections between these different ideas and core mathematical ideas. Representations can play a critical role in supporting the communications of ideas (to make connections) in classrooms and meaningfully bridge and add extra meaning to the numerical problem solving, however, the ways in which representations are currently used in textbooks will need to be reconsidered in order to support these new purposes.

Endnote

1. When pictorial representation is used but does not support actual problem-solving process (e.g., a large picture of an apple is shown for a problem that discussed grocery shopping), it was not considered to be a representation that supported problem solving.

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BASIC MATHEMATICAL COMPETENCIES IN PROBLEM SOLVING SITUATIONS THAT INVOLVE DECISION-MAKING

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The main purpose of this study is to document and analyze high school students' competencies exhibited while working on models and modeling activities. Students worked a set of tasks that involves decision-making strategies and the use of proper representations to communicate and support their responses. Results indicate that the context and structure of the tasks helped students look for approaches and resources that often are not part of the curriculum they have studied and also recognized the need to constantly examine and refine their initial approaches to the tasks.

The objectives of school mathematics helping students develop powerful conceptual tools to build and work in systems that are progressively more complex in our present world (NCTM, 2000) which tend to revert the anumerical behavior that according to Paulos (1988) increase the vulnerability of the citizens before different influences of the natural and social environment that surrounds them. On the other hand, there are mathematical thought processes that are not classified within the mathematics curriculum in the place they should be (Doerr and English, 2003; Santos, 2002). Among those processes we could mention ponder, order, control, describe, experiment, classify, assess, select, organize and transform complete sets of data instead of specific data.

Models and Modeling (Lesh and Doerr, 2003) is a theoretical perspective that tries to rationalize the mathematics problem solving, and the teaching-learning processes of mathematics. The main idea in the *models and modeling* approach is that individuals give sense to their experiences in terms of the models that they have previously built. The concepts of *models and the modeling cycles* are important in this perspective.

"Ends-in-View-Problems" are relevant to the present world (Lesh and Doerr, 2003). "Ends-in-View-Problems" are those that give students specific criteria to generate useful, complex and multisided products that go beyond the information they received. Among the different products this type of problems generates are the models and the systems to assess. On the other hand, *decision-making* is one of the relevant problematic situations for future citizens. Decisions are constantly made during the day and decision-making if fundamental in science.

Methodology

Problematic Situations

A set of nine problematic situations were designed, but five of them required an assessment instrument before decisions were taken. An example of one of those situations is included below, such as these was presented to students.

The Zapatas have to move.

The Zapatas, the parents, a 16-year-old daughter and an 11-year-old son, have to move because of the father's job. The options the company gives them are: Aguascalientes, Tijuana, Ciudad Juárez, Saltillo, Distrito Federal, León, Guadalajara, Morelia, Puebla y Cancun. The family now lives in the Metropolitan Area of Mexico City (suburbs located in the boundary of Mexico City and the State of Mexico). The Zapatas, mainly worried because of their children, would rather have their new home in a "quiet" place. To make their decision, they got the following table in which the main problems in the different cities (depending on the number of inhabitants) are recorded.

MAIN PROBLEMS FOR INHABITANTS IN 12 CITIES

CITY	PO	INS	UNE	CRI	VIO	PUSER	DRAD	POL
Aguascalientes	8	29	38	10	2	4	7	3
Tijuana	16	29	6	21	10	4	11	2
Ciudad Juárez	15	22	3	18	18	2	21	1
Saltilllo	5	29	5	18	13	7	22	1
Distrito Federal	12	32	10	27	8	6	4	1
Metropolitan Area	12	20	10	24	10	6	13	3
León	17	12		19	10	4	21	1
Guadalajara	18	34	16	15	9	2	5	5
Morelia	18	21	12	20	12	2	9	2
Monterrey	18	29	18	16	8	7	9	5
Puebla	20	24	9	18	6	11	5	5
Cancún	20	26	10	17	2	19	10	1

Where: PO: Poverty; INS: Insecurity; UNE: Unemployment; CRI: Criminality; VIO: Violence; PUSER: Public Services; DRAD: Drug Addiction; POL: Pollution. (Each percentage corresponds to a city and to a problem. The number indicates the percentage of inhabitants who consider that problem as the most important.)

Help the Zapatas by pointing out which the best three options are! Write a letter to them explaining in detail how and why you chose those cities. Illustrate your decision with graphs or diagrams, yet with enough reasons so that they can choose the best option.

In these problems, essential mathematical ideas are centered in notions that include pondering, quantifying qualitative information, ordering, selecting and adding (combining, joining) ordered amounts, and constructing classifications. The problems are thought to elicit the multifactorial ordering notion to make a decision. It can also be stated that all of them are isomorphic from a mathematical viewpoint, what changes is the context of the problematic situation.

Sample Studied

The school where this test was applied is a public school that is part of the Colegio de Ciencias y Humanidades of the UNAM. 51 junior high school students participated, 37 were girls and 14 boys. They were all between 15 and 16 years of age. They constituted a typical school group and said group was chosen without taking into consideration any particular criterion.

Sources and Analysis of Information

While the students were solving the problems their oral participation was recorded and later transcribed. In addition, written records of each of the work groups they kept for each of the situations they considered were also collected. On the other hand, the professor-researcher took notes of all those aspects he considered relevant during all the work sessions. Consequently, data analyzed came from transcripts of the recorded participation, the written work kept by the groups and the notes taken by the researcher. At first, six transcripts (supported by the researcher's notes and the student's written work) corresponding to the solving process carried out by six small groups on six different tasks were analyzed. This analysis was focused on the numbers and the relationship among them the students considered, on the different ways in which they interpreted the tasks, on their mathematical reasoning to focus the problem, and on the representations they used. This analysis identified the different interpretation cycles of each of the small groups through each of the tasks. Secondly, the analysis was focused in describing the variations between the different models that were developed by the students for each for each and every one of the nine tasks. Products obtained in the student's written work together with the field notes of the small groups were analyzed to identify the differences between the different models that were constructed.

Results

- Some students begin by supposing which they consider the best option is. In this case during the modeling process the actions are oriented towards assessing, proving or justifying what they propose.
- Most of the students do not take into account the conditions in which the need emerges when they make their decision.
- The students asses which the relevant characteristics of the options are in respect to the conditions that surround the need.
- When a characteristic has almost the same values for a certain number of options (which might be all of them) the students consider it irrelevant for the decision in the options in which said characteristic is present.
- The students, once the characteristics of the option have been pondered, eliminate those that are not relevant.
- When the values of the characteristics are given in a qualitative form, students transform it into a quantitative one constructing measuring scales to carry out said transformations.
- Students order numerical data collections.

Some Observed Limitations

There are basic competencies in which the students have certain limitations. Those that were noticed are:

- They have difficulties in interpreting what certain numeric values included in the table represent:
- They have difficulties in applying the order relationship when the numeric values included in the table are written as decimal numbers;
- They have difficulties in ordering the numerical values of the columns when said column is "big" (for example, three columns by twelve rows); this might possibly because they get "confused" when they see "many" numbers and they then do not know what to do;
- They do not recognize the particularities surrounding the need, possibly because they do not pay attention to the statement that describes the problem and they focus their attention more on the values included in the table.

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STUDENT BEHAVIORS AND WAYS THEY ARE ASSOCIATED WITH THE PROBLEM SOLVING PROCESS

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We analyze data from classrooms to build upon pilot research in which we found an association between certain types of student behaviors and the development of understanding. The pilot focused on an after school program in a suburban setting. In this case, we focus on several inner city classrooms. Our findings indicate that there are many similarities. These results may have important implications for understanding how student behaviors evolve over the course of a particular problem solving experience.

This research is one component of a larger longitudinal study in which University researchers partner with teachers in a large and densely populated urban school district to provide professional development (Schorr, Warner, Gearhart & Samuels, 2007). In the present study, we analyze data collected from two classrooms in an effort to build upon research gleaned from a pilot study that took place in a middle-class suburban after-school center (Warner, 2005). In the pilot, the behaviors of the students were examined as their mathematical understanding of a particular idea grew (using the Pirie/Kieren model for the growth of mathematical understanding). Results suggest that there appears to be a relationship between certain types of student behaviors and the growth of a particular strand of ideas. These behaviors include, for example, building and linking representations, reorganizing and questioning your own and/or others' ideas, setting up hypothetical situations, etc.. In the current research, we test these hypotheses in 2 other classroom situations. Our results suggest that such similarities do take place, along with a few differences. This has important implications for understanding how student behaviors evolve over the course of a particular problem solving experience.

Theoretical Framework

The Pirie-Kieren model for the growth of understanding (Pirie & Kieren, 1994) provides a framework for analyzing student growth in understanding, via a number of layers through which students move both forward and backward. Pirie (1988) discussed the idea of using categories in characterizing the growth of understanding, observing understanding as a whole dynamic process and not as a single or multi-valued acquisition, nor as a linear combination of knowledge categories. Pirie & Kieren (1994) illustrate eight potential layers or distinct modes within the growth of understanding for a specific person, on any specific topic: primitive knowing, image making, image having, property noticing, formalizing, observing, structuring and inventizing. In Warner (2005) a relationship between student behaviors and movement through layers in the Pirie-Kieren model was reported on. The results were based upon detailed case studies of three middle-school students' problem solving efforts as they solved a series of combinatorics tasks with the same or similar underlying structures. A summary of the observed behaviors and the associated layers in the Pirie-Kieren model follows:

- There was a general marked decrease in students re-explaining, questioning and/or using their own or others' ideas, with this being the most frequent behavior occurring when the students were working in the inner-layers or early stages of understanding.
- Students moved to new representations at a fairly constant rate throughout the process, with instances of this behavior occurring in each of the layers.
- There was a slight increase in students reorganizing or building on their own or others' ideas, beginning with a high frequency in the inner-layers of understanding.
- There was a general marked increase in students linking representations to each other, with a low frequency in the inner-layers of understanding.
- There was a marked increase in students setting up hypothetical situation and connecting contexts, with almost no instances of these behaviors in the inner layers.

These findings suggest that as understanding grows, there is a general shift in behaviors such as students questioning each other, toward those involving the setting up of hypothetical situations, linking of representations and connecting of contexts. The present study allows us to investigate whether or not these same associations occur in the context of two inner-city classroom settings. As part of the current study, we also revise and refine our associations to better account for the observed behaviors.

Methodology

Background

The research reported on in this study takes place in the largest urban school district in the state of New Jersey.

Data

As with the pilot study, the data that forms the basis for this study consists of at least six months of videotaped sessions (regular classroom sessions in the present study). In each case, at least two video cameras captured different views of the students' group work, students' presentations, etc. All student work was collected and descriptive field notes were compiled.

Analysis

In this study, the investigation of a task, over several classroom sessions, is analyzed using observations, field notes and videotapes. In both classrooms, we analyze the students investigating a combinatorics problem involving a mathematical structure of the form: $n(n - 1)/2$. The problem essentially is to find the fewest number of calls that needs to be made when 15 people call each other only once. It is also important to note that the students in each classroom explored a task with the same underlying structure (the "handshake problem", NCTM, 1989), at least 4 months prior to these sessions.

Selected episodes were identified based upon student(s) understanding in the outer-layers of the Pirie-Kieren model. We traced these moments back in time in an effort to document specific behaviors or actions on the part of students (see bullet points above) that appeared to contribute to movement through the layers of the Pirie-Kieren model. Detailed summaries and transcript were compiled.

Results and Discussion

Our analysis, which is thus far consistent with the pilot study, suggests that a number of student behaviors appear to be associated with the growth of mathematical ideas in specific ways. For example, in one particular case, which was typical of many others, an eighth grade student began her problem solving experience (for a task similar in structure to one that was used in the pilot) in the image-making layer (doing something to create an image). There were many instances in which she explained, used, and questioned her own and others' ideas. Others also questioned her. During the early stages of understanding, it is to be expected that students would in fact, question each other, and build representations that others don't understand at the time, thereby resulting in repeated questioning and requests for explanations and clarifications.

As this student moved from the image making layer to the image having layer (reaching a "don't need boundary" where she is no longer tied to the action), we began to see more instances of her moving to new representations and reorganizing and building upon other's representations. We suggest that one explanation for this is that she wanted to refine her representation in order to make it more useable to her, and in order to address the challenges of her peers. This is consistent with activity in this stage of understanding, which is still in the process of formation until the learner has an 'image.'

In the process of moving from image having to property noticing (connecting her images to each other), we began to see more instances of her linking representations to each other and setting up hypothetical situations. These behaviors began to increase as she moved to, and began working in, the formalizing layer (creating a "for all" statement). As an example, she began to set up many hypothetical situations until she began to realize how the ideas she had been considering would correspond to any case, for any number. We suggest that when working in the formalizing layer, a student seeks to "tie up loose ends" as she links representations to each other in an effort to justify her formal statement.

In our findings, there wasn't a decrease in students re-explaining, questioning and/or using their own or others' ideas, as students moved to the outer-layers or later stages of understanding (which was the case with the pilot study). In both of the classrooms, we found that the teachers' actions may explain this inconsistency. For example, each teacher spent a good deal of time highlighting selected students' ideas and encouraging them to present their ideas to the class after they moved to the outer-layers of understanding. They also encouraged the students in the "audience" to ask questions. In one of these classrooms, in particular, the justification questions asked by students in the audience appeared to heavily influence the students' move to another outer-layer of understanding.

Conclusion

It is our hope that by documenting the relationship between these behaviors and the development of understanding, at least for this type of mathematical problem, we can shed more light on the problem solving process in general, and when one might anticipate the occurrence of specific behaviors as students are working on mathematical problems.

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THE ESSENCE OF MATHEMATICAL PROBLEM SOLVING: A DELPHI STUDY

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Under investigation are the components that are emblematic of mathematical problem solving. Participants, N=22, 13 men and 9 women, completed three rounds of a Delphi Survey which was created to investigate experts' opinions on mathematical problem solving (MPS). Experts agree that MPS is an identifiable process and has identifiable descriptive characteristics.

Mathematical Problem Solving in Literature

When mathematical problem solving (MPS) is defined, it is described as a verb or noun. As a verb, mathematics educators describe what problem solvers are doing as a process. As a noun, mathematics educators describe characteristics of curricula inherent in MPS tasks. With myriad definitions, a Delphi Study was conducted to solicit opinions of experts throughout the world to reach consensus on what constitutes MPS.

As a process, students must be engaged in solving a novel task or one for which the solution is not known prior to starting it (Carpenter & Moser, 1983; NCTM, 2000; & Wheatley, 1999). The novelty of the situation may suggest that implementing a simple algorithm will not be sufficient to successfully solve the problem. Some mathematics educators suggest that for MPS to occur, iterative cycles must take place in an attempt to create a model (Doerr & English, 2003; Lesh, et al, 2000). This process means that problem solvers continually work to create a mathematical model to explain the situation and it has been referred to as expressing, testing, and revising (Lesh & Zawojewski, in press) which may occur to approximate an adequate mathematical model. Other mathematics educators suggest that metacognition must occur for problem solvers to truly be engaged in solving a problem (Despete, Roeyers, Buysse 2001; Metallidou & Efklides, 2000; Teong, 2003). Another process in which problem solvers may be engaged when solving a problem is representation (Moreau & Coquin-Viennot, 2003; NCTM, 2000; Reusser, 1989; Staub & Reusser, 1995). In MPS, representation means that an external method may be adopted to represent something. As an example, first-grade students may use pictographs to pictorially represent the number of boys and girls in their classroom. In so doing, students gain an alternate perspective of what the various numbers represent.

MPS has also been described as a list of characteristics such as realistic to problem solvers' lives (Lesh, et al., 2000). Creating equitable kick ball teams for recess is realistic to a class of third graders, but creating a budget for a trip from Missouri to Oregon on a Conestoga wagon in the mid 19th century is not realistic. A grouped list of characteristics often mentioned regarding MPS is interesting, challenging, and/or problematic to the students (Hiebert, et al., 1997; NCTM, 2000; Resnick & Ford, 1981). These descriptors are closely tied to realistic, because students may be unlikely to be interested, challenged, or to find the task problematic if the task is not realistic. A final characteristic of MPS worth mentioning is that more than one successful solution may be created by the problem solvers (Hiebert et al., 1997; Hiebert & Wearne, 1996; NCTM, 2000). As an example, one group of problem solvers might use the arithmetic mean to calculate an average while another group may find that the standard deviation is more informative to solving the same problem.

Methods

A website was designed by a mathematics educator with expertise in instrument design. The instrument was comprised of two prompts, "What is your definition of MPS?, and "What is your definition of a mathematical exercise?" MPS is the focus of this paper. Following responses to the two open-ended prompts, the qualitative data was analyzed and made into Likert items which had a scale of 1-4 (1 indicates never and 4 indicates always). Subsequently, the second and third round of data collection took place to reach consensus among participants. Nearly all participants, 17 of the 22, were full professors and had more than 50 publications. Moreover, several participants were current or former presidents of national or international mathematics education groups.

A Delphi study has two pieces of data which are instrumental to interpreting the results. The first piece of data is a grouped median and it illustrates the rating of the topic (e.g. 1-4). The second piece of data is an interquartile deviation, IQD, and it shows whether or not agreement was reached by the experts. In a Delphi study, agreement is statistically achieved when the IQD is less than one-tenth of the total Likert Scale. In this study there were four ratings from which to choose so the IQD must be < 0.4 for agreement. For the sake of these results, the second piece of data has been omitted because only those data points that have less than a 0.4 IQD have been reported. The IQD is calculated by the formula (Q_3 - Q_1) $\div$ 2 where Q_3 represents the third quartile and Q_1 represents the first quartile.

Results

Grouped medians of Likert ratings reveal the rating of the topic among experts. Participants agreed to the following topics:

A. For problem solvers to successfully complete a problem solving task, they must
- engage in cognition (4.0)
- seek a solution to a mathematical situation for which they have no immediately accessible/obvious process or method (3.78)
- seek a goal (3.75)
- mathematize a situation to solve it (3.24)
- define a mathematical situation or goal (3.13)
- create assumptions and consider those assumptions in relation to the final solution (3.06)
- revise current knowledge to solve a problem (3.0)
- engage in iterative cycles (2.94)
- create a written record of their thinking (2.88)
- create mathematical models (2.83)
- create new techniques to solve a problem (2.78)
- communicate ideas to peers (2.73)

B. Problem solving activities
- DO NOT lend themselves to automatic responses (3.94)
- can be solved with more than one approach (3.18)
- promote flexibility in thinking (3.18)
- can be used to assess level of understanding (3.06)
- can be solved with more than one approach (3.0)

- require the implementation of multiple algorithms for successful solution (2.94)
- can be purely contrived mathematical problems, puzzles, or games of logic (2.94)
- can be solved with more than one tool (2.89)

Implications are that teachers and curriculum coordinators could use this data to identify the types of tasks they are implementing in their courses. Moreover, the data compiled, may be used to create a rubric to analyze tasks currently used by teachers.

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TRACING AND EXTENDING HIGH SCHOOL STUDENTS' PROBLEM SOLVING APPROACHES BASED ON THE USE OF DYNAMIC SOFTWARE

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This study documents problem solving approaches that high school students develop as a results of using systematically Cabri-Geometry software. Results show that the use of the software becomes an important tool for students to construct dynamic representation of the problems that were used to identify and examine different types of mathematical relations. In particular, there is evidence that the use of the software allowed students to engage in a line of mathematical thinking in which they constantly reflected on the plausibility of conjectures and ways to support them.

Research programs in mathematics education, curriculum proposals or instructional practices all involves the design or construction of tasks or problems that are used to explore or promote students' development of mathematical thinking (Schoenfeld in press, NCTM, 2000). The identification and evaluation of mathematical qualities that make those tasks worthwhile to use often demand that researchers or teachers to examine hypothetical approaches or solutions' paths that students may consider while working on the tasks. In addition, when students work on those tasks there always appear new ideas that often become a source of new approaches or extensions to those tasks. In particular, the use of computational tools seems to offer students the possibility of formulating questions to examine the tasks from diverse perspectives or angles. For example, the use of dynamic software often allows the students to represent the task dynamically and use that representation to identify relations or conjectures that can be explored numerically or geometrically. To what extent does the use of the software help students identify and visualize mathematical relations related to the problem and ways to support them? What types of problem solving strategies and mathematical arguments are enhanced when students use the software in their problem solving approaches? To what extent do initial or potential learning trajectories, identified by the research team, get modified, adjusted or enhanced during the students' actual process of working on the tasks using dynamic software? These are relevant questions used to frame the process of development and implementation of series of tasks in which students had the opportunity to use systematically dynamic software. Thus, we are interested in documenting and analyzing students' types of representations, questions they pose and pursue, and arguments they use to support conjectures during their approaches to the tasks.

Theoretical Framework

This study is part of an ongoing research program that aims at documenting and analyzing high school teachers and students' use of computational tools in mathematical problem solving. Thus, a theoretical construct used to design and select series of tasks was an adjusted version of what Simon and Tzur (2004) call hypothetical learning trajectory. Here, we distinguish aspects that helped us organize the study: The students' learning goals, the type of tasks used to promote students' development of those goals; the hypothetical process that students' may be involved during the learning process; and ways for teachers and students to reflect on the development of a new problem solving tool or knowledge. This view is consistent with what Clements and Sarama (2004) characterize as learning trajectories:

> We conceptualize learning trajectories as descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain. (p. 83)

The use of dynamic software during the construction and discussion of hypothetical students learning trajectories and students' actual use of the tool became an important component of the study. Here, we recognize the importance for teacher and students to transform an artefact, in this case the Cabri-Geometry software, into a problem-solving tool as a process that involves thinking of the problems or tasks in terms of the facilities and constrains provided by the tool. Guin and Trouche (2002) recognize that this process is complex and involves aspects related both to the actual design features of the tool and also to the cognitive process involved in students' appropriation of the instrument to solve problems (the development of instrumentation schema). That is, it is crucial to document the way students develop and use a style of work in the context of a new tool (Cuoco, 2002). In this context, it became important to pay attention to the design limitations associated with the use of the software that might have interfered with students' work.

Research Design, Methods and General Procedures

The study involves two related phases: The design and selection of series of tasks in which a research team (a math educator, two graduate students and two high school teachers) worked on the identification of hypothetical learning trajectories associated with each of the selected tasks; and the actual implementation of the tasks in which 12 high school students participated in a 3 h weekly problem-solving sessions during one semester. The dynamic of the sessions includes students working on the task individually, in pairs, presentations of their work to all participants, and plenary discussions orchestrated by the teacher. All the pair students' interactions were recorded, students' presentations and plenary discussions were video-taped, students' handed in their initial individual work and pair work (including their electronic files). Data analyzed in this report come from the pairs' presentations to the session and plenary discussions. Here, we focus on analyzing the students' approaches to a task that included two problem-solving sessions. To sketch the hypothetical learning trajectory of this task, the research team chose to present this task as a sequence of squares (without particular dimensions) to explore initially the students' ways to support their responses. Later, they were asked to approach related cases to develop a method or tool (including an argument to support it) that could be used in other figures (rectangles, parallelograms, and hexagons). Relevant contents that appear in this task includes area concept, triangles congruence, triangle' area formula; mathematical arguments (proofs); and the use of proper notation.

The initial problem: Two farmers, John and Paul, will seed a piece of land that has a square shape. The plan is to divide the land in such a way that each farmer will get the same area. Figure 1 and 2 represent the two initial ways that they were considering to divide the land. A neighbour suggested that they could take any point on any side of the land and draw a line that passes by the centre of the land (square), then the line divide the land in two regions that have the same area (Figure 3). Does the neighbour' procedure always work? Justify your response. How is the neighbour's method related to the farmers' procedure? Justify your response.

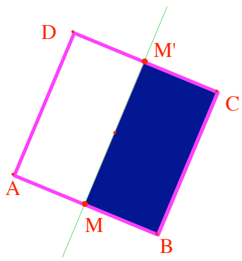

Figure 1: M and M' are midpoints of side AB and CD respectively

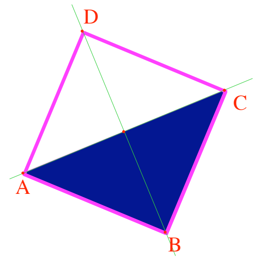

Figure 2: AC is a diagonal of square ABCD

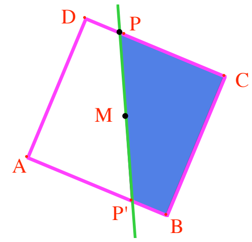

Figure 3: Point P is any point on one side of ABCD, point M is the centre of the square

Presentation of Results and Discussion

Five pairs of students used their rulers to measure the side of the square and argued that in figure 1 and 2 the square was divided into two "equal" rectangles and two "equal triangles. Only one pair did not use a ruler and mentioned that in both cases the square was divided in two congruent figures. It was interesting to observe that in the third case (Figure 3) all students initially used the software to check numerically whether the line PP' divides the square in two regions of equal areas. When they moved point P along the sides of the square they noticed that the other cases (figure 1 and 2) appear when P becomes the midpoint and a vertex of the square.

So, at this point all students were convinced that the line passing by the centre of the square divides it in two regions with same areas. They used visual, rotational arguments (rotating quadrilateral P'BCP around point M, 180 degrees), and congruence arguments to support their claims. For example, a pair of students, which happened to be the same students who did not use the ruler initially, showed that triangles PMC and P'MA are congruent (SAS) which implies that quadrilaterals PDAM and P'BCM have the same area (using here that the diagonal AC divides the square in two congruent triangles) (Figure 4).

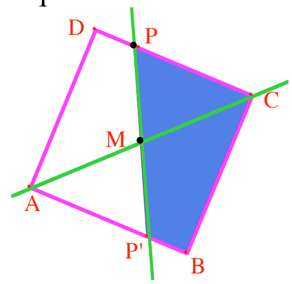

Figure 4: Searching for arguments to support their conjectures.

The students' presentation (pairs) to the whole session helped them discuss the importance of presenting various types of arguments and their relations. At this point, the teacher suggested another way to divide the land: Select any point P inside the square and join point P with the square vertices to draw triangles APB, BPC, CPD, and DPA. Based on this construction, a farmer will get the shaded part and the other the white part (Figure 5). Does each farmer get the same area? Justify your response.

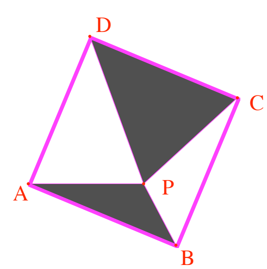

Figure 5: Point P is any point located inside the square.

All students used the software to check whether the teachers' method divided the square in regions of equal areas. By moving point P inside the square and calculating the shaded regions and the square areas (directly using the software), they observed that the shaded area was half of the square area.

Here, students used diverse arguments to support the conjecture. One involves drawing perpendicular lines from point P to the sides of the square and noticing that the square was divided in four rectangles, HPEC, EPFD, FPGA and HPGB (Figure 6). They argued that each of the rectangles includes two triangles (shaded and white) that have the same areas. Other approach focused on calculating the areas of the triangles and observing that the sum of the areas of the shaded triangles is half of the area of the square (Figure 7).

$$A(\Delta CPD) + A(APB) = \frac{d(DC)\times d(EP)}{2} + \frac{d(AB)\times d(GP)}{2} =$$

$$= \frac{d(AB)\times (d(EP+PG)}{2} = \frac{d(AB)\times d(BC)}{2}$$

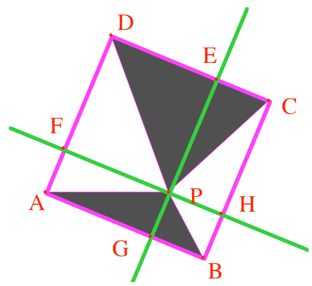

Figure 6: Comparing rectangles areas

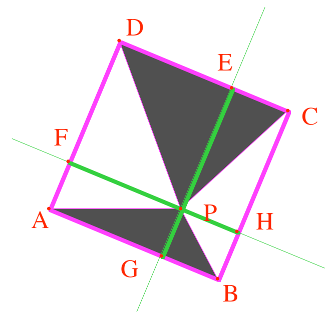

Figure 7: Calculating the sum of the shaded areas

During the presentation of these approaches to the session, a student posed a question: Is there another method to divide the square different from the teacher's procedure? They were asked to explore other options and Hugo using the software proposed to draw parallel lines to DC and AD passing by P. These lines intersect the sides of the square at point Q, R, S and T. Then the area of the quadrilateral QRST is the same as the area of the white region (Figure 8).

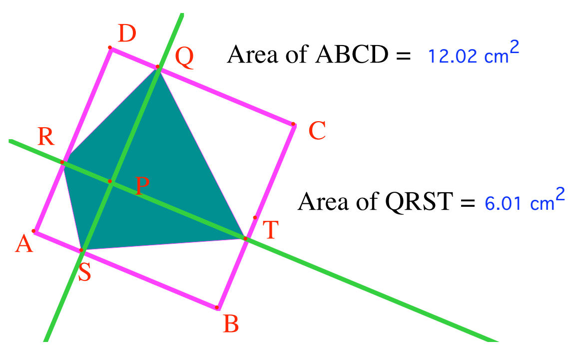

Figure 8: Area of QRST is the same as the area of the white region.

Students recognized immediately that Hugo's method also worked since they had used similar construction previously in figure 6 and used the same argument to show that the shaded area was the same as the white region (Figure 8). It was evident that the use of the software became a problem-solving tool for students at this stage. Here, the teacher challenged the students to examine whether the methods used to divide the land could also be used for other shapes of the land. For example, can you show that the procedure functions to divide rectangles, parallelograms, pentagons, etc?

Here, students realized that when the land's shape is a parallelogram, which is a more general case than a rectangle, the methods used to divide the square also worked (Figure 9 and 10).

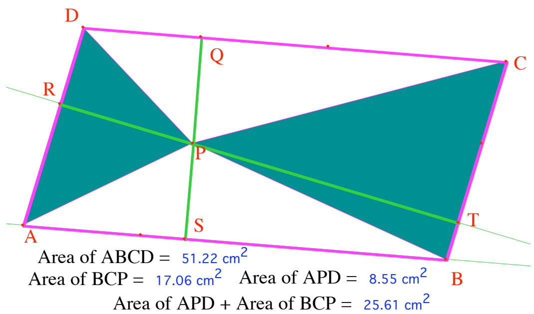

Figure 9: Area of APD + area of BCP = half of area of ABCD

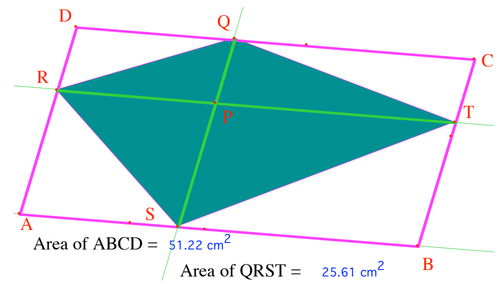

Figure 10: Area of QRST is half of area of ABCD

In addition to checking (using the software) that for different positions of point P both methods to divide the land in equal areas work, students also presented a formal argument to support their responses. In general terms, they observed that in figure 9, area of ABCD can be expressed as $d(AB) \times d(QS) = d(AD) \times d(RT)$ while the sum of the areas of triangles ADP and BPC can be written as:

$$\frac{d(AD) \times d(RP)}{2} + \frac{d(BC) \times d(PT)}{2} = \frac{d(AD)(d(RP) + d(PT))}{2} = \frac{d(AD) \times d(RT)}{2}$$, and this expression is half of the area of ABCD. Similarly, students argued that, in figure 10, lines RT and QS which are parallel to sides AB and BC divide the quadrilateral ABCE into four parallelograms (RPQD, PFCQ, ASPR, and SBTP). The diagonals of each parallelogram divide each parallelogram in two congruent triangles. Therefore, the shaded area is the same as the area of the white region.

Other cases that students explored includes a hexagon. Here, they drew using the command "regular Polygon" one hexagon and a point P inside of it. Then they drew perpendicular segments from P to sides CB, AB, FE, and DE. Here they found that for different position of point P, area of GHIJ was half of the area of the hexagon (Figure 11).

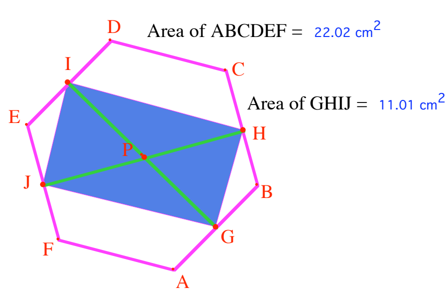

Figure 11: Does the method work for a hexagon?

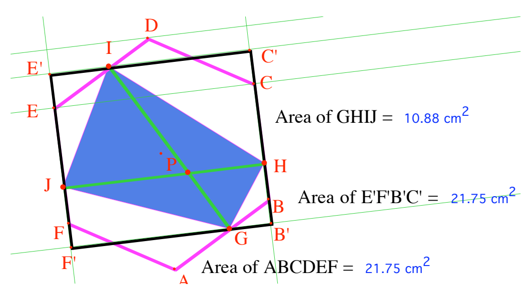

Figure 12: Looking for an argument to support the conjecture.

While looking for argument to show that area of ABCDEF is half of area of GHIJ, some students found that the area of rectangle E'F'B'C'(constructed by taking point E', as the intersection of the perpendicular line to line EF passing by point I; point C' as the intersection of that perpendicular with line BC, B' as the intersection point of line BC and the perpendicular to BC passing by point G; and F' as intersection of that perpendicular with line EF) is the same as the original hexagon (figure 12). Here, the students' goal was to check the conjecture: the area of rectangle E'F'B'C' is the same as the area of the hexagon. In addition, with the use of the software, some students found that the intersection points (U, V and T) of lines ED and BC; BC and AF and ED respectively are the vertices of equilateral triangle UVT (Figure 13).

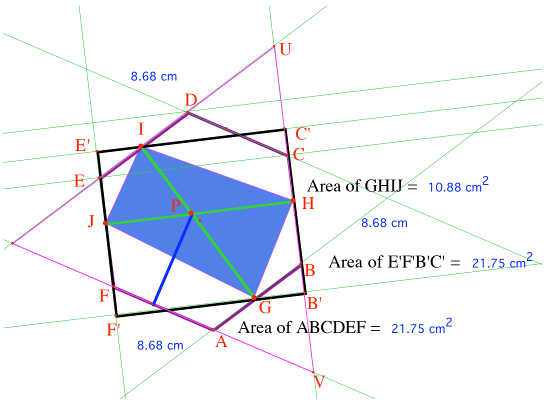

Figure 13: Looking for other relations around the hexagon ABCDEF.

Here, students also showed that in this case a line passing by the centre of the hexagon and a point on a side of it divides the hexagon in two regions of equal areas. They also were asked to justify the existence of the equilateral triangle and the possible relation between the height of that triangle and the sum of the perpendicular segments drawn from point P to each side of the triangle. Again the use of the software became important for them to identify relations to support findings.

A Variation Connection

The use of the software led the students to explore relations (invariants) dynamically, for example, while they were moving the point P (figure 6) inside of the square, they observed that the area of quadrilateral QRST was constant; however its perimeter changed at different position of P. Here they posed a question: **Where should point P be located to identify the inscribed quadrilateral with the minimum perimeter?** They also used the software to respond to this question. Figure 14 shows that when point P is situated at the centre of the square, then the quadrilateral QRST becomes a square with the minimum perimeter.

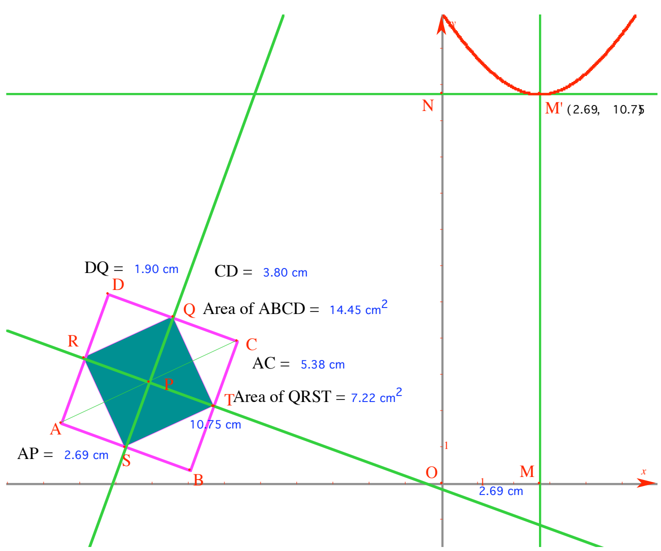

Figure 14: When P is located at the centre of the square, the quadrilateral QRST has the minimum perimeter.

Remarks

The construct “hypothetical learning trajectory” was used as a framework by the research team during the design of series of mathematical tasks to promote students’ mathematical thinking. Discussing the potential associated with each task becomes relevant to identify concepts, resources and problem solving strategies that students could use during their interaction with the tasks. It was noticeable that the students’ use of dynamic software evolves from considering static representation initially to dynamic representation of the task. This process became important for students to explore and pursue questions that were not initially considered by the research team. For instance, students examined the behaviour of elements of those dynamic representations to identify invariants or other relations that resulted from moving objects within the representation. Thus, the students’ use of the software requires that the teacher to be flexible and reflective to direct what students represent and explore. In addition, they searched for arguments to support diverse conjectures that they had checked numerically with the use of the software. As a consequence, students recognized and value the importance of presenting geometric or formal arguments to “prove” what they had visualized and checked numerically. Thus, the use of the software becomes important for students to explore initially numerically the behaviour of a particular relation and later to look for proper information to justify or support that behaviour or result. For example, in Figure 6 they observed that when point P was moved inside the square the area of the shaded region was always half of the square; then they focused their attention on finding an argument to support this conjecture. Using the software allowed them to explore whether the methods used to divide the square could also used to divide other related figures. That is, there is evidence that the use of the software provided them ways to initially formulate a conjecture and later to search for an argument to support it. In this process, new mathematical relations appeared as a result of adding and moving mathematical objects within the configuration. At this stage, students were aware that any mathematical relation that they had found needed to be justified in terms of providing a mathematical argument. Thus, students got engaged in activities that guided them to the search of those arguments. In short, the use of the software offers students ways to represent mathematical tasks or objects dynamically and while moving particular objects within that representation, students could visualize and formulate relations or conjectures that eventually need to be justified.

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YOUNG LATINO STUDENTS' LEARNING IN PROBLEM-BASED REFORM MATHEMATICS CLASSROOMS: DEVELOPING MATHEMATICAL THINKING AND COMMUNICATION (1)

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This study explores how Latino first grade students develop mathematical problem solving and communication in their native language. Problem types came from Cognitively Guided Instruction (Carpenter et al. 1999) and were embedded in students' cultural and linguistic experiences. Findings show students solved a wide range of CGI problems and developed confidence, flexibility, and sophistication in their strategies and explanations.

Problem solving and communication are at the center of reform mathematics education (National Council of Teachers of Mathematics, 2000). Word problems embedded in culturally and linguistically familiar situations allow young children to use what they know about the world to make sense of mathematics and learn with understanding (John-Steiner & Mahn, 1996; Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Hiebert & Carpenter, 1992). Students from diverse backgrounds, however, may not have equal access to reform mathematics curricula (Lubienski, 2000; Carey, Fennema, Carpenter, & Franke 1994). Latino immigrant children in U.S. schools deal with issues of culture, class, and language that affect their opportunities for full participation in problem solving activities (Trueba, 1999). This study explores what first grade, Spanish-speaking, Latino immigrant students can accomplish when they have repeated opportunities to solve problems and communicate their thinking in a native-language learning environment. Specifically, we show how children talk, draw, and write about word problems based in familiar contexts. Problem types used for the study were drawn from Cognitively Guided Instruction [CGI] (Carpenter, Fennema, Franke, Levi, & Empson, 1999).

Research Design and Methods

Researchers and teachers collaborated to design and implement problem solving lessons in two first-grade bilingual classrooms. For a period of one year we engaged in co-teaching mathematics lessons in Spanish with the purpose of developing students' problem solving strategies and their abilities to communicate their mathematical reasoning. In these lessons we worked with the teachers to contextualize word problems for young students (Carpenter et al., 1999).

Two bilingual first grade classes from an elementary school in a city in the Southwestern United States participated in the study. The school population is identified as Hispanic 86.3%, Native American 6.4%, Anglo 4.3%, African American 1.5%, Asian 0.8%, and other 0.9%. The majority of the students in this school speak Spanish, come from low socioeconomic backgrounds, are Mexican immigrants, and 99.1% of the students receive free or reduced price meals.

Data were collected on a purporsefully selected group of eight focal students, two boys and six girls. The criteria to select the students were: 1) they had participated in a CGI problem solving study the previous year (Turner, Celedón-Pattichis, Marshall & Tennison, in press), and 2) their first grade teachers agreed to receive classroom-based professional development on the CGI approach to teaching mathematics.

Data collection focused on how first grade students learn to solve CGI word problems and communicate their mathematical thinking. This process involved pre and post student assessments in the form of individual video taped clinical interviews where students were asked to solve CGI word problems using manipulatives or paper and pencil and to explain their solutions. The problem types were similar to those used by Carpenter et al. (1993) and included Join, Separate, Compare, Multiply, Divide, and Multi-step problems. Focal students were also videotaped three times solving CGI problems in pairs and in groups of four to observe the effects of peer interactions on students' problem solving strategies and communication. Weekly field notes recorded focal students' words and actions during mathematics lessons and their drawings were collected from these sessions.

Data analysis involved open coding and axial coding (Creswell, 1998) of students' strategies, verbalizations and drawings. Codes were consolidated into categories and themes.

To ensure that teachers had support in problem-solving lesson development, researchers and teachers debriefed after each weekly lesson to discuss students' reactions to the day's problems, strategies students used to find solutions, the obstacles students encountered, how students communicated their thinking, and how student understanding could be assessed. Based on these conversations, the lesson for the next week was planned. These debriefing sessions were audio taped.

Preliminary Findings

The data from this study demonstrate that young Spanish-speaking immigrant students can successfully engage in key processes of reform mathematics when they have the opportunity to make sense of word problems by building on what they already know culturally and linguistically about the world. We found that:

- Students developed confidence and flexibility in problem solving and successfully solved a range of problem types not usually introduced in first grade including multiplication, division, compare and multi-step problems.
- Students developed increasingly sophisticated ways of talking about their mathematical understanding in their native language, Spanish, and for some, in their second language, English.
- Students' conceptual development went hand-in-hand with more flexible thinking and their ability to use more than one method to find a problem solution.
- Students demonstrated growing networks of mathematical understanding, but this growth was nonlinear with many concepts still partially or superficially developed and connections among concepts weak.
- Students' pictorial representations of problem solutions gave a valuable window into student thinking and showed both insights and gaps in student understanding.

Implications and Recommendations for Future Research

When students had repeated opportunities to solve problems and explain their thinking, they developed a flexible set of strategies and approaches that showed they were learning mathematics with understanding (Hiebert & Carpenter, 1992). This level of success would not have been possible without access to their native language and culture to make sense of the mathematics involved (Trueba, 1999). Although students were highly successful, the findings also show that all students, even the most advanced, need multiple exposures to concepts and repeated opportunities to practice with a variety of problem situations so that their growing knowledge base can become more secure. Along with multiple problem solving opportunities, teachers should give students multiple opportunities to demonstrate their thinking in order to truly uncover the depth of student understanding.

This study confirms the importance of exploring students' mathematical thinking and problem solving development. Specifically, the longitudinal data show the central role of non-linear concept formation among primary grade students and the importance of developing students' mathematical discourse to help them organize and consolidate their thinking (NCTM, 2000). Because representation gives valuable insight into student thinking, further research is needed to provide more evidence on how students represent increasingly complex problem types.

Endnote

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INVESTIGATING STUDENTS' MOVEMENT TO GENERALIZATION AND ABSTRACTION USING DIENES' "LEARNING CYCLE"

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Introduction & Theoretical Framework

Generalization can be described as: "…deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly identifying and exposing commonality across cases, or lifting the reasoning or communication to a level where the focus is no longer on the cases or situation themselves but rather on the patterns, procedures, structures and the relationships across and among them" (Kaput, 1999, p. 136). Students' abilities to construct generalizations are often linked to their engagement in the process of abstraction (Davydov, 1990), which provides them with a means to move beyond the context of a given problem and transfer their knowledge to new problem situations. For the purposes of this study, it was useful to consider Dienes' notion of the 'learning cycle' (Dienes, 1970) in order to examine the ways in which students engage in generalization and abstraction during mathematics learning. According to the six stages within the "learning cycle", students first engage in a pattern of preliminary, unstructured exploration with a given concept, extend their learning through consequent, more structured explorations, and later, test the applicability of the concept in new situations before it becomes a 'functional' idea to them. Using Dienes' ideas as a lens, we document and examine students' ability to move beyond the mathematical context of a combinatorics task toward generalizations, which in this case, involves exponential, quadratic, cubic and other functions of that nature, and their graphs.

Methods

For this poster session, we analyze video-data of ten ninety-minute sessions in an eighth grade inner-city mathematics classroom, while students worked on a problem-solving task.

Preliminary Findings

One finding demonstrates that after students had found local solutions to a specific problem, a student-posed hypothetical situation (one that extended the problem beyond the specifics of the original task), highlighted by the teacher, contributed to students' movement to generalizations. The students found patterns as a result of unstructured exploration. For example, as the students were exploring exponential functions on the graphing calculator, they noticed that as their bases increased, the graph of the function remained in the first quadrant. One particular student posed the question about how to move their functions into other quadrants, which appeared to contribute to the students' exploration of other exponential functions, and later, functions with a variable base and constant exponent. Consequently, students developed formal statements to describe the behavior of their functions. Moreover, as time elapsed, the students began to pose more of their own hypothetical situations, thereby prompting themselves to extend their ideas through additional self-structured explorations, and ultimately, were able to use the new mathematical concept in a functional way.

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MATHEMATICAL PLAY AND STUDENT AFFECTIVE EXPERIENCE

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When a student has experienced consistent pedagogies in school mathematics for a number of years, s/he develops particular beliefs, attitudes, and emotions for mathematics and mathematics classes. As mathematics teachers enact teaching practices different from students' prior experiences with school mathematics, for example, a less teacher directed and more student centered pedagogy, it becomes increasingly important to understand the transitions students must make sense of to successfully engage in these mathematics courses. When expectations for student engagement are differently defined, students' affective experiences likely come to play a crucial role in their efforts to make sense of the new experience. Further when middle grades students, who necessarily experience a transition in the mathematics content, are confronted with a shift in pedagogy, supporting optimal experience (Schiefele & Csikszentmihalyi, 1995) becomes an important concern for the teacher.

The study described in this poster presentation was conducted to address these concerns and to test the conjecture that allowing students the opportunity to engage in mathematical play (Holton, Ahmed, Williams, & Hill, 2001; Steffe & Wiegel, 1994) could promote productive affective experiences during such a transition. Technology tools providing a microworld for the student's activity were incorporated with the intention of supporting students' efforts to engage in mathematical problem solving with a playful orientation. As the teacher of the seventh grade mathematics course studied, I used achievement goal theory (Ames, 1992) in order to choose participants who engaged in activity with varying goal orientations. Data sources included student artifacts, structured and semi-structured student interviews, teacher/researcher field notes, and video recordings of student activity with technology tools during the sixth and seventh months of the academic year. Using students' emotional experiences as an entry point to the analysis of students' mathematical play and affective experiences provided a unique insight into their mathematical play, problem solving, and the transition in experiencing school mathematics students endeavored to make sense of. Opportunities to play and to use technology tools proved productive for many students' affective experiences on multiple occasions, but not for all students. Findings and implications suggest how students with differing goal orientations engaged in the course and offer thoughts for teachers who hope to employ a more student-centered pedagogy and/or incorporate technology tools into school mathematics.

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ORAL RETELLINGS: THEIR NATURE AND USE IN SOLVING WORD PROBLEMS AMONG THIRD GRADERS

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Rationale

Much of the difficulty children encounter with word problems may be attributed to mathematical language, which differs from the language they encounter in other contexts (e.g., Kliman & Richards, 1992). Both the vocabulary (Paramar, Cawley, & Frazita, 1996), and the semantic structure (LeBlanc & Weber-Russell, 1996) may influence problem difficulty.

For many years, oral retelling has been an accepted strategy for developing and assessing reading comprehension (e.g., Gambrell, Pfeiffer, & Wilson, 1985), yet only a few studies (e.g., Vershaffel, 1994) have applied oral retelling to word problems. Cutler and Monroe (2006) found oral retellings to enhance problem solving among sixth grade students. We wondered if oral retellings would also aid younger students in solving word problems.

Research Questions

To explore the effects of oral retelling on third graders' ability to solve word problems, we addressed the following questions:

1. After instruction with oral retelling, to what extent do third graders use this strategy in solving word problems when they are not required to do so?
2. What is the nature of the oral retellings that third graders use in solving word problems?

Method

The study employed pre-treatment interviews, a sequence of researcher-developed lessons, and post-treatment interviews of the same children. The 8 students interviewed were teacher-selected as representative of the mathematics abilities and backgrounds of students in the class.

1. During *pre-treatment interviews*, students were asked to solve several multiplication and division word problems and urged to explain their thinking.
2. *Fifteen lessons* of 30-40 minutes were conducted over a 4-month period. These lessons included (a) a brief review of why and how to use oral retellings, (b) guided practice in using oral retellings for a "problem of the day," (c) a challenge to solve the problem in at least two different ways, and (d) opportunities for 2 or 3 students to present their solutions to the class, followed by a class discussion of their strategies.
3. During *post-treatment interviews,* students were asked to (a) explain what strategies they generally used in solving word problems, (b) solve several multiplication and division word problems and explain their thinking, and (3) orally retell at least one of the problems before solving it.

Findings

Of the 8 third graders interviewed, 6 retold a specific problem completely and accurately and solved the problem correctly. Although most students interviewed stated that they used oral retelling in solving word problems, particularly difficult problems, they did not spontaneously

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use this strategy during the interviews. However, they frequently used what we refer to as *partial retellings*. Often they repeated the numbers given in one or both conditions, sometimes giving the nouns as well: “10 rows of flowers,” “five cookies in six bags.” Some restated the problem question; others framed their own questions to clarify a problem condition or to check their understanding. Their spontaneous use of language did not seem to be related to the formal instruction they had received in oral retelling, as neither the frequency nor the sophistication of these partial retellings increased from pre-treatment to post-treatment interviews.

Implications for Future Research

Additional research needs to be conducted to examine the point at which oral retellings, as described in the research literature, become initially helpful in solving word problems. In addition, the nature and role of partial retellings as a word problem solution strategy should be a rich field for investigation.

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STUDENTS' MATHEMATICAL PARTICIPATION IN COOPERATIVE GROUPS

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The National Council of Teachers of Mathematics (NCTM) (1989; 2000) has called for changes to education that incorporate cooperative group work. In implementing such ideas into the K-12 classroom, it is important to examine the types of interactions that students take part in within the aforementioned groups. This examination is important in identifying the types of participation that may help or hinder learning within cooperative environments. The purpose of this study is to refine an existing coding scheme (Barnes, 2003) in order to apply it to a follow-up study.

The study conducted for this poster session focused on ninth grade Algebra students' participation within a cooperative task at a Midwestern high school. The students completed a three session open-ended task which required them to work as a group to solve a problem concerning exponential functions.

The mode of inquiry utilized for this study was videotaped sessions of students working in aforesaid cooperative groups. Videotapes will be analyzed based on a coding scheme created by Barnes (2003) while examining calculus students in a high school classroom. Codes will be categorized in order to reveal emergent patterns of participation which arose from the cooperative task.

Results of the study will include the adapted coding system based on the specific environment in the given Algebra classroom. Codes will be applied to data from a pilot study in order to further refine them for a larger study.

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STUDYING A UNIVERSITY-SCHOOL PARTNERSHIP MODEL OF PROFESSIONAL DEVELOPMENT

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This presentation, jointly prepared by university educators and teachers, will describe outcomes from a model of school-based professional development developed by Rutgers faculty in mathematics education for university-school partnerships where teachers study the lessons they teach and the mathematical thinking of their students within the structure of off-campus graduate courses. The courses are located in middle schools in two New Jersey school districts: Monroe Township, once rural but now suburban, and Plainfield, an urban community. The shared context of the professional development at the two sites is their goal of deepening teachers' attention to the specific mathematical thinking and activity of their students and enabling them to better meet their students' needs while simultaneously engaging as learners themselves of the mathematics that they are expected to teach. This model has facilitated articulation across grade levels in that Monroe involves middle school and high school teachers focusing on lessons from the Integrated Mathematics Program (IMP), and Plainfield involves teachers from elementary school and middle school focusing on lessons from the Connected Mathematics Program (CMP2). For the Plainfield site, in particular, this presentation builds a "teacher as researcher" research initiative that has been supported by two NSF initiatives (REC-0309062 and ESI-0333753).

Dual Research Perspectives and Guiding Questions

1. For the university educators: What evidence, if any, is there in teachers' reflections and actions of a shift from surface characteristics about the activity toward a closer attention to students' thinking and subsequent implications for classroom instruction?
2. For teachers as researchers: (a) What particular mathematical strategies and representations do we observe among students as they engage in solving mathematical investigations that embody concepts that are basic to the district's curriculum? (b) What teacher decisions, questions, and interventions do we observe during implementations of the lessons and how do they affect the students' mathematical thinking?

Context and Methodology

Within the context of the off-campus graduate courses the teachers analyze particular strands of mathematics from the IMP or CMP2 curriculum, and are further informed by selected literature from research as well as state and national standards. Plainfield has adopted CMP2 as their middle-school curriculum, which permits close study of samples of their students' work for the group to identify specific concerns and develops "research lessons" specifically focusing on these needs. Monroe is in the process of transitioning from more traditional texts to reform curricula, thus their teachers are forging new ground in exploring use of the IMP materials. Common to both sites is that the lesson is implemented in one or more of the teachers' classrooms as a part of the course. A plan for the lesson is prepared cooperatively, including anticipated strategies and representations to come from the students and possibilities for teacher questioning that might enhance the activity without directing the students. Members of the class are released to observe the first implementation of the lesson as it is facilitated by a volunteer from the group with his or her students, videotaped for closer study and followed by a discussion of the students' mathematical activity, which is also videotaped when possible. Subsequent implementations are carried out by other members of the class, also videotaped and observed by the instructor and others of the group. Each participant documents any classes observed as well as developing a reflective analysis of the session that he/she facilitates. These data, along with artifacts from each session and samples of student work are collected as a class portfolio. The data provides opportunity for further study by both the university researchers and the teachers as researchers that build on earlier findings (Alston, Potari &Myrtil, 2005; Maher, 1998; Martino & Maher, 1999).

Findings To Be Shared

Documentation from the portfolios, samples of student work, and critical segments from the classroom implementations will be analyzed and organized to provide a narrative of the cycles of development and implementation of particular lessons. The intention is to provide evidence of the teachers' deepening attention to their students' thinking and to identify issues that are common to the two sites.

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TWO DIFFERENT TYPES OF PROBLEM-SOLVING METHODS IN EXPOSITORY TEACHIG IN HONG KONG EIGHTH-GRADE MATHEMATICS CLASSROOMS

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One Hong Kong classroom from the Learner's Perspective Study (Clarke, Keitel, & Shimizu, 2006) is analyzed to examine two different types of problem-solving methods across sixteen consecutive lessons during expository teaching. The study explores how the teacher portrays two different types of problem-solving methods to his students and how these different methods affect students' understanding of mathematical concepts.

The expository teaching style, typically utilized in whole class instruction, is still prevalent in Hong Kong. This type of instruction is described as ineffective for learning of mathematics conceptually. Hong Kong students are depicted as passive learners from a Western view (Watkins, & Biggs, 2001). However, Lopez-Real and Mok (2001) argue that mathematical ideas are explained in depth during the expository instruction in Hong Kong classrooms creating opportunities for students to learn conceptually.

In this study, one Hong Kong classroom selected from the Learner's Perspective Study (LPS) (Clarke, Keitel, & Shimizu, 2006) is analyzed for the purpose of examining how expository teaching portrays two different types of problem-solving methods in class:

(1) If the teacher emphasizes utilizing an efficient method (easiest way or the simplest way to solve) regardless of connecting concepts
(2) If the teacher focuses on using an efficient method (looking for the best method to produce a meaningful solution) related to mathematical concepts.

A comparative case study method is employed in the study to explore how the teacher portrays two different types of problem-solving methods to his students and how this influences students' understanding of mathematical concepts. Sixteen consecutive lessons transcripts are analyzed in terms of what counts as a mathematically noticeable solution in teacher's explanation; Does expository teaching focus on the efficient or the effective the problem-solving method? The lessons are examined using a coding scheme to identify instances of the efficient and effective methods during expository teaching in Hong Kong classroom with regard to mathematical concepts. Student post-interviews, simulated recall interviews, are investigated for students' ability to solve problems using the methods and students' comment upon classroom events during the interview. This analysis provides information about the impact of teacher's portrait of mathematically significant solutions on students' understanding of mathematical concepts.

The study pursues the results which will reveal expository teaching in Hong Kong classrooms in terms of what counts as a mathematically noticeable solution and its impact on students' learning of mathematics conceptually.

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FIDELITY OF IMPLEMENTATION: TEACHERS' INSTRUCTIONAL MOVES IN THE CONTEXT OF A STANDARDS-BASED CURRICULUM

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Recent calls for research on the impact of the Principles and Standards for School Mathematics (NCTM) indicate a need to better understand the relationship between teachers and curricula. This paper examines 2nd grade teachers' implementation of Standards-based whole number lessons in terms of an enactment's fidelity to two curricular forms: the literal and the intended curricula, and then in terms of teachers' instructional moves. In so doing, we explore interactions between teachers' instructional moves and students' opportunities to learn mathematics. We conclude by considering how curricula support or fail to support teachers' instructional moves and choices.

Standards-based mathematics curricula are viewed by many as engines for reform, for they integrate the new standards for teaching and learning into mathematics lessons. For this reason and as argued by the Mathematical Sciences Education Board (2004), it is important to measure the extent to which teachers adhere to the content and ideas within such curricula. Despite efforts to measure fidelity levels (e.g., Chval et al., 2006), we know little about the extent to which curricular modifications stray from the core ideas of teaching and learning espoused in the materials. Drawing on data from the *Math Trailblazers* Whole Number Study, this paper explores the relationship between an enactment's fidelity ratings, described in more detail below, and teachers' instructional moves. In so doing, we explore the ways in which teachers' *instructional moves*, i.e., the instructions, questions, and guidelines teachers put forth during lessons, are indicative of *instructional choices*, i.e., teachers' decisions about how tasks and activities should unfold during lessons (e.g., Lampert, 2001).

Background

Given the integral role of teachers in curriculum use, research on fidelity is shifting its focus from assessing the extent to which teachers implement curricula to understanding how teachers use curriculum materials to create learning experiences in classrooms (Snyder et al., 1992). Moreover, some researchers have begun to attend to differences in teachers' orientations toward curricula (e.g., Remillard, 2004), while others have explored factors that impact how teachers read and use curriculum materials (Floden, 1980; Manouchehri & Goodman, 1998) or how the inherent qualities of curriculum materials influence teachers' reading and understanding of the curriculum (Ben-Peretz, 1990). Furthermore, as noted by many (Clark & Elmore, 1981; Lampert, 2001), teaching is a complex activity during which teachers face decisions and challenges that influence the course of instruction, and ultimately, students' opportunities to learn. Moreover, to support students' engagement in tasks a variety of pedagogical skills are necessary. For example, resisting the persistent urge to simply tell and direct students how to work on the task can provide students with enough time to think

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through what they are asked to do (Donovan et al., 2000). Anticipating student responses and having an awareness of common errors can help teachers effectively respond to students' during discussions (Fennema et al., 1993). Employing such instructional practices, however, requires extensive and demanding work on the part of teachers.

Methodology

Interviews, videotaped classroom observations, surveys, and teachers' responses to classroom video excerpts constitute the data sources for the *Math Trailblazers* Whole Number Study. Our analysis of the classroom observations, the focus of this paper, followed a protocol for examining an enactment's fidelity to two curricular forms, the literal curriculum and the intended curriculum.[1] To analyze videotaped classroom observations, we transcribed the videotapes and then partitioned the transcripts into segments, with each segment beginning and ending with a shift in activity. Each segment was coded to indicate which opportunities to learn arose, arose in a limited manner or failed to arise during the enacment. To characterize students' opportunities to learn we developed nine subcodes, with five subcodes addressing students' opportunities to reason, and four addressing students' opportunities to communicate about mathematics. Coded observations were then given two summary ratings, with one rating indicating the extent to which the enacted lesson aligned with the lesson description in the written materials, i.e., a *fidelity to the literal curriculum* rating, and a second rating indicating the extent to which the intended opportunities to learn arose during the enactment, i.e., a *fidelity to the intended curriculum* rating.

Research Results

In this paper, we focus on the instructional moves observed in four Grade 2 teachers' enactment of two lessons, *Base-ten Subtraction* and *Zoo Lunches*. The lessons *Base-ten Subtraction* and *Zoo Lunches* were designed to support students' exploration of mathematical operations in particular contexts. Thus, these lessons align well with the curriculum's approach to whole number operations, described within the *Math Trailblazers Teacher Implementation Guide* (TIG) as consisting of two components. The first component deals with the use of representations "that highlight the attributes of the numbers and operations" (TIG, p. 205). The second deals with the use of context-based problems which students are asked to solve "before they learn formal, paper-and-pencil procedures" (TIG, p. 205).

In *Base-ten Subtraction* the students, having had prior experiences representing and adding numbers with base-ten pieces, are asked to use the base-ten pieces to solve a series of subtraction problems. Thus, the lesson is intended to create a context in which students can invent strategies for using a representation of number, namely base ten pieces, to explore the operation of subtraction. Questions such as, "what do I do when I run out of bits," are meant to arise naturally from the children's mathematical activities.[2] In *Zoo Lunches* the students are given data from Chicago's Lincoln Park Zoo and are asked to determine, given a specified shipment of food, how much food a given group of animals might receive. Thus, in *Zoo Lunches* students are given context-based division problems, prior to the introduction of formal division algorithms, with the intent of creating opportunities for students to draw on their knowledge of number and invent arithmetic strategies.

Fidelity Rating	Low Literal	Moderate Literal	High Literal
Low Intended			*Base-ten Subtraction*- Mrs. Smith *Zoo Lunches* – Mrs. Miller
Moderate Intended			
High Intended		*Zoo Lunches* – Mrs. Olivier	*Base-ten Subtraction*- Mrs. Patton

Table 1. Fidelity Table

The classroom observations we will discuss in this paper (see Table 1) were selected to illustrate the instructional moves observed in enactments rated as high or moderate fidelity to the literal lesson. In other words, we chose enactments where we observed no major alterations to the lesson steps described in the written materials. These enactments, however, showed a great deal of variation with respect to the level of fidelity to the intended curriculum, and therefore, the degree to which the intended opportunities to learn arose.

As indicated in Table 1, Mrs. Smith's[3] enactment of *Base-ten Subtraction* was rated as low fidelity to the intended to the curriculum. This rating indicates that many of the intended opportunities to learn were not observed during the lesson enactment. Analysis of the data indicate that Mrs. Smith made several moves (i.e., she put forth instructions, questions, and guidelines) during the beginning of the lesson that ultimately derailed the lesson, i.e., the teachers' moves created contexts for learning that differed from those intended. For example, rather than create an environment in which students might explore using base ten pieces in subtraction contexts, Mrs. Smith chose to lead the students through tasks with step-by-step instructions. These instructions began with Mrs. Smith stating which number to represent (the minuend), having students state the number of skinnies and bits needed, stating the subtrahend, and directing the students while they removed the appropriate number of bits and skinnies. Thus, the students' work with subtraction was highly structured by Mrs. Smith's prompts and questions, as illustrated below.

Teacher: I want you to show me the number fifty-six. Show me fifty-six. Fifty-six. Look at your neighbor; see if your neighbor's matches yours. Looking good. Looking good. And does yours match mine? Fifty-six.

Students: Yeah.

Teacher: Five skinnies, how many bits?

Students: Six.

Teacher: Six, okay.

Furthermore, when Mrs. Smith's students began work on the first subtraction task that required regrouping, rather than allow students to explore and discuss how to regroup in this context, Mrs. Smith chose to continue to structure the students' work. In this case, rather than ask for the number of bits to be removed, Mrs. Smith immediately pointed out that there was a "problem." Consequently, the question "what do I do if there are not enough bits?" did not arise from the children's activities as intended, but rather was posed directly as a teacher-initiated question. Moreover, Mrs. Smith funneled students' responses towards a teacher-selected answer, as illustrated below.

[Task: 84-25]

Teacher: If I have eighty-four, right now, can you reach down and take, well you tell me. How many ones do I have to take for this problem?

Students: Five.

Teacher: Five. If you have four ones on your mat right now, can you reach down and take five?
Students: No.
Student: No, it's impossible.
Teacher: It is impossible. You can't do it. So, what is a, what's our solution? How are we gonna fix this problem? What are we gonna do with this problem? What, what's, what could we do? […]What do you think they could do? Let's remember, what does one of these skinnies equal?
Students: Ten.
Teacher: It equals ten. What would happen if I took this skinny and I broke it up into bits? Does anybody get an idea of what I can do? Could you go to a pretend bank? […] If you took this and you took it to an imaginary bank, what would the person give you back for it? If you were just exchanging it?

This enactment lies in contrast to Mrs. Patton's, whose high fidelity rating to the intended curriculum indicates that many of the intended opportunities to learn arose during the lesson. Analysis of the data indicates that Mrs. Patton's instructional moves played a different role than Mrs. Smith's during the lesson. For instance, during whole class discussions Mrs. Patton repeatedly posed questions aimed at exploring students' mathematical thinking rather than at guiding students' toward a specific answer.

Teacher: Okay. Somebody knows another way to do it? A different way? No? This is it? Those are your favorites?
Eric: I did the Two Hundred chart, but different.
Teacher: How did you do it differently on your Two Hundred chart? Okay, tell me.
Eric: I, I started at, I, I did it like Avi but you go up, uh, from, instead of going down you go up.
Teacher: Oh. On Two Hundred chart, yes definitely, you need to move because...
Eric: No, I meant, like Avi, like Avi, but instead of moving from, from...
Teacher: Seventy-eight?
Eric: Seventy-eight to down. I went from thirty-six to up.
Teacher: From forty-two up?
Eric: Yeah.
Teacher: Okay. Show us how you did it? That's good. That's good too. Do you, do you understand what he's saying?
Students: Yeah.
Teacher: Okay. Lead us. Tell us how you have done it and we will do it, we'll follow you.

Furthermore, when Mrs. Patton's students began work on subtraction tasks requiring regrouping, the question of "what do we do if there aren't enough bits?" was a response to students noting, as opposed to the teacher, that they had encountered a problem. Moreover, the answer to this question was elicited from students and thus, a result of student thinking.
[Task: 52-14]

Teacher: Seth?
Seth: Take away one skinny.
Teacher: We can take away one skinny. Okay, I will cross it out. We will take it away. Now? (A few students raise their hands)
Eric: It's a tricky one, but I can do it.
Teacher: (To Eric) Yeah? You found it is tricky? (To Seth) Do you think it's tricky?
Seth: Mm-hmm.
Teacher: Why?

Seth: Because you can't take four (unintelligible).
Teacher: We don't have four bits, right? We need to take four bits and we don't have it. So what can we do? (Students are raising their hands.) What can we do Eric?
Eric: You take away two from the ten. You take away two from here (points to drawing of a skinny) and leave that.

As shown in the excerpts, the distinction between Mrs. Patton's and Mrs. Smith's enactments are, in some ways, subtle. For example, Mrs. Smith posed questions to students about the tasks and their thinking, but not nearly to the same extent or in the same manner as Mrs. Patton. While Mrs. Smith provided highly structured guidance for students' work and engaged in dialogues predominated by known-answer and funneling questions, Mrs. Patton engaged students in dialogues rich with exploratory questions aimed at eliciting students' ideas and approaches.

In the enactments of *Zoo Lunches*, there are similar differences in terms of the instructional moves teachers employed. Mrs. Miller's enactment of *Zoo Lunches* was rated as low to the intended curriculum, indicating that many of the intended opportunities to learn did not occur during the lesson. Analysis of the data indicates that Mrs. Miller made several moves during the beginning of the lesson that led to fundamental shifts in the tasks involved. For instance, Mrs. Miller initiated *Zoo Lunches*, a lesson aimed at supporting students' invention of arithmetic strategies, by repeatedly reminding the students to carry out a strategy she modeled for the class, as illustrated below.

Teacher: The problem is each day the zookeeper receives sixteen carrots, okay. Everyday she gets sixteen carrots. She wants to give each of the four apes the same number of carrots. How many carrots will each of the apes get? So you're going to take sixteen carrots and share them with four apes. Do you remember; how do we show our four apes? What did we do to show our groups? Mona?
Mona: We drew circles.
Teacher: Yeah. We said like here's one, here's two, three, four and then we started handing them out. We didn't just start saying one, two, three, four, five, six; we gave each ape one. […] Okay so start handing out your sixteen carrots. Can you start handing them out? Start handing them out. Can you start handing one out to everybody? We're going to hand them out until we get to sixteen. Okay, so one. [To student] Oops, does that guy get two before anybody else gets another one?

Mrs. Olivier's lesson enactment, in contrast, was rated as high fidelity to the intended indicating that many of the intended opportunities to learn arose during the lesson. Analysis of the data indicates that Mrs. Olivier made several moves during the lesson that fostered situations in which students could invent strategies to solve context-based division problems. For instance, Mrs. Olivier initiated the lesson by posing a mathematical situation for which students had no known solution strategy, as illustrated below.

Teacher: I'm going to read the problems and I want you to use your paper to, um, help you solve the problem. You can draw pictures. You can do other things to help you figure out the answer. Harry?
Harry: Can we do it with our partner?
Teacher: And, yes, I'd like you to work with your partners also. But, before you begin you have to look at the problem because it's real important. Okay. Each day the zookeeper receives sixteen carrots. She wants to give each of the four apes the same number of carrots. How many will each ape receive? Okay. Now remember there are sixteen carrots and there are four apes, and each ape is going to get the same number. So I would like you to work with your partner to try to solve that

problem. And I'm going to call someone up to explain it. Okay. Is there anyone who doesn't understand the problem? Okay. You can begin.

The distinctions between Mrs. Miller’s and Mrs. Olivier’s instructional moves at the beginning of the lesson are, like those of Mrs. Smith and Mrs. Patton, somewhat subtle. Both Mrs. Miller and Mrs. Olivier set up contexts where students could represent a problem situation and determine how much food each animal would receive. Mrs. Miller’s students, however, did this by employing a teacher specified-strategy, while Mrs. Olivier’s students were given an opportunity to develop strategies of their own. Furthermore, these differences where indicative of differences we observed throughout the enactments. Mrs. Miller’s instructional moves tended to structure students’ work in ways that ultimately reduced a task’s mathematical complexity. In contrast, Mrs. Olivier’s instructional moves tended to support students’ mathematical explorations and thus, their engagement in tasks more demanding that those seen in Mrs. Miller’s enactment.

Linking Teachers’ Instructional Moves and Choices

With the examples provided, we explored how teachers’ instructional moves, i.e., the instructions, questions, and guidelines put forth by the teacher, can impact the types of opportunities to learn that arise during an enactment. For instance, in the enactments we contrasted one can see significant differences in terms of students’ opportunities to select, compare and contrast strategies, as well as differences in students’ opportunities to describe and clarify either their own or others strategies. Teachers’ moves are also important to consider because they are indicative of teachers’ instructional choices, i.e., teachers’ decisions about how tasks and activities should unfold during lessons. For instance, consider the two enactments rated as low fidelity to the intended curriculum. Mrs. Miller’s decision to begin the lesson with specific instructions on how to distribute the animals’ food is indicative of an instructional choice: namely, the instructional choice of replacing open-ended tasks with practice tasks (e.g., tasks where students repeat a specified procedure). Similarly, Mrs. Smith’s moves, e.g., leading students to a particular “problem” and then guiding them to a pre-selected solution, are indicative of an instructional choice: namely, the instructional choice of introducing students to a regrouping strategy, as opposed to creating a context where students must develop strategies if they are to solve problems. Since these instructional choices were at odds with the curriculum’s approach to whole number operations it is not surprising that they hindered rather than supported the intended opportunities for students to learn mathematics.

Discussion

If one takes into consideration that all four teachers followed the literal curriculum, i.e., the written lesson steps, yet varied with respect to their level of fidelity to the intended curriculum, then the degree to which the curriculum materials inhibit or support teachers’ instructional choices comes into question. For instance, the Unit Resource Guide (URG) for *Zoo Lunches* states, “Present these two problems and have children show their solutions by making drawings and writing number sentences” (p. 44) The URG, however, makes no clear reference to the importance of having students invent strategies to solve problems, nor does the URG contain information for teachers on how to help students make sense of the potentially varied and, at times, erroneous strategies. Making reference to the importance of affording students’ opportunities to invent strategies may help teachers better understand why it is important that they do not alter this component of the lesson. Furthermore, teachers’ inclusion of strategies in their set up of tasks severely limits the need for discussion and consensus building. Being without information about the interplay between students' opportunities to select strategies and the development of whole class discussions that explore students' mathematics in meaningful ways, teachers may fall back on instructional practices that are at odds with the curriculum's approach to whole number operations, especially if appropriate practices conflict with legitimate concerns, e.g. "students might not tell in a way that is productive for the rest of the class or in a way that moves the class forward" (Lobato et al., 2005). Thus, one must ask, given the teachers' adherence to the literal curriculum, would the teachers have enacted the lessons differently if such information were included in the URG?[4] Such questions highlight the need for work exploring interactions between teachers and curricula that simultaneously consider the ways teachers use curricula (e.g., Remillard, 2004) and the affordances of the curricula, i.e., the ways curricula support teachers' instructional moves and choices.

Endnotes

1. For more information about the protocol see Brown, S., Ditto, C., Kelso, C. R., & Pitvorec, K. (2006) Teachers' Implementation of Standards-based Elementary Whole Number Lessons. S. Alatorre, J. L. Cortina, M. Sáiz, A. Méndez (Eds.) *Proceedings of the Twenty-eighth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.*

2. The term *bits* refers to small solid cubes used to represent ones. The term *skinny*, used later in the article, refers to larger solid pieces that resemble a stack of ten bits.

3. All names are pseudonyms.

4. We acknowledge that curriculum developers are limited in the amount of information they can provide if they are not to produce lessons that are weighty tomes.

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CHANGING IDEAS OF RATIONAL NUMBERS AS MEASURES: A CASE OF ELVIO

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This study is part of an ongoing NSF-funded investigation into middle school students' changing ideas of rational numbers over a three-year period. Elvio was chosen and analyzed over the two years for changing ideas of rational numbers as measures. Results showed a change in his thinking when approaching measurement problems, away from classic part-whole notions to those of measurement.

Past research has documented middle school students' difficulties interpreting rational numbers as measures, specifically when dealing with perimeter and length (Martin & Strutchens, 2000). Only 39% of the eighth graders could estimate length of one object correctly using another, and only 21% could predict the perimeter of a quadrilateral using the given unit of length (National Center for Educational Statistics, 2005).

In part, to address these difficulties, previous foci have been students' knowledge of rational numbers in developing curricula. They emphasized fractions, ratios, and proportions (Lesh, Post, & Behr, 1988). Despite these efforts, they did not analyze student growth longitudinally over several grade levels (Carraher, 1996).

This investigation is part of a larger NSF-funded research project that investigates students' conceptions of rational numbers over a two-year period. In particular, we describe one student's changing ideas of rational numbers as measurement over a two-year period.

Theoretical Framework

Lamon (1999) noted five interpretations of rational numbers as ratio, part-whole, quotient, operator, and measurement. The measurement interpretation of rational numbers includes identifying some length, defining a unit, and measuring the length with the unit via iteration (Lamon, 1999). For example the fraction 2/3 can be interpreted ,in terms of the measurement subconstruct of rational numbers, as a length of 2 (1/3 units). Defining the unit in some way is termed unitization. Unitization is the "cognitive assignment of a unit of measure to a given quantity", rational numbers that represents the measurement then change according to this new unit (Lamon, 1999). For example, when measuring a length of a table as 2 meters, 2 meters can be seen as 2 (1-meters), or be unitized differently as 4 (1/2 meters).

Students' ideas about measurement change over time with exposure to classroom practice and the mathematics in their daily lives (Lesh, Post, & Behr, 1988). While a specific, generalizable learning trajectory, or finite set of learning trajectories, is desirable (Carruthers, et al., 2007), it is outside the scope of our analysis.

Methods

This study took place in an elementary and middle school in an urban area in the southwestern United States. The school was predominately Hispanic, with over 90% of the students receiving free or reduced lunch. The students were taken from math classes and interviewed once every three weeks concerning a variety of rational number concepts. Their

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.

math classes occurred five times a week for 50 minutes a day, and interviews typically lasted forty to fifty minutes. Interviews were supplemented with classroom observations twice a week.

A cohort of 31 students were observed and interviewed over the two years, from sixth to eighth grade. Data collected means to document their notions of rational numbers and how those notions change over time. For this study, we selected one student, Elvio (pseudonym), based on the clarity with which he expressed his ideas, and analyzed his changing measurement thinking during interviews over the two-year period.

Results

Initial Stage

Elvio began by approaching measurement problems using part-whole strategies. Rather than iterating units to measure he focused his efforts on counting parts, relative to a whole. For example, Elvio, when asked to compare 1/2, 2/4, and 4/6 he drew the following fraction bars:

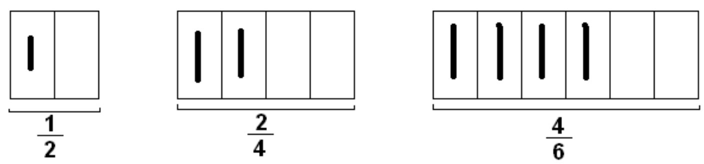

Figure 1. Elvio's part-whole fractions.

When comparing three fractions Elvio consistently used three different wholes. He focused on counting the number of parts relative to the wholes (each of a different size), rather than iterating the unit some number of times. This was indicative of part-whole, not measurement, thinking (Lamon, 1999).

Intermediate Stage

Elvio, over time, began to use a consistent unit, or whole, to compare fractions using a part-whole partitioning strategy. For instance, when he was asked to compare _ and 2/3 he drew two fraction bars of the same length, and partitioned them into two and three equal pieces, respectively (see Figure 2). He then counted the number of pieces appropriately and compared the size of each shaded portion. On another occasion, when he was asked to find how many lengths of 1/3 of a yard fit into 2 yards he drew the representation in Figure 3.

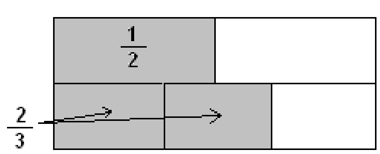

Figure 2. Elvio's comparing of fractions.

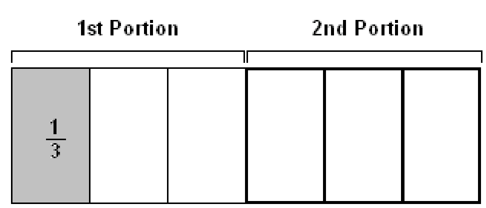

Figure 3. Elvio's finding of 1/3.

Initially, Elvio drew the first portion of the rectangle, divided it into three equal pieces, and shaded one piece and labeled it "1/3". He then drew the second portion and also divided it into three equal pieces. Then he counted the number pieces to arrive at his answer of six. Elvio used a part-whole partitioning strategy to find 1/3 as a part of the whole, one yard. Although he does not consider 1/3 as a length measure, he iterates the part. By iterating the part, he shows limited measurement understanding because 1/3, as stated, *is not* treated as a length measure but as a portion of the whole.

Current Stage (Last Stage Over Time Period)

Elvio was asked to locate _ on a number line as shown in Figure 4. He determined the size of his unit by comparing the distance from 7/2 to the next given mark on the number line, then iterated the unit of _ backwards from 7/2 to 6/2, 5/2, 4/2, 3/2, 2/2, and to _ to correctly find his answer. Elvio's focus was on the length of _ rather than it being one part of two in some whole.

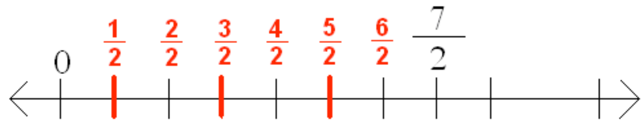

Figure 4. Elvio's strategy for locating _ (Elvio's work is in the red).

Discussion

Elvio's interviews over the two-year period showed a change in his thinking when approaching measurement problems, away from classic part-whole notions to those of measurement proper. Initially, he lacked concept that required a consistent unit. This was especially problematic when comparing fractions, even via part-whole strategies. He was able to compare counted parts of wholes of one fraction to another, but the part-whole ratios did not reference a consistent whole, so the relative size of the fractions was not comparable. Over time, his thinking included consistent wholes in part-whole comparisons, and later a consistent length as a unit of measure.

Iteration was also an initial difficulty for Elvio. He did count a subset of the total number of pieces in each whole, representing his part-whole fraction, but did not iterate a unit to measure a length. Specifically, the 2/3 in Figure 2 was not interpreted as 2 (1/3 units) but instead as two parts of three total parts in the whole. At the end of the two years, he acquired an understanding of iteration. He was not only able to iterate by _ units backwards from 7/2 (see Figure 4) but also remarked to the interviewer, "if I go one forward I will have 8/2."

Initially unitization occurred in Elvio's strategies, but not in a consistent, useful way. At the intermediate stage he engaged unitization, but only as a strategy that established consistent part-whole units, or wholes, which facilitated his part-whole comparisons. In the current stage, he determined his unit as a length. While this does constitute unitization, it may have created problems for him if his unitization had not conveniently measured the entire length without a remainder. The interview data did not provide information about unitization of different lengths as units so how Elvio would have responded to a remainder is unknown.

Elvio's thinking, as he more commonly applied measurement strategies such as unitization and iteration, demonstrated the strategies emphasized by Lamon (1999) within the measurement subconstruct of rational numbers. Elvio, however, does not yet exhibit the simultaneous reunitization of continuous, connected units (Bright, et al., 1998).

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ELEMENTARY TEACHERS' MATHEMATICS TEXTBOOK TRANSFORMATIONS IN TERMS OF COGNITIVE DEMAND OF PROBLEMS IN FRACTION UNITS

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This study explored elementary teachers' transformations of textbooks into practice in teaching fractions in the context of recent efforts to reform mathematics education. This study focuses on the first chain of teachers' transformations for classroom instructions—that of selecting problems with their textbooks. Factors that support and constrain teachers' textbook transformation approaches are also explored. This study revealed three problem selection patterns in terms of cognitive demand. Three influential factors, textbook cognitive demands, curriculum policy, teachers' view on textbooks were identified.

This study examined teacher's textbook transformations in terms of cognitive demand when teachers select their problems with their textbooks. Research on teachers' textbook use and influential factors has been done over the course of two decades and has provided a substantial number of categories of teachers' textbook use patterns and factors that influence them (e.g., Freeman and Porter, 1989; Kauffman, 2002; Schmidt, Porter, Floden, Freeman, & Schwille, 1987; Stodolsky, 1989; Sosniak and Stodolsky). However, most of the previous studies on textbook use focused on the maximal extent of coverage, such as to what extent teachers use textbooks in planning and teaching school subjects. They do not help us understand how teachers use their textbooks to provide different students' learning opportunities.

Professional Standards for Teaching Mathematics (NCTM, 1991) articulated that students' opportunities for learning are created by, what Stein and Smith (2000) called, cognitive demands of task, "the level and kind of thinking required students to successfully engage with and solve the classroom problems". Although researchers have documented mathematics instructional changes with different frameworks, such as, "new mathematics topics," "variety of manipulatives," etc, to gauge teachers' implementation of reform ideas, opportunities for student learning are not created simply by putting students into groups, by placing manipulatives in front of them, or by handing them a calculator. Rather, it is the level and kind of thinking in which students engage that determines what they will learn (NCTM, 1991).

The purpose of this study is to examine elementary teachers' textbook transformation when selecting problems in terms of cognitive demand and its influential factors. This study focused on the topic of fractions. The detailed research questions are as follows:

1. *What textbook transformation patterns, in terms of cognitive demand, do elementary teachers exhibit in their problem selection?*
2. *What kinds of the factors influence teachers' problem selection with textbook in terms of cognitive demand?*

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Conceptual Framework

Cognitive Demands of Problems

Problems here mean a mathematical object in the textbook and in teaching meant for the students to figure out and solve. *Cognitive demands of problems* mean "the kind and level of student thinking required when (students) engage in problems. Two different levels of cognitive demands—high vs. low—identified by Stein and Smith (2002) were used in this study.

Factors Influencing Teachers' Textbook Transformation

To explore factors that influence teacher's textbook use, this study employed three-level factors: *individual-level*, *contextual-level*, and *teachers' opportunity-to-learn factors.* Table 1 shows three levels of factors and the sub-categories this study explored.

Table 1. Factors that influence teachers' transformations

Factor-level	Sub-category
Individual-level factors	
	Teacher use of student objectives
	Teachers' skills and knowledge
	Beliefs about teaching and learning
	View about textbooks or textbook use
Contextual-level factors	
	District textbook policy
	Teacher perception of test (e.g., MEAP test, NCLB)
	Teachers' perceptions of the students' mathematics ability
Teachers' opportunity to learn	
	Professional development opportunities
	Teachers' perceptions about cognitive demands of textbook lessons

Methods

This study originally employed a mixed method design that combines both quantitative (survey) and qualitative approaches (interview & observation) to explore elementary teachers' use patterns. However, due to space limit, this paper addressed only one method, quantitative method.

Participant

A total of 169 teachers participated in this study from second through sixth grade. This is convenient sample. Participants were recruited through Master courses at Mid-Western University and professional development programs in the U.S. Data was collected from 2006 summer to 2007 fall semester.

Data Instrument

The survey was first developed based on the previous studies (e.g., Horizon Research questionnaires (2003) and then the pilot study was conducted. Survey were revised based on their feedback and reformatted the survey to facilitate easier reading. The revised survey is comprised of five parts: (1) background information, (2) teachers' perceptions of the cognitive demand of their textbooks, their problem selection and their questions, (3) individual-level factors, (4) contextual-level factors, (5) teachers' opportunities-to-learn factors (see Table 2). Teachers' textbook transformation patterns were measured using teachers' perception of the kinds and levels of problems presented in their textbook lessons and in their problem selection. They were asked to indicate the frequency of the various types of problems presented in their textbook and those used in teaching. Influence of individual-level factors, contextual level factors, and teachers opportunity to learn factors are measured by asking teachers to how much they would agree or disagree with each of the provided statements.

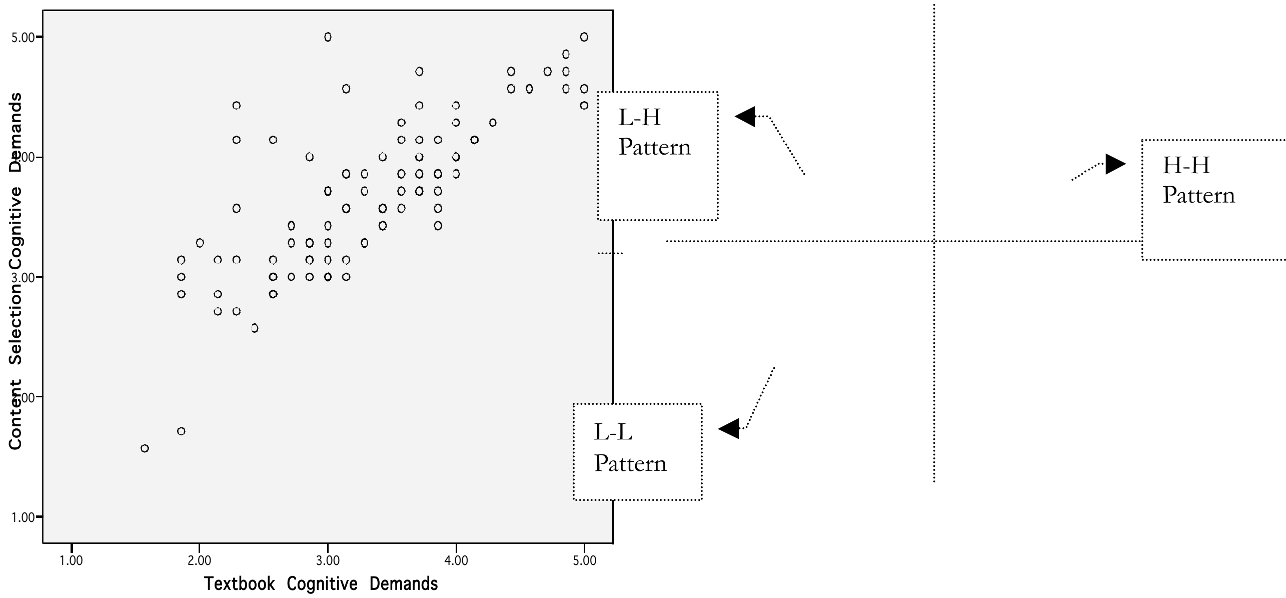
Data Analysis

After the preliminary statistics using the Statistical Package of the Social Science (SPSS), scatter plots were first used to get sense of the relationship between different variables and textbook transformation patterns were identified based on the average means of scale. Then regression analysis were completed for two purposes: (a) to examine relatively important influences among the three groups of factors and (b) to identify the most influential group of factors among individual-level factors, contextual-level factors, and teachers' opportunity-to-learn factors.

Results

What textbook transformation patterns, in terms of cognitive demand, do elementary teachers exhibit in their problem selection?

To find the relationship between cognitive demands of textbook problems and selected problems in their teaching, this study used the scatter plot. Figure 1 shows the linear relationship between cognitive demands of textbook problems and those in teaching, indicating that the higher cognitive demand of textbook problems teachers indicates, the higher cognitive demand problems teachers use. From this trend we can draw two different transformation patterns, which are *H-H pattern*, teachers who selected high cognitive demands problems with high cognitive demands textbooks and *L-L pattern*, teachers who selected low cognitive demands problems with low cognitive demands textbooks.



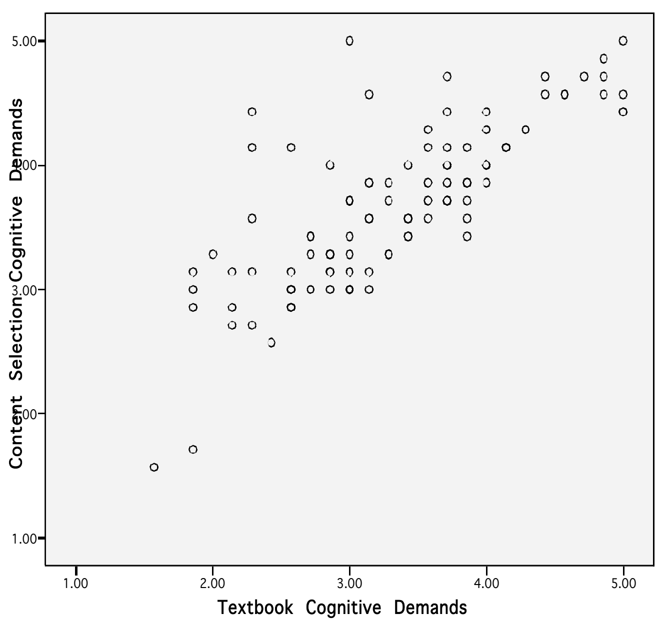


Figure 1 Scatterplot of Textbook Problem Cognitive Demands against Cognitive demand of Problems in Teaching

Figure 1 also shows that there are variations, indicating that there are other possible patterns. To explore precisely textbook transformation patterns, the average rating scale (m=3) were used. For example, if the average composite score of textbook cognitive demands problem is greater than 3, it was considered high cognitive demands textbook; otherwise low cognitive demands textbook. Three patterns are identified as shown Figure 1.

1. *What kinds of factors influence teachers' problem selection in terms of cognitive demand?*

This study used a hierarchical regression to explore this question. Based on the literature review, the order of entering three groups of factors was decided. Since individual level factors (e.g., teachers' knowledge) are considering important factors, I first entered individual level factors. In the second step, learning opportunities factors were added. Then contextual level factors were added to create a full model that explains teachers' higher cognitive demands content selection.

The results show that while student objectives among individual-level factors contributed significantly to the prediction of the teachers' problem selection while teachers' knowledge and beliefs did not have direct effect. Second, the addition of factors related to teachers' opportunity to learn highly improved the prediction of teachers' cognitive demands problem selection (24%_68%). Particularly, teachers' perception of textbook cognitive demands had a direct and significant effect, but the effect of student objectives decreased and became insignificant. When contextual level factors were entered, curriculum policy (groups with curriculum policy vs. no policy) was significantly associated with teachers' problem selections, meaning that teachers select slightly high cognitive demands contents when there is curriculum policy. This study has implications to policy makers, curriculum developers, and teacher educators.

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FRACTIONS ON THE NUMBER LINE: THE TRAVEL OF IDEAS

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This study analyzes the emergence and travel of ideas about fractions on the number line in three upper elementary school mathematics classrooms. The findings indicate that students' public displays of ideas shifted during the course of the lesson, classrooms varied in their treatment of the number line, and shared discourse within classrooms was rooted in different understandings.

A principal canon of inquiry-oriented mathematics classrooms is that teachers use student thinking as an instructional resource. In their instructional moves, teachers pose problems, solicit contributions, and orchestrate discussions in ways that lead students to make public their ideas and put their ideas in relation to those of others. In this process, students' mathematical ideas emerge and become elaborated, sometimes refined, and often transformed in classroom communities. Over the course of such classroom discussions, ideas may be taken up or rejected, valued or devalued, and understood differently by individual students in the community. The travel of ideas may take varied trajectories, and these trajectories may vary across classrooms implementing the same lesson. Thus, not only may the ideas that are taken up by students differ across classrooms, but also students within a given classroom may understand the same ideas in varied ways. Understanding the emergence and travel of ideas – what travels and how -- is basic to understanding learning and teaching in inquiry-oriented classroom communities. It also requires new ways of extending and elaborating empirical and conceptual frameworks.

The purpose of this paper is to present an analysis of the travel of mathematical ideas in three upper elementary school classrooms, each implementing the same lesson sequence involving fractions and number lines. In the paper, we focus on a single lesson across the three classrooms. The lesson presented a non-routine Problem of the Day (PoD) that involved the identification of a fractional point on a number line. In the problem, the number line was partitioned into unequal intervals, and the 'big idea' of the lesson was to support students in understanding the function of partitioning the line into equal intervals in conceptualizing the fractional value of a point. Our focus on the travel of ideas is important, particularly for understanding emergent mathematical environments and learning opportunities in such inquiry-oriented classrooms.

Though existing empirical approaches to the study of teaching and learning have made important contributions, they reveal in only very partial ways processes whereby ideas travel. For example, quantitative efforts have made use of rating scales to capture the depth of coverage of subject matter and the extent to which teaching builds on assessment of student thinking (e.g., Gearhart et al., 1999; Saxe, et al.,1999). Though these studies point to general processes that enhance learning in inquiry-oriented classrooms, they do not reveal well processes whereby ideas travel and become elaborated or put aside. Qualitative analyses describe with greater texture the interplay between learning and teaching in classrooms,

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including teachers' instructional decisions (Ball, 1993) and representational contexts (Cobb, 2002; Lampert, 2001; Sfard, 2002). But these studies typically do not follow the emergence and transformation of mathematical ideas in classroom life over a lesson, series of lessons, or even longer durations of time.

The Focal Lesson

The multi-phase lesson structure is depicted in Figure 1. It consisted of initial independent work on the problem, whole class and small group discussions, whole class problem resolution, and ended with independent work on problems that were similar to the PoD.

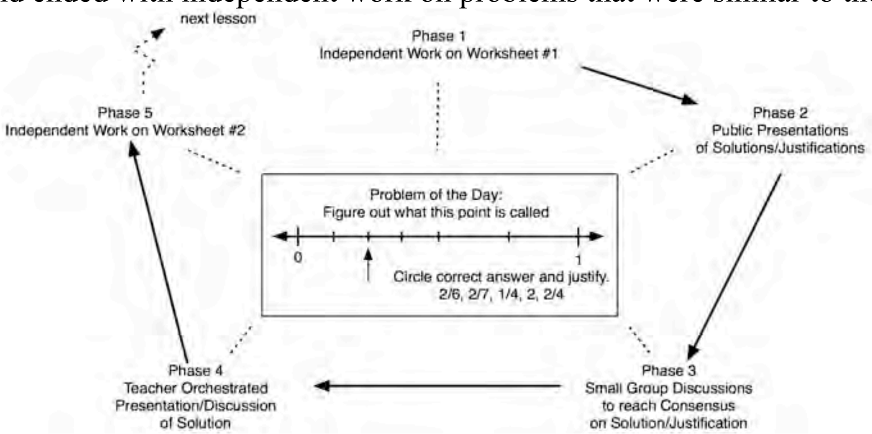

Figure 1. Lesson structure for the PoD: Identifying a point on the number line

Methods

Three teachers implemented the hour-long lesson in their untracked mathematics classrooms (1 fifth grade, 2 sixth grade) in an urban school district. We videotaped whole class and small group discussions, and interviewed a subset of students after class about the character of change in their thinking from their initial independent work to their current thinking. We also queried students both via written responses and in interviews about ways in which peers and the teacher may have influenced shifts in their thinking about the problem. In addition, we made use of a sociogram to identify students that were seen as particularly competent in mathematics, perceptions that we suspected might influence uptake and travel for some students.

Results/Discussion Points

Drawing upon our framework for studying the travel of mathematical ideas within classrooms (Saxe, Earnest, & Shaughnessy, 2007; Saxe, Shaughnessy, Earnest, & Cremer, 2007), we triangulated our data sources. In this process, we first coded worksheets to determine shifts in students' approaches to solving the PoD, and we followed up these analyses with an examination of (a) case studies of uptake of ideas in whole class and small group discussion, (b) the relation of uptake to issues of social position (based upon sociogram data and self reports influence). The following are a subset of our findings and discussion points:

1. *Initial Student Ideas:* As expected, at the beginning of class, some students viewed the PoD as an incompletely partitioned number line, and added or removed hash marks to create intervals of equal length. But others made sense of the number line with fraction names like "2/6," "2/7," "2/4" and "2," responses that suggested whole

number reasoning (counting marks) and idiosyncratic understandings of number line conventions (confusions of the role of the zero, hash marks, or arrows).

2. *Travel of Ideas:* Students' public displays of ideas shifted during the course of whole class and small group discussion. Students' approaches to naming points on the number line were taken up, resisted, elaborated and coordinated in different ways in each of the three classrooms. Additionally, students varied in their approaches to treatment of 2/8 as a potential answer. Some students developed geometric arguments (removing hash marks) and other students developed arithmetic arguments (reducing) to argue that 2/8 = _.
3. *Social Influences on Ideas*: Analyses of student reports about who influenced shifts in their thinking and sociograms that revealed beliefs about their peers' mathematical competence provide a basis to identify social sources of change – who, if any one, influenced change in thinking and why.
4. *Comparative Analyses:* In all classrooms, the number line was a focus of discussion; however, students' use of the number line to solve the PoD varied by classroom.
5. *Shared discourse and individual understanding:* In all classrooms, students and teachers used collective language about fractions, points, and hash marks. But interviews with students revealed that what was shared in ostensibly coherent back-and-forth dialogue was rooted in different understandings.

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STUDENTS MAKING SENSE OF MODELS FOR FRACTION MULTIPLICATION AND DIVISION

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This report will describe the findings of a new two-year curriculum research project studying student understandings of operations involving fractions, decimals, and percents. The focus of the project is on middle school students in urban classrooms to help them meet the 6-8 National Standards for fractions, decimals and percents as stated in the NCTM (2000) *Principles and Standards for School Mathematics*. Although the curriculum covers many topics, the focus of this report will be on fraction multiplication and division strand and how the presentation of this strand has evolved throughout the various stages of development. Insights into student understandings as they interact with different models of this difficult topic will be presented.

Many middle school students are able to apply the algorithm for multiplying fractions, but most are unable to describe the reasons why the algorithm works. Even fewer are able to make sense of the division algorithm. Previous work on addition and subtraction of fractions (Cramer & Henry, 2002) show how sustained uses of manipulative models develop mental images that support students as they make sense of these operations. There has not been as much work done in the area of multiplication or division of fractions. We will present a variety of models for multiplication and division of fractions and how students made sense of each of these models. We will also describe how students attempt to use mental images of these models to make sense of the standard algorithm for multiplying and dividing fractions.

Theoretical Framework

The Rational Number Project (RNP) http://education.umn.edu/rationalnumberproject is a cooperative, multi-university research project funded from 1979 - 2002 by the National Science Foundation (with the exception of one year). The Rational Number Project is most known for its investigations into students' initial fraction learning. Initial fraction learning includes developing meaning for fractions using a part-whole model; constructing informal ordering strategies based on mental representations for fractions; creating meaning for equivalence concretely; and adding and subtracting fractions using concrete models.

The theoretical model we use to guide the curriculum development is the same model used to guide the work of the RNP. We strongly believe that the reason the RNP fraction curriculum has been so successful with students is that the lessons emphasize translations within and between modes of representation with an emphasis on extended use of concrete models. In this translation model, five modes of representation are identified: real-world situations, manipulatives, pictures, spoken symbols, and written symbols. Conceptual understanding is dependent upon students having experiences representing mathematical ideas in each of these modes and translating within and between modes. A translation is the reinterpretation of a concept from one representation to another. For example, when multiplying fractions, a student might be asked to model a story problem using a concrete model and to describe how her actions with the

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manipulatives model the action in the story. She might be asked to record these actions using symbols; she might be asked to create another story problem for the same fraction multiplication problem. The constant movement and intellectual activity in the model reflects a dynamic view of instruction and concept development.

The seminal publication for this research on fraction learning was published in the *Journal for Research in Mathematics Education* (JRME) (Cramer, Post, & delMas, 2002). In this study, students' achievement on initial fraction ideas was contrasted between students using a commercial curriculum with students using the RNP fraction curriculum. The study involved 1600 students; 66 fourth and fifth-grade classrooms were randomly assigned to treatments. Results revealed significant differences in favor of RNP students across several strands. Qualitative data showed differences in students' thinking – RNP students' thinking was more conceptually based while student thinking among those using a commercial text was more procedural in nature.

RNP research documents effective methods for developing initial fraction concepts at the fourth and fifth grade. Much less is known on how to move students beyond this beginning level to master the fraction, decimal and percent goals for grades 6-8 as stated in the National Standards. Little research has been done to understand students' thinking about multiplication and division of fractions; research also is limited on identifying effective instructional strategies for developing meaningful understanding of these topics. Student performance data on state and national assessments shows we need to address this area.

State and national assessments measuring fraction, decimal and percent understanding has consistently shown that students struggle to learn these concepts in a meaningful way. Perhaps as educators we have underestimated the complexity underlying working with these numbers. Perhaps in our effort to move towards proficiency with symbols, we have neglected developing meaning for fractions, decimals and percents as numbers themselves, resulting in too many students operating on them in a rote manner; this, in turn, limits their abilities to estimate reasonable answers and to apply these skills to solve problems embedded in realistic contexts.

The state of Minnesota was ranked first among eighth grade mathematics performance on the 2003 NAEP. But our own state tests paint a less optimistic picture among students of color. Eighth grade students in Minnesota are required to pass a basic skills test in math and reading in order to graduate. The Basic Skills Test specifications indicate that about 30% of the test involves fractions, decimals and percents. While 81% of the white eighth-grade students passed the test in 2005, only 35% of African American students passed; only 46% of Hispanic students passed; only 47% of American Indian students passed (Minnesota Department of Education, 2005). Currently, over 16,000 students still need to pass the Basic Skills Test to graduate. Most of these students are students of color. This is only one indicator of the achievement gap in Minnesota. Other indicators show that Minnesota has one of the largest achievement gaps in the nation.

While the need to improve instruction is needed for all students, the need is greatest for students who traditionally under-perform on state and national tests. The goal of this project is to improve student learning in this content strand, particularly among students of color.

Methods

The teaching experiments were all conducted in an urban school district in Minnesota. Both of the schools that participated in the teaching experiments serve diverse student populations. The curriculum contains 34 lessons on fractions, decimals and percents, 12 of these lessons deal specifically with fraction multiplication and division.

All of the students in the study were exposed to the entire RNP Level One curriculum (Cramer, Behr, Post, & Lesh, 1997) in a previous year or they completed a ten-day review packet of the Level One curriculum weeks prior to the introduction of the new curriculum. A pre-test was given to each student and taped interviews were completed with at least six students in each classroom before the teaching experiment began, twice during the experiment, and once after the teaching was completed. All students took two tests during the experiment and one post test. Student work during the lessons was collected and student comments were recorded in a field notebook throughout the entire teaching experiment.

The first draft of the curriculum was developed and field tested in a school with 28 sixth-grade students. The results of the teaching experiment were used to revise the curriculum. Comments from an expert panel were also used in the writing of the second draft. The second teaching experiment was completed with 24 sixth-grade students at a different school. The results of this teaching experiment were used to write a third draft of the curriculum. The third draft of the curriculum will be pilot-tested during the 2007-08 school year in over 20 urban classrooms.

Results and Discussion

The preliminary findings of this project indicate that students are able to make appropriate connections among manipulative models, real world situations, mental images, and symbols when they work with addition and subtraction of fractions. The mental images support these students as they develop and make sense of the formal rules for fraction addition. Developing mental images related to multiplication and division of fractions proved to be much more complex.

The student work collected during the teaching experiments and interviews helped sort and classify the various ways students think of fraction multiplication and division. We were able to document some of the ways that students make sense of the varied situations related to these operations. We found how sequencing of models for multiplication and division relate to student sense making. We are finding that certain models are helpful for students as they learn about situations involving fraction multiplication and other models help develop the formal algorithm. We are also finding other models that can be used after the introduction of the algorithm that help students make more sense of multiplication and division situations involving fractions.

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THE DEVELOPMENT OF NUMBER SENSE AND EFFICIENT STRATEGIES

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This study examines third, fourth and fifth grade students' recorded strategies of a two-step problem. The students written recordings of strategies differed across grade levels. The fifth grade students used a less conceptually complex strategy compared to the lower grade students. Implications for the development of number sense and efficiency are discussed.

Theoretical Framework

Number sense is a central aspect of understanding number and operations (NCTM Principals and Standards, 2000). Furthermore, the NCTM principals and standards (2000) document states that students must be capable of using efficient strategies. In other words; simply memorizing strategies for solving problems without understanding is not adequate. Students must use their number sense to figure out efficient strategies to solve problems. This means that students can use a variety of formal and informal written and mental methods to find solutions to problems (Anghileri, 2000).

We are currently involved in a professional development project in a western state. One of our goals is to help teachers facilitate number sense development in their students. We also want to help students use more efficient strategies as well. Therefore, we developed a 10 question test to understand the types of strategies that children use to solve basic problems involving number operations. Our larger goal is to use the analysis of student strategies to plan our professional development sessions with the teachers. In this paper, we present an analysis of one test item that we found interesting. The problem that is analyzed is a two step problem. The questions that we are investigating are: What kinds of strategies do children in grades three to five use to solve a two step problem. What can we learn about the problem type and children's development of number sense?

Method

Students in grades 3 to 5 were asked to solve the following problem:

> There were 15 students in the bus. The bus stops and drops off 5 students. Then it stopped again and drops off 3 more students. How many students are still on the bus after the second stop? Solve the problem and show your work.

The participants in the study are third, fourth and fifth grade students from a western state. The data represents a total of 274 students. Fifty-seven students were third graders, 89 students were fourth graders and 128 students were fifth graders. This sample represents 15 classes in rural settings in a large geographic area. The test was administered by a project team member in each classroom. The data was analyzed based on the solution strategies that emerged. Two types of data emerged. Students either used numeric strategies involving digits or they used pictorial representations such as drawings, tally marks. We coded the data as follows based on the strategies that emerged:

Code	Student Cognitive Strategy	Explanation of strategy
N1 (numeric)	a-b=c ; c-d=e	Solve problem in two steps as two separate subtraction problem using numerals.

N2 (numeric)	a-b-c=e	Repeated subtraction using numerals.
N3 (numeric)	a-(b+c)= e	Add the total quantity that needs to be subtracted together then subtract from whole using numerals.
Pictorial		
P1 (pictorial)	a-b=c ; c-d=e or a-b-c=e	Solve problem in two steps as two separate subtraction problem using pictorial representation.
P2 (pictorial)	a-(b+c)= e ; a-b-c=e	Repeated subtraction using pictorial representation. Or added the total quantity that needed to be subtracted from the whole. (1) Hard to distinguish which cognitive strategy used therefore we coded these strategies as P2

Table 1:Coding Used

Results

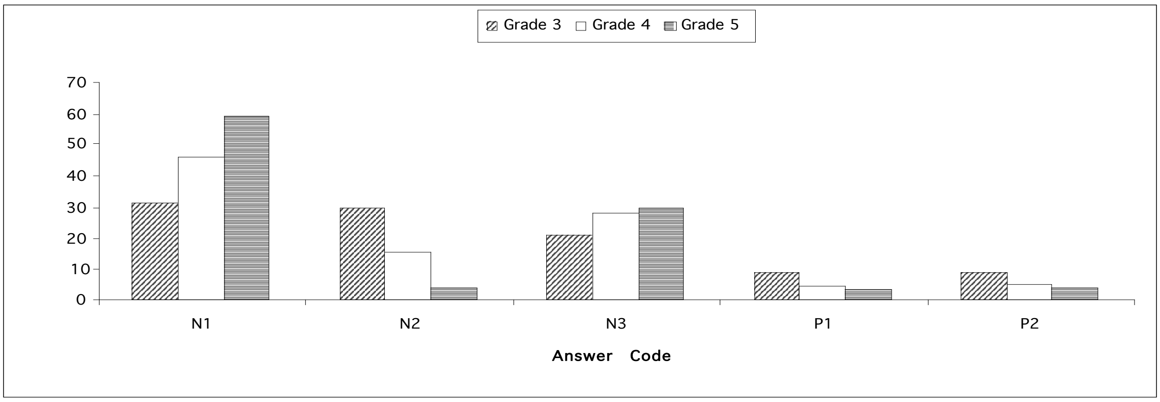

Table 2: Graphical Representation of Strategies Used across Grade Levels

Answer Code	Grade 3	Grade 4	Grade 5
N1	31.6	46.1	59.4
N2	29.8	15.7	3.9
N3	21.1	28.1	29.7
P1	8.8	4.5	3.1
P2	8.8	5.6	3.9

Table 3: Percentage of Strategies Used for Each Question Type

A high percentage of third, fourth and fifth grade students were able to successfully solve this problem. The pictorial strategies mirrored the numeric strategy. The data in tables 2 and 3 represents the distribution of the different types of strategies recorded by students. Almost two-thirds of fifth graders (59.4%) used the numeric strategy a-b=c and c-d=e. The recording of students strategies indicate that the problem was noted as two separate subtraction problems. Only 3.9% of fifth graders recorded the solution to the problem as repeated subtraction a-b-c=e (N2). About a third (29.7%) of fifth grade students used the a-(b+c)=e (N3) strategy. Almost half of the fourth grade students (46.1%) used the N1 strategy. 15.7% of fourth grade students used the N2 strategy. Almost a third of the fourth grade students (28.1 %) used the N3 strategy. Third grade students' strategies were evenly distributed among the different types of strategies as indicated in table 3.

Discussion

Majority of students in this study were successful in solving the problem correctly. The word problem or the numbers in the word problem was not difficult for students. However their strategies differed across the grade levels. What we found surprising is the recordings of fifth grade students used in N1 strategy is conceptually simpler than the other strategies used. More fourth grader used this strategy than third graders. We conjecture that this trend indicates that even though the strategy was less sophisticated it was actually a more efficient strategy. For example, it is easier to break the problem into two smaller subtraction problems and also keep track of their recordings. It is also likely that students are also using familiar calculation strategies to solve the problem. Furthermore, students are likely using formal notation that they have been taught to record their answer. A repeated subtraction problem recording (N2) a-b-c is a less conventional strategy. The N3 strategy is more conceptually complex. As students use more efficient recordings, it is likely that the strategies are less conceptually complex.

The problem posed in this study could lend itself to the development of more sophisticated mathematical ideas as indicated in the solution types that emerged. For example:

a-b=c and c-d=e	involves conceptualizing the problem as a two step problem involving subtraction
a-b-c = d	involves understanding repeated subtraction and also that the associative property cannot be used.
a-(b+c)=d	strategy lends itself to exploring the distributive property and the order of operations

The data in this study did not reveal if students understand connections between these recorded strategies. For example, a-b-c=a-(b+c). The limitation of this analysis is that we examined the recorded strategies of students. In addition, the question itself is limited in the complexity of the numbers used. We found very few studies in the research literature that actually addresses two step problems. Therefore, further research is needed to fully understand the potential of using two step problems to support student thinking and development of number sense.

Endnote

*1. The test item was developed by the Northeastern Nevada Mathematics Project team: project Teruni Lamberg, Bob Quinn, Dave Brancamp, Sharon McClean and Gini Cunnigham.

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THE IMPACT OF MEASUREMENT MODELS IN THE DEVELOPMENT OF RATIONAL NUMBER CONCEPTS

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This study investigated the models students used to solve problems involving the comparison of rational number quantities. One hundred ninety-one fifth-, sixth- and seventh grade students at two research sites were given three rational number problems and their responses were analyzed to determine what diagrammatic representations were used to support students' reasoning for their answers.

This study investigated whether students used models to solve problems involving the comparison of rational number quantities, and, if so, what type of models they used. The research questions were: How does a measurement perspective for the development of rational number affect students' conceptual understanding? How does a measurement perspective for the development of rational number affect the representations used by students? Does an emphasis on the concept of unit impact children's approach to comparing numbers?

Theoretical Framework

The researchers for this study are involved in a research and development project, Measure Up (MU) (Dougherty, 2003), at the Curriculum Research & Development Group at the University of Hawai'i. The MU approach to elementary mathematics challenges the traditional notion of what constitutes the basics of early mathematics learning. While mathematics in the early grades traditionally begins with counting, number recognition, and simple computations, the MU program starts with an emphasis on the structure of mathematics through measurement and algebraic thinking. In grade 1, students compare continuous quantities such as volume, area, mass and length through direct and indirect measurement to develop understanding of relationships, operations, and unit. In each successive grade level, mathematical ideas are linked to measurement, including the development of rational number.

MU is based on a Vygotskiian distinction between spontaneous and scientific concepts (1978). Spontaneous or empirical concepts are developed when children abstract properties from specific experiences or examples. As an example, spontaneous concepts progress from natural numbers to whole, rational, irrational, and finally real numbers, in a very specific progression. Topics are taught within each number system, and often not connected across systems. Scientific concepts, on the other hand, are developed from generalized experiences with properties, and specific instances are then explored. The scientific approach focuses on real number concepts first; then specific cases related to natural, whole, rational, and irrational numbers are studied. Davydov (1975) conjectured that a general to specific approach in the case of the scientific concept was much more conducive to understanding than using the spontaneous approach. This challenges the essence of what has constituted elementary mathematics and sets MU apart from other curriculum development projects.

The concept of unit is fundamental to measurement and plays a central role in MU, particularly in the development of rational numbers. The researchers were interested in the

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degree to which MU students' experiences with unit and the use of measurement models were reflected in their understanding of certain rational number concepts. Additionally, we wondered whether MU students understood concepts of rational numbers differently than students who have not learned rational numbers through measurement. Others (Sophian et al., 1997; Stafylidou & Vosniadou, 2004) have noted a relationship between students' errors in solving problems involving comparing fractions and experiences they have with whole numbers. For example, when asked to name a number between $\frac{1}{5}$ and $\frac{2}{5}$, students often say there is none, which is attributed to over-generalizing a pattern known to work with whole numbers while not recognizing the density of the number. Sophian, however, posits that if students worked with whole number in ways that emphasized an understanding of unit, they might not be as limited in their solutions.

Methods

One hundred ninety-one students in grades 5 through 7 at two research sites, the University Laboratory School (ULS) and Connections Public Charter School (CPCS), were given three problems to solve: (1) Which fraction is closer to 1: $\frac{5}{6}$ or $\frac{6}{5}$? (2) Are the fractions $\frac{6}{9}$ and $\frac{10}{15}$ equivalent?; and (3) Name a fraction between $\frac{1}{10}$ and $\frac{2}{10}$. Students were instructed to explain their reasoning and "support their answer" for each question.

All students completed the three-problem assessment in their mathematics class. Twenty minutes were allowed to complete the tasks. The responses were scored independently by two researchers. A score of 1 was given for responses that were correct and included correct explanations. A score of 0 was given for all other responses. Any discrepancies in scoring were discussed by the researchers until agreement could be reached. Data was also collected on the use of models employed. This paper discusses the issues of correctness and model use.

In addition to scoring responses and categorizing models used, twenty-four students were selected for individual interviews within a day of completing their written assessment. Six of the interviewees were fifth-grade MU students. Six were sixth-grade students who had been in MU and six were sixth-grade students who had not been in MU. None of the six seventh-grade interviewees had been in MU. Students were selected to represent low, middle and high achievement levels as determined by their teachers. A semi-structured interview protocol was used. Each student was provided a copy of his/her written solution and asked to explain the work. This data is still being analyzed and is not reported in this paper.

Results and Discussion

The data below indicate the percent of correct responses per question. While there is very little variance in percent correct across grade levels, there seems to be an ordering of the difficulty of the questions. Of those who answered correctly, the fifth-grade students (all MU) were more likely to have used a model, and the use of a model decreased from grades 5 to 7. None of the seventh-grade students in the study had been in MU.

Percent correct per question	Question 1	Question 2	Question 3
Grade 5 (N = 35)	20	26	46
Grade 6 (N = 73)	27	41	43
Grade 7 (N = 83)	23	39	46

Of those correct, percent who used a model	Question 1	Question 2	Question 3
Total	46	25	24
Grade 5	86	67	50
Grade 6	50	30	19
Grade 7	26	9	16

We classified the models students used into five categories: area, not circular; area, circular; linear; linear, area [e.g., a rectangular area model that used only linear attributes]; and discrete. The table below contains the percent of students who used these models on each question. It is interesting to note that the type of model per question varied by grade level. For example, 40% of students in grade 5 used a strictly linear model for question 1 while only 3% and 1% of sixth- and seventh-grade students, respectively, used this model. In general, more grade 5 students used models on each question, and use of models decreased in grades 6 and 7. For example, only 3% of seventh-grade students used a model on question 2.

A discussion of comparison of use between MU and non-MU students as well as results from the 24 interviews will be reported upon the completion of the analysis.

Percent of students who used each model on Question 1	Area, not circular	Area, circular	Linear	Linear, area	Discrete	No model
Grade 5	9	9	40	0	0	42
Grade 6	10	12	3	23	0	52
Grade 7	0	7	1	5	0	87
Percent of students who used each model on Question 2	Area, not circular	Area, circular	Linear	Linear, area	Discrete	No model
Grade 5	3	17	31	6	3	40
Grade 6	12	4	3	19	0	62
Grade 7	0	0	1	2	0	97
Percent of students who used each model on Question 3	Area, not circular	Area, circular	Linear	Linear, area	Discrete	No model
Grade 5	0	6	37	3	0	54
Grade 6	3	3	7	7	1	79
Grade 7	1	2	12	4	0	81

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UNDERSTANDING OF FRACTIONS THROUGH THE USE OF FAIR TRADES: DEVELOPMENT AND IMPLEMENTATION OF A HANDS-ON UNIT ON FRACTIONS

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Many elementary and middle school teachers have expressed frustration with students' understanding of fractions. This topic seems to be a stumbling block for many children as they progress in mathematics. Numerous reasons for such difficulties have been offered by the research community, such as the material being taught too abstractly, too procedurally, and outside of any meaningful contexts (National Research Council, 2001). Steffe and Olive (2002) detail some of the problems observed in students' understanding of fractions, and suggest that students' difficulties stem from learning fractions using rote memorization of procedures without a connection "between what is taught and their informal ways and means of operating" (p. 128).

The activities developed for this research project use the idea of fair trade with pattern blocks as a common theme throughout the unit on fractions. This study reports on the development of the hands-on unit as well as two sixth grade classes' development of fractional understanding.

Objectives

The primary research goal in this project was to investigate the outcomes of teaching a six-week hands-on unit on fractions to sixth graders. The sub goals of this project were to (1) determine students' initial strengths and misconceptions about fractions, (2) identify elements of the unit that address such misconceptions in fraction knowledge, and (3) determine increase of knowledge of fractions.

Theoretical Framework

The framework used in this study is based primarily on constructivist theory, with a specific focus on the work done by Simon, Tzur, Heinz, and Kinzel (2004) on mathematical conceptual learning. Their perspective reflects three key principles of constructivism: 1) Mathematics is created through human activity; 2) existing knowledge affects the assimilation of future understanding; and 3) learning results from the adaptation of one's existing conceptions. They elaborate on a mechanism for conceptual learning that builds on Piaget's theory related to disequilibrium and reflective abstraction. These ideas have been incorporated in the design of the fraction activities, and have been revisited during the analysis phase.

Developing the Unit on Fractions

Part of the inspiration for the development of the activities used in the project can be attributed to Van de Walle and Lovin (2005). The approach to fractions teaches conceptual understanding through the following means:

- The approach is hands-on. Pattern blocks are used as the foundation for understanding all aspects of fraction concepts as well as computations.

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- The instruction progresses through three distinct phases of learning and understanding fractions: 1) visual representation of fractions using pattern blocks; 2) developing fractional "number sense" (independent of the visual representation); and 3) developing the algorithms for addition, subtraction, multiplication, and division of fractions using pattern blocks and pictures.
- Real-life problems add a context to the operations. Students are given word problems involving fractions and are asked to make sense of the word problem through use of pattern blocks and their own, created models.
- The idea of "fair trade" to create equivalent fractions is the critical theme woven throughout each of the lessons in the unit on fractions. This theme is used to give meaning to the procedures related to fraction operations, and sets this unit apart from other hands-on units on fractions.

Mode of Inquiry

The study took place with two sixth-grade classrooms in a public school in the San Francisco Bay Area. This school was purposefully selected based on the researcher's interactions with one of the teachers (the second author) during several years of content-based professional development institutes. The second author had experienced frustration in teaching fractions to her sixth graders, and had expressed a desire to improve her conceptual development of fractions. The two of us met weekly for two months during the summer to develop the activities and to create an implementation plan. At the time of implementation, we team taught the lessons over a six-week period.

Data and Analysis

Data Collected

Data collection included pretests, posttests, quizzes, homework, and field notes on interactions with students during activities, including individual and whole class discussions. In addition, each of the sixth grade classes in the school took part in a common exam on fractions after the unit had been taught. The exam questions were produced by experienced mathematics teachers in the district. This data was used as a comparison to other classes who had studied fractions in a more traditional manner, and had not taken part in the unit on fractions described above.

Analysis

Qualitative data was analyzed with reference to the framework by Simon et. al. (2004). T-tests were run on the pretest and posttest scores to determine significant differences in understanding based on written tests.

Results and Discussion

During the teaching of the unit on fractions there were many opportunities to observe students' developing understandings of fractions. In agreement with other studies, students did not seem to have a strong foundational understanding of fractions upon which to build (National Research Council, 2001; Steffe and Olive, 2002). We found the following to be true for many of our students: 1) Students have not yet come to understand that a larger denominator represents a smaller fractional piece. 2) Students understood that the denominator tells them how many groups to break the whole into, yet many students didn't realize that the groups have to be of equal size. 3) Students lacked a number sense about the size of fractions, which resulted in difficulties when ordering fractions. 4) Students demonstrated an inability to extend their understanding of whole number operations to fractions.

One of the tasks addressed the common misunderstanding represented in #2 above. Students were given a picture created by pattern blocks and were asked if it were possible to create one fifth of the picture. (This was not possible using the pattern blocks.) Several students chose blocks of unequal size to demonstrate how it could be done. For example, one sixth grader represented the answer of one fifth using five pieces—four blue rhombuses and one green triangle. This child experienced no cognitive dissonance between his answer and his concept of fractional parts, because his understanding of fractions did not yet require the parts to be equal. For the second part of this task students were asked to "prove," using fair trades with pattern blocks, that their answer was correct. It was at this point the student encountered a situation that caused him to reflect on his understanding of fractions.

The issue described in #4 above was demonstrated by some students by the inability to use pattern blocks to solve problems such as, "find 1/5 of 20", although they could solve problems such as, "find 1/5 of 5". Steffe and Olive (2002) refer to *prefractional concepts*, one of those being the belief that one-fifth refers to "five single elements in a collection or to five parts of a continuous whole" (p. 129), rather than "five composite units of indefinite numerosity" (p. 129). Building upon the research, it is also important to view this limited understanding of fractions in light of students' prior experiences with models of division. What we found in our interactions with sixth graders was that the students who struggled to find 1/3 of 12 with pattern blocks also had trouble when asked to divide 12 into 3 groups. This problem uses the sharing interpretation of division. Students' difficulties with fractions seem to be connected to their lack of facility with the different models of division. Students with these difficulties were presented with problems that encouraged them to broaden their understanding of a denominator to represent a certain number of equal *parts* rather than the limited view of a denominator representing a certain number of equal *pieces*, with the number of pieces equal to the denominator.

One of the surprising strengths that developed over the course of the unit was in students' abilities to explain mathematics conceptually. Using the pattern blocks and fair trades in their explanations, students were able to explain why fractions were equivalent, for example. Students also developed their strengths in solving word problems. Since students had spent six weeks using pattern blocks to develop their conceptual understanding of fractions, they were able to use these same skills to create models to represent word problems involving fractions. Results from a pretest showed that only a few students could solve word problems involving fractions. Results from the posttest show a marked improvement in students' ability to approach, and correctly solve, word problems involving fractions. Furthermore, students did not express the usual fear of word problems during this unit. Students actually exhibited more confidence in solving the word problems than they did with a straight computation problem involving arithmetic of fractions.

For the second author (the classroom teacher), the most impressive results were from the school wide test on fractions that was given at the end of the unit. Analysis has revealed a difference in favor of the group that took part in the hands-on fraction unit.

Summary

Over the course of the six-week period, students demonstrated growth in their conceptual understanding of fractions. Many of the students' misconceptions were addressed by the tasks that were presented in this study; these tasks encouraged students to further refine and adapt their conceptions of fractions. The use of pattern blocks and the concept of fair trade have shown to help students attach meaning to the formerly abstract procedures related to fractions.

Moreover, pretest and posttest results indicate an increase in student achievement within the group. Comparisons based on the school wide test on fractions favored the group that took part in the hands-on fraction unit.

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USING INNOVATIVE FRACTION ACTIVITIES AS A VEHICLE FOR EXAMINING CONCEPTUAL UNDERSTANDING OF FRACTION CONCEPTS IN PRE-SERVICE ELEMENTARY TEACHERS MATHEMATICAL EDUCATION

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We examine elementary school teachers' prior understandings of concept of fractions within different contexts. We argue that teacher' understanding on fractions significantly improves when they are offered opportunities to investigate fractions in new context, specifically, in the situations when they cannot rely on familiar procedural approaches. This leads to cognitive conflict when approaching problems involving both, continuous quantities and wholes consisting of several objects. These situations arise when problems related to Egyptian fraction are introduced to the pre-service teachers. Investigations of these problems have led to deeper conceptual understanding of fair share concept, fractions comparison, relations between different wholes, fractions part-part-whole representations (bigger fraction represented as sum of smaller fractions), connection of different fraction representations with numerical measure (number of cuts), and- to understanding of why Egyptians used unit fraction representation.

Introduction and Theoretical Perspective

Despite recent reforms in K-12 mathematics curriculum, fraction concept and understanding of rational numbers remain one of the weakest points in pre-service teachers' mathematics education. Many researchers agree that the root of the difficulties in teaching and learning fractions lies in the fact that fractions comprise a multifaceted construct (Brousseau, 2004; Kilpatrick et al., 2001; Lamon, 2001).

First examples of fractions in the history can be found in Egypt from around 3,000 B.C., where principally unitary fractions were used in sophisticated problems. This representation (as a sum of unit fractions) is described, in detail, in the Rhind (Ahmes) Papyrus, the most extensive Egyptian mathematical papyrus (Boyer, 1991). According to the papyrus, this was a method recommended, e.g., for dividing loaves of bread between several people. Most algorithms for computing Egyptian fractions described in the literature are quite complicated.

The question why Egyptian fractions were used in the first place has puzzled many mathematicians. For example, according to Hoffman (Hoffman, 1998), when R. Graham (who wrote his Ph.D. dissertation on unit fractions) asked Andre Weil this question, A.Weil answered, "They took a wrong turn".

Several simple activities were described in the literature and successfully used in middle school grades (Bentley, 2004; Oliver, 2003, O'Reiley, 1992; Gardner, 1978). However, they are used quite rarely, and very little related curriculum material exists. We argue that there is need for expanding research in this area. As pointed by David Lingard, "the use of unit fractions by the early Egyptians goes largely unnoticed. If ever there was a topic ripe for exploration, with a well documented historical background, which would help many learners to gain a deeper appreciation of the arithmetic of fractions, then this is surely it" (Lingard, 1999).

Authors have developed a new approach to teaching Egyptian fractions and general procedure for the algorithm of finding unit fraction representation. They also showed that in most contexts Egyptian fractions representations given in the Rhind Papyrus provide an optimal solution (smallest number of cuts). This approach has led to the development of new exploration activities appropriate for incorporating into mathematics and mathematics methods classes for pre-service elementary teachers. This activity has become a focus of our study with pre-service elementary school teachers learning how to teach mathematics.

Method, Participants, Procedures and Results

The sample of the study consisted of 75 pre-service teachers involved in the study of mathematics and mathematics teaching methods. Both control (60) and treatment (15) group of students were engaged in learning fraction concept during two months period. Control group received the instruction, focusing on both conceptual and procedural aspects of fractions derived from a standard textbook for math methods. Students in the treatment group also studied conceptual and procedural aspects of fractions derived from a standard textbook, and, at the same time were involved in mathematical project investigations in the context of word problems related to the Egyptian fractions. Pre-service teachers' investigations of these problems have led to deeper conceptual understanding of fraction concepts compared to the control group.

At the end of the cycle both control and treatment groups' teachers designed and implemented lessons in elementary classrooms. Specifically, treatment group designed and implemented lessons related to Egyptian fractions. The lessons captured through field notes were analyzed and compared. Pre-service teachers' reflections, learning journals, and pre- post-surveys were collected and analyzed. The results indicated a statistically significant improvement in posttest scores for the treatment group. Student interviews and surveys indicated that bringing new contexts in teaching and learning of fractions (1) helped students to see the practical applications of fractions in the new light, (2) illustrated that addition of fractions can be illustrated without common denominator procedure, and (3) made study of mathematics more interesting and meaningful for pre-service teachers.

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A BASE-TEN PACKING INVESTIGATION AND ITS INFLUENCE ON WHOLE-NUMBER COMPARISON SUBTRACTION METHODOLOGIES

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This poster session reports on the findings of a research study that examined second grade student approaches to comparison whole-number subtraction problems. Research on students' difficulties with the operation of subtraction as well as place-value understanding is well documented (Fuson, 1992). This study seeks to examine the effects of connecting a base-ten packing investigation directly to comparison subtraction situations.

The NCTM, in its *Principles and Standards (2000)*, states it is "essential that students develop a solid understanding of the base-ten numeration system and place-value concepts by the end of grade 2." (p. 79). The Council also points out that by allowing students to work in ways that have meaning for them, teachers can gain insight into students' developing understanding and give them guidance (p. 85). Research shows that students also develop understanding of place value through invented strategies (Fuson et al.,1997). When subtraction problems arise in meaningful contexts, students invent ways to solve them that incorporate and deepen their understanding of place value, as well as their facility with the operation of subtraction itself. Fosnot and Dolk discuss the power of base-ten packing investigations in helping students to develop a deep understanding of our place-value system as well as create meaningful strategies for adding and subtracting multi-digit numbers (2001).

In this study, a second grade class of twenty students participated in a two-week investigation of base-ten packing problems in which they eventually subtracted in comparison situations using invented strategies. In the investigation, students first worked to determine how many ten-packs and how many single pieces of bubble gum were needed to fill various bubble gum orders. Eventually they worked on combining orders (addition) and determining what was needed for partially filled orders (subtraction). An assessment consisting of three contextualized subtraction questions was given to all county students and the responses of the 20 students were compared to those of 100 others who took the assessment. The analysis shows that there was a significant difference in the approaches taken between the students who experienced the investigation and those who did not. While the majority of the students in the control group used either the traditional subtraction algorithm or simply provided an answer with no explanation, a plurality of the students in the case study group began with the subtrahend and took leaps of tens and single units to get to the minuend or began with the minuend and removed the subtrahend in tens and single units to get to the answer. The results show that case study students were more flexible in their approach to subtraction problems and that they more often considered the difference between the minuend and subtrahend as a distance. The finding suggests that targeted subtraction instruction using a packing investigation has an effect on student conception of comparison subtraction.

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BUILDING A MORE PROFICIENT MODEL OF RATIONAL NUMBER UNDERSTANDING

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Rational numbers have a variety of interpretations making them difficult for children to learn. Lamon (2001) described five fraction representations for alternative instructional approaches to the often used Part/Whole model. Just as Lamon noted that as rational numbers need to be taught and understood as a system of contexts, meanings, operations and representations, so too should children's perception, understanding, and conceptual coordination be assessed in multiple ways. My research explores how students' understanding of fractions is revealed in their drawings and original story problems. Analysis of their responses utilizing Lamon's framework revealed misunderstandings arising from confused interpretations of fraction representations.

At a public school in Northern California, I individually interviewed six teacher-chosen fifth-graders and asked a series of questions about the problem _ + 2/3 and its solution. Students were asked to add the fractions symbolically, to draw the parts and solution, to estimate the answer, and to explain if their drawing matched their symbolic answer. They then represented the problem and solution on a number line, and finally wrote a story problem with addition.

On the surface, my study showed results similar to Peck and Jencks' study twenty five years ago where students were "terribly confused" about fractions (1981). In my study, only one out of six had a firm understanding across all representations of fraction addition. Most had difficulties with their story problems, both in set-up and solution. However, I also found that even when students were unsuccessful with symbols, they had at least a partial mathematical understanding as expressed in their drawings and stories. Drawing helped some students realize that their algorithm solution was not correct. Two students could see by their pictures that the answer was slightly more than 1, showing an understanding that the incorrect symbolic solution did not indicate.

When I examined the student generated story problems, I found that their authors used one of, or a combination of, the five Lamon fraction interpretations: part /whole comparisons with unitizing, operator, ratio and rates, quotient, and measure. All but two had some notion of part/whole within their story. One used only part/whole, correctly representing and solving the problem. One used part/whole correctly but could not complete the solution. The other four used an incorrect combination of fraction representations. For example, one student said, "Molly has _ cookie and she has 2/3 (2 friends and one is missing) How much would Molly have to break it up in?" She interpreted 2/3 as "two out of three" and then used that quantity as an operator. Also apparent in the story is the idea of sharing, a construct often associated with fractions. As Peck and Jencks pointed out, children make sensible mistakes, that is, their mistakes have sensible origins (1981).

By utilizing students' writing and drawing as evidence of their thinking, the level of their existing understanding can be determined. This analysis offers a proficient analytical model

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using the role of representation to determine student understanding of rational numbers. As educators, we can build upon this information to enhance conceptual fraction understanding.

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FROM RESEARCH TO PRACTICE: TEACHING FRACTIONS CONCEPTUALLY

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What happens when sixth graders, with a background of learning fractions very procedurally, experience a hands-on, conceptual approach to learning fractions? That was the question explored by a middle school mathematics teacher and a university mathematics educator when developing and co teaching a unit on fractions. This poster will describe our process of collaboration in the development of the unit, and will present the evolution of the activities over the course of the year.

Development of the Activities

This project began in the summer of 2005, at which time we met to discuss areas of concern in the teaching and learning of fractions, and to develop a hands-on approach that would directly address many of these student misconceptions. The constructivist framework that builds on the ideas of disequilibrium and reflective abstraction (Simon, Tzur, Heinz, and Kinzel, 2004) was incorporated in the design of the fraction activities.

This approach to fractions teaches conceptual understanding and encourages the learner to make sense of the properties of fractions. Introductory activities build on the concepts of multiplication and division of whole numbers to develop the concept of a fractional part, and make liberal use of two models of division: partitive (sharing) and measurement (repeated subtraction). The instruction progresses through three distinct phases of learning and understanding fractions: 1) visual representation of fractions using pattern blocks; 2) developing fractional "number sense" (independent of the visual representation); and 3) developing the algorithms for addition, subtraction, multiplication, and division of fractions using pattern blocks and pictures. Real-life problems are included to encourage students to make sense of problems through use of pattern blocks and their own, created models. These created models are then linked to the traditional algorithms. Part of the inspiration for the development of the activities used in the project can be attributed to Van de Walle and Lovin (2005).

Implementation of the Activities

In the fall of 2005, we team taught the lessons over a six-week period. Initial assumptions about prior understanding of fractions led to numerous adjustments in the lessons, primarily to reintroduce basic concepts. Once student misconceptions were exposed, the subsequent lessons were altered to focus on establishing these appropriate conceptions. One of the surprising results of this unit was related to students' ability to solve word problems. Throughout this unit students were encouraged to use student-generated strategies to make sense of their math calculations in a meaningful way. Students actually gained confidence in their ability to solve the word problems correctly.

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Implications

During the implementation phase of this project it became apparent that it is indeed difficult to teach students conceptually after years of procedural learning. At the sixth grade level, although understanding of fractions was low, their knowledge of procedures was strong enough to give students a false sense of security. During the six-week period we were able to see substantial improvement in students' understandings of fractions. Yet we were not able to address all of the misconceptions that students had accumulated over the years of working with fractions. In order for this approach to be more successful, it needs to be implemented when students originally encounter fractions, and built upon each year. Subsequent implementations of these activities occurred in grades two, three, and five. The final result of this project was an alignment of fraction content with state grade level standards and NCTM Standards, so that the activities could be used at multiple grade levels.

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A RESEARCH-BASED MODEL FOR ANALYZING THE QUALITY OF ONLINE ASYNCHRONOUS MATHEMATICAL DISCOURSE

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The purpose of this paper is to describe a model for analyzing the quality of mathematical discourse in online asynchronous discussions. Using the constant comparative methodology for developing theory, the analysis revealed five hierarchical stages for coding individual messages of the mathematical discourse: participation, contribution, deliberation, justification, and expansion. The paper also discusses implications of the findings and recommendations for research.

Introduction and Theoretical Framework

Student participation in mathematical discourse is a foundational idea of the *Principles and Standards for School Mathematics* (NCTM, 2000). As students propose methods for solving tasks, question each other's ideas, and make conjectures based on investigations, they are engaged in processes that facilitate their mathematical thinking and learning. Research demonstrates that although teachers play a key role in the discourse process as they learn when to intervene, students must take on more responsibility for contributing to the discourse development (Cobb, Boufi, McClain, & Whitenack, 1997; Groves & Doig, 2004; Hufferd-Ackles, Fuson, & Sherin, 2004; Nathan & Knuth, 2003; Sherin, 2002; Simonsen & Banfield, 2006).

An online setting in which mathematical discourse is prevalent provides a natural venue for investigating mathematical discourse and learning more about students' role in the discourse process. In fact, in online courses taught using an asynchronous communication teaching strategy, students are virtually forced into engaging in some level of mathematical discourse as a course requirement (Simonsen & Banfield, 2006). The quality of individual students' contributions to the discourse, however, is not well documented.

Researchers have tried to make sense of asynchronous discourse threads in many ways including examining systematic construction of knowledge (Gunawardena, Lowe, & Anderson, 1997), construction of professional content knowledge (Bairral & Gimenez, 2002), and patterns in interactions that constitute classroom cultures (Sandovel, Lozano, & Trigueros, 2006). Moreover, research has shown that asynchronous discussions can enhance higher level communication (Jarvela & Hakkinen, 2002) and aid students in processing course information at fairly high cognitive levels (Hara, Bonk, & Angeli, 2000). Yet, little research exists on the quality of asynchronous mathematical discourse.

Gunawardena, et al. (1997) provided a model for examining the social co-construction of knowledge in computer conferencing. This model was developed for analyzing global online debate with a distinct emphasis on cognitive conflict. In the researcher's past experience in using the model to analyze mathematical discourse it was evident that a mathematics-based model would better capture the nuances of the discourse (Simonsen, Luebeck, & Bice, 2007). Thus, the primary purpose of this paper is to describe a research-based model for analyzing the quality of individual messages of the mathematical discourse in the context of online asynchronous discussions in a statistics course for teachers.

Methods

Setting

Using a completely asynchronous model of distance delivery, one of the researchers developed and implemented an online graduate level statistics course to meet the needs of place-bound mathematics and science teachers. During its annual offering over a six-year period, the course consistently had a maximum enrollment of 25 students. The class membership consisted of secondary mathematics teachers preparing to teach AP statistics or secondary science teachers interested in the design and analysis of data-driven classroom experiments.

Procedures

The statistics course instructional format was a completely asynchronous model of distance delivery using the WebCT course management system. The structure of the eight-week course consisted of a series of online, multi-element lessons organized in parallel with the AP statistics curriculum and the required textbook (Moore & McCabe, 2006). Students turned in assignments every Monday and Thursday, took weekly quizzes, and made at least three required weekly postings to the asynchronous discussions. The required asynchronous discussions between students and instructors served as the main instructional component of the course. Nine topics that corresponded to the online lessons provided the organization for the discussions: (1) graphical and numerical summaries, (2) design, (3) sampling, (4) sampling distribution of the mean, (5) confidence intervals, (6) hypothesis tests and t-tests, (7) least squares and regression, (8) comparing two means, and (9) ANOVA. Within each lesson, discussion postings were grouped as threaded discussions. Participants were required to monitor all threads. The course discussant also monitored the discussions while using the "withhold" discourse intervention strategy for online asynchronous discussions (see Simonsen & Banfield, 2006). The discussant was minimally responsive, and informed the students that information was being purposely withheld in order to let the discussion develop. Students also had the opportunity to work with each other or the instructor privately. These interactions, however, were quite limited given the course design. Students' required contributions to whole class discussions earned credit, but not their private e-mail discussions with the instructor.

Data Analysis and Model Development

The research database consisted of the archived asynchronous discussions from one offering of the course. Although 1018 discussion messages occurred during the course, messages not directly involved with one of the statistical content discussion areas (117) or those posted by the course discussant (68) were not used. The researchers analyzed 833 messages using the constant comparative methodology for developing theory (Bogdan & Biklen, 1992). Merriam's (1998) "category construction" provided assistance in giving structure to the stage development. The development began by sorting messages into categories while looking for key issues, recurrent events, and questions posed. Two graduate students (one in mathematics education and the other in statistics who was also the trained course discussant) assisted in initial category development by meeting weekly with the researcher who was also the course instructor. Continuous refinement of the stages generated two distinct but corresponding models: one for coding *statements* and another for coding student *questions*.

The two authors conducted the second phase of model development. After using the two models (statement and question) to code the first and largest set of discussion threads, the researchers combined the two models into one. The primary reason for this change was the realization that punctuation marks did not always appropriately identify whether a message was a statement or a question. Continued discussions prompted stage reconstruction and refinements.

The researchers used the individual message as the unit of analysis while recognizing the importance of the entire thread as a context for examining the messages. This decision was based on the recommendation of Garrison, Anderson, and Archer (2001) in that a single message represents a complete collection of one writer's thoughts. Though a single message may contain sentences representing various stages of the model, each message was associated with a single numeric code representing the highest stage reached for that message. The interrater reliability for the coding of the remaining eight sets of threads ranged from 75% to 88%, with an average of 80%. When disagreements occurred, researchers discussed the coding until reaching a consensus.

Results

The Model

The resulting model consisted of five hierarchical stages for coding individual messages within a given discussion thread: participation, contribution, deliberation, justification, and expansion. Multiple descriptive statements delineate each stage of the model (see Table 1).

Stage	Identity	Description
I	**Participation**	A. Statements of thanks, affirmation or agreement without elaboration. B. Simply restating the question or statement. C. Assignment clarification. D. Asking a knowledge question (list, tell, locate, name…) E. Answering a knowledge question incorrectly (no justification given).
II	**Contribution**	A. Adding to the discussion (including restating the idea in their own words, correcting an error, references to text, websites, calculator/technology use (what technology to use, not the helpful procedure of how to use it), attachments, data sets, real-world idea and extraneous variables) without elaboration. B. Providing an initial answer to a homework problem without elaboration. C. Contributing classroom or philosophical statements that may not be directly related to the discussion of the concept. D. Answering a knowledge question but no justification given. E. Asking a comprehension or application question (explains, interpret, discuss, solve, show, illustrate…) F. Answering a comprehension or application question incorrectly (no justification given).

III	**Deliberation**	A. Answering a comprehension or application question but no justification given. B. Identification of misconception but no justification given. C. Elaborating on a new description, definition, real-world idea, classroom or contextual example to help with the deliberation process but no thoughtful justification given. D. Attempts to provide justification to any question or concept but with obvious uncertainty. E. Attempts to provide justification to a knowledge, comprehension or application question or concept but is incorrect. F. Provide helpful description of a procedure (telling how) but not justification of the conceptual understanding of the procedure (not telling why) (including detailed calculator steps). G. Asking an analysis question (analyzes, compare/contrast, investigate, distinguish, examine…) H. Answering an analysis question incorrectly (no justification given).
IV	**Justification**	A. Thoughtful justification of their own understanding of the concept. B. Thoughtful justification of another's misunderstanding of the concept. C. Providing justification to a knowledge, comprehension, application, or analysis question. D. Asking a synthesis or evaluation question (create, invent compose, predict/infer, formulate, judge, debate, prioritize…). E. Answering a synthesis or evaluation question incorrectly.
V	**Expansion**	A. Adding a new contextual example and elaborating on it to enlighten the discussion. B. Examining the knowledge from a new perspective (including the classroom perspective) and elaborating on it to enlighten the discussion. C. Making connections between multiple concepts and elaborating on it to enlighten the discussion. D. Elaborating on a synthesis question and taking the discussion of the content to new level.

Table 1. Model for Coding Mathematical Discourse

Sample phrases provide further guidance for characterizing individual messages with respect to the model. In participation messages, students' inputs were more general in nature as they acknowledged others' contributions or asked for assignment clarifications. For example, "I agree with you. The answer is …" or "How did you get your answer?" In the contribution stage, students' messages contained helpful information but without elaboration or posed questions at a comprehension or application level. Examples included phrases such as "I found the example on page 77 helpful when trying to find the probability." or "How do we interpret the Bonferroni results?" At the deliberation stage, students shared their initial thoughts about the discussion topic but did not justify their thinking or were unsure how to proceed. For example, "This is how I was thinking about approaching this design question …" or "Some times I think that the data can be analyzed using the hypothesis tests and other times I think they cannot." Justification messages showed students explaining and defending their mathematical thinking. For example, "I think this is a stratified random sample since …" or "I understand confidence intervals in this way… because…" In the expansion stage, students identified and elaborated on new contexts and new perspectives. Examples included phrases such as "If we take the idea of confidence intervals one step further …" and "Let's look at simple random sample in another way."

Application of the Model

Researchers used the resulting model to analyze the individual messages within a mathematical discourse thread. An example of a thread with the respective coding follows. In this example, one student posted a message asking for assistance on an assigned homework problem in which she had to state whether the given percentages were statistics or parameters. The homework problem clearly stated that the Tennessee STAR experiment randomly assigned children into two different treatments, regular or small class, for high school. It then stated that 40.2% of blacks in small classes, and 31.7% of blacks in regular classes actually ended up taking the ACT or SAT college entrance exams.

Message no. 416 Posted by Ann Tuesday, June 28 3:19pm

Okay I have a question on 3.64. I understand how to determine whether it is a statistic or a parameter, that depends on whether it's a sample or a population. What I'm not sure about on this problem is whether it's a sample or a population. The word experiment kinda makes me think it's a sample, therefore they are statistics since these percents are from the students that were in the experiment. I just want to make sure I'm thinking correctly! Thanks!

STAGE III - Deliberation

Message no. 424 Posted by Beth Tuesday, June 28 6:09pm

Now you have me thinking! I had one answer but because of your question I am changing it! Another way to think of parameter vs. statistic is that a parameter is "a number that describes a population" and a statistic is "a number computed from the data." p.240.

So I think 40.2% was computed, here's my thinking. There was data on who took the ACT or SAT tests and this data was broken down by what type of class they were in (small classes or regular classes). So 40.2% and 31.7% were calculated from this data. What do you think?

STAGE IV - Justification

Message no. 429 Posted by Curt Tuesday, June 28 10:32pm

Using the fact that a statistic refers to a sample (parameter-- whole population) I agree (I think) that the two percentages are referring to the samples.

STAGE II - Contribution

Message no. 436 Posted by Dawn Wednesday, June 29 8:59am

I looked at the data as talking only about the students in the STAR program - the 40.2% and 31.7% are from just that group of students and not the population as a whole. So I think these are statistics.

STAGE III - Deliberation

Message no. 448 Posted by Eve Wednesday, June 29 6:12pm

A couple thoughts on this question...

1. I don't think that the ability to compute a number dictates whether that number a statistic or a parameter. Although it's unusual to have enough data to compute an exact parameter, it's possible. For example, in problem 3.63, 56% of registered voters had voted. This percentage is a parameter because it refers to the entire population (registered voters). It was computed because they had data on every single registered voter. However, when I look at the full definition that the book gives of parameter, it says that "in practice we do not know its value."
2. I don't think the problem gives us enough information to determine whether these percentages are statistics or parameters. The key (as someone else mentioned) is what the population is. If researchers are only focusing on the children in the STAR program, then they are the entire population and both of the figures would be parameters. If researchers are focusing on all school children, then the children in the STAR program are only a sample of the entire population, and both of the figures would be statistics.
3. I have a general frustration with this type of question -- it seems to me that everyone who has posted in this discussion has a good understanding of the difference between a sample and a population, and the difference between a statistic and a parameter. However, the lack of clarity of the question makes me feel like we just need to flip a coin to see what the text writer wanted for a correct answer.

STAGE V - Expansion

Once familiar with the model, the coding of the individual messages was often straightforward. For example, Ann's message was clearly an example of initial thoughts in a deliberation process. Also, most Stage V messages were not difficult to detect. Though not difficult to detect Stage V messages required more discussion than other level messages simply to make sure that the contents warranted the high ranking. Eve's message contribution was an example of expansion as she provided a new example and elaborated on it to enlighten the discussion. Some messages required more effort to understand the context of the writer's thoughts or the content of the discussion thread's preceding messages. For example, if taken in isolation Curt's message may be considered deliberation; however, his response was simply a restatement of the ideas in his own words. Perhaps even more challenging was Dawn's message. It was evident that Dawn justified that the STAR program was a sample; therefore, the percentages were statistics. However, her justification was more of an elaboration of Ann's ideas and did not constitute thoughtful justification.

Discussion and Directions for Future Research

Ultimately the goal of this research is to evaluate the quality of mathematical discourse. With respect to asynchronous communication mathematical discourse corresponds to discussion threads. The resulting model was designed to analyze the quality of individual messages within a given discussion thread. Further investigation is needed to determine the relationship between the coded messages and the quality of the discussion thread. Does a discussion thread containing an expansion (Stage V) message represent quality mathematical discourse? If students' justification of ideas is the goal, then perhaps threads containing justification (Stage IV) messages indicates quality mathematical discourse? What is the role of deliberation (Stage III) messages? Perhaps quality mathematical discourse is related to the number of distinct participants posting deliberation (Stage III) messages or above?

The ability to document quality mathematical discourse in a discussion thread has direct implications for online teaching practices. Do discussion threads based on homework problems or group projects tend to prompt higher levels of mathematical discourse? Will discussion threads that begin with a deliberation (Stage III) message engage a higher percentage of participants? What does it mean if the majority of the discussion threads are of low quality? Answers to these types of questions will challenge instructors to decide how best to create an online course designed to facilitate mathematical thinking and learning.

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CHARACTERIZING ELEMENTARY SCHOOL STUDENTS' EXPLANATIONS OF THEIR THINKING

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The purpose of this study is to characterize elementary school students' explanations for their solutions to whole number computation problems. The analysis of 4th and 5th graders' work showed that students were more effective at describing their solutions, but were challenged to provide both complete descriptions and justifications as they struggled with the roles of representations and taken-as-shared knowledge.

Theoretical Framework

Research has shown that establishing a classroom environment where students are expected to communicate their mathematical thinking by describing their ideas, listening to each other, questioning each other's ideas, and providing reasonable arguments to convince each other helps students see that mathematics makes sense. It further helps students develop the ability to reason by justifying their thinking (e.g., Ball & Bass, 2003; Lampert, 2001; Martino & Maher, 1999; Wood, 1999; Yackel & Cobb, 1996). Less is known, however, about how the thinking and sharing that takes place in these kinds of discussions transfers to contexts where students must explain their thinking in writing.

In considering the ways students explain their thinking, it is our contention that there is an important distinction to be made between describing one's thinking versus justifying it. We define a *description* as relaying the steps of a procedure one follows. For example, to solve 45 – 17 = ___, one might use a strategy that breaks apart the subtrahend in order to do the subtraction in smaller parts. A description of that strategy might be: "I first subtracted 15 from 45, which gave me 30. Then I subtracted 2 from 30 and I got 28." We consider a *justification*, on the other hand, as relaying reasoning about why a procedure is mathematically legitimate. For the strategy above, a reasonable justification might be: "This works because for the original problem I needed to take 17 away from 45, but instead I started by taking 15 away from 45. Then I needed to take 2 more away. As long as I end up taking 17 away altogether from 45, it doesn't matter in how many parts I do it."

Research has shown that when discussing solutions it is more common for teachers to ask students to describe their thinking rather than explain it (Fraivillig, Murphy, & Fuson, 1999; Grant & Kline, 2002, 2004), and one would expect students' written work to reflect this emphasis. In this study we explore the written work generated by students experiencing a curriculum that focuses on reasoning and explanations of thinking in order to analyze the nature of the explanations students produce.

Methodology

The data for this study were gathered from one elementary school building as part of a larger study on the impact of implementing a new curriculum, *Investigations in Number, Data, and Space (Investigations) (TERC, 1998),* on both teachers and students. Classroom practice and teachers' use of the curriculum materials during a pilot phase and the first year of implementation were the foci of the larger study. The data included videotaped classroom observations and interviews with teachers. Student work was also collected to study student progress and inform discussions about implementation.

In *Investigations* there are typically two main formal assessments in each unit along with a collection of more informal checkpoints. For these informal checkpoints, students work either with their peers or individually while the teacher observes, listens, and documents information on the students' performance. The two formal assessments are embedded in the curriculum materials as part of the students' daily tasks. For these assessments, students are typically expected to work independently on the task and are prompted to show their solution methods using representations and/or explain their thinking in writing. For this study, we chose to utilize the formal embedded assessments in the curriculum to analyze student explanations rather than develop additional assessments in order to collect authentic data on the impact of the curriculum.

We collected assessments from all students in grades 1 - 5, approximately 230 students. One assessment task was chosen from the number units taught in the beginning of the year, and one assessment task was chosen from the number units taught towards the end of the year. The focus of this paper is on the results of the fourth and fifth grade student work, where expectations were higher for both descriptions and justifications. The assessment tasks for fourth and fifth grades are shown in Table 1. Both of the assessment tasks for the fourth graders involved solving a multiplication problem in two different ways. The first assessment task for the fifth graders was about finding factor pairs of a number and the second assessment included a multiplication and a division problem. All of the tasks required students to write about how they solved the problems and to explain their thinking.

	First Number Unit	**Second Number Unit**
Fourth Grade	Solve 27 × 4 in two different ways. After each way, write about how you did it. Be sure to include: what materials you used to solve this problem; how you solved it; and an explanation of your thinking as you solved it.	Solve 34 × 12 in two ways. Write about how you solved it. Be sure to include: what materials you used to solve this problem; how you solved it; and an explanation of your thinking as you solved it.
Fifth Grade	Find some factor pairs of 1100, and write about how you found each factor pair.	Solve the following problems and write about how you solved them (record each step of your strategy) 42 × 51, 374 ÷ 12

Table 1. Assessment Tasks for Fourth and Fifth Graders

The analysis of the explanations was an iterative process. Initially, the student work was categorized according to their descriptions and grouped into three categories: explicit, valid, and complete description of the strategy; valid description of a part of the strategy; and incorrect, vague, or not matching the strategy. Students' justifications were analyzed by following a similar method, and the following features--meaning of the operation, reasoning behind the steps of the strategy, and connections between the steps and the original problem--were searched for to identify partial and full justifications. Finally, the justifications were analyzed to identify missing components in order to ascertain issues relevant to the knowledge that may have been taken-as-shared by the students.

Results and Discussion

The analysis of the student work indicated that the majority of the students communicated their thinking about solving computation problems by using representations, such as numbers, number sentences, and drawings. Both describing and justifying were challenging for students, but as shown in Table 2, they were better at describing than justifying their thinking. There were 324 responses in total as some of the tasks required two separate solution methods. Of those responses, 124 included full or partial descriptions of their work, while 49 included partial justifications as well. Of the 69 responses that included full descriptions, 30 of those also contained a partial justification. Of the 55 responses that had partial descriptions, 19 had partial justifications. There were no responses that included both full descriptions and full justifications.

	Partial Justification	No Justification
Full Description	30	39
Partial Description	19	36

Table 2. Number of Responses Including Descriptions & Justifications (324 responses total)

Relationship Between Descriptions and Justifications

Full Description and No Justification

The example shown in Figure 1 is indicative of those responses that included a full description of their work with no justification. This student clearly described the steps of her strategy. However, she did not include justification for the steps taken. She does not explain, for example, why putting 34 × 2 and 34 × 10 together makes 34 × 12. Therefore, this work was considered as having a full description without justification.

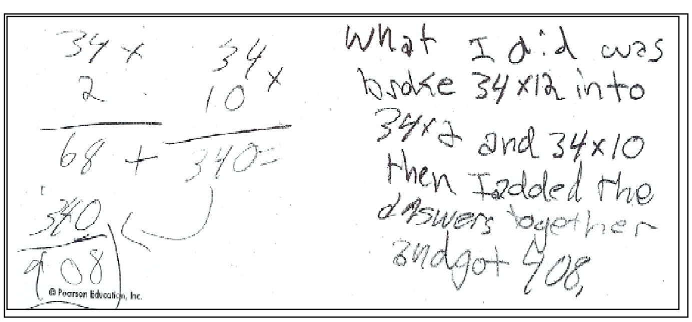

Figure 1. Full Description and No Justification

Partial Description and No Justification

Overall, 36 of the responses were considered as having a partial description with no justification and the example shown in Figure 2 is indicative of such responses. This student's representation suggests a utilization of the partitive meaning of division as the student interprets the divisor as a number of groups (as represented by the circles). The written explanation also provides some description about how the student solved the problem as she suggested that she drew 12 circles and she multiplied 30 by 12, presumably by placing 30 in each circle, which gave her 360. However, she did not describe the step of adding 12 and whether she essentially thought of placing one more in each circle, nor did she explicitly explain what R2 meant. Therefore, this response was considered a partial description, since it included a valid description of only a part of the strategy. In addition, the response was considered as having no justification, because it didn't address any of the three components

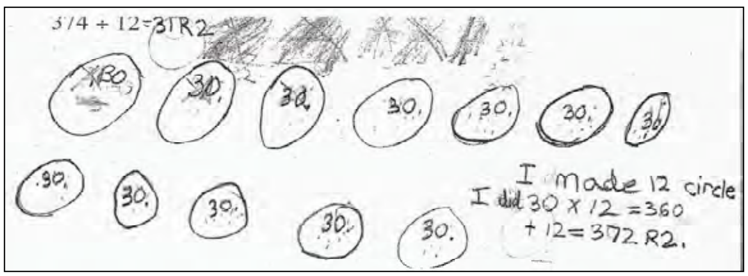

Figure 2. Partial Description and No Justification

Full Description and Partial Justification

Students who partially justified their strategies could usually demonstrate the meaning of the operation in their responses, but the reasoning behind the individual steps of their strategies and the connections between the steps and the original problem tended to be missing. For example, in Figure 3 the student shows work that involves solving $27 \times 4 =$ ___ by counting by fours 27 times. The student explains that he used the 100's chart and continued counting beyond the chart by adding two more groups of four. Since the student described each step taken in the strategy, this written work was considered as a full description. Also, the meaning of the original problem, that 27×4 represents 27 groups of four, was implied in the statement, "I counted four 27 times." Furthermore, his statement "I just added two more" implied that he added two more groups of four. However, neither the reason for why this strategy made sense nor the connections between the individual steps and the original problem were explicitly stated (e.g., including a statement about how there were 25 groups of 4 in 100, and that 2 more groups of 4, which is 8, would make 27 groups of 4). Therefore, the justification was considered as only a partial one.

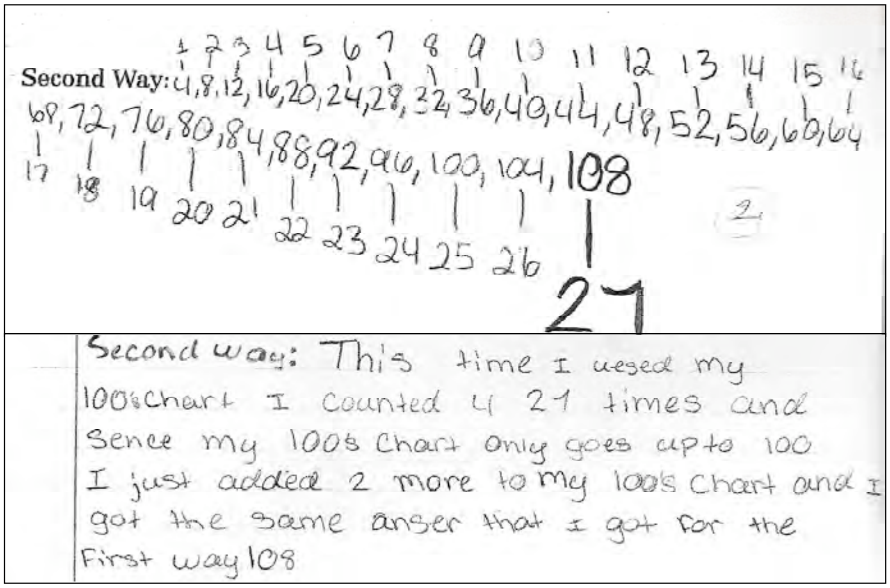

Figure 3. Full Description and Partial Justification

Partial Description and Partial Justification

In the next example shown in Figure 4, it is interesting to note that while the solution is incorrect, both the description and justification of the student's thinking provide important information. The description is only a partial one, because some, not all, of the steps of the strategy were explicitly stated. For example, drawing 12 circles, placing 10 in each circle, and repeating this 3 times was included in the text, but the description of some of the other steps was omitted, such as how the total came to 32 with a remainder of 1. Similarly, it is only a partial justification because there is no explicit discussion of the use of the partitive meaning of division, that is, sharing 374 things among 12 groups and finding out how many things each group receives. In addition, the statement "ten twelve times was not over 374" justifies why the student kept adding more cents into each circle. However, the connections among other steps and the connections between the steps and the problem were not justified explicitly.

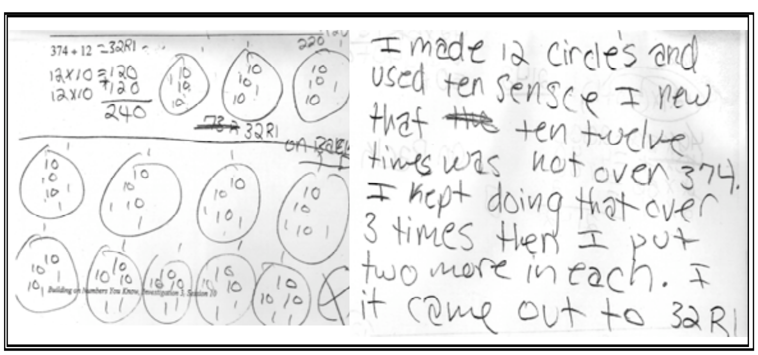

Figure 4. Partial Description and Partial Justification

Given that this was the students' first year using a curriculum that encouraged reasoning, it was not surprising that they were challenged to justify their thinking. Beyond this "experience" issue, two additional factors seemed to influence the nature of students' justifications. One is the assumption made about what can be counted as taken-as-shared knowledge. In other words, as students work on justifying solutions to computation problems over and over again, they and the teacher may either publicly agree or assume that certain ideas or steps in a strategy do not have to be justified any longer. A second factor may be the negotiation of what information is provided by one's representations versus words. We questioned which aspects of the students' explanations may have been present or missing as a result of these issues.

Using Basic Facts or Basic Calculations (Such as Multiplying by 10, Doubling, or Adding)

The majority of the students who provided partial justifications did not include written explanations for how they found the answers to the calculations they used in their strategies. The examples shown in Figures 1, 2, and 4 illustrate this issue. All three strategies included steps where students multiplied numbers by 10 (e.g., $12 \times 10 = 120$), doubled numbers (e.g., $34 \times 2 = 68$), and added numbers (e.g., $120 + 120 = 240$). However, none of those explanations included reasoning for these calculations. It might be the case that students already established this kind of knowledge as taken-as-shared in their classrooms and did not consider a justification for each calculation necessary. There were also cases where students used basic facts, such as $7 \times 4 = 28$, and none of the students provided a justification for how they knew that 7 times 4 was 28. Given the fact that these were fourth and fifth grade students, it might be reasonable that explanations for these steps were omitted from their written justifications.

In addition, on the first assessment task in fifth grade, where students found factor pairs for 1100, there were several students who included statements that may have been based upon taken-as-shared knowledge. Some examples included the following: "I found this factor (1×1100) because anything times 1 equals itself"; and "I split 1100 in half and got 2×550"; or "11×100 works, because 10 or 100 times anything just adds one or two zeros to the end." While these are valid statements, they are not justified. However, it is reasonable to think that these ideas may have been discussed in these classrooms and students may have believed that they did not need to provide justifications for those statements in their written work.

Connections Between the Steps of a Strategy and the Original Problem

Interestingly, none of the students provided a justification for how the steps of their strategies were related to the original problem they were solving. A significant example of this was not justifying why it makes sense to add partial products together to get the solution to a multiplication problem. While several students described how they found the partial products, and stated that they added these together at the end, none of them explained why it made sense to add them all. For example, in Figure 1, the student described how he solved $34 \times 12 =$ ___ by breaking 12 into 10 and 2, finding 34×2 and 34×10, and then adding those two quantities together. However, he did not justify why putting 34×2 and 34×10 together makes 34×12. This would require utilizing language about the meaning of multiplication, for example, that he was adding 2 groups of 34 to 10 groups of 34 to get 12 groups of 34 altogether, a seemingly important statement in justifying this strategy.

Negotiating the Role of Representations

It was most often the case that students used their pictorial representations as a way to describe their work, but did not often refer to them in justifying their strategies. For example, in Figure 2 where the student drew 12 circles and placed 30 in each circle to solve 374 divided by 12, she simply wrote "I did 30 x 12 = 360." She did not access that representation to explain why making 12 circles or why placing 30 in them made sense. It may be the case that students viewed these representations as an avenue towards solution, and connected them with any wording they would include to describe their solution, but did not necessarily view them as something that required justification. In other words, in the students' minds, the picture might just speak for itself.

Conclusions

In writing justifications for computational problems, depending on the grade level, it would be reasonable to establish norms for not including reasoning for each single calculation that takes place in a strategy. However, it does seem important to expect students to include reasoning about both the meaning of the operation and the connections between the steps of the strategies and the original problem in their justifications. These expectations need to be explicitly discussed with students by comparing and contrasting a variety of explanations one might provide for different strategies.

The fact that students were challenged to both describe and justify their work in this study is not surprising given the fact that they were in their first year of a curriculum that encouraged these kinds of explanations. While it was not part of our study to analyze classroom practice, it may also be the case that the nature of the discussions around student thinking may have been tentative as teachers were just learning ways to engage effectively with their students' ideas. With that said, the ways in which students struggled with their explanations provides important information for teachers and curriculum developers alike on what to be cognizant of when developing students' reasoning. Describing their work involved a negotiation between the use of representations and written text to explicitly communicate how they arrived at their solutions. And determining appropriate justifications involved knowing what to support and what to assume as "taken-as-shared" knowledge. These are difficult decisions for students to make and trying to make public what is private is incredibly demanding on both students and teachers. It is our contention that working on what makes both a satisfying description and justification of one's thinking is critical to realizing the full potential of developing students' understanding.

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HIGH SCHOOL TEACHERS' COGNITIVE SCHEMES SHOWN IN PROBLEM SOLVING APPROACHES BASED ON THE USE OF TECHNOLOGY

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This study documents the type of proof schemes that high school teachers developed and used in problem solving scenarios that involve the use of dynamic software (Cabri-Geometry). Research questions that helped organize and structure the development of the study include: (i) To what extent does the high school teachers' process shown to pose questions or formulate problems influence their ways to validate mathematical relations or conjectures? (ii) What types of problem solving strategies do the participants use to identify and support conjectures that emerge as a result of constructing and examining dynamic problem representations? Results indicate that the subjects' use of dynamic software to represent mathematical objects and situations dynamically not only favors their ways to formulate conjectures; but also the schemes' construction to support and validate those conjectures.

How does a mathematical relation emerge? What does it mean to prove or demonstrate a particular mathematical relation? What types of arguments are important to validate a mathematical conjecture? How visual, empirical, geometric, and analytic arguments are used to validate mathematical relations? To what extent do the systematic use of dynamic software favor or enhance a particular ways of reasoning and thinking about proofs' construction? The discussion of these types of questions sheds light on the complexity involved during the subject's construction of mathematical arguments and the relevance of problem solving approaches that promote the teachers and students' use of technology to foster both the formulation of relations and the search for arguments to support mathematical conjectures. Those problem-solving experiences involve the subject's direct participation in formulation of questions or problem posing activities, the development of problems solving strategies and the use of different artifacts, including computational or digital tools, to represent and explore mathematical ideas or problems.

It is common to associate the term "mathematical proof" to the development and presentation of deductive arguments, based on a set of propositions, to support results or mathematical relations; however, the process of proving involves more than only the use of logic or formal arguments; it includes for the subject to be convince himself/herself initially and to convince others about the viability and validity of the conjecture or mathematical relation to be proved (Harel & Sowder, 1998). What does it mean for the subject to be convinced about the validity of a particular mathematical relation? We argue that it means the opportunity for the subject to explore conjectures or mathematical relations in terms of visual and empirical explications that often rely on measuring figures or attributes (areas, perimeters, lengths, etc.) and moving objects and observing patterns of particular behaviors. Since the use of technology seems to facility the representation and exploration of mathematical situations, then it is important to investigate the extent to which the use of particular tools helps teachers and students develop ways of reasoning that favor the use of distinct arguments to validate and prove mathematical conjectures or results.

Thus, in this study, we are interested in documenting and analyzing the process exhibited by high school teachers to construct dynamic representations of situations that lead them to formulate and examine conjectures and ways to support them. The research questions used to guide and structure the development of the study were:

1. To what extent does the high school teachers' processes shown to pose or formulate questions influence their ways to validate mathematical relations or conjectures? Here, there is interest to document how the participants' construction of dynamic representations of situations helped them to initially pose questions that eventually led them to identify and explore mathematical conjectures. Similarly, we focused on analyzing ways in which the participants look for arguments to support those conjectures. In particular, we identify and discuss the proof schemes that emerged through the participants' use of dynamic software.
2. What types of problem solving strategies do the participants use to identify and support conjectures that emerge as a result of constructing and examining dynamic problem representations? Here, we focused on documenting the types of problem solving strategies (examining particular cases, looking for patterns, using coordinate system, and finding objects' loci) used to solve problems and construct arguments and proofs.

Conceptual Framework

An important feature in the process of learning mathematics is the construction of a line of thinking in which the learners have the opportunity of using their previous knowledge to identify mathematical relations and to provide arguments to support results. Harel and Sowder (1998) distinguish two related aspects that are relevant during the subjects' construction of proofs or arguments to justify conjectures: The subject self-convincement stage in which he/she is convinced that the conjecture is valid and make sense to him/her; and the need to persuade others about the validity of that conjecture. That is, one is an individual recognition and the other a community acceptance. Harel and Sowder (1998) also identify seven types of sub-categories of proof schemes: (a) ritual proof scheme in which the subject's convincement is based on accepting the form rather than the content or argument; (b) authoritarian proof scheme in which the subject's convincement is based on arguments or affirmations presented by an authority (teacher, textbook, or expert); (c) symbolic proof scheme in which conviction is based on symbolic manipulations without explicit explanation of the meaning attached to those manipulations; (d) perceptual proof scheme in which conviction is based on using rudimentary mental images that lack actions to anticipate results; (e) inductive proof scheme in which conviction is achieved through the use of quantitative evaluations; (f) transformational proof scheme in which the subject relies on goal-oriented operations on objects to anticipated results; and (g) axiomatic proof scheme which is also a transformational proof that relies on the use of axioms and established definitions.

Thus, each proof scheme becomes relevant to explain the cognitive process embedded in both the subject's own convincement about the validity of mathematical relations and the subject process to convince others about the pertinence, meaning, and proof of that conjecture.

We also argue that the cognitive process involved during the construction of mathematical arguments to support relations can be traced or explained in terms of the subject's ways to formulate and pursue significant questions (Santos-Trigo, et al., in press). Thus, problem solving approaches that encourage students/learners to formulate, examine, and support conjectures might help them value the use of distinct types of arguments to justify results and conjectures. Santos-Trigo (2007) illustrates ways in which high school teachers and students can transform typical textbook problems into nonroutine problems when they construct various representations (including dynamic representations) of those problems and look for distinct ways to approach them. In particular, the use of computational tools (dynamic software for example) seems to offer proper conditions for the learners to pose and examine questions that lead them to formulate and later support conjectures. In this context, we are interested in documenting the extent to which the categories identified by Harel and Sowder (1998) can be used to explain the subjects' construction of arguments within a problem solving environment that promote the use of dynamic software.

Participants, Design, and Procedures

Seven high school teachers participated in a weekly 1.5 hr problem solving sessions during one semester. However, we focus on the work shown by three of those participants because their approaches to the tasks are representative of the group's work. The aim of the sessions was to work on a series of tasks that involves the construction of geometric configuration, using Cabri-Geometry software to identify and support mathematical relations or conjectures. In general, the pedagogic approach that consistently characterized the development of the sessions included:

1. The responsible or coordinator of the sessions introduced a task to the participants and explains to them ways to work and report their work.
2. The participants worked on each problem individually and later they had opportunity to present and examine their work within the group.
3. At the end, each participant handed in a report that included electronic files and written comments and observations that appeared during their individual and collective participation.

To analyze what the three participants showed during their problem solving approaches, we focus on those tasks that involve the construction of geometric configurations that were used to identify and discuss mathematical relations. The initial tasks and instructions that the participants received to construct those dynamic configurations included:

1. Given a line and a point that does not belong to that line, construct an isosceles triangle with one side lying on the line and the third vertex the given point that is not on that line.
2. Draw a square given one of its vertices, and the middle point of one of the side of the square that is not adjacent to the given vertex.
3. Draw a tangent circle to two given circles.

Data used to analyze the participants' approaches to the tasks come from electronic files, written reports, and field notes taken by the sessions' coordinator during the problem sessions. The first goal was to analyze the extent to which the proof schemes identified by Harel and Sowder (1998) consistently appear in the participants' performances. In addition, the identification of problem solving episodes (Schoenfeld, 1985) became important to identify the type of strategies used to identify, construct, and support mathematical relations.

Presentation of Results and Discussion

There is evidence that the use of the software became important for the participants to initially identify key elements which they used to construct a dynamic representation of the tasks. Thus, dragging particular points or objects within the representation was an important activity that helped them detect invariants or conjectures. For example, Ann approached the first task (isosceles triangle) by drawing a line l and point C out of that line. She chose point P on line l and drew a circle with center at point C and radius the segment CP. Thus the triangle PCQ is isosceles by construction. In this case, Ann observed that she could draw a family of isosceles triangles when point P is moved along the line l and asked at what position of P the triangle PCQ becomes equilateral? (Task 1.1) (Figure1a). Hugh drew line l and point Q on that line and a circle with center at point Q and radius QC (C is not on line l). Then he drew the perpendicular bisector of segment QC, located point S and asked: What is the locus of the point S when point Q is moved along the line? (Figure1b). He observed that the locus of point S was a line and verified this assertion by selecting two points on the locus and drawing a line passing by those points and observed that it overlaps the locus.

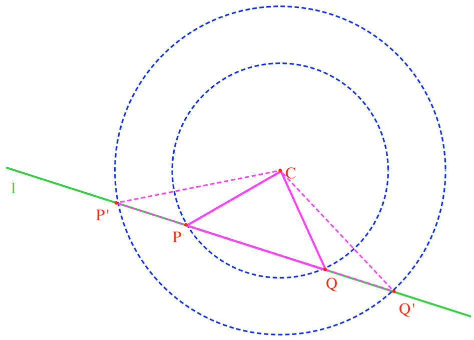

Figure 1a: At what position of point P, triangle PCQ becomes equilateral?

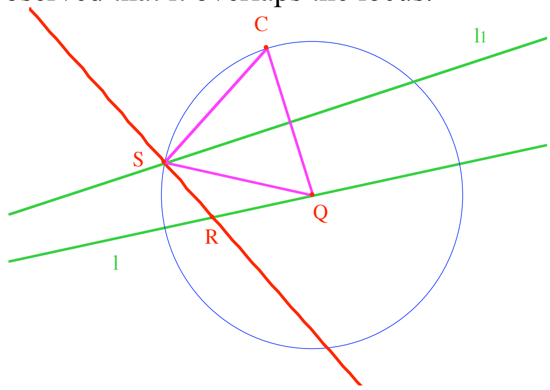

Figure 1b. What is the locus of the point S when point Q is moved along line l?

Hugh noticed that the locus intersects line l at point R (Figure 1b) and then he drew segment CR and a circle with center at point C and radius CR, and located point T to construct the equilateral triangle. He used the software to measure the angles in order to verify the measure of each interior angle was 60 degrees (Figure 2).

What types of proof schemes (following Harel and Sowder, 1998) did the participants utilize to convince initially themselves and later to convince others about the pertinence and validity of their results? Table 1 shows a summary of the type of proof schemes used by the three participants.

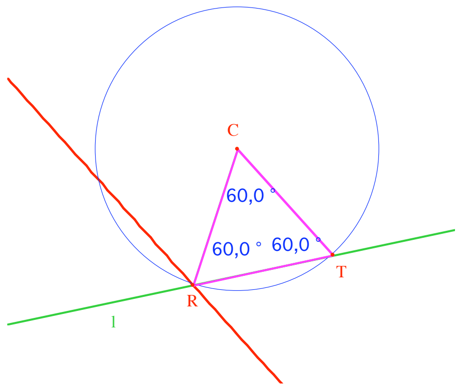

Figure 2. Equilateral triangle.

	Task			
Participant	1	1.1	2	3
Ann	d, g	d	N	d, f
Mary	e, g	e, g	d, e	d
Hugh	e, g	d, e	d, f	d

Type of proof scheme: (a) ritual, (b) authoritarian, (c) symbolic (d) perceptual, (e) inductive, (f) transformational, (g) axiomatic, (N) problem not solved.

Table1: Proof schemes used by the participants

Mary constructed an equilateral triangle by drawing a line L, and a perpendicular to L passing by point C. This perpendicular line cuts line L at M. Then se drew line l_2 and points D_1, D_2 and D_3 such as $MD_1 = D_1D_2 = D_2D_3$. Then drew segment CD_3 and parallel lines to this segment passing by points D_2 and D_1. The latter intersects line MC at T. She drew a circle with center point T and radius TC. This circle intersects line L at points A and B. Here she stated that triangle ABC was equilateral. How did Mary convince herself that the triangle she had constructed was equilateral? Mary, as the other participants, used the software initially to measure the angles in order to verify if they measured each 60 degrees (Figure 3a). When the participants exchanged ideas and discussed their approaches with the whole group, they recognized the importance of providing other type of evidence to show that, in this case that the triangle was equilateral. For example, Mary at the end of the session in her report wrote: CM is perpendicular to L (by construction) and T divides segment CM into a ration 2:1. Let h be equal to TM, then CT = AT = BT = 2h, this is because CT, AT and BT are radii of the same circle (Figure 3b). Triangle AMT is right triangle, therefore, MA = $\sqrt{3}h$; similarly CA = $2\sqrt{3}h$, and CB = $2\sqrt{3}h$. As a consequence triangle ABC is equilateral. A key idea used in Mary's report is the identification of point T (center of the circle). Her construction was based on assuming the existence of the equilateral triangle and to identify its relevant properties. That is, she used the properties to guide her construction.

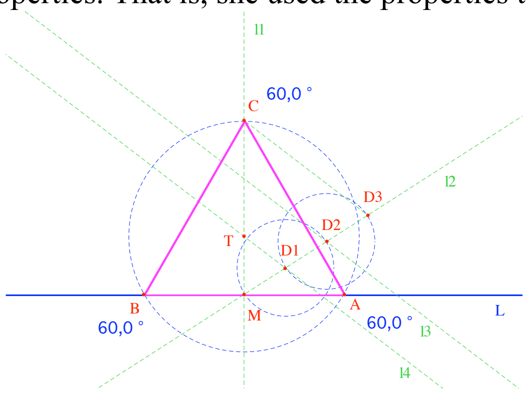

Figure 3a. Constructing an equilateral triangle.

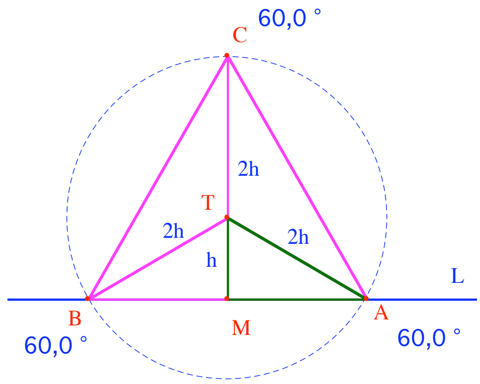

Figure 3b. Providing an argument to validate the construction.

Hugh observed that the locus of the perpendicular bisector when point Q was moved along line l (Figure 1b) seemed to be a parabola. His first strategy to convince himself that the locus was a parabola was to use the software command (conic) to visualize if five points on that locus determined that conic (parabola) (Figure 4a). At this stage, he was convinced that the locus was a parabola; but he was aware that it was important to think of other types of arguments. Later, he chose point P on the locus and drew a perpendicular line to l that passes through point P. This perpendicular line intersects line l at point R. Then, with the use of the software he measured distances PR and PC and observed that for different position of point P the distances were the same (Figure 4b). Here, Hugh assumed that line l and point C were the directrix and focus of the parabola respectively and used the software to verify the definition of this conic.

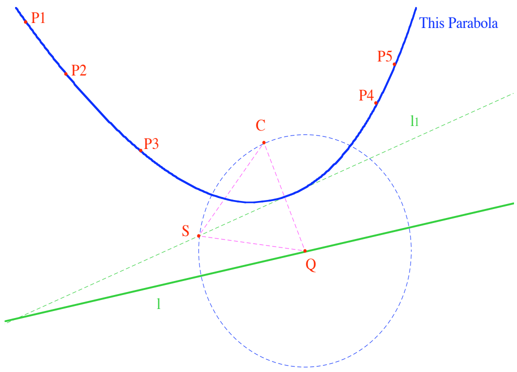

Figure 4a. Perceptual scheme to validate the existence of a parabola.

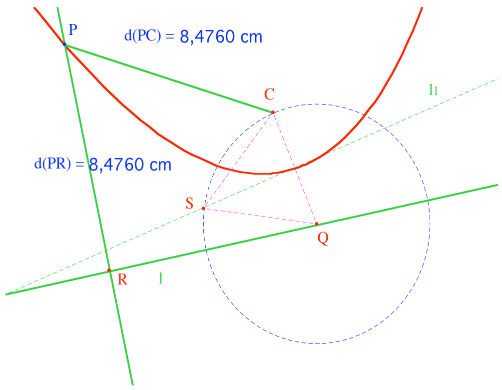

Figure 4b. Empirical validation of the definition of a parabola.

The use of the software also allowed the participants to display transformational proof schemes. For example, Ann approached the task that involved the construction of a tangent circle to two given circles by initially focusing on a partial solution. That is, given the circles with centers at point A and B respectively, she situated points R and Q on each circle. Then she drew lines AR and BQ and observed that for certain positions of these points the lines get intersected at point C. She drew a circle with center at point C and radius CR. This circle is tangent to circle with center at point A (Figure 5a) (partial solution). Ann noticed visually that when point R is moved along the circle there was a point on circle with center B at which the circle with center C is tangent to both circles. To justify this construction, Ann argued: For certain positions of point R the circle with center C does not intersect the circle with center B while for other positions of point R the circle intersect that circle at two points (Figure 5b), then there should be a position for R in which the circle intersects the other at only one point. That is, there must be a position for point R in which the circle with center at C is also tangent to the circle with center at point B.

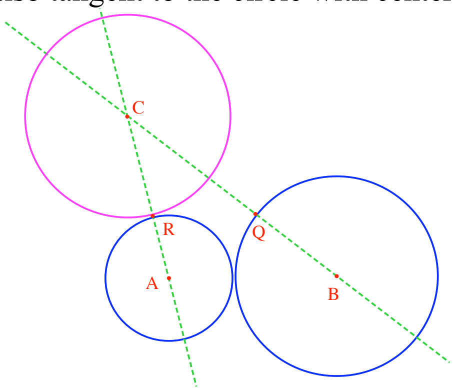

Figure 5a. A partial solution to the problem.

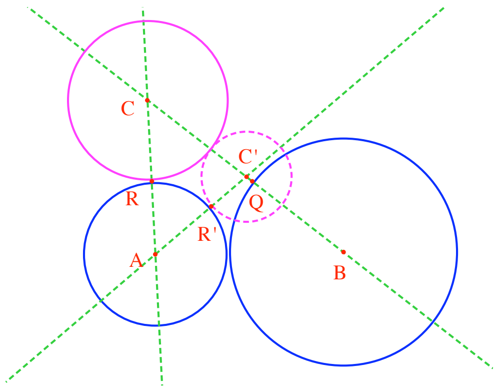

Figure 5b. Dragging point R around circle with center A to find the solution.

Another example of the appearance of a transformational scheme is shown in Hugh's approach to the construction of the square. He constructed a family of rectangles holding the condition that P was one vertex and Q the middle point of the opposite side (Figure 6a). When point N is move along the circle, he observed that one element of that family of rectangles represented the solution of the problem. To justify his method to identify the square, Hugh argued that when point N is moved along the circle, the family of the generated rectangles holds initially that the length of segment PN is less than the length of segment PR

(Figure 6a); however, for certain positions of point N it appears that the length of segment PN is greater than the length of segment PR (Figure 6b). Therefore, there should be the case in which both lengths are equals.

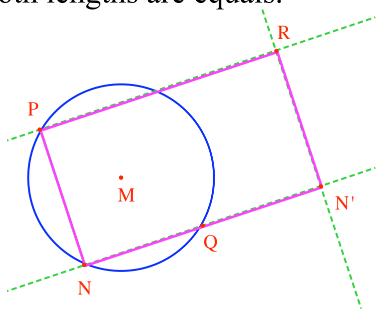

Figure 6a. M is the middle point of segment PQ and N' symmetric to N with respect to Q.

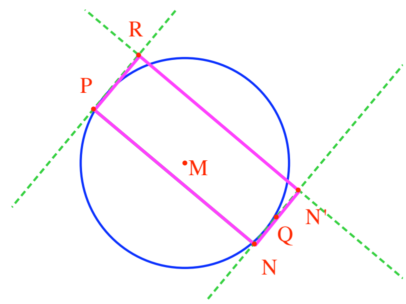

Figure 6b. The length of segment PN is greater than the length of segment PR.

Final Remarks

There is evidence that the use of the software helped the participants construct dynamic representations of mathematical objects that eventually became a source or departure point to formulate questions and problems. What relevant features characterize the participants' process to construct a dynamic representation of the situation? It was observed that the participants started to analyze the situation in terms of geometric properties and translated this information into the construction of objects that eventually could be moved and observed components' behaviors. For example, when Hugh situated point Q on a line and point S be part of a circle, he was aware that when moving point Q on line l, it was important to follow the path left by point S and the perpendicular bisector of segment CQ. Indeed, tracing those loci led them to construct the equilateral triangle and to identify the locus of the perpendicular bisector as a parabola. At this stage, Hugh directed his attention to finding distinct types of arguments to support his finding. Again, the use of the tool was relevant to explore a quantitative approach (measuring distances and angles) to initially verify the properties of those loci.

Although some of the proof schemes identified by Harel and Sowder (1998) seemed to appear in the participants' problem solving approaches, there is evidence that with the use of the tool, the participants can move from visual, empirical and perceptual approaches to more formal or deductive schemes. In addition, the participants' process of posing problems helped them to initially be convinced that the problem or question and associated conjectures were relevant and needed to be explored or supported. As a consequence, it was natural to think of different ways to support their responses.

Finally, the use of the tool seems to enhance problem solving strategies that include (i) assuming the problem solved and then to identify properties to construct a dynamic representation; (ii) representing and solving the problem partially and then examining the representation by moving some elements within the representation to find the complete solution; and (iii) using the tool "locus" to observe the behavior of some elements of the representation to solve the problem or to formulate other questions or problems.

Acknowledgment

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TRACING MIDDLE-SCHOOL STUDENTS' CONSTRUCTION OF ARGUMENTS

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A group of twenty-four middle-school students from a low, socioeconomic community worked collaboratively on open-ended problems involving fractions during an informal after-school program. This paper describes the forms of reasoning that emerged as the students co-constructed arguments and provided justifications for their solutions.

Although the goal of building students' mathematical reasoning is a priority among educators and policy leaders, according to results disclosed by the Seventh Mathematics Assessment of the National Assessment of Educational Progress (NAEP), students have difficulty explaining and justifying their solutions in ways that are mathematically adequate (Arbaugh et al., 2004). Other researchers have found that participating in discussions about mathematical ideas leads to mathematical learning (Balacheff, 1991; Cobb et. al., 2001; Maher, et. al., 2006). The purpose of this research is to build a deeper understanding of how reasoning can develop in middle-school students by showing the variety of ways they reason in problem solving.

Theoretical Framework

Thompson (1996) defines mathematical reasoning as "purposeful inference, deduction, induction, and association in the areas of quantity and structure" (p.267). Yackel and Hanna (2003) extend this definition to recognize the social aspects of reasoning and describe it as a communal activity in which learners participate as they interact with one another to solve (resolve) mathematical problems. They stress that in a supportive environment, all students, as early as elementary school, have the potential to make and refute claims and participate in inductive and deductive reasoning. However, Yackel and Hanna (2003) also emphasize that we are only beginning to understand how students' mathematical reasoning develops and what environments can support this development. Our study addresses these issues. We placed students in an environment where collaboration and justification were encouraged and documented the development of their reasoning over time. In particular, the question that guided our work is, in developing a community of learners: *What forms of reasoning emerge as students provide justifications for their solutions?*

Method of Inquiry and Data Source

This research is a component of a larger, ongoing longitudinal study, Informal Mathematics Learning Project (IML), conducted through an after-school partnership between a University and school district that is an economically depressed, urban area and whose school population consists of 98 percent African American and Latino students. This report focuses on the development of reasoning of middle-school students. We report on the first cohort of students, 24 sixth-graders, who, over five, 60-75 minute sessions, worked on fraction tasks, interacted with peers, and had ample time to explore, discuss and explain their ideas. Students had available

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.

Cuisenaire rods to work on their tasks; they were invited to collaborate, and they were encouraged to justify and make sense of their solutions

Video recordings and transcripts were analyzed using the analytical model outlined by Powell, Francisco & Maher (2003). The video data were described at frequent intervals; critical events (episodes of reasoning) were identified and transcribed, codes were developed for flagging for solutions offered by students and the justifications given to support these solutions. Arguments and justifications were coded according to the form of reasoning being used (direct; by contradiction; by cases; using upper and lower bounds) and as valid or invalid (did the argument start with appropriate premises and was each deduction within the argument a valid consequence of previous assertions).

Results

Examining the data across sessions, we found: (1) there were instances in which direct reasoning, reasoning by contradiction, upper and lower bounds, and case-based reasoning were used correctly by students in our study; (2) for many tasks, multiple arguments were presented correctly using different forms of reasoning; (3) students challenged one another's arguments that began with inappropriate premises, when the method used to draw conclusions was not sound, and when the conclusions that students drew led to contradictions; (4) challenges sometimes led the community to tighten or refine previous incomplete arguments; (5) the frequency in which students questioned or challenged each others' arguments increased over time.; and (6) all of the students whom we focused on in our analysis actively participated in this process of argumentation. Each of these points is discussed below.

Forms of Reasoning

In Session 2, we see an example of direct reasoning in an episode where a student poses the following challenge*:* What is the number name for red (when blue is named one?

Dante: I said that two whites equals up to the red and then there's one white would be one-ninth and two whites would be two-ninths.

Brittany: I think it's two-ninths because um three, I mean one red and one white equals green and if you take away um and um if you take away that one you're still going to have red and two of the whites make one red.

In the same session, we observe reasoning by contradiction in which R1 poses the challenge: If I call the blue rod one, I want each of you to find me a rod that would have the number name one-half. In response, Chris explains that there is not a rod that is half of the blue rod because, "*There's nine little white rods you can't really divide that into a half so you can't really divide by two because you get a decimal or remainder so there is really no half, no half of blue because of the white rods*".

Reasoning by cases is also observed in the same task in which Justina explains: "*I was just making half of the color rods, I just made this picture, so like um, half of the orange was yellow, half of the brown was purple, half of dark green was light green, and the same for those two.*" At the overhead, Justina presented her diagram and drew all of the rods with halves along side (e.g., two yellow rods lined up next to an orange rod, and said that she just wanted to indicate the set of rods for which there were halves in the set.

For arguments with upper and lower bounds, we present an example by Dante who explained that the purple rod and the yellow rod are not halves of the blue rod. Dante said, "I*t's not half, it's just equal, it's not half because yellow is bigger than the purple*". He held up the yellow and purple rod to show this. R1 asked him how much bigger the yellow rod was and Dante replied, "*by one white, one white piece*." Dante continued, "*We tried all we can because if usually for the blue piece, it would usually be purple or yellow but yellow would be one um one white piece over it and the pink would be, I mean purple would be one white piece under it*".

Multiple Arguments

For the problem: *If I call the blue rod one, I want each of you to find me a rod that would have the number name one-half,* eight students produced arguments that represented the four forms of reasoning. Three of these arguments are highlighted below.

Reasoning by Contradiction: Chris explained that there is not a rod that is half of the blue rod because, "There's nine little white rods you can't really divide that into a half so you can't really divide by two because you get a decimal or remainder so there is really no half, no half of blue because of the white rods".

Reasoning by Cases: At the overhead, Justina presented her diagram. She drew all of the rods that have a half next to the two rods that make up the half (two yellow rods lined up next to an orange rod, two purple rods lined up next to a brown rod, two light green rods lined up next to a dark green rod and two red rods lined up next to a purple rod). She explained, "*I was just making half of the color rods, I just made this picture, so like um, half of the orange was yellow, half of the brown was purple, half of dark green was light green, and the same for those two*".

Reasoning by Upper and Lower Bounds: Chanel stated that the blue rod does not have a half "*because it needs one more white rod*". She displayed a model of the blue rod lined up next to nine white rods and a model of the blue lined up next to a purple rod and a yellow rod. When asked how she knew that the length of the yellow rod is not half of the length of the blue rod, she stated, "*Because blue, this is blue and the yellow is a little, is, the yellow is a little bit more than a half and the purple is shorter than a half"* and showed a model of the two rods next to the blue rod.

Students Challenging Each Other

For the problem of finding a rod whose length was equivalent to half of the blue rod, Michael and Shirley identified the purple and yellow rods as being equivalent to half of the blue rod. In response, Dante and Chanel explained why they were not correct.

Michael	Purple and yellow
Dante	No, yellow is not
Michael	I said purple and yellow
Chanel	No [she has lined up two yellow next to the blue]
R1	Chanel doesn't agree with you, why not Chanel?
Chanel	Because if you put two purple together its still smaller than the other, than the blue [she built a model of two purples lined up next to blue and points to the space that is remaining]
Michael	Purple and yellow [he has a model of a purple and yellow lined up next to the blue]

Dante	I know, I know, there is none
Shirelle	Yellow and purple can make a half [she has a model of a yellow and purple lined up next to the blue]
Dante	But they're not the same length
Michael	That what I been saying yellow and purple, yellow and purple, yellow and purple, but y'all won't listen
Dante	Yeah but um there not really actually halves
R3	Why?
Dante	Because the yellow is bigger so it's uneven
R3	So it would be uneven
Dante	So there's no half
Chanel	And the yellow takes up more space than the purple and to be halves they should be the same

During session 5, students were asked to name the white rod when the orange rod was named one. In response, Herman named the white rod ten. Dante corrected him and named the white rod one-tenth, comparing the task to the previous challenge of naming the white rod when the blue rod was named one. Herman disagreed with Dante, and R1 asked Dante to explain his argument again. He used Herman's model of ten white rods lined up next to the orange rod and explained that if he took ten rods away, he would be left with one-tenth. He continued to add the rods back on, counting by tenths, up to ten-tenths.

Students were asked to find a number name for the red rod when the orange rod was named one. Chanel named the red rod one-half and R1 asked the class what they thought about this. Dante replied, "*I disagree because, because the red, the red shouldn't be a half because the yellow is a half of orange, so how could red be a half of orange if it takes five of them instead of two. So that's why I disagree*".

Challenges Refining Incomplete Arguments

In Session 2 students attempted to solve the problem of finding a rod equivalent to half the length of the blue rod. The arguments produced by Michael, Shirelle, Dante, and Chanel illustrate this point. For example, Michael and Shirelle insist that the yellow and purple rods could be named one-half (when the blue rod was named one), Dante and Chanel used different types of reasoning to contradict their statements.

Another example is provided by Chris who built a model of nine white rods lined up next to the blue rod.

Chris	If you take out four that's an even number but if you put the four back … that's not a half because it's only nine and nine is an odd number.
Danielle	This one is even too. What did she say the blue one was?
Chris	One, so you can't find a half of the blue one because if you put all white you only have nine so for nine you can't really do it.

Brittany built a model of the following combinations of trains of rods lined up next to the blue rod: yellow and purple, red and black, light green and dark green, brown and white, and two purple and white.

Brittany	But I can explain it because I can say you can't do it because if you put a black and then a red you can't do it because they are not a half and they are not the same size
Chris	And also if you put the white ones you it's an odd number which is nine and you can't do it.

A student lined up two purple rods next to the blue rod and Jeffrey said that she needed a white rod.

Chris	See that's not a half so you can't use blue, it's an odd number
Danielle	So you can't use this one, one, two, three, so for the whole thing you can't use it because of the white ones, you can't really divide an odd number…..If you had like, say if you had two yellows that would equal the blue right? That would equal the blue that's one-half, no I know it wouldn't equal the blue I'm just saying if the yellow did go into half the blue.
Chris	Overall you can't do it because if you use a white one it is an odd number so you can't divide by two.
Jeffrey	Unless you get a decimal or a remainder.
Chris	And you wouldn't be able to do it anyway because none of these are even.

Jeffrey and Danielle then began to reason using upper and lower bounds using the purple rod as the lower bound and the yellow rod as the upper bound.

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Danielle	If only we had the two yellows but the yellows were shorter.
Jeffrey	If purple was bigger.
Danielle	Yeah if the purple was bigger, then the green was kind of shorter so that the same color green could fit on it.
Jeffrey	If the purple would have been a little, like half of the white, it would have been good.
Danielle	Or if the light green was um, it was, yeah like a little bigger, then you could only have one up there or two.
Jeffrey	Bigger than the purple one has to be like .
Danielle	But the purple has to be bigger, the purple has to be bigger equally because if you take another purple you're going to have to add a white, yeah you're going to have to add a white, see it don't work
Chris	The thing we should say is that since we put the white cubes and we got an odd number then if you have an odd number you can't divide by two so you get one-half so you get a decimal or a remainder so you can't really divide it, right? [He points to his model of nine white rods lined up next to the blue rod]
Jeffrey	Yeah

Chris I got it already, see you can't do it since you have one blue and you get nine white cubes so you can't do it because you can't divide an odd number into two, I mean by two

The Frequency of Challenging Each Other Increased Over Time

In the process of integrating their ideas with the ideas of others, students corrected each other. The occurrences of corrections increased gradually; in the first three sessions combined, there were seven instances of the questioning or challenging of students' arguments; however, in the fourth session there were 14 such instances and 19 in the fifth session. As students became more comfortable in the learning community, they felt safe correcting one another and challenging each other's ideas.

Full Participation

Table 1 highlights the eight students involved in the analysis and displays the occurrence of and types of reasoning that they engaged during the five sessions.

Forms of Reasoning	Chanel	Dante	Michael	Shirelle	Ian	Chris	Brittany	Jeffrey	Danielle
Direct	17	15	5	1	7	7	2		1
Contradiction	1	2				2			
Upper/Lower Bounds	2	1						1	1
Cases	1								

Table 1. Forms and occurrences of reasoning over the five sessions

Discussion and Implications

Our results indicate that the middle-school students in the sessions generated mathematical justifications using the various forms of reasoning in order to convince their peers of the correctness of their solutions. Also, they sought to understand and proceeded to challenge the justifications of others. These results support the idea that given a supportive environment, all students can and do make and refute claims and participate in inductive and deductive reasoning (Yackel and Hanna, 2003). The results are consistent with the findings of previous longitudinal studies[i] that document the fact that students are able to use convincing arguments and various forms of reasoning in the development of mathematical ideas. This study differed from the previous studies in three ways. First, the students in the previous longitudinal studies had been working together for years in an established mathematical community (Maher, 2005); the first five sessions documented in this paper, consisted of approximately six hours of contact with us and students began to engage in mathematical discourse as early as the first session. Second, the students in this study were in middle school and from a disadvantaged, urban environment. Finally, this study was conducted in an informal, after-school environment rather than in the context of the mathematics classroom. The data in this paper also suggest ways in which a culture of reasoning can develop. These include, but are not limited to, the following: providing students with open-ended tasks and sufficient time to investigate and re-visit tasks, with minimal researcher/teacher interventions; beginning in small groups and then opening up discussion to the larger community; researcher/teacher modeling listening to student explanations; inviting students to explain and justify their reasoning; having materials available so that models can be built; and providing a safe, supportive environment for collaborative learning.

Endnote

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TRACING STUDENTS' USE OF MEANINGFUL REPRESENTATIONS IN THEIR DEVELOPMENT OF COMBINATORIAL REASONING AND JUSTIFICATION

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This study documents the mathematical development of a group of eleventh-grade students who built representations to solve challenging combinatorics tasks and then refined and linked those representations in order to develop an understanding of the relationship among the tasks, the combinatorial idea of (m choose n), and Pascal's Triangle. The students' use of representations was critical for their development of an understanding of combinatorics, a topic that has been noted to cause difficulty for students. This suggests that ideas of combinatorial notation should not be imposed on students, but rather students should be given time to develop these ideas in environments that encourage them to recall, produce, refine, and connect representations.

Purpose and Theoretical Framework

The National Council of Teachers of Mathematics [NCTM] is explicit about the importance of students learning discrete mathematics, including combinatorics: "As an active branch of contemporary mathematics that is widely used in business and industry, discrete mathematics should be an integral part of the school mathematics curriculum…" (2000, p. 31). Research indicates, however, that combinatorics is a field that most students find very difficult (Batanero, Navarro-Pelayo, and Godino, 1997; Eisenberg and Zaslovsky, 2004; Hadar & Hadass, 1981). Many students associate combinatorics with negative experiences calculating permutations and combinations, often confusing one with the other (Sriraman and English, 2004). One explanation offered for this difficulty may be the manner in which combinatorics is traditionally taught—with teachers' demonstrated use of conventional, symbolic notation and algorithms. "Students often face an overwhelming difficulty when studying combinatorics. They are given many formulas (with little justification), they attempt to memorize these formulas (with no relational framework), and the resulting confusion can be disastrous" (Schielach, 1991, p. 137).

This research is based on the view that when students are presented with challenging problem-solving tasks in an appropriately supportive environment, and have opportunities to build, modify, and connect representations for their ideas, their thinking and reasoning also develop (Davis & Maher, 1997; NCTM, 2000; Tarlow, 2004; Warner & Schorr, 2004). According to Davis (1984), the building of mental representations is the foundation of doing mathematics. Mental images formed by individuals are used in building representations of mathematical ideas. After students have built their own representations for a problem task, they seem ready to listen to the ideas of other students (Maher & Martino, 1992). In doing so, their ideas may be challenged or supported. The resulting interactions may lead students to reject, modify, or strengthen an original argument. As learners cycle between representations in building justifications for their ideas, new knowledge is constructed. "It is important that students have opportunities not only to learn conventional forms of representations, but also to construct, refine, and use their own representations as tools to support learning and doing

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mathematics" (NCTM, 2000, p. 68). In this manner, students develop understanding by building upon their experience, rather than by being told by the teacher. The purpose of this research is to investigate the following question: Within the context of problem-solving situations that involve combinatorics, what is the nature and the role of students' representations as they build solutions and justifications for their ideas?

Methods

Background, Setting, and Subjects

As part of an ongoing longitudinal study (1) involving the development of students' mathematical ideas, initiated in 1989, students have been engaged in problem-solving explorations, working together to find solutions to problems and to build justifications for their ideas. Combinatorics activities were presented beginning in grade two, before formal class instruction of algorithms. In grades three through five, they explored the tasks that are the basis of this study, the Tower and Pizza Problems.

At the time of this component of the study, nine eleventh grade students, fifteen and sixteen and years old, investigated combinatorics tasks in six eleventh grade after-school sessions, which were not part of their regular school mathematics instruction. Five of these students were a subset of the original group that had been involved in the longitudinal study in grades one through eight.

The students worked together in pairs or in small groups, and each session lasted approximately one and one-half hours. Students were invited to explore ideas, develop representations, invent notations, make conjectures, devise strategies, test their methods, discuss their ideas with their peers, and to justify their solutions. The teacher's role was to step back to give students freedom to pursue their ideas, to observe the students' work, and to listen carefully in order to decide when an appropriate intervention was necessary. An appropriate intervention on the part of the teacher might be to ask a question or to pose a modification of the task to encourage students to explain their ideas and reasoning. Problems, or similar ones, would later be revisited, so students would have the opportunity to think about their ideas over time.

Tasks

The tasks that provide the basis for this study are the Tower Problem, the Pizza Problem, and extensions of these problems. In the Tower Problem, students are asked how many towers of a certain height they could build from Unifix cubes (for example, 4 cubes tall) when there are two colors to choose from. In the Pizza Problem, students are asked how many pizza choices a customer has if there are a certain number of toppings to choose from. Further, both tasks challenge students not only to enumerate all possible combinations, but also to provide a convincing argument that all possibilities have been found. The demand for justification, rather than simply for an "answer," sets the stage for building representations through which lines of reasoning can be proposed and then explored, argued, and modified. The Tower and Pizza Problems have isomorphic mathematical structures, and their solution can be represented by a generalization that may be justified by either a proof by cases or a proof by induction.

Data and Analysis

At least two cameras were used to videotape each session. One camera focused on the actions of the students; the other camera focused on the students' written work. The videotapes, students' written work, field notes, transcriptions, and analyses for each session provide the data for this research.

A qualitative methodology for data analysis was employed (Powell, Francisco, & Maher, 2003). To manage the large amount of data that were analyzed, a visual representation of the flow of students' ideas and justifications was developed. Combinatorics ideas under consideration, contributions made by each student or partnership, and teacher/researcher interventions were charted with corresponding time codes. Students' representations, strategies, justifications, connections, and interactions, as well as the role of the teacher/researcher were coded, and the codes were used to identify and trace the students' use of representations in their development of combinatorial reasoning and justification.

Results

When the students investigated the Tower and Pizza Problems in the elementary grades, they began with concrete representations. They built towers with Unifix cubes and drew pictures of pizzas. As the students continued to explore the problems, they modified their representations, which became more symbolic. They drew grids to represent towers, labeled with single letters to indicate the color of each cube, and used letter codes and lists of topping combinations to indicate pizza choices, with varying degrees of organization. Space limitations prohibit a detailed report on the students' initial work on these combinatorics tasks in the early grades. During the conference session, the students' use of representations to develop ideas in combinatorics in the early grades will be examined in greater detail.

This report focuses on the mathematical ideas of a group of four students, Robert, Stephanie, Shelly, and Amy-Lynn, in an eleventh grade after-school session, during which they revisited the Pizza Problem (given four toppings) that they had investigated six years earlier in grade five. When presented with the Pizza Problem, Shelly, Stephanie, and Amy-Lynn discussed the fact that they "just did this in school, combinatorics stuff," and after using their calculators to try to solve the problem, they said it was "so pathetic" that they couldn't remember the last section they did in math class. For approximately ten minutes, the students tried to recall and apply an algorithm and then abandoned those attempts.

Shelly: I don't know if it's a factorial or combination. I don't know if you would just do, like, five factorial plus four factorial plus three factorial plus two factorial plus one factorial, 'cause, okay. So far, do you have five? Ah, this is so confusing. I don't know how to explain it. I don't want to do it [this way].

Solving the Problem and Justification Using Proof by Cases

Stephanie suggested that they "plot out the pizzas" like Shirts and Pants or Towers [referring to problems that they had explored in the study several years earlier], and Shelly recalled that they had used a "tree-diagram type thing." Stephanie and Shelly then drew tree diagrams to represent topping combinations. After doing so, Stephanie said that she didn't "know how to organize it [her tree diagram]," because she had duplicated topping combinations. A pepperoni-mushrooms-sausage pizza is the same as a sausage-pepperoni-mushrooms pizza, so “we’re just going to have to go back and, like, cross things [duplicated combinations] out, when we’re done.”

As the students continued working, they revised their representations. They used symbols—letter codes—instead of full names to represent the toppings. In addition, since they had recognized the need to “organize” their pizzas in order to avoid duplicating topping combinations, they revised their tree diagram. They decided that “plain” is not a topping “because a pepperoni is [the same as] a plain with pepperoni,” and they controlled variables when determining topping combinations in the rows of their tree diagram. In their revised representation, the pizzas were organized by cases according to the number of toppings --0, 1, 2, 3, 4 toppings-- for 1, 4, 6, 4, 1 topping combinations, a total of sixteen pizzas.

Figure 1. Revised tree diagram with pizzas organized by cases

Stephanie and Shelly used their revised representation to find the solution to the problem and then to justify their solution with a proof by cases. Further, they recognized these numbers as a row in Pascal’s Triangle.

Shelly: One, four, wait a minute. One, four, six, four, one, so the next one will be one. This is the

Stephanie: The Triangle.

Shelly: The Triangle.

Stephanie: Yeah. So the next one is one, five, ten, ten, five, one.

Stephanie and Shelly connected the pizzas to Pascal’s Triangle. Now, to find a solution for the last part of the task, they did not draw a tree diagram; they drew Pascal’s Triangle and used the next row to determine the number of possible pizza choices if an additional topping, anchovies, were available. The researcher asked for an explanation.

Stephanie: Those are the anchovy pizzas [pointing to the 1 5 10 10 5 1 row on her drawing of Pascal’s Triangle].

Researcher: So tell me what these numbers mean [pointing to that row].

Stephanie: [pointing to each number in turn] One plain pizza, five pizzas with one topping, ten pizzas with two toppings, ten pizzas with three toppings, five pizzas with four toppings, one pizza with all five.

Stephanie pointed to their drawing of Pascal’s Triangle and said, “Well, now, that was pretty easy. Because look, now we know what all the pizzas [for any number of available toppings] are.”

Explaining Addition on Pascal’s Triangle Using Pizzas

The researcher asked the students how they drew Pascal’s Triangle so quickly, and Stephanie explained how to add numbers in a row to produce the numbers in the next row.

Now, the researcher posed an extension to the original problem and asked the students to explain the addition on Pascal's Triangle in terms of pizzas.

Researcher: You know the kinds of pizzas they are [points to the 4 on Pascal's Triangle]. And you know the kinds of pizzas they are [points to the 6]. And you know the kinds of pizzas they are [points to the 10].

Stephanie: Uh-huh.

Researcher: I'd like you to explore why that [addition] works with the pizzas, and we're going to leave you alone. Do you understand my question?

Shelly: Uh-huh

Stephanie: Uh-huh.

Researcher. Okay. [Researcher leaves].

The students suggested strategies: making another tree diagram or using Pascal's Triangle to consider addition on other rows and looking for a pattern. They then discussed the idea of addition with actual pizzas.

Stephanie: It's like you're materializing a topping, ya' know? Like I can't add plain to pepperoni and make sausage all of a sudden, and that's how one and three make four, ya' know? I don't understand. I just can't get past the fact that you can't make a pizza out of other pizzas.

The students then considered another strategy to solve the problem.

Stephanie: I think maybe if it was applied to something else I could look at it differently. Be, like, oh right."

Robert: Apply it to towers.

Stephanie: I knew someone was gonna say that. I figured it would be them [the researchers], though. All right, so let's apply it to towers then.

The students discussed drawing or building towers, but dismissed the idea because "we're gonna get done, and we're gonna be, like, oh well, now I understand it with towers, but I have no idea how to do it still [with pizzas]."

The students retrieved their drawing of Pascal's Triangle and again linked the numbers on the Triangle to the corresponding topping combinations. They then agreed that a pizza in the next row can be "just the pizza…with the [new] topping thrown on it." Now the students were able to use pizzas to explain the addition rule for Pascal's Triangle. Stephanie explained the addition of 3 [pizzas with one topping] + 3 [pizzas with two toppings] on the third row = 6 [pizzas with two toppings] on the fourth row.

Stephanie: So then here, um, you have 6 pizzas with two toppings. Now you already have 3 pizzas with two toppings. So these 3 pizzas with one topping get an extra topping added on.

Researcher: Okay.

Stephanie: So these become 3 pizzas with two toppings. And then 3 pizzas with two toppings plus 3 pizzas with two toppings equal 6 pizzas.

Generalization and Justification Using Inductive Reasoning

Robert generalized the solution for the number of topping combinations as 2^n, for n available toppings, but he did not justify his generalization. He explained that it was based on a doubling pattern that he observed when he summed the numbers in the rows of Pascal's Triangle, and he "just remembered something with towers that we did: to find total combinations was two to the something." Later, when Stephanie explained how pizzas could be moved to two different places on the Triangle—in one move they stay the same and in the other move they get a new topping added on—Amy-Lynn connected this two with Robert's 2^n, to provide a justification for his generalization.

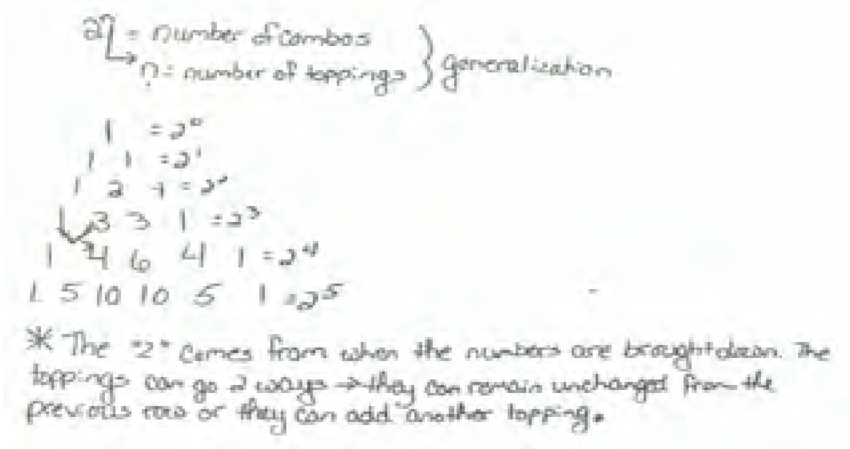

Figure 2. The students' generalized solution and justification

Connecting Pascal's Triangle, Towers, and Pizzas

The researcher asked the students if they remembered the towers. All four students said that they did.

Researcher: Can you imagine what these might mean, what these numbers might mean, with respect to towers then [pointing to the 1 3 3 1 row on the students' drawing of Pascal's Triangle]?

Stephanie: Like if we have blue and red, this is 1 tower with all blue. I don't know how high it would have to be. And this is 3 with

Robert: Two blue one red.

Stephanie: Thank you, two blue one red. And then this is 3 with two red one blue. Oh, and then this is 1 all red. So, that would, I guess they are three high.

The students then explained the addition—1 + 3 = 4— on Pascal's Triangle using towers.

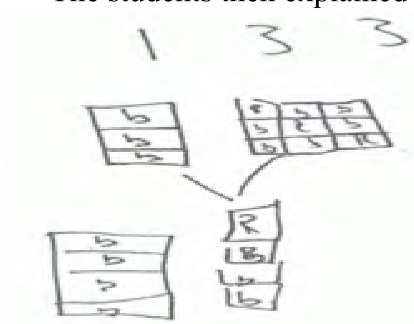

Figure 3. Using towers to explain addition on Pascal's Triangle

Stephanie: You have the 1 three high with all blue. And then you have the 3 with one red, so you have: red blue blue, blue red blue, blue blue red. And then these two [pointing to 1 and 3] make, this 1 is four blues. Okay, and these two together [pointing to 1 and 3] make, um, the one with four, the 4 with one red. So this 1 gets a red added on because it's already got three blues. So it can't have any more blues. And then these 3 all get

Shelly: A blue added on.

Stephanie: A blue added on to it.

The researcher asked the students if the Pizza and Tower Problems were the same or if they coincidentally had the same answer. Shelly replied that it's easier to explain the "two thing [2^n] with this [towers] 'cause there's only two colors…'cause with all those toppings, it throws you off…you expect like eight hundred pizzas." The researcher continued to question the students.

Researcher: Is there a way of thinking about the pizzas another way so the toppings don't

Robert: Yeah.

Researcher: Robert.

Robert: Toppings is the height. Like four toppings would be a tower four high.

Shelly: Uh huh.

Robert: And then the two colors would be with or without toppings.

Conclusions

The pizzas and towers became metaphors for thinking about important combinatorial ideas. As the students explored the tasks, they retrieved, built, refined, and linked representations. They developed a progression of representations that became increasingly symbolic and abstract, and they moved back and forth between their representations as they developed their ideas. This ultimately aided them in finding a solution to the Pizza Problem together with a justification; they organized their pizzas by cases according to the number of toppings and used a proof by cases to justify their solution. They then connected their cases to the numbers on Pascal's Triangle and explained the addition on Pascal's Triangle using pizzas. They also noted the doubling pattern as the number of available toppings increased and generalized the solution to the problem as 2^n for n toppings, which they justified by linking their representations and using inductive reasoning. Finally, they used their representations to connect the Tower and Pizza Problems, the combinatorial idea of (m choose n), and Pascal's Triangle. In this manner, the students developed important ideas and connections in combinatorics and justification for their ideas.

For these students, the properties of combinations grew from very concrete images, such as pizzas and towers. Once these properties emerged in Pascal's Triangle, they linked these images intro a larger framework. The students' use of their representations was critical for their development of an understanding of combinatorics. This suggests that ideas of combinatorial notation should not be imposed on students, but rather students should be given time to develop these ideas in environments that encourage them to recall, produce, refine, and connect representations. By examining the nature and role of the students' changing representations, educators may better understand students' use of representations to develop and justify their ideas. It is suggested that this has important implications for educators who wish to incorporate combinatorial reasoning and justification into the curriculum in a meaningful manner.

Endnote

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DIFFERENCES BETWEEN MATHEMATICS MAJORS' VIEWS OF MATHEMATICAL PROOF AFTER LECTURE-BASED AND PROBLEM-BASED INSTRUCTION

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This study investigated prospective mathematics teachers' views of proof in a course utilizing problem-based instruction. A Mathematical Proof Survey (MPS) was developed and used to assess whether a problem-based instructional approach affected the perspectives of proof and pedagogical views held by undergraduate mathematics majors, including secondary teacher candidates, enrolled in lecture-based and proof-based mathematics courses. The results showed that the treatment was effective in enhancing the undergraduates' views of mathematical proof, although not to a degree that was significantly more effective than the lecture-based instruction. However, the students in the problem-based sections developed significantly more process-oriented pedagogical views of mathematical proof than did students in traditional lecture-based sections.

Background and Objectives

The purpose of the research presented in this paper is to investigate undergraduate mathematics majors' views of mathematical proof in a course taught using the modified Moore method (MMM), a teaching style that promotes a problem-based approach. In the MMM, the emphasis in teaching proof shifts from presenting a series of formal proofs to encouraging students to go though the proving process (Cohen, 1982). Researchers and mathematics teacher educators have reported that prospective teachers' views of mathematics and their mental models of teaching are developed by their learning experiences as mathematics students (Ernest, 1989). It is well-documented that incomplete conceptions of mathematical proof are held by both secondary mathematics teachers and undergraduate mathematics majors, who are usually educated through traditional lecture-based instruction during their undergraduate studies (Jones, 2000; Knuth 2002).

Methodology

The following research question frames this study: Are the views of proof of undergraduate mathematics majors who experience MMM in an undergraduate mathematics course different from those who experience traditional lecture-based instruction in the same course? In order to answer this question, we adopted a pretest-posttest control-group design. The study was conducted with 61 undergraduates enrolled in four MMM sections (treatment group) and two lecture-based sections (control group) of number theory courses in a large state university.

The Mathematical Proof Survey (MPS) was developed for this study to assess participants' perspectives and pedagogical views of mathematical proof. It was modeled after the Views About Science Survey (VASS) (Halloun, 1997) and the Views About Mathematics Survey

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.

(VAMS) (Carlson, 1997). It consists of 15 items related to perceptions of mathematics and proof (Dimension 1) and pedagogical views of proof (Dimension 2). Each item consists of two contrasting alternative views of proof. Respondents weight the views using an eight-point Contrasting Alternative Design (CAD) instrument. Mathematicians' responses to MPS were used to calibrate the experts' perspectives and pedagogical views of mathematics and proof. Students' responses were classified into three categories: humanistic/process view; mixed view; and absolute/product view. Students' responses to each item were quantitatively analyzed by assigning a score as follows: humanistic/process view = 2; mixed view = 1; absolute/product view = 0. Possible maximum total score for 15 questions in the survey was 30 points.

Results

Table 1 presents means, standard deviations and *t* tests between pre and post scores on MPS within each of the MMM and control groups. As shown in Table 1, overall undergraduate students' scores on MPS increased in both MMM sections and control group sections after taking the proof-based course, which implies that undergraduate students benefit from a proof-based course with respect to their views of mathematical proof. The paired-samples *t* test analysis showed that a significant change in scores from pre-test to post-test occurred in MMM sections while no significant change was found in the control group sections.

Table 1. Means, standard deviations, and t tests between pre and post scores within groups

Group	*N*	Pretest Mean (SD)	Posttest Mean (SD)	*t*
MMM	33	19.00 (4.93)	21.15 (5.26)	3.25*
Control	28	19.07 (4.52)	19.57 (5.65)	0.53

*p < .01.

Table 2 presents means, standard deviations, adjusted means, and the Analysis of Covariance (ANCOVA) *F* test results for the overall posttest scores in the Perception and Pedagogy scales. No significant differences appeared between the groups at the end of the semester. When examining students' scores by scale, the ANCOVA test analysis showed that no significant difference was found in Perception scale between posttest means of MMM sections and posttest means of control group sections. However, the ANCOVA results showed that MMM students' scores changed significantly in Pedagogy scale, indicating that MMM students' pedagogical views were significantly more process-oriented after the course than were those of the control group.

Table 2. Means, standard deviations, adjusted means, and ANCOVA *F* test results for Perception, Pedagogy scale and total

Scale	Maximum	MMM	Control	*F* (1, 58)

	Possible	M (Adjusted M)	SD	M (Adjusted M)	SD	
Perception	12	8.39 (8.45)	2.22	8.75 (8.68)	2.32	.18
Pedagogy	18	12.77 (12.69)	3.94	10.82 (10.92)	4.33	5.33*
Overall	30	21.15 (21.18)	5.26	19.57 (19.54)	5.65	2.25

*$p < .05$.

These results suggest that experience with proof and instruction in such a problem-based courses may provide opportunities for undergraduates to develop more humanistic and process-oriented views of proof.

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WHEN STUDENTS PROVE STATEMENTS OF THE FORM (*P* _ *Q*) ⇒ (*R* _ *S*)

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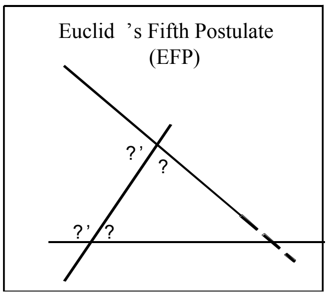
We explore the way that students handle proving statements that have the overall structure of a conditional implies a conditional, i.e. (p _ q) ⇒ (r _ s). Students recruited a proving frame from their experience, which was insufficient for the complexities of the statement. This led them to start with the totality of (p→ q) in ways that were problematic.

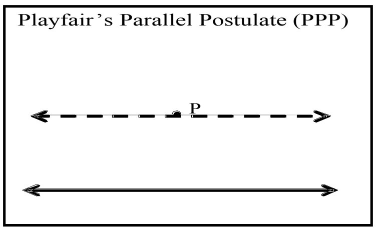
The purpose of this paper is to explore the ways in which students prove statements that have the overall structure of a conditional implies a conditional, i.e. (*p* _ *q*) ⇒ (*r* _ *s*). This logical structure occurs often in statements to be proven at the university level. For example, since the definition of A is a subset of B (A⊆B) is a conditional statement (x∈A implies x∈B), then a simple set theory statement such as "If A⊆B, then A∪B⊆B." has this logical form. Another instance of this logical structure occurs when proving the induction step in a proof by induction, i.e. If a conditional statement is true for k terms, the same conditional statement is true for k+1 terms. We will examine the situation of students working to prove one direction of the equivalence of two forms of the parallel postulate of Euclidean geometry.

The research literature indicates that undergraduate students struggle with proof writing (e.g. Weber, 2001), understanding the logical structure of the mathematical statements (Selden & Selden 1995; Dubinsky & Yiparaki, 2000), and completing induction proofs (Brown, 2003; Harel, 2001). This paper adds to that literature by describing students' proving using a logical structure that is common in mathematical problems at this level, but which has not been directly addressed in previous work.

Methods

The data for this study was collected as part of a semester long teaching experiment (Cobb, 2000) in an upper division geometry course. Data consisted of videotape recordings of each class session, and copies of students' written work. During small group discussion there were two cameras each focused on a different small group. The first group of students' was Alice, Emily and Valerie. The second consisted of Andrea, Nate, Paul and Stacey. The curriculum consisted of a series of activities in which students would need to defined, conjectured and proved results in geometry on the plane and the sphere (Henderson, 2001).





Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.

This study focuses on one day late in the semester in which students were asked to prove one direction of the bidirectional statement Euclid's Fifth Postulate (EFP) implies Playfair's Parallel Postulate (PPP). Henderson (2001) states EFP as, "If a straight line crossing two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended indefinitely, meet on that side on which are the angles less than the two right angles," and PPP as, "For every line and every point not on the line there is a unique line through the point that does not intersect the original line." The instructor told the students that the two postulates are equivalent and gave them the option to "*use*" EFP in order to prove PPP or vice versa. In her introduction of the task the instructor drew the two figures shown while explaining each of the theorems. The teacher's initial drawing showed only the part of the statement that was given. For example for the PPP picture initially she just drew the bottom line, l, and a point P not on that line. However, when she explained the conclusion of each statement she completed the picture (as denoted by the dashed line) and these completed pictures were left on the board for students to reference. In addition, the two pictures in the book for these two statements were completed pictures similar to what the teacher had drawn.

Results and Discussion

Since EFP and PPP are both conditional statements, "EFP $\Rightarrow$PPP" has the overall structure of $(p \rightarrow q) \Rightarrow (r \rightarrow s)$. Note that for some types of statements of this form it may be possible to start with the initial statement p $\rightarrow$ q and formulate a series of equivalent statements to turn this into r $\rightarrow$ s. More often, for proofs of statements with this logical structure the most efficient strategy is to begin with the givens of the second conditional, the "*r*," and work through what is known using $p \rightarrow q$ in the middle of the proof to reach the conclusion of the second conditional, the "*s*." In this paper we give illustrative examples demonstrating ways of proving evoked for students by this logical structure.

The two groups began their discussion with the intent to prove EFP $\Rightarrow$ PPP. Students recruited a proving frame from their experience. By frame we mean a conventional and schematic organization of knowledge (Fauconnier & Turner, 2002). In this case, we believe the students are functioning from a frame in which something (EFP) implies something else (PPP) similar to familiar problems of the type, given *p* prove *q*. As noted above, this could be effective if they treated EFP and PPP as conditional statements. Instead, both groups of students appear to treat EFP as a chunk of knowledge where the implication involved in EFP is ignored. In the first group, Alice laid out her frame early in the discussion.

Alice: So Okay, so. So when it says, prove EFP implies PPP or PPP implies EFP that means whichever, like if we're doing from this one to this one, we get to say this is true so we have to prove this. […]

Emily: How do you get from one to the other? [inaudible] run into problems.

Valerie: I don't know --

Emily: 'Cause basically if you prove one to prove the other, you have to get from one to the other.

There are two main struggles that students have in trying to apply their traditional proving frame to the special situation of the conditional implies a conditional. In one scenario, as seen with Stacey below, the student believes since EFP is given that *both* parts of EFP are given. In other words, as researchers we note that Stacey's assumption is logically equivalent to *p* and *q* instead of *p* implies *q*. An example of this occurred near the beginning of the second group's discussion.

Paul: Well, if you assume the first one [EFP] would there be three cases that _ + _ < π , _ + _ = π, or _ + _ > π and then the uniqueness part of it would be proved by the _ + _ = π and in that case they wouldn't meet. Well, that wouldn't be unique either because _ + _ > π wouldn't meet either.

Andrea: Well, if _ + _ > π, then they would intersect on the other side .

Nate: Then you can look on the other side .

Paul: The other side, yeah.

Stacey: But if we are assuming the whole thing though, we are assuming they are less than π.

Paul: Okay, well I'm confused.

Stacey's statement seemed to suggest that she was assuming *both* the premise and the conclusion of EFP, further evidenced by her statement a few minutes after this exchange, "Cause it's all assumed. This whole thing is. We are assuming $\alpha + \beta < \pi$. We don't care if it's equal to π." At this point, the group was unsure of how to interpret Stacey's assumptions.

In the second scenario, the students looked at the completed picture for EFP (see Figure) and tried to work from it, to the picture for PPP (see Figure). Each of the two groups spent some time considering a sketch like the EFP Figure and considering what would happen if the top line were to be in different positions. They envisioned three cases as stated by Paul in the previous transcript. In the first group we see Emily make this explicit.

Emily: Cause basically, say you're given this picture [indicates EFP picture]. [Alice: M'hmm.] and you choose a side and you add it up and it is greater than 180. Okay, fine.

Valerie: So that means they don't intersect on that side.

They continued by discussing the other two cases including the case where $\alpha + \beta = 180$ and thought that this line parallel to the bottom line would be the unique case that was indicated in the picture for PPP. So in this way they tried to start with the picture of EFP and turn it into the picture for PPP.

The second group also worked from the EFP picture instead of the setup for PPP. During the proving process the group became confused as to which direction they were in fact proving. While clarifying the direction of the proof, Nate started working from the premise of PPP. Paul finally realized that the group had been working from the complete EFP picture.

Nate: Cause here, we start out, we start out with something that looks like this, right? L and P. [...]

Paul: Yeah, I think we were starting with this drawing [EFP] instead of starting with that drawing [premise of PPP] in our head. That's what I was doing anyways.

Nate: Right and that's what had me confused over here when she was pointing out how to do it and I couldn't tell you guys why we couldn't just do it that way…

The complexity of a conditional implies conditional statement requires that students have a robust proving frame. In our data both groups assumed EFP as a whole. This took the form of assuming *p* and *q* of EFP or assuming that the completed EFP picture was given. These two problematic scenarios hampered their efforts to prove PPP, given that they did not start with the premise of PPP. Since the conditional implies conditional logical structure is inherent in many proving tasks, particularly induction, it is valuable to examine the difficulties encountered by students related to this structure.

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IN LINE WITH STUDENT REASONING: A RESEARCH METHODOLOGY WITH PEDAGOGICAL POTENTIAL

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This case study present an analysis that (a) assesses how students interpret the number line representation and (b) considers instruction involving non-normative treatments of a representation as an opportunity for students to learn. In this paper, I present data documenting student response to non-normative treatments of the number line that problematize the function of intervals and tickmarks. This case study is an analysis of one student's understanding of representational features and a shift in this understanding. Results of this interview study confirmed that students often inaccurately interpret the function of intervals and tickmarks when attempting to identify unknown points on the number line.

Objective

The number line appears throughout elementary schools and serves as a public display of the continuous nature of the number system. The tickmarks and intervals of the representation often adorn walls of the classroom, and curricula incorporate number lines into a variety of mathematical topics at different grades. Yet despite its omnipresence, students often lack appropriate opportunities to consider the full mathematical meaning in the representation. The relative sizes of intervals are rarely problematized for students, the result of which is that students are not confronted with what is relevant or irrelevant in determining the location of points on the number line. The objective of this paper is to explicate an effective method for assessing elementary school students' understanding of the mathematics represented through the number line. An assumption underlying this study is that non-normative treatments of the representation yield generative ideas for students (for an application with fractions, see Gearhart et al., 1999). The efforts reported here reflect an attempt first to accurately assess how students interpret intervals and tickmarks and their functions, and then create opportunities for students to learn the mathematics underlying the number line. In this paper, I present data documenting student response to non-normative treatments of the number line that problematize the function of intervals and tickmarks.

Theoretical Framework

The design and analysis presented here make use of a sociocultural perspective on education. This follows an educational philosophy that individual understandings are situated and co-constructed within some instructional context in which particular social practices constrain and qualify what information is displayed and valued, and individuals use previous understandings in order to engage with ideas (Vygotsky, 1986). A robust analysis of learning must include the greater context in which a student is situated. This view of learning, based in sociocultural theories of cognitive development, framed the design and the subsequent analysis of data described below. For the analysis presented here, I argue that explicating and understanding any assessment and learning that is taking place must consider evidence of the student's thinking in conjunction with prompts designed to elicit particular kinds of information at specific times.

This study builds on previous contributions by using a form-function framework (Saxe, 2004) to analyze cognitive development. This framework considers the relationship of *forms*, which may be a word form or a written representation, to *functions* that these forms serve both individually and collectively; these functions may vary across individuals, while at the same time there may be a culturally normative function associated with a given form. In instructional settings, the relationship of forms to mathematical functions provides an assessment of students' understandings of a representation. In this case study, I consider two forms, tickmarks and intervals, and how one student does or does not make use of them in order to perform the function of naming a point on the number line.

Research has demonstrated that the number line is a particularly useful tool for exploring the topics of rational numbers (Saxe et al., 2007) and algebraic reasoning (Carraher, Schliemann, Brizuela & Earnest, 2006; Moses & Cobb, 2001), and is considered to be a powerful context for teaching and learning (Wu, 2002). Nevertheless, students have varied understandings of which features of the number line are relevant or irrelevant in order to interpret the location of rational numbers, or that, for example, points on the number line mark a relative magnitude from zero rather than marking discrete values. For a student learning how to engage with the number line, what informs the relative salience of particular representational forms over others? The premise of this study posits that forms such as tickmarks or intervals may have varying functions for different students, just as a function, such as locating a point on the number line, can be interpreted using different forms.

Methods

The case study here comes from a series of interviews involving a sequence of eight number line problems. Video data and written work presented here document a Grade 5 student working with two of the eight problems in the sequence. The case comes from a larger study involving one-on-one interviews with fourteen Grade 5 students in an urban public school in the San Francisco Bay Area. The premise underlying the problem sequence is that if any two points are marked on the number line and assigned a value, one can determine the numerical value of any point on the line (for example, see Figure 1).

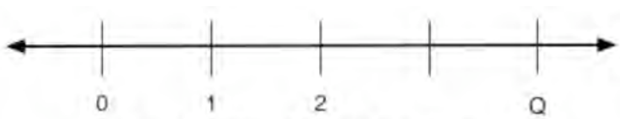

Figure 1: The Q Number Line

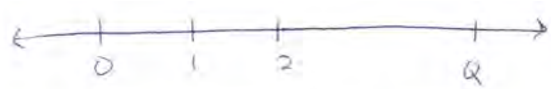

Figure 2: The Q Number Line with a tickmark omitted

With each problem in the sequence, students were provided with either (a) counter suggestions, or (b) unequal intervals problems and counter suggestions. Group A students received prompts that pose a different response that another student gave for the same problem (e.g., could Q ever be 3?). This type of prompt attempts to elicit further information on how a student is using tickmarks and intervals, and also whether a student would continue to provide their initial response or shift to a different value or multiple values for a point such as Q based on the counter suggestion. Group B received prompts involving a redrawn number line, such as that in Figure 2, which presented the same problem again with unequal intervals. Students receiving this prompt also received counter suggestions to first establish whether an unknown point, such as Q, can stand for a value or values other than 4 on this new number line. The case study presented here involves a student in Group B that received problems involving both unequal intervals and counter suggestions. In addition to this case, I present data across students in order to evaluate the utility of the unequal intervals group as compared to the counter suggestions group.

Analysis and Results

The case study here involves one Group B student, Linette, working on the second and sixth problems in the eight problem sequence. Each of these problems demonstrates the potential of the number line representation. The analysis of the Q number line, the second problem in the sequence and shown above in Figures 1 and 2, demonstrates a method for using the number line as a tool for assessment. In Linette's work on the P number line, shown later in Figures 5 and 6, the analysis demonstrates how engagement with this non-normative treatment of the representation provided her with an opportunity to shift to a more normative understanding of the mathematics implicit in the representation.

Method for Assessment: The Q Number Line

With the problem of identifying the value of Q (Figure 1), Linette responded immediately that Q was 4 on this number line, as did all 14 students in the study. In fact, one might expect a typical fifth grade student to easily identify the value for Q in such a problem. Based on her response, however, her method for determining the solution remained unclear. In order to better understand how Linette perceived the function of the various representational forms, the interviewer first probed further by providing a counter suggestion of 3. In doing this, the interviewer already knew he would employ the unequal intervals prompt by redrawing a similar number line that omits a tickmark between 2 and Q; he wished to assess before this whether Linette would entertain the notion that Q can be 3 on the original number line. Linette, given the number line in Figure 1, indicated that Q cannot be 3.

> I: And can you go 0, 1, 2, 3? *<I indicates where the Q is when he says 3, skipping over the unnamed tickmark.>*
>
> L: Um. I don't think that could be, skipping a number. And it would be out of order.

Linette accurately rejected the idea that Q could be 3. Both of her responses thus far—that Q can be 4 and refuting the counter suggestion that Q can be 3—were correct; nevertheless, it was unclear how Linette interpreted the function of naming points on the number line. Is the location of a point determined by the number of tickmarks or by the spacing of those tickmarks? The interviewer drew a new number line that looked quite similar to the original. This time, however, there was a tickmark omitted between 2 and Q (Figure 2), which was an effort to have Linette explicate the function she saw these various forms as showing. Her response to the value of Q on this new number line would shed light on *how* she completed the function of naming a point on the number line and using *which* forms.

> I: What if the number line looked like that? Could Q be 3 on that one?
>
> L: Yeah!
>
> I: Why could Q be 3? Why on this one, and why *not* on this one?
>
> L: Well, these things …
>
> I: The tickmarks?

L: Yeah. They show that there's a number there, I think. So there's no number here. So it could be 1, 2, 3 <*L indicates individual tickmarks as she counts*>.

While Linette immediately responded correctly when presented with the initial number line in Figure 1, she shifted her answer for the non-normative number line with unequal intervals. For her, the tickmarks were clearly more salient in order to name a point: "*They* show that there's a number there." Across the 14 interviews, she was not alone. *All* students initially answered correctly (open response) to this problem that Q was 4 before receiving further prompts. A striking finding in the data emerged from differences between groups A and B.

Figure 3 is a diagram depicting responses on this problem over the 14 students interviewed. All students correctly answered that Q was 4 (question type 1), and then were divided into Groups A and B. Group A, counter suggestions, had 6 students total. Four students continued to respond correctly while two students changed to an incorrect response. Group B, unequal intervals and counter suggestions, had 8 students total, including Linette. Of these students, only 2 continued to respond that Q was 4, while 6 students, Linette included, changed to an incorrect response.

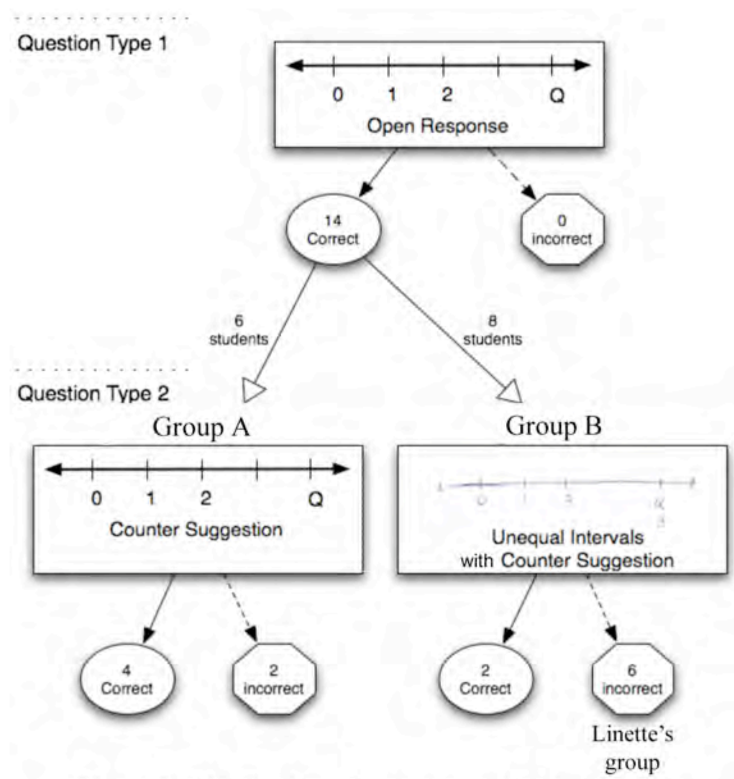

Figure 3: Distribution of responses on Q Number Line

As shown in the bottom row of Figure 3, 4 students from Group A and 2 students from Group B continued to respond correctly that Q was 4. However, the four correct responses from Group A may have been false positives; a student answering correctly in Group A may or may not have been using interval size in a normative way, whereas a student in Group B answering correctly must have been.

What emerges from a closer look at the data, as in the case of Linette, is that a comparison of correct and incorrect solutions is problematic across the two prompts. The four students with correct answers in Group A were never confronted with unequal intervals; it is therefore unclear just *how* they evaluated the function of the tickmark. The two students from Group B answering correctly showed a normative understanding of the function of the interval; the omission of the tickmark was irrelevant in the function of naming a point on the number line. Group B's six incorrect responses show that students used the tickmark with a counting function without regard to the distance between tickmarks. The unequal intervals prompt elicited this non-normative understanding.

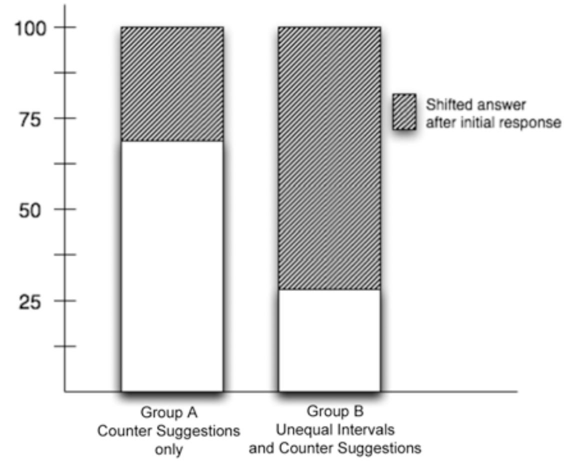

Figure 4: Responses to the Q Number Line

These findings for the Q number line are presented as percentages in Figure 4. The shaded portions of the bars indicate frequency of students shifting after the initial response of 4. As Figure 4 illustrates, students in Group B were much more likely to shift their response than students in Group A. This shouldn't be viewed as a misleading prompt, but rather as a more telling indication of how a student understands the number line. Students shifted their answers based on how they understood the representation. Group A students that continued to answer correctly may or may not have had a normative understanding of the function of intervals. The prompt they received was unable to elucidate this.

The number line with unequal intervals provided a more explicit view on the function of these forms. As Linette talked through her thinking during the interview, she used her fingers to indicate two tickmarks at a time, as if she were isolating each interval. The interviewer pointed this out to her and asked her what she was doing. In response to the interviewer's probe, Linette brought up the idea of adding an additional tickmark between 2 and Q.

I: It looked like you were measuring.

L: Oh. <*Looking at the number line*> And then it gets really big <*indicating the interval between 2 and Q*>. So you might need to add another one <*tickmark, indicating between 2 and Q*>. I don't know.

Linette had responded to the new number line that Q could be 3, but then seemed to be confronted with what she was saying when the interviewer points out her use of fingers to measure. She suggested adding an additional tickmark to the new number line, the result of which would be the original number line with equal intervals. Sensing that Linette seemed unsure if Q could be 3 on this number line or not, the interviewer asked Linette for two different arguments to illuminate how she viewed the representational forms.

I: Here's what I want you to do. I want you to make an argument why it [Q] *could* be 3, and then I want you to make an argument for why it *couldn't* be 3.

L: Okay. So it could be 3 because these stand for a number. And so if there's no number there. I mean, if there's no tickmark right there [between 2 and Q], so then that could stand, that would stand for 3 because it, it wouldn't stand for 3 if that, if there was a tickmark was right there.

I: Okay. And now make a case for why Q could *not* be 3.

L: Okay. Q could not be 3 because the number line maybe is about, like, how, like, even. Well, like, it has to be even in between, otherwise it could be like one, two, threeee <*drawing out the way she says 'three'*> and it would go really long for 3. So they'd have to make one [tickmark] in between these [2 and Q] two. Otherwise it would be a really weird number line because I've never really seen one that goes like that.

Both tickmarks and intervals play a role in making the final case for why Q cannot be 3. Whereas unit intervals were irrelevant to Linette when the problem was first posed, students like Linette ideally move towards a normative understanding that the unit intervals are necessary to identify any point and, once the unit interval is determined, the addition or omission of a tickmark does not in fact inform the naming of a point.

Had the interval form on the number line not been called into question with the unequal intervals prompt, Linette would have had no reason to assign to it any particular functional meaning. Her case provokes the question of how to consider the role of instruction. A part of the utility of the number line is in the manner in which mathematical information is communicated, a part of the reason it has been proliferated in classroom practice. Assuming that Linette's construction of the number line forms is not uncommon, instruction must consider how to utilize various representational forms and functions in a way that moves students towards a normative mathematical understanding of the representation that aligns with accepted conventions.

Opportunities for Learning: The P Number Line

I again use the case of Linette to illuminate the potential of non-normative treatments of representations in providing opportunities to learn. The case here involves the sixth problem in the sequence. In this problem, students once again identify an unknown value on the number line, though the process to determine this unknown is more difficult than in Figure 1. In this problem, students must first determine what the unit interval is before determining the value of the unknown, which in this case is P, as depicted in Figure 5.

Figure 5: The P Number Line

Figure 6: Adding another tickmark

The approach that each of the students in the interview study chose (though not the only approach for determining the value of P) was to first determine the value of the first unnamed tickmark before solving for P. In the case of Linette, she quickly answered that P was 75.

L: Um, 75.

I: And tell me how you thought about that.

L: Because there's nothing right there <*points to the unlabeled tickmark*>, and I saw the 50 right there. So I remembered what half of 50 was, and that's 25. So I just added another 25 to 50, and that's 75.

As in the case of the Q number line, Linette provided a correct initial response. Once again, the interviewer attempted to assess how Linette was using forms on the number line to serve particular functions. The interviewer first asked, given the number line in Figure 5, whether P could ever be 100. Linette responded that P could not be 100 in that case.

While she has responded accurately, the interviewer added a tickmark (hand drawn during the interview) to see if Linette would change her response. Figure 6 shows an image of the number line with the additional tickmark drawn between 50 and P. The additional tickmark is hand-drawn to be somewhere in between 50 and P so that the newly created smaller intervals are juxtaposed next to the larger intervals from 0 to 25 and 25 to 50.

I: On *this* number line, could P be 100?

L: Um. Could P. Let's see. Okay. <*whispering*> 25. 50. 75. 100. Yeah.

I: So would that work?

L: Yeah.

Similar to her response on the Q number line, Linette seemed to continue with an adding on strategy, this time by 25s. With the addition of an extra tickmark, she clearly used these tickmarks as counting markers without regard to the intervals between tickmarks. On this problem, however, Linette's reaction then became distinctly different from her response on the Q number line. Right after responding that P would be 100 with an additional tickmark, she rejected this response and revised it back to 75, providing a justification for it.

L: Wait! No. Because those [spaces] aren't even. This [space] is the same as that <*referring to the spaces between 0 and 25 and then 25 and 50*>, that's fine. But then these are half of that <*referring to the two intervals between 50 and P and comparing them to the first two intervals*>. So if you wanted to make that, you'd have to be even. I mean, fair.

Unlike her response for the Q number line, Linette began to think anew about the function of particular forms in the representation. The sequence of similar problems provided an opportunity for Linette to restructure and reorganize how she used particular forms on the number line to mediate her response to a given task.

Conclusion

In order to explicate the utility of non-normative representations, I focus on the use of unequal intervals in particular because of the dual role served in assessment and learning. In this, the role of instruction is key; it is through instruction that the non-normative treatment of a representation yields a juxtaposition among representational features. In the case of Linette, this representational juxtaposition became explicit when she counted on the number line: "One, two, threeeee." By having spaces that are clearly unequal, the juxtaposition more easily became a focus of discussion and reflection (Figure 7).

L: Well, like, it has to be even in between, otherwise it could be like one, two, threeee and it would go really long for 3.

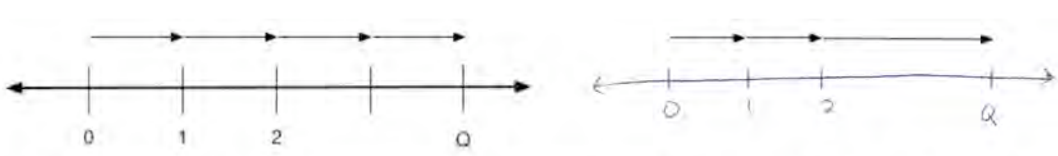

Figure 7: Juxtaposing the intervals on each number line by positioning them side by side

In the case of Linette, her elongated "threeee" is a result of the smaller intervals juxtaposed next to the larger intervals, as shown in Figure 7. When engaging students with non-normative representations, these juxtapositions can arise from interviewer or instructor prompts, or from students' own reflecting on representations that have been manipulated to be non-normative.

Whereas the use of the number line pervades elementary curricula and foreshadows work in secondary and tertiary mathematics, students rarely have opportunities to consider the generative mathematical ideas. With instruction involving non-normative treatments of the number line, students have opportunities to grapple with and eventually make explicit mathematical meanings. In turn, student response provides education theorists and practitioners structured apertures onto student understanding of such representations. To an adult discerning the mathematics involved in the Q number line, the question of what is a correct or incorrect value for Q, or what is relevant or irrelevant in answering this, is trivial. However, as Linette's case study demonstrates, understanding the mathematics involved in finding the value for Q is far from trivial. The result of these non-normative treatments of the number line holds both methodological and pedagogical potential in research and in classrooms.

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DEMONSTRATING MATHEMATICS EDUCATIONAL NEUROSCIENCE

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There has been a resurgence of interest regarding the role of neuroscience in informing education and mathematics education in particular. This latter interest has been cultivated in part by research in cognitive psychology and cognitive neuroscience research in mathematical cognition and learning. There have also been cognitive psychologists and cognitive neuroscientists, and some educational researchers who are applying methods of cognitive neuroscience to mathematics education research. This paper demonstrates some preliminary results from one such effort, capturing a bona fide "aha" moment.

There has been a resurgence of interest regarding the role of neuroscience in informing education (e.g., Blakemore & Frith, 2005; Byrnes, 2001), and in mathematics education (e.g., Campbell, 2006a, 2006b; Iannece, Mellone, & Tortora, 2006). This latter interest has been cultivated in part by research in cognitive psychology (e.g., Campbell, 2004), and cognitive neuroscience research in mathematical cognition and learning (Dehaene, 1997; Butterworth, 1999). Cognitive psychologists and cognitive neuroscientists (e.g., Ansari & Dhital, 2006; Szücs & Csépe, 2004), and some education researchers (e.g., Campbell with the ENL Group, 2007; van Nes & Gebuis, 2006) have now begun applying methods of psychophysiology and cognitive neuroscience to matters that should be of genuine interest and abiding concern to mathematics education. This paper presents data and results from one such initiative.

Theoretical Framework and Methodology

Our theoretical framework is predicated on the embodied view of cognition, as articulated by Varela, Thompson, & Rosch (1991) and others. A fundamental entailment of embodied cognition is that changes in lived experience will manifest through changes in bodily state in various ways that may be more or less obvious. Accordingly, a major task of mathematics educational neuroscience is to establish such connections, and thereby provide more evidence-based ground to our study of mathematical cognition and learning.

It follows that augmenting mathematics education research with endocentric data sets such as electroencephalography, electrocardiography, respiration rates, etc., along with eye-tracking, pupillary responses, and so on, will provide deeper and better understandings of the psychological aspects of teaching and learning mathematics and the implications thereof. At a very basic, coarse-grained level, it will prove a significant advance for mathematics education research to have more evidence-based measures that can better distinguish amongst, let's say, perception, concept formation, and reasoning — augmenting traditional exocentric data, and thereby lessening the need for speculation in the interpretation of traditionally relied upon audiovisual data sets toward the formulation and substantiation of cognitive models.

Data Acquisition, Analysis, and Discussion

Using a paradigm from Dehaene, Izard, Pica, & Spelke (2006), this paper presents data from one such effort that demonstrates our theoretical framework and methodology. Figure 1 is an image from a video clip capturing an "aha" moment. Data are integrated, synchronized, and coding proceeds in accord with analysis of these data in a simultaneous, or triangulated, manner. Our aim in this particular study is to identify and distinguish between pattern recognition and image based reasoning punctuated by moments of insight, through observations of brain behavior and other embodied modalities.

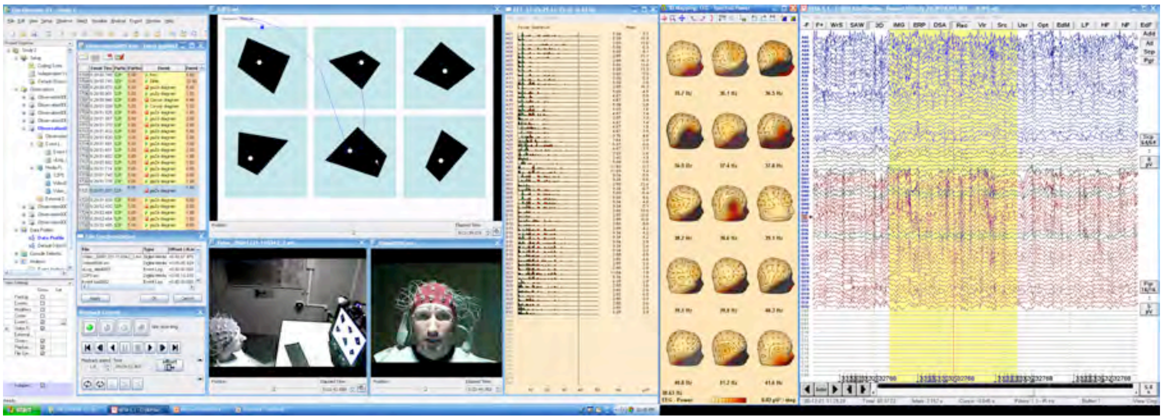

Figure 1. An integrated and time synchronized data set capturing an "aha" moment

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INNOVATIVE DATA COLLECTION TECHNOLOGIES FOR DESIGN EXPERIMENTS

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New audio/visual technologies coupled with new computing and network technologies for data collection and management have potential to support education researchers in their work. In the context of a design experiment involving an eighth grade class, this paper compares the benefits and drawbacks of traditional videography for data collection with a new system we call Embedded Information and Communication Technology. The use of such new systems may affect the types of research questions that researchers might formulate.

In formulating a study, researchers consider theoretical frameworks and methodologies that are amenable to their research questions. Data collection capabilities constrain these choices, often implicitly. For instance, when investigating student learning in the context of a classroom, design experiment methodology may use video as a main source of data on student learning (Roschelle, 2000). Traditionally, audio/visual data is captured with a camera and, possibly, with an external microphone. While effective, this method, as with all others, influences what information is collected and how it is analyzed. Emerging audio/visual (A/V) technologies, coupled with new computing and network technologies may provide alternatives for how educational researchers can conduct their research. For our design experiment with an eighth grade class, we were able to capture the classroom data in a new university research facility, adjacent to a middle school. The specially designed classroom provides the research team with the ability to record both whole class and small group work at optimal resolution with a high degree of flexibility, and little intrusion into the teaching/learning context. It integrates 12 digital cameras in the ceiling and 12 hanging microphones with a control room adjoining a large classroom that can be divided in half by a motorized wall. Thus, two different instructional settings can be recorded at the same time. From the control room, camera angles are adjusted to capture data that are deemed critical to document the learning and teaching of the design experiment. These data may be student groups, white board and smart board artifacts, and teacher's instructional actions (Mojica, Lambertus, Wilson, & Berenson, 2007). Up to 6 separate audio/video streams are mixed and recorded simultaneously. After completion of the recording session, digital video and audio input are

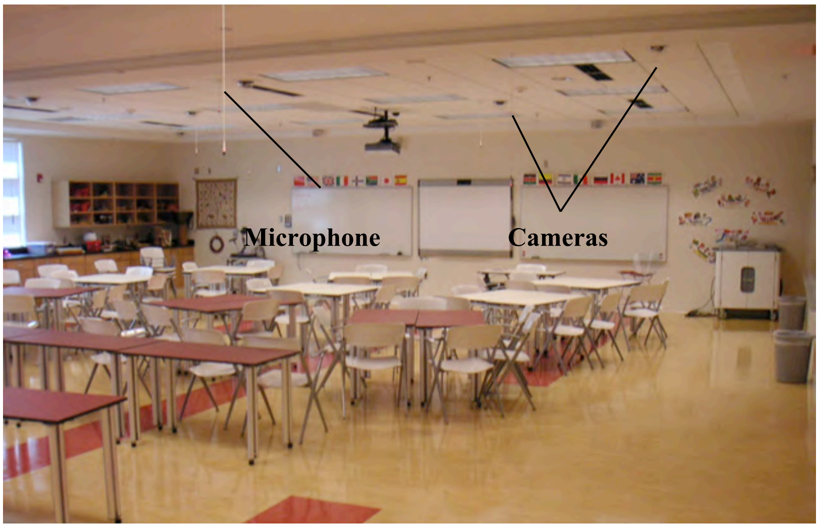

Figure 1. Embedded Information and Communication Technology (ICT) Classroom for Teaching Experiments.

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno

integrated and connected to a computer server for data transfer. We call this strategy Embedded Information and Communication Technology (ICT). There is tremendous potential for these technologies in educational research. For example, one of our research goals was to store all of our data in a virtual data library where eight researchers in the group had access to a very large amount of data via password. The ease of data storage with ICT helped us realize this goal. Careful consideration needs to be made both in terms of the use of the equipment and recognition of its influence on research methodology. In the context of a teaching experiment (c.f., Berenson, Mojica, Wilson, Lambertus, & Smith, 2007), this paper offers a comparison of traditional A/V data collection strategies with an Embedded ICT strategy and reports on different considerations that should be taken by researchers when deciding to employ these alternatives based on our experiences with ICT.

Traditional Videography

One of the benefits of traditional A/V data collection is the observer-participant status of the videographer. Since the collector is a part of the context of the classroom, he or she has an awareness of student and teacher actions and may anticipate and respond according to their own understanding of the research study and their status as a participant in the classroom (Powell, Francisco, & Maher, 2003). Further, he or she can take multiple perspectives of the students and teacher and adjust their recording accordingly. However, there are drawbacks associated with this collection strategy. By being a part of the context, the videographer inherently alters the context that he or she is seeking to observe. A single camera is limited to one focus at a time. Increasing the number of cameras in the room to increase the number of views concurrently increases the number of operators and the level of intrusion into the classroom. Because the camera is operated by an individual—either hand-held or on a tripod—points of view are typically limited from two to six feet off the ground. This provides a natural point of view but sometimes can limit the capture of activity happening on tables encircled by students. In raw form, data is available only on tape and a significant amount of processing time is required for digitization on computers to expand accessibility to all members of a research team.

Embedded ICT

Embedded ICT can address some of the constraints of traditional video data collection. The absence of videographers and cameras at eye level in the classroom may help researchers maintain ecological validity. Whereas traditional strategies tended to have one focus, the embedded ICT strategy affords multiple foci with minimal cost in terms of labor or intrusion into the classroom. Student interactions, student/teacher interactions, teacher actions, and individual students' work can all be captured in multiple, simultaneous streams. A single ICT operator can simultaneously view and record all of the recorded streams, because camera control is managed through an internet-based web interface. Another alternative is to use additional camera controllers to manage subsets of cameras on laptops in the control room or in the classroom. The classroom control of cameras is accomplished with almost no intrusion into the students' space.

While the increased number of data streams can greatly increase the amount of data being collected, it is crucial that one individual acts as primary "director" of the recording process to make sure that, collectively, the recorded streams stay focused on the research questions of interest. For this teaching experiment, we used our least experienced researchers as those directing the cameras and our most experienced researchers took field notes. In fact, the decisions concerning what data to capture are more crucial to the success of the experiment than the skill required to take the field notes. The sophistication and the multiple data streams require a much higher level of research skill than we anticipated.

The ceiling mounted cameras and microphones have both strengths and weaknesses. The ceiling location is minimally intrusive and provides superior points of view for whole class and small group work at tables. However, it is not a particularly natural angle of view and sometimes loses facial expressions of those working at tables. In addition, the hanging microphones cannot be quickly "pointed" at an emerging event the way a camera mounted microphone or boom mike can capture sound.

Because the audio/video streams are being stored on hard disks in digital form, the transfer to other computers on the network is relatively straightforward. However, as with all digital video, conversion to other formats (e.g., burning MPEG2 DVDs) can be time intensive. Similarly, the increase in the number streams being recorded can be both a blessing and a curse. With the possibility of six simultaneous streams being recorded, the research team can quickly be buried in analysis work. A clear set of goals as to how the audio/video data are going to be used to answer the research questions is important when it is one operator with one camera in the classroom. It is that much more crucial when choosing how many video streams are going to be used with the Embedded ICT system.

Conclusion

With much of educational research, the questions one would like to ask are in tension with the technology available to capture the data needed to address them. Embedded ICT opens doors as to what kinds of research questions can be asked in a classroom teaching experiment. The system outlined in paper and in use by the research team has demonstrated a number of ways in which a shift in A/V and ICT technology deployment can expand the ways in which educational researchers can conduct their work. There are interesting challenges associated with this new system, as well.

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RECONSTITUTION OF THE CONCEPTUALIZATION OF A MATHEMATICS SITUATION

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In order to explore the potential of Wenger's theoretical framework for the regular mathematics classroom and refine it, a situation in mathematics was conceptualized through collaboration between a graduate student and an in-service teacher. We analyze the process of elaboration as taking place in different contexts *from the theoretically-based conception up to the experimentation in class. This invited us to consider unforeseen ways for understanding how part of the network in which research, teaching, and students' experimenting in the classroom comes together in mathematics education.*

Two years ago, a student came to graduate studies in mathematics education (Maheux, in preparation) with the purpose of investigating Wenger's framework for the design of communities of practice (Wenger, 1998). Wenger's framework already inspired many researchers (see Boylan, 2004; Boaler, 2002) and was used here to conceptualize mathematics classroom situations. Nevertheless, we have found that, up to its experimentation with students, a situation goes through multiples re-inventions. In this paper, we focus on these inventions in order to understand what was involved in the whole process of conceptualization.

A Qualitative/Interpretative Standpoint Taking Account of Teachers' Perspectives

Ethnology of education (Woods, 1990) suggested working from a grounded perspective (Strauss and Corbin, 1990) from which meaningful elements or interpretative structures can emerge and be interpreted in order to constitute or enlighten a theoretical framework useful in a broader context. Being deeply rooted in context, ethnography requires taking into account of the role of every participant in action (Denzin, 1997). In that way, teachers' perspectives, which have shown to be a central component to consider for research related to practice (Schön, 1987; Bednarz et al.,2001), clearly need to be regarded.

This research took place between May 2005 and January 2007. At the beginning of the process, a few mathematics classroom situations were conceptualized during monthly meetings, gathering the graduate student and his two research directors. These situations were presented as 'ideas' to three public high school teachers who discussed them together as to the relevance of the situations for their own classrooms. A teacher volunteered to design and experiment, together with the graduate student as the researcher, one of the situations. All five classrooms sessions were recorded on video, while the researcher, acting as observer, took notes. A few students were interviewed before and during the study in order to understand the experiment from their point of view. From the very first idea to its experimentation with students, we reconstruct the whole development of the conceptualization of our situation in a richly descriptive narrative way (see Jaworski, 1994), and that reconstitution was afterward inductively analyzed.

Conceptual Framework Underlying the Analysis

The Webster dictionary defines the word *context* as "the interrelated conditions in which something exists or occurs". From a situated perspective, Lave (1988) has argued that a context can be defined in terms of the structuring resources that frame the ongoing activities of persons acting. These resources, that may support activity or interfere as constraints, are associated to the tasks and to the environment in which they take place, as much as to the individuals that experience them. That is why Lave talk about a mutual constitution between a "constitutive order" (form by a dialectical relation between culture and a given political economy and social structure) and the "experienced lived-in world" (as a person reconstruct for herself any ongoing activity regarding to her feelings, values, knowledge, etc. and in relation to her own activity in the actual setting in which they occur) (Lave, 1988). These structuring resources can be associated with various sources: knowledge, motivation, body, social relations, time, space, artifacts and so on (Jonnaert et al., 2005). These concepts of context and structuring resources were used to analyze our data.

Results

Seeking for structuring resources during the process of conceptualization of a situation in mathematics education leads us to divide it into episodes associated to different *contexts* in which intervene various structuring resources. "*For the research context"* occurs while the situation was mainly thought of 'by the researchers' and 'for research': while the 'world of research in mathematics education' was at the core, nevertheless, influences from classroom perspectives were already perceptible. "*For teachers' context*" appears when a very different version of the situation was built to prepare its presentation and discussion with teachers, conducting us to carefully choose what was proposed. "*With a teacher context*" happens to be the one of the close collaboration between the in-service teacher and the graduate student who worked together in order to conceptualize the situation that the teacher would experiment in one of her classrooms. "*In classroom context"* gives great importance to the students, now intervening directly in the situation, prompting the teacher to reinvent it in the very course of action. Finally, a "*While implementing context"* has been identified, since, after each class, the teacher and the researcher kept working on the situation to decide what would be done in the upcoming one.

The analysis of the process of invention taking place in these five contexts tends to characterize the contributions from what we called three communities of practice (Lave and Wenger, 1991): the world of research, the teacher's community and the students' community, each engaging in structuring resources in its own way. It also casts light on the richness of the structuring resources involved in this process of invention of a situation: *Intentions* (the project that motivates, what is to be realized), *Principles* (values, conceptions, what is cared about), *Ways of doing things* (strategies, action-knowledge), *Constraints* (obstacles or limits, what frames are imposed), *Roles* (what is expected from each other, social positions) and *Facilities* (what is available for action or understanding). These families of resources clearly informed us from a contextualized perspective on this process itself, all along the continuous re-invention of the situation. It also evokes and extends what we can see in recent research on teachers' professional knowledge (engaged in such a process). In addition, it gives us an interesting viewpoint on students' experiments in the mathematics classroom. Moreover, an analysis in terms of affordances and/or tensions between these families of resources and communities of practices might provide unforeseen ways of understanding, despite its complexity, how a part of the network in which research, teaching, and students' experimenting in the classroom comes together in mathematics education.

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FOSTERING MORE COMPLEX WAYS OF KNOWING IN COMMUNITY COLLEGE PRECALCULUS STUDENTS

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This case study of two community college precalculus classes examined relationships between classroom community, communication, and students' personal epistemologies (their ways of knowing and ways of learning). Within each class, we analysed classroom social norms, roles of instructors and students, instructors' portrayal of mathematics, and levels of communication. One instructor fostered a discussion-based class; however, he maintained authority for learning. Students in this class evidenced some change in their ways of learning but little changed in their absolute ways of knowing. The second instructor lead a didactic class that included efforts to challenge students' absolute ways of knowing. Students in the second class demonstrated some change in their ways of knowing but little change in their received ways of learning.

Mathematics education researchers have recently increased their focus on students' personal epistemologies (their ways of knowing and ways of learning) and how personal epistemologies may be influenced by the classroom as a community of practice (Boaler & Greeno, 2000; Solomon, 2006; Speiser, Walter & Glaze, 2005; Zohar, 2006). This case study of two community college precalculus classes examined relationships between characteristics of classroom community, communication, and students' personal epistemologies.

Theoretical Framework

The roles and social norms afforded to students in mathematics classrooms affect student learning. To understand the diverse range of forms of classroom communities, we begin by contrasting didactic and discussion-based classrooms. Within didactic classroom communities, students experience mathematics as a set of procedures, disjointed from real life (Boaler & Greeno, 2000). *Received knowers* believe knowledge to be certain and therefore accept knowledge from authorities (Belencky, Clinchy, Goldberger, & Tarule, 1986/1997). In didactic classes, students are not expected to discover or connect mathematical concepts on their own because mathematical authority resides with the instructor and textbook. In contrast to didactic classrooms, discussion-based classrooms are often characterized by small group and whole class discussions of open-ended problems and relationships become central to learning (Boaler & Greeno). Student roles consist of contributing to shared understanding and mathematical authority resides in students' own sense-making.

One of the most salient differences between didactic and discussion-based classes is the nature of communication that occurs. To examine communication within the two classes, we used a framework developed by Brendefur and Frykholm (2000) which describes four levels of communication. The least interactive level, *uni-directional communication*, is characterized by instructor lectures and closed questions; student answers do not provide information about student thinking. At the next level, *contributive communication* defines discussion that occurs when the instructor or students assist other students, or when students present their solutions, but classroom social norms do not oblige students to listen to each other. *Reflective communication* refers to discussions in which the instructor and students

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reflect on ideas offered by students who use these as opportunities for deeper explorations. This may happen as students attempt to justify or refute conjectures and is supported by sociomathematical norms in which students are expected to listen to each other and respond thoughtfully. Finally, *instructive communication* occurs when instructors provide situations that stimulate students to reconsider their current understanding of the concepts. The resulting communication informs the instructor of students' current conceptions and affects their instructional decisions.

Students' ways of knowing affect their beliefs about instructor and student roles and social norms (Baxter Magolda, 1992; Yackel & Rasmussen, 2002, Solomon, 2006). Conversely, social norms related to mathematical interaction (sociomathematical norms) affect student beliefs about their roles and classroom social norms (Cobb and Yackel, 1995; Solomon) and characterize their ways of knowing. The present study employed a framework of college students' ways of knowing developed by Baxter Magolda (1992). The lowest level of knowing in this framework, absolute knowing, includes receivers and masters who believe knowledge is certain and gained from authorities, while higher levels of knowing, such as independent and contextual knowing are consistent with current recommendations that students listen to their peers, explore, and become validators of mathematical ideas (National Council of Teachers of Mathematics, 2000). In order to develop more complex ways of knowing, students' current ways of knowing must be confirmed and challenged (Baxter Magolda). In this article we consider how characteristics of discussion-based and didactic classroom communities either confirmed or challenged students' ways of knowing.

Research Design

The emergent perspective guided the research design, data collection, and analysis. This perspective coordinates sociocultural and constructivist theories of learning by examining the social interaction in classroom communities, the mathematical development of the community, and students' individual mathematical constructions (Cobb & Yackel, 1995).

The present cases included two community college precalculus classes in the northwest United States studied during the summer quarter of 2005. We selected instructors, Mr. Reilly and Mr. Anderson (pseudonyms), based on their efforts to establish community and relationships with students, levels of classroom interaction, and reputation for strong mathematical knowledge and standards. Their classes had 14 students and 13 students, respectively. Questions guiding the inquiry included (1) What is the nature of the classroom community and how does it develop? (2) What is the nature of instructor and student interactions related to mathematical activity?

Data sources included a student questionnaire containing questions that elicited student preferences for their roles, their instructors' roles, and their peers' roles during class, in addition to questions about how students best learned mathematics. Other data came from student and instructor interviews, fieldnotes, and classroom observations. Each class was observed and videotaped 19 of their 31 meeting days. Researcher fieldnotes and classroom observation forms were completed for each observation. The first ten video-taped lessons from each class were transcribed, and information from fieldnotes and classroom observation forms were used to determine segments of other observations to transcribe.

Preliminary data analysis began when the first data were collected and continued systematically throughout the study; this analysis informed the selection of interviewees, interview questions, and interactions to examine (Tobin, 2000). After the questionnaire and after each observation or interview, data were coded into categories (e.g., participants' expectations for community and roles, the nature of social and sociomathematical norms and how they developed, participants' roles as they developed during the course and the relationship between roles and ways of knowing). Sub-categories were developed from the above-described research-based frameworks. For a fine-grain analysis of patterns, differences, and changes across the semester, we continued to create and modify categories to code and index references to various aspects of community and norms. We used data matrices and concept maps to understand the data within the categories (Miles & Huberman, 1994) and qualitative software to organize and reduce data in text form (*N6*, QSR, 2002).

Findings

Students' Initial Expectations

Students responded to the questionnaire on the first day of class. The majority of students in both classes responded that their mathematics instructor's role was to show them procedures while explaining clearly. This idea threaded through responses to several of the questions. To the question, *How do you best learn mathematics?* the majority indicated they learned mathematics best when someone showed them how or explained. When asked specifically about things the instructor could do during class to help them learn, 20 of the 27 students indicated that the instructor should do examples or explain clearly or in detail. Daniel's response, "Make sure they've answered my question so that I understand it" displayed the common idea that students' understanding was a result of the instructor's explanations.

Students also had specific ideas about their roles and the roles of their peers. In response to what types of input they usually offered during class discussions, 15 of the 27 students wrote they would ask questions, while only five responded they would offer other types of responses. Responses to whether listening to other students' questions or explanations helped them learn also indicated that students believed their peers' roles were to ask questions, but it was the instructor's role was to explain; 22 of the 27 responses specifically indicated students liked to hear other students' questions. Kathy explained, "Listening to their questions and hearing the teacher's explanation helps." Although students appreciated the ideas introduced by other students' questions, they wanted clear explanations from the instructor.

Student responses to the questionnaire indicated they expected passive roles in mathematics classrooms. While most students liked class discussions, fewer were willing to participate, and those who were willing perceived their roles as providing correct answers or asking questions, rather than offering ideas to be explored by their peers. Twenty-four of the 27 students indicated stances of absolute knowing in at least half of their responses.

Maintaining Absolute Knowing in a Discussion-based Class: Mr. Anderson's Class

Mr. Anderson established a discussion-based classroom environment, providing opportunities for group work during 14 of the 19 observations. He valued students' interaction with each other and believed discussing mathematics would promote learning.

> I'll put a problem on the board, or say, "Work on this problem, work together if you want or bounce ideas off each other; or check your answers; convince each other that you're right or wrong," sometimes I use that phrase. But sometimes they still want to get stuck in their little world; they interact with me, as a class, but … they're nervous about sharing with their fellow students. (Mr. Anderson Interview, July 5)

In spite of these values and providing opportunities for students to work together, he did not require students to work together. Throughout the term the same three students always worked alone while the other ten students often worked with peers. Mr. Anderson explained how to find answers, asked questions, and pointed out mistakes as he walked around.

Students who worked alone or in groups received his individual attention so they did not need to work with others. The following example reveals the nature of support Mr. Anderson offered during group work:

> Mr. Anderson: First thing, I see sum, you did product. [Thomas erases something on his paper]. So, sum is going to mean add. … Now did we square the second number? There you go, now we have the sum of the squares. [Mr. Anderson continues walking around the room.]
> Carol [to Mr. Anderson]: Does that look right?
> Mr. Anderson: Yeah, that looks great, keep going.
> Carol: So now I just [inaudible]?
> Mr. Anderson: Yeah, now you can either factor it, or use the quadratic formula, or complete the square. [He continued to walk around]…Oh, oh not just two. Yeah, *x* plus two…Square, not square root. (Observation, June 28)

We observed variations on Carol's question, "Am I doing this right?" throughout each group work. Thus, while the environment provided opportunities for students to become relational agents, Mr. Anderson's responsiveness to student preferences as absolute knowers allowed students to depend on him rather than build mutual accountability with their peers.

The other major activity besides group work was whole-class discussion. Mr. Anderson rarely lectured without encouraging student participation by asking questions, providing opportunities for students to ask questions, and providing long wait times. He further encouraged student participation by using their ideas to determine the direction of discussion, explaining and providing new examples until students indicated they understood. But, his responses did not challenge students' beliefs that it was the instructor's role to understand and explain.

The affordances of this discussion-based classroom affected some students' ways of participating in class. Three students who responded on the questionnaire that they preferred to work alone regularly worked with others. Two who said they liked to work with others worked alone, most likely because they were physically isolated from others in the large room and were not required to work with others. Students who worked with others were more likely to speak up during whole-class discussion even if they responded on their questionnaires that they would not participate, while those who worked alone did not participate even though they responded on their questionnaires they would. Thus, group work seemed to support students' willingness to participate in whole group discussions.

Social norms and roles affected the levels of communication in this class. The average number of lines coded at each level of communication in days 2-5 of the term were 52% uni-directional and 48% contributive. No reflective or instructive communication was observed during the term. While Mr. Anderson used student ideas and questions to introduce new problems and examples, a component of instructive communication, these were not coded instructive communication because students continued to rely on instructor explanations and did not necessarily modify their current understanding but accepted explanations without reconsidering their old conceptions.

Evidence of this type of learning arose in an interview with Sarah. The class had graphed circles written in the form $(x-h)^2+(y-k)^2=r^2$ earlier in the week, then discussed transformations of graphs. Finally, on the day of this interview they discussed the graph of quadratic functions written in the form $y=a(x-h)^2+k$.

> When we were doing the vertex, how it's the negative, [pointing to $y=2(x-3)^2-4$ in her notes] well the three and the negative four. Well, I just assumed from circles, doing *h*, *k*, that it would be the opposite, so [the vertex would be at] three, four. So that's what I

> was thinking, and so when he first wrote that down I was like, well that doesn't make sense, but then after we wrote down what we needed to do to the parabola [shift right three and down four] then I was like, of course, if you're starting at zero, zero as your starting point and you move, you know, to the right three and then down four, and if I graph that, then that's where the vertex would be, so I had to write this down, yeah, cuz when he first said that I was like, well wait a second that's kind of opposite from what we were doing with circles, cuz circles you do the opposite signs as the center. (Sarah Interview, July 13)

Sarah initially compared the form of the quadratic function to a rule for locating the center of a circle from its equation, apparently connecting the use of the letters h and k. However, when the rule did not work and Mr. Anderson justified the location for the vertex using transformations, she said it made sense. That is, Sarah said it "made sense" when she knew which rule to choose based on Mr. Anderson's explanation; she did not pursue her initial idea by questioning why the rule for equations of circles seemed different from the rule for transformations. Thus, "making sense" takes on different meanings for different ways of knowing. In Sarah's case, it meant knowing which rule to choose based on a single explanation from an authority whereas to contextual knowers, it means resolving perturbations in their own ideas while taking into consideration others' ideas.

Fostering More Complex Ways of Knowing in a Didactic Class: Mr. Reilly's Class

Mr. Reilly maintained a didactic form of teaching throughout the course. Mr. Reilly stayed at the chalkboard coordinating all communication and did not provide opportunities for students to work together in class. Mr. Reilly planned and executed each lecture carefully, developing concepts by connecting to previously discussed ideas. However, students did not work together and did not respond to each other during whole-class discussions. In contrast to students' expectations, he rarely worked examples, but modelled and discussed a disposition to do mathematics by solving problems in more than one way, discussing strategies, using multiple representations, and emphasizing the need to explore. He explicitly discussed metacognitive practices, reflecting on or asking students to reflect on both processes and results. For example, throughout verifying an identity, he asked students to give reasons why they selected certain procedures and emphasized they should have mathematical reasons. After verifying the identity, he asked "Could we have done it better? Let's do it again and see if we can do it better." The next day, after verifying another identity using students' suggestions, he said, "Look at and think about what you did," and pointed out where they had reversed a step on the first verification (Observation, July 13).

Mr. Reilly also portrayed mathematics as uncertain and developed by humans throughout history while trying to solve problems. In particular he described his perspective of mathematics as different from the authors' of the textbook, and concluded, "Don't think of math as this eternal unchanging entity completely divorced from opinion." Throughout the term, he incorporated stories about the history of the mathematics and real life applications into the lessons. Students clearly valued these elements and suggested it helped them connect to the mathematics:

> I had no clue of all those things in math; you can actually teach the background of where it came from. There's books you can even read. Well nobody's ever done that in any math class I've ever had in my life. It's just, this is math, this is what we do... I think it's wonderful. (Natalie Interview, August 10)

Portraying mathematical knowledge as uncertain and created by people challenges students' absolute ways of knowing: "Clarifying that the information presented comes from a particular perspective and is generated by other human beings is essential for students to begin to see themselves as capable of forming their own perspectives" (Baxter Magolda, 1992, p. 278).

Because of learner and instructor roles, communication in this class remained uni-directional and contributive. In days 2-5: 64% of communication was coded uni-directional and 36% contributive; no instances of reflective or instructive communication were observed. Uni-directional communication included short-answer questions and once a student answered correctly, Mr. Reilly accepted the answer as if the whole class understood. For instance, throughout one whole-class discussion focused on solving a problem, Mr. Reilly emphasized problem-solving techniques of understanding the problem, having a plan, and reflecting on the answer. His questions at the beginning of the episode encouraged contributive communication, however he did not pursue suggestions that would be unproductive. While students offered several suggestions of sides and angles he could find, he said they needed to have a plan. As he used some of their suggestions to write a plan, he labeled the sides or angles without calculating, and pointed out that while they could find certain lengths and angles, many of the results would not get them any closer to finding a solution to the problem. After finding an answer, he reflected on the process and whether or not the answer was reasonable. The goal of this activity was to explicitly discuss processes of mathematical problem-solving. But, students did not have the opportunity to discuss and work on problems during class without Mr. Reilly's authority guiding the process. It was apparent that students realized the necessity of changing the way they approached learning mathematics, but that they did not know how. During our interview I asked Steve what he was going to do to get ready for the next test:

> [Laughed] All the things that have been taught to me in my previous math classes, which are probably all the things that I shouldn't be doing, because I've been kind of bred to do things a certain way. And I think he's trying to break that mold that I've been put in and I see that in him. But I'm not sure how to at this point unless I took the teacher home with me. (Steve Interview, July 21)

While Steve recognized and appreciated that Mr. Reilly expected him to learn differently in this class, he did not develop strategies for learning in new ways.

Discussion

The two cases indicate that in order to understand relationships between classroom characteristics and students' personal epistemologies, it is important to examine finer-grained characteristics beyond the broad strokes of either *discussion-based* or *didactic.* Some of the domains we sought to better understand included learner, peer, and instructor roles; the certainty by which mathematics is portrayed; and types of communication.

Student and instructor roles during whole-class discussion were more alike in the two classes than different. In both classes, students gave short answers and almost no explanations, and most students usually only answered if they thought they knew the right answer. The instructors supported these roles since they made very few attempts to encourage students to explain or to influence the nature of student explanations. Instructors answered all questions with thorough explanations rather than shift the responsibility for thinking back to the students, and also interpreted and expanded on students' short answers rather than ask the students what they meant. Learner, peer, and instructor roles were consistent with the preferences students indicated on their questionnaires. Because the social norms and roles maintained authority with the instructors, communication during whole class discussion in both classes remained unidirectional and contributive.

Student roles differed in some ways in the two classes. In Mr. Anderson's class, students asked more questions and provided more answers and ideas than in Mr. Reilly's class.

However, not all students chose to work together and Mr. Anderson aided this choice by answering their individual questions and not requiring peer interaction. In addition, students who worked together still asked Mr. Anderson if they were correct and relied on him to explain the mathematics. Thus, while discussion-based classes offer opportunities for students to become relational agents, helping each other understand mathematical concepts (Boaler & Greeno, 2000), such environments do not necessarily foster this type of relationships since students and instructors may not change their beliefs about their roles and the source of knowledge.

Mr. Reilly's portrayal of mathematics challenged student beliefs about the certainty of mathematical knowledge while his emphasis on understanding concepts, attention to problem solving strategies and metacognition, and his use of real life applications challenged absolute ways of knowing. Yet, students were largely observers of Mr. Reilly's engagement with the mathematics rather than members of the community whose ideas were elicited and valued, and contributed to their peers' understanding. Mr. Reilly's case illustrates how students can begin to change their beliefs about ways of knowing mathematics without changing their received ways of learning.

This study indicates that to foster more complex ways of knowing, instructors need to create discussion-based environments and challenge students' belief in the certainty of mathematical knowledge. However, these elements are not sufficient. Instructors must also challenge student beliefs about learner, peer, and instructor roles, and support students' own construction of mathematics and ways they interact with their peers and instructors. More research is needed to understand how these elements can be incorporated into mathematics classes and how they influence learning and ways of knowing.

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"JUST DON'T": SUPPRESSING AND INVITING DIALOGUE IN THE MATHEMATICS CLASSROOM

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Responding to concerns raised by grade 11 mathematics students, we examined a broad set of mathematics classroom transcripts from multiple teachers to examine how the word 'just' was and could be used to suppress and invite dialogue. We used corpus linguistics tools to process and quantify the large body of text, not to describe the nature of the discourse, but rather, in the tradition of critical discourse analysis, to prompt reflection on a range of possibilities for directing classroom discourse. Drawing on Bakhtin's distinction between monoglossic and heteroglossic utterances, we found that the word 'just' acted as a monoglossic tool, closing down dialogue. However, we propose that 'just' can also be used as a heteroglossic tool as it can focus attention and thus invite dialogue.

> *Just* is OK for students to use. Teachers shouldn't use *just*. Teachers JUST shouldn't do it.
>
> [W]hen [teachers] use 'just' it's kind of an aggressive word. It's kind of like they just use 'just' because they don't want to explain why it is. They just say, 'It's just that.'

These two utterances come from grade 11 mathematics students who were conversing with their classmates about their language practice. In their animated discussion about the word 'just', some expressed concern that 'just' implies simplicity and might frustrate students who may not find the process so simple – e.g., "And you just change it to two square root five." These students' exploration of the word revealed how they felt about it and the classroom dynamics it represents. (For more detail on this conversation with students, see Wagner, 2004.) Taking their perspectives seriously, we looked for evidence of their concerns in a large body (corpus) of mathematics classroom discourse collected in 2005-06. The word 'just' was the 27th most common word used in this body of classroom interaction (4672 unique words): nine times more common than 'multiply', four times more common than 'why', twice as common as 'because'. We asked:

- How is the word 'just' used in mathematics classroom discourse?
- What can we learn about the way students and teachers relate to each other by looking at its use in practice?

It may seem somewhat frivolous for educators to obsess about one word to initiate reflective practice, but the vehemence with which the students in the above-described situation asserted the significance of their concerns surrounding the word and their tenacity to sustain conversation about it over two months justifies serious consideration of the word. Perhaps research questions too often arise out of the experiences of educators and thus ignore the questions raised by mathematics students or assume the questions are the same.

Interpretive Frame

Following traditions of critical discourse analysis (CDA), our approach "includes linguistic description of the language text, interpretation of the relationship between the […] discursive processes and the text, and explanation of [their] relationship" (Fairclough, 1995, p. 97). Consideration of possible meanings of language can raise awareness of interpersonal dynamics. Quantification can help us recognize common discourse patterns that may help us notice such dynamics in our own practice. We promote change in practices by making common practices seem strange and not innocent. Our synthesis of corpus linguistics and CDA methodologies (traditions that criticize each other, e.g. Stubbs, 1996 vs. Chouliaraki & Fairclough, 1999) sees the value of corpus linguistic tools to draw attention to particular instances of language for denaturalization and consideration of alternative practice.

As noted by Chouliaraki and Fairclough (1999), CDA researchers need to be aware of the social 'problem' that drives and informs their interpretation. Our concern (sense of a 'problem') is for mathematics students' positioning in their classroom discourse, especially the aspects noted by students in Wagner's (2004) conversations mentioned at the beginning of this article—the students' concern for implications of simplicity and their opportunity for agency. The word *'positioning'*, as described by Harré and van Lagenhove (1999), refers to the way people use action and speech to arrange social structures. For example, there are language forms that a teacher can use to structure a social arrangement that resembles the physical arrangement common to many classrooms—students sitting apart from each other beneath the teacher who stands front and centre.

Though the traditions of mathematics classroom discourse already position students and teachers in certain ways relative to each other, discursive moves within particular instances of the discourse substantiate, and have the potential to alter, these structures. With our interest in positioning, we find significance in a distinction made by White (2003) in his "appraisal linguistics," which draws on Bakhtin's (1975/1981) notion of heteroglossic interaction as opposed to monoglossic utterances. White suggests that linguistic resources can be "broadly divided into those which entertain or open up the space for dialogic alternatives and, alternatively, those which suppress or close down the space for such alternation" (White, 2004, p. 259). The content of speech can invite or suppress expressions of agency, but this can also be done with the form of the speech, the medium through which content is indexed.

White's (2003) interpretive frame has not been applied to studies of the word *just* in general or in specific contexts, but it relates to other linguists' studies of the word. Aijmer (2002), for example, noted the restrictive nature of *just* when it is used as an adverb. It closes off aspects of potential dialogue. More significant to our analysis, she and others note the power of the word in persuasion. Weltman's (2003) study of political discourse demonstrated how *just* was used to 'justify' the refusal to give explanation. Such refusal defies a strong social expectation identified by Grice (1975): his maxim of quality describes the expectation for adequate evidence. Linguists and others also note less overt ways in which *just* represents closed dialogue. Weltman showed how it represents repression, "nudging the conversation away from certain sensitive matters" (p. 351), and Spruiell's (1993) consideration of the word in psychoanalysis reminds us that such repression may not always be conscious. Whether the repression is rhetorical or subconscious, there are consequences—dialogue is suppressed. Most significantly for mathematics teaching, explanations are suppressed.

Data Sources and Analysis

The larger data set from which we draw includes 148 classroom observations (1) from eight mathematics classrooms (grades 6 through 10) in seven schools. The teachers in these classrooms were purposefully selected to vary gender, context of teaching situation, certification level, years of teaching experience, and so on.

For this paper, we draw on the classroom observations from January because the discourse patterns were fairly stable by then. We analysed 184,695 words including 931 uses of the word 'just'. Our description of language practice used corpus analysis—the quantification of utterances from a large corpus (body of discourse). Our corpus is reasonably-sized for oral text, It is larger than the oral corpus analyzed by Tagliamonte's (2005) investigation of *just* and other 'discourse particles' in general speech.

Wordsmith Tools 4.0, a corpus linguistics software, generates concordances, in which all the uses of a particular word are listed, as shown in this sample excerpt.

uestions on it? € FS: No. € T: Okay, so I'm	***just***	gonna go around and around as best as I can,
we'll talk about surface area and volume in	***just***	a second. Okay? Alright, ahm, now, if we thi
he slant height. Okay? So it's got two parts,	***just***	like that had two parts, they are the base plus
Area of a rectangle? € FS: I'll think about it.	***Just***	tell me the surface area. € T: I'm working on
are you looking at? € FS: For any of them. I	***just***	don't know what the formula is. € T: Well, w

A research assistant and one of the authors used a spreadsheet for independent coding and then discussed and achieved consensus on discrepancies. We began by looking at possible meanings of 'just' in the range of its uses by attempting to replace 'just' with other text to understand the position it occupies. This form of representation supports our interest in encouraging the consideration of alternative practice. Wordsmith also calculates collocations – words that are commonly located together (co-location) – a tool useful later in the analysis.

Findings

Shades of Meaning

In the presentation we will address all the shades of meaning we found; in this paper we focus on the three most prevalent forms, which include the two most relevant to the students' concerns. In the corpus' most common usage of 'just' (28% of occurrences), it is an adverb relatively synonymous with 'simply'. For example, in the teacher utterance, "After we've changed it to an improper fraction we *just* plug straight through," listeners may hear "we *simply* plug straight through" – it is simple. Significantly, this most common shade of meaning is the one that dominated student concerns about being positioned as powerless (Wagner, 2004).

Another common usage (21% of instances) is relatively synonymous with 'only', as in "I want to go over *just* one type of each problem." This usage was not discussed by the students who were concerned about 'just' and seems relatively innocuous in terms of personal positioning in the classroom. However, many of the instances we coded with this shade of meaning were relatively ambiguous: whether we replace 'just' with 'simply' or with 'only' the utterance makes sense but means something considerably different. For example, "we just do 11 through 29" could mean that it will be simple and unproblematic to complete these problems for homework, or it could suggest recognition of the (perhaps considerable) amount of work in the entire problem set by restricting it to 11 through 29. For this instance, contextual clues led us to code this with the second interpretation, but it is important to be aware that the first interpretation and others were possible for students in this classroom.

A powerful usage of 'just' includes situations that represented varying degrees of frustration (22%). A strong degree of frustration can be seen in "Well, you *just don't* want to have two that are […] exactly the same lengths." Here 'just' could be replaced with an expletive or an expression with similar meaning: "You really, really don't want to …". In a usage that expressed more mild exasperation, "Don't look, just put your name down," 'just' might well be replaced with the aside: "do it without asking why." This usage can also represent gentle encouragement as in "Hit, just hit *enter*," which was a teacher's reply to a student wondering how to do something with his calculator. This utterance is similar to "Trust me. Hit *enter* and don't worry about why yet," but students may read greater exasperation than the teacher intended. These instances are often ambiguous. The teacher telling his student to hit enter could be pleading that this is a simple thing to do, representing the most common shade of meaning. To distinguish among these levels of frustration, interlocutors depend on paralinguistic cues, which may also be read in various ways.

23% of the instances confounded our coding. Often this ambiguity seemed to be a result of the speaker's frustration, as in "Oh, I know … It's just, as we started this …" These cases of extreme ambiguity, in addition to the kinds of instances of ambiguity we have described above, in which we coded a particular meaning based on contextual clues, prompt us to ask, What is the experience of the listener? Listeners typically judge intentions and meanings quickly and subconsciously. All these shades of meaning relate to each other in some way and all the meanings ought to be taken as activated to some extent every time the word 'just' is used. In the next section, we draw on linguistic literature to interpret some of these shades of meanings that we think are especially important to mathematics teaching and learning. These interpretations led us to further investigate the corpus, paying particular attention to some of the differences between teachers' use and students' use of the word 'just'.

Suppression and Invitation

When 'just' implies simplicity it is a monoglossic tool. It suggests that thought is not necessary, and thus there is no call for others to respond. Such implications are qualitatively different when they are made in imperatives or in statements. When it is in the imperative, it calls for performance not reflection (e.g., "Just solve the equation"), and it directs others to follow only an authorized path. When it has a personal subject (e.g., "and then I just solve the equation"), it is less directive, but still significant. In such statements, it is *reflecting*, instead of *directing*, a monoglossic relationship structure. Such uses suggest that there is nothing to discuss: equation-solving is unremarkable. In this case, the simplicity-suggesting sense of the word relates to the meaning synonymous with 'only': don't think, only act (or watch/listen). In addition to making reflection seem redundant, 'just' positions a reflective listener as incapable: "just/simply solve" suggests that a person who has to think about how to solve the equation is no expert.

When 'just' replaces 'only' it limits, but not necessarily as a tool for the monoglossic. The speaker refers to distinctions being made, opening up the possibility for others to make a different kind of distinction.

In our view, the strongest monoglossic shade of meaning of 'just' is the directive: "Do it without asking why," which is suggested in most of the instances in which 'just' marked a sense of frustration. As stated above, even the simplicity-suggesting form discourages reflection and personal agency, but 'do it without asking why' is explicit.

Suppressing dialogue is an act of power. Thus we think educators should ask: Am I suppressing dialogue in my mathematics classrooms and, if I am, for what purposes? To begin to address this question using other educators' practice, we will turn to distinctions in the form of the text in our corpus. However, it is important to acknowledge that tools for the monoglossic need not suppress dialogue. Any of these tools can be 'retrospectively dialogic' (using White's term) because fighting against alternative positions can, in fact, draw attention to alternative positions and thus open up reflection and its potential to underpin acts of personal agency. For example, when a teacher said, "Okay, that's just kind of a personal preference. Some people like to solve them vertically…," he was pleading with his students to accept multiple approaches to solving this kind of equation. In cases such as these, the speaker uses 'just' to say how important an idea or approach is to him, he is begging for complicity, and thus positions himself as a supplicant and his listeners with power.

Furthermore, we would not argue that all uses of monoglossic tools are wrong. As we will discuss below, we are not hoping that educators simply eradicate 'just' from their vocabularies. Rather, we want to consider how the word can be used thoughtfully.

Participants' Agency in Suppression and Invitation

In the analysed corpus, students and teachers used 'just' with fairly equal frequency (1:195 words for students and 1:196 for teachers). Though there is much the same about the teacher and student uses of 'just', it is important to remember that student language choices are socialized by their teachers' constructions: as explained by Bakhtin (1953/1986), participants use each other's words in any discourse. Nevertheless, we begin to see distinctions when we look at the words connected to 'just' using collocation charts. The most frequent L1 collocates (words appearing one position to the left of 'just') for both teachers and students were the personal pronouns, 'I', 'you' and 'we', as shown in this excerpt from the top of the collocation chart (at right).

Word	L5	L4	L3	L2	L1	Total Left
YOU	17	13	16	41	83	170
I	8	7	16	16	83	130
WE	6	3	2	14	46	71
IT'S	2	7	3	2	36	50
I'M	1	2	2	1	23	29

'You just' and 'I just' are the most common pairing, with 83 instances each. These numbers increase as we include derivatives (e.g. I'm) and L2 collocates which allow forms such as "I am just."

The frequency of these personal pronouns prompted us to distinguish between teachers and students because these words draw attention to distinctions in role. Limiting to L1 personal pronouns, 35% of teachers' subjects were second person (you, you're, you'll), 41% were first person singular (I, I'll, I'd, I'm) and 25% first person plural (we, we're, we'll). For students the percentages were 40%, 44%, and 17% respectively. In half of the classrooms considered, students did not use the first person plural with 'just' at all.

The proportional similarity between 'I just' and 'you just' for teachers and students may indicate complicity. We assume that the teacher has more power, and that students (positioned as novice) mimic the form (and content) of their teachers' utterances, but students do carry some power in structuring discourse norms. The strong difference between teachers' and students' use of 'we just' may support the assumption that teachers carry more authority in the structural formation of the discourse. Teachers feel justified speaking on behalf of the local classroom community, but students do not share this sense of authority. Pimm (1987) and Rowland (1992; 1999; 2000) have noted that teachers use 'we' to represent the voice of the mathematical community. Our corpus shows mathematics teachers using 'we' in this way when it precedes 'just', and does not show students using it in this way.

Positioning the Classroom Participants

The frequency of personal pronouns preceding 'just' also prompted us to consider the verbs modified by just. In mathematics, processes have significance. The most common verbs to follow 'just' in the R1 position were 'go' (12%), 'do' (8%), 'say' (7%), 'have' (7%), 'want' (6%), 'make' (5%), 'write' (4%), 'put' (4%), 'take' (4%) and 'think' (4%).

The most common of these are extremely vague. What did it mean when the teacher said, "Just do it one step at a time"? The verb 'do' describes action but it could describe any action. The teacher implies that there is no need to explain. 'Go' and 'have' are similar. Many 'just' + verb combinations include a condensation of meaning that makes assumptions about the listeners. We concentrated on instances of 'just' in the context of talk about mathematics rather than other, nonmathematical topics (e.g., "just wait [for the bell]"). Schleppegrell (2004) listed 'density' as a distinction of the academic register. We ask whether 'just' along with the verbs it modifies is an example of this condensation of meaning. A simple vague expression – "just go" – carries with it many meanings, but such vagueness seems to be a special kind of condensation that is different from conciseness. We suggest that further research address the vagueness inherent in 'concise' condensation. Nevertheless, Rowland (2000) makes clear the value vagueness can have in mathematics classrooms. Vagueness can invite reflection. And it can close it down.

The less vague common R1 collocates of 'just' can be considered in terms of Rotman's (1988) classification of imperatives used in mathematics discourse. He distinguishes between 'exclusive' verbs, which describe action that can be done independent from others (e.g., write, calculate), and 'inclusive' ones, which require dialogue (e.g., explain, prove). We would expect the actions 'just' modifies to be exclusive rather than inclusive, because inclusivity opens up space for dialogue and 'just' tends to close down such space. Most of the verbs commonly modified by 'just' were exclusive and monoglossic. Because a student can 'want', 'make', 'write', 'put' or 'take' something independent from relationship, these verbs are categorized as exclusive. It is more complex to consider the remaining common verbs, 'say' and 'think'. What does it mean to *just think*? We have suggested that 'just' implies a rejection of reflective thought. Though we might categorize thought as exclusive action because it can be done alone, prompting students to think is certainly not monoglossic.

Significance for Reflection

This data raises many questions. We see that the more questions we pursue in the corpus, the more questions are raised leading us to look more and more at individual utterances in their contexts, which makes for increasingly complicated calculations and decreasingly statistical significance.

This draws attention to our intentions. Statistical significance is less important for us than it is for linguists whose work is to document discourse practices. Our aim is to describe the state of mathematical discourse only to the extent necessary to prompt reflective awareness. We encourage our readers to note the questions our data and interpretation raise, to ask these questions of their own practice and to apply them to classroom research contexts. We do not want to say, "This is how mathematics discourse is." Rather, we want mathematics educators to ask, "What is my discourse like?" and "How might I change it to reflect my intentions?"

It is probably evident that we lean toward promoting heteroglossic discourse as opposed to monoglossic discourse. However, we recognize that particular discourse moves that can be characterized as monoglossic have their place. As we have noted, utterances can be retrospectively heteroglossic, opening up a possibility for dialogue by suggesting that there is only one way of doing or seeing something. Furthermore, one of the teacher's primary roles is to direct attention appropriately. Closing down dialogue in one area opens the possibility for focused dialogue in another. Gattegno (1984) noted that such stressing and ignoring is commonplace and he often asserted and demonstrated that these processes are especially important in mathematics. He claimed that stressing and ignoring is in fact the process of abstraction.

There are various alternatives available to a teacher who wants to direct attention to a certain area and away from other concerns. One can say explicitly, "Don't think about [some thing]," rendering it quite impossible to avoid thinking about the thing. Alternatively, one can employ subtle tools of the monoglossic, like the word 'just', to direct attention away from some processes and thus invite attention to other processes. This kind of subtlety is powerful, as it invites general dialogue focused in a particular way and also because of its potential for structuring a monologic environment in which student agency is suppressed. The power is in the subtlety.

Tools for the monoglossic are especially powerful in environments structured with significant positioning distinctions. Mathematics classrooms are just such places.

Endnote

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MAY I HELP YOU? PUBLIC SERVICE AS A FACTOR IN CAREER SELECTION

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Although many talented students leave the disciplines of mathematics and science, women do so at a higher rate than men. In mathematics, the under-representation of women is most significant at the graduate level and in the workforce. The purpose of this study was to investigate why women mathematics majors choose to study mathematics at the undergraduate level and what factors influence their decision whether or not to continue with a career in this field. We found that many of the participants desired a career in which they felt like they were helping others and making a positive impact on society. Therefore, teaching mathematics in the context of social justice and educating students on mathematical careers that have a direct positive impact on society may be useful means for attracting and retaining women in the field of mathematics.

Many researchers have noted the under-enrollment of talented students in the Science, Technology, Engineering, and Mathematics (STEM) fields. Many students who demonstrate potential success in these fields choose majors in non-STEM fields and a higher percentage of students switch from STEM fields to non-STEM fields than the other direction (Seymour & Hewitt, 1997). It has also been noted that women are less likely to enroll in a STEM major (Clewell & Campbell, 2002) and those who do, tend to change their majors to non-STEM disciplines more frequently than men do (Seymour & Hewitt, 1997). The low percentage of women earning STEM degrees at the university level then transfers into low participation in the workforce.

Although the number of women in STEM fields is low in general, in the field of mathematics, the under-representation of women is most significant at the graduate level and in the workforce. Over the past decade, women have consistently earned approximately 47% of the bachelor's degrees in mathematics; however, women only earned 41% of the master's degrees and 25% of the doctorial degrees in mathematics during that same timeframe (NCES, 1995-2005). Furthermore, the proportion of women recent-PhDs currently being hired as faculty in mathematics departments is lower than the proportion of women receiving doctorial degrees in mathematics (Gordon & Keyfitz, 2004). This trend of low participation exists in other mathematics-based careers as well.

Some possible reasons suggested in the literature as to why women are less likely than men to pursue the field of mathematics include: lack of confidence (Davenport, 1997; Seegers & Boekaerts, 1996), social and cultural influences (Fox & Soller, 2001; Sadker & Sadker, 1994), and mathematics curriculum (Becker, 1995; Boaler, 2002). Most of this research, however, has been conducted with the purpose of exploring why women do *not* choose to study mathematics. Considering the statistics given above, however, a substantial number of women currently *do* choose to study mathematics at the undergraduate level. These women simply are not choosing careers in mathematics. Therefore, it is also important to consider *these* women's experiences with mathematics.

The purpose of this study was to investigate why women mathematics majors choose to study mathematics at the undergraduate level and what factors influence their decision of whether or not to continue with a career in this field.

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Method

The participants in this study were eight undergraduate women mathematics majors at a large public university in the Midwest. Five of the participants were planning to become high school mathematics teachers; three of them were not. (We initially intended to have five participants from each category, but due to certain restrictions outside the control of the investigator, we were only able to secure three participants from the second category.) Each of the participants had either junior or senior class standing.

The main data source for this project was a series of three 90-minute interviews conducted with each of the participants. We utilized the interview technique known as in-depth, phenomenologically based interviewing. "In this approach interviewers use, primarily, open-ended questions. Their major task is to build upon and explore their participants' responses to those questions" (Seidman, 1998, p. 9). We followed the Three-Interview Series protocol as suggested by Seidman (1998). During the first interview, we collected information about each participant's mathematical life history. The second interview focused on each participant's experiences thus far as a mathematics student at the undergraduate level. During the third interview the participant was asked to reflect on the meaning of their experiences and to discuss their future career and life goals. Each interview was audio and video-recorded.

After the interviews were transcribed, we began analyzing the transcripts looking for dominant themes. The video-recordings of the interviews were watched while reading the transcripts in order to capture any non-verbal communication that might be relevant to interpreting meaning within the participants' stories. Due to the qualitative nature of the data, we did not approach the data with pre-determined categories developed in another context. Rather, we allowed the categories to emerge from and be grounded in the data itself (Strauss & Corbin, 1998). Once tentative categories were formulated, the interviews were reread and recoded with respect to these categories. After the interviews were coded and the categories were refined as necessary, we utilized the coded data to draw conclusions with respect to the research questions.

Results

Most of the women in this study developed their interest in mathematics as a young child. Family members (fathers, in particular) played a large role in these women's interest in mathematics. All of the participants considered themselves to be good at mathematics and had been in advanced mathematics classes since middle school. Many of these women, however, excelled in a variety of disciplines; some of them graduating toward the top of their high school class.

There were a variety of reasons that these women chose mathematics as a major. Many of the participants spoke of always liking mathematics and because they were good at it, it seemed to be the logical choice of major. Some chose mathematics because, as future teachers, they felt it would be marketable. Others talked about the benefit of not having to write as many papers in mathematics as they would in other disciplines. Two of the participants originally had chosen majors in the sciences, but found their college mathematics classes more enjoyable, so they switched their major to mathematics. While none of the participants had switched from a non-STEM major to mathematics, two of them were double-majoring in mathematics and Spanish.

All of the participants talked about the strange reactions they frequently get when they tell people they are majoring in mathematics. These reactions often suggested that these women must be unusually smart or that they must be "geeks" (i.e. have no social life). Many of the participants, however, both like and dislike these reactions. They often spoke about how they enjoy the prestige of earning a degree in mathematics. Many of them claimed that one reason they chose to pursue mathematics is because it is considered to be a challenging subject. Furthermore, they also like not fitting "the image." For many of these women, both those going into teaching and those who were not, majoring in mathematics meant to them that they were breaking the gender boundaries.

Despite choosing mathematics as a major, none of the participants in this study knew of many careers available to them with a degree in mathematics. Several of the participants actually spoke of how limiting their degree was and claimed that this would be an obstacle to someone wanting to earn a degree in mathematics. One student claimed:

> If you're not going to be a teacher, that's just an obstacle in itself. I mean, how many jobs are really out there for people who get math degrees? Like, there's some, but it's not like. You know, like it's very specific things you can do… You can't just get any job and use what you learned with your math degree.

All five participants who were earning a degree to become secondary school teachers stated that they had chosen the profession of teaching first, and then decided they wanted to teach mathematics. Two of them even claimed they had made their decision to become teachers as early as third grade. One participant suggested a variety of reasons for choosing this profession, yet the other four mentioned only one reason. They chose teaching because they want to help people and want to make an impact on society. One future mathematics teacher stated:

> [My parents] were good role models in trying to decide what to do and knowing to help people and stuff like that. So that's why I kind of think of teaching as my method of, uh, helping people I guess.

Another claimed:

> [Teaching is] contributing to that experience for other people. I think that's like a way you could have a good impact. Like that could be done with my life. I could be like, "Hmm, I helped all these people!"

One participant discussed how while growing up she had always wanted to become an engineer and design airplanes or sports cars, however, after learning of one of her best friend's life-stories during her junior year of high school, she changed her future career aspirations:

> S5: I really wanted to be an Aerospace Engineer. Cuz I love, I just. Okay, there's just *really* cool shapes that you can make using like math. And like planes, you know, the Stealth Bomber. And like I just; that stuff is so interesting to me. I would love to be able to like design or to just make a design plausible. Like, take someone's cool design that looks the most sleek and awesome, like a Viper or something… And figure out how to make it plausible, by using math.
>
> I: Right.
>
> S5: That's what I wanted to do for a long, long time.
>
> ….
>
> S5: At that point in my life I had talked to a friend. Like this is totally separate. But I had talked to a friend who, umm, who was, umm, a Korean adoptee, and. Cuz she was adopted from Korea. She, I think she only lived there for like three months. She was very, very young when she was adopted… Umm, but she told me, you know, her story and what happened and I just thought, you know, there are so many people out there that have this amazing potential. Like [she] is an amazing Clarinetist… I think she might have been student president for a little while. Umm, she's absolutely gorgeous. Stunning.

Stunning. Like one of the most beautiful people I've ever met. And, umm, she's a great actress and she was band, she was the drum major. Yeah. She's really, really an amazing person, and if she hadn't been adopted. You know what I mean? Like if she hadn't had all the opportunities. And I just feel like everybody has that potential. It's just that people need to cap the opportunities. And so, I wanted to become a teacher.

I: Okay. And so, your friend's story is what ultimately influenced that?

S5: Yeah.

I: Wow!

S5: And I, I kept the math thing, cuz I, you know, cuz I love math and because it's marketable. (laughs)

At a later point during the same interview, this student expanded further on her decision to leave her dream of becoming an engineer in order to become a teacher:

S5: I guess it's more of a moral thing. Like I went to the auto show, the Detroit auto show, last July or January when it was, but um. I *love* those cars. *That's* what I wanted to be a part of. I wanted to be a part of making that car. But, then I see like, what about. I see all the kids that are there and I'm like, "What about the kids' education?" You know? I mean I care about that more. That's so much. I don't want to promote something like owning a nice car. I mean, I'd rather teach people how to get a good education and make good decisions for themselves and inspire them, you know. And to like, you know, challenge themselves. That is so much more important to me than a car, you know. So um, that's what changed my mind.

Although this student had a real passion for engineering, when it came to making future career decisions, she found that her passion for helping others took precedence.

Those participants who were not interested in becoming secondary school teachers, in general, were less certain about their future careers. One of these participants, however, had already decided to leave the field of mathematics in order to pursue a career in non-profit work. She explained her decision as follows:

> If I were to [become a mathematician], it wouldn't be satisfying to me… It doesn't seem very rewarding or fulfilling. If you spend your whole life like trying to come up with some new equation, you know, it's cool because you're still doing something, but I don't know. I need more of a reward than that… I guess knowing that you've helped like one person is like enough for me… Going into the non-profit is like - for me it's like doing social activism. Like, trying to change the system, the standards. Trying to like bring some sort of justice to people that deserve it.

Furthermore, this student claimed that she did not believe that her knowledge in mathematics would play a role in her future career in non-profit work, other than if she became an accountant for a non-profit organization.

Although the other two participants not planning to become secondary school teachers did not speak directly about desires to have a career involving public service, both demonstrated subtle inclinations towards this effort. One of the two has decided that after graduation, she will spend two years in the Peace Corps. Following that experience, she hopes to move to Washington D.C. to pursue work with the government to "help find terrorists." Although she never directly spoke about wanting to make a positive impact on society, involvement in the Peace Corps and working to help strengthen national security could both be considered careers that entail public service.

The final participant not intending to become a secondary school teacher is an extremely motivated mathematics student who had begun taking graduate level mathematics courses during her junior year in college. As such, she has decided to complete a program offered by her university that will allow her to earn both a bachelor's and a master's degree in mathematics in a five year time period. After that, she plans to enter a Ph.D. program either in mathematics or in mathematics education. Upon completing her Ph.D., she intends to work either at the NSA, a large business, or a college. Although this student has conducted undergraduate research in mathematics and is interested in continuing with mathematics research, she explained that her interests really vary:

M3: Um, my interests kinda go into very different directions. Um, I'm very interested in the education; why students reject math type thing and what you can do to get them interested again, at like the elementary school level. But I could never work with kids. Then again I'm also interested a lot in [mathematics] research… I'm actually kind of interested in [education] at the high school level too but I couldn't work with them either… I don't have the, the patience. I mean I'm sure I could and it'd be fine and eventually I'd get over myself and suck it up but I'd just rather not.

I: Okay. Where do you think that interest stemmed from?

M3: I don't know. It's just something I've always been pondering in the back of my mind. When I hear people talking about it I get interested and listen to what they have to say.

Although this participant is quite interested in doing research in mathematics, she also expressed interests in helping keep young children interested in mathematics, and is considering earning a Ph.D. in mathematics education as a result. Furthermore, this participant hypothesizes that women are advancing in some of the sciences faster than mathematics, because the ability to help people is more evident in these fields:

> I think a lot more women are becoming doctors, instead of nurses. Which is going to help significantly. That's probably the easiest field for them to advance in. Because they are still helping people. So that's the easiest one to see. Um, and a lot of women that get encouraged in mathematics immediately go into teaching. Which is fine… There are a lot more women math teachers now [than 30 years ago].

Discussion

The results of this study suggest that many women want careers where they can help people and positively influence society as a whole. Considering national statistics on the employment of women in specific fields can further support this theory. In the year 2004, although women constituted 46.5% of the U.S. workforce, 61.1% of those employed in "community and social services occupations" were women. Furthermore, when removing the category of "clergy" from this classification, the percentage of women increases to 71.6%. The numbers are similar for the classification of "education, training, and library occupations" in which women constitute 73.4% of those employed in these occupations, with the subcategory of "postsecondary teacher" demonstrating the least amount of female involvement at 46.0% (BLS, 2005). Therefore, women constitute a substantial number of those employed in these service-oriented careers.

During that same year, however, only 29.4% of all "physicians and surgeons" and 29.4% of all "lawyers" were women. While it appears that women are not entering these service-based careers in very high numbers, these statistics do not tell the complete story. For example, during the years 2000-2004, women earned 44.4% of the M.D.'s and 48.0% of the law degrees (LL.B. or J.D.) granted within the United States (NCES, 2005). Therefore, although historically women did not choose these careers at very high rates, women today are entering these fields at nearly the same rate as men.

There is a growing national concern, however, that not enough U.S. citizens are entering mathematics-based careers (Kuenzi, Matthews, & Mangan, 2006). The results of the current study suggest that one factor that may influence women's career choices is their desire to have a career that involves public service. Therefore, it is possible that by teaching mathematics in the context of social justice, as described by Gutstein (2003), we may be able to interest more women in the field of mathematics. Furthermore, when promoting mathematics-based careers to undergraduates, it is important to emphasize careers that involve public service.

Further Comments

It is also interesting to note that, despite the fact that national statistics published by the NSF and NCES do not consider mathematics teachers as having mathematical careers, none of the participants in this study entering the field of teaching considered themselves to be leaving the field of mathematics. To the contrary, they saw themselves as choosing a non-traditional field and "breaking gender boundaries." This disparity solicits discussion as to what constitutes a career in mathematics and whether the current definition utilized by national statistical agencies is appropriate.

It is recommended by the Committee on the Undergraduate Program in Mathematics (CUPM) of the MAA that, in addition to fulfilling the standard requirements for a mathematics major, majors preparing to be secondary school teachers must also "learn to make appropriate connections between the advanced mathematics they are learning and the secondary mathematics they will be teaching," know about the history of mathematics, and be able to use technology in order to conduct mathematical modeling (CUPM, 2004, p. 52). Such a program would provide prospective secondary school teachers with a stronger mathematical base than standard mathematics majors. If the degree that one must earn in order to enter a specific career is at all indicative of the field with which the career belongs, the profession of being a secondary education mathematics teacher ought to be considered a career in mathematics.

Future Research

Although we find the patterns identified in this study to be fairly compelling, it is important to note that this is a small study based on the responses of eight participants, all attending the same university. Furthermore, this university is known throughout its state as having a strong teacher education program. Students within the state who are passionate about becoming teachers often choose to earn their degree from this university. Consequently, it is important for us to determine if our result that women entering the field of teaching mathematics at the secondary school level often choose teaching prior to choosing mathematics can be generalized to students at other universities. Hence, we are beginning to interview women at other universities within and outside of the Midwest in order to determine whether the results we have found can be generalized. We also intend to make this study longitudinal by interviewing all participants three to five years after their initial interviews to determine the career paths these women ultimately choose.

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REPRESENTATIONS AND PARTICIPATION: LINKING MATHEMATICAL AND SOCIAL RELATIONSHIPS

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This paper explores the idea that intertwining significant mathematical relationships with classroom social structures might support the development of shared understanding, particularly by using the social as both a scaffold for drawing attention to, and a resource for building on, the mathematical. In the case described here, students' affiliations with different representations in the context of their respective contributions to collective classroom discourse helped to resolve a mathematical debate, and to build a shared insight into the relationships among multiple representations of a function.

Students' mathematical activity in classrooms is certainly always tied to social relationships; with luck, a reasonable share of their classroom social interactions will involve the joint exploration of mathematical relationships. This paper seeks to explore intersections between social and mathematical structures in classroom activity, with a particular eye toward the ways those intersections might constitute resources for supporting student learning. Following a brief treatment of relevant theoretical perspectives, I describe a study conducted in a classroom where the social and the mathematical were provocatively and productively intertwined, and present a detailed analysis of one exemplary lesson.

Theoretical Framework

The perspective developed in this paper draws heavily on two other accounts of the relations between social and mathematical practice. Yackel and Cobb's (1996) notion of sociomathematical norms provides a powerful framework for characterizing the points of intersection between students' participation in classroom interactions, and in mathematical activity. On this account, classroom communities develop and maintain collective, local guidelines regarding appropriate ways to contribute to mathematical discussions. Students' understandings of these sociomathematical norms serve as resources through which they come to participate with increasing autonomy in classroom inquiry. In a related way, Stroup, Ares and Hurford's (2005) work on networked devices explores the ways mathematical ideas can structure classroom social activity. They argue that the individual and group-level mathematical objects that students can create in a device network support new forms of classroom participation and collective mathematical activity. The dialectic between these new mathematical and social spaces opened up by generative designs for networked activity provides a resource for re-imagining and transforming mathematics teaching and learning.

Each of these perspectives relies on similar insights into the potentially productive links between the social structures of classroom interaction and the normative and conceptual dimensions of classroom mathematical activity. The premise of this paper is that in certain cases, those links can also serve as resources not only for organizing or reshaping mathematics pedagogy, but also for making important mathematical relationships and structures salient in ways that might support rich opportunities for learning. In particular, I take up the notion of multiple linked function representations (Kaput, 1989) as a set of conceptual objects to be distributed among multiple learners in classroom mathematics activity in order to foster both collaborative mathematical discourse and meta-representational insight. Put more succinctly, I explore the conceptual consequences and the instructional potential of linking different students with different representations of a collectively engaged mathematical function. Importantly, Yackel and Cobb's account of sociomathematical norms and Kaput's work on multiple representations reflect different theoretical perspectives on learning. While the former draws on a sociocultural framework, in which learning is characterized by students' changing participation in relation to the mathematical practices of a classroom community, the latter emphasizes students' developing understanding of mathematical concepts. Greeno (1997) has proposed an approach to integrating these sociocultural and cognitive accounts of learning; Sfard (1998) likewise argues that these perspectives are more productively viewed as complementary metaphors than incommensurable theories. In a similar spirit, this paper attempts to integrate these perspectives by examining the ways social and mathematical relationships overlapped in one classroom.

Methods

This paper draws on a detailed examination of a single episode in a high school mathematics classroom. Students were 11th and 12th graders participating in the fourth year of a reform mathematics curriculum. As part of a larger study, this classroom was videotaped daily for an entire year of instruction. Observation of classroom sessions, review of the daily videos, and formal and informal interviews with the teacher and students all contributed to the development of an account of learning and teaching practices and classroom norms in this setting. The episode presented in this paper was selected as particularly illustrative of a classroom sociomathematical norm related to the linking of participation and representation during student contributions to discussion, and for a novel moment of collective classroom discovery emerging from that linkage. This episode took the form of a whole-class debate, lasting nearly 30 minutes, about the relationship between the functions $y=x^2$ and $y=x^2+1$. All audible student and teacher contributions to this discussion were transcribed, and all mathematical work either written on the board or projected on an overhead display from a graphing calculator was likewise captured from the video record. Together, these data were analyzed to examine the ways each instance of student participation in this episode involved the use of one or more representations (symbolic, tabular, graphical) of a relevant function in order to contribute to resolving the mathematical problem under discussion. This analysis included coding each such student contribution with regard to the representation on which it relied most heavily, along with more detailed examinations of the mathematical features of those contributions as they built on or refuted those of other students, and of the gradual process through which the class moved from a mathematical dispute to a collective interpretation.

Analysis

A key feature of this teacher's instructional approach involved regularly asking students to demonstrate solutions, pose conjectures, explore ideas or advance arguments through brief and impromptu presentations from the front of the room, usually through the use of either dry-erase markers on a white board or a projected graphing calculator display. One sociomathematical norm governing these student contributions to collective problem-solving discourse involved the use of different representations as resources. Students were encouraged to draw from an array of representations, and some even came to identify themselves in relation to a particular representation—so that a student might even explain for their choice of representational mode by saying things like: "I'm a graphing person."

Linking individual participation with representation in this way appeared to serve as a classroom resource both for resolving mathematical debate, and for developing collective understanding of the links among multiple representations. In the episode examined in detail for this paper, 11 students made a total of 13 successive presentations from the front of the room regarding the problem under discussion. Of these contributions, summarized in Table 1, 7 were coded as relying primarily on graphical representations, 3 on tables, and 3 on symbolic expressions. The prominence of graphical representations reflected the nature of the debate; some students thought that the graph of $y=x^2+1$ had a "different shape" from $y=x^2$, while others asserted that the two functions were identical but for a vertical shift. The use of a public graphing calculator display in some students' presentations only strengthened the support for the incorrect interpretation, and resolving the debate involved a careful weaving among the symbolic, graphical and tabular perspectives by several students in order to reconcile the correct interpretation of the vertical translation with a confusing graphical display.

Student	Contribution	Representation
1	Added one to values in Table, "makes it steeper"	Table
2	$y=x^2+1$ can't touch x-axis because $0^2+1=1$	Symbolic
3	$f(0)=0+1$ so graph through origin incorrect	Symbolic
4	Erases graph that looks least like x^2—too skinny	Graph
5	"2^2+1…doesn't change the shape, it just changes the height."	Symbolic
6	Graphs both functions on calculator, argues they're different	Graph
7	Graphs x^2+10	Graph
1	"it's almost the same shape, but it's…compressed some"	Graph
8	"it's the same exact graph" "all the y-coordinates are up one"	Table
9	"it only looks squinched on the calculator"	Graph
10	Compares differences in table	Table
11	"If you just draw it by hand… they stay just one apart."	Graph
7	Traces projected curve, moves projector	Graph

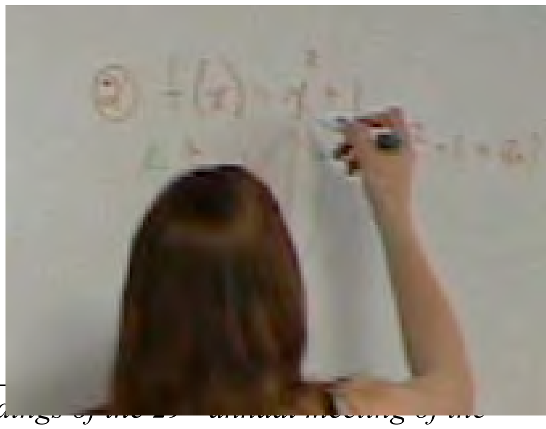
Table 1. Student Presentations and Representations

In the analysis that follows, excerpts from four of these student presentations will be examined in greater detail—one each from the symbolic, graphical and tabular modes, followed by an argument that attempts to make connections between multiple representations.

Symbolic

Students used symbolic representations to make two key points. The first was that unlike $y=x^2$, the graph of $y=x^2+1$ could not touch the x-axis. The second point was that because the two functions differed only by a constant, their respective graphs had identical shapes. Student 5 elaborates the latter argument in this excerpt:

Student 5: The thing that makes it this shape, is, um, would be the x squared [indicates expression]. It's the x-squared part. Because when you do, like, if this was two [touches x-axis at x=2], it's four here [draws the point (2,4)], and negative two is negative four here [draws the point (-2,4)]. So it's the same shape, and then if you have a negative, er, two squared plus one [indicates expression again], it doesn't



change the shape. It just changes the [moves hand up and down to show vertical translation]...the height.

Figure 1. Symbolic representation

Importantly, this student illustrates her argument through repeated references to a graph, including drawing two points on the board. But I have coded that argument as symbolic because she relies on references to the algebraic expression as the evidence for her assertions about that graph. In particular, she stresses that "it's the x squared part" of that expression that "makes it this shape," so that adding one "doesn't change this shape."

Graphical

The next student to come up disagreed with that conclusion, and illustrated his counterargument by plotting several points associated with the two functions, sketching the graphs, and then asserting that this work "shows that it's different". The sketches failed to provide clear support for or against his claim, so the teacher suggested that he graph both functions on a calculator projected onto the board. This prompted an extended class discussion about the appropriate window dimensions in which to view and compare the graphs. After adding the function $y=x^2+10$ in order to provide a clearer contrast, they eventually settled on the display shown in Figure XX. The following excerpt illustrates the way several students interpreted this graphical display:

Student 1: I think it's like, it's almost the same shape, but it's kind of like, compressed some, so like, even if it's way, like if you have another one way up here, like x squared plus like 20, it's gonna go out to the same thing.
Student 7: Right, but the…
Student 1: Like the same lines. Like, all three will go up eventually, looking like really close.
Teacher: So you're saying, now, I'm going to rephrase what he's saying, and you tell me whether this is correct. You said that eventually it's going to look like that, but at the bottom, it's scrunched differently.
Student 7: Yeah.

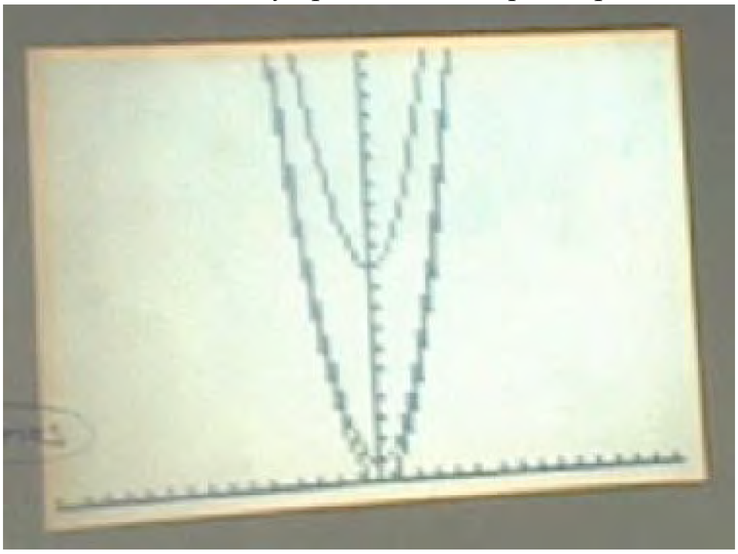


Figure 2. Graphical representation

These students agreed that while these parabolas have "almost the same shape," and share similar end behavior—they "go out to the same thing," they were "compressed" or "scrunched" differently at the bottom. This misunderstanding of the relationship between the functions appears to be a consequence of a sort of optical illusion produced by comparing the vertical spacing between the curves near the respective vertices with the horizontal spacing at other points.

Tabular

The next student to come up used tabular representations to challenge this interpretation of the two graphs as having different shapes. This presentation opened with an exchange between the teacher and student about the latter's choices of representational medium and mode:

Teacher: Ok. So [Student 8] wants to go up.
Student 8: Oh, not on the calculator, but on the…
Teacher: Ok, you want to talk. Go ahead.
Student 8: Yeah, just on the t-table.
Teacher: Fine.
Student 8: Ok, so for this one, this is…[draws a table of values for $y=x^2$]. And for this one, it's just…[draws a table of values for $y=x^2+1$].
Seated Student: I am a graphing person.
Student 8: And, it's like, it's the same exact graph, it's just that all the y-coordinates are up one…this two is just up one from one, and this four is just up to five, it's just the whole thing is just up one.

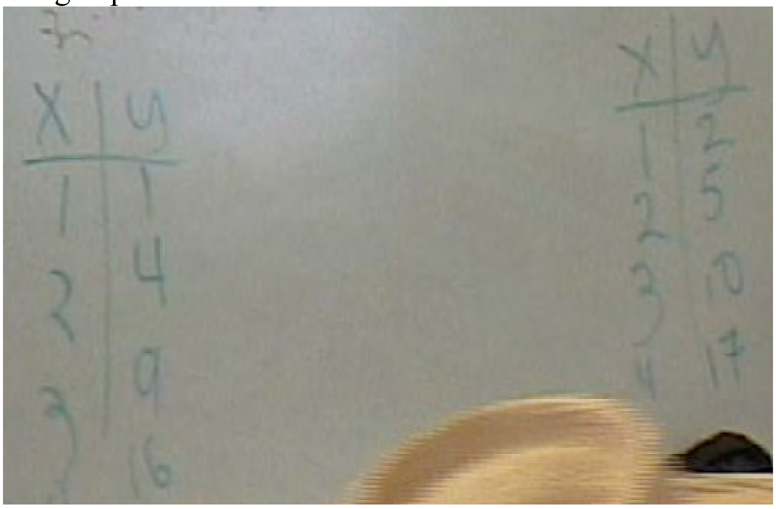


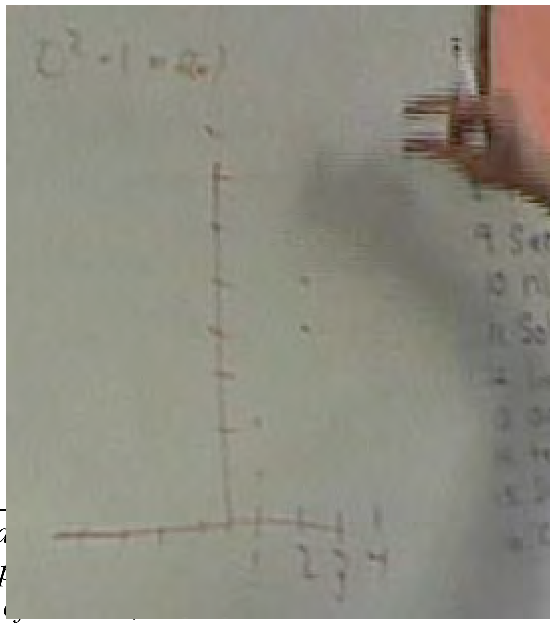
Figure 3. Tabular representation

This student expresses her preference for the table rather than the graphing calculator as a resource for making her argument. In apparent response, a student seated near the camera asserted aloud that he was "a graphing person"—a moment illustrative of the classroom sociomathematical norms regarding multiple representations. Meanwhile, Student 8 used the tables she had drawn to demonstrate that the second function was indeed just a vertical translation of the first—"the whole thing is just up one."

Linking Representations

Presented with these symbolic and tabular cases for the two functions having the same shape, and the apparently clear graphical demonstration of their difference, the class remained divided about the correct interpretation. The next excerpt proved pivotal in resolving that division, and I will argue that it achieved that resolution precisely because it involved drawing connections among multiple representations:

Student 11: If you just draw it by hand, you see that it's the same, like, from the little table, because, like, if you just do it yourself, you just go, like, one, two, three, four [counts off as he scales and labels x-axis]…And then up one, two, three, four, five, six, seven, whatever [scales y-axis]. And then you just, like, here it's, so you know it starts here, and then like, on the first one you're here [makes a mark on the graph at (1,1)], and then you're at four [marks the point (2,4)]…up here. And on the second one you'd, so you move up here [marks (1,2)], and



then you're at five [marks 2,5]. And you just keep going and they're, they stay just one apart [holds his index finger and forefinger a unit apart].

Figure 4. Vertical shift

This student began by emphasizing the connections between the graphical—"if you just draw it by hand"—and the tabular—"it's the same [as] the little table." To illustrate these connections, he plotted the same points Student 8 had identified as ordered pairs in her t-tables for the two functions. Just as Student 8 had stressed that "two is just up one from one, and this four is just up to five," and that "just the whole thing is just up one," Student 11 emphasized that the points from the respective functions "stay just one apart." He then carefully sketched the two curves and again emphasized that there was "just one [vertical] difference all the way up." In doing so, he effectively linked the one-unit difference that his classmates had identified in the symbolic expressions and tables of the two functions to their respective graphs. As student 11 concluded his explanation, the remaining students who had thought the shapes of the two graphs were different acknowledged that they had now been convinced otherwise.

Discussion

This debate was both fueled and eventually resolved by the distinct affordances and limitations of these different representations. Students who drew on the symbolic and tabular modes were able to clearly explain and illustrate the one-unit vertical translation from $y=x^2$ to $y=x^2+1$, but that vertical shift was more difficult to observe in the graphical displays provided by hand-drawn sketches and the graphing calculator. Student 7, in fact, went to the front of the room to add the graph of $y=x^2+10$ to the calculator display because she thought it would help student 6 to see that the curves had the same shape, but concluded after seeing the graph by agreeing with students 1 and 6 that the parabolas were "scrunched differently" at the bottom.

These apparently contradictory interpretations of these different representations of the same functions posed two conceptual problems for the class. First, the class could not, prior to Student 11's presentation, come to agreement about which interpretation of the relationship between the shapes was correct. And second, they could not resolve the apparent discrepancy between what they expected to be equivalent representations. Student 11's linking of arguments from the table and graph provided a way to solve both these problems simultaneously. He gave a convincing answer to the question of whether the two functions had the same shape, and he demonstrated that that answer was indeed supported by both representational modes.

My point here is not to overemphasize the significance of Student 11's presentation relative to those made by other students. In fact, the presentation's effectiveness appeared to be supported less by the clarity of the speaker or the depth of the insight than by the progression of other student contributions on which it built. In other words, the link between the graphical and tabular modes was less a consequence of Student 11's unique perspective than it was an emergent feature of several successive presentations from distinct representational perspectives. While Student 11 made reference to the tables drawn by Student 8, his presentation was clearly focused on the graphical mode—as in every other student presentation, a single representational mode predominated. But by highlighting a connection between his graph and Student 8's table, and by finding a way to interpret the graph that aligned with other students' interpretations of the tables and symbolic expressions—as demonstrating the identical shapes of the two curves—he helped to complete the class's gradual weaving together of these multiple representational threads.

Ultimately, the episode supported a view of the links among multiple representations precisely because each student—in keeping with a classroom sociomathematical norm—stayed in a single representational mode, as agreement among representations emerged as a necessary corollary to agreement among students who had been on opposite sides of a debate. In terms of the dialectic proposed by Stroup et al., the social space defined by this succession of student presentations with the support of representational tools helped to delineate a corresponding mathematical space. As students brought their respective views of the two functions into alignment, they drew attention to the ways the different representations underlying those views were likewise aligned: the links among multiple function representations were made visible through transactions among multiple social actors. At the same time, the mathematical links among the different representational artifacts invoked by students provided resources for negotiating those transactions.

Conclusion

I conclude by suggesting that this classroom episode may point toward some valuable implications for the design of mathematical learning environments. Work such as Kaput's attests to the promise of multiple linked representations in computer-based learning environments for supporting student understanding of complex mathematical concepts like function. But that promise may be much more likely to be fulfilled if the links among those representations are socially as well as technologically mediated. To that end, my own design-based research efforts are currently devoted to investigating the potential of linking mathematical and social structures as a principle for designing collaborative problem-solving activities that utilize classroom computer networks (White, 2006; 2007). These designs connect the relationships among a set of mathematical objects, including but not limited to multiple representations, with the pedagogical organization of relationships among multiple students. I hope that these designs will not only make rich classroom conversations like the one reported here more common, but also contribute to broader theorizing about the relationship between the social and mathematical dimensions of classroom activity.

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STUDYING THE AFFECTIVE/SOCIAL DIMENSION OF AN INNER-CITY MATHEMATICS CLASS

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We study affect and its associated social and cognitive aspects in an urban classroom, as students working in small groups express themselves as they solve mathematical problems. We focus on several students and their interactions with their peers, the teacher, and the influence of the environment in which they live. We propose several "archetypal affective structures" for which we describe evidence from the analysis of classroom videotapes.

This report is based on one component of a larger study presently under way (1), investigating the occurrence and growth of urban students' *powerful affect* in relation to the learning of conceptually challenging mathematics. By powerful affect, we mean affective structures that contribute to mathematical engagement, persistence, problem-solving success, and achievement (McLeod, 1992, 1994; Goldin, 2000, 2007). These structures include certain patterns of emotions, attitudes, beliefs, and values (DeBellis and Goldin, 2006). In the broader study (see Alston, Goldin, Jones, McCulloch, Rossman, & Schmeelk, 2007, for additional description), we approach the challenging and significant problem of characterizing affect, together with its associated social and cognitive aspects, in the context of several mathematics classrooms as students are working in groups and expressing themselves spontaneously as they work. In this research report, we focus on one group of urban students, and their interactions with their peers, the teacher, and the environment in which they live. As a result of our preliminary analysis, we propose and describe several ``archetypal affective structures'' for which we see evidence.

Theoretical Background and Framework

We draw on ideas from several disciplines, including sociocultural and situated learning theories (e.g., Wertsch, 1985; Brown, Collins, & Duguid, 1989), cognitive science (e.g., Greeno, Collins, & Resnick, 1995), and mathematics education (e.g. Schoenfeld, 1992; Lesh, Hamilton, & Kaput, 2007). We extend this literature by attending to specific challenges faced by students in inner-city middle schools (Dance, 2002, Anderson, 1999) as they work on mathematics problems.

In addition, we refer to a model of attention proposed by Cohen (1978), wherein attention is a limited commodity that needs to be allocated among tasks. One adaptive strategy is to give tasks deemed to be instrumental to attaining high-priority goals greater attention than tasks associated with lower priority goals. When people anticipate that an adaptive response is needed, they will "activate a monitoring process that evaluates the significance of the stimulus and/or decides on appropriate coping responses" (Cohen, 1978 p.3). For instance, a stimulus occurring for many youngsters living in urban environments is *danger* that can arise from an insult (tacit or explicit) by another youngster, or from an act that makes one look wrong or foolish, so that one loses "face." We propose that this is true for most or all human beings, regardless of the environments in which they live; however, our particular focus here is on urban youth. According to Anderson, "life in public often features an intense competition for scarce social goods in which 'winners' totally dominate 'losers' and in which losing can be a fate worse than death." (p.37). Anderson notes that an important aspect of the "code of the street" is to not appear weak and/or perceived as a "loser". Hyper-vigilance for unpredictably-occurring incidents in which students' honor or "face" will be challenged during group discussions may then draw students' attention away from the mathematical ideas behind the task under discussion. McLaughlin (1993) quotes an inner city youth worker who describes his objectives for his peers this way: "I want these youngsters to 'duck the bullet' – not just the bullets that come from the gun but from the verbiage of peers" (p.38).

Regardless of the conditions in which students live, we know that urban students *do* become invested in doing mathematics, and some achieve great success (e.g., Schorr, Warner, Samuels, & Gearhart, 2007; Silver & Stein, 1996). We therefore believe that it is important to understand the circumstances that surround this achievement, so that many more students can have similar opportunities. In our work, we explicitly reject a "deficit model" of urban education. A main purpose of the present study is to better understand the constellation of affective/social/cognitive structures that surround the development of mathematical success in urban mathematics classrooms.

Research Questions and Methods

The research questions of the overall study include: How do teachers contribute qualitatively to creating an emotionally safe classroom environment for urban students to explore conceptually challenging mathematics? What are the affective and cognitive consequences for urban children learning mathematics, including students' social interactions, emotional states, and mathematical learning?

Subjects: The 8th-grade classroom that is the focus of this research report is one of three that comprise the broader study. It is located in a school in one of the largest NJ cities. The school is classified as "low income", with 100% of the students being members of minority groups – 93% "African American" and 7% "Hispanic". Students in the observed classroom worked in small groups varying in size from three to five students. The composition of the groups sometimes changed from session to session, affording us an opportunity to observe selected "focus students" interacting with different students and in different groups.

Procedures: Classroom interactions of the entire class and of the focus students were videotaped using 3 separate cameras. Classes were observed in 4 "cycles" over the course of the school year, with each cycle spanning two consecutive days. Prior to the start of each cycle, an interview was conducted with the classroom teacher to ascertain her plans for the lesson and her ideas about what she expected to happen when she taught the lesson. A follow-up interview with the teacher took place after each cycle. After each cycle, focus students were interviewed using a visually stimulated recall protocol. Clips containing what the researchers believed were key affective events were selected from the videotapes and shown to the interviewee, to evoke the student's perspective on what had transpired and the emotional responses the student recalled having had. All videotapes were transcribed, and students' written work on the problem was scanned. Data were coded and interpreted using four analytical "lenses," including: the flow of mathematical ideas, the occurrence of key affective events, social interactions among the students, and teacher interventions.

Results and Some Illustrative Data

We have formulated, based on our preliminary analysis of data, the concept of an "archetypal affective structure." This refers to a recurring pattern, inferred from observing the classroom and interview tapes, that is a kind of *behavioral/affective/social constellation.* Included are characteristic patterns of behavior (speech, gesture, movement, posture, and emotional expression) indicative of affective pathways (sequences of emotional states), accompanying internal "self-talk," and characteristic social interactions. These have important cognitive implications, and can entail important day-by-day choices, with powerful consequences – positive or negative – for mathematical education.

The term "archetypal" is intended here to suggest the idealized and, we believe, universal nature of these patterns, abstracted from the observed instances of their occurrence. In this sense, it is quite the opposite of "stereotypical" behaviors or characteristics, which so often characterize the deficit models that we have rejected.

Our preliminary analysis has led to the identification of several such structures in the context of conceptually challenging mathematics. Three examples are described here; for each, we describe (idealized) behavior and associated self-talk, with an illustration.
(I) "Don't Disrespect Me"; (II) "Check This Out"; (III) "Stay Out of Trouble"

I: "Don't Disrespect Me": The essential ingredient in this archetype is the experience of a challenge that poses a threat to one's actual or perceived safety or well being, and therefore must be resisted. The process of resisting elevates the struggle above the task, which serves as the arena for the struggle. The need to maintain face is paramount.

Behavior: Someone disagrees with my idea. I interpret that as a social *challenge* (self-talk: "He thinks my idea is wrong," together possibly with, "He wants to make me look bad and make himself look good"; note this is different than just disagreeing with my idea) → (self-talk: "I may be seen as weak or a loser") therefore feeling *threatened* → (self-talk: "People seen as weak get picked on or harassed by others") therefore experiencing *fear* → (self-talk: "How dare he disrespect my ideas") therefore feeling *anger* → (self-talk: "I can't let him get away with that") therefore aggression (fighting or arguing with the other person) → (self-talk: "What does he know about math anyway") therefore feeling *contempt* (for the other person and his ideas) and unwillingness to consider the ideas.

The following excerpt describes a highly impassioned discussion of mathematics and how it develops. The students are working on a problem that asks them to find the maximum area for a cow pen given a fixed perimeter (they are told that the farmer purchased 100 feet of fencing). Dana [all names are fictitious] has been working productively with her group, often engaging the others in mathematical exchanges about the solution process. After working on the problem for about one period, the teacher asks the students to prepare a poster documenting their solution strategies. The students are then encouraged to walk around the room to observe and comment, in writing and verbally, on other groups' work.

It turns out that Dana and her group have been calculating the area incorrectly (instead of multiplying length by width, they multiply length***s*** [side a + side c] by width***s*** [side b +side d]). This process is displayed on their poster. Another group notices this as they walk over to the table where Dana's group's work is displayed. The excerpt below occurs when Dana learns that members of the observing group think that the work her group did was wrong. She engages in a conversation with three young men, Shay, Ghee and Larry, about her group's solution. She vehemently defends the work of her group (which she later admits may have been incorrect). Both Dana's and Shay's facial expressions and bodily movements become increasingly combative as the conversation goes on

[Dana]: … where it is wrong? [Shay]: Cause you put, when you finding, um, the area, you timesed the width times the (inaudible) [Dana]: All right, but we timesed all that up. [Shay] But you're not supposed to. [Dana]: Alright, but we did it though, so … [Shay]: But you're not supposed to, so it's wrong. [Dana]: No, it's not wrong. Actually, no it's not. [Shay]: It's wrong, it's wrong, IT'S WRONG! [Dana]: No, it's not. [Ghee]: How is it wrong? [Shay]: Look at that (showing on calculator), that's what y'all got, 1600? [Dana]: Yes. [Shay]: That's what y'all got? [Dana]: Yes. [Shay]: So the width times the 40. [Dana]: Well, we didn't do that, it equals 400, but we didn't do it. [Shay]: 40, yes, that's how you do it. 40 times 10, that's how you get that. [Dana]: Well, we didn't do it that, so, oh well. [Shay]: That's the right answer.

Shay continues to try to convince Dana that her method is incorrect, using 40 as the length and 10 as the width. Ghee, another student who had been working with Dana also engages in the conversation. [Dana]: Oh well … [Dana]: We already know 40 times 10 equals … [Shay]: That's what your area is. Ok, that's what your area is. [Ghee]: 40 times 10 … You said 40 times 10? 40 times 10, 40 and 10 … none of that add up to 100. [Larry]: Yeah, so … [Shay]: You add … you add this, that's 40. That's 80 right there [referring to 40 + 40]. And that's 100 [adding 10 + 10 to the 80 feet]. Yeah, so you don't know what you're talking about. That's the perimeter, that's the perimeter. [almost shouting, and moving closer to Ghee's face as he points to the work] And area is 400. [Ghee]: No, 80…that's exactly what we did. [Shay]: And that was wrong. [Dana]: 80 times 20 is not **wrong** from 100. [raising her voice and speaking with conviction to Shay] (unknown) 80 plus 20 is 100. [Dana]: **Thank you! Thank you**, 80 times … I mean, 80 plus 10 … I mean, 20, is 100. [Ghee]: So now who got ripped? Shut up, man! [Dana]: So wait … 80 … [Shay]: Exactly, tell 'em you gotta multiply length times width. [shouting at Dana] Not what y'all did. [Dana]: That's what we did! ! [shouting at Shay] That's what we **diiiid!** [Ghee]: That's exactly what we did (makes angry motion)

When Dana was asked to watch a video of the above interaction and reflect on it, she notes that being wrong makes her uncomfortable, especially when other students are trying to *prove* her wrong, but acknowledges the possibility she was wrong.

[Interviewer] So what do you think? [Dana] That was a funny clip. [Int] How were you feeling during that? [Dana] Mad angry. Everything. [Int] Well, why? [Dana] Because, Shay was proving me **wrong**. [Int] So how did he prove you wrong? [Dana] He was **trying** to prove me wrong. Maybe he was right, and then maybe I wasn't right. Or maybe I was **right**, and then he wasn't right. I don't know. Cause he multiplied the length and the width, but I multiplied both of the lengths and both of the widths. So … [Int] So what do you think? [Dana] That … maybe his answer is wrong. Maybe he multiplied the length and the width, because on the paper it said multiply the length, and not the width. [Int] So what do you think? What did you understand from that day? [Dana] That, maybe I was wrong. I don't know whether I was wrong or right. I was just, that day. He was, I couldn't say nothing to him cause he, I was mad at him. [Int] Were you feeling comfortable that day? [Dana] No [Int] Why? [Dana] Cause he was trying to prove me wrong. [Int] Do you feel that way often? Does this happen often? [Dana] Yes. [Int] So you're uncomfortable often? [Dana] Not all the time. But when I'm right, I'm not uncomfortable. But when I'm wrong, when they try to **prove me** wrong, I'm uncomfortable. [Int] That's interesting. So when it is that, when are the moments when you get mad? When what's happening? [Dana] When they, when people in my group really aren't doing nothing at all, that makes me mad. [Int] Is there anything else that makes you mad? [Dana] Uh.. yeah, when people try to prove **me** wrong too.

In this interview, Dana said that she was uncomfortable "…when people try to prove *me* wrong." Notice that she chose to use the word "me" rather than saying "when people try to prove my *idea* wrong". We infer that she took it personally, which supports our contention that her primary focus was a need to save face. The motivation to maintain face seems to become stronger than the motivation to engage in mathematical inquiry.

II. "Check This Out." The essential ingredient in this archetype is the occurrence of a realization that solving a problem is related to learning something that can have a payoff—now or in the future. This motivation to engage with the task can lead to intrinsic interest in the intellectual aspects of the task and/or extrinsic interest in the potential payoff of doing the task now or at some later time. Initial task compliance (self-talk: "I will do the math problem that teacher has assigned") → (self-talk: "This looks like it might be interesting" therefore *reflection* (thinking about the task) → (self-talk: "Maybe knowing this can help me in some way" therefore *curiosity* → (self-talk: "Let me see what I can do with this problem" therefore *engagement* with task (start to work diligently at the task) → (self-talk: "This is actually interesting") therefore *heightened interest* → (self-talk: "I am good at this") therefore *pride*, and changed self-concept.

To illustrate this archetype, we report an instance (during another class session) where we infer that Shay has set in motion a "Check This Out" affective structure for himself as well as for several other students working with him in his group. The task asks which of several "DJ Packages" is the best value, in terms of cost per hour of DJ time. In contrast to the "cow pen" task above, Shay sees its relevance to a task he may someday encounter. He has become engaged with the task, and open to new information that may help him arrive at a more accurate solution. In one instance, he expresses an idea that a girl in his group believes is incorrect. As he listens to her idea, there are no external signs indicating any defensiveness on his part or hers. Rather, he is willing to consider her views, and adjust his ideas accordingly. His willingness to consider her views and change his mind encourages her to become even more engaged with the problem as they engage in a productive mathematical dialogue. As they do this, another student who heretofore had not been engaged becomes curious and starts to look at what Shay is doing with his solution. Shay's interest in the problem appears to stimulate his classmate to also become involved.

The following dialogue involves Shay, Gina, a girl in his discussion group, and the Teacher (Ms. B.). [Teacher]: The Y represents what? [Gina]: The number of hours [Teacher]: (to Gina) So can you label X and Y for me? [Shay]: (engaged in the task, points out what he thinks are X and Y) [Gina]: I think it's supposed to be the other way around. [Shay]: Yah, Yah, Yah (acknowledging that Gina was correct).

Here is a potential "branch point," where Shay could either remain engaged, or could have become defensive and attempted to save face, as in the *Don't Disrespect Me* archetype. A non-defensive presentation on his part in turn enabled Gina to comment on his mistake in a non-threatening way. Shay then continues to participate enthusiastically, writing information on his paper. He seems engaged in the process, willing to listen to feedback from groupmates and willing to change his answer based on this information. We conjecture this engagement was set in motion by his desire to *Check This Out.* His openness encourages Gina to become even more engaged, and the two of them get involved in a productive mathematical dialogue. As they do so, Tyrone, another student in the group who heretofore had not been engaged, becomes curious and starts looking at what Shay is doing with his solution.

The retrospective interview then suggests Shay's developing self-confidence: [Interviewer] ok, all right, umm so how are you feeling when … when you are explaining your work and (inaudible)? [Shay] Like I could do it, but I don't know how to explain it.

III. "Stay Out of Trouble" The essential ingredient in this archetype is the need to take care of oneself and avoid entanglements that will lead to trouble. Argument (between 2 other people – the conversation is getting heated, and could end in a confrontation) → (self-talk: "This situation has become dangerous and someone may end up in trouble") therefore *fear* → (self-talk: "If I get in trouble in school it will jeopardize my security, or present or future chances for getting ahead") therefore *feeling threatened* about the future → (Self talk: "If I don't get involved and don't say anything, they will ignore me and I won't have any problem with them" therefore *withdrawal* (an attempt to hide and not be noticed, so as not to attract the anger of one of the protagonists) →(self-talk: "If I get too involved, and my answer is different, it may threaten someone else and they will get angry at me leading to trouble")

therefore affective *numbness* → (self-talk: “I don’t care about what’s happening in this argument anyway”) therefore *disinvestment* in the mathematical task.
Will, a member of Dana’s group, short in stature, recently immigrated to the United States from Guyana. He was often quiet during the discussions that took place in the group, and at times, was actually admonished by his peers in the group for not participating. One group member told the teacher that Will was unwilling to share his ideas, that he wrote his ideas on a paper but later crumpled it up and threw it away. When the teacher asked Will to share his ideas, he did so very quietly.

Interviews and careful examination of the classroom video revealed that Will had a good understanding of the mathematics involved in the problem. Further, in his interview, he stated that he liked being in the United States because it gave him an opportunity to get a good education. In the retrospective interview, it appeared that Will believed his solution and the solution chosen by the group were equally likely to be correct. However, he was reluctant for the group to choose his solution, for fear of backlash. [Will]: I won’t get mad because they didn’t choose mine because mine could have been wrong, and theirs could have just been right. So, if they had chosen my wrong one, and the right one they tossed it away, they might’d get mad at me. So I just left it like that, so we didn’t know which one was right, so we just basically chose one.

Will also expressed concern about arguments. When the interviewer asked Will how he was feeling when he crumpled his paper, he said that his “level of happiness went down.” The interviewer asked him why he crumpled up the paper. [Will]: I didn’t like, saying anything. [Int]: Why not? [Will]: Because, it might just cause an argument in the first place. [Int]: And how do you feel when there’s an argument? [Will]: I don’t like arguing with people, because mostly, they become more like a fight. [Int]: Interesting. [Will]: And, if it comes to a fight, you just get suspended. And maybe, they just kick you out of school.

Based upon our analysis, we suggest that Will felt that the situation could be dangerous and potentially lead to a fight. Consequently, Will hung back and only shared his ideas quietly and unassertively, ideas which had he been willing to defend might have changed the direction of the group’s solution to a more productive path.

Discussion

These are just a few of a larger set of archetypal affective structures that we identified. We hypothesize that much of the strongly expressed affect seen here is connected with the need for maintaining “face,” which is a key element in the street culture of the inner city. At times, it engulfs and overwhelms all else during mathematical group discourse, focusing students’ attention on “looking good” as a higher priority than discovering mathematical patterns or achieving success.

A longer-range goal of this type of analysis is to provide a framework to describe some possible teacher interventions that produce movement toward mathematically powerful affect. For example, in a group situation where some students are not participating, we observe teacher interventions that contribute to the students becoming involved. The atmosphere changes from latent resentment at an unfair distribution of work, with consequent disinvestment from the task, to a feeling of justice and fairness, with consequent ability to pay attention to intellectual and practical aspects of the task. By identifying archetypal affective structures, and becoming able to recognize them in classroom discussions of mathematics, we hope to better help teachers create affective/social contexts in which difficult situations occur with decreased frequency, and productive situations are maximized.

Endnote

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THE IMPACT OF MOMENT-TO-MOMENT DISCOURSE MOVES ON OPPORTUNITIES TO LEARN MATHEMATICS

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Different opportunities to learn mathematics are created by habitualized, moment-to-moment choices teachers make in scaffolding mathematical conversations. This study uses discourse analysis to identify and analyze the patterns of discursive moves in two seventh-grade math classrooms. Differences in norms for participation and the kinds of intellectual work teachers required in their classrooms were reflected in the patterns of follow-up moves the teachers employed.

As learners construct meaning, their participation in activities and within communities mediates their ways of thinking and learning (Lave & Wenger, 1991; Wertsch, 1991). From this point of view the kind of classroom interaction between students and teachers is critical to how and what children learn. Because the primary mode of interaction in schools is discourse, it is an important factor in the teaching and learning of mathematics. It is both essential and unavoidable in classrooms. However, many studies of classroom interaction take the larger class organization as the unit of analysis and investigate the impact of various activity structures, tasks, curricula, and differing classroom norms on student learning and reasoning (Hiebert & Wearne, 1993; Saxe, Gearhart & Seltzer, 1999; Yackel & Cobb, 1996). Though important, these macro-level factors tell only part of the story. They fail to identify the discursive mechanisms in real-time interaction that constitute learning.

This research posits that different opportunities to learn mathematics are created by habitualized, moment-to-moment choices teachers make in scaffolding everyday conversations in math classes. Over time, ways of responding "in the moment" are internalized and eventually become recognized and accepted ways of doing things in the classroom community. Thus, variation in the normative routines and kinds of talk in classrooms is likely to affect learning. It is these differences in discursive moves that I will identify and analyze in this paper.

Conceptual Framework

The theoretical foundation of this line of inquiry includes the situated nature of learning (Brown, Collins & Duguid, 1989; Lave & Wenger, 1991), the relationship between thinking and speech (Vygotsky, 1986), and the role of discourse and dialogicality in learning (Bakhtin, 1981). As the hallmark of interaction, language held particular interest for Vygotsky because of its mediational potential to transform future thought and action. The act of organizing vague, disconnected ideas through discourse can help produce meaning for oneself and others by making utterances objects of reflection. Ideas become public and concrete providing opportunities for reflection, analysis, and joint consideration. Mercer explains, "We acquire ways of using language that can shape our thoughts. These ways of using language provide us with frames of

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reference with which we can ‘recontextualize’ our experiences” (1995, p.75). The process of schooling is one of the means by which we acquire these “ways of using language.”

Because classroom discourse is broad and complex, I will focus on one aspect of discourse and its relationship to students’ opportunities to learn: teacher follow-up of student responses. This is based on Mehan (1979), Sinclair & Coulthard (1975) and Wells’ (1996; 1999) description of the widespread IRE/IRF (Initiation-Response-Evaluation/Follow-up) patterns found in schools. Teacher follow-up is also one of the primary means by which teachers scaffold learning. Choices made in the follow-up slot can constrain and enable the kinds of classroom discussions and learning that occur. My question, then, is do differences in follow-up impact students’ opportunities to learn?

The pervasive tendency in American classrooms is for participants (primarily teachers) to use the follow-up slot exclusively for evaluative purposes; hence the E in IRE patterns (Mehan, 1979). When following-up to evaluate, the evaluative move signals the end of the discussion; there is no space for a productive exchange to develop. What happens when the purpose of follow-up is to elaborate, or, better yet, have students do the elaborating? Does this affect student learning?

Previous research suggests it does (Haneda, 2004; Hiebert & Wearne, 1993; Nassaji & Wells, 2000; Nystrand et al, 1997). By encouraging students to verbalize thinking, follow-up moves can support coherence and clarity in thinking, help students organize and restructure new information into prior experiences, create opportunities for participation and engagement, and increase metacognitive awareness (Haneda, 2004; vanZee & Minstrel, 1997; Webb, 1991, Wells, 1999).

Methods

This study was conducted as part of a larger program of research, Scaling Up SimCalc™(Shechtman et al, 2005). The Scaling Up study implemented a new, technology-rich curricular unit on rate and proportionality in seventh-grade classrooms across the state of Texas. It used realistic problem contexts and simulations of motion to teach the foundational concepts of variation and covariation. Approximately 100 classrooms were involved in the larger experimental study, half of which were using the SimCalc™ curriculum and the other half using their normal curriculum to teach a three-week unit on proportional reasoning. Video data of the implementation of the SimCalc™ unit was collected in multiple classrooms during the 2005-2006 academic year. The data for this paper comes from video footage of the same lesson from two teachers who were teaching the SimCalc™ unit for the first time.

Both teachers in this study taught seventh-grade mathematics in small cities in Texas and had been teaching for over twenty years. Additionally, both teachers had relatively large gains in student learning as measured by a pre-and post-test on rate and proportionality. Teacher M had an average gain score of 10.83 points, and Teacher N had an average gain score of 6.92 points (out of 30 possible points). The mean and standard deviation of the gain scores for all teachers implementing SimCalc™ (n=48) were 5.77 ($\bar{x}$) and 1.85 (s). These teachers were selected primarily because their students were successful learners (as measured by the assessment), and I wanted to investigate the kinds of classroom discourse that supported this kind of learning.

I used a microanalytic approach to analyze the moment-to-moment differences in the implementation of the same lesson in these two classrooms. Drawing on the qualitative techniques of discourse analysis (Mehan, 1979; Schiffrin, 1994; Sinclair & Coulthard, 1975; Wells, 1999), I used transcripts of whole-class discussion to identify normative interaction patterns and categorize follow-up moves. Specifically, my research question was: What are the patterns of teacher follow-up and how do they impact students' opportunities to learn mathematics?

Coding Scheme

In order to systematically investigate patterns of discourse, I relied on Gordon Wells' (1999) analytic framework of the co-construction of dialogue. I define a *move* as the smallest building block of an interaction. It can be a statement, a question, or an answer; others might define this as an utterance or a turn. Every move can be categorized as an initiation (I), a response (R), or a follow-up (F), and a series of reciprocally-bound moves is called an exchange. At minimum, an exchange consists of an initiation and a response. As an example, consider the following excerpt of a classroom discussion in Table 1.

Line #	Discourse	Coding		Prospectiveness
 1 2	*Sequence 1* Teacher – How can you figure out the speed guys?	Exchange 1	Initiation	D
3	Student 1 – Uh.		Response	G
4	Student 2 – Distance divided by time.		Response	G
5 6 7	Teacher – Ok. So let's take a look at table (c.), 25 meters divided by 5, Suzy, gives me what?	Exchange 2	Follow-up Initiation	A D
8	Suzy – Five.		Response	G
9 10 11	Teacher – Five. So that means he is running 5 meters per second right?	Exchange 3	Follow-up Initiation	A D

Table 1. Excerpt of classroom discourse

The first exchange, in lines 1-5, consists of an initiation, two responses, and a follow-up move. The second exchange, in lines 6-9, also follows the typical IRF pattern common in classroom discourse. Above the level of exchange is a sequence, the unit containing a series of topically related exchanges. The sequence captured in Table 1 is a discussion of how to calculate the rate of a runner in a simulation.

In addition, each move can be further categorized with respect to its *prospectiveness*, or the degree to which a response is expected. Following Wells, the three levels of prospectiveness I use are Demand (D), Give (G), and Acknowledge (A). The initiating question in sequence one is a Demand (D) since the teacher is requesting information and the probability of a response is high. The student responses (on lines 3, 4 and 8) are Gives (G) since they provide information. The prospectiveness of Give moves is less than Demand moves since they may or may not evoke a further response. And the "Ok" (line 5) along with the revoicing of "Five" on line 9 are Acknowledge moves (A). By themselves they have low prospectiveness along with all other moves falling into the

Acknowledge category (yes/no responses, evaluations, expressions of thanks, etc.) because a subsequent response is unlikely.

In my coding of the transcripts, I first grouped turns of talk into sequences and categorized each move as an initiation, response or follow-up. Then I classified each move according to its prospectiveness (D, G or A). After coding all of the content-related episodes in the transcripts, I did a frequency count of the total instances of teacher follow-up for each classroom. As subcategories of follow-up, I also counted the number of Demands, Gives and Acknowledges that occurred in the follow-up slots. Because a follow-up move can fulfill more than one function, teachers often gave an Acknowledge move in combination with a Demand or Give move. As an example see lines 5-7 in Table 1. I counted this as a single instance of follow-up and as both an Acknowledge and a Demand (where the Demand move served to extend the sequence with an initiation).

Findings

Prospectiveness is a useful construct because it helps to identify both who is doing the intellectual work in the classroom and the kind of work being done. For example, classrooms with a large proportion of follow-up moves in the Acknowledge category create environments where answers are given priority over reasoning. Further, relying primarily on Acknowledge moves perpetuates the teacher's status as the authority for determining what is mathematically correct. In contrast, Give and Demand moves shift some of this responsibility back to students by suspending judgment and leaving concepts *under construction*. Here either the teacher (Give category) or the student (Demand category) makes connections, provides examples or counterexamples, supplies missing information, or reformulates a previous response.

The following chart displays the variation in the prospectiveness of follow-up moves for the two teachers in this study.

Prospectiveness	**Proportion of Occurrence**	
	Teacher M	Teacher N
Demand	.559 (80/143)	.588 (60/102)
Demand w/ Uptake	.45 (36/80)	.317 (19/60)
Give	.462 (66/143)	.353 (36/102)
Give w/ Uptake	.515 (34/66)	.222 (8/36)
Acknowledge	.545 (78/143)	.706 (72/102)

Table 2. Prospectiveness of Teacher Follow-up

Both teachers followed-up by requesting information (D) from their students about 60% of the time. However, there were differences between the proportions of follow-up involving Give and Acknowledge moves. To tease out this difference, the categories of Demand, Give and Acknowledge were further separated into subcategories and compared across teachers. One contrast that emerged was within the subcategory of Demand and Give moves that involved uptake. Of the 80 Demand moves Teacher M used, 36 (45%) involved uptake. Contrast that with 19 out of 60 for Teacher N (32%). This same trend is seen in Give moves: Teacher M used uptake in 52% of follow-ups that "gave information" whereas this occurred in only 22% of Teacher N's Give moves.

What does this mean? An additional clue might lie in the proportion of times Teachers M and N used a rebroadcasting move in their follow-up (note that rebroadcasting is an echo or exact repetition of what was just said, which, in essence, functions as a positive evaluation of a previous response placing it in the Acknowledge category). One potential reason Teacher N used Give moves less frequently is because her preferred mode of interaction was to instead rebroadcast the student's response (she did this in 51% of all of her follow-ups and Teacher M 33%) and either end the sequence or continue on by asking another question in a recitation-script (Tharp & Gallimore, 1988) kind of fashion. Because moves in the Acknowledge category are usually evaluative and end discussion, it is not surprising that Teacher N has a smaller mean sequence length (the average number of moves in a given sequence). She averaged 7.5 moves (I, R, or F) per sequence whereas Teacher M averaged 13.5 moves per sequence.

Differences in uptake, rebroadcasting, average sequence lengths, and the prospectiveness of follow-up moves between the two classes influenced students' opportunities to learn mathematics in the following ways: 1.) expected student roles and modes of participation, 2.) the teachers' willingness to listen to students and take their ideas seriously, and 3.) the teachers' approaches to whole-class discussion.

Teacher M framed discussions using core mathematical concepts to organize the lesson trajectory. Discussions had a sense of cohesion, and procedures and formulas were introduced as a natural part of answering central questions related to the big ideas of speed and rate. Teacher M allowed her students to play an active role in discussion and knowledge-generation through uptake of their ideas. She probed their thinking and allowed their observations and conjectures to be the impetus for further class discussion. She also encouraged students to test their thinking, reasoning, and problem solving with one another. For example, in the discussion of a position-time graph, one student thought two piecewise linear functions depicted different routes for two vehicles instead of different speeds. Instead of correcting her, Teacher M let the class struggle with this interpretation (the sequence was 54 moves long) asking the student first to explain her reasoning and then providing her classmates the opportunity to share their thinking.

Teacher N was more task-oriented and preferred to present mathematics as a series of small, manageable tasks and calculations. This preference was reflected in a higher proportion of Acknowledge moves and less use of uptake which resulted in a smaller mean sequence length. As an example, consider the following excerpt (see Table 3) from a whole-class discussion of speed in Teacher N's class. The class had been viewing a simulation and its corresponding position-time graph of a trip from Abilene to Dallas that both a van and a bus had made.

Turn	**Speaker**	**Text**	**Move**	**Prosp**
		Sequence 34		
150	Teacher	Can you see when you replay the simulation when the bus started to slow down?	I	D
151	Student	Yes.	R	G
152	Teacher	Right before it got to the end, correct?	F	G
		Ok at what?	I	D
153	Student	140.	R	G
		Sequence 35		
154	Teacher	Ok. So now let me ask you some questions. What, how long did	I	D

		it take them for this trip? Raise your hand. How long did it take em for this trip? Chris.		
155	Chris	Three hours.	R	G
156	Teacher	How do you know it went three hours?	I	D
157	Chris	Cuz it ends, the line ends at three.	R	G
158	Teacher	Very good,	F	A
		cuz the line for the bus or the van?	I	D
159	Mult S's	Both.	R	G
160	Teacher	Both end at what?	I	D
161	Mult S's	Three hours.	R	G
162	Teacher	Three hours. So you know it took them three hours.	F	A
		Sequence 36		
163	Teacher	Was the speed of the two vehicles the same?	I	D
164	Mult S's	No.	R	G
165	Teacher	No.	F	A
		You know that by how?	I	D
166	Student	They're not the same.	R	G
167	Teacher	Because of the (pause) graph right? And because of also running the simulation. They didn't stay right beside each other, did they?	F	G
		Sequence 37		
168	Teacher	Ok, umm, how far did they go in 3 hours? Raise your hand. How far did they go in three hours? Charles?	I	D
169	Charles	** (unintelligible)	R	G
170	Teacher	How'd you know that?	I	D
171	Charles	**	R	G
172	Teacher	180 miles where? (pause - no student response).	I	D
		On the graph, right?	R	G
		Sequence 38		
173		Ok. And let's see if you can calculate the speed. Go through, run your simulation and see if you can calculate the speed for the van and then calculate the speed for the bus and then also calculate the speed for the bus when it started to slow down since you guys have told me it started to slow down.	I	D

Table 3. Discourse Excerpt from Teacher N's Class

Sequences 34-38 were orchestrated to accomplish the teacher's goal of calculating the speeds for the van and bus. Notice that this larger goal was not mentioned to the students until the last sequence. Instead, it remained hidden in a series of loosely-related questions (from the students' perspective) that ultimately identified the necessary components for calculating speed. Additionally, as seen in this excerpt, little control or responsibility is given to the students as the teacher generated and controlled the topics for discussion. Conversation was tightly scripted with little room for student ideas to shape the discussion. It is as if she had a predetermined path that she was leading her students down: she may have heard their responses but did not incorporate them substantially into the flow of the lesson. In fact, there was very little of substance for her to incorporate since students' roles were limited to paying attention, performing basic calculations, recalling formulas, and answering yes/no or short-answer questions.

Through subtle differences in discursive patterns, students in the two classrooms were socialized to interact, think, and respond in different ways. Variation in uptake and organization of discussions (i.e., mean sequence length and the frequency of rebroadcast moves) created different expectations for participation and opportunities to learn. Consistent follow-up patterns in Teacher M's classroom helped create a community that was more closely aligned with practices and beliefs valued by mathematicians and math educators. These included the beliefs that knowledge is co-constructed, learning is participatory, and mathematics is created through a process of argumentation and negotiation. Because they were significantly involved in doing substantive mathematics, Teacher M's students were more agentic and empowered learners, successful at posing and answering mathematics questions and critically analyzing their own and others' thinking.

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"WHAT COUNTS AS MATHEMATICAL ACTIVITY AND WHO DECIDES?": CHALLENGING THE DISCOURSE OF MATHEMATICS IN MATHEMATICS EDUCATION

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Research suggests that it is important to leverage the home and local discourse practices of non-dominant students to create more equitable and meaningful mathematics classrooms. Gee's (1990) notion of Discourse and its relation to power suggests that this may be difficult to do without first restructuring the predominant Discourse of mathematics, which takes as central the practices of mathematicians and successful mathematical test-takers to the learning of mathematics. Our research indicates that by perpetuating culturally based distinctions between math and non-mathematical activity and reasoning based in the dominant mathematical Discourse, teachers (and students) necessarily marginalize alternative discourses that may support broad-based mathematics learning.

Students' discourse practices vary widely in mathematics classrooms with diverse populations of learners (R. Gutiérrez, 2002a; Moschkovich, 2002). Research suggests that it is important for the discourse practices that students bring from other contexts such as home and local communities be leveraged to support their participation in classroom mathematical activity (Cobb & Hodge, 2002; Hand, 2003). However, it is unclear what this entails for the development of equitable teaching practices. One interpretation rests upon the assumption that teachers often miss important mathematics in the linguistic practices of ELL students, and suggests that teachers provide opportunities for students to negotiate how they participate in mathematics (Moschkovich, 2002). Another interpretation is that teachers need to honor students' non-dominant discourse practices by constructing a hybrid discourse structure for classroom mathematical discourses (K. D. Gutiérrez, Baquedano-López, & Tejada, 1999). In this paper, we draw on Gee's (1990) notion of Discourse to argue that it is difficult for teachers to engage in either practice without first becoming aware of how the dominant Discourse of school mathematics plays out in their classrooms. Our research indicates that by perpetuating culturally based distinctions between math and non-mathematical activity, the Discourse of mathematics makes it challenging for teachers to recognize alternative discourses of mathematics learning embedded in students' sociocultural practices.

R. Gutiérrez (2002b) argues that as an increasing number of non-dominant individuals enter the field of mathematics the field itself will begin to change. This claim is consistent with a situative view of learning that holds that knowledge is interwoven in the practices of communities, which comprise particular values, beliefs and ways of participating in activity. As newcomers to a community successfully negotiate participation by drawing on their repertoires of practice (K. D. Gutiérrez & Rogoff, 2003) from outside communities, the community necessarily adapts and changes to reflect these new practices. We propose that this argument can be applied to the mathematics classroom as well. As students from diverse backgrounds introduce different forms of participation into the mathematics classroom, the norms for what it means to do mathematics in the classroom may shift as well. However, this shift can only occur when the existing power structures are disrupted to allow new hybrid practices to form (Engeström, 2001; K. D. Gutiérrez, Baquedano-López, & Tejada, 1999).

Mathematics has a significant amount of power in K-12 education, as a 'gatekeeper' to higher education, as a perceived proxy for 'general intelligence', as a marker of international dominance, and as one of the most salient aspects of the achievement gap. As such, the Discourse of mathematics carries tremendous weight in society, and perpetuates itself by virtue of this power. Gee (1990) argues that "each Discourse protects itself by demanding from its adherents performances which act as though its ways of being, thinking, acting, talking, writing, reading, and valuing are 'right', 'natural', 'obvious', the way 'good' and 'intelligent' and 'normal' people behave" (p. 191). The following studies reveal aspects of how the Discourse of mathematics is perpetuated in local classrooms, and constrains teachers (and students) from building on alternative forms of mathematical practices, reasoning, and communities.

The first study examined how teachers in two reform-based mathematics classrooms treated different forms of student participation. Historically, analyses of student participation in reform mathematics classrooms have tended to focus on students' participation in classroom mathematical practices. Less is known about how students come to be engaged in these practices, and whether the practices themselves shift as students negotiate diverse forms of participation in taking them up. This paper argues that one reason for this is that teachers (and mathematics education researchers) tend to view classroom learning through the lens of a particular Discourse of mathematics, which often leaves unexamined broader sociocultural processes that permeate the learning environment. This study illustrates how the teachers differently negotiated the Discourse of mathematics in their classrooms, with the aim of engaging traditionally underserved students.

The study was part of a broader one that investigated the implementation of reform mathematical practices in three ninth-grade mathematics classrooms in an urban public high school (Hand, 2003). The school and teachers were selected for their success in producing strong mathematics learners from diverse cultural and ethnic backgrounds (Boaler, 2006; Horn, 2002). The teachers in this mathematics department also utilized a common set of pedagogical practices and an in-house mathematics curriculum that drew on reform mathematics principles. The significant overlap in instructional practices among the classrooms afforded an opportunity to tease out differences in the classroom discourse structures.

The study drew on methodological tools from an ethnographic perspective on classroom discourse to capture both classroom mathematical activity *in situ* (Gee & Green, 1998), and the meanings made of this discourse by local participants over the course of a school year. Research methods included documenting over one hundred hours of classroom activity across the three classrooms through electronic and video records, student and teacher interviews and surveys, and six days of intensive student shadowing. A multi-level interaction analysis was conducted to identify the norms and practices that were functioning to shape mathematical activity in each of the classrooms, and the nature of students' mathematical engagement, which were then triangulated by coded interview and student shadowing transcripts. One aspect of the analysis focused on how student activity was framed within classroom discourse structures, and the implications this had on the type and level of student engagement.

The classrooms were significantly different in this regard. In one of the classrooms, students' activities were treated as being either mathematical or social -- the latter positioned as being detrimental to the students' progress. As the teacher stated, "nothing should get in the way of the mathematics" (class observation, 2002), suggesting that mathematics learning is predicated on a distraction-free environment. While a number of the students in this classroom engaged deeply in classroom mathematical discussions, a number were also positioned and positioned themselves as resistant to them.

In contrast, the second classroom was highly dynamic and supported broad-based participation by a range of learners. The students' activities were continually re-negotiated and re-framed by the students and teacher as a part of what it meant to do mathematics. When one student argued that her test grades indicated that she was poor at mathematics, the teacher asked if tests measured what she knew and could do mathematically. When another student got frustrated trying to explain the meaning of x^3 to his group and started to give up, the teacher used the context of baseball to remind him that persistence pays off. In words of the teacher, it was important to support students in "whatever is needed to get to math" (interview, 2002).

These findings reveal how each classroom negotiated the Discourse of mathematics learning, linked to the values and practices of individuals who are successful in mathematics (e.g., mathematicians) and on achievement tests (e.g., college students). The discourse practices in the first classroom reflected this broader Discourse in emphasizing that concentration and focus lead to progress in mathematics learning. The discourse practices in the second classroom often challenged (and rejected) the Discourse of mathematics. Instead, priority was given to helping students who struggled in mathematics build stronger relationships both with each other and with mathematics.

Teachers' discourse practices are examined again in a study that addresses the perceived disconnect between reasoning inside and outside of the mathematics classroom (1). The study was designed to elicit the teachers' perceptions regarding the logic and mathematical reasoning of their students in three environments: mathematics classrooms; non-mathematics classrooms; and outside of school. It was hypothesized that if mathematics teachers believe student strengths in other settings can be utilized as points of access to mathematics content, what they consider an acceptable contribution to the classroom mathematics discourse could change in ways that allow more students to participate in constructing meaningful mathematics.

Ten teachers with experience teaching middle-school mathematics as well as other middle-school subjects (3 to 19 years) were audio-taped as they participated in one-on-one semi-structured interviews (Wengraf, 2001) (2). The interviews consisted of examples of informal mathematics reasoning as well as problems appropriate in content to a middle-school mathematics classroom. Participants were asked to explain the strategies used by a child street vendor (Saxe, 1988) and a fisherman (Nunes, Schliemann, & Carraher, 1993) in their everyday mathematical activities, as well as solve two problems (one created by the participant) in multiple ways: an informal strategy modeled from the Saxe and Nunes, Schliemann and Carraher research, a strategy they would expect to see and a strategy they would prefer to see their students use in their mathematics classroom. Analysis of the interview data focused on which aspects of mathematical strategies teachers' value, as evidenced by their explicit and implicit judgments.

First, the teachers recognized all strategies as being valuable, but tended to favor traditional algorithms for the purposes of speed and success on standardized measures of mathematical achievement and future mathematics classes. This is not necessarily surprising since the teachers' schools use mathematics reform curriculum, but teachers continue to be responsible for preparing students for future algebra curriculum that is perceived to be more traditional, as well as remain accountable to state and national tests.

Nearly all participants, despite their stated willingness to accept alternative strategies within their classrooms (although sometimes only as an access point to teaching students a more formal algorithm), struggled with interpreting the informal reasoning within the Saxe (1988) and Nunes, Schliemann, and Carraher (1993) scenarios. They tended to pull the numbers straight from the problem and ignore context, immediately explaining the basic operations that were taking place rather than discussing the situation or tools that were available or unavailable. However, once participants deciphered the strategies being used in the scenarios, they felt that the street vendor and the fisherman showed good "number sense" in a way that a traditional algorithm would not necessarily show.

One way to interpret these findings is of conflict between two Discourses -- the traditional mathematics Discourse and what is characterized here as the *sense-making* mathematical Discourse. The traditional Discourse values speed, accuracy, and conciseness within the mathematics classroom; the fulfillment of these constituting sufficient understandingThe sense-making Discourse is conceptualized as primarily valuing student understanding and supporting student-created strategies that increase such understanding, with the conciseness of the strategies being a secondary concern.

The traditional Discourse is represented in a quote from Karen, one of the teachers, who explained why she wouldn't accept the "chunking" strategy in her classroom,

> Because of the speed…I think they understand it, and I think that's great, but…standardized tests are—not that I'm basing it off that, but they're timed…so it's good that you understand it, but you gotta… you have to know more than that.

Note the references to speed as in opposition to understanding.

In the sense-making Discourse, understanding and accuracy go hand-in-hand. Student-developed reasoning strategies are prized first and foremost, with accuracy and efficiency being valued as another step in the process. In her interview, Lindsey described the interplay between student understanding, efficiency, and accuracy within her classroom:

> So, we'll present different strategies, and um, we'll talk about efficient ways and accurate ways, and we'll have the kids choose whichever way makes the most sense to them…. There's no one way to do math—kids need to realize that.

Student understanding is valued as the first step, with students being considered as responsible for determining which strategy is personally best for them.

The participants moved in and out of the two Discourses throughout the interviews. To illustrate the layering of these two Discourses the concept of *lamination* is adopted from Goffman (1981; cf. Jurow, 2005): the simultaneous presence and salience of both for the teachers in their conceptualization of student mathematical reasoning. The key contribution of *lamination* to this analysis is the tension created by the dominant mathematical Discourse as teachers reflect upon student understanding, efficiency, and accuracy: the participants, regardless of which Discourse they most identify with, must frame their own values within the traditional values of the dominant Discourse. Most participants spoke of preferring the values ascribed to the sense-making mathematical Discourse, and yet struggled to negotiate this discourse within the dominant mathematical Discourse.

The third study illuminates the natural hybridity of classroom discourse by attending to student talk around mathematical activity. The paper draws on the results of a semester long study in a pre-calculus/calculus undergraduate discussion section. The section was created to support the development of relationships and a sense of belonging among students typically underrepresented in STEM fields. The aim of the study was to understand the intersection of students' social and academic discourses as they unfolded in the mathematics classroom. This paper reports on one aspect of the analysis: What are students' perceptions of the form and function of non task-related talk during mathematics small group? Findings from this study broaden existing conceptions of what is mathematics discourse.

In attempting to merge students' social and academic lives, the site served as a key setting to explore the complex and multifaceted nature of students' talk as they engaged in group work around the mathematics tasks. Research methods included field notes from over thirty classroom observations, three student surveys, video and audio recordings of both whole class and small group discussions, and student and instructor interviews. Open-ended survey instruments asked 16 students about their perceptions of non task-related talk during group work and their academic and social relationships with their peers both inside and outside of the classroom. In-depth interviews allowed six participants (two groups) to elaborate on their survey responses and discuss their perceptions of the purposes and effects of task (and non task) related talk. These students were also asked to comment on two video recordings taken earlier in the week that captured their group interactions and talk with peers and the instructor during mathematics class.

The extensive use of group work in this class both promoted and underscored students' social and academic relationships with their classmates (Lotan, 1997). Hence, looking at how students negotiated social and academic talk in group work guided the analysis of this study. Two specific aspects of the analysis included: (1) fluency in the social discourse of the group and/or classroom, and (2) the relation between fluency in the social discourse and participation in mathematical activity.

At the beginning of analysis, open coding was used to code field notes, survey responses, and video content logs and transcripts. The identification of categories of talk as social and academic was coupled with analysis of the purposes of talk. The coding of talk quickly moved beyond a dichotomous categorization, as distinctions between on task (academic) and off task (social) talk (as have been traditionally regarded) were rarely well defined by participants or my observations. In thinking of mathematics as a collection of practices, to say that talking about the weather was a non-mathematical practice seemed to go against what students' responses to survey items, interview questions, and classroom interactions suggested. Hence, students talking about the weather could be considered non task-related, but as a discourse practice it at times was encompassed in what it meant to do mathematics.

The findings of this study are threefold: a) developing a sense of comfort with group members was often a precursor to progressing on mathematics activity, b) students' perceived non task-related talk to have multiple purposes during group work, and c) students perceived a threshold whereby non task-related talk during group work became problematic.

The students reported that feeling comfortable with group members was important for mathematics discussions, since it facilitated the acts of challenging each others' ideas, engaging in deeper mathematics discussion, and helping, listening, and seeking help from peers. A bi-directional relationship between comfort and non task-related talk emerged from the data; students perceived non task-related talk as both a means to and a product of group comfort.

Non task-related talk served a multitude of purposes in the groups' work. Some of the affordances described by students included: better quality work, better understanding (for everyone), it makes math fun, leads to comfort in discussing math, serves as a break from math, facilitates discussion, and helps keep the group working at the same pace. One student went so far as to say, "If you don't have anything in common or anything to talk about SOCIALLY, then you sure as hell aren't going to be able to communicate mathematically." To this student and to his peers, engaging in non task talk and building comfort was important (or even essential) to making progress mathematically.

Despite the perceived benefits and purposes of non task-related talk to students, many indicated that there is a vaguely defined point when social talk can distract a group from progressing on the mathematics task. For some students, the determining factor had to do with the duration of non task-related talk; for others topic and/or composition of group was a determining factor. The study showed that students were not always successful at negotiating task and non task-related talk within their groups; successful ways of negotiating different forms of talk deserves further attention.

Findings show that students perceived non task related (social) talk as important (and even necessary) to advancing on the mathematics activity. Students' talk and their perspectives on the nature and function of different forms of talk blur the lines between what has traditionally been deemed acceptable as part of the mathematics and school Discourses.

Combined, three studies provide insight into the perpetuation of the Discourse of mathematics in the local contexts of mathematics classrooms and the categories for mathematical activity, reasoning, and talk that organize mathematical activity within them. These categories necessarily limit what some teachers recognized as significant and productive to students' motivation and success in learning math. Alternatively, teachers and students who implicitly or explicitly disregarded, contested, or even re-framed the distinctions that were handed-down to them (*e.g.*, social/mathematical, efficient/sensible) created space for more inventive and meaningful ways to learn mathematics. By revealing how the Discourse of mathematics operates in both K-12 and undergraduate mathematics classrooms to constrain the non-dominant discourse practices that support students' mathematics learning, this study contributes to an understanding of the complex issues surrounding mathematics reform and equitable mathematics learning.

Endnotes

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2. The first three participants were considered pilots as interview protocol constantly evolved. The remaining seven were interviewed using flexible but finalized protocol, and their data was the subject of analysis.

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COOPERATIVE GROUP WORK AND SELF-EFFICACY BELIEFS OF FEMALES

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Gender inequity has long been an issue in mathematics education. Advancing females' education and opportunities for careers in science, technology, engineering and mathematical (STEM) fields historically has been overlooked. In the past two decades, mathematics education has made great strides in decreasing gender inequities (National Center for Education Statistics, 2005; Tate, 1997). Initially, research centered on the differences in mathematics achievement between females and males (Fennema & Sherman, 1976; Fennema & Sherman, 1977). Current research indicates changes in females' access to higher level mathematics and an increase in societal acceptance of females in STEM fields has diminished the earlier differences in mathematics achievement between males and females. Moreover, no differences are found in complex problem solving or conceptual understanding (Hyde & Linn, 2006). Considering this information, research is needed to examine the reasons why females are still not persisting in mathematics and related fields.

This study is currently being conducted in a mid-Western high school algebra I classroom, focusing on the self-efficacy beliefs of females and their participation in cooperative group work. In this short research report, I will present preliminary results of the study of the self-efficacy beliefs of females and how these beliefs are related to participation within cooperative groups. The question guiding this study is: How are high school females' self-efficacy beliefs affected by their participation in cooperative group work in mathematics class?

The framework for this study is a combination of the situative perspective (Lave & Wenger, 1991) and Bandura's social cognitive theory (Bandura, 1989). The situative perspective allows for the examination of females changing participation within their cooperative groups. Social cognitive theory compliments the situative perspective in that it allows for the individual examination of self-efficacy beliefs, a person's perception about their ability to perform a given task (Bandura, 1989). The merging of these two perspectives will allow for the examination of the social environment of cooperative mathematics while honing in on the individuals' perceptions about their ability.

Preliminary results will be obtained through the following modes of inquiry (1) videotaped sessions of cooperative work, and (2) surveys pertaining to self-efficacy beliefs of females. Video data will be coded using a coding system derived from Barnes (Barnes, 2003). Barnes' coding system was modified to focus on algebra classrooms as opposed to calculus lessons. Survey data will be analyzed to identify females holding high and low efficacy beliefs. The two sources of data will later be cross-analyzed to identify potential trends in the data which may reveal the relation of self-efficacy beliefs and participation in mathematics.

This short research report fulfills the goals of PME-NA in the following ways: (1) The study examines the relationship between motivational aspects of education in conjunction with cognitive aspects of mathematics, and connects the classroom experiences with the psychological experiences of group work, and (2) the study will glean information on the failure of females to persist in mathematics and related fields as related to psychological variables such as self-efficacy.

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ENGLISH LANGUAGE LEARNERS: DEVELOPING MATHEMATICAL UNDERSTANDING

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Today mathematics reform efforts face the challenge of raising the performance levels of students and are forced to examine how to accomplish this within growing culturally and linguistically diverse populations (Grant & Lei, 2002; Gebhard, 2002). Analysis of the National Assessment of Educational Progress (NAEP) indicated that English language learners (ELLs) do not achieve at the same level as their English speaking counterparts (Rampey, Lutkus, & Dion, 2006). In 2005, 71% of the non-ELL eighth-grade students were at the basic level or above compared to only 29% of the ELL eighth-grade students who were at the basic level or above. The report documents that both ELL and non-ELL student mathematics achievement has slowly increased since 1996 and the achievement gap between ELLs and non-ELLs was slightly reduced. But, with only 6% of the ELL eighth-grade students reaching proficiency, inequities are significant.

Cocking and Mestre (1988) suggested that the language used to describe mathematical ideas in English has specific interpretations and is problematic for students with limited understanding of English. Thus, achievement gap may not be due solely to difficulties with math concepts or to second language acquisition but to a complex interaction of the two. Students interpret math related words and symbols based on previously constructed understanding. When students are also learning English, the process of acquiring new mathematical understanding is mediated through two sets of meanings developed in their native tongue and English. Translation alone, without attention to prior knowledge of math concepts is not effective. Helping students who are ELLs develop a rich understanding of mathematics is challenging because everyday and abstract ideas may have been internalized in one or the other language, based on individual experiences. Therefore it is extremely difficult for teachers to identify the prior knowledge upon which they should build.

This research report is part of a longitudinal study that examines the how ELLs assimilate mathematical understanding while acquiring English language skills. The purpose of this report is to describe the mathematical ideas that fourth- and fifth-grade ELLs understood after using *Investigations in Number, Data, and Space* (Akers et al., 1997).

Theoretical Framework

Theorists (e.g., Cobb, Wood, & Yackel, 1970; Kumpulainen & Mutanen, 2000) suggests that social activities influence the learning of mathematics as individuals negotiate meaning through interactions. From the perspective of symbolic interactionism (Blumer, 1969), an individual interprets another person's words and gestures to create meaning. Shared meanings are constructed through a dynamic process of creating and re-creating meanings as individuals interact with each other. People attach particular meanings to words, symbols, and gestures through these social interactions (Voigt, 1996). As the individual interacts with others (e.g., teachers, peers, parents) words may acquire a single meaning or multiple meanings. Attention to the meanings that words and symbols acquire is particularly important in mathematics because many mathematical words also have a common usage that is quite different from their mathematical meaning.

Methods

This study relies on qualitative methods to investigate fourth and fifth grade English language learners' (ELLs) mathematical reasoning. The authors used the Colorado state assessment framework to select released questions from national tests to create an assessment with ten multiple choice and constructed response questions. This assessment met the state's achievement expectations for the fourth and fifth grade students, which included the following topics: patterns with whole numbers, fractional models, measurement, geometry, interpretation from graphs and tables, and algebraic reasoning. The authors created a scoring rubric by considering likely student responses and assigning a level-of-understanding score of "0" (no grasp of the concept), or "1/2" (some grasp), or "1" (a sound understanding) for each test question.

The assessment was administered during the spring of 2006 to all student in one fourth- and two fifth-grade classrooms. Twelve fourth-grade ELLs and 29 fifth-grade ELLs were individually interviewed during the following week. Students reflected about their mathematical thinking for all of the questions, regardless of whether or not they correctly answered these questions. Detailed field notes recorded students' mathematical thinking during the interview. The performance of the students was statistically analyzed using ANOVA to determine whether there were achievement differences between the two groups of students. Qualitative analysis is in progress to describe the mathematical reasoning utilized by these students. This analysis will be further collapsed using a conceptually-ordered cross-case matrix (Miles & Huberman, 1994).

Findings and Discussion

Findings from this study are preliminary. The ANOVA test indicated no significant differences between the fourth- and fifth-grade ELLs. With sixty percent of the students in the elementary school classified as ELLs, the elementary school staff was concerned about the complexities of increasing student achievement when students have limited proficiency in English. Adler (2001) described some of the challenges that teachers and students face when students are learning mathematics in a language different from their home language but offered no instructional practices to help teachers meet these challenges. In a review of bilingual literature, Moschkovich (2002) found that Latino students' with low mathematics achievement in mathematics may involve the comprehension of English vocabulary or the translation of English words into mathematical symbols. This suggests that like the teachers in the elementary school in this study, indeed, many classroom teachers struggle to discern whether ELLs become low performers as a result of limited English proficiency, as a result of instruction practices, or both. The qualitative analysis of students' responses which this study will report is of interest to researchers interested in untangling this very complex web of learning English and mathematics simultaneously.

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EXPLORING LATINO STUDENTS' UNDERSTANDING OF MEASUREMENT ON NAEP ITEMS

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This study explores Latino students' thinking when solving NAEP measurement problems via task-based interviews. Our study is grounded on a combination of sociocultural and cognitive perspectives in which multiple resources were considered when students represented their mathematical ideas. We discuss themes that emerged from the data, some challenges that students experienced, and the richness in their thinking across three items.

This exploratory study seeks to complement large scale studies that have been done on the NAEP (e.g. Abedi & Lang, 2001, Lubienski, 2003) to shed light on Latino students' thinking as they solve measurement problems from NAEP. Lubienski (2003) reported that in the 2000 NAEP, the greatest gap between white and black students and between white and Hispanic students occurred in the measurement strand. Our effort was to gain a better understanding of how working-class Latino students thought about some of these NAEP measurement items. The goals of this study were (a) to uncover challenges that selected NAEP measurement items raise for a small group of Latino students (b) to document their interpretations of the problems; (c) to understand their reasoning and communication of their solutions; and (d) to investigate the role that language plays in their thinking process of procedures and concepts, especially if the students' first language is not English.

Theoretical Perspectives

This study is part of a larger research agenda that looks at the interplay of mathematics, language and culture among Latino students. Our perspective is essentially a combination of a sociocultural perspective and a cognitive perspective (Brenner, 1998; Civil, 2006; Cobb & Yackel, 1996). On one hand, we used task-based interviews, which are typical of cognitively-based studies; on the other hand, our analysis of these interviews is guided by a "situated-sociocultural view of mathematics cognition, language, and bilingual learners" (Moschkovich, 2002, p. 196). Further, the work of Bielenberg and Fillmore (2005) has informed us of critical language features needed for discourse development, such as academic English vocabulary, common academic English structures, and such language functions as explaining, defending, and discussing.

By utilizing task-based interviews, researchers are able to describe and assess mathematical thinking as well as the interplay of contextual and social factors, observed behaviors, and inferred cognitions. The use of language is central to our study in that we are interested in the students' use of language, both everyday and academic language, as they interpret the tasks and explain their thinking; in some cases, our students are bilingual, but more proficient in one of their two languages. Because language structure of assessment items can present added cognitive demands for the students, especially ELL students (Campbell, Adams, & Davis, 2007), it is pertinent to take into consideration the students' multiple resources for their communication of their reasoning in solving the problems. Moschkovich (2002) points out that communication is multifaceted involving gestures, expressions, and objects as resources to simultaneously

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communicate mathematical ideas, and they are especially crucial for students who are less proficient in English. Thus, in our approach for this study we focus on the students' use of various resources, including non-verbal and linguistic resources.

Modes of Inquiry

We conducted and videotaped task-based interviews with twenty-eight Latino students in grades 4 through 8. The students were attending elementary and middle schools in predominantly working class / low income neighborhoods. The first phase included 11 students working on various measurement items taken from the NAEP; the second phase had 17 students working on these three problems: (1) "if both the square and the triangle above have the same perimeter, what is the length of each side of the square?" (Lengths of the triangle are given as 4, 7 and 9); (2) a problem in which the student is given cutouts of a triangle and a square and is asked to compare their areas; and (3) a problem asking students to find the area of a trapezoid ABCD (made of a rectangle and a right triangle). The area of rectangle BCDE is given; and the lengths ED and AE are given.

The students first solved the problems independently and then they were asked to explain their thinking in their solutions. After they finished their explanations, we asked probing questions based on their responses. In some cases, the students' interactions with the researcher prompted them to revise their initial solution. The video tapes of the students were first analyzed by problem. We paid attention to the students' thinking that was observable as they worked on the problems independently. We then looked across the different problems for common themes. In the next section we address three of the themes: (1) linguistic demands; (2) reliance on visual cues; and (3) area and perimeter confusion.

Results

Linguistic Demands

Linguistic complexity can stem from the written form of the problem, especially for English learners. For example, in the perimeter problem (1), the problem reads, "If both the square and the triangle above have the same perimeter, what is the length of each side of the square?" One of the fourth grade students interpreted the "if" statement as "they do not have the same perimeter." When probed, she said "but they do not because it says **IF** (emphasis added)." This child was interpreting the "if" statement as a negation statement, therefore, the square and the triangle could not possibly have the same perimeter. Her facial expression indicated that she was faced with conflict. By the student's interpretation of the written language used in this problem, it is impossible to assess her mathematical understanding of two shapes having the same perimeter.

We found that the students who were successful in solving these problems were able to work at a more abstract level with the measurements given and had moved beyond the concrete level of counting. For example, a successful 6th grade student simultaneously represented his manipulations of the concrete shapes in problem (2) with rectangles and with the equation 2P=2N ("P" represented the triangle and "N" represented the square). He reasoned that if 2P=2N, then half of each rectangle is the triangle P and the square N, therefore, P=N and the areas were equal. Successful students were able to explain their reasoning with more linguistic precision than those who struggled, who tended to use lots of pronouns with no clear referents.

Reliance on Visual Cues

Some students relied on the drawings as their key source of information, yet these were not drawn to scale. A few students requested a ruler for finding the lengths of the sides for some of the figures. Regarding the first problem (finding the length of one side of a square with the same perimeter as a triangle), most students were able to arrive at the correct solution, but those that did not arrive at the correct solution relied on approximating the side of the square visually to be 4 since it looked like one of the sides of the triangle that was labeled 4. In this same problem on perimeter, some students who were incorrect relied on the use of grids to find the perimeter, and understandably, this strategy can be tied to reform curriculum that emphasizes meaning making, but in this example, we found that such drawings by the students interfered with their reasoning. In fact, this reliance on the grid deserves further attention, as we found it to be not always helpful and part of the confusion with area and perimeter that we address next (also, see Kamii & Kysh, 2006, for their insights on counting squares on a grid to find area of shapes).

Confusion of Area and Perimeter

We consistently found that when the problem presented both square units for area and linear units for lengths of sides of a figure, that students confused the concepts and procedures for finding the area or perimeter. In the first problem on perimeter, most students had no trouble with finding the solution since the measurements were given in linear units. But multi-step problem (3), involving both linear and square units, was more challenging. The 8th and 6th grade students who were able to solve it, effectively combined their knowledge of the formulas with the given information to solve the problem.

Closing Thoughts

We found that some of the challenges that students faced were related to their interpretation of the problem due to the linguistic demands, their confusion of area and perimeter, the lack of problem-solving strategies used, their reliance on concrete and visual approaches, and their representations of the mathematics. However, we found that some students were resourceful in their use of tools and problem-solving strategies and in drawing on previous knowledge, but could not always clearly express their thinking about the problems. Our findings show that most students were able to participate meaningfully in mathematical discourse at some level about their solutions and convey their thinking through various resources.

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IN COLLABORATION: A CASE STUDY OF STUDENT PARTICIPATION IN A COOPERATIVE LEARNING SETTING

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This study explores student interactions in a yearlong cooperative learning Algebra class. Preliminary results of the on-going qualitative study identify patterns of participation that impact individual and group engagement in math. The paper argues for a deeper understanding of specific behaviors in groups that delineate opportunities and challenges to learning math in a collaborative setting.

The case study described in this paper explores how peer collaboration shapes student participation and learning in math. The study builds on prior research that suggests cooperative learning (small group learning) is considerably more effective in advancing student achievement than interpersonal competition or individual effort (Johnson et al., 1981). Students also report increased social and academic support in cooperative settings (Angier & Povey, 1999; Leikin & Zaslavsky, 1997), greater intellectual and social autonomy, and greater responsibility for their learning (Yackel et al., 1991). And yet more recent studies of student interactions during collaboration show greater variation in outcome than prior research would suggest (e.g., Barron, 2003; Sfard & Kieran, 2001). This case study extends this latter work and closely examines the inner-workings of groups in a first year Algebra class designed for cooperative learning. The study asks: *How does peer collaboration shape individual participation and learning in math?* The study finds collaboration to be a support and challenge to individual learning, especially for adolescents who feel academically and socially judged by their peers. This paper describes preliminary patterns in peer interactions that impact individual participation. The paper argues for a deeper understanding of specific behaviors in groups that delineate opportunities and challenges to learning math in a collaborative setting.

Theoretical Framework

In focusing on student interactions and outcomes of participation, this project draws on two main theoretical positions: 1) feminist social science, and 2) symbolic interactionism. This stance shifts what might otherwise be an analysis of individual student characteristics to the analysis of student interactions, and what meaning is attributed them.

Feminist social science argues status and difference are made relevant by individuals in interaction and in institutions. As West & Zimmerman argue in the case of gender, "While it is individuals who do gender, the enterprise is fundamentally interactional and institutional in character, for accountability is a feature of social relationships and its idiom is drawn from the institutional arena in which those relationships are enacted" (1987, pp136-7). In this case study, this means students define difference within the larger cultural institution of schooling. Adolescents mark the margins and mainstream of acceptance through dress (e.g., baggy pants or skinny jeans), affiliation (e.g., jock or mathlete), speech (e.g., native English speaker or English language learner), and so on. Thus an understanding of how these culturally laden status markers are invoked and enacted in student interactions, how students describe and locate themselves in

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relation to these status markers, and what consequence they have during collaborations, frames this research.

The second theoretical stance of this case study is the sociological paradigm of symbolic interactionism. Symbolic interactionism suggests that human interactions express a natural and rational connection of individuals to their peer communities. Two theoretical underpinnings of symbolic interactionism are *intersubjectivity* and *reflectivity*. *Intersubjectivity* refers to the way "humans derive their (social) essences from the communities in which they are located…there can be no self without the (community) other" (Prus, 1996, p.10). *Reflectivity* suggests people become objects of their own awareness through interaction with others, which then informs their own behavior (Ibid). In this project, these codependent processes of being and reflecting capture the watchful ways students collaborate. Moreover it suggests students express through language and behavior, a vigilant appraisal of themselves in relation to others that matters for their participation and learning. For example, when a student asserts herself in one cooperative learning group and suppresses herself in another, these behaviors are not easily dismissed as inconsistency in participation. Rather, they may be reexamined as an intelligible jockeying for position among different sets of peers.

Methods and Data

This case study takes place in a multi-ethnic public suburban high school in the Western United States. The focus class of 21 students represents the lowest track in math offered at the school. The students are all ethnic minorities, two-thirds of who are female. The teacher is trained in cooperative learning as an instructional method in math, and has also trained teachers in its use, across the United States. Students in the focus class were observed for 26 weeks over the 2006-7 academic year. Students were randomly assigned to groups every two weeks to complete academic tasks; one group was selected for observation during each cycle. This design allowed for an examination of patterns in student participation as the group composition changed each cycle, over time. The full corpus of data includes observational field notes, accompanying audio recordings of groups, student interviews, in-class surveys, student journals, student work on tasks and assessments, weekly progress sheets, individual state and national test scores, and student demographic information.

Results and Discussion

This paper presents three preliminary constructs that emerged in the analysis of observational field notes, interviews, and student surveys. As such, this paper does not analyze student learning outcomes, though that is the aim of future work. The forthcoming examples illustrate how collaborating with peers serves as an opportunity and challenge to individual participation.

Anointing occurs when a group develops the tacit understanding that one student is most expert. Anointing is indicated by several participatory changes: group members consult one student exclusively, communication among the un-anointed breaks down, and the anointed becomes sole arbiter of mathematical methods and solutions, whether right or not. Becoming the "anointed" is an enduring status marker within a group but not across group compositions. Anointing is a detriment to individual learning and participation as peer scrutiny weighs heavily on the anointed, while opportunities to demonstrate competence is limited for the un-anointed. Anointing endorses a view of math as the singular practice of a special few.

Lone Ranger-ing occurs when a student circumvents the group norm of 'stay together' or 'help & ask for help' by attempting to complete group tasks alone. The term lone ranger is taken from the popular 1940s-50s TV show about a masked ranger pursuing criminals through the Wild West. The term is meant to provoke an image of the individual maverick – one who purposely defies convention in order to achieve a desirable end. While lone ranger-ing serves as an opportunity for students to demonstrate and develop individual competence, it also serves to truncate the learning and opportunities for participation of others. Operating as a lone-ranger without sanction confirms a view of math learning as a solely individual accomplishment.

Aligning occurs when students take up another's idea or position in opposition to others in the group. Aligning often occurs when students disagree on a method or solution. However, aligning also occurs as a gesture of protection, sympathy, or solidarity, as when a student feels another is being unduly scrutinized by the teacher or a peer. Aligning also serves to regulate action in the group, as when two students tell a third to slow down. Alliances are not typically enduring; they shift over time, within a task, by individual need, or by group configuration. Aligning generally serves to promote individual learning and participation as students experience having their ideas supported by others in the group. Aligning, especially around content, offers a view of math as a social practice where justifying, negotiating meaning and reasoning are a group accomplishment.

Revealing occurs when a student makes evident an aspect of their social network, personal life, or cultural heritage to better their position in the group. Revealing occurs in multiple ways: through obvious display (e.g., dress, eye make-up, etc.), in response to direct questioning, or as an indirect response to others. Revealing often occurs when students have either never worked with one another before, or are in a new group configuration. Revealing is most often coupled with evaluations by peers that subsequently shape that student's participation. When revealing occurs spontaneously, it often serves a specific interactional function: forming common ground, establishing reputation, locating oneself in a specific context, staving off criticism, mitigating known differences, repairing a misunderstanding, and so on. Students "reveal" in a variety of ways and with different effects on their participation that indicate their positioning among peers.

In summary, these preliminary themes suggest student behaviors and interactions are an important site for evaluating the opportunities and challenges of working collaboratively with peers. Identifying and describing these and other participatory patterns helps make visible how peer collaboration shapes student participation and the possibilities for learning in math. This then offers teachers a way to identify, evaluate, and problem-solve student experiences of cooperative learning by identifying how (and which) students are most vulnerable or empowered by its use.

Implications

Bringing to bear the theoretical traditions of feminist social science and symbolic interactionism offers a fresh perspective to the understanding of individual participation in a cooperative learning math environment. By focusing on student interactions, and by identifying patterns of participation, this work suggests how teachers and students might similarly examine and problem-solve peer collaborations to promote academic success for all.

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INVESTIGATING PEER AS "EXPERT OTHER" DURING SMALL GROUP COLLABORATIONS IN MATHEMATICS

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Significant emphasis has been placed on the importance of collaborative learning in mathematics amongst peers (National Council of Teachers of Mathematics/NCTM, 2000). Barnes and Todd's (1978) seminal research on peer collaborations suggests that students are more likely to achieve fruitful discourse (i.e., open, collaborative discussion, and argumentation) when they are able to take independent ownership of their learning and this occurs *outsid*e of the immediate range of the teacher. Sfard, Nesher, Streefland, Cobb, and Mason (1998), however, point out that students need to be in the "presence of a more highly structured awareness" (p. 49), which is described as "carefully constructed tasks, exposition, and *people* [italics added]" (p. 49). Adler (1998) concluded from her study of small group settings that the inner dynamics of the group often reflected a small class setting where one student takes on the role of the teacher. Yet Mercer (1996) contends that peer discourse is reflective of *symmetrical* relationships; thus, the role of "teacher" may only be illusionary. This work examines peer as "expert other" during collaborative inquiry in mathematics.

Theoretical framework

This research draws from the theoretical assertions of Vygotsky (1962; 1978). Vygotsky proposes that knowledge is a reconstruction emerging externally from interactions with people, activities, and cultural artefacts (i.e., either tools assisting the external to be internalized or signs supporting internal activity aimed at self-understanding and mastery). With respect to knowledge, Vygotsky states that "the path from object to child and from child to object passes through another person" (p. 30) or a more 'expert other' through *scaffolding.* Scaffolding occurs within a symbolic space (Lerman, 2001, p. 103) conceptualized by Vygotsky (1978) as the *zone of proximal development* (ZPD), which is "the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (p. 86).

Methodology

Peer collaborations were analyzed using *video study methodology* (Kotsopoulos, 2007). Data collection occurred during the 2005-2006 school years, in an eighth grade classroom (34 students). Data sources included: in-class observational field notes taken during non-collaborative learning and teaching; student interviews; one teacher interview; 38 hours of video and audio taped peer collaborations, over three different mathematical tasks, emerging authentically from the teacher's own programming; student artifacts; sociometric questionnaires; focus group session transcriptions. Video and audio data were transcribed and analyzed.

Results

Predominantly, students were *not* observed helping one another understand the mathematics in any of the observed tasks despite requests for assistance from students within the group. There was extremely limited evidence of an “expert-other” emerging from within the various groups to support another’s learning even when students who where experiencing difficulties made their challenges explicit to others. Students’ conceptual challenges were made evident within the groups in three ways: (1) the student did not construct the assigned model, (2) the student’s calculations were incorrect, and/or (3) the student talked aloud in attempts to elicit support. On most occasions when students had incorrect calculations or answers, it was observed that remediation primarily consisted of dictating the correct answer to the student – a form of directive – object level discourse. On very few occasions, students asked leading questions like, “Did you divide?” “Did you multiply?” or “did you forget the cash reserve?” Despite this type of questioning, the answer was immediately dictated aloud to the student. There were no subsequent checks following the oral dictation to determine if the solution was correctly recorded. Students did not take it upon themselves to question each other’s understanding when incorrect answers were provided. Some students did not notice other students’ struggles.

In the present research, an obvious “expert other” does not actually emerge in the student groups to provide scaffolding for other group members. Thus, this research calls into question the potential of a peer to act in the capacity of “expert other” in a group setting. This finding also calls into question the ways in which “expert other” is defined. Loose definitions of “expert other” as simply the most capable individual in a setting are questionable. I contend that “expert other” has particular and specific skills and knowledge in relation to a discipline, topic, or field. This does not suggest that in education *all* teachers have the necessary skills and knowledge to function as an “expert other” in all areas of curriculum. A teacher may, however, be a more probable candidate as an “expert other” than simply the most capable student in a given set of students.

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LANGUAGE USED BY PARENTS AND CHILDREN WORKING ON MATHEMATICAL TASKS: A STUDY OF GENDER DIFFERENCES

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This research report examines communication patterns between parents and children working on three different mathematical tasks. Among ideas being investigated is the extent to which there are gender differences in use of cognitively demanding language among parent-child dyads. The report briefly discusses theoretical foundations of the project, validation of survey instruments, and differences in communication patterns.

This paper presents preliminary analysis of data collected as part of a three-year project *The Role of Gender in Language Used by Children and Parents Working on Mathematical Tasks* funded by the National Science Foundation to investigate the following research questions: 1) To what extent are there differences in the use of cognitively demanding language among four types of child-parent dyads (daughter-mother, son-mother, daughter-father, son-father) working together on mathematical tasks in number, algebra, and geometry? 2) To what extent are there gender-related differences in children's self-efficacy in mathematics and parents' competence beliefs for their children's success in mathematics? and 3) What are the relationships among (a) parents' competence beliefs for their children's success in mathematics, (b) children's self-efficacy and interest in mathematics, and (c) cognitively demanding language used by children and parents when working together on mathematical tasks?

The research questions are based on theories of the role of gender on children's self-efficacy (Bandura, 1997; Pajares, 2002; Zimmerman, 2000), parents' competence beliefs for children (Eccles, Frome, Yoon, Freedman-Doan, & Jacobs, 2000) and how these affect cognitively demanding language (Tenenbaum & Leaper, 2003).

Theoretical Framework

Crowley, Callanan, Tenenbaum, & Allen (2001) found that in parent-child conversations involving interactive science museum exhibits, girls were one-third as likely to hear explanations from parents. Furthermore, 22% of the explanations given boys were causal, versus only 4% for girls. Boys and girls were not significantly different in whether they initiated engagement. Tenenbaum and Leaper (2003) investigated parents' teaching language during science and non-science tasks among middle class families. Their findings indicated fathers used more cognitively demanding speech with sons than daughters when working on a physics task, but not on a biology task. Extending into mathematics, this project is studying language used by children and parents as they work on tasks in three content areas: 1) number and operations, 2) reasoning and algebraic thinking and 3) spatial sense and geometry. Based on prior research that reported gender differences in geometry (Baenninger & Newcombe, 1995; Casey, Nuttall, & Pezaris, 2001) and on the reasoning (Fennema, Carpenter, Jacobs, Franke, & Levi, 1998) it is hypothesized that the types of mathematical tasks will produce gender differences in the cognitively demanding language used by children and parents.

Methods

The project involves one hundred third and fourth grade students and their parents from public schools in Hawai‘i that serve ethnically diverse populations with 50% to 80% low socio-economic status. Each parent-child dyad is videotaped while working on three ten-minute mathematical tasks. Each task has multiple methods of solution and was designed to provide for a high level of interaction between the parent and child. Prior to working the mathematical tasks, both parents and children completed surveys to gather data on competency beliefs, self-efficacy, and value/usefulness of mathematics as well as demographic information.

Instrument Development

Parent and child surveys were developed using data from the Childhood and Beyond Study (GARP, 2006 by Jacqueline Eccles and others. Reliability (Cronbach's Alpha) tests were employed on surveys collected from 66 students and 44 parents from three elementary schools to examine internal consistency. Initially, a nineteen-item survey with six constructs (ability, values, usefulness, interest, parent involvement and effort) was proposed. However, reliability results indicated that only two of the six constructs were reliable ($\alpha > 0.70$). After reducing the number and regrouping the items, the reliability of the remaining constructs, 1) self-efficacy; 2) value/usefulness and 3) competency beliefs, improved ($\alpha > 0.84$).

Transcribed video data are coded with a project-developed instrument consisting of three main categories: 1) getting started, 2) discussion mode, and 3) vocabulary usage. Getting started focuses on reading and discussing the task and the discussion mode involves questioning, directing, and correcting by the parents along with follow-up questions and statements. Also coded is the child's reasoning as prompted by parent's statements or questions and reasoning initiated without parent's prompting along with the level of mathematical vocabulary used by the parents and children.

Data Collection and Analysis

The data collected include 1) parents' belief in children's competence and value/usefulness of mathematics, 2) children's self-efficacy for and value/usefulness of mathematics, and 3) cognitively demanding language used by children and parent and interactions within child-parent dyads. Results from the parent and child surveys are analyzed to determine relationships between the parent and child responses. The use of cognitively demanding language is coded and analyzed using the qualitative data analysis software ATLAS.ti. Analysis includes examining relationships between parents' competence beliefs, children's self-efficacy and cognitively demanding language used by children and parents.

Results and Discussion

Two transcribed segments of videotaped sessions between a daughter-father dyad and a son-mother dyad illustrate the type of gender-related differences found. The dyad is responding to the following questions on the algebra task cards that focus on patterning:

1. Tell each other the pattern you notice.
2. Describe what Train 4 would look like. Build Train 4 on your work area.
3. In this pattern, if a train has 6 triangles, which train number would that be? How many squares are in this train? Talk about how you know.

Father (F) – Daughter (D)	Mother (M) – Son (S)
F: Oh, cool, ok, so, we have to look at the different patterns we see and then we have to make what we think train 4 would look like. Do	M: What pattern do you notice?

you see a pattern? Do you see anything between train 1 and train 2 that looks kinda similar? Or…	
D: Yep.	S: The…..I only notice it has, each train has 2 squares and 1 triangle and it multiplies 1 by 1, 1, 2, 3.
F: What?	M: OK Mmm
D: How you make this. They are all the same and how you make train 4 is like…	M: Let's do the task number two.
F: But, before you make train 4, try and explain to me what is the pattern that you see? Right? What's the difference between train 1 and train 2?	M: OK and task number 3.
D: This is just going the same way, it's the same pattern but it has one extra.	M: So what would the train number be?
F: Right. Ok, and then train 3…	S: It would be six and.
D: Has…	M: How many squares would be in that train?
F: how is it different from train 2?	

The two transcript excerpts contrast examples of conceptual questioning and perceptual questioning by parents. The father (F) working with his daughter (D) asks conceptual questions focusing on relationships and more abstract ideas while the mother (M) working with her son (S) asks perceptual questions which primarily require one word answers.

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LATINO FAMILIES INVOLVEMENT IN THEIR CHILDREN'S MATHEMATICS EDUCATION

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This study is grounded on a sociocultural perspective and focuses on the role of Latino families in their children's mathematical learning. Through parents' voices we explore possible ways of inclusion that may allow for Latino families to overcome social and educational exclusion. Latino mothers explain strategies to counteract this exclusion using resources they find in their communities.

This study focuses on Latino families' involvement in their children's mathematics teaching and learning. Our goal in this paper is to try to understand the nature of this involvement from a sociocultural point of view. Through parents' voices we explore possible ways of inclusion that may allow Latino families to overcome social and educational exclusion. We examine the kinds of strategies that Latino mothers in our study used to overcome some of the barriers that affect their children's learning of mathematics.

Theoretical Framework

Drawing from the sociocultural approach (Cobb & Yackel, 1996; Nasir & Hand, 2006), we assume that learning is a social and cultural phenomenon that cannot be understood only through the analysis of cognitive processes, but also it is a process in which contextual aspects of the student intervene. We understand learning as a reality in which school, teachers, students, families, and the social context in which they live in are intertwined (Elboj, Puigdellívol, Soler, & Valls 2002). Whereas other studies have looked at educational practices inside schools, hence focusing on interactions among students and teachers in the classroom, our focus is on the parents' perspectives of their children's learning of mathematics. This approach builds on prior work we have done in this area (Civil & Andrade, 2003; Civil & Bernier, 2006; Civil, Planas, & Quintos, 2005).

Methods

Our research takes place in two elementary schools located in working-class, primarily Latino neighborhoods in the Southwest of the U.S. We center on the voices of the mothers that participated in our study so that they are the ones who explain the difficulties they faced as they tried to help their children learn mathematics and what resources they drew from to overcome these difficulties. This is consistent with the use of a critical communicative framework (Gómez, Latorre, Sánchez, & Flecha, 2006). From this methodological point of view we emphasize the interpretations that the people directly involved in the research, that is the parents themselves, make of the dynamics that explain how Latino families take charge of their children's mathematics education. As Hidalgo (2005) writes, "I take the position that the more one knows about the contexts of Latino/as lived experiences, the better one may understand their processes of adaptation and change" (p. 378).

We individually interviewed sixteen mothers from the two schools. We transcribed and analyzed all the information collected by video and/or audio recording. In order to analyze the information we used grounded theory (Glaser & Strauss, 1967) with the purpose of inductively bringing up the themes the mothers considered to be relevant.

Results

In this paper we report on the three main barriers that the mothers mentioned and on the kinds of strategies that they used to overcome them.

Differences between Schooling in Mexico and in the US

The participating mothers in this study pointed out that they encountered many barriers when trying to help their children with mathematics homework. One of the difficulties relates to the differences between Mexican and U.S. educational systems, as the quote below shows. One of the mothers, Lucrecia, is reflecting on when her daughter first started attending school in the U.S. upon her arrival from Mexico:

> Lucrecia.- It seems like she gets a little upset, at the beginning when it was very easy for her because she already knew it then it was like if, and the teacher used to say that she got distracted since she had already studied that, so she knew it already, that's why she didn't take it seriously, and so it seems like she got upset.

Tensions between Different Forms To Do Mathematics

This is in part related to the previous barrier. Several mothers commented on how they learned mathematics when they went to school and that this was very different from how their children are learning now. This mother, for instance, comments on the conflict with her daughter when she tries to help her with her homework:

> Laura: With my daughter Yahaira, they taught her one way and they taught me a different way, but it's the same answer. It's what I tell Yahaira, I explain it to her one way, and they explained it to her a different way. I tell her, "but look, it's the same answer."

Issues of Language

Some mothers mentioned that their limited knowledge of English made it hard for them to help their children with homework: "Every once in a while I cannot explain (it) to her because I know almost no English" (Selena). Another side of this language issue is when the children themselves do not know English well. In some cases this leads to resistance and rejection towards the school, as this mother recalls when they first arrived to the U.S., "It was all in English. And then I started being proactive and, I said, no, poor children, because they were traumatized. They didn't want to go to school anymore, they wanted to go back to Mexico, and it was everyday that they cried…"

Strategies To Overcome These Barriers

The mothers in our study were resourceful at finding ways to help their children learn mathematics. In general this desire to "help their children" is a source of motivation for the mothers to look for resources in their communities. Some mothers we interviewed take mathematics classes to get to know the local educational system. Other mothers study English; they ask their own children to translate the homework questions for them, or they look for help from the teachers or other people in the community, as we can see from the following quotes:

> Jacinta. - But on Thursday, I am going to take a math class because there are problems that I don't know how to solve. And the teacher invited me on Thursday afternoon at five to teach me what they are teaching to him, so that I can help him.
> Loli. - Now I know little bit more the procedures and what she is teaching, and I feel more, like with more freedom to help her, because before… "You are not helping me. You better move away, Mum. You better go away." And yes, it is true; if I don't help her, why be a hindrance? But now I understand better than before… How did I do at the beginning? I had spider webs because I couldn't get anything in any way, now I started to remember a little, all because [these classes].

Some mothers looked for support through their networks (family, friends, community), for example by sending their children to other friends' houses (or family' houses), who already know English well, to work together on the homework. Other mothers made use of the local community centers (e.g., Boys and Girls club) to seek assistance with homework for their children: "sometimes when she [her daughter] is not able to do it, she calls her cousin. Or she also brings the homework to the club." [Noelia]

Discussion

Our study adds to the work that highlights the resources, the knowledge, the caring, and the resiliency present among Latino working-class parents (Delgado Gaitan 2001; Valdés 1996), but with a specific focus on mathematics education. We are therefore guided by a theoretical approach that highlights the understandings these families have that allow them to advocate for their children. Our analysis shows that cultural elements such as language and the educational experience in another country are key aspects that affect how much mothers can help their children. In the case of the Latino families in our study, we observe that not knowing English or how to solve a specific mathematics homework problem may constitute obstacles for the collaboration between parents and children. This makes the school-family connection more difficult to attain and, according to other studies (Elboj et al, 2002), this connection is one of the key components for academic success. Yet, our study also shows that Latino mothers actively look for ways to fight against these difficulties that are of a more structural nature. Through community resources (attending workshops, English classes, mathematics classes, civic centers in the community, etc.) and their own families / friends (asking other family members for help on English translations, sometimes even their own children), those parents work towards overcoming these difficulties and try to look for ways of inclusion.

Acknowledgment

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REAL-WORLD CONTEXTS AND CLASSROOM CURRICULA: RELEVANCE AND MATHEMATIZABILITY

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This practitioner research considers how real-world problem contexts can support mathematics teaching and learning. Examining both a specific curriculum unit in depth and the growth of the practitioner's curriculum over time, this report articulates and applies frameworks of relevance and mathematizability to problem contexts and centeredness, individualization, re-presented complexity, and congruence to curriculum.

This short oral report summarizes the findings and understandings emerging from three years of practitioner research on the role of real-world context in mathematics curriculum. As a high school mathematics teacher, the researcher has engaged his students in a variety of projects with real-world problem contexts. This report addresses the following research question: "What features of problem contexts and classroom curricula engage students in building deeper understanding of both mathematics and the real world?"

Theoretical Framework

Mathematical understanding is connecting mathematical ideas to each other and to other knowledge (Hiebert et al., 1997). Depth of understanding correlates with the depth, number, and type of connections students make with mathematics. Real-world problem contexts, such as currency for place value, can help students to make connections. Many of these problem contexts, however, can be "foreign" to students. In addition to failing to acknowledge cultural resources, this mismatch also inhibits students' mathematical learning. School mathematics can instead "center" on students' lived experiences (Tate, 2005). This centricity of curriculum with students' lives is consistent with culturally relevant pedagogy (Ladson-Billings, 1995).

The process by which real-world problems are analyzed and solved entails mathematization. Mathematics can further serve as a tool for advocating for change when contexts of social injustice are brought into the classroom (Gutstein, 2006). Students may still fail to connect procedural school mathematics with problems set in real-world contexts (Boaler, 2002). In a different sense, mathematization could refer to the study and formalization of everyday cultural practices using a mathematical lens, an ethnomathematical perspective (Barton, 1996).

Classroom Setting

The data for this study come from the researcher's classroom in a school that serves recently immigrated English language learners in New York City. Students come from twenty-two countries and speak twelve different languages; 71% of students qualify for free or reduced lunch. The school is part of a growing network of schools dedicated to integrated content and English language instruction, project-based instruction, experiential learning, and support for students' first languages. Students take classes in blocked groups with an interdisciplinary team of teachers. To encourage peer support for English language development, students stay in the same "institute" for two years so that the classes in this study consist of both ninth and tenth graders.

The researcher develops some of his classroom work with a collaborative group of teachers and a university researcher. In *Centering the Teaching of Mathematics on Urban*

Youth, participating secondary mathematics teachers meet monthly during the school year and intensively during the summer to share their experiences as teachers as well as to learn more about how to incorporate real-world social justice contexts into their classroom teaching.

Methodology and Data Sources

This qualitative action research study begins out of intrinsic interest, but then moves into a more instrumental view in the development of a set of dimensions with which to classify curricula according to the relevance and mathematizability of the problem contexts (Sagor, 2004; Stake, 2000). Classroom curriculum and student work and reflections are coded using open and focused coding (Emerson, Fretz, & Shaw, 1995). The objective of this coding is to find instances of relevance and mathematizability and then to develop subcodes which identify more specific features of *curriculum.*

The data for this study come from multiple sources:

- Curriculum: projects, handouts and other materials from instructional units.
- Teacher fieldnotes
- Students' written reflections: daily, at the end of each unit, project, and semester.
- Student work: daily products collected in folders, drafts, and final products

The researcher also collaborates with a university researcher who visits the classes during certain projects. The researcher engages in reflective discussion about his ongoing work with the visiting researcher and with the other participating teachers.

Results

In the highlighted project, students constructed cartograms based upon current data on their native countries to investigate proportionality. Through this project, students make connections: within mathematics; between mathematics and the world; with each other, their native countries, and the world; and between variables and questions within the world.

This in-depth analysis of a specific curriculum unit yields a description of features of curriculum, which are the ways in which students are presented tasks and support for exploring real-world problem contexts. As far as a problem context is relevant to students, the curriculum may explore topics which are *centered* on students' everyday lived experiences and require them to produce *individualized* final products. Insofar as a problem context is mathematizable, the curriculum that implements it will need to *re-present complexity* in the real world and employ mathematical methods which may vary in terms of *congruence* with the practices in the real-world context. An elaboration of these four features, as outlined in Table 1, is the main result of this study.

These features are further applied to other curriculum units, such as ones about taxation, text-messaging systems, community mapping, the consumer price index, technical analysis of stock prices, and structural package design. The researcher's curriculum demonstrates a trend toward greater prominence of these features over time. These features are further related to students' expressions of interest as measured by reflections and evaluations.

Feature	**Questions and Characteristics**
Centeredness	To what extent does the curriculum take students' everyday lived experiences and interests as a point of departure to pose tasks which draw upon students' cultural resources and knowledge?
Individualization	To what extent are students tasked with creating products which are unique and reflect their own individual perspectives and knowledge?
Re-presented	How does the curriculum explicitly make assumptions, simplifications,

Complexity	omissions, or other modifications in order to present students with a problem to which mathematics can be more easily applied?
Congruence	How closely and authentically do the mathematical processes used in the curriculum parallel how people in the real-world context approach the problem or task?

Table 1. Elaboration of features of curriculum.

Discussion

If these curricular features are indeed central to using real-world contexts which are both relevant to students and mathematically rich, then it is still unclear *how* teachers learn to develop curriculum units with these features. What knowledge about students and dispositions toward students' cultural knowledge and resources facilitate the development of curriculum with these features? From the point of view of developing curriculum, where is it more useful to start—the mathematical content or the real-world contexts?

As a first attempt, these features have been introduced as questions which can be answered descriptively for each project. A further step would be to use these categories to develop a rubric for evaluating curriculum. How might these rubrics apply to commercial curricula which claim to draw primarily upon real-world contexts in the teaching of mathematics?

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RESEARCHING TEACHER LEARNING: CENTERING THE TEACHING OF MATHEMATICS ON URBAN YOUTH

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This paper describes research that fosters, documents, and analyzes a collaborative project between a university researcher and middle and high school mathematics teachers. An operational framework of centering the teaching of mathematics is provided. The paper provides a theoretical analysis of how mathematics teachers, as individuals and as a professional community, learn to center their mathematics teaching on students in an urban setting
.

Theoretical Framework

We begin with Ladson-Billings's (1995) specific construct of culturally relevant pedagogy (CRP) as a framework for equity pedagogy, with its three attributes: it i) emphasizes students' academic success, ii) encourages the development of cultural competence, and iii) facilitates development of critical consciousness. We connect this definition of CRP to the content area of mathematics, to arrive at a three-pronged framework of *centering the teaching of mathematics on urban youth,* borrowing Tate's (1995) notion of centric instruction: 1) students' academic success is emphasized through the advancement toward understanding of and proficiency with mathematical concepts and skills, 2) cultural competence is fostered through the inclusion of aspects of the lived experiences of students and their communities as contexts for mathematization, and 3) critical consciousness is developed through the use of mathematics to addresses relevant, sociopolitical themes. While we recognize that the principles of centering teaching of mathematics could be applied to any community of students, we include the term "urban" to specifically denote a diverse, densely populated setting.

The inherent challenges of planning mathematical instruction centered on students demand collaborative efforts between practitioners and researchers. This project utilizes a model of teacher learning community (McLaughlin & Talbert, 2006).

Methodology

This paper focuses on the following two sets of research questions: 1) How do teachers translate goals of centering the teaching of mathematics on urban youth into their daily practice? 2) How does a structure of a professional community, organized around the goal of learning to center their teaching of mathematics on urban youth, contribute to teacher learning? Participants include ten middle or high school mathematics teachers, each with five or fewer years of teaching experience who teach at middle and high schools in three of New York City's five boroughs. Data includes audiotapes of two 5-day summer workshops, teacher written reflections, audiotapes of two years of monthly meetings, as well as field notes from individual classroom visits and individual interviews by the researcher.

Results

Mathematics in Community

One of the goals of the summer workshops was to facilitate teacher learning about how to develop contexts for mathematical learning that would be relevant to their own students.

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno

Accordingly, over the two summers, the teachers participated in two versions of a community walk along nearby census tracts. In the first summer, teachers were given a map of a census tract and directed to map a route and collect information about that tract according to categories like housing patterns or businesses present in the tract, or according to teachers' own interests, or, according to categories that might emerge from the walk itself. In the second summer, teachers were again given maps of nearby census tracts, but were asked to choose alternate, ethnographic ways of gathering information about that neighborhood. Subsequent to the physical activity, in both summers, teachers were directed to various electronic sources of data about that tract, supplied by national or local agencies. This community-walk activity was posed as a potential mechanism for their own learning about their students' communities. However, the primary interpretation of this activity by participants over two years has been to take the concrete notion of mapping as a basis for curricular projects that connect mathematical ideas to local communities.

One important concern that quickly emerged among the teachers is the location and identification of mathematical concepts that could be associated with the urban theme of community. Critical questions related to local statistics immediately surfaced, such as the racial or linguistic categories used by the US Census in contrast with the teachers' knowledge about the demographics of the local community. For instance, one teacher noted that the Census reports a percentage of Black individuals in a tract, their walks and local knowledge attested to the presence of a diversity of Black individuals, some African Americans, some from the Caribbean, a diversity lost by the general Census category. Another teacher focused on finding and analyzing the ratio of public play space to the number of residents in the tract, and comparing that ratio with a more affluent neighborhood. Another teacher examined the multiple perspectives of a neighborhood's diversity, by creating a flip-book that used the mathematical concepts of area and proportion to demonstrate different social or economic characteristics.

Housing also emerged as a theme of interest, with questions about ratios of owners to renters, changes in rent prices over time, development, and consequences of gentrification. A third mathematical theme connects to the field of graph theory by examining the efficient counting of paths or cycles on a map, the equitable location of important resources in a community, or the strategic, competitive location of businesses or facilities. A fourth theme related to geometric concepts of the location of the population mean or median of an area and the possible significance of these locations. The next step in the teacher learning process involved the actual implementation of these projects, and the next section describes two such classroom projects.

Classroom Projects

One of the projects was titled the Community Mapping Project, done with 9th and 10th graders, at a school for English Language Learning students. In this project, the students conducted individual community walks and gathered data about their own neighborhoods according to variables of their own interest. They constructed scale maps of their own communities, and retrieved and analyzed local Census data. Each student also compared some aspect of his or her neighborhood with that of a classmate and made predictions about the future of their neighborhood.

A second year-long project was created around the theme of housing and construction in a 7th grade class at a K-12 independent school. The project began with a focus on the destruction of housing by Hurricane Katrina and a class visit to a Habitat for Humanity worksite. Students were later asked to describe the racial diversity of their neighborhoods and were then directed to investigate the corresponding Census data. The students later researched property values in the local region as well as typical salaries to better understand the economic meaning of a local home. They worked on the topic of percentages by studying simple interest in the context of applying for fictitious mortgages to fund a proposed housing project. Then students then embarked on an architecture design project which emphasized the topics of rational numbers and measurements in designing scaled floor plans as well as scaled styrofoam 3-D models of their floor plans, all according to designated budget.

Discussion

One point of tension that has emerged out of this study is the question for teachers of what drives the project, the math goals or the urban theme itself. One can begin with a set of mathematical concepts and develop a corresponding project, or one can start with an urban theme and find ways to read that theme with the mathematics in one's curriculum. A second tension that has emerged in this process is a struggle over the issue of relevancy, as well as the related, but slightly different, issue of authenticity. How can a teacher determine whether a theme is relevant to her students, especially given that she does not belong to the students' communities?

A third point of tension stems from the challenges of urban diversity when it comes to identifying communities because of the way these communities are layered, even within a single neighborhood. This, of course, extends to using mathematics to analyze relevant social or political themes – relevant to whom? Urban diversity also complicates notions of race. As a teacher explained: "I've always thought as far as racism or prejudice is concerned it was whites against everyone else. But some of the things that are wedging between Blacks and Dominicans and Dominicans and Puerto Ricans and Puerto Ricans and Blacks and Blacks and - I'm amazed. I'm like, "Wow!""

A fourth point of tension relates to the current climate of high stakes testing. The teachers in the research group continue to feel challenged by the gatekeeping exams that focus solely on students' knowledge of classical mathematics. And so, while they agree in spirit with the other two goals of culturally relevant pedagogy, cultural competence and students' critical consciousness, they express concern that the real social justice issue might be getting the students to pass those particular exams.

Finally, a fifth tension which has emerged from the data, is teachers' fear of "double oppression." Since the participating teachers are outsiders to their students' communities, as they try to develop students' critical consciousness by using mathematics to analyze the social or political conditions of their students' lives, they fear that this could be experienced by the students as further oppression. As one of the participating teachers asked, "If you consistently point out injustice to kids, don't they lose hope? Are your students, as immigrants, more hopeful about changing this country than my students who have had family in this country for generations and are still living in poverty?"

We do not interpret these tensions as an indication that centering the teaching of mathematics on urban youth is intractable. Instead, we view this it as a multi-dimensional challenging task, especially for new teachers. Our project is still underway, and we plan to share further findings as we proceed.

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RETHINKING EQUITY IN MATHEMATICS EDUCATION: A POSTMODERN PERSPECTIVE

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This theoretical essay examines the achievement gap as a construct made more real by continual research about it and suggests genealogy as a possible way of working toward equity concerns.

Much of the equity-oriented work in mathematics education has been focused on the achievement gap – explaining it, measuring it, and theorizing about ways to close it.

Postmodern philosopher Michel Foucault described this constant reiteration of a particular idea as a will-to-truth. Using the phrase "the achievement gap" reiterates a certain set of power relations, which inscribe certain truths, but not others, "in social institutions, in economic inequalities, in language, in the bodies themselves of each and every one of us" (Foucault, 1980, p. 90). The constant repetition of the phrase "the achievement gap" works to create a real phenomenon. An ERIC search for documents with the phrase in the abstract turned up more than 500 texts; a Google search produced over a million hits. Measuring the achievement gap etches this particular relation of power into our social institutions, our language, and our bodies more deeply. What we measure (alienation, attitude and math test scores), how we measure (multiple choice, surveys, short answer) and what we report (scores by race and gender) are technologies that allow educators to exercise power relations. They remind some individuals of their inferiority, and others of their superiority. NCTM (2005, p. 4) links the achievement gap to "racial, ethnic, linguistic, or socio-economic status," and because of the continual reiteration of these relations of power, readers know -- without being told -- which groups are on the low and high ends in each of the named categories.

These technologies – the methods of measurement, the content being measured, and the way scores are (or are not) disaggregated – are not innocent, neutral or natural; they do not simply measure what is true; they produce it. Typically, scores are not reported by income-level, educational attainment of parents, hair color or height. We choose which categories to make important. In one study on stereotype threat, researchers told European American males that Asian Americans typically outperform whites in mathematics. Then researchers then gave these men a challenging math test. These participants performed significantly worse on the test than men who had taken the test without having this stereotype invoked beforehand (Aronson et al, 1998). For minority students, the act of taking a standardized test may work to invoke stereotypes about performance even without an explicit reminder because phenomena like the achievement gap are so widely accepted as real. Thus, the achievement gap works in two ways in these situations. First, its acceptance as a real phenomenon impacts student performance on tests, and second, the tests then go on to produce evidence, in the form of test scores, that the phenomenon is, in fact, real.

This relationship is also an example of what philosopher Ian Hacking called "the looping effects of human kinds," (Hacking, 1999). As social scientists, the categories we use to describe those we study create kinds of people, and unlike species of plants or animals, human kinds (minorities, females, urban children) can learn about their kind and can change their behavior as a result. In the mathematics education literature today, many human kinds exist. Researchers discuss African-American students, Latinas/Latinos, girls, students with disabilities, English language learners, urban students, and others. As intuitive as these categories may appear, they were not always available. Early research in mathematics education tended to differentiate students in far fewer ways than we do today. Edward Thorndike (1922) referred only to "the pupil." For instance, we should try to find problems "which not only stimulate the pupil to reason, but also direct his reasoning to useful channels" (p. 20). Similarly, other writers who discussed problem solving in the early part of the 20th century tended to see students in more unitary ways than we do today. William Brownell (1938) did not discuss differences among students, except in regard to ways that they had been taught.

Today, when individuals recognize and read about their kinds, they must find ways of living in response to the categories. Stereotype threat isn't possible without the stereotype, and research explicitly focused on kinds of students who have trouble in mathematics can, without intending to, force individuals to "fit or get away from the very classification that may be applied to them" (Hacking, 1999, p. 34). To address these concerns, equity-focused researchers in mathematics education may want to consider adopting new stances in their writing. One possible way of framing such work would be to draw on the genealogical traditions developed by postmodern theorists. In discussing the differences between ethnography and genealogy, Erica McWilliam (2003) wrote that one of the greatest challenges of adopting a genealogical perspective is letting go of the role of advocate. As an example, she noted that deciding to ask the genealogical questions "Why bullying now?" and "How bullying now?" rather than the ethnographic questions "What is bullying?" and "How do we stop it?" can be unsettling (McWilliam, 2003, p. 60). In similar ways, the genealogical questions of "How has it become possible to think about home culture as different from school culture?" or "Why has it become common to explain students' schooling experiences in terms of race?" might seem like less compelling ways of working toward an equity agenda than the ethnographic questions about why students with certain demographic characteristics are succeeding or failing in mathematics. The object of genealogical work is not to document the present through the mobilization of evidence, method and theory, but to bracket the present through argument so that an object of study – such as the achievement gap -- appears contingent, permeable and historically dependent. Rather than reinscribing the importance of current social categories, the goal is to make it possible, however briefly, to imagine the world otherwise.

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SOCIAL JUSTICE AS MOTIVATION FOR CHOOSING A MATHEMATICS PROGRAM: TEACHERS' PERSPECTIVE

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Teaching mathematics to foster students' understanding of social justice issues is growing in mathematics education. However, results from an initial analysis of our investigation reveal that some teachers choose mathematics programs based on their social justice motives, their desire to preserve students' academic and social self-concept, and to foster students' overall developmental success.

Mathematics is utilized by a growing number of educators as a vehicle to help students develop awareness and agency about social justice issues (Gutstein, 2007; Gutstein & Peterson, 2005). However, there are also social justice issues imbedded in the reasons teachers choose certain mathematics programs, their practice of teaching mathematics, and teachers' understanding of how mathematics attainment impacts students' developmental trajectories. During an Ontario Ministry of Education (OME) funded research project "Studying the Approaches of Mathematics Instruction" (SAMI), analysis of teacher interviews revealed that some teachers choose mathematics programs based on their personal social justice beliefs.

Theoretical Perspective

Social justice in education commonly focuses on reducing educational inequalities that arise from factors that impede the educational and developmental success of students (Cook-Sather, 2007). Freirean pedagogy espouses the socio-political empowerment of students through education (Freire, 2000), and empowering mathematics involves students' gaining power through the practice of mathematics (Stinson, 2004). However, these emancipatory pedagogies assume a certain level of student engagement. One reality is that mathematics is a "gatekeeper" subject that is used to academically track and socially stratify students (Stinson, 2004), and as such there exists groups of students who may be prevented from reaping the full developmental benefits of mathematics attainment. However, if teachers understand how mathematics impacts students' developmental trajectories, it may be that we will begin to positively impact the mathematic attainment of students and foster their developmental success.

During the analysis of our data, we observed the emergence of four themes that related to teachers' motivation for selecting mathematics programs. These emergent themes include teachers' beliefs about: (1) how innumeracy impacts students' levels of participation in society; (2) educators' personal experiences with a pivotal teacher who influenced their learning of mathematics; (3) desire to foster the learning of struggling mathematics students as they may be the most vulnerable members of society; and (4) the intentional selection of mathematics educational programs which provide incremental success and accessibility to mathematics students of all ability levels. These distinct categories form the structure of this paper and will be presented as manifestations of teachers' larger social justice motives in teaching mathematics.

Method

SAMI is a large, on-going OME research initiative evaluating the structured pedagogical approaches enacted in elementary mathematics programs in Ontario, Canada. One of the goals of the SAMI project is to compare pedagogical approaches that share a focus on explicit instruction. This process was initiated with inquiry-based open-ended interviews of four grade 4 and grade 5 mathematics teachers. During the first phase of coding, we discovered a social justice theme that emerged from four distinct, but related, categories. The initial exploratory phase of this study was interview-based. We selected mathematics teachers from public and private schools who held differing pedagogical approaches to instruction. Recruited participants were asked to complete a questionnaire relating to teacher beliefs and instructional practices. The questionnaire was then used during semi-structured interviews to guide researchers in probing teacher-participants' deep belief structures. The interviews were transcribed and coded using an inductive coding process (Maxwell, 2005). Once coded, the data was organized by teacher cases, and cases were analyzed for similarities and divergence. The cases were then categorized by common themes.

Results

We selected two teachers, *Sidney* and *Epstina*, that best illustrate teacher-participants' social justice motives for selecting mathematics programs. Sidney is a teacher in a private school, while Epstina teaches for a public school; both are grade 5 teachers. Throughout the interview process both participants acknowledged that their personal experiences with innumeracy resulted in their commitment to helping students who struggle with mathematics. "I was a struggling student myself" revealed Epstina. As the interview progressed Epstina shared how she struggled with mathematics as a child, and how her quest to learn mathematics as an adult led her to understand that as a teacher of a gatekeeper subject, she plays a pivotal role in the developmental trajectories of students who struggle with mathematics.

In a related theme, both teacher-participants identified having a pivotal mathematics teacher who had succeeded in teaching them mathematics as adults. Sidney encountered this educator in Teachers College, while Epstina befriended a mathematician and subsequently learned mathematics as a result of this friendship. Both participants reported the belief that having struggled as children, and having subsequently learned mathematics as adults helped them to understand the incremental nature of mathematics, and that they both believed that this experience provided them with metacognitive insight into the challenges that some children face when learning mathematics. Further, both teachers believed that their personal experiences fostered their ability to successfully teach students who struggle with mathematics.

It is important to note that Sidney's and Epstina's choice of mathematics programs were not made independent of pedagogical considerations. Both teachers reported that the programs they chose utilized a non-authoritative incremental approach to mathematics instruction. The teachers identified that they chose a mathematics program based on the organizational structure of the instructional materials as the mathematics program divided complex concepts into related, incremental steps, and had a particular emphasis on single and double-step solutions, which resulted in struggling students having increased opportunities to succeed in mathematics.

Both participants also offered that they believed that the materials they chose helped struggling students develop foundational knowledge about mathematics through multiple opportunities to practice skills as these mathematics programs reinforced effective strategies, which in turn impacted students' motivation to learn and engage in mathematics, and overall contributed to students' sense of self-competence in the domain of mathematics.

Another pedagogical feature that influenced mathematics program choice was the graphic design of the materials. Sidney and Epstina both noted that while other mathematics materials were filled with colourful graphics and visual representations, the program's minimal use of language and plain black-and-white graphics aided students in sustaining attention and "on-task" behaviour. Sidney offered that teaching mathematics from programs and materials that are not "language laden" "visibly lifts a burden from students who struggle with language" and allows these students to experience success and become excited about and engaged in mathematics.

An unexpected psycho-social theme emerged when the participants independently noted that the mathematics programs they chose possessed the feature of variable entry, and that this feature fostered an equitable classroom learning environment as students of all ability levels worked in the same work-book. Participants noted that this program feature has particular relevance for struggling students as it allows these children to maintain their self-concept as an equal member of the classroom, as their personal struggles in learning mathematics are not "made visible" as they are when lower-ability students are required to work out of books that are physically different from books used by other class-mates. Both participants noted multiple times that choosing mathematics programs and materials that are inclusive to all ability levels fosters a reduction in social exclusion among students and ensures that "no-one gets left behind".

Discussion

Our initial purpose for conducting this phase of the SAMI study was to understand teachers' perspectives of enacted instructional approaches. However, our analysis revealed an emerging theme about the social justice motives that influence how teachers choose mathematics programs and materials. Although we initially speculated that the teachers' selection process of mathematics programs was based on the instructional components inherent in the programs, we have found that some teachers also base their decisions on how programs foster mathematics success for students who are struggling, and how these programs increase learning motivation, and positively preserve students' academic and social self-concept. This is illustrated in the cases of Sidney and Epstina, as both participants articulated that they choose mathematics programs to help struggling students experience success in mathematics as they were concerned about the developmental trajectories of their students. This social justice theme was an unexpected finding in our study.

While we will proceed with our larger research questions, we have been made aware of this unique, emergent theme, and are now considering conducting another smaller phased study examining this particular aspect of mathematics. This smaller study may help us determine both the enacted similarities or differences of instructional practices and the degree to which enacted practice originates out of teachers' developmental social justice motives.

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THE CO-CONSTRUCTION OF ARGUMENTS BY MIDDLE-SCHOOL STUDENTS

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The importance of students building arguments to support their solutions to problems is undisputed. However, there is a need for research into the process by which arguments are built by middle school students, particularly from urban communities. This report provides insight into student's co-construction of arguments by urban, low income, sixth-grade students. Over five sessions, there was a move from re-iterating ideas of others to expanding on ideas, and finally to jointly co-constructing solutions.

Participating in mathematical discussions and reasoning about mathematics requires that students be afforded the opportunity to share and to discuss their ideas with others (Lampert & Cobb, 2003). In their *Principles and Standards for School Mathematics*, the National Council of Teachers of Mathematics (2000) emphasizes the importance of communication for students to develop mathematical understanding. They recommend that instruction be designed to enable students not only to share their ideas in a mathematical community but also to analyze and evaluate the thinking of other members.

The link between communication and mathematical understanding has been identified by others. Skemp (1979) defines relational understanding as the ability to utilize general relationships to form more explicit rules or procedures; he defines formal/logical understanding as the ability to combine mathematical symbols and notation with mathematical ideas and indicates that this connection leads to mathematical understanding. He suggests that formal/logical understanding occurs when students use their relational understanding to convince others in a community. Mathematical discourse has a central role in the building of mathematical understanding. Collective mathematical understanding, according to Martin, Towers, and Pirie (2006), occurs when a group of learners are engaged in a mathematical task. They assert that collective understanding emerges from the social context of the learning environment. This implies that the growth of mathematical understanding can be explained, at least in part, by the co-actions of group members, rather than individual understandings. According to Martin, et al (2006), co-acting is a process in which an individual's mathematical ideas and actions are adopted, built upon, and internalized by others, thus becoming shared understandings rather than being limited to the individual. Interest in studying the extent to which mathematical discourse can lead to the co-construction of arguments and justifications in problem solving by middle-school aged students resulted in a study designed to investigate the following research question:

What evidence, if any, is there that students integrate the ideas of others into their justifications?

Theoretical Framework

Guiding this study is the idea that in the activity of group problem solving, individual students call upon and use certain representations to communicate their ideas (Davis & Maher, 1997). Through discourse about the representations that are used (e.g. words, model building, drawings, and symbolic representations), ideas are made public and subject to scrutiny by other members of the community. Students in the process of explaining and justifying a solution to a problem often revisit, review, and perhaps revise and/or extend their original ideas, and further build their understanding. In making ideas public, different forms of justification emerge (Maher & Davis, 1995). As members of the community have access to the ideas of others, there is opportunity to act upon these ideas in such a way as to influence their development. This study is based on the view that collective mathematical understanding is built as learners co-construct ideas and problem solutions. The process involves reiterating, redefining, and expanding on shared statements or ideas. It also involves questioning members of the community in order to verify ideas and representations, gain information and facilitate the understanding of others; and correcting each other as a means of leading to mathematical argumentation. While working in the mathematical community in which this co-construction occurs, students advance their own understanding and that of their group by the posing of their own challenges.

Methodology

This study is a component of a longitudinal study, conducted through an after-school program partnership between a university and urban school district. The purpose was to explore how mathematical reasoning develops in middle-school age students over time and under certain conditions, such as, working on challenging tasks, interacting with peers, having time to explore, and explain and discuss ideas. The focus is on the ideas that students bring in connection with their communication with each other. Data come from video recordings of the first five sessions (60-75 minutes), over a three-week period, in an after-school program. Data analysis follows procedures consistent with qualitative and video-based methodologies (Powell, Francisco & Maher, 2003). Twenty-four sixth graders, African Americans and Latinos, were participants. They worked on open-ended mathematics tasks dealing with fractions. Four cameras videotaped the students, three focusing on particular groups and the fourth *roving* or attending to overhead presentations. Codes for the co-construction of ideas emerging from the analysis of the data were organized into the following categories: building on other's ideas, questioning others, and correcting others. As the data were analyzed sub-codes emerged.

Results

The table below summarizes the co-construction of problem solutions across the five sessions. Based on the initial ideas expressed, students' building on the ideas of others was relatively stable across sessions. However, there is an interesting trend of increase in students questioning and correcting of ideas by the third week. In Session 1, the majority of the interactions involved restating a student's argument. There was one instance of a student questioning another student and this was to seek verification. In Session 2, students continued to expand upon, redefine, and reiterate their classmate's ideas. The questions asked during this session were mostly seeking verification, such as, asking classmates to repeat or clarify a statement. There were two questions flagged as seeking information. In this session students began to correct each other. During Session 3, mostly, questions involved verification; however, there were three instances of questioning that involved students seeking information and two instances in which students created and posed new challenges to the class. The number of corrections that students made of others' work was five. In Session 4, the majority of interactions were flagged as building on and expanding ideas posed by others. The mode of questioning shifted from seeking verification to facilitating each other's understanding through information seeking questions. Four students posed challenges; one posed two challenges. Students corrected each other 14 times. By the fifth session, students backed up their corrections with their own models, warrants and explanations. Questions were geared toward understanding each other's models and reasoning. As the students built on each other's ideas, questioned and corrected each other they began to co-construct arguments. This was evident in the back and forth nature of the dialogue; one student made an idea public, another added to it and soon it became a collective argument.

Student Co-Construction of Problem Solutions					
Date of Session	**No. of Students**	**Initial Idea Expressed**	**Student Builds on Ideas**	**Student Questions Idea**	**Student Corrects Idea**
11/12/03	7	5	17	1	0
11/13/03	11	5	22	8	2
11/19/03	9	4	16	19	5
11/20/03	13	7	21	24	14
12/03/03	9	11	30	28	19

Discussion/Implications

Over the five sessions, the frequency and nature of students' co-constructions changed. There was a move from re-iterating ideas of others to expanding on ideas, and finally to jointly co-constructing solutions. Implications for this work suggest the importance of providing conditions and context for students to work together in building solutions to problems in actual classroom settings.

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UNDERSTANDING URBAN AND RURAL KENYAN STUDENTS' OUT-OF-SCHOOL MATHEMATICS PRACTICE

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This study examined 36 urban and rural Kenyan students in grades six and eight to understand their perceptions of their out-of-school mathematics practice. We found that these students' perceptions could be classified as one of the six fundamental mathematical activities (Bishop, 1988) and was also connected to their perception of whether they learned mathematics outside school.

Purpose of the Study

Students gain mathematical power when their in-school mathematics experiences build on and formalize their knowledge gained in out-of-school situations, and when their out-of-school mathematical experiences apply and concretize their knowledge gained in the classroom. Furthermore, an important part of mathematical experiences in school is the guidance and structure that teachers provide to help students make connections among mathematical ideas. By building on the mathematical knowledge that students bring to school from their everyday experiences, teachers can encourage students to (a) make connections between in-school and out-of-school mathematics practice in a manner that will help formalize the students' informal knowledge, and (b) learn mathematics in a more meaningful, relevant way. "Mathematics teaching can be more effective and will yield more equal opportunities, provided it starts from and feeds on the cultural knowledge or cognitive background" of the students (Pinxten, 1989, p. 28). In order to help students connect and learn from doing mathematics in school and doing mathematics out of school, we need to know how students use and perceive how they use mathematics in everyday situations. To that end, the research study discussed here was conducted to gain some insight into how students perceive that they use mathematics in out-of-school situations.

Perspectives and Guiding Frameworks

Our general aim in this study was to develop a better understanding of mathematics practice in everyday situations. To this end, we focused our attention on standard six and standard eight students' mathematics practice in out-of-school situations. More specifically, our goal in this study was to gain insight into how these students' *perceive* that they use mathematics in out-of-school situations.

Our analyses of these data are framed by hybridity theory (Bhabha, 1994), in the sense that people in any given community draw on multiple resources or funds to make sense of the world (Moje et al., 2004). We suggest that students, whether in school or out of school, draw on everyday and academic mathematical knowledge as they attempt to make sense of things happening around them in the world through their ways of knowing, reading, writing, and talking—what Gee (1996) called Discourses. Moje et al. discussed the construction of "'third space' that *merges* the 'first space' of people's home, community, and peer networks with the 'second space' of the Discourses they encounter in more formalized institutions such as work, school, or church" (p. 41, emphasis in original). We conjecture that it is in third space where students' in-school and out-of-school mathematics practices may enable them to become more powerful mathematically in both practices.

Methods and Data Sources

The research study reported here consisted of three phases. In the first phase, 18 standard six and 18 standard eight students in Kenya were interviewed to determine their perceptions of their everyday mathematics practice. Twelve of the students from each grade level lived in and attended a primary school in an urban area, while the other six lived in and attended a primary school in a rural area. The respondents were asked questions such as "How do you think you use mathematics outside school?" "How do you see other people using mathematics outside school?" "What do you think mathematics is?" and "Do you think you know any mathematics that you didn't learn in school?" The interviews were audiotaped and transcribed.

During the second phase, the 36 respondents kept a log for a week and recorded their use of mathematics outside the classroom. The directions were simply to describe how and where they used mathematics. Students were given a set of log sheets to record each day for five days. We analyzed the interview responses and the logs for the purpose of classifying mathematics usage. In the third phase, we interviewed the original 36 respondents again, asking them to (a) clarify things they had written in their logs, (b) expand on earlier responses, and (c) discuss any insights they gained while keeping the log. Again, the interviews were audiotaped and later transcribed. The second interviews occurred approximately two weeks after the first interviews.

We analyzed the data through an inductive process (Strauss & Corbin, 1998), looking for units and categories of mathematics usage in out-of-school situations, changes in the students' perceptions after keeping a log, and any evidence of students making connections between their in-school and out-of-school mathematics practice. After analyzing the students' views of what mathematics is, as well as whether the students stated that they knew mathematics that they did not learn in school, we then compared these views to the students' interview statements and log entries and looked for patterns that emerged.

Findings

Perceptions of Mathematics Usage

In studying different cultures, Bishop (1988) argued that there are six fundamental activities that "are both universal, in that they appear to be carried out by every cultural group ever studied, and also necessary and sufficient for the development of mathematical knowledge" (p. 182). Mathematics, as cultural knowledge, "derives from humans engaging in these six universal activities in a sustained, and conscious manner" (p. 183). Through the interviews and log sheets, we found evidence that the mathematics students perceived they used outside school can be classified as one of the six activities that Bishop (1988) has called the six fundamental mathematical activities—counting, locating, measuring, designing, playing, and explaining. Furthermore, we found some evidence of all six activities.

All of the 36 students (100%) reported examples of what we classified as counting. Measuring was another activity that had the majority of students reporting examples (67%). Fourteen percent of all students reported data that we classified as playing. However, this was distributed unequally between urban and rural students with only 4% (1 student out of 24) of urban students connecting playing with using mathematics outside school, while 33% (4 students out of 12) of the rural students noted that they used mathematics when playing. Using Pearson's chi-square significance test, we found that these differences are statistically significant with a probability value of .02. Activities that we classified as designing were mentioned least often, with only two students reporting these. Both of these students lived in an urban area. No rural students mentioned activities that we classified as designing.

Perceptions of What Mathematics Is

As mentioned in the description of the methods, the students were asked what they think mathematics is during the interviews. In an earlier study of grade six and grade eight students in the United States, Masingila (2002) found that only students with what she classified as "broader views of mathematics" (p. 37) perceived themselves as using mathematics in locating, playing, or explaining activities. Those students with narrower views of mathematics mentioned only activities that Masingila classified as counting, measuring, and designing. Masingila defined students' views of what mathematics is as narrower if they were more limited in scope; "they either involved viewing mathematics as a subject to be learned in school …, as a set of rules or principles …, or as numbers" (p. 36). The students that Masingila classified as having broader views of mathematics had views that included "the idea that mathematics involves of way of thinking" (p. 36).

In analyzing the data for this study, we did not find the same relationship. Out of the 36 students, we classified only 19% (7 students) as having a broader view of mathematics. In general, we found students' expressed views of what mathematics is to be closely tied to arithmetic as used in school.

Perceptions of Learning Mathematics Outside of School

As part of the post-log interviews, we asked students "Do you think you know any mathematics that you didn't learn in school? If so, how did you learn it?" Forty-two percent of all students (15 students out of 36) responded affirmatively. This was distributed unequally between male and female students, as well as between urban and rural students. While the percentage of affirmative responses is distributed fairly equally when considering the data by grade level (44% of standard six students responded affirmatively; 39% of standard eight students responded affirmatively), the differences are quite a bit larger when considering the data by gender (25% vs. 63% for females vs. males) or by location (33% vs. 58% for urban vs. rural). Using Pearson's chi-square significance test, we found that the differences by gender are statistically significant with a probability value of .02.

Perhaps the most intriguing finding was that out of the 12 respondents reporting activities that we classified as involving explaining, playing, locating, or designing, nine of these were students who answered affirmatively to the question about whether they think they have learned mathematics outside of school. Thus, 75% of the students reporting non-counting, non-measuring activities were students who stated that they think they have learned mathematics in out-of-school settings. We infer that although almost all of these students (10 students out of 12) did not state a view of mathematics that we classified as a broader view, they were in fact broader in their view of how they perceived themselves using mathematics outside school and were open to seeing that they may have learned mathematical ideas outside of the classroom.

Discussion

Many, if not most, children in Kenya are required to complete chores at home that involve them in working with plants and animals, and other chores such as cooking and cleaning. The students' comments from the interviews and logs reflect this engagement with their local environment. However, this involvement with the local environment does not appear to influence the students' perceptions of their out-of-school mathematics practice, where two of the mathematical activities (counting, measuring) were each reported by more than 65% of the respondents but the other activities (explaining, playing, locating, designing) were each reported by fewer than 20% of the students. Parallel to this, we classified the vast majority (81%) of students' views of mathematics as narrower. We speculate that the *second space* of these students—the Discourses they "encounter in more formalized institutions such as work, school, or church" (Moje et al., 2004, p. 41)—situates mathematics as something that is done in school, with certain procedures and rules. These students may very well be mathematically powerful in their out-of-school mathematics practice but they may not count that as mathematics due to the disconnect between the Discourses of everyday and school.

Moje et al. (2004) argued that "academic texts can limit some students' learning as they struggle to reconcile different ways of knowing, doing, reading, writing, and talking with those that are privileged in their classrooms. School texts can act as colonizers, making only certain foreign or outside knowledges and Discourses valid" (p. 43). Perhaps for these students, and for many students learning mathematics in formal school settings, what they perceive as mathematics is confined to academic mathematics (which is often perceived by elementary students to be arithmetic) and does not include a broader view of mathematics such as a way of thinking and making sense of data. We believe it is important for researchers and teachers to work to support students in the construction of third space where their in-school and out-of-school mathematics practices may enable them to be more powerful in both practices.

WEATHER AND RUSSIAN DOLLS: STUDENTS NEGOTIATING WHAT IT MEANS TO DO MATH

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Distinctions have often been made between what constitutes social and mathematics talk. In this study, undergraduate students' talk and their perspectives on the nature and function of different forms of talk blurs the line between what is math and non-math talk. The construct of lifeworld is used to illustrate how students came to negotiate what it meant to do mathematics.

Theoretical Framework

Calculus has long been a gate-keeping course for students in college (Adelman, 2006). More troubling yet is that certain student populations remain underrepresented in STEM majors that require success in calculus. Hypotheses have been made about why non-dominant students have been less successful in college mathematics; the achievement gap has been explained from different perspectives. Some perspectives have viewed low achieving students as having a deficit in their preparation, cultural background, and family support. Other perspectives have seen the problem as being the setting of the mathematics classroom, in which non-dominant students' academic and social lives have been polarized (Fullilove & Treisman, 1990). Still, other perspectives have located the problem in classroom Discourses (Gee, 1990), where non-dominant students' practices and knowledge are positioned as estranged from the classroom discourse (Cobb & Hodge, 2002; K. D. Gutiérrez & Rogoff, 2003) and what it means to do mathematics.

Lave and Wenger's (1991) notion of communities of practice and legitimate peripheral participation provide insight into the inequitable opportunities to meaningfully participate in mathematics that students currently underrepresented in STEM have. Lave and Wenger and the situative perspective view learning as contextual and knowledge as embedded in social relationships and interactions within the social environment. Such a perspective is a useful lens for understanding how students' practices get positioned. Furthermore, focusing on social interaction is fundamental to understanding how students come to participate in communities of practice: "participation in an activity system about which participants share understandings concerning what they are doing and what that means in their lives and for their communities" (p. 98).

This study illuminates the natural hybridity of classroom discourse by attending to student talk around mathematical activity. The paper draws on the results of a semester long study in a pre-calculus/calculus undergraduate discussion section at a Midwestern university. The section was created to support the development of relationships and a sense of belonging among students typically underrepresented in STEM fields. The aim of the study was to understand the intersection of students' social and academic discourses as they unfolded in the mathematics classroom. This paper reports on one aspect of my analysis: What implicit and explicit assumptions and beliefs guide students' expectations for effective group work?

In attempting to merge students' social and academic lives, this site served as a key setting to explore the complex and multifaceted nature of students' talk as they engaged in group work around mathematics. In this paper I draw on Habermas' (1987) notion of *lifeworld*, to illustrate how students come to negotiate what it means to do mathematics together when norms for productive group work are not made explicit.

Methods

Research methods included field notes from over thirty classroom observations, three student surveys, video and audio recordings of both whole class and small group discussions, and student and instructor interviews. Open-ended survey instruments asked 16 students about their perceptions of non task-related talk during group work and their academic and social relationships with their peers both inside and outside of the classroom. Interviews with six participants (two groups of three) served as an opportunity for students to elaborate on survey responses and discuss purposes and effects of task (and non task) related talk. Additionally, students were asked to comment on two video recordings from earlier in the week that captured their group interactions and talk with peers during mathematics activity.

At the beginning of analysis, open coding was used to code field notes, survey responses, and video content logs and transcripts. The identification of categories of talk as social and academic was coupled with analysis of the purposes of talk. The coding of talk quickly moved beyond a dichotomous categorization, as distinctions between on task (academic) and off task (social) talk (as have been traditionally regarded) were rarely well defined by participants or my observations. In thinking of mathematics as a collection of practices, to say that talking about the weather was a non-mathematical practice seemed to go against what students' responses to survey items, interview questions, and classroom interactions suggested. Hence, students talking about the weather could be considered non task-related, but as a discourse practice it was encompassed at times in what it meant to do mathematics.

Next, discourse analysis (Gee, 2005) was conducted on video clips used during student interviews. Habermas' *lifeworld* construct became useful in the analysis of video transcripts in attempting to support the themes that emerged from other data sources. Interactions where students could be described as talking, acting, and making claims in the lifeworld were identified: the times people are acting, talking, and making claims as "everyday" people, not experts or specialists. For example, instances of laughter or jokes were considered lifeworld talk. In another instance, a group was trying to solve a trigonometric problem that required multiple use of the half angle formula; one student told her peers that the equation was like "Russian dolls" (referring to the nested nature the solution).

Results

Two main findings emerged from the study. First, developing a sense of comfort with group members was often perceived by students as a precursor to progressing on mathematics activity. Secondly, when expectations for group work were not made explicit by the instructor, students negotiated norms that were guided by their social perceptions of peers.

Feeling comfortable with group members was important for mathematics discussion since students said it facilitated the acts of challenging each others' ideas, engaging in deeper mathematics discussion, and helping, listening, and seeking help from peers. A bi-directional relationship between comfort and non task-related talk emerged from the data; students perceived non task-related talk as both a means to and a product of group comfort. When non task-related talk was referred to as a product of comfort, the conceptualization of comfort was vaguely defined. For some students, comfort was just a feeling that you either had or did not have; a friend was defined by one student as someone who laughed at her comments. In video tapes and hypothetically, students responded that they could identify comfort, but how it developed was variably and vaguely defined.

Guidelines from the instructor for group work did not extend beyond telling students that they should work together in their groups on the specific tasks; further expectations for group work (or even what working together looked like) were not made explicit. Hence, students had ample space to make inferences about beneficial group practices. In addition, they negotiated their own expectations for group work. Students often interacted with one another as they would in the *lifeworld*. Students seemed to be using their lifeworld ways of knowing and being to negotiate ways of working together on mathematics. For example, students perceived the use of non task-related talk as an important group practice. A sub-analysis question hence became: What are students' perceptions of the form and function of non task-related talk during mathematics small group? The findings were twofold: a) students' perceived non task-related talk to have multiple purposes during group work and b) students' perceived that there was a threshold to when non task-related talk during group work became problematic. More details of these findings can be found in *"What counts as mathematical activity and who decides?": the Discourse of mathematics in mathematics education* (PME 2007 Proceedings).

Discussion and Implications

This study supports research in the field of mathematics education that maintains that students' Discourses can be drawn on to provide more opportunities for students currently underrepresented in mathematics to participate meaningfully in mathematics classrooms. In this study, talking, acting, and making claims in the *lifeworld* were central for the groups to be able to engage in mathematics activity; non task-related talk was seen as a tool to build and foster group comfort. In the groups discussed here, the students' use and value of non task-related talk during mathematics activity created more opportunities for students to be seen as competent group participants. By operating most of the time under *lifeworld* expectations and norms, students were able to admit that they did not always have answers to mathematics problems, and yet they could still contribute to the group's mathematical progress in meaningful ways.

Although not always encouraged by the instructor, the setting created a space for students to negotiate their own ways of working together. In this space, social relations and ways of being came to influence the ways students worked on math together in important ways; the setting opened up a space for students to introduce new ways of participating in groups and doing math together.

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WORDS, THOUGHTS AND ACTIONS: EXAMINING THE POTENTIAL IMPACT OF A SHARED LANGUAGE OF MATHEMATICS PEDAGOGY

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One hallmark of a professional community is its shared discourse. The unique language and discursive practices of a community provide a means by which members of that profession communicate about objects and practices (Gunnarson, 1997; Wenger, 1998) and (re)create a world view. Through language use, certain aspects of the environment are highlighted and relationships among objects, practices and people are implicated (Clark, 1996). At the present time, the shared professional discourse of the mathematics education community lacks the capacity to describe the core of its work— mathematics pedagogy. To strengthen mathematics teaching and learning, we contend that it is critical to develop a shared language of mathematics pedagogy (Hiebert, Morris & Glass, 2003; Siemon et al., 2004). This shared language may facilitate discussions of practice, support teacher learning, and afford conceptual tools that teachers can draw upon as they organize mathematically rigorous lessons and reflect on their teaching.

Theoretical support for these assertions and this inquiry is rooted in sociocultural theories of learning. Vygotsky (2002) and others recognize the importance of language: it mediates thought, directs one's attention, and guides one's analysis on events. By naming objects or practices, we gain the ability to hold them in our mind and deliberately reflect on the object or practice itself. Similarly, Wenger's (1998) work on communities of practice highlights the importance of physical and conceptual tools, such as named practices, in mediating meaning and opening opportunities to negotiate meaning for individuals within a community.

To explore the feasibility and impact on practice of a shared language of mathematics pedagogy, the authors and five secondary mathematics teachers collaborated to a) identify a set of deliberate pedagogic actions that teachers use to support students' mathematical thinking during whole-class inquiry, and b) develop a conceptual model that described the role of the teacher in supporting such discussions. The set of pedagogic actions (purpose a) included strategies such as *promoting others' perspectives*, *exploring incorrect solutions*, *seeking new representations*, and *highlighting*. The conceptual model (purpose b) comprised five nested components of the teacher's role that were necessary to support whole-class discussion. *Valuing* was at the core, whereby the teacher had to value the student and students' thinking. Other components included *publicizing* student ideas, supporting *broad-based participation*, *assessing*, and *guiding the mathematics*. This work was done primarily by analyzing videotapes and transcripts of grades 7-10 math lessons during the summer, 2006 (25 hrs). During the academic year, there were three additional meetings and the first author followed up with four of the teachers to examine the ways the conceptual tools resulting from this collaborative experience were (and were not) taken up (Adler & Reed, 2000) in their daily work. The following questions guided the inquiry:

1. What role do secondary mathematics teachers perceive for the shared language of mathematics pedagogy (the specific strategies and conceptual model) in supporting and enhancing their daily work?
2. What happens as this shared language of mathematics pedagogy meets up with the demands of daily practice?

In this paper, we report primarily the findings related to the first research question.

Data Sources and Analysis

Data for this study comprised teachers' written reflections from the summer and academic year meetings on the value of the shared language we had developed as well as the idea of a shared language in general; fieldnotes from conversations with the teachers after lessons; and a formal interview with each teacher. All interviews and meetings were audiotaped. Interviews and portions of the meetings were transcribed. Data collected from each participant were analyzed both individually and across cases (Miles & Huberman, 1994) using standard techniques of qualitative analysis (Glaser & Strauss, 1967). Themes were developed regarding the perceived value of the shared language and the purposes they saw for the language, the specific pedagogical actions, and the conceptual model in their practice.

Results and Discussion

We briefly describe two of the cases and then discuss implications of the work.

Nancy and Ken, who had 25 and 7 years of teaching experience, respectively, taught at different schools in the same regional district. Ken, a high school teacher, reported a very positive influence of the shared language of mathematics pedagogy on his daily practice. He reported a heightened awareness of what he was doing in the classroom and described how the language had given him new tools to understand his teaching. Although he reported that fundamentally it did not change his approach, he felt that he had reconceptualized portions of his pedagogy and he came to understand certain strategies and aspects as more important than he had in the past.

The increased awareness and tools for reflecting on his teaching Ken reported had led to opportunities to learn about his practice and from his practice. He noted being able to better discern patterns across classes, patterns that he would not have recognized previously. For example, during the fall interview he said:

> But now I think I'm more aware that I'm doing it, and I don't know, … maybe I feel like I do it more, but maybe I feel like I do it more because I know what it's called. And I now can label it, what I do. … Versus in the past, I've done it, but I don't label it in particular ways, so I don't realize it. … [W]e want to connect past learning with our students. I need to start connecting past teaching within what I do, and I think, doing this allows me to. 'Cause then I see that in one class I did something, and in another class, I did something. It's the same strategy, I've connected them together, which I might not have seen before cause if I'm doing fractions here, and I'm doing law of cosines here, and I'm factoring in another class, I can be doing the same strategy and not seeing the connection, but now I do.

Like Ken, Nancy, a middle school teacher, reported increased awareness of her actions during class discussions. Although Nancy stated that the named pedagogic moves were not new to her practice, she now understood them in a new way and ascribed a new role for them in her teaching. One specific example she noted was *exploring incorrect solutions.* She had done this in previous years, but now she used this strategy more frequently and deliberately, and felt she had a new appreciation for how it supported students' learning. Nancy also described how she had reconceptualized her practice and ascribed a new role to various strategies.

> There's a whole new role for it. It really and truly is a whole new role for it. .. And really, you know, like bringing something to the forefront that somebody has come up with. And really *highlighting* what's going on there and …[y]eah, we talked about it before, but it never was-, it's more important for me to do this now. And it's more important to me to *promote somebody else's perspective*. It's…more important for me to look for an *alternative approach*. I'm seeing a value for those; I'm finding it more

important. It wasn't that I didn't do it before, I don't think I did it *enough*. … There's a new meaning…. It's not *brand* new, but, there is something new about it.

Discussion and Implications

From this preliminary study, we conclude that a shared language of mathematics pedagogy can be productive in supporting teachers in engaging students in mathematically focused discussions and in helping them think about (and potentially change) their own practice. All teachers reported thinking about their teaching in new ways, and three of the four teachers reported significant shifts in their practice along some dimensions. Although not included above, two of the four reported specifically learning more about student thinking as a result as well.

The results presented here echo Siemons et al.'s (2004) work with elementary teachers. The teachers reported that the development of a shared language of numeracy strategies, generated by watching colleagues teach small groups, was powerful because it supported conversations about practice in ways that had not occurred previously.

It is important to note that the powerful shifts the teachers described were primarily linked to how they conceptualized and reflected on their practice. Our data revealed conceptual shifts more than the development of new strategies. These teachers reported using most of these pedagogical strategies in practice previously, but now had a new understanding of the role they played and how they fit together in their practice. It seems that the availability of this language afforded them the opportunity to reconceptualize aspects of their practice as well as develop a heightened awareness or level of metacognition as they were teaching.

Given the exploratory nature of this study, many questions remain. We hypothesize that the analysis of the videos involved conversations among the teachers that uncovered teachers' "inner speech" (Vygotsky, 2002), resulting in a shared language of practice that seemed to influenced cognitive shifts. Further research is required to determine whether a shared language of mathematics pedagogy is more valuable for those already at a particular level of sophistication in their practice.

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COFLECTION IN PROFESSIONAL DEVELOPMENT: COLLECTIVE INQUIRY INTO EQUITY IN MATHEMATICS EDUCATION

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Underrepresented students' performance is significantly poorer than their White counterparts in mathematics and professional development is seen as the key mechanism for addressing the achievement gap. The contemporary model establishes effective professional development as school-based, collaborative, inquiry-oriented, teacher-directed, subject matter specific, ongoing and sustained, and focused on student learning and performance goals (Hawley & Valli, 1999). This model, however, does not provide a way for teachers to directly address equity issues. We build on the contemporary model for professional development by proposing a theoretical construct, coflection, as a process or mechanism that can facilitate the substantive treatment of equity issues in teaching and learning mathematics. In elaborating this construct, we aim to bridge two bodies of literature – equity in mathematics education and professional development.

Coflection (Skovsmose and Valero, 2001), a knowledge generating process, comprises features we have identified as (a) collective, (b) deliberate, and (c) critical. Equity issues become the core of professional development experiences which are located in epistemic processes. Through coflection teachers generate local knowledge about students' socio-economic backgrounds, the cultural knowledge they bring to school, curricular/instructional policies and practices and the like. We use the assessment of students' mathematical thinking as the site for explicating the features of coflection. Assessment is a rich site for coflection because teachers' ongoing examination of student thinking and its role in teaching and learning holds great potential for improving the quality of students' mathematics teaching and learning experiences (Steinberg, Empson & Carpenter, 2004). The poster provides guiding questions and strategies for coflective processes as teachers critically examine student work.

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ETHNOMATHEMATICS OF DANCE

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The proposed poster will contribute innovative research methodology and present mathematical information about dance. Both pieces will be of immediate use to educators viewing the poster as well as contributing to long-term positive trends in the field. Mathematical structures characterizing dance traditions will be displayed, resulting from the author's in-depth interviews of dance exponents from two or more of the following groups: Alaskan Native, Native American, African, African Diasporan, urban youth subcultures, and Latin American cultures*. The interviews are part of a larger curricular development project in which Culturally Situated Design Tools (Eglash et. al, 2006) have been developed in collaboration with experts in numerous areas of cultural knowledge; the interview data will be used in development of new Tools involving dance.

Two questions underlie the interview project:

1) What kinds of mathematics inhabit the specialized forms of movement of a dance tradition?

2) How can one research this question without repeating the mistakes historically plaguing other forays into mathematical interpretation of areas outside the dominant domains of mathematical knowledge?

In some studies of the mathematics of cultural activities, the researcher holds to a preconceived notion of what mathematics might be found therein, causing two major problems. First, the activity can become merely a context on to which to project the researchers' own mathematical perspective. Much of multicultural mathematics as taught in classrooms today has been critiqued as being resultant of this process. Second, when researchers have a narrow view of what mathematics might be found, they often conclude that the culture exhibits no mathematics at all. This type of oversight has been described extensively by Eglash in the case of European interpretations of sub-Saharan African cultures (e.g. Eglash,1999).

Research methods characterized by greater collaboration between the researcher and research participant, in which research is considered an iterative process of coming to accurately represent the reality of the participant in the terms of the study, characterize advancements in qualitative research methods (Merriam, 1998). This project brings into the dialog on educational reform the method, not just of teaching, but of gathering information to be taught. When the knowledge which is brought into schools is gathered in a collaborative process, one which honors the knowledge of the community, it can be expected that this positive, respectful dynamic will influence the nature of classroom learning as well.

A protocol to guide the interviews has been developed and successfully piloted with an exponent of African Diasporan dance, being found to elicit extensive relevant mathematical information. To learn (of) the knowledge found in a particular community, it is essential to ask questions in the language of that community (Solano-Flores and Nelson-Barber, 2001). Therefore, questions on the interview protocol have been worded in appropriate dance language, making use of the author's perspective as an exponent of one dance tradition and student of several others. An iterative process is followed wherein the researcher periodically shows her diagrammatic notes to the interviewee to confirm whether she has correctly captured the dance movements in the process of translation into mathematical framing, revising as necessary. Analysis of the evolution of the protocol and interview process to meet the needs of use in a variety of dance cultures will be an important component of the project's findings.

Endnote

*The project is ongoing and will eventually incorporate each of these categories and possibly others; as the author is currently in the process of soliciting interview candidates, it is unknown which sub-sample will be completed during the time frame covered by this presentation.

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INVESTIGATING AFRICAN AMERICAN STUDENTS' IDENTITY AND AGENCY IN A MATHEMATICS AND GRAPHING CALCULATOR ENVIRONMENT AT A LOW-SES SCHOOL

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Race and socioeconomic status (SES) are equity factors that have long been associated with the disparities and achievement gap amongst students in mathematics. Although there has been a multiplicity of meanings of the term equity in relation to mathematics learning, the general consensus has been the acknowledgement of the existence of this achievement gap. Moreover, technology has been one of the tools recommended for achieving equity in the mathematics classroom (NCTM, 2000). The integration of technology in the mathematical learning of students has, however, added an important aspect to the issue of mathematics equity pertaining to access and use of technology. This research study focuses specifically on graphing calculator technology because it is more accessible to students.

To ensure that access to and use of graphing calculators in the mathematical learning of students does not result in another dimension of inequity, it is important to explore the mutual interaction between the students, their teacher, peers, and the technology, within specific contexts. It is through this interaction that students position themselves within the classroom and use the technology to shape their identities and agencies as learners and doers of mathematics. This focus on social relations and interactions views as important the complexities of race, racism, culture and class in influencing students' mathematical experiences. Indeed, while there is a great deal of research on equity in mathematics education that has been conducted to highlight disparities and limited persistence along racial lines, there has been a general lack of theorizing about race and racism and their relationships to mathematics learning and participation (Martin, 2006a, 2006b).

This study investigates issues of African American students' identity construction (racial, mathematical, and otherwise), how these identities shape each other, and the sense of agency exhibited in the process within a mathematics and graphing calculator environment. This is reflected by the kinds of roles and interactions that these students assumed and the ways in which they used the graphing calculators as tools for mathematical learning. Further, this study is influenced by a sociocultural perspective and Holland, Lachicotte, Skinner and Cain's (1998) framework of figured worlds, positioning and authoring.

This study is being conducted at a low–SES school with two different 11th grade classes of students taught by the same mathematics teacher. Data, that are currently being collected through both qualitative and quantitative means, will be reported in this poster. The quantitative part comprises of a Likert-scale survey instrument, while the qualitative part comprises of classroom observations and semi-structured interviews. The survey instrument will first be administered to all students in each classroom community. This will be followed by ten classroom observations and three interviews with a representative sample of four African American students from each classroom.

To construct an understanding of the modes of graphing calculator use, I will use both deductive and inductive coding scheme by initially coding the data according to Doerr and

Zangor's (2000) modes of graphing calculator use as a tool, and then looking for data that do not fit into these codes. The classroom observations will give insights into the interactions of students amongst themselves, the teacher, and with graphing calculators, issues of power and positionings manifested in those interactions, and the classroom norms negotiated by the students and the teachers. In analyzing students' interaction with the graphing calculator, I will draw upon Goos, Galbraith, Renshaw and Geiger's (2003) metaphors of technology as master, technology as servant, technology as partner, and technology as extension of self, that describe the varying degrees of sophistication with which students and teachers interact with technology.

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LINKING DATA, VALUES AND CONTEXT IN AN ALGEBRA WRITING ASSIGNMENT

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Contextualized mathematics pedagogies seek to make mathematics meaningful to students. We take the position that a strong presentation of contextualized mathematics lessons will articulate both mathematical and interdisciplinary, contextual knowledge in significant ways (Staats, 2007). Still, fundamental questions remain: when students respond to a contextualized mathematics lesson, how do we know when it is meaningful to them, and how do we know when they have done a good job of coordinating values or personal meanings and mathematical understandings?

As part of a class exam, undergraduates in a developmental algebra class wrote a values-based response to an article in which Jeffrey Sachs (2005) argues that a donation of $3 per year from each of the 1 billion people living in high-income countries would be sufficient to control malaria in Africa. Students were directed to support their opinion with relevant data and computations. The primary data source was Sachs' article, but students were also encouraged to use the United Nations Development Programme's Human Development Indicators (2005), a fact sheet on infectious diseases distributed in class, and any other source for which they could provide a citation. Two raters coded the written responses of 41 students for nine categories involving argument type, use of evidence and presentation of context (e.g., detailing public health issues in particular countries, relationships among nations or foreign aid as an economic investment) (Marshall & Rossman, 2006). The initial level of agreement was 0.70 using Cohen's kappa coefficient. We found several notable results. Nearly all students' arguments could be described by four categories, each associated with types of data provided in class. Students structured their arguments around the type of data available to them. Chi-squared tests of independence indicated a significant relationship between students' written portrayal of an detailed context and students' use of data from sources outside of the assigned article. There was also a significant relationship between the use of an detailed context and the use of mathematics to support the argument effectively.

Contextualization in students' mathematically-oriented writing is not directly an expression of mathematical knowledge, and so it is somewhat unexpected that it is associated with the use of mathematical evidence and with effective argumentation. Discursive coherence may be a contributing factor. Contextualization in writing may help students produce a coherent answer as they weave together multiple data sources or try to intentionally link values to mathematical calculation. Attending to students' use of contextual details in mathematical writing may help teachers assess the degree to which socially-engaged mathematics is meaningful to students.

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MATHEMATICS SELF-EFFICACY AND MOTIVATION OF HIGH SCHOOL STUDENTS

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The purpose of this study is to investigate the attitudes and motivation of high school students toward mathematics course-taking. Two high school math teachers have noticed that more girls are taking higher level math courses, and are outperforming boys in achievement tests. Their observation has led to the following research questions: *Is there a trend towards more girls taking more math classes? Are there gender differences in confidence of doing mathematics, intrinsic value, and perceived usefulness of doing mathematics?* Thus, the study was designed to better understand the course taking pattern of high school students with more female participation.

A survey was conducted at two high schools in suburban towns located within 50 miles of a large research university. The survey instrument consisted of 51 questions that are mostly in a 5-point Likert Scale format, similar to the questions developed by Wigfield and Eccles (2000). They have tried to explain students' choice of tasks and persistence in those tasks through expectancy-value theory and ability beliefs. Similar to ability beliefs is self-efficacy, which is another construct that is included in the survey due to its influence on how students choose their activities and courses. Bandura defined self-efficacy beliefs as "people's judgments of their capabilities to organize and execute courses of action required to attain designated types of performances" (1986, p.391).

In this study, self-efficacy beliefs are further organized into the students' level of confidence, the degree to which they value mathematics, and perceived usefulness of mathematics. The confidence construct measured how well they are doing in their math courses *(_=0.883; 7 items)*. The construct for intrinsic value included questions on the importance of getting good grades in math classes and in being good at problem solving *(_=0.814; 6 items)*. The utility construct measured how they thought about the usefulness of math in everyday life *(_=0.698; 3 items)*.

Based on the students' previous and present math courses, two ability categories were formed: high ability and normal/low ability. Similarly, the students' self report of the math course that they would like to take next year indicated their persistence, or lack of motivation to persist. From the factor analysis, the factor scores were obtained and then used as the response variables for the generalized linear model with two fixed effects: gender and the ability-persistence category. MANOVA revealed a significant multivariate effect for the ability-persistence category. Regardless of ability, there is a significant gender difference in the confidence variable for the group of non-persisters.

This study can be helpful in studying gender equity in the classroom, since attitudes, persistence and motivation for taking mathematics courses are related to ability group, not just gender. In the case of high ability girls who are not persisting to take more math classes, it is important for teachers to pay attention to them because these girls fail to realize the value and usefulness of the course. As for boys who have normal to low ability in math, teachers should encourage them to persist in taking more math courses.

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'SHOW ME YOUR MATH': INVITING CHILDREN TO DO ETHNOMATHEMATICS

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From ethnomathematical conversations with Aboriginal elders in Eastern Canada, we concluded that community children could do more authentic ethnomathematics than we could. This inspired a regional fair using the internet – school children from all levels would display the mathematics they see around them. The students' displays can be used as a teaching resource and have further benefits for researchers.

As part of a large-scale project investigating mathematics and science learning in informal contexts in Atlantic Canada, we have been interviewing Aboriginal elders to identify some of their everyday practices (both traditional and current) that could be deemed mathematical. This typical approach to ethnomathematics research (c.f. Powell & Frankenstein, 1997) relies on Bishop's (1988) definition of mathematical activity (practices that involves counting, measuring, locating, playing, designing or explaining) and on the assumption that any mathematics is an artifact of a particular culture.

In reflection on this research, we saw ourselves as mediators, interfering to some extent with the intended process – the elders' knowledge and experience being communicated to the community children. We realized that the conversations would be more authentic if the children themselves talked with elders and others to find mathematics in traditional and modern community practices.

Aboriginal communities across Canada connect via the internet for various purposes. This infrastructure is often used for 'contests' in which people share their stories and knowledge with the people in other communities. We are using this medium and this emerging tradition to draw school children into ethnomathematics. The results of the students' work become available to all – an excellent resource for any teacher who wants students to see how mathematics is used to address real issues.

Our original intent was to go beyond typical ethnomathematical practice to consider differences between school mathematics and community values and intentions. The 'Show me your math' contest provides an excellent setting for the expression of values and intentions. We record conversations amongst teachers planning local events related to the contest, and amongst community members deciding criteria for judging the contest entries.

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STUDENT RESPONSE TO A "SOCIOCULTURAL ISSUES IN MATHEMATICS EDUCATION" GRADUATE COURSE

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Teacher education, both undergraduate and graduate, provides vital opportunity for preparing teachers to teach today's diverse student body more effectively. Many educators and researchers contend, however, that teacher education devotes insufficient (and sometimes inappropriate) attention to issues of equity and diversity in teaching and learning (e.g., Grant & Gillette, 2006; Sanders, 2002), This may be particularly so in mathematics education (e.g., Brewley-Kennedy, 2005), problematizing application of the National Council of Teachers of Mathematics' (2000) Equity Principle.

This purpose of this research was to investigate the impact of a new "Sociocultural Issues in Mathematics Education" graduate course conducted in Spring 2007 at a major public university in the West. The course addressed issues related to gender, race/ethnicity, social class, language background, and exceptionality with some attention to a variety of relevant topics such as ability grouping/tracking, test bias, and social justice. The 11 students enrolled in the course were a fairly even mix of master's and doctoral students and U.S. and international students. Course participants completed entry and exit surveys regarding the course topic. The surveys consisted of open-ended questions, such as what students considered to be the main equity/diversity issues in mathematics education today and how we can address them (entrance survey) and how student thinking had changed as a result of having taken the course and what students suggested as the most positive course aspects and ideas for improvement (exit survey). Through multiple readings and adjustments, conceptual categories were constructed within the written comments to address the research question.

The data from this study show that students had had very little prior preparation in equity/diversity issues in mathematics education and that their minimal background tended to be superficial and relate mainly to social identities most commonly discussed in relation to mathematics (in particular, gender). Students indicated that the course had made substantial positive changes in their knowledge base, in both depth and breadth, and that this had impacted the classroom practice of those who were currently teaching. Students continued to see gender as an important equity concern in mathematics education, but the greatly expanded repertoire of topics/issues they identified as important clearly reflected the course material. Course aspects deemed most effective were those that facilitated in-depth understanding of current research (e.g., weekly readings, writing responses, and in-class and online discussions). Because students expressed a strongly positive response to the course, they had little suggestions for improvement. The most substantive of these were to incorporate field trips (e.g., to the campus Disability Resource Center and a local "diverse classroom") and to include more diversity among guest speakers (e.g., someone for whom English is a second language to discuss his/her experiences learning mathematics). The research findings may imply, as one participant stated, the need for all math teachers and math teacher educators to take such a course.

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SUPPORTING GIRLS' PARTICIPATION AND EFFICACY IN A DISCUSSION-INTENSIVE MATH CLASSROOM

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In a discussion-intensive math classroom students often engage in challenges and disagreements with classmates (Wood, Williams, & McNeal, 2006). Lampert et al. express concern that female students may avoid the social discomfort of disagreement in order to preserve relationships with peers, rather than pursuing discussion and promoting mathematical understanding (Lampert, Rittenhouse, & Crumbaugh, 1996). Boaler (1997, 1998) provides further insight into girls' experiences as participants in mathematical discourse, describing how girls value experiences that offer an "open, reflective, and discursive approach" to learning mathematics. These studies suggest that, while mathematical discussion can and should entail disagreement, disagreement can create social discomfort for some students. Considering these findings, there is a need to understand more about how discursive approaches are experienced in different ways for different girls. Boaler (1997) found very few researchers consult girls, she argues for "the importance of giving voice to girls' concerns" (p. 303). Her research opens an avenue of inquiry into girls' experiences in mathematics through listening to the voices of students.

Building on prior research in sociocultural theory and affect in mathematics this study explores girls' underlying feelings about participation in a discussion-intensive math classroom. Using qualitative data from individual interviews with 15 middle school girls, this study analyzes the range of girls' feelings about mathematics through the students' own words. This poster displays a thematic content analysis of three representative students' reports that revealed how students' feelings about functioning in a discursive classroom and working with classmates are varied and quite complex. This complexity illuminates the ways in which public display of help seeking, understanding, and making mistakes impacts students. These girls' differing feelings about interacting in the classroom when they needed help understanding impacted their beliefs about what it means to work with classmates and the teacher.

These girls' thoughts suggest (a) teachers look closely at interactional messages students give and receive about accuracy, understanding, and help-worthy problems; (b) certain kinds of math have higher status than others, as constructed by students in the classroom; (c) some girls' feelings about collaboration are tied to feelings of efficacy; and (d) getting help, getting an answer wrong, and not understanding are all experiences in collaboration which create specific and noteworthy vulnerabilities for students.

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THE NON-MATHEMATICAL ASPECTS OF MATHEMATICAL PROBLEM SOLVING IN URBAN CLASSROOMS

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"Being hard like a "gangsta" is viewed by several youths…as inextricably linked to being Black or Latino, male, low-income, and from a tough urban neighborhood." (Dance, 2002 pps. 4-5). Closely related, Anderson (2000) notes that certain types of behaviors, or "codes" often govern the actions of the youngsters—both in and out of school. "At the heart of the code is the issue of respect – loosely defined as … being granted one's 'props' (or proper due) or the deference one deserves," (Anderson, 2000 pps 33-34). These "codes" include mannerisms, gestures, and facial expressions indicative of street cultures (Dance, 2002). The main question that we address is: when and how do students' enact, through gesture or argument, these types of street "codes" in the context of solving mathematical problems? This research is part of a larger study that seeks to identify how certain aspects of the urban environment influence the mathematical problem solving behavior of children (see Epstein, et al. in press).

Methods

The study takes place over a one-month period, which involved one visit a week (for 60 minute sessions) in a diverse, low performing, eighth grade inner-city classroom. The visits are part of a professional development project in which the classroom teacher & district teacher leader participated in professional development sessions with researchers, both at the University and within the context of their own classrooms. At least two video cameras captured different views of the teacher interventions, students' group work, students' presentations, etc. All student work was collected and descriptive field notes were compiled by at least two researchers. After each lesson, the participants would "debrief" to discuss key ideas relating to affective issues, mathematical ideas, and other relevant issues.

Preliminary Results

Below, for brevity's sake, we report on one episode in which the students were given a task involving maximizing area, given a fixed perimeter. They were asked to work in groups, and then share ideas at the overhead projector. Several students argued about the meaning of area and perimeter. The mathematical discussions that arose from these varied perspectives were at times heated, and appeared to be confrontational in nature. The following is one such example:

[J]: Calm down! [T]: It didn't say the area! [J]: I know…You just said…Remember you just said…I know… *(Clapping hands)* [T]: But you just said…You just said it's not the area *(gestures with hands and looks at E)*. [E]: We're looking for the area *(hits hands together)*. [Teacher]: Okay class! Let's be civilized. [E]: I say 64 is not the area… [N]: We is *(sic)* civilized. We is *(sic)* having a debate with numbers. [E]: 64 is not the area!!

Throughout the exchange, the students appeared to use hand gestures and facial expressions that emphasized their commitment to an idea. The teacher's comments indicate that she felt that they were acting "uncivilized". These behaviors, while benign in nature in the context of this lesson, could easily be misconstrued by an outside observer, or a teacher, unfamiliar with, as Anderson describes, ‘the code of the street’. We speculate that by better understanding such behaviors, we may gain deeper insight into how to create and maintain a productive classroom culture.

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REIFYING TEACHERS' TACIT KNOWLEDGE ABOUT TEACHING: CLOSING THE GAP BETWEEN THEORY AND PRACTICE

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Teachers learn about teaching from their teaching experiences and their level of teaching expertise is considered a function of these experiences. This learning takes place within the iterative practice of teaching and results in the development of teacher knowledge which, in turn, influences the practice of teaching. But what is the nature of this knowledge – how does it develop and how does it change? In this paper I first argue for, and then provide evidence of, a mechanism through which teachers' knowledge develops and changes. I argue that through the mechanism of reification teachers' knowledge and belief about teaching and learning can rapidly undergo substantial revision leading to equally rapid revisioning of practice.

Consider the case of John:

> *Anyone who knew John when he first became a mathematics teacher eight years ago would say that he has come a long way. He is no longer the idealistic, naïve, teacher he was at the beginning of his career. John is now savvier in the ways of teaching. Although not a specialist in mathematics, over the last eight years John has managed to acquire a large repertoire of ideas and practices that he relies heavily on in his daily teaching of mathematics. These ideas and practices have been gleaned from textbooks, teachers' guides, colleagues, workshops, but mostly from his experiences in the classroom. But there is an incongruity within John. John identifies himself as being a bit of a traditionalist when it comes to teaching mathematics – he espouses the virtues of drills, skills-based assessment, believes in the transmission model of teaching and learning, and bemoans the problems of the 'new new math' movement. Despite these dispositions, however, many of John's favourite lessons and instructional strategies can best be described as being steeped in the traditions of the reform movement.*

How is it possible that such an experienced teacher can embody such contradictions between his *knowledge* and his *practice*? What is the *condition of John's knowledge* that allows for such contradictions? How has this contradiction developed, or perhaps more relevant, how has an agreement between *knowledge* and *practice* failed to develop? In this paper I look more deeply at John, and other teachers not too dissimilar from him, whose teacher knowledge has not fully developed through their experience as teachers, but whose knowledge does develop more fully when put in a situation wherein they were required to reify (1) their knowledge, and then act (and enact) that knowledge. My thesis is that through this process teachers' knowledge and practice can develop.

Theoretical Framework

Teachers learn about teaching from their teaching experiences (cf. McClain & Cobb, 2004; Kennedy, 2002; Ma, 1999; Mason, 1998; Shulman, 1986) and their level of teaching expertise is considered a function of these experiences (cf. Berliner, 1987). This learning takes place within the iterative practice of teaching and results in the development of teacher knowledge. Leikin (2006) identified three dimensions of this teachers' knowledge: kinds of teachers' knowledge, sources of teachers' knowledge, and conditions of teachers' knowledge.

KINDS OF TEACHERS' KNOWLEDGE – is based on Shulman's (1986) classification of teacher knowledge and consists of *subject-matter knowledge* (SMK), *pedagogical content knowledge* (PCK), and *curricular content knowledge* (CCK). Subject matter knowledge is teachers' knowledge of mathematics, pedagogical content knowledge is knowledge of how mathematical content is received by students, and curricular content knowledge is knowledge of curricula and approaches to teaching mathematics.

SOURCES OF TEACHERS' KNOWLEDGE – is based on Kennedy's (2002) classification and consists of *systematic knowledge*, *prescriptive knowledge*, and *craft knowledge*. Systematic knowledge is developed through studies of mathematics and pedagogy, prescriptive knowledge is developed through institutional policies, and craft knowledge is developed through classroom experiences.

FORMS OF KNOWLEDGE – is based on the differentiation between teachers' *formal knowledge* (Atkinson & Claxton, 2000), their *intuitive knowledge* (Fischbein, 1984), and their *beliefs* (Scheffler, 1965). Formal knowledge is knowledge that consciously guides practice, intuitive knowledge is knowledge that subconsciously guides practice, and beliefs is *subjective* knowledge that consciously and/or subconsciously guides practice.

These three dimensions can be represented along three dimensions of a Cartesian coordinate system (see **Figure 1**).

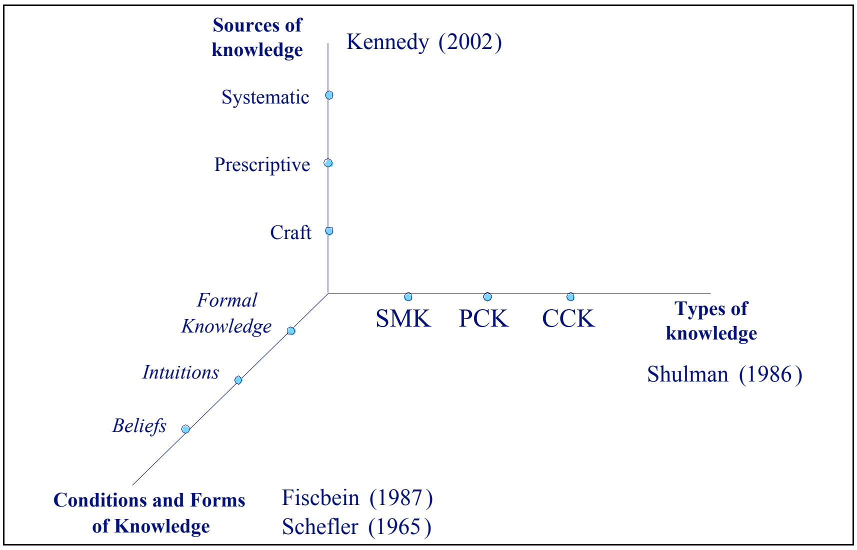

Figure 1: Three dimensions of teachers' knowledge (from Leikin, 2006)

The thesis of this paper is situated within the third dimension – the conditions of teachers' knowledge in general, and the transformation in the conditions of teachers' knowledge in particular. The conception of this dimension, as described above, is not without challenges, however. The simultaneous partitioning of the conditions of knowledge across the knowledge/beliefs divide and the conscious/subconscious divide is somewhat problematic.

To begin with, at the level of teachers' action the distinction between knowledge and beliefs is not so clear. In general, knowledge is seen as an "essentially a social construct" (Op 'T Eynde, De Corte, & Verschaffel, 2002). That is, the division between knowledge and belief is the evaluations of these notions against some socially shared criteria. If the truth criterion is satisfied then the conception is deemed to be knowledge. But when teachers operate on their knowledge the distinction between what is true and what they believe to be true is not made. Leatham (2006) articulates this argument nicely:

> Of all the things we believe, there are some things that we "just believe" and other things we "more than believe – we know." Those things we "more than believe" we refer to as knowledge and those things we "just believe" we refer to as beliefs. Thus beliefs and knowledge can profitably be viewed as complementary subsets of the things we believe. (p .92)

Thus, for the purposes of examining teachers' practice I do not make the distinction between beliefs and knowledge. I do, however, make the distinction between what teachers know/believe at the *conscious* level, and what they know/believe at the *subconscious* level. In part, this difference can be summarized by Green's (1971) distinction between evidential and non-evidential beliefs. Evidential beliefs are formed, and held, either on the basis of evidence or logic. Non-evidential beliefs are grounded neither in evidence nor logic but reside at a deeper, tacit level. So, I reinterpret Leikin's (2006) description of the *conditions of teachers' knowledge* as being comprised of the *conscious* knowledge/beliefs that guide their practice, and the *subconscious* knowledge/beliefs that guides their practice. This reinterpretation can be used to describe the discordance between John's practice and his espoused stance on teaching (presented in the introduction). What John espouses is informed by his *conscious* knowledge/beliefs whereas what he does is informed by his *subconscious* knowledge/beliefs. In John's case, what is needed is better articulation between the *subconscious* and the *conscious*.

Wenger (1998) provides us with the language of reification to articulate the movement of knowledge/beliefs from the *subconscious* to the *conscious*. For Wenger (1998) reification is "the process of giving form to our experiences by producing objects that congeal this experience into *thingness*" (p. 58). This congealing of experience can be the movement from tacit knowledge/beliefs to explicit (conscious) knowledge/beliefs, the transformation of abstract thoughts into concrete ideas, or the articulation of fleeting notions into tangible statements.

Methodology

Participants for the portion of the study presented here are 18 inservice teachers working in two teams (Team A – 10 teachers, Team B – 8 teachers) in two different school districts. Although working in different districts, both teams were formed for the same purpose – to collaboratively design numeracy tasks to be used district wide to assess the level of numeracy of grade 5's and grade 8's. The teachers involved in this project ranged in age from 25 to 63 (average was 36.4 years), and ranged in teaching experience from 1 year to 36 years (average was 8.2 years). What did not range, however, was their expertise in mathematics. Like John, none of the 18 teachers in this project has a specialization in mathematics – they are all generalist teachers. And like John, there exists some incongruity within each teacher between their espoused views of teaching and learning and their practice. In fact, John is an amalgamation of these 18 teachers, constructed to exemplify the participants as a whole (Leron & Hazzan, 1997).

The research was conducted over the course of the four, four hour long, planning and implementation meetings allocated for the task design project. The nature of the meetings is summarized below (2):

Meeting 1 – co-construct a shared understanding of numeracy and begin to design tasks for immediate pilot testing.

Meeting 2 – debrief the pilot testing of tasks, refine tasks, begin to discuss the logistics of scripting/administrating (3) tasks, and prepare to re-pilot test the tasks.

Meeting 3 – refine tasks, finalize scripting/administration of tasks, and prepare to administer the task as an assessment.

Meeting 4 – mark the tasks and do a post-mortem on the process.
Each of these meetings was facilitated by the author.

The data for this project consists of field notes collected during the meetings, transcriptions of audio recordings made at select times, and artifacts produced in the meetings. In each of the four meetings the participants were engaged either in discussion (small group and large group) or in creating artifacts (definitions, tasks, scripts, layout, etc.). The data for this paper come from the first of theses meetings.

Results and Discussion

The first meeting with each team began with a prompt for them to think about what numeracy is. In both cases the teams responded with the ideas that numeracy must include the rapid recall of arithmetic facts, fluency with arithmetic algorithms, and good number sense. Although not explicitly stated, there was a sense that this list of characteristics of numeracy was not only *necessary*, but also *sufficient*.

When prompted to think about the qualities of a successful (numerate) student (4), however, the nature of the responses changed. Now, the list of characteristics included notions such as: perseverance of effort, a willingness to engage, an ability to transfer knowledge to new contexts and novel problem solving situations, a broader awareness of mathematics around us, creativity and flexibility in thinking, and an ability to explain their thinking. When asked to synthesize these ideas into a definition the two groups produced the following:

Numeracy is the willingness and ability to apply and communicate mathematical knowledge and procedures in novel and meaningful problem solving situations.

Numeracy is not only an awareness that mathematical knowledge and understandings can be used to interpret, communicate, analyze, and solve a variety of novel problem solving situations, but also a willingness and ability to do so.

These definitions are a long way from their initial musings about what numeracy is. Initially dominant in the conversations were their *conscious* (traditional) ideas about what it means to 'know' mathematics. I argue that the emergent ideas were not new knowledge/beliefs, but rather the explications of previously tacit knowledge/beliefs that had built up from their experiences with teaching. The act of first verbalizing and then synthesizing these tacit ideas reified their good experiences with teaching and with students and moved them into their consciousness. In so doing, they "projected their knowledge/ beliefs into the world" (Wenger, 1998, p. 58).

When asked to begin to think about designing a task that could assess students' numeracy abilities the conversations were dominated by the content of the newly created definition. In particular, comments regarding the need for creating a task that presented a *novel and meaningful problem solving situation*. This can be seen in a comment by Joanne (taken from the field notes):

... the problem has to be new to them. It has to be something they have never seen before otherwise we won't know if they are numerate. ... are they figuring it out or are they just going through some steps they have learned before?

Joanne is not only referencing to the definition ('*new to them*') but she is talking about numeracy ('*if they are numerate*') as if it has "a reality of its own" (Wenger, 1998, p. 58).

The initial attempts to produce such tasks (although somewhat unrefined) were far removed from the traditional 'skills' view of numeracy that the participants presented at the beginning of the meeting. There were tasks built around planning for a party, designing a set of stairs, designing an enclosure for a pet, organizing a rock-paper-scissors tournament, and selling cell phones (see table 1 for the initial crafting of these tasks). These tasks all had the characteristic that they would be new to the students. They also had the characteristic that they were projects more so than single lesson tasks for assessment. When asked about the scale of the tasks the teams were generally not clear on how long each task would take, nor were they overly concerned about it (5). This can be seen in the comments of Phil (transcribed from a conversation):

We know it's big. But what are we going to do? It needs to be a bit of a challenge for them, so we have to give them time to think. We want to make sure there is something for them write about the problem at the end, something for them to explain to us, so it needs to prompt that. And it needs to work well with groups.

This last point about group work was an interesting development in the task design phase that appeared among groups working in both teams. When asked about the need for group work the general reply was that numeracy included an ability to communicate, so a person to communicate with was requisite. This can be seen in the comments of Jacki working on the cell phone task (transcribed from a conversation):

It just makes sense to us. I mean, numeracy is not just about doing, its about being able to communicate what you are doing. So, we just figured that the best way to help this along is to get them talking.

This comment by Jacki brings to the fore another emergent theme – the theme of *process*. The participants began thinking about the process of solving these problems more so than the final answers that the students would eventually arrive at. This theme along with the others became even more prominent in the second meeting when they began discussing how to script the administration of the tasks for other teachers.

Again, I argue that the emergent knowledge/beliefs exemplified above is the result of the explication of tacit notions and desires about teaching and learning mathematics that have accumulated from their experiences (both positive and negative) in teaching mathematics.

Table 1: Initial numeracy task ideas

<table>
<tr><td>Plan a Party
You are planning a class party. There are 24 students in the class and you have been given a budget of $70 to spend on pop, chips, and pizza.
2 litres of pop @ $1.79
1 large bag of chips @ $3.45
1 large pizza (10 slices) @ $8.50
How would you spend the money?

The party will be held in the activity room which needs to be furnished with chairs and tables. The school has a whole bunch of card tables (20 of them), which are square tables that can seat four people (one on each side). These tables can be used separately or be put together to form larger tables. Using as many tables as you would like:
a) come up with as many different configurations as you can to seat all 24 students at one large table
b) come up with several different configurations to seat all 24 students at two or more different tables (as long as the tables are the same size)
c) come up with one interesting configuration of your choice to seat all 24 students</td></tr>
<tr><td>Designing Stairs
You have been asked by your father to design a set of stairs that will go from your deck down to the grass in your back yard. The deck is 243 cm high and there is a window 310 cm from the deck that you do not want to block with the stairs. This means that the stairs can touch down at any point before reaching the window. Your father also informs you that there are some building guidelines that you must adhere to:
• Each tread must be at least 22 cm deep.</td></tr>
</table>

• Each riser must be between 16 cm and 20 cm tall. • Each step must be exactly the same size. Design a set of stairs keeping the above-mentioned constraints in mind.
Pet Pen You are the proud owner of a pet __________. You need to design a pen for your pet such that the perimeter of the pen is exactly 6 meters. Decide which pen is best for your pet and why.
Rock, Paper, Scissors 1. You are going to play a variation of the game *Rock, Paper, Scissors*. The variation allows you to play it with three people. Decide among yourselves how score will be kept so that the game is fair.

Table 2: An example of a refined numeracy task

GIVING OUT BONUSES

You are the manager of the Teen-Talk Cell Phone company that employs a number of independent sales people to sell their phones **seven days a week.** These sales people work as much or as little as they want. As a sales manager you don't care how much they work, but you do care how much they sell. So, to motivate them to sell more you give out bonuses based on how productive they have been. There are two bonus plans:

- the top producing individual receives $500.
- the top producing team **shares $500 in a fair manner**.

However, there are also two problems:

- different people have different ways of reporting their productivity.
- the individual sales teams don't have the same number of people on them.

Based on the information provided in the table below, **who should get the bonuses this month, and how much do you think they should get? Justify your answers in writing.**

Sales Person	Team	Sales Reported for the Month of April (30 days)
John	A	300 cell phones sold this month
Peter	B	An average of 56 cell phones sold every 5 days
Lewis	A	An average of 10 1/3 cell phones sold each day
Amy	A	598 cell phones sold in the last 60 days
La Toya	A	An average of 98¾ cell phones sold every 10 days
Jennifer	B	An average of 11 4/15 cell phones sold each day
Steven	B	An average of 55 cell phones each week
Fantasia	C	4113 cell phone sold in the last year
Diana	C	An average of 10.05 cell phones each day
Matthew	D	An average of 10.87 cell phones each day
Camille	D	An average of 9 1/6 cell phones each day
Jasmine	C	267 cell phones this month

Conclusions

Teachers learn from teaching. What they learn, however, is not always made explicit to them. Teaching experiences may accumulate in disjunctive ways at a very tacit level. The catalyst for unifying and explicating these disjoint and implicit experiences is not always clear. Certainly the culture existing within the school setting has been shown to galvanize some of these experiences into concrete notions (Goos, 2006; Karaagac & Threlfall, 2004). In this paper I offer a different mechanism for galvanizing teachers' experiences. First, teachers' knowledge/beliefs are moved from the subconscious to the conscious through the act of reification (Wenger, 1998). But reification is more than just an explication of tacit knowledge/beliefs. It is also putting that knowledge/beliefs out into the world as if they have a reality of their own. Acting, and then enacting, this knowledge/beliefs through task design and then delivery of these tasks serves to further galvanizes this knowledge/beliefs. Although not comprehensive, and far from conclusive, I have introduced the idea of reification and enactment into the discourse of *Learning Through Teaching* in general, and into the area of *conditions of teachers' knowledge* in particular.

Endnotes

1. In this paper *reify* and *reification* is used in the tradition of Wenger (1998) rather than in the tradition of Sfard (1994). McClain & Cobb (2004) make a similar notification of distinction.
2. This is really just a summary. There are contextual details about this process, and subtle differences between the two groups that, although relevant, space constraints do not allow for.
3. This included discussions around time allocation, the role of group work, the provision of graphic organizers, and the use of writing prompts.
4. This prompt is considered a *critical question*. Research into critical questions is part of the larger project within which this paper is situated.
5. After pilot testing, refining, and re-pilot testing the tasks became much more manageable in scale. See table 2 for an example of a refined task that was used for large scale assessment.

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TEACHING UNHEARD VOICES: STUDENTS AT RISK IN MATHEMATICS (1)

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Intermediate students at risk in mathematics were studied using an intensive case study approach to examine the range of issues. As well, teachers drawn from three Northern Ontario school districts were surveyed to assess their beliefs and practices regarding students at risk. Findings consistently showed that these teachers tended to fall back on traditional practices when working with students at risk in mathematics, and seemed particularly reluctant to adopt reform-based teaching methodologies with such students.

Rationale and Context

Mathematics reform is often viewed as particularly crucial for students at risk, and best practices for working with such students in mathematics are well-defined (for example, NCTM, 2000). Research on the level of implementation of reform based programs in schools with a large high risk population does exist from other areas (e.g., Balfanz, MacLiver & Byrnes, 2006) but research relating to the special needs of Canadian students particularly in Northwestern Ontario is limited. The current project attempted to probe the needs of intermediate students at risk in mathematics in the Northwestern region of the Canadian province of Ontario by observing their daily classroom experiences over a full (four month) semester. Observations of teacher practice were also recorded, which were extended via a written survey to a larger sample of teachers.

Framework

Achievement in mathematics may be influenced by the developmental process of learning itself (Geary 1994, 2000). Other factors may include motivation, attitude and confidence related to mathematical ability (Hannula, 2006; Phillips, 2005). The teacher may also influence the level of success a student may attain with respect to mathematics (Balfanz et al., 2006). The exact needs of students at-risk are not clearly defined (McFeetors & Mason, 2005), but the success of students at-risk can be linked to their motivation which may in turn have the potential to direct behavior (Hannula, 2006). Hence it is beneficial for the teacher to understand a student's motives if they are to fully understand their actions (Hannula, 2006). Many students at-risk appear unmotivated and do not have the drive to persist. Their often lacking levels of participation may be related to their needs (Sullivan, Tobias, & McDonough, 2006).

Students have the need for identity, independence and social acceptance (Sullivan et al., 2006). For example, if the teaching methodology is one in which much routine and rote learning takes place, the need for independence cannot be met, and in classrooms which do not foster communication and group activities, students cannot fulfill a need for socialization or develop a sense of acceptance (Hannula, 2006). As well, students are more likely to practice incorrect methods with this type of rote learning especially if they are working individually (Woodward & Brown, 2006). Thus, more practice and volume alone does not necessitate success in mathematics (Woodward & Brown, 2006).

Students who are placed at-risk for developing mathematically have most likely been subjected to previous negative schooling experiences, and such experiences are likely to affect their present and future learning of mathematics (McFeetors & Mason, 2005). The negative experience has the potential to begin a cycle which is difficult to break (Marchesi, 1998). Hence, student confidence and self esteem appear to significantly contribute to potential lack of success.

The behavior expressed by students as a result of the perception that school does not meet their needs is often the refusal to participate in the learning that the curriculum offers (Daniels & Arapostathis, 2005). Many of these students figure out very early on in their schooling that they are uninterested, and decide to reject what they are being taught for various reasons (Marks, 2000, cited in Daniels & Arapostathis, 2005).

When students' needs are being met, they will be more engaged and motivated, which will better enable goal attainment (Sullivan et al., 2006; Daniels & Arapostathis, 2005; Hannula, 2006). Consequently, behavior and negative attitudes should improve and better mathematics learning will be possible (Sullivan & McDonough, 2006; Daniels & Arapostathis, 2005; Hannula, 2006).

Many of the recent curriculum changes in Ontario were inspired by the National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics* document (NCTM, 2000). The vision described by the NCTM is one in which all students have access to learning mathematics no matter what skills they possess (NCTM, 2000). Teachers are to provide rich learning experiences for students, helping students work through mathematical problems with multiple perspectives (NCTM, 2000).

Acceptable ways of interaction and means to support or refute mathematical arguments must be developed and supported. To achieve effective learning in a group environment, students need to have some common ground to refer to and must be able to relate to one another socially (Wood et al., 2006). In reform-based classroom environments, students are typically required to support their answers with an oral presentation defending the strategy they chose, which underscores the importance of being able to effectively interact in a social setting (Wood et al., 2006). Students are not able to switch back and forth between a student centered and teacher centered classroom; the transition to a student centered learning environment must be gradual and consistent (Huhn, Huhn & Lamb, 2006).

Unlike the traditional perception, much research indicates that students at-risk will not profit from rote learning or procedure practice alone (Fleener et. al, 1995, Huhn et al, 2006, VandeWalle & Folk, 2005). Instead, their successful learning is dependant upon manipulating concrete objects, exploration and active problem solving (Fleener et. al, 1995; NCTM, 2000). Intermediate students in particular need to be actively involved in their learning, and be provided with many hands-on, relevant and engaging learning experiences to better support the development and retention of knowledge (NCTM, 2000; The Ontario Curriculum, Grades 1-8 Mathematics, 2005). Hence it would be fair to conclude that traditional practices are particularly damaging for students at risk, while reform-based practices might be particularly beneficial.

Methodology

The study involved teachers and students from three school boards. A mixed methods design was used to probe individual students as well as examine teachers' perceptions across a broader spectrum. Fifteen students in grade seven to nine were initially chosen for study based on the recommendations of their teachers. Three of these students are described here, chosen to exemplify the range of observed issues. As well 66 teachers answered a written survey related to their perceptions and teaching practices for students at risk (2).

The case study students were observed during their mathematics classes in entirety two to three times each weekly during the fall 2006 school term. As well as watching and documenting the actual lesson, the research assistant worked individually with the students each day.

Classroom Case Study Observations

The following case studies describe three teachers and three students. Space limits our discussion, but further details and more examples are available in the full report (3). All names used are pseudonyms.

Classroom A - Grade 7 Mrs. Adams

Mrs. Adams was a very experienced, confident teacher who has been teaching for over 30 years. Each of Mrs. Adams's math lessons tended to follow a similar format. First, the students began by "reflecting". For example, on one particular day, the students were to write down as many strategies as possible to find the area of irregular figures. After the reflection any homework questions were typically taken up with the class. Errors did not appear to be used as opportunities for learning, and correct answers were emphasized. It was also observed that the time allowed for student answers was very short in general. Rarely did she wait for more than one student to raise their hand before selecting a student to answer.

In the majority of the lessons observed, the students were asked to copy a short note from the board which was followed by examples and homework questions. The students primarily worked independently and were told that if they needed assistance they could ask the teacher one question and they were allowed to ask a friend for help, but again, only once.

Brian had been struggling with the material at the grade 7 level and receiving failing grades. Mrs. Adams placed him in a "foundations" curriculum in which he worked independently in a 60 page workbook at a grade 5/6 level. Her explanation was that he had too many gaps in his learning and that this would benefit him by filling in these gaps. An Educational Assistant was occasionally present but she admitted that she struggled with the material herself. Mrs. Adams was never observed to ask Brian how he was progressing or to offer him help. However, she did collect and grade the workbook periodically.

Brian appeared to be a Caucasian boy who spoke English as his first language, and was a very well behaved student in the classroom environment. He regularly attended school although he seemed unmotivated during math class. Brian did not seem to possess a great deal of self confidence and frequently suggested that he equated getting the right answer with being 'smart'. For example, when working independently on a question, the researcher often heard him muttering under his breath, "Is that right? Am I smart?"

It appeared to us that one of the greatest factors inhibiting Brian's success in mathematics was his difficulty with reading. For example, a question which began with, "A patio will be made with square stones …" was read by Brian as, "A *potato* will be made with square stones…". The researcher was able to point out the reading error by saying the mispronounced word out loud and having Brian repeat the question. In the patio example, after meaning was made, the problem was discussed and the researcher helped to create a visual image for him, Brian was able to complete the problem with very few prompts. On another day, once a question he couldn't answer initially was read to him and he discussed what it was really saying with the researcher, he drew a picture on his own and said, "This question is easy!" When asked why he could not complete it in the first place he said it was because he "…did not understand it". It may well have been that it was the written text (rather than the mathematical ideas) which he did not understand. It is possible that remediation in reading might have been sufficient to support higher achievement in math as well, as much evidence exists in our field notes that indicate Brian was able to negotiate the mathematics in both the workbooks and even in the regular class once he understood the written text.

Classroom B – Grade 8 Mr. Brown

Mr. Brown had been teaching for seven years. He often started his lessons by welcoming the students and telling a math related anecdote. He attempted to use examples connected to student interest, such as the salary of a baseball player. This introduction was followed by a short, concise black board note with relevant examples and then time for the students to practice their work was provided. As well, Mr. Brown used manipulatives such as integer chips and snap cubes. However, his lessons also remained relatively teacher directed.

Mr. Brown was dealing with a classroom with a significant number of extreme behaviour issues. In fact he seemed to be in a constant battle with student behaviour and had many students who appeared highly disinterested with the subject matter, and who were very reluctant to ask questions or to speak when given the chance. Students frequently interrupted and disrupted the class. Eventually, misuse of manipulatives resulted in some being withheld. It was very difficult for the teacher to have the students actively involved in the lessons.

Early in the fall Mr. Brown appeared to have every intension of using currently recommended teaching strategies and reform-based learning. However, Mr. Brown appeared unable to implement these methodologies and stated that they were unrealistic in his current classroom, even though he appeared aware of the needs of students at-risk in terms of hands-on learning and manipulative use. We felt that this example illustrated a classroom environment which was seen to not only have a potentially debilitating effect on other students, but in fact on the teacher himself.

A number of students in this class freely self-identified themselves to the researcher as being of Aboriginal descent. Diane (a pseudonym) attended school on a reserve about 300 kms away from her present school, from kindergarten to grade 3. She missed a lot of grade 3 because she was “visiting her sister”. A new school was attended for grades 3 and 4, followed by a move to another reserve for grades 4 to 6. She cannot recall why but knows that she “missed a few months of school” in grade 6. During grade 6 her family moved again and she has attended this school since. This means that Diane attended four different schools in less than nine years. As well, she was absent from school nearly 30% of the fall term, and she was not the only student in the class for whom this was the case.

The classroom included a number of troubled students, including one who admitted to the researcher that she used drugs and alcohol on a weekly basis. This student, for example, regularly called out in class or pounded her fists on the desk, adding to the extreme number of distractions. As well, the classroom was in an open concept school and was next to another grade 8 class separated only by two small blackboards. Diane often seemed distracted by the noise levels around her.

Diane said math made her feel “stupid, frustrated and dumb”. She could recall experiences only in primary level math where she enjoyed the use of concrete materials as well as having positive experiences with teachers. Currently, Diane was easily frustrated and would often just stop doing a question and would not ask for help. Diane felt she would not succeed before she even attempted particular mathematics problems. Even when Diane had completed a question correctly, she thought that it was wrong. Extremely low self-concept and unwillingness to ask for help seemed to be her greatest challenges.

Classroom C – Mrs. Chase Grade 9 Applied

Mrs. Chase had been teaching for three years at this particular high school. Typically, Mrs. Chase's lessons involved well organized but extremely lengthy formal lessons (up to 45 or 50 minutes of a 75 minute period). When the students had the opportunity to give answers and contribute to the lesson, the response time given was generally only as long as it took for the first student to raise their hand and participate. Mrs. Chase gave many step-wise procedures to the students. Homework practice also tended to be mostly procedural practice, and few contexts of any sort were evident.

The students were often not attentive throughout the lengthy lesson and were observed talking to other students and not copying the note as they were instructed. Following the formal lesson homework was assigned. The students usually had an ample amount of class time to complete it, although they did not always use this time to do so. It seemed as though many were just waiting for the lesson to be over so they could ask to leave to use the washroom.

Mrs. Chase attempted once during the research to try a reform style group task. She did not lead up to notions of group work, manipulative use, or problem solving in any way before the lesson so it was an abrupt change of style, which is difficult for students (Huhn et al, 2006). She was very dissatisfied with the lesson as many students were off task, did not complete it, and misused the manipulatives. This was the first time she had used manipulatives with the class. After that experience, she did not use any form of manipulative again.

Mrs. Chase attempted to use a real world context once when explaining the concepts of area and perimeter. She related perimeter to home improvements, like measuring baseboards, and area concepts were used for discussing painting walls and knowing how much carpet to buy. On this particular day, the classroom behavior was observed to be much better than any others. Perhaps this is not coincidental and that there was a link between the relevance in the lesson and at least somewhat improved behavior.

Susan was chosen as a case study student and a typical day with Susan was as follows [from researcher field notes]:

> *Susan is in the hallway as soon as class begins, just minutes after the bell has rung at 1:00 pm, and is apparently going to the washroom. The lesson continues and Susan comes back to class at 1:25 pm. She leaves again for whatever reason and returns about 10 minutes later. Upon her third arrival to the class, she is not paying attention as the lesson goes on and continues to make conversation with the students around her.*

Susan described a troubled childhood with many moves. Susan said that she did not see the point of math and did not understand why the students needed to go through all of this "stuff". She said that you should just learn what you needed for a particular job and that would be sufficient. However, Susan wanted to collect unemployment insurance instead.

Yet there were many instances in which Susan demonstrated ability as well as potential. For example, instead of using the formula sheet, she intuitively re-created the formula for the area of a triangle. Susan managed to pass the course, but Mrs. Chase felt that this would not have been possible without the individual support of the researcher for such a substantial period.

Teacher Survey and Discussion

The teachers surveyed were consistently able to identify the common issues for students at risk as found in the literature and evident in the case studies. These included behaviour issues, poor motivation, poor problem solving skills, reading issues and gaps in previous knowledge. However, teachers did not generally describe *teaching* in a manner consistent with best practices as described in the literature. They identified using personal teaching practices consistent with what we observed in the case studies, which were generally highly traditional, procedural, and teacher directed, particularly in Classrooms A and C, in which students were mostly expected to work on decontextualized tasks without social interaction or other motivating aspects.

Descriptive statistics, ANOVA, and independent t-tests were used to examine the survey data. A more extensive analysis is available in the full report as cited previously. The grade seven to 10 teacher sample [N=66] was relatively balanced in terms of grades taught and experience levels. Teachers in the sample generally described using traditional practices *more* to help students at risk. Teacher directed instruction was the most favored approach for teaching a student at-risk. More than 50% of teachers reported using teacher directed instruction more for students at risk, and less than 5% used it less for these students. Extra help during seatwork was selected by 82% of the respondents as "usually" used to as a technique to help a student at-risk in mathematics, and the low variance found for giving at risk students extra help during seatwork indicated general agreement on this strategy.

Teachers generally reported that they used rich tasks and projects either less than or to the same degree as they would for a student who is not at-risk. The vast majority - over 70% - reported using tasks *less*. Only 4.5% reported using a rich task or project more frequently for a student at-risk. The use of projects or tasks had the lowest mean and the least amount of variance of the teaching approaches suggested on the survey. This suggests that these teachers consistently responded by saying they do not favor projects or tasks for these students.

These findings point to the crux of our work, already glimpsed in the case studies, and solidified with the quantitative data. Even though best practices for students at risk as described in the framework indicate the importance of active, hands-on, engaging, interactive and involved learning rather than traditional and teacher directed instruction, more than half of teachers surveyed reported using teacher directed instruction *more* than with other students, and most used tasks less.

Consistent in the case studies, these results indicate an area of significant concern which also points to the crucial need for increased in-service training and support. Even though teachers surveyed identified weak problem solving skills as an area for concern, they report doing *less* problem solving with such students. Even though the students we observed typically could not focus for long periods, they were being subjected to *more* teacher directed instruction. These environments potentially continue to perpetuate the cycle of disinterest, boredom and frustration, eventually manifested in significant classroom misbehavior which we also observed.

Conclusions

Although many issues consistent with the literature were observed with the students in the case studies, none of the students demonstrated 'only' mathematical issues in isolation from behavior, motivation, or other issues such as home life, attendance or reading difficulties. While it is unclear which issues came first, there is an indication in many of the cases that a lack of satisfaction with mathematical learning may exacerbate inappropriate behavior in class.

Both the case studies and the survey indicate that the teachers studied were coping with students at risk by falling back on those practices they knew best – teacher directed instructional periods of varying lengths accompanied by extra help on individual homework. They were reluctant to use investigations, tasks and other student-centered or reform-based techniques. Needs for social acceptance and independence were generally not priorities. Mathematical topics, particularly in the grade 9 classes, were decontextualised, formal, and not engaging. Students responded by off task behavior and mis-behavior, and little engagement was evident. Most students were very reticent to ask questions or seek assistance. The data gathered and presented in this paper consistently points to a significant disconnect between best practices for students at-risk as described in the literature, and what was observed and reported by teachers to be happening in actual classrooms in the study. We believe this is an area of grave and significant concern which warrants further study and points to a crucial need for training and support.

Endnotes

1. We acknowledge gratefully the funding for this project from the Northern Ontario Educational Leaders (NOEL) through the Ontario Ministry of Education, as well as the support of the University of Manitoba CRYSTAL project funded by the Natural Science and Engineering Research Council of Canada (NSERC).

2. This survey can be found at http://noelonline.ca/depo/fdfiles/KajanderNOEL%20FINAL%20report%20APRIL%2030%202007.pdf

3. More detailed examples and transcripts of interactions can be found in the full research report available at the site listed in endnote 2.

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A COLLABORATIVE TO STUDY BELIEFS OF MATHEMATICS TEACHER EDUCATORS

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Seven novice mathematics teacher educators (MTEs) collaboratively explored their beliefs about mathematics teacher education. Although we shared and valued many of our peers' beliefs, we found differences in central beliefs systems. We discuss similarities and differences as well as implications of these findings for the preparation and continued professional development of MTEs.

A transition from K-12 teacher to teacher educator is complex and is a relatively unexplored area of research (Cochran-Smith, 2003); however literature is emerging regarding the development of mathematics teacher educators (MTEs) (e.g., Tzur, 2001; Van Zoest, Moore & Stockero, 2006; Zaslavsky & Leikin, 2004). There is abundant research about mathematics teachers' beliefs and how such beliefs influence practice within K-12 mathematics classrooms; it is a natural extension to investigate MTEs' beliefs and their impact on teaching practice. This study is a collaborative effort of a team of seven novice MTEs at six different institutions who conducted self-study research that entailed reflecting upon and explicating their beliefs about mathematics teacher education to one another. The purpose was to bring forward such beliefs, identify commonalities and differences, and develop a "collective wisdom" of beliefs of MTEs.

Theoretical Perspective

We acknowledge the "messiness" of defining what we mean by belief and any distinctions from what might be meant by knowledge (Parajes, 1992). We adopt a "belief is in the eye of the beholder" stance; any definition of belief or what is meant by knowledge *is* a belief and each participant has defined these constructs as a part of the study. However, we have agreed upon key assumptions about beliefs. For example, as novice MTEs, we see ourselves as constructing a personal pedagogy of teacher education (Tillema & Kremer-Hayon, 2005) within mathematics education and recognize that our beliefs in several areas influence our instructional decisions. We assume that beliefs are "predispositions to act" (Rokeach, 1968); thus we used our instructional practices as starting points for examining our beliefs. Beliefs can be held implicitly and are organized as a system in a way that is meaningful to the individual (Green, 1971; Leatham, 2006). Beliefs are inferred and cannot be directly observed (Green, 1971; Parajes, 1992); our goal is to make our individual beliefs explicit to ourselves and then to others for examination.

Methodology

We are a group of 7 MTEs who are either tenured, or in tenure-track faculty positions at six U.S. universities. All of us completed our PhDs at the University of Georgia in the year 2000 or later. Our subsequent interactions have brought forth conversations highlighting the successes and struggles in our new roles as MTEs and researchers. These conversations, our desires to collaborate and our experiences of studying the beliefs of mathematics teachers motivated this study.

Our work began by each of us engaging in self-reflective analysis activities to develop a concept map as a pictorial representation of his or her beliefs. Accompanying this map, we wrote a narrative that provided illustrative examples from practice. Guiding this process were the questions: What are my beliefs and what examples can I provide to support my claims? How do my beliefs relate to one another? Furthermore, we contend that attending to professional dilemmas and solutions helped us to connect our beliefs to our teaching (Lunenberg, & Willemse, 2006; Tillema & Kremer-Hayon, 2005). Consequently, we also reflected on our struggles and how we have worked to resolve them. Examples of data sources included syllabi, materials that were generated for the promotion and tenure process, and lesson plans from our teaching.

We then exchanged our concept maps and narratives in groups of two or three. Each group engaged in dialogue to understand the perspective of the other, and wrote notes to document our conversations. All seven then participated in a conference call wherein we discussed and documented commonalities and differences of our beliefs and experiences. This conversation continued via email during subsequent days.

Results and Implications

We found that we valued each other's beliefs about mathematics teacher education, and that the process of engaging in these discussions prompted each of us to then re-assess our beliefs and practices. As we sought to explicate our beliefs and to compare and contrast them with those of our peers, we saw areas of strength and weakness; perhaps most important, we saw in others' systems of beliefs many connections with our own.

For example, Andy and Signe's discussions about Signe's focus of creating a safe environment for her pre-service teachers (PSTs) so that meaningful interactions could occur challenged both Andy and Jennifer to consider how and why similar notions were or were not in their respective belief maps. As we engaged in our discussions we realized that that some of these beliefs, while shared, were not necessarily core beliefs that explicitly informed each MTE's instructional practices. This result has implications for mathematics teacher education in that although MTEs are frequently in agreement with one another about important ideas, the degree to which we emphasize these ideas varies significantly because of differences in core beliefs. The questions arise: Does this matter? Do we need a "common core" across the work of MTEs in order to truly impact K-12 mathematics learning?

Another important idea that emerged from our discussions initiated from Keith and Jennifer's dialogue about their maps and narratives. Both maps emphasized PSTs (rather than K-12 students) as learners. Jennifer explained that after her doctoral program, when she began as a teacher educator, she centered on teaching PSTs *about* the child as learner rather than recognizing the PST *as* the learner, despite the fact that her dissertation was about PST learning. Keith noted that he always saw the teacher as the learner, and in fact began his doctoral program with the intent of learning about teaching teachers.

In contrast, Wendy, Nickey, and LouAnn identified formative assessment of K-12 children as a prevalent theme across their maps. They used the concept of formative assessment as a vehicle for teaching PSTs relational understandings of mathematics as well as about mathematics teaching and learning. Andy's notion of the importance of PSTs creating models of students' mathematics also served this purpose. Engaging in formative assessment is also central to Signe's instruction, as she strives to learn from her students (PSTs). The point here is that Signe's formative assessment is OF the PSTs, whereas the others seemed focused on formative assessment of K-12 students.

This is not to say that Wendy, Nickey, LouAnn , and Andy are not thinking about PSTs as learners. They all focused on how PSTs should learn mathematics via the process standards during their PST education programs. Andy discussed the need to problematize teaching too, including problem solving related to modeling students' mathematics. Wendy, Nickey and LouAnn seek to prepare their PSTs to listen to their students and to assess students' thinking in order to inform their own instruction. But to what extent is Andy making models of his PST's thinking about mathematics teaching, and to what extent are Wendy, Nickey, and LouAnn listening to their PSTs and assessing the PSTs thinking about mathematics teaching in order to inform their own instruction? This focus on PSTs as learners of *teaching mathematics*, and not only learners of *mathematics* stood out to us as a significant distinction. Tzur (2001), Van Zoest et al. (2006) and Zaslavsky & Leikin, R. (2004) also bring forward this distinction when considering mathematics teacher educator development.

A corresponding implication for mathematics teacher education is that much initial and continuing development of mathematics teachers is explicitly based in research –research about how children learn mathematics. To what extent are our teacher preparation programs based on research on how teachers learn to teach mathematics? In what ways do we make this foundation explicit? And, in what ways can we make this explicit when considering mathematics teacher educator development?

Another significant implication for MTE development which arises from this work is the importance of personal reflection and dialogue of examining what drives our practices. Just as mathematics teachers need safe environments in which to reflect on beliefs and practices, MTEs need to be afforded the same opportunities. Perhaps this should be a major component of doctoral programs in mathematics education. Furthermore, MTEs are a diverse group of professionals from a number of arenas including district leaders and mathematics department chairs within school districts, staff developers within education centers, and instructors within alternate certification programs. Our field must consider ways to support reflective instructional practices of MTEs in all such settings.

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ALGEBRA TEACHERS' PERCEPTIONS OF TEACHING STUDENTS WITH LEARNING DISABILITIES

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Sixty-three Algebra I teachers in 27 school districts in a southern US state were surveyed to determine their perceptions of teaching students with learning disabilities (LD) and factors that might affect these perceptions. The results indicated that Algebra I teachers do not have favorable perceptions of teaching students with LD. Gender, collaboration with a special education teacher, and number of students with LD in the general education classroom were found to significantly affect teachers' perceptions of teaching students with LD.

The Individuals with Disabilities Education Act (1997) guarantees students with disabilities a free and appropriate education and requires that all students receive instruction in the least restrictive environment, which in many cases is the general education classroom. Concurrent with IDEA, the No Child Left Behind Act (2001) has significantly raised standards for all students, including those with LD. This challenge is perhaps greatest in states where students are required to pass Algebra I and a high school graduation exam in order to receive a high school diploma. Algebra poses a significant challenge to students with LD (Gagnon & Maccini, 2001); as a result, students with LD often have high failure rates in Algebra I (Carnine, 1997). However, research on teachers' perceptions of teaching students with LD has been conducted, for the most part, without regard to content areas. This study was designed to examine Algebra I teachers' perceptions of students with LD and factors that might contribute to these perceptions.

Method

One hundred seventy Algebra I teachers from 74 public high schools in 27 school districts within a southern U.S. state were surveyed. The survey included demographic questions and a 16-item Likert scale on teachers' perceptions of teaching students with LD. Many of the survey items were adapted from the Regular Education Initiative Teacher Survey (Semmel, Abernathy, Butera, & Lesar, 1991). The scale ranged from 4 to 1 (4 = strongly agree, 3 = agree, 2 = disagree, 1 = strongly disagree, N = no basis for opinion). Validity was established via a formal review by several outside experts in both mathematics education and special education. To establish reliability, a pilot study of 46 participants was conducted with Algebra I teachers in several school districts not within the survey population. The Cronbach's alpha coefficient for the pilot study was .8708. Sixty-three of the 170 Algebra I teachers returned their survey resulting in a 37% response rate.

Data Analysis and Results

Teachers' Perceptions of Teaching Students with LD

To analyze Algebra I teachers' perceptions of teaching students with LD, descriptive statistics were used. A majority of the algebra teachers had favorable perceptions (i.e., mean

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greater than 2.6) on 5 of the 16 survey items. Ninety percent of the teachers agreed that they were comfortable collaborating with the special education teacher, and that they had the primary responsibility for the achievement of all students (with and without LD) in their classrooms. Seventy-five percent agreed that students with LD had a basic right to receive their education in the general education classroom. Sixty percent of the teachers were comfortable implementing IEPs, and agreed that students with LD did not cause behavior management problems. Algebra I teachers were evenly split (mean between 2.4 – 2.6) on two of the survey items: (a) that special education teachers provided adequate support and (b) that students with LD having a sense of belonging when placed in the general education classroom.

Algebra I teachers indicated non-favorable perceptions on 9 of the 16 survey items (i.e., mean less than 2.4). Eighty-three percent of teachers disagreed that their initial teacher training program adequately prepared them for teaching students with LD. Seventy-five percent of teachers reported that the stigma many students with LD experience of being “dumb,” “different,” or a “failure” was not reduced by inclusion and disagreed they had enough planning time to meet the needs of students with LD. In addition, approximately 60% of the teachers indicated that in inclusive classrooms, (a) they did not have time to meet the state curriculum goals for all students, (b) that adequate resources for students with LD did not exist, (c) the self-esteem and academic achievement of students with LD did not improve in general education classrooms, (d) they do not have the knowledge and skills necessary to teach students with LD, and (e) the achievement of students without LD decreased in inclusive classrooms.

Relationship between Perceptions and Factors

Chi-Square was used to analyze the data to determine the relationship between algebra teachers’ perceptions of students with LD and demographic factors. Significant relationships occurred between the Algebra I teachers’ perceptions of students with LD and (a) gender, (b) number of students with LD in the classroom, and (c) collaboration with the special education teacher. No relationships were found between perception and (i) teacher experience, (ii) highest degree earned, (iii) number of college courses taken or (iv) number of workshops attended that addressed teaching students with LD.

Gender

The chi-square analysis indicated a significant difference (_^{2} = 5.91, p = .015) between males and females concerning the item stating students with LD have the right to receive their education in the general education classroom. Eighty-nine percent of females agreed with this item whereas only 63% of males agreed. In general, females had more favorable mean scores than males on 14 of the 16 items.

Number of Students with LD

Significant differences were found between the number of students with LD in the classroom and four items on the survey. Algebra teachers with more than six students with LD agreed that (a) inclusion improves the self-esteem of students with LD (_^{2} = 4.35, p = .037) (52% compared to 26% for teachers with less than 6 students), (b) adequate resources exist to meet the needs of students with LD (_^{2} = 6.61, p = .010) (52% vs. 20%), (c) inclusion does not take time away from state curriculum goals (_^{2} = 4.35, p = .037) (52% vs. 26%), and (d) students with LD lose the stigma of being "dumb, different, or failure" when place in the general education classroom (_^{2} = 4.77, p = .029) (32% vs. 10%).

Collaboration with Special Education Teacher

There were significant differences between collaboration with special education teachers and five items on the survey. Algebra I teachers who collaborated at least once every 2 weeks with a special education teacher agreed that (a) inclusion improves self-esteem of students with LD (_^{2} = 5.74, p = .017) (48% vs. 16% of teachers with less collaboration), (b) adequate resources exist to meet the needs of students with LD (_^{2} = 9.31, p = .002) (48% vs. 6%), (c) adequate support exists from the special education teacher (_^{2} = 7.02, p = .008) (63% vs. 26%), (d) students with LD experience more academic success in general education (_^{2} = 5.97, p = .015) (46% vs. 12%), and (e) they are comfortable implementing IEPs (_^{2} = 6.27, p = .012) (73% compared to 39%).

Discussion

Unlike most studies of general education teachers perceptions of inclusion, Algebra I teachers expressed more negative perceptions of inclusion (Bender, Vail, & Scott, 1995; Scruggs & Mastropieri, 1996). In this study, females had more favorable perceptions of teaching students with LD than males. Previous studies relating gender and inclusion have been mixed; males tend to be more confident in their abilities to meet the needs of students with LD (Boyer & Bandy, 1997), but other studies have shown males with less favorable perceptions of inclusion (Avramidis, Bayliss, & Burden, 2000). Algebra teachers who collaborate more frequently had more favorable perceptions of students than teachers who collaborated less frequently. Collaboration is important since research indicates that students with disabilities can make significant academic gains when the general and special education teacher collaborate effectively (Klingner, Vaughn, Hughes, Schumm, & Elbaum, 1998). Finally, algebra teachers with six or more students with LD had more favorable perceptions of teaching students with LD than teachers with less than six students with LD. More research needs to be conducted to explain these findings. A limitation of this research is that generalizations of findings may be weak given the small population size and low response rate.

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MATHEMATICS AFFECT: SYMPTOM NOT CAUSE. KEY IS UNDERSTANDING THE STUDENT'S MATHEMATICS SELF IN RELATIONSHIP

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In my dissertation project I explored, with college statistics students, the hypothesis that affective problems, found to figure significantly in achievement, are not causes of mathematics distress but rather symptoms of students' mathematics relationality. My study, piloting a relational counseling approach integrated with constructivist tutoring, provided support for my hypothesis and effective ways to help students repair and develop their mathematics selves.

The research described here grew out of a troublesome sense that the cognitive and affective expressions that we, in the learning center context, see in struggling college mathematics students may be *symptoms* rather than *causes* of a student's real difficulties. Predominant in practice in my field, I saw a disjointed or narrowly focused approach to mathematics affect (e.g., on learned helplessness or locus of control or anxiety or …) and treatments often only tenuously linked to students' mathematics competence development. I became certain that this was not adequate to understand how college students undertake to learn mathematics nor to help them recover in time to succeed in their current course. When a student like Jamie, who reported extreme mathematics anxiety, "hid" in class and avoided accessing help despite the fact that she was repeating the class, how should I interpret such an emotional and counterproductive behavioral response? Surely I needed a fuller understanding of her history and its effects on her present mathematics experience in order to understand what I, as a mathematics learning specialist, could do to help her and what she needed to change in order to succeed this time.

Could it be that students with otherwise adequate mathematics skills and aptitudes are limited by unconscious forces linked to earlier mathematics learning experiences that cause them to repeat counterproductive practices? How might poor preparation interact with a student's developing mathematics self to affect her approach in the current course? I was driven to look beyond mathematics anxiety and learned helplessness research, into the realm of psychotherapy.

Theoretical Framework

The central hypothesis that emerged was that affective problems—beliefs, attitudes and emotions, found to figure significantly in mathematics achievement (cf. Ma, 1999), are not causes of mathematics distress, but rather, symptomatic of student's mathematics relationality.

In recognizing the need to address root causes of mathematics affective problems, I had returned to theorists of mathematics affect such as McLeod (1992) and looked more closely at their endorsement of classical Freudian-type analysis and counseling approaches albeit for cases of extreme mathematics emotionality (see McLeod, 1992, citing Tahta, 1993). Weyl-Kailey (1985) uses Freudian psychoanalytic techniques in a clinical setting to probe and remediate puzzling mathematical behaviors as she uncovers and treats related psychological disturbances. Weyl-Kailey and others found that attention to students' unconscious motivations gives insights that other approaches do not. Because my interest was in the mathematics mental health[1] of ordinary students, not just those with extreme difficulties, I had earlier rejected psychoanalytic theory. But I now saw the promise of psychoanalysis in its attention to the unconscious and the present effects of the past on *everyone*. Indeed McLeod (1997) noted with interest Buxton's (1991) suggestion that some struggles of ordinary students with mathematics might well be understood in terms of Freud's concept of the superego. I resolved to explore Freud's theory and the theories that evolved from it.

The work of Stephen A. Mitchell (1988, 2000) emerged as highly relevant to my research because it used a form of relational conflict psychotherapy derived from Freudian psychoanalysis to help ordinary adults who had goals that were not being fulfilled because they were embedded in conflicting relational patterns with themselves and their significant others (both internal and external). In 1988 Mitchell integrated the three major relational strands of psychotherapy that emerged from Freud's classical psychoanalysis. Each of these strands emphasized one dimension of what Mitchell termed as a person's *relationality* or her current behavior that is the outcome of the development of her self, her external and internalized objects, and her interpersonal attachments (Mitchell, 2000). When I considered these dimensions in the context of a student's mathematics learning experience, I interpreted them as follows:

1. Mathematics self or selves (cf. Kohut, 1977);
2. Internalized mathematics presences or objects (cf. Fairbairn, 1952); and
3. Interpersonal mathematics relational or attachment patterns (cf. Bowlby, 1973).

Understanding a student's mathematics relational dimensions and how they interact with one another to express her relationality might provide the insight into the origin and development of her puzzling behaviors, affect, and conflicts that I was seeking.

Methods

To investigate my hypothesis, I piloted a *brief* relational mathematics counseling approach with students taking a summer introductory-level statistics course taught at a small, urban, commuter state university in the Northeast. I was embedded in the class as a participant observer. Ten of the original 13 students signed up for individual mathematics counseling. I identified, adapted, and developed instruments and approaches to explore students' mathematics learning; their history, beliefs, attitudes, and emotions about mathematics learning; and their relational patterns as they participated in course. Some of these instruments (e.g., My Mathematics Feelings) I administered pre and post to the whole class; others (e.g., the College Learning Metaphor, JMK Mathematics Affect Scales) I administered during counseling sessions. Each session was audiotaped and two classes were videotaped. I analyzed the transference-countertransference interactions between the participant and me, classroom observations, mathematical products, assessment and survey data, and transcripts of sessions using an integrated cognitive-relational analysis approach that included mathematics educational analysis (e.g., of class tests, the Algebra Test, Sokolowski's (1997) adaptation of Brown, Hart, & Kuchemann's (1985) Chelsea Diagnostic Algebra Tests, my Arithmetic for Statistics Assessment), and relational episodes analysis (Luborsky & Luborsky, 1995). Using these approaches I developed a case study of the class as a whole, in-depth case studies of three of the 10 participants, and shorter case analyses of the other 7 participants.

Results and Discussion

I found that, indeed, when I looked only at student responses to *Feelings* and *Beliefs* surveys, and observed and asked participants about them, the picture was confusing. For Jamie mathematics anxiety seemed to be a central affective issue but it did not debilitate her during exams and was not central to her vulnerability to failing the course. It became clear to me, looking through a relational framework, analyzing her metaphor ("storm" during which she would hide) and the transference and countertransference in our relationship, that Jamie was an adequately prepared mathematics student with an undermined mathematics self. She had "bad" teacher internalized presences, and damaged teacher and mathematics attachments; Jamie was a Category II student who avoided one-on-one contact with tutor and instructor and did not get the help she needed when she needed it. Her social/public mathematics anxiety was a symptom of these relational problems; the resolution required mathematical and relational attachment repair—I helped Jamie become conscious of her adequate mathematics competence and the relational roots of her anxieties (including a 5th grade teacher who "yelled') and avoidant behavior and gave her relational assignments. She accessed help, reattached to mathematics and mathematics teachers, her metaphor changed to "partly sunny day...I can go out in it…" her anxieties abated and she earned a B+ in the course.

Each participants' cluster of mathematical and affective symptoms differed from the others but there were three distinct groupings of student, each needing different types and intensity of support, relatively easily discerned from their relational patterns linked to mathematics competence (preparation with respect to the current course). Category I students with sound mathematics selves were well-prepared with sound *self esteem*; Category II students with undermined mathematics selves had adequate preparation but compromised ***self esteem*** cf. Jamie); Category III students had underdeveloped mathematics selves characterized by underpreparation and related low *self esteem*. I found a further useful sub-categorization of Category II and III students into those with malleable versus inflexible relating patterns.

This study does indeed provide support for my hypothesis that mathematics affect might more usefully be regarded as symptomatic of students' underlying mathematics relationality, than as a cause of their mathematical distress. Further, the framework of mathematics relationality provides a way to understand and prioritize support for students to repair and develop their mathematics selves and succeed in their course.

Endnote

1. The term Sheila Tobias (1993) uses in *Overcoming math anxiety* (Revised ed.). New York: W. W. Norton & Company to describe a person's "willingness to learn the mathematics [she] needs when [she] needs it" (p. 12), which I expand to include her mathematics cognitive functioning.

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USING CHALLENGE-BASED MODULES TO SUPPORT CHANGE IN PRE-SERVICE TEACHERS' BELIEFS ABOUT LEARNING AND TEACHING MATHEMATICS

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This pilot study investigates the use of a challenge-based module as a support for changing pre-service teachers beliefs about mathematics teaching and learning. The module focuses on an instructional sequence that was taught in a first grade classroom over a five-week period and includes video-recordings of classroom episodes and of pre- and post-interviews conducted with the students.

Most pre-service teachers, through their own K-12 and university mathematical experiences, have developed the belief that mathematics is merely a set of rules for manipulating symbols (Thompson, Philipp, Thompson, & Boyd, 1994). This "classroom mathematics tradition" has shaped pre-service teachers' "construction of scientific or mathematical knowledge by constraining what can count as a problem, a solution, an explanation, and a justification" (Cobb, Wood, Yackel, & McNeal, 1992, p. 575). Thus, mathematics educators should do more than try to inform pre-service teachers about content and pedagogy. In addition, they must guide pre-service teachers toward beliefs and practices consistent with current reform efforts, where teaching mathematics is a complex process in which teachers are required to assimilate knowledge from a variety of disciplines (National Council of Teachers of Mathematics, 2000).

The technological tool we have developed to support the pre-service teachers' learning consists of an authentic challenge-based case (Bransford, Brown, & Cocking, 1999; Brophy, 2000; Schwartz, Lin, Brophy, & Bransford, 1999) in which they can investigate the process of supporting students' learning of significant mathematical ideas. The Patterning and Partitioning module comprises a series of challenges for pre-service teachers to explore. The module focuses on an instructional sequence that was taught in a first grade classroom over a five-week period and includes video-recordings of classroom episodes and of pre- and post-interviews conducted with the students. The goal of instruction in these lessons was to enable the students to develop a relatively deep understanding of elementary number concepts, with an emphasis on relationships between numbers up to 20. In the course, we coordinated the use of this module with message board discussions (Bringelson & Carey, 2000). The use of the message board allowed the pre-service teachers to continue discussions initiated in class sessions as well as to debate new ideas that emerged.

This paper reports findings from a pilot study conducted during our summer elementary mathematics methods course. The study investigates the viability of using the Patterning and Partitioning module as a support in developing shifts in pre-service teachers' beliefs about mathematics teaching and learning. "For the purposes of investigation, [beliefs] must be inferred" (Pajares, 1992, p. 315). Although past studies (cf. Collier, 1972; Fennema, Carpenter, & Loef, 1990) have implemented Likert-type scales to collect quantitative data about beliefs, there are many limitations in using such an instrument to collect belief data since participants are limited in their response possibilities. "For example, each participant might interpret the wording of items in different ways or think about different situations when responding. Likert items typically are generic and not context specific. The items are often leading in nature so that participants might agree strongly to an item, but might not have considered it important otherwise" (Wilson, 2006). To overcome some of these difficulties, we used the case-based survey instrument developed by the Integrating Mathematics and Pedagogy (IMAP) research group at San Diego State University (http://www.sci.sdsu.edu/CRMSE/IMAP/pubs.html). The survey solicits participants to "use their own words to react to, or answer questions about, learning situations. Although this format does not remove the need to draw inferences, it reduces it" (Ambrose, Philipp, Chauvot, & Clement, 2003, p. 2). The IMAP survey instrument measures seven specific beliefs relating to three different categories of mathematics instruction: "Beliefs about mathematics", "Beliefs about learning or knowing mathematics", and "Beliefs about children's (students') learning and doing mathematics" (see IMAP website given above).

The participants of this study were the pre-service teachers taking the Elementary Mathematics Methods course during the first summer session of 2006. Data were collected before and after the implementation of the web-based module. The case-based survey instrument was administered to seven female pre-service teachers during the class period prior to introduction of the first web-based module as a pretest and the class period after completion of the module as a posttest. Naturally, the pre-service teachers were asked not to collaborate on their responses. The survey responses were blinded so that coders were not able to determine whether the responses were from pre or posttests. The data consist of responses for seventeen rubrics (see the IMAP web address given above) for each of the pre and posttests for the pre-service teachers. Prior to coding any of this data, the researchers completed the practice tests provided on the IMAP website to ensure proper coding techniques. Each of the seventeen rubrics were then coded by the researchers and compared to increase interater reliability. If a difference in scores arose, the researchers revisited and discussed the response and came to an agreement on the score. After each individual rubric was coded, the researchers used the Rubric of Rubrics found in the IMAP scoring manual to find the overall scores for each of the seven beliefs.

Data from pre and post surveys were analyzed to identify trends. Researchers tested the total scores in each section for reliability and for significant changes in pre-service teacher beliefs. Results indicate that the items on this survey were reliable as shown through a Cronbach's Alpha of .903. Pre and post group's scores were analyzed with an ANOVA. Items such as effect size and observed power were considered to determine the significance of each shift in scores. Participants displayed a shift in beliefs for all survey sections. The most significant shifts occurred in results for beliefs three and four. Belief three had a level of significance of .02 as determined through Levene's test of equality of error variances. Sections three and four measure beliefs about knowing and learning mathematics. More specifically, belief three measures the priority a participant places on understanding content rather than procedures. Belief four had a level of significance of .016. This section also measures beliefs regarding the use of procedures in the mathematics classroom. Participants responded to this section in the pre-test with an emphasis on utilizing procedures during lessons. The post-test results indicate that participants now place more of an emphasis on developing conceptual knowledge rather than procedural knowledge during mathematics instruction. When identifying correlations between beliefs from the post results, beliefs three and four were found to have a Pearson correlation of .944 with a level of significance of .001, which would indicate that when a participant scored high on belief three, they would also score high on belief four. These results indicate that the Patterning and Partitioning module could play a significant role in changing pre-service teacher beliefs about learning and doing mathematics in the elementary classroom.

Given the variation in the pretests and posttest scores, it is evident that pre-service teachers' beliefs changed over the course of interacting with the modular media case. These promising results call for a more in-depth study on the use of this and other modular cases to support pre-service teacher beliefs about mathematics teaching and learning. However, it is important to note that many other variables might have played a role in changing pre-service teachers' beliefs. For example, since the same survey was used for both the pre-test and posttest, testing effects may well account for a portion of the results of this study, even though three weeks separated each administration. There is also a limitation to the types of data analysis techniques that were appropriate to use for the number of participants in this study.

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ACTIVITIES, APPRECIATION, AND ABSTRACTION: HIGH SCHOOL MATHEMATICS TEACHERS' BELIEFS ABOUT TEACHING/LEARNING GEOMETRY

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Studying teachers' beliefs is important for understanding mathematics teachers' behaviors in the classroom. (Cooney, Shealy, & Arvold, 1998; Ernest, 1989; Raymond, 1997; Thompson, 1992).

One cannot research beliefs without first defining or characterizing what one means by the word "beliefs". I have adopted the characterization of knowledge and beliefs suggested by Furinghetti and Pehkonen (2002) that consider two types of knowledge: objective and subjective. Objective knowledge has to be true whether proved by experiment and/or socially accepted; subjective knowledge is knowledge constructed by an individual. Therefore, belief is taken as subjective knowledge.

Charalambous, Philippou, and Kyriakides (2002), Ernest (1989), Raymond (1997), and Thompson (1992) researched teachers' beliefs about the nature of mathematics. All of these studies were investigating global beliefs about mathematics. Törner (2002) proposed the term "domain-specific beliefs" when referring to beliefs about different mathematical domains such as geometry.

Charalambous et al. (2002) analyzed data from 229 questionnaires which resulted in a five-factor model of teachers' viewpoints of mathematics. Ernest (1989) distinguished three philosophies of mathematics that occur in the teaching of mathematics: the instrumentalist view, Platonist view and the problem solving view.

I used a questionnaire to investigate high school mathematics teachers' beliefs about the nature of geometry and its teaching and learning. The sample included 520 high school mathematics teachers from four countries-The United States, Canada, Australia, and The United Kingdom. I used factor analysis to enhance my understanding of the part of my data set that was composed of 48 Likert type statements. Through principal components analysis with varimax rotation I was able to extract a three factors model which I have identified as:

Factor 1 -- A disposition towards activities
Factor 2 – An appreciation of geometry and its applications
Factor 3 – A disposition towards abstraction

I was able to divide the respondents into eight groups depending on whether the respondents' had a positive or negative score on each factor.

In this presentation I will share the results of the factor analysis and how I used the results to initiate an intervention in the classroom of one of the respondents from Group II: positive, positive, negative. This means the teacher had a positive disposition towards activities, a positive appreciation of geometry and its applications and a negative disposition towards abstraction. The teacher was uncomfortable about teaching proof in her classroom. After the intervention the teacher was more willing and even happy about teaching proof.

A SYNTHESIS OF RESEARCH ON TEACHER BELIEFS, DISPOSITIONS, AND EMOTIONAL INTELLIGENCE

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Many state and national accreditation agencies, such as NCATE require assessment of dispositions of teacher candidates. Mathematics education has a history of related research on teacher beliefs. In this synthesis of literature on dispositions, teacher beliefs, and emotional intelligence, connections will be drawn between the various areas of research. Research on emotional intelligence from the fields of medicine and business will also be examined to provide other perspectives.

Many state and national accreditation agencies require institutions to assess teacher candidate dispositions. According to the National Council for Accreditation of Teacher Education (2006), dispositions are "the values, commitments, and professional ethics that influence behaviors." As a result, most teacher preparation programs must identify instruments to use to assess dispositions, even though several researchers (e.g., Damon, 2005; Maylon, 2002) have warned of possible dangers of using dispositions to screen teacher candidates. Although a variety of assessment methods exist for dispositions and for beliefs, methods vary widely, and none is commonly accepted.

Mathematics education has a history of related research (Thompson, 1992; Philipp, 2007), including the impact of beliefs on teaching and the characteristics of effective teaching (e.g., Evertson, Emmer, & Brophy, 1980). Another strand of research attempts to identify effective teaching practice through contrasting novice with expert teachers (e.g., Livingston & Borko, 1990). Several other influential case studies have focused on teacher beliefs (e.g., Cooney, 1985). Much of this work is based on more general research on beliefs (e.g., Green, 1971).

In this synthesis of literature on dispositions, teacher beliefs, and emotional intelligence, connections are drawn between these various areas of research. Research from the fields of medicine and business are also examined to provide other perspectives on the challenge of measuring and influencing beliefs, dispositions, and emotional intelligence.

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CHARACTERISTICS AND BELIEFS OF ELEMENTARY SCHOOL TEACHERS AND MANIPULATIVES USE

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The National Council of Teacher of Mathematics (NCTM, 2000) states, "students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge" (p. 20). As suggested by Hiebert et al. (1997) there are different tools that teachers can use to help students develop mathematical understanding, and one of those tools is physical materials, such as base-ten blocks or fraction tiles. Although the use of manipulatives has been found to help students' understanding of mathematics (Clements, 2007), teachers have been found to differentially use manipulatives in their instruction. Some of the factors affecting manipulative use discussed in the literature are grade level (Weiss, 1994), teachers' background characteristics (Opdenakker & Van Damme, 2006) and teachers' beliefs (Moyer, 2001).

This study investigated the relationships between teachers' grade level, background characteristics, and beliefs about manipulatives, and the frequency with which teachers use manipulatives as a part of their instructional strategies. We analyzed data collected from 530 in-service elementary school teachers. Regression analyses were used to model teachers' use of manipulatives. Overall, teachers' grade level and beliefs were found to be the most important predictors of how often elementary school teachers use manipulatives in their mathematics instruction.

The findings form this study have important implications for both pre-service teacher education as well as in-service teacher professional development. Mathematics teacher education programs should provide opportunities for teachers to reflect on their instructional beliefs through first-hand work with activities involving manipulatives. Through these experiences, teachers may come to recognize the value of using appropriate manipulatives for *all* grade levels of students and ultimately incorporate them more frequently into their mathematics instruction to help their students reach mathematical understanding.

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MATH CONTENT COURSES FOCUSED ON CHILDREN'S THINKING: ADDRESSING PRESERVICE TEACHERS' EFFICACY AND BELIEFS ABOUT MATHEMATICS

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This study examines how the children's thinking approach used in mathematics content courses is connected to the self-efficacy and beliefs preservice teachers have about mathematics. Surveys completed by control and experimental groups from a large mid-west university indicate that preservice teachers' beliefs and efficacy can be altered using such an approach and that efficacy and beliefs about mathematics are related.

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A PARTNERSHIP BETWEEN A MIDDLE SCHOOL MATHEMATICS TEACHER AND A UNIVERSITY RESEARCHER CENTERED ON CONTENT AND TEACHING

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This case study examines the evolution of a partnership between a middle school mathematics teacher and a university researcher as we discussed content and teaching. Resolutions of content-teaching tensions moved the partnership through three stages, from a focus on the content, to discussions of lesson design as the teacher gradually adopted the Connected Mathematics curriculum. Our goals changed from individualistic to a more shared vision derived through implementation of activities/problems and focusing on student thinking. The shift in the partnership can be attributed to the adoption of the Connected Mathematics curriculum by the teacher and the effective management of the dialectic tensions of acceptance/judgment, dependence/independence, affection/instrumentality and expressiveness/protectiveness by both partners.

The current push for reform mathematics seeks to break away from the traditional "banking" concept of education (Freire, 1970) where the students are viewed as receptacles that need to be "filled" by the teacher via showing and telling, and instead move towards an approach informed by constructivism (von Glasersfeld, 1995). There is a call for students to actively construct their knowledge as part of a classroom community by engaging in challenging problems, inventing procedures, justifying the validity of these procedures and communicating these ideas to peers (NCTM, 2000; Simon, 1997).

Teaching with the reform vision outlined above is a challenge for teachers and calls for a deep and flexible knowledge of mathematics which goes beyond what teachers learn in their pre-service teacher education (NCTM, 2000). There is a need for ongoing professional development that might include a number of models that focus on various aspects such as, teacher knowledge (e.g. Developing Mathematical Ideas seminars [Cohen, 2004]), instruction, (e.g. Japanese Lesson Study [Fernandez, 2002]), student thinking (e.g. Cognitively Guided Instruction [Fennema et al., 1996]), and a combination of instruction and student thinking (e.g. Purdue Problem-Centered Mathematics Project [Cobb, Wood, & Yackel 1990]). All these examples of professional development involved prolonged interactions between teachers and researcher(s), but there is a lack of an in-depth examination of the collaboration between the researcher(s) and teachers and the evolution of this collaboration over time. This exploratory qualitative case study (Merriam, 1988) examines the simplest case (one-on-one) of collaboration between a middle school mathematics teacher and a university researcher as we come together for the ongoing professional development of the teacher. The study examines how the partnership evolves over time and its influence on the teacher's planning and teaching.

Theoretical Framework

Collaborations between schools and universities have evolved in various ways and two major divisions, cited in the literature, are based on the organizational structure of the collaborations like Professional Development Schools (Handler & Ravid, 2001) and others that focus on the relationships in the collaborations such as symbiotic partnerships and organic partnerships (Whitford, Schlechty, & Shelor, 1987). Symbiotic partnerships

(Goodlad, 1988; Whitford et al., 1987) involved unlike partners who came together for the mutual satisfaction of their goals. Further, each partner displayed enough selflessness to ensure the satisfaction of the others goals. Schools and universities, like all institutions, deteriorate when things are left unattended, and Goodlad (1988) suggested symbiotic partnerships between these institutions as a way for renewal through varied expertise and a continuous flow of new ideas.

Organic partnerships involved partners who performed unique functions but had common goals. The common goals were made explicit and neither partner 'owned' the goals. This differed from symbiotic partnerships, where the goals were owned by one of the partners. Whitford et al. (1987) pointed to professionalization of teaching as an example of an issue that was 'boundary spanning' and could foster organic partnerships. Here the two institutions could be jointly responsible for the development of teachers from the recruitment, selection, preparation to the ongoing professional development.

Collaborations between members of the university and schools are a challenge to build and sustain as learning relationships have been traditionally hierarchical rather than collaborative (Olson, 1987). To face this challenge, Cole and Knowles (1993) assumed that the teacher and researcher should negotiate all stages of the research work to reflect true collaboration. However, others (Clark et al., 1996) feel that by involving the teachers, with their busy schedules, in all aspects of the research, benefited the researcher more than the teacher since publications are viewed more favorably in the university than in the school. Instead Clark et al. (1996) argued that true collaboration arose out of dialogue between the partners that led to understanding of each others roles. They argue that there should be a push for shared understanding rather than shared work.

Tensions are a common theme in collaborations between unlike partners and resolving these tensions can be critical in moving the partnership forward. Some studies (Freedman & Salmon, 2001) have examined the evolution of partnership through the lens of dialectical tensions of acceptance/judgment, dependence/independence, affection/instrumentality and expressiveness/protectiveness (Rawlins, 1992). The dialectic of acceptance/judgment referred to the tension between evaluating and holding the other partner to standards or accepting them with their strengths and weaknesses. The freedom to be independent but also to be there in times of need for the other partner is the dialectic of dependence/independence. Affection/instrumentality referred to the rendering of help for the other partner as opposed to expecting the favor returned. The efforts that the partners make to balance restraint and candor, so that it furthers trust, is captured by the dialectic of expressiveness/protectiveness. Day (1991) cited attributes like a caring nature contribute to resolving tensions and sustaining longer ethical collaborations where concerns of the teacher are addressed.

Methods

A qualitative case study (Merriam, 1988) approach was used to work, over a period of three months, with a mathematics teacher (Linda) in a middle school in the south west of the country. Linda was part of a middle school cohort in a large National Science Foundation funded project at the university. She was introduced to me (the researcher) by one of the Principal Investigators of this research project.

Meetings with Linda involved classroom observations in her mathematics class (her 7th grade class for two months and her 6th grade class for another month). Immediately after the classroom observation, Linda and I met, during her planning period, to have our discussions. I collected data for a total of 35 classroom observations and 32 planning period discussions respectively. Linda's instructional moves in the classroom and the group interactions of the students at my table were the major focus of my classroom observations. Three major content areas namely, probability, functions, and statistics were discussed in the planning period discussions, prior to the implementation of these topics in the classroom. The planning period generally started with posing a problem or activity for Linda and then discussions took place around issues that came up. The problems were chosen from the *Connected Mathematics Project* (CMP) curriculum and other resources (e.g. *Mathematics for Elementary School Teachers* [Bassarear, 2005]). Lesson planning was also done at appropriate times and discussions of the students thinking also entered into the conversations. Field notes were taken of all classroom observations and discussions in the planning period were audio or video recorded or both. All our discussions in the planning period (a total of 32) were transcribed and data analysis was done in NVivo7 (QSR International, 2006) by isolating themes that reflected engagement and interactions of Linda and myself. The content-teaching tensions (discussed below) were identified as a key component in describing the stages of the partnership.

Results

Based on our interactions, the partnership evolved in three stages. The first stage was characterized by our discussions around the mathematics content. I introduced Linda to probability, which was a topic she would be teaching in the coming weeks. We discussed problems from the seventh-grade *Connected Mathematics Project* (CMP) curriculum and *Mathematics for Elementary School Teachers* (Bassarear, 2005) that emphasized problem solving and conceptual connections. Linda would attempt a problem and then we would have a discussion based on her work. The level of difficulty of the problems was such that they could be adjusted to the level of the students, if Linda chose to do the activity/problem in the class. For example, the Tile problem involved finding the fraction of tiles of a color in a bucket, given that there were tiles of three colors. Linda first did this activity as part of our discussion in the planning period and later chose this activity to introduce her students to the concepts of experimental and theoretical probabilities and their connection. In the first stage I controlled the agenda of our discussions and Linda participated in the activities/problems. She would share her thinking, answer questions I posed, and inquire about mathematics that was unclear. I dominated the first lesson that Linda planned at this stage (Tile problem) by reminding Linda of the important aspects of the problem that she should ensure her students understood. Linda did not have ownership over the activity and relied on remembering the important aspects as she attempted to transfer these to her students. After debriefing this lesson, I suggested that Linda do another activity with her students to ensure that they grappled with the concepts of experimental and theoretical probability. Linda’s lessons at this stage were typically outlined based on the Performance Objectives (POs) of the State Standards as directed by the school district; student thinking was not a prominent focus although we would discuss the performance of the students on an activity.

The second stage was characterized by the appearance of content-teaching tensions in our discourse. This tension represented a sudden change in our conversation, initiated by Linda, from a discussion about the mathematics to her immediate planning needs. For example, in the following conversation about word problems, I emphasized the need for procedures in the promotion of algebraic thinking and Linda suddenly changed the topic of discussion to her planning needs.

Anthony : … just doing the word problems and knowing how to represent them won’t get you to the solution because you still have to solve, you have to manipulate the [expressions](Linda interrupts)

Linda: Now going back to…the lollipop problem. If I were to do that on Monday, just give them some fun before I leave…I mean it’s basically what we have done.

But how can I set it so that it is adding something else to it?

The above discussion is representative of the sudden change in our discourse from a discussion of mathematics content towards the specific planning of a lesson that Linda wanted to teach. This interaction contrasted with similar interactions in the first stage where Linda would not interrupt our discussions with planning needs. With the appearance of tensions the discourse moved towards teaching and sometimes there was a tendency to dwell on logistical issues of implementation. In these instances, I tried to ensure that we refocused on the mathematics content. I did this by discussing the mathematics that arose in an activity/problem that Linda wanted to discuss. In general, the mathematical issues were intertwined with discussions of planning and this was also a characteristic that was present in the next stage of our partnership. At this stage Linda anticipated me to provide guidance in teaching issues that arose in the class. This would be different in the next stage as Linda took a more proactive role in thinking and resolving teaching issues that arose in her class.

The third stage was characterized by a move towards resolution of the content-teaching tensions as Linda assumed a proactive role in controlling the agenda. She adapted problems for her class and reflected on the difficulties that arose in the class discussions. I assumed the role of being a resource for Linda by suggesting activities, supporting her decisions, giving feedback for her ideas, and answering her questions. For example, in our discussion on Statistics, Linda expressed a desire to do an activity with the students that would build on their previous lessons on various data representations. We decided to brainstorm and came up with a number of activities that Linda evaluated by considering 'what if' moves (The 'what if' move involved Linda reflecting on the activity from the students' point of view and conjecturing about possible difficulties the students would encounter). Finally, Linda adapted the Hat problem to the Shoe problem. The Hat problem involved a person starting a hat shop who had to determine the number of hats he should purchase, given that hats came in lots of a thousand. The Shoe problem was similar, but involved shoe sizes of middle school students.

The planning of the Shoe lesson contrasted to the role that Linda assumed in the planning of a lesson in the first stage where she accepted my suggestions without a lot of discussion. Further, in this stage the activities were not adopted directly from the curriculum but were adapted to suit Linda's and the students' needs.

The activities that Linda assigned from the CMP differed from the textbook problems that the students were doing before the study, as a result there were student difficulties that had to be resolved. In the previous stages Linda would seek my guidance in resolving these issues but in the third stage Linda attempted to resolve these issues on her own. Reflecting on some of the challenges of the Shoe problem, Linda related the following:

L: Then I'm thinking to myself, how much of it do I do as a whole group because then they [the students] are just going to emulate what I'm doing and not going to think about it themselves. So I kind of just played with it all day long. Some classes I went as far as making a frequency chart.

This episode was typical at this stage in the partnership when Linda was actively reflecting on the students' thinking and the amount of guidance that she was providing them. This differed to her reaction in the second stage when she anticipated me to provide guidance in resolving teaching issues that arose in the class. Tasks were just not selected if they just satisfied the POs, like the first stage; instead the tasks were also analyzed more closely from the students' point of view and implementation in the classroom.

Discussion

The evolution of the partnership in this study through the three stages, reflects an evolution from a symbiotic partnership to an organic one (Goodlad, 1988; Whitford et al., 1987). The partnership began as a symbiotic one with Linda's goals being to further her understanding in the mathematics that she taught and my goal was to expose Linda to the important mathematical ideas and examine the evolution of our partnership as we had these content discussions. These goals are reflected in our interactions at the first stage where the mathematics content was at the core of our discussions. We both made efforts to ensure that the other person achieved their goals. Gradually, as Linda chose to adopt the CMP curriculum (even though this was not the original goal of my study), she had more pressing questions about the activities and implementing them in the classroom. This can be seen in the content-teaching tensions that arose in the second stage of the partnership. Our content discussions were interrupted by Linda who wanted to discuss a specific activity she wanted to implement in her classroom. In the third stage, Linda and I both focused on the CMP curriculum and the issues that arose in the classroom during implementation. Thus we moved to an organic partnership with the common goal being the implementation of the curriculum and reflecting on the related classroom issues that arose as a result. In the process, Linda focused a lot more on the student thinking and chose to implement activity/problems could further their thinking.

Key Elements for the Movement

The first stage of this study was important in setting the pace for the rest of the study. Linda was introduced to the mathematics content through various resources that included the CMP curriculum. As Linda grappled with the mathematical ideas, she realized the potential of the curriculum for her students. Linda saw how multiple POs were addressed, simultaneously the conceptual understanding of the students could be developed, and connections could be made between and within topics. Thus she felt that her students would have the opportunity of understanding by building and strengthening their internal networks (Hiebert & Carpenter, 1992). Linda also saw the value of these activities in preparing the students for the assessments and addressing the time pressures that she faced in covering the content. She felt the problem solving activities would be more engaging to the students than their regular textbook exercises. The first stage went a long way in establishing the credibility of the CMP as a valid curriculum in her class. Further, I was also there to help her with the initial start up and address problems that arose. The first stage was crucial in Linda choosing the CMP curriculum and as a result determined the path that our partnership took over time.

Another important factor for the progress in our partnership was the collegiality in our relationship facilitated by effective management of the dialectic tensions (Rawlins, 1992). Tensions of acceptance/judgment arose in our discussions of the mathematics content may have been perceived by Linda as an evaluation of her mathematical knowledge and there was potential for our partnership to stall. Later in the study Linda mentioned that she was reluctant to talk too much in the classroom as she was afraid of making an error in the mathematics in front of me as she perceived me as the 'expert'. In our interactions around the content, I made an effort not to appear to be judging Linda's mathematics knowledge. As a result, I would provide a lot of guidance if Linda expressed difficulty with the content. In the last stage, with our improved understanding of each other, I waited longer for Linda to come up with solutions independently. As our discussions shifted to teaching and the student thinking, Linda gained more confidence as she could contribute a lot to the discussions. Two factors that could have contributed to building trust and acceptance in Linda were (a) acknowledging errors that I made in the mathematics content; and (b) recognizing that she knew more about her classroom and students. I encouraged her to share her knowledge of the classroom and students as part of the discussions and explicitly recognized her contribution.

Linda and I entered into the partnership by choice and there was freedom of independence but also a commitment to some level of dependence on each other. Linda felt free to cancel our session if she had other work that needed attention and I accepted these instances as part of the study. This went a long way in establishing a trusting relationship and managing the dependence/independence tensions. I was there to support Linda when she wanted suggestions or asked me to comment on activities/problems that she chose for the class. I could also depend on Linda for allowing me to work with her and she was also open to implementing our discussed ideas in the classroom. Thus we could both express our dependence and independence in this partnership.

The dialectical tension of affection/instrumentality features prominently in general in the work between teachers and researchers, as there is a tendency to view the teacher as being useful in getting the research done. In our partnership, Linda asked me to outline the benefits that she would get in working with me and I gave two namely, learning more mathematics and supporting her in classroom. These were the conditions on which the study started and trust was built by adhering to these conditions during the course of the study. In thinking about this dialectic, I questioned myself after the appearance of the content-teaching tensions in the second stage and decided to deviate from my research agenda in an attempt to resolve these tensions. Thus Linda took control in the final stage and our collaboration grew. Although it was difficult to initially resolve stresses about this particular research model; this partnership allowed me to take risks and support our relationship professionally.

The dialectic of expressiveness/protectiveness played out with more protectiveness in the first stage and more expressiveness in the last stage as we established trust. In the first stage it was better to exercise restraint with respect to Linda's mathematical knowledge so that there was a degree of trust established before I could be expressive about errors that she made. The wait time, that I allowed for Linda to solve a problem independently in the after class discussions, increased as the study progressed. Later in the partnership I could be more candid in my suggestions about the mathematical issues and her teaching in the classroom.

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Conclusions

The partnership highlights the importance that a reform curriculum or new innovation could play in mediating the relationship between a teacher and researcher. Balancing the dialectic tensions (Rawlins, 1992) contributes to relationship building and provides the language for expanding the construct of collegiality in collaborative research. In this study the teacher chose to adopt the reform curriculum over the traditional curriculum and was the only teacher to do so in the school. This has potential for further study as she influences the other teachers and could provide the blue print for ongoing professional development.

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AUTHENTIC CONTEXTS FOR MATHEMATICS LEARNING (1)

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In this work we examined the ecology of classroom interactions in the presence of case based tasks between two cohorts of prospective secondary mathematics teachers enrolled in a college geometry course. Using a teaching experiment methodology, and two case based tasks as research instruments, we traced the typology of the teachers' discourse in two different courses and at two different academic semesters. Our data suggests that cognitive modeling by the facilitator is a crucial ingredient in focusing teachers' analysis of mathematics.

The idea of grounding the education of teachers in the study of contexts from classroom events as a means to engage them in subject matter and learning has become a popular trend in mathematics teacher preparation (Merseth, 2003). Despite widespread endorsement of this approach by educators, there is little knowledge about what educators might expect in the way of prospective teacher learning or conditions that the use of case based tasks might impose on classroom interactions and instruction. In this work we examined the relationship between case based tasks and mathematical activities of two cohorts of prospective secondary mathematics teachers. Our data collection and analysis focused on one broad question:

> *What impact do the case based tasks have on classroom interactions when used as instructional tools in a content course required for prospective secondary mathematics teachers?*

Theoretical Considerations

Our work is grounded in a situated cognition approach to learning (Lave & Wenger, 1999), which assumes knowing, understanding, and thinking happens in socio-cultural contexts and as individuals work on real problems that provide an intellectual space for learning and inquiry (Rogoff, 1990). To situate learning about teaching means creating environments in which the teachers will experience the complexities of teaching by analyzing vignettes from real classroom events (Merseth 2003), or examining teaching actions and decision making in the presence of learners' cognition and thinking (Lampert, 2001). In doing so, teachers create knowledge out of the raw materials of practice (Ball & Bass, 2003) while reflecting on teaching actions. Such experiences enable the teachers to move beyond book knowledge to develop the more sophisticated practical knowledge that plays such an important role in the teaching process. Kirshner and Whitson (1997) highlighted the value of situating the learning of teachers in the world of the classroom as a means of creating a rich environment for professional discourse. The authors asserted that case based learning provides specific benefits: the opportunity to step back from the crush of the classroom experience; the ability to provide a group of teachers with one common experience which can be explored from multiple perspectives; and the ability to provide a robust learning environment that can be explored in nonlinear ways. In this endeavor the tasks used play a major and crucial role in the environment as they provide the medium for development of learning. Scholars have argued that considering the host of issues that might be addressed in

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case based tasks their utility in mathematics teacher education need to be explored (Merseth, 2003).

Methodology

Using a teaching experiment methodology (Cobb & Steffe, 1983), data were collected over the course of two academic semesters in two different sections of a course titled Modern Geometry, an advanced mathematics course. Each teaching experiment lasted 4 weeks of instruction (6, 75 minutes long sessions). Two case studies were used in each of the teaching experiments. Discussion of each case took three class periods. The order in which the cases were used was deliberately changed to trace any changes in the participants' behaviors or their interactions with one another that could be attributed to the particular mathematical concepts embedded in the task. In each class, two video cameras were set to capture both large group interactions as well as two targeted small groups. The small groups were selected randomly and the same groups were videotaped throughout the teaching experiment.

In examining the participants' activities and interactions, we considered three specific aspects of classroom life. These included: (1) Group interactions-- Ways in which the participants interacted with each other; (2) Participants' interactions with the task--Issues that the participants raised about and/or extracted from the activity; (3) Facilitator's (Teacher-researcher's) interactions with the participants—modes of intervention and participation required for facilitating group learning. At the first cycle of analysis we looked for patterns and themes that emerged from each of the teaching experiments first. We then combined the themes and coded the videotapes again to identify the frequency of occurrence of each target behavior, activity, and mode of interaction on the part of teachers and the facilitator.

Results

Analysis of data revealed significant changes in the participants' foci of analysis, modes of peer interactions, and depth and extent of their mathematical inquiry as they moved from the first case analysis episode to the second during each teaching experiment. Changes also occurred in the nature of the facilitator's interventions in class and the roles she had to adopt as she interacted with individuals in both small and large groups. Tables IA and IB summarize the typology of the participants' comments during the case study analysis sessions.

Table IA: Typology of Teachers' Comments

		Case I	Case II
Eliciting Guidance	Statement of need for additional guidance on pedagogical decision	M=5 SD=4.6	M=32 SD=4.2
	Statement of need for additional information on curriculum	M=12 SD=4.2	M=25 SD=0.04
	Statement of need for additional guidance on mathematics	M=9 SD=1.08	M=89 SD=7.09
	Statement of need for additional information on learners and how they learn	M=7 SD=1.2	M=33 SD=2.4
Declarative Statements	Statements of concern about the ability	M=0	M=12 SD=1.6
	Statements of concern about the ability to make sense of children's work	M=0	M=18 SD=.54
	Statements of concern for knowledge about appropriate decision making	M=0	M=36 SD=2.8
	Statements of concern about finding appropriate resources	M=0	M=28 SD=1.88
	Statement of concern about the quality of their teacher training	M=5 SD=.98	M=29 SD=1.8

Table IB: Typology of Teachers' Comments

	Pedagogical				*Mathematical*		
	Hypothesizing about appropriate instructional strategies (without referencing mathematics)	Analysis of the impact of instruction on children's work (no references to mathematics)	Description of evidence used to support decision-making	Analysis of impact of children's work on instruction: Focus on methodology	References to mathematical theory in analyzing children's work	Analysis of the impact of children's mathematical work on instruction: Focus on mathematical	References to connections among mathematical solutions or ideas
Case I	M=67 SD=10.1	M=43 SD=8.09	M=0	M=21 SD=4.8	M=2 SD=.1	M=8 SD=0.9	M=0
Case II	M=14 SD=2.3	M=12 SD=1.4	M=51 SD=4.8	M=14 SD=2.6	M=87 SD=6.7	M=45 SD=4.4	M=97 SD=10.07

During the first case analysis experience, the participants tended to show less interest in mathematical problem solving. They made quick suggestions about what a teacher might do in class without referencing mathematics or the mathematical work of the children. Initiatives to justify their suggestions regarding pedagogy or to debate opposing viewpoints offered by different group members were rare. The activities of the facilitator during this episode consisted of confronting participants' ideas, and setting a social context for peer deliberations and mathematical inquiry. In doing so, she challenged the participants' assessment of children's work, and asked that they must justify their suggested pedagogical moves by grounding their ideas in mathematics. The need for modeling effective modes of analyzing a case became vital. Hence, the facilitator spent approximately one class period modeling ways in which children' solutions could be unpacked, linked to one another and connected to a mathematical structure. She supported her conjectures about ways in which a teacher could organize her instruction with a rationale grounded in theory as well as the data illustrated in the case.

During the second case analysis session, the participants attempted to use the thinking process modeled by the facilitator in the first case study session. They explained their reasoning, tried to reach consensus on their analysis and asked probing questions from each other and the facilitator regarding relevant mathematical structures and pedagogical theories. The participants demonstrated the tendency to focus on making sense of children's solutions and to construct justifications for their decisions. They exchanged ideas about how a solution might be interpreted, what experiences could have contributed to the constitution of the solution method, and what additional questions the teacher could ask to guide the thinking of individuals. The facilitators' comments during the second case analysis episode were of an instructive nature. She shared mathematical information, explained connections among the children's ideas and provided mathematical detail, per participants' request. Additionally, she offered suggestions for how different ideas could be used to generate additional problems for children's inquiry.

Case I

During the first case analysis, the participants' interactions with the task followed three cycles: (1) Pedagogical theorizing; (2) structured problem solving, (3) reflecting. At the first cycle, the participants entered the discussions focused on sharing ideas about how a teacher might proceed with the lesson in each respective case without referencing mathematics. Motivated by the facilitator's request, in the second cycle the participants began to unpack the mathematical content of the case. In Cycle 3 the participants' primary focus was on articulating concerns about their own mathematical knowledge and ways in which their own knowledge could impact their work in the classroom.

Interactions with the Task and Peers

In both sections, during the first case analysis session the participants entered the discussions with a focus on pedagogical theorizing. They shared ideas about what the teacher must do in class independent of a deep analysis of the solutions children had presented. While we had anticipated that the participants might argue that there was insufficient data on children to make a reasonable recommendation for teaching; they only demanded background information on the textbook the teacher used, and the mathematical topics she would have to cover in class next. Indeed, during this phase the participants seemed reluctant to engage in discussions about mathematics. They made brief and trivial references to children's solutions and characterized them as right or wrong. Their assessment of the merit of children's work was grounded in the mathematics they recognized or could easily make sense of. This appeared to be an acceptable practice since none of the participants publicly objected to what was shared in each group. While the participants made suggestions about how the teacher might proceed with her lesson, their suggestions were not responsive to children's work or draw from them. In hypothesizing about how a teacher might proceed with a lesson, participants rarely made any references to mathematics. Plans for highlighting the links among different solutions and/or connecting various solutions that children had offered were also absent. Neither one of the teachers tried to formalize or justify their assessment beyond stating their personal preferences regarding the classroom environment they felt were conducive to building children's confidence.

The participants also showed the tendency to be non-critical in their interactions with one another. Public attempts at challenging peers' assessments or pedagogical solutions they offered were infrequent. While during the small group work the participants spent time sharing stories about teaching, and multiple challenges they anticipated in accommodating the diverse backgrounds that the students might bring to a learning environment, their dialogue did not reflect a desire to seek solutions to such problems. The facilitators' comments during this phase were aimed at structuring both mathematical activities and the group interactions by confronting the participants' assessment of children's work and the suggestions they offered. In doing so, she attempted to channel individual and group interactions.

Adopting the role of a modeler, and choosing one of the methods which the participants had labeled as "incorrect" as an example, the facilitator spent approximately 40 minutes of the second class session outlining her interpretation of one child's method, and ways in which it connected to different mathematical concepts. She identified the utility of the same method when solving specific types of mathematical problems, and commented on additional questions the teacher could ask the child to advance his thinking. As such, she structured group interactions and coached the group towards a deeper level of analysis.

Following the modeling episode the participants examined the case again in small collaborative groups with an attempt to find connections among different ideas that children had posed. A structure for group deliberation was also set to reach consensus on their assessment of children's work as well as approaches to pedagogy. Group deliberations became more visible during this cycle of the participants' interactions with the task and as they were faced with the challenge of justifying their choice of pedagogical solution. The participants' interactions with the facilitator became paramount at this phase as group members requested her to validate accuracy of their ideas and to provide explanation on children's work.

During the large group discussion the participants shared their ideas, while the facilitator helped formalize and synthesize both mathematical and pedagogical aspects of the task. A major component of the participants' discourse in this phase included articulation of concerns about their own knowledge of mathematics. Comments reflecting self concerns and doubts about ability to teach, including, "How can I teach this stuff to kids, when I have so much trouble doing it myself?" and "I am not sure I could do this without help," were expressed in each class.

Case II

During the second case analysis, the participants' interactions with the task followed three cycles: (1) mathematical problem solving; (2) mathematical problem posing and problem defining, (3) pedagogical inquiry. At the first cycle of interaction, the participants entered the discussions focused on mathematical sense making. They shared their interpretation of what each child had suggested in each respective case and asked for alternative ways in which a solution could be assessed. The second cycle included an extended problem posing session, during which the participants posed and examined mathematical extensions that could be used based on each child's strategy. During the third Cycle the participants seemed focused on learning about school curriculum and learning theories that accounted for how individuals learned mathematics. They eliciting information on ways to enhance their own teaching knowledge and expressed an interest in learning the subject matter in ways more compatible with methods they had experienced during the teaching experiment.

Interactions with the Task and Peers

During the second case analysis, the participants entered the classroom discussions focused on trying to understand children's mathematical work. The group activities included scaffolding what each child had suggested, what each solution meant mathematically and revealed about the child's knowledge and what he knew. This discussion took the entire class session. The second class session in each teaching experiment was devoted to an extended discussion about connections among various solutions children had posed. The participants articulated ways in which the teacher could extend children's thinking by her choice of extensions and through problem posing. Participants' debates focused on defining the major instructional dilemmas the teacher faced and appropriateness of extensions different individuals presented. Questions and discussions at this phase concerned how the curriculum could be sequenced so to optimize learning of all children. During this phase the participants frequently asked questions that manifested their tentative status of knowing and desire to learn, including: "Is this going to work?" "Would it be better to tell them or to let them explore first?" The participants asked that the facilitator must comment on their analysis and to offer guidance on the accuracy of their analysis. They elicited information on whether questions they had suggested were developmentally appropriate.

During the second case analysis episode the participants objected to arguments, considered more cases, and made and requested more references to mathematical theory as they debated connections among various solutions depicted in the case. They also elicited alternative interpretations of the of children's work they had examined. These episodes not only evidenced a deeper mathematical analysis on their part but also their desire to engage in problem solving. The facilitator's prominent role during the second case study analysis session was one of an information provider and explainer. The participants requested that she provided insight solutions they had difficulty understanding. Questions concerning ways in which growth in mathematical thinking of children could be measured were also asked.

Discussion

In this work we had set out to study the utility of case based tasks for mathematical learning of secondary teachers when used in a content course designed for teachers. Our data indicate that the prospective teachers' desire to engage in mathematical problem solving was motivated by the participants' concern for learning about teaching and improving their own ability to accommodate the mathematical needs of the children. Indeed, the intensity of peer interactions, collaborative decision making, and joint problem solving increased as the participants became familiar with both the instructional expectations and the parameters of the tasks. Both the quality and quantity of peer interactions enriched the participants' mathematical work, inquiry, and pedagogical problem solving. This development permitted the facilitator to focus her attention on modeling and guiding group inquiry as opposed to motivating participation and building group structure.

Our findings suggest that case based tasks can enhance content preparation of mathematics teachers given adequate modeling is provided for them in instruction. This is a particularly important issue since despite wide endorsement of the use of case based tasks in mathematics teacher preparation little is shared relative to ways in which cognitive apprenticeship might be utilized in instruction when implementing such task. There is tremendous need to deepen understanding of the role of facilitator in such as setting so to assure the integrity of both mathematics and pedagogical inquiry is maintained in the process.

Endnotes

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CONTEXT-BASED PROFESSIONAL DEVELOPMENT IN MATHEMATICS EDUCATION: ANALYSIS OF A COLLABORATIVE RESEARCH PROCESS AIMED AT SUPPORTING IT

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The analysis of a two-year collaborative research project involving elementary and high school teachers and mathematics consultants is presented. We analyzed data from the regular meetings between researcher and practitioners, group interviews and individual self–reports in order to characterize the collaborative research process supporting professional development of practitioners. Such a process, strongly context based, sheds light on a new way of approaching continuing education for mathematics teachers.

This project is part of a research program focusing on professional development of teachers involved in participatory research studies. Data collected at two different sites for collaborative and action research projects involving school personnel have been analyzed with regard to various components of professional development, including learning taking place, changes made to professional practices and development of collective competencies promoted by the research (Savoie-Zajc & Bednarz, in press). In this paper, we will focus on the analysis of the process supporting the professional development of practitioners at the collaborative research site. The emerging analysis of the underlying approach developed together with practitioners helps to define a certain concept of context-based professional development for teachers.

Professional Development Issues in Mathematics Education

Many studies in mathematics education show the limits of top-down models for continuing education (Bentley, 1998; Bednarz, 2000; Bauersfeld, 1994)) and point out the necessity of developing different approaches. Such alternative approaches have to focus more on teachers'practice, so as to explore and better understand mathematics teaching and learning in context (Lave, 1996, Ball & Cohen; 1999, Ball, 2000, 2002, Ball & Bass, 2003) and to take account of the complexity of mathematics education practice (Bednarz, 2000; Bednarz, Perrin, &Glorian, 2005). From that point of view, continuing education approaches are strongly grounded in context, focusing on professional knowledge developed by practitioners in a situated practice (Lave, 1988). It is from such practice-related knowledge that new knowledge in mathematics teaching, embodied in action, could emerge. Among these alternative approaches, collaborative research opens the way by taking teachers' professional practice into consideration in the professional development process and in building new knowledge linked to this practice. Indeed, collaborative research addresses the issue of the relationship between professional practice (in this case, the practice of teachers who teach mathematics) and the research in a specific area (in this case, research on mathematics teaching), referring to the reciprocal enlightenment that each is able to provide the other (Schön, 1983, 1987; Curry & Wergin, 1993). This new way to approach research provides the basis for our collaborative model (Desgagné et al., 2001).

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Theoretical Framework: Our Model of Collaborative Research and Its Foundations

For researchers working in collaborative research, the construction of teaching situations (developing problems, activities, classroom interventions, etc.) inevitably proceeds through an understanding of the practice within which it develops. Since such understanding is considered as shaping the construction of these situations, collaborative research is founded on the idea that the practitioners are essential to the process of producing knowledge. This perspective on the role of practitioners brings into play a teaching actor who is assumed to be reflective and knowledgeable, as well as a competent, situated social actor (Giddens, 1987), that is, a source of professional ingenuity (Dubet, 1994). So the idea of collaborative research is based on the idea of knowledge to be constructed, related to practice, taking into account not only researchers but also teachers' experiences and knowledge. The reflexive activity at the heart of the model has a dual function. It is an opportunity for professional development through reflection, with the objectives of clarifying, making explicit, improving and understanding practice, and ultimately contributing to its restructuring. It is also a research opportunity as the zone of interaction between researchers and practitioners provides data helping to analyze a specific object of interest regarding practice-related knowledge. This reflexive activity entails a process of co-construction. In the reciprocal negotiation between teachers and researchers, Davidson, Wasser, & Bresler (1996) speak of creating a shared interpretive zone), a series of arguments are developed regarding the ways the various parties ascribe meaning to an object of common interest. Collaborative research also entails bringing together what Lave and Wenger (1991) call communities of practice, within which these researchers and practitioners operate. To be a researcher or a practitioner is to belong to a group engaged in shared practices whose members (in the ethnomethodological sense) construct between themselves ways of acting and thinking about their everyday affairs according to their own constraints and resources. As such, collaborative research advocates the bringing together of two worlds and a cross-fertilization between them, in order to construct knowledge resulting in an informed, even enlightened practice and perhaps the emergence of a new community.

The Collaborative Research Context

As researchers in mathematics education, we were interested in the very challenging gap that occurs between elementary to high school, in relation to the specific subject matter content taught at the two levels, and the ways it is approached at each level. A group of ten participants from the same school board were invited to join a collaborative research project focusing on the transition from elementary to high school mathematics. The group was composed of eight teachers (three from high school and five elementary teachers), and two mathematics consultants (one for each school level). The research project ran from August 2004 to June 2006. Building on their practice, their observations and their concerns about students' math difficulties, the participants focused on topics they wanted to work on to better understand these difficulties and uncover possible strategies to make the transition from elementary to secondary math easier for students. At this exploratory stage, participants defined an object of investigation occurring at the point where a research question, such as the teaching issues surrounding the elementary-secondary transition in mathematics, intersects with a practice-based question originating in practitioners' observations and concerns related to their practice. This co-situation was the basis for a research project focusing on two specific issues at the heart of this transition: arithmetic skills and problem solving.

Methods of Data Collection and the Framework for Analysis

Practically speaking, the research was shaped through regular meetings between the researcher and practitioners, meetings that enable the creation of an interpretive zone around the practice being investigated (arithmetic skills and problem solving approaches in the transition from elementary to high school). The joint process took place for two years and entailed a one-day reflection every month, conducted so as to encourage continuous dialogue between classroom experience (on problem solving or arithmetic skills) and a review of participants' experiences. Teamwork revolved around the participants' accounts at each level of their in-class activities, the questions they asked, their observations and records of statements from the students etc. This review of experimentations served as a starting point for reflection. All these meetings were recorded and transcribed. We conducted a qualitative analysis of the interactions between the researcher and the teachers during these regular meetings in order to characterize the process of collaborative research in which the professional development of teachers took place. Group interviews and written self-reports, conducted on two occasions (at the end of the first and second years), where teachers and mathematics consultants reflected on the meaning they gave to their experience, were also used, for a cross analysis of the process (a triangulation with different data sources).

The collaborative research approach is framed by what we could name an "explicit contract" which in this case consists in working together with practitioners on an issue related to the transition from elementary to high school mathematics, particularly on problem solving and arithmetic skills. Nevertheless, the way in which this issue is investigated in situ falls within the implicit. Our analysis aims at grasping that implicit in order to better understand and eventually characterize the teachers' professional development approach set up in the collaborative research.. The theoretical concept of "collaborative contract," inspired by the concepts of "didactical contract" (Brousseau, 1988; Schubauer-Leoni, 1988), "communicational contract" (Giglione, 1987) and "pedagogical contract" (Filloux, 1974) was used in this perspective to shed light on the way *interactions are implicitly regulated* during the process..

Results: The Reflexive Contract at the Heart of the Collaborative Research Approach

A qualitative analysis by inductive coding (Strauss and Corbin, 1990) of meetings' transcripts led us to identify a number of themes that actually characterize the regulation that took place in the collaboration group. In what follows we present in detail one of the theme which resulted from the analysis so as to help understand how we proceeded.

An Example of Group Regulation: "Justifying One's Point of View" Theme

This vignette is extracted from the transcript of the forth meeting (which occurred in January 2005). In it, practitioners and researcher have a discussion on students' strategies when solving the problems proposed to them at both levels. We start with L. (a high school teacher) who comments the strategies used by some of his students on the following problem: "At the end of the hunting season, we ask Nemrod how many hares and pheasants he has shot. Teasing us, he answers that he got 19 heads and 54 legs. Can you tell how many hares and pheasants he has killed?". Since one student choose, as a strategy, to draw legs and heads and connect the whole in order to find the answer, the interactions goes as this:

- L: Drawing, you know, is a strategy that may lead to a good answer, but it is not the one we will favor in high school…

- R (researcher): That is why I've asked you about it, what will you be doing with such strategies?
- L: In an exam, I would be inclined to, yes, I mean he has a correct answer, but the strategy… (implying that it might not be the best one)
- J (an elementary teacher): His strategy wouldn't be the good one?
- L (obviously embarrassed): If I… For myself, I do not like drawings. I'd prefer that they use a mathematical calculation.
- R: Why don't you like it?
- L: Well… When we have… now that there are 19 heads and 54 legs… But if we had 250 legs… you can't do it anymore.
- S (another elementary teacher): That is the question I wanted to ask all of you. At high school level, you do expect students to use calculation, that 's what I was telling to my students, I gave them examples like yours: if you have a thousand, what will you do? A time comes when you have to depart from drawings.
- R: I agree with you [to S. and L.] that when you have big numbers, you need to drop the drawings. But in order to understand the problem, I have noticed that when I tried that problem with students, they had more control over it when they used drawings than when they just work with calculations. They have more control over the structure of the problem, on how it works. I understand that you want to move to calculations, but for the understanding of the problem, the drawing is interesting.
- S: As for me, when they use drawings, I always ask them to also do a calculation, because I find that when they explore the problem by means of drawings, most of them can solve it successfully… but a time comes, I do agree with L., you are in grade 9, 10, 11, they are about to quit school and get a job, you are no longer making drawings.
- J: Unless it is a problem of that kind [showing another complex problem].
- R: When it is a complex problem, yes, I do draw to understand what it is about.

[and further in the transcript]

L (referring to another problem, based on logic): There you needed to draw. If you did not draw, you could not find the answer.

In this discussion, participants address one of the key issues of elementary to high school transition in problem solving: the legitimacy of strategies used by children at elementary school and accepted as such by teachers, and which are no longer valid at high school level. In that respect, arises some sort of break between the two levels in the way of approaching problem solving in classroom. The use of drawings, not favored at high school (according to L.), almost leads to a clash in the group during the interactions, when J.'s reaction ("His strategy wouldn't be the good one?") clearly embarrassed L.

In this case, the way in which the researcher and the group thereafter regulate the interactions do inform us about an implicit contract. In order to allow a construction around that issue, the researcher encourages from start the expression of L.'divergent perspective ("That is why I've asked you about it"). By so doing, she then allows the discussion to go on, asking participants to justify their points of view ("Why don't you like it?") with the purpose of opening up the dialogue rather than closing it in letting participants only state cut-and-dried opinions. The conversation then leads to an analysis that permits the co-construction on that issue: the limits of the drawing strategy (when the size of numbers varies), its interests (as far as understanding the problem is concerned) and relative character according to the problem to be solved (for some problems drawing is essential, and for complex problems drawing may be helpful).

A first theme comes out then of the analysis of that episode (that theme will be afterwards confirmed by the analysis of other episodes extracted from other meetings' transcripts). We labeled it "Justifying one's point of view, while interacting, about the validity of a certain way of approaching mathematics teaching" (number 2 in Figure 1). By this, we see that in collaborative research (that's what came out of a posteriori analysis of meetings' transcripts), one can't lay on opinion to construct knowledge: one has to refrain from expressing unwarranted statements, even when only reporting one's way of doing in the classroom. Cut-and-dried opinions, unwarranted judgments, especially over someone else's practice, must be avoided. Participants are invited to adopt an analytic posture, trying to explain, analyze, comprehend and give meaning to what is contributed in the group. Participating in this collaborating research thus means to go deep into the understanding of what is been put forward in interactions, of the rationality that underlies it. Constitutive elements of the reflexive contract associated with that theme can be designated as "Argue on what is proposed to the group" and "Try to explain and understand what underlies." A similar analysis of other vignettes led us to throw light on the implicit contract that characterizes the collaborative research approach involving a researcher, teachers and school advisor.

Eleven Themes That Characterize Our Collaborative Contract

Eleven themes (see Figure 1) emerged from this qualitative analysis of the meetings transcripts, illustrated by different vignettes. These themes, and their components (not specified in the schema, space being reduced), *allow us to characterize the reflexive contract, according to two dimensions*: *contributing to the development of action-knowledge in mathematics teaching around a common issue* (the development of arithmetic skills and problem solving in the transition between elementary and high school in mathematics); *and maintaining a dialectic between action and reflection in this process.*

The reflexive contract which emerged from this analysis can be explained in terms of the process of co-construction of a certain action knowledge in mathematics teaching, *a co-construction approached in a manner we describe as "complexification".*

Co-construction, a central idea in collaborative research, implies the existence of *common achievements* (the development of teaching situations related to problem solving or arithmetic skills so as to support for students a better transition from elementary to high school mathematics), but these common achievements are rooted in participants' individual experiences, as shown by the analysis of the contract (see for example themes 3, 6, 5, 1, 2). Thus, the idea of co-construction highlights individual and collective dimensions of the undertaking. *One can thus think that the participants develop, in a contextualized way,, competences related to the functioning of a co-construction group.*

The knowledge co-constructed refers to an action-knowledge in mathematics teaching, the development of situations, interventions strategies, ways of acting in classroom, of approaching problem solving, of developing arithmetic skills, of looking at students' productions. *Finally, this way of approaching the co-construction is characterized in terms of complexification of this teaching knowledge* in mathematics throughout the collaboration process (see for example themes 7, 9, 11, 8, 4). From inside the reflexive contract, complexification thus implies to reflect on mathematics teaching and learning by adopting an analytic viewpoint, try to understand underlying assumptions, foresee various possible choices, open up to other possibilities, moderate, refine, restructure *building upon* one's own practice, and not an entrenchment on unique and fixed choices.

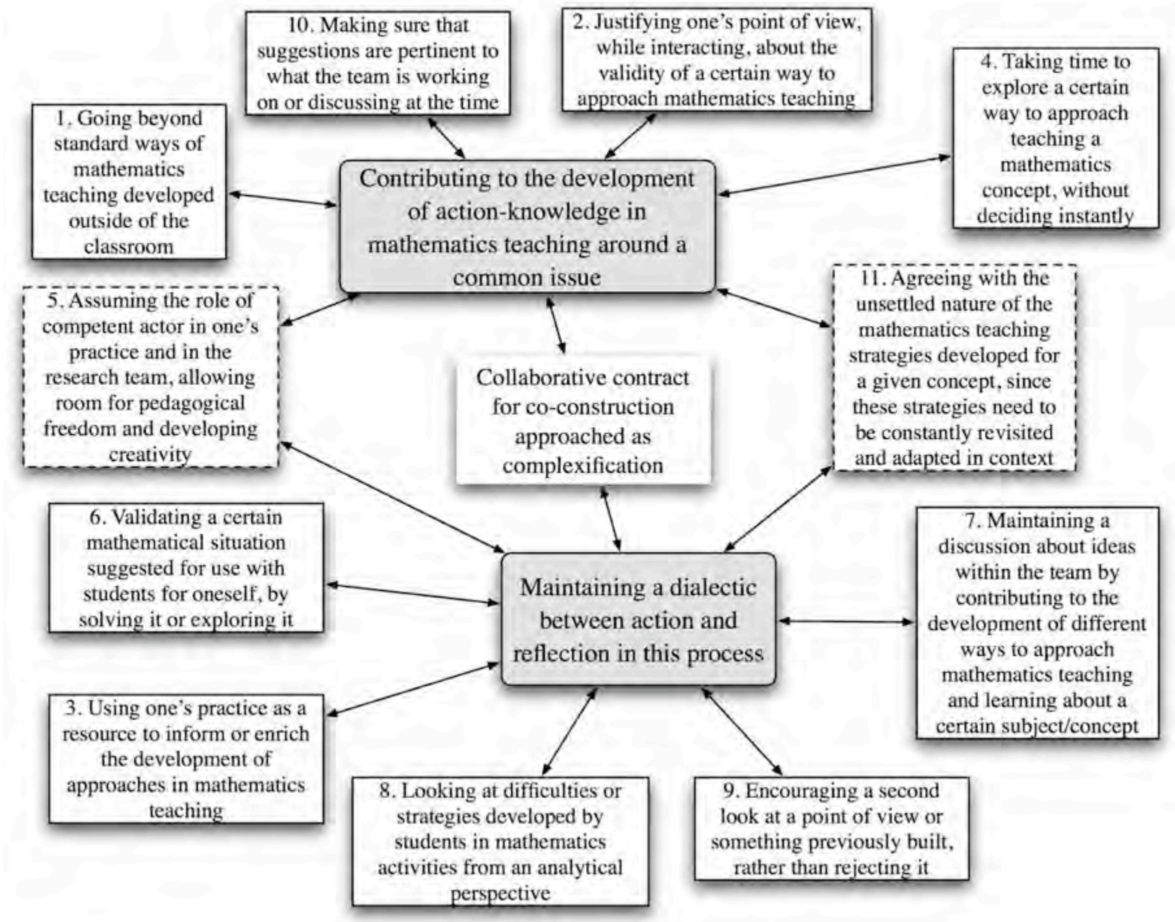

Figure 1. Reflexive contract in the collaborative research approach

Conclusion

The analysis of the interactions shows that teachers and mathematics consultants involved in this process develop professional competencies related to the functioning of a co-construction group and to a certain way of approaching this co-construction, enlightened by the different components of the implicit contract. This process of complexification was confirmed by the analysis of group interviews and self reports (when the participants spoke of the approach, the way they lived it, the role of the researcher or their role). The use of metaphors by which practitioners illustrate how they view their participation in the group is in that respect meaningful. For example, a participant talked about a trip over different countries (a focus on an opening up to contributions from various points of view) while another suggested the metaphor of waves in a swimming pool (to point out the complexity of teaching which can't be"solidified" but require to be constantly questioned, revisited). As our analysis of those late data as well as the continuation thereafter of our research project show, that complexification leads to a broadening of the practitioners' "world of possible" in their interventions at different levels, in their class, in their vision of the other level, but also in their broader intervention in the educative community.

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DOING MATHEMATICS IN PROFESSIONAL DEVELOPMENT: THEORIZING TEACHER LEARNING WITH AND THROUGH SOCIOMATHEMATICAL NORMS

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This research report discusses the work guiding Researching Mathematics Leader Learning (RMLL) (1), a 5-year research project intended to investigate mathematics PD leaders' understandings and practices associated with developing mathematically rich learning environments. In this report, we will focus on how we are elaborating the construct of sociomathematical norms with leaders as a way to understand the workings of a mathematically rich environment. Distinguishing social norms from SM norms is an important dimension of our work with leaders. We have learned to attend to the language that we use carefully and to pay attention to how the leaders themselves engage in mathematics in order to understand how they make sense of the construct of SM norms.

Few take issue with the fact that high quality teachers make a difference in children's learning, so it stands to reason that having high quality professional educators (i.e., leaders) is important for teachers' learning. How we engage in mathematical work with teachers in professional development (PD) provides significant models for teachers for what it means to engage in mathematical reasoning. The PD context is one place where teachers have opportunities to learn what it's like to develop mathematical habits of mind such as generalizing, proving, engaging in argumentation, and connecting representations to their symbolic equivalents. Yet there is a dearth of research on how we structure and lead mathematical work with teachers (Even et al., 2003; Stein et al, 1999). And we have much to learn about how engagement with mathematical reasoning in PD helps teachers learn how to create such environments for their students and to think about the mathematical work that gets accomplished in the classroom. Cohen & Ball (1999) note that, "ironically, while the role of the teacher educator is critical to any effort to change the landscape of professional development… there is little professional development for professional developers" (p. 26). In a study of the California Mathematics Projects, Wilson and Berne's (1999) review suggested that the mathematics often gets negotiated away in professional development. This may be a direct result of the lack of PD for leaders.

This research report discusses the work guiding Researching Mathematics Leader Learning (RMLL) (1), a 5-year research project intended to investigate mathematics PD leaders' understandings and practices associated with developing mathematically rich learning environments. In this report, we will focus on how we are elaborating the construct of *sociomathematical norms* with leaders as a way to understand the workings of a mathematically rich environment. (In the context of our work, we use the term PD leaders because they are often teachers who also do professional development work rather than district level professional developers or university personnel.)

Sociomathematical Norms in Professional Development

Sociomathematical (SM) norms are the specific ways students (translated to PD—teacher participants) engage in mathematical work in the classroom (or PD). These norms govern the ways people engage with mathematics (Yackel & Cobb, 1996). SM norms include things like what counts as an acceptable mathematical explanation or what constitutes a mathematical justification. These norms are negotiated either explicitly or implicitly by teachers and students (in PD, leaders and participants) through social interactions. To build SM norms more likely to promote mathematical understanding, researchers suggest that teachers need to pay attention to the mathematical work of their students (Kazemi & Franke, 2003; Sherin, 2001). Similarly, our focus is on how leaders learn to pay attention to participants' mathematical thinking and how their learning connects to fostering productive SM norms in mathematics PD.

Exploring mathematics with groups of people is inherently a cultural practice. As communities of people work together over time, normative behaviors and practices are developed. Several factors shape these practices—the students, the specific contexts, the mathematical content, as well as what is valued and defined as competent participation in mathematics class (Lampert, 2001). The same must also be true for PD practices. Wilson and Berne (1999) and Lord (1994) suggest that what is typically defined as competent participation in PD is engaging colleagues in social pleasantries and avoiding dialogue that may prove uncomfortable for participants. Moreover, Remillard and Rickard (2001) suggested that teachers inquiring into practice and digging into mathematical ideas were not typical norms in the PD seminars they facilitated unless teachers were provided support and scaffolding. While social norms, such as being polite and considerate of colleagues, are often the focus of teachers and PD facilitators, they do not expressly support the deepening of mathematical knowledge.

The negotiation of SM norms is part of enculturating participants into particular ways of reasoning in mathematics. These ways of doing mathematics may, or may not, be compatible with participants' ways of knowing and being in the world. As a result, facilitators must be cognizant of the tensions that may exist between the cultural ways of engaging in the world that participants bring to learning opportunities and the practices practitioners and researchers believe are important for fostering deep understandings of mathematics (Forman, 2004; Ladson-Billings, 1995). An additional issue at play here is that leaders' capacity to cultivate deep understanding has everything to do with their ability to know what and how to press for mathematical understanding. Moreover, as leaders work with colleagues in PD, tension is likely to arise as they focus on teachers' mathematical understandings and uncover teachers' mathematical confusion. Leaders are also likely to grapple with navigating the use of errors, negotiating participants' social and intellectual status and connecting mathematical work done in the PD context to work that teachers do with their students.

Method

Seminars for Leaders

To provide some context for our research, we briefly summarize the PD for leaders. They participated in a series of seminars to focus their attention on SM norms and the cultivation of mathematically rich environments. The seminars were organized around a set of videocases showing facilitators leading mathematical activities with teachers. Each seminar included the following set of activities:

- Working on mathematical tasks with consideration of the mathematics in the task and how teachers might approach it
- Viewing videos of professional development involving teacher discussion of the mathematical task
- Discussing the videos to consider what mathematical explanations teachers are offering to one another and how the group engages in mathematical reasoning
- Engaging in a series of "Connecting to Practice" activities designed to help leaders apply some of the ideas in designing and facilitating mathematics professional development and to consider implications for their own work
- Reflecting through journal writing
- Doing homework activities involving examining the ideas from the seminars in leaders' own work or in their observations of other leaders.

Through these experiences, we attempted to make explicit both the mathematical content and the way teachers engage in doing mathematics. The mathematical content of the seminars emphasized algebraic ideas, particularly how to create arguments for the generalized case of a solution.

For this research report, we focus on one particular session in which leaders viewed and discussed a video from a professional development session called *Janice's Method.* In the video, Janice shares her way of solving 92-56. She solves the problem by subtracting 60 from 90 to get 30 and then adding 6 back on to get 36. As she explains how she knew to add 6 back on, she explains, "so 90 is two away from 92. So I needed to recover that two. And 56 is four away from 60, so I needed to recover my four." The facilitator in the video case presses Janice to explain what she means by "recovery" and a discussion unfolds in the group to discuss why Janice's method works and how it can be generalized to other subtraction situations as well as discussions about what subtraction means as an operation. This videocase is used because it is an opportunity to discuss the facilitator's role in opening up discussion to develop conceptual explanations.

Participants

Data from this report come from two different groups of participants. The first group was a highly experienced group of teacher leaders (n=11) with whom we piloted our materials (will be referred to as the pilot group). The pilot group was most experienced participating in and facilitating the *Developing Mathematical Ideas* (Schifter, Bastable, Russell, 1999) materials, typically with seven to nine years of experience. The second group was a less experienced group of teacher leaders (n=13) working in a district in the southwestern United States, typically with one to three years of experience facilitating PD (will be referred to as the SW group). The SW group had also participated in professional development often tied to the district math curriculum, assessment, and a range of other activities such as mathematics study groups and lesson study. The vast majority of leaders in both groups had experience teaching elementary school. A few had experience in middle or high school. The pilot group worked with the *Janice's Method* once (as the third in a sequence of four videocases over 2 months) while the SW group worked with it twice, once at the beginning and once at the end of their participation (in a sequence of eight videocases over the course of an academic year).

Data Sources and Analysis

We documented the leaders' experiences in the seminars by videotaping and transcribing each session with two video cameras, which enabled us to capture both whole group discussion and at least two small group discussions for each activity. All work generated by participants were kept in a journal and used for analysis. Participants also completed on-line surveys before and after the seminars to provide us with both demographic information and to reflect on their experiences. For the purposes of this report, we focused our analyses on one session, *Janice's Method*, common to both the pilot group and the SW group. Qualitative data analytic techniques were used to compare the perspectives and engagement of the two groups.

Findings

Our elaboration of SM norms in the PD context has led us to focus, as initial entry into these ideas, on the nature of explanations. Our deliberations about how to focus on adults' ways of engaging with mathematics have led us to a two-tiered approach. First, we pose questions that focus leaders on describing in detail (1) what mathematical explanations were offered in the group regardless of how and who did the explaining. With that discussion as a foundation, we look again at the case materials to deliberate (2) how the group being observed engaged in mathematical explanation. This second question then allows us to press leaders to pay attention to the nature of questioning, the treatment of errors and confusion, the locus and distribution of kinds of mathematical questioning, and the consequences of presenting particular explanations or raising particular questions to the groups' mathematical work (in other words, what mathematical work did the group accomplish). Our focus on the nature of explanations has led us to unpack four relevant practices: (1) sharing, (2) justifying, (3) responding to confusion and errors, and (4) questioning (see Table 1).

Distinguishing social norms from SM norms is an important dimension of our work with leaders. We have learned to attend to the language that we use carefully and to pay attention to how the leaders themselves engage in mathematics in order to understand how they make sense of the construct of SM norms. Our analyses thus far have led to the following two assertions.

1. *The pilot group's greater experience participating in and facilitating PD focused on mathematical reasoning and students' thinking was reflected in the way they engaged in the mathematical tasks themselves and how they analyzed the videocase with respect to SM norms.*

   There were important differences between the way the pilot group and the SW group engaged in the mathematical task itself during the leader seminar. In preparation for viewing the videocase of *Janice's Method*, both groups did the mental math task, 92-56 themselves.

After generating various methods for solving the problem, the leaders were asked to think about how the solutions worked and what connections they could see between them. Two things happened among the pilot group. One small group felt that they were so familiar or "saturated" with this math task that they could not engage in discussing and comparing the mathematical solutions. This seemed a potential hazard of doing mathematics that is quite familiar to leaders. Others, however, in the pilot group engaged with the mathematics for its own sake. They used the solutions generated by the group as a springboard for further mathematical inquiry. For example, they investigated how to model subtraction as an operation (as removal versus difference) on the number line and whether their approaches would work with decimals. In contrast, the SW group used their small group time to connect the solutions they shared as a group back to the work of classroom teachers or what students might do or struggle with. We considered why we might be seeing this difference between the pilot and SW group and conjecture that it might have to do with how leaders interpreted the intellectual work of doing mathematics in PD. To press for explanation and justification among colleagues may involve issues of risk (not wanting to convey any confusion or misunderstanding) or an expectation that everyone can naturally deduce the mathematical significance of solutions. As one participant explained to her small group, "When Cathy (Carroll) asked me how I did it, I wasn't expecting to explain how. I thought that once I explained what I did, I had already done it. I wasn't thinking like I would think [about] it when I was teaching. If I was teaching, I would have prepared for my explanation in some way." This same participant reflected later in the session noted that she has been tentative to push mathematical ideas when facilitating PD. We found that our explicit attention to how teachers engaged in mathematical reasoning collectively in the video allowed leaders to think about the mathematical purpose of highlighting, selecting, and sequencing particular solutions in a group discussion (Stein et al., 2006).

2. *There was growth in the SW leader group's attention to how participants' in the videocases engaged in mathematical discussions. However, learning to elaborate the ideas behind sociomathematical norms is not a trivial endeavor.*

We engaged in lengthy discussions in our research team about when and how to introduce the idea of SM norms. Although Cobb, Yackel and colleagues draw attention to things such as what counts as an acceptable or sufficient explanation, we discovered that even experienced leaders react to these terms in ways that might obscure their intent. For example, as we discovered in piloting, the words *acceptable* and *sufficient* led some leaders to focus on the absolute correctness or validity of solutions rather than on what becomes negotiated in a particular context. This may be due to the everyday understandings of the terms acceptable and sufficient.

The term *"norm"* turned leaders' attention to social norms that are general to any group that is trying to work together. The SW group for example generated the following list of group norms they thought were established in the *Janice's Method* videocase: not interrupting, valuing process, no cellphones, no sidebars, open to hearing ideas. One participant began her small group's discussion of what group norms may have been in play in the videocase by stating, "When we set our norms, we draw a picture of Norm. Is that what you guys do?" to which others nodded. Social norms, are of course, important in understanding how a group functions, but they do not necessarily give much purchase in thinking about what mathematics gets accomplished and how that was achieved. However, we did not think that we could introduce the term sociomathematical norm without a set of meanings and activities to link it to. We also used the phrase "norms for mathematical reasoning" interchangeably with SM norms, thinking that the latter may be excessive jargon for a non-academic group. For this reason, after much deliberation within our own research team and discussions with the pilot group, we settled on experimenting with an initial question, "How is the group engaging in mathematical explanation?" When prompted with this question, the SW group did focus more on the mathematical work being done in the videocase. The discourse in the SW group shifted over the course of the seminar series from a more heightened focus on the ideas of individuals in the videocase (and whether they did or did not understanding something) to more concern with how teachers in the videocase engaged with one another's mathematical thinking. We then attempted to link their discussions about how the group engaged in mathematical thinking with questions about how those discussions provided evidence for the norms for mathematical reasoning (or SM norms) that were established or being negotiated in the videocase. We are now in the process of analyzing how the groups made sense of the idea of SM norms, how they connected their ideas to their discussions about how teachers engage in mathematical reasoning, and what impact their thinking about SM norms had on their planning for and facilitation of PD.

Table 1: Norms for Explanation That Support Teacher Learning in Professional Development

	Productive Social Norms	Productive Sociomathematical Norms
Sharing	• Teachers and the PD leader listen respectfully to one another • Teacher share solutions or strategies • Teachers work together to find solutions to problems • Multiple solutions may be explored	• Sharing has a purpose of extending and deepening mathematical thinking • Sharing consists of explanations that emphasize the meaning of mathematical ideas • Mathematical connections among solutions, approaches, or representations are explored
Justifying	• Teachers describe and give reasons for their thinking	• Justifications consist of a mathematical argument • Justifications emphasize why and how methods work
Questioning	• Both teachers and the PD leader pose questions • Questions support multiple voices and ideas	• Questions push on deepening understanding of mathematical ideas
Confusion/ Error	• Confusion and error are accepted as part of the learning process • Teachers are not put "on the spot" over incorrect answers • PD leader encourages teachers to clarify their explanations	• Confusion and error are embraced as opportunities to deepen mathematical understanding—comparing ideas, re-conceptualizing problems, explore contradictions, pursue alternative strategies

Conclusion

For both the pilot group and the SW group, introducing and making sense of sociomathematical norms allowed them to think explicitly about how teachers engaged in mathematics during PD. How leaders engage with and take up the idea of SM norms is naturally tied to their understandings of the purpose of doing mathematics in PD (see also Elliott & Kazemi, 2007). We have reported here the beginning of our analyses of how leaders made sense of collective mathematical activity. How leaders take these ideas up as they facilitate PD is a major concern of our research project and will be the focus of future analyses connecting their participation in these leader seminars to their own work as PD facilitators.

Endnote

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EXPLANATIONS FOR THE FINNISH SUCCESS IN PISA EVALUATIONS

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At the University of Helsinki, we decided to put together a book "How Finns Learn Mathematics and Science?" in order to give a commonly acceptable explanation to our students' success in the international PISA evaluations. The book tries to explain the Finnish teacher education and school system and Finnish children's learning environment at the level of the comprehensive school, and thus give explanations for the Finnish PISA success. The book is a joint enterprise of Finnish teacher educators: there are altogether 40 authors. The explanations for success given by the authors can be classified into three groups: Teacher and teacher education, school and curriculum, and other factors, like the use of ICT and a developmental project LUMA (1). The main result is that there is not one clear explanation, although research-based teacher education seems to have some influence. But the true explanation may be a combination of several factors.

The Finnish students' success in the first PISA 2000 evaluation was a surprise to most of the Finns, and even people working in teacher education and educational administration had difficulties to believe that this situation would continue. Finland's second success in the next PISA 2003 comparison has been very pleasing for teachers and teacher educators, and for education policymakers. The good results on the second time waked us to think seriously on possible reasons for the success. Several international journalists and expert delegations from different countries have asked these reasons while visiting in Finland. Since we had no commonly acceptable explanation to students' success, we decided at the University of Helsinki to find it out in the form of a book in which Finnish teacher educators and researchers present their views on essential features of Finnish mathematics and science education, and implementation of the education policy. So far no such critical and analytic overview has been published.

Theoretical Framework

There is very little rigid research done on what really happens in mathematics and science classrooms in Finland. Therefore, the chapters of the book at hand present to a certain extent only the views of Finnish teacher educators and researchers in mathematics and science education. Research done is more generally on school teaching (e.g. Komulainen & Kansanen, 1981; Syrjäläinen, 1990; Patrikainen, 1997). Up today there are only very few descriptions on teaching and learning mathematics and science in Finnish classrooms (e.g., Norris & al., 1996, Maijala, 2007). Of course, there are more research done on teachers' and pupils' beliefs and views on teaching and learning (e.g., Perkkilä, 2002) or on special teaching and learning interventions (e.g., Viiri, 2000).

PISA Evaluation

The PISA programme aims at assessing young people's skills, knowledge and competencies from the perspective of future learning demands. PISA assesses 15-year-olds' performance in three main domains: reading literacy, mathematical literacy and scientific literacy. The PISA programme involves surveys to be conducted every three years with

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alternating prime domains. In 2003 the prime domain was mathematics, in 2000 reading literacy, and in 2006 science. In the first two evaluations, Finnish pupils were ranked at the top (cf. Lie & al., 2003; Mejding & Roe, 2006). The results of the third evaluation will be published in December 2007.

Development of Finnish Education Policy

There are three leading principles in the educational policy of Finland: One is the commitment to a vision of a *knowledge-based-society*. This vision can be found also in the national documents published in the 70s, where implementation of common comprehensive school (Committee Report, 1970) and university level teacher education (KATU Project, 1978) were presented. Another long-term objective of Finnish education policy has been to raise the general standard of education and to promote *educational equality*. Basic decisions towards this direction were made during the 1970's with the other Nordic countries when a change to a comprehensive obligatory school system was decided (Committee Report, 1970). In Finland, local authorities have strong autonomy, a lot of freedom, power and responsibility. This movement was strengthened in 1994 curriculum (NBE, 1994). Consequently, the third general education policy principle in Finland is the *devolution of decision power* and responsibility at the local level.

The Book "How Finns Learn Mathematics and Science?"

After the Finnish success in the PISA evaluations, several international journalists and expert delegations from different countries have asked the reasons for our success while visiting in Finland. Since we had no commonly acceptable explanation to our pupils' success, we decided at the University of Helsinki to find it out in the form of a book "How Finns Learn Mathematics and Science?" (Pehkonen & al., 2007).

The Structure of the Book

The chapters are organized into three sections: General aspects, Mathematics, and Science. Each section consists of 5–6 chapters written by the teacher education specialists around Finland. After each section, there is a synthesis written by an international or national specialist who has not been involved in writing of the chapters.

The first section "General aspects" contains six chapters. In chapter 1 the researchers who have participated in the PISA and TIMSS studies discuss the reasons for Finnish students' high achievement in mathematics and science in the recent international assessment studies PISA 2000, PISA 2003 and TIMSS 1999. In chapter 2 the Finnish comprehensive school system is described in general and the educational tasks of mathematics and natural sciences in detail. In chapter 3 an overview of planning, organising and evaluating of mathematics and science teacher education in Finland is given. In chapter 4 the Finnish learning environments in mathematic and science are described; the authors refer to social, psychological, and pedagogical contexts in which learning occurs and which affect students' achievement, attitudes, and beliefs. Chapter 5 focuses on gender differences: Although in the Finnish society equity between genders is relatively good, there are gender differences in students' attitudes towards these subjects; different career choices in mathematics and science are no longer compulsory. In chapter 6 the authors consider the additional factors that influence school teaching and learning in a more restrictive way like INSET training for mathematics and science teachers and the extensive science and mathematics program (LUMA) (1) launched by the Finnish Ministry of Education in 1996.

The second section “Teaching and learning mathematics” includes five chapters. Chapter 7 introduces what is meant with problem solving in mathematics education and how it is manifested in the Finnish curricula, textbooks, lessons and assessment. In chapter 8 the authors tell about the special characters of Finnish primary teacher education and also about its development during last decades. Some alternative teaching methods that have been delivered for more than twenty years for mathematics teachers of the comprehensive school are introduced in chapter 9. In chapter 10 some essential features of the mathematics teaching and assessment in Finnish primary classrooms are outlined. Chapter 11 gives an overview of technology-based activities in the Finnish mathematics education based on official documents that illustrate measures that have been trigged by more or less administrative projects, and research articles made in the university within mathematics teacher education.

Section three “Teaching and learning science” contains also five chapters. Chapter 12 concentrates mainly in analysing the teaching methods of chemistry and physics and discuss on what has actually been going on in the science classrooms. Chapter 13 focuses on how to meet the challenge of making science and technology more attractive by introducing them in context-based educational settings. Chapter 14 concentrates on modelling and practical work as instructional approaches at lower secondary level. Chapter 15 focuses on the science teaching at primary level and on the class teacher as teaching science. In chapter 16 the national ICT strategies from 1986 to 2000 and their implementation are analysed and related to core curriculum development, development of software and learning environment, teacher education, as well as research activities in the field of ICT use in Finland.

The Focus of the Paper

In the paper we try to single out reasons for Finnish PISA success in mathematics and science as a common understanding of Finnish teacher educators – here the third strand of PISA *literacy* is not at all dealt with. The research question could be formulated, as follows: What reasons do Finnish teacher educators give for the Finnish pupils’ success in international PISA comparisons?

Methods

When planning the book “How Finns Learn Mathematics and Science?”, we (the editors) sketched a title for each chapter and discussed together who might be proper persons to write them. The idea was that in each chapter there are at least two authors, in order to increase discussions and reflections when writing and thus result better papers. Half a dozen of first contacted persons did not accept the invitation, and they were substituted with other ones. The first idea was to have a doctoral degree mainly as a demand for the authors. The first contacts with the authors happened in spring 2005, and the writing began in autumn 2005.

When the authors were asked to write a paper of about 20 pages with the given title, they were explained the purpose of the book with the following sentences: “*The book “How Finns Learn Mathematics and Science?” has two aims. It tries to explain the Finnish school system and Finnish children’s learning environment at the level of the comprehensive school. Therefore, it describes the development of 30 years of the school system, teaching methods in mathematics and science, teacher education system, and factors that might influence teachers and thus teaching. Research can influence school teaching with a long delay, and therefore it is not here in a main focus.*” In connection of the first reading of the manuscripts, the authors were asked to summarize at the end their views on the reasons of the Finnish PISA success.

The book is a joint enterprise of 40 Finnish teacher educators. As we focused on finding warranted reasons, we looked for persons who had experience both in teacher education and research. Thus, we asked all Finnish teacher educators in mathematics and science to co-operate. The authors represent all Finnish universities, especially their teacher education faculties. Almost all professors and docents working in mathematics and science education in Finland are involved in the writing process. The authors have also peer reviewed each others' chapters and, therefore, the book presents a "national view" on mathematics and science education and their teacher education.

There are three sections (General aspects, Mathematics, and Science) in the book. We decided to have a synthesis of the chapters in the section. The original idea was to find two authors who were not involved in writing, one from Finland and one from abroad. We expected that such an international specialist has visited Finland several times and therefore, had some understanding on our system. Only in section Mathematics one case our idea completely succeeded, in two other cases the persons asked refused, and we had difficulties to find a substituting person.

From each chapter, we have selected the conclusion part, and from the conclusion part the sentences that describe the authors' understanding on the reasons of the Finnish PISA success. When having all these sentences, we grouped similar explanations together, and discussed our results as long as we could agree one common solution.

Results

Here we have gathered the success explanations given by the different authors of the book. These are grouped into three sets according to their explaining factor: teachers and teacher education, school and curriculum, other factors (especially ICT and LUMA). Each of these sets is discussed briefly.

Teachers and Teacher Education

The Finnish citizens value education, school and teachers in general. According to our educational policy both primary and secondary school teachers are educated to master level programmes in universities. Teachers at lower secondary schools are typically specialized in two subjects. A feature of Finnish teacher education is a *research-based approach* that has been adopted in teacher education as a main organising theme, emphasizing teachers' pedagogical thinking, i.e. reflecting on their own teaching in school. One example of the use of research in teacher education is to develop teachers' pedagogical subject knowledge. Through in-service training, teachers might get new ideas how to teach in an innovative way.

This high level education allows teachers to have a lot of *freedom and responsibility*. Freedom and responsibility means that teachers are, for example, responsible for developing the curriculum for their courses, choosing the teaching and evaluation methods based on the national guidelines and also selecting the learning materials. There are no inspectors, no national evaluation of learning materials, nor national assessment. In other words, Finnish teachers are educated to be *autonomous and reflective academic experts*. Most Finnish teachers are devoted to their work.

School and Curriculum

The authors involved in PISA research conclude that the good results of Finnish pupils should be taken as recognition of the *high quality of Finnish schools*. One factor behind the good PISA results seem to be the Finnish *curriculum planners' scenario* of the future of mathematics and science teaching and learning that were given already before the beginning of the 1990s; these have been coherent with the PISA framework. The Finnish curricula (NBE 1985, 1994) contained many *novel aspects*. PISA studies asked for pupils' abilities to use their knowledge in different situations, interpret tables, graphs and other kind of scientific presentations, use the scientific language, not only quantitative knowledge e.g. equations. Application of knowledge and problem solving skills has been an essential part of Finnish comprehensive school education. In Finland we don't have a common science, but separate physics, chemistry and biology from grade 7.

The alternative teaching methods like Models from everyday life, Activity tasks, Mathematical modelling, Learning games, Problem solving, Investigations, and Project work will surely develop pupils' skills to solve such tasks as in the PISA tests. In teaching science, the approach is subject-oriented both in primary and lower secondary level; therefore, teachers may transmit more the nature of science. Also experimentation in science is an essential part of the Finnish comprehensive school curriculum. Modelling is appreciated in Finnish science classrooms, it can be considered as an important step for understanding the nature of scientific processes and knowledge that were in turn among the main objectives of PISA 2003 assessment criteria. Thus, the national curricula have strongly affected teaching methods in schools.

Other Factors (ICT and LUMA)

One of such factors is the development of the whole country into an information society, i.e. improving all citizens' possibilities to use information and communication technology (ICT). Therefore, the use of *ICT in school has an impact* on pupils' general ability to deal with information. About ten years ago, the Finnish government launched the joint national program in mathematics and science teaching (LUMA, Heinonen 1996). The *LUMA program* created substantial and exhilarating climate for science and mathematics education.

Discussion

Already before this book, there have been papers that have tried to describe and explain the Finnish results in the PISA tests. One has to mention first of all the publications given by the Finnish PISA researchers after the first PISA testing (e.g. Välijärvi & al. 2002). They explain our success "with comprehensive pedagogy, students' own interests and leisure activities, the structure of the education system, teacher education, school practices, and Finnish culture". These are coherent with the explanations given by the authors of the book in question. After the second PISA results that were published at the end of the year 2004, the explanations were similar (Kupari & Välijärvi 2005).

Simola (2005) has developed an explanation for the PISA success through analyzing teaching and teacher education in a historical and sociological framework, and he gives several general historical and political reasons for the success, such as a homogeneous society (lack of minorities), consensus developed during the Winter War (1939-40), and rapid development from a poor agrarian state to a modern welfare democracy. Some earlier officers of the National Board of Education have also published their views on the PISA reasons (Aho & al. 2006). They ended up with four broad conclusions: (1) Comprehensive school that offers all children the same top quality, publicly financed education. (2) Education reform has been evolutionary rather than revolutionary. (3) Success of the education system is politically, culturally and economically intertwined with other sectors. (4) In a stable political environment education reforms have based on long-term vision, hard work, good will and consensus.

In an international conference on mathematics education Björkqvist (2006) has described his understanding on the reasons for the Finnish PISA success. He raised an important component that is not discussed in the book at hand. Special education is in Finland strongly involved with the ordinary education, and thus it offers learning opportunities also for low-attainers – only two per cent of Finnish pupils are in special teaching institutes. Those who take part ordinary education in the comprehensive school have carefully-tailored support that correspond pupils' needs (cf. also Vauras 2006). The relatively small deviation of Finnish PISA results can be understood on these supports of lower-attaining pupils.

In a science education conference Lavonen (2006) has listed the following reasons: information society, equality in education, devolution of decision power and responsibility changing to local level. Furthermore, there exists a paper on factors influencing mathematics teaching during the last 30 years (Pehkonen 2007, Pehkonen & Seppälä 2007) that offers a holistic view on the development of mathematics teaching from the beginning of the comprehensive school to nowadays.

Concluding Note

Summarizing, we may state that there is no clear single explanation, but the true explanation might be a combination of several factors. Nevertheless, the idea of research-based teacher education seems to be in the core of explanations (cf. Jakku-Sihvonen & Niemi 2006). Additionally, we would like to emphasize the importance of the Finnish pupils' success in literacy, since many PISA problems are such that good understanding in reading is important.

Endnote

1. The acronym LUMA comes from Finnish language terms: **Lu**onnontieteet [Natural Sciences] and **Ma**tematiikka [Mathematics])

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LOCATING PROFESSIONAL DEVELOPMENT WITHIN INSTITUTIONAL CONTEXT: CASES FROM THE MIDDLE GRADES

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This paper presents analyses of data collected during two multi-year collaborations with middle-grades mathematics teachers. Although our primary goal at each site was to support the teachers' reconceptualization of their practice, analysis of our work revealed significant differences across the two sites. Working to understand these differences led us to delineate the characteristics of the institutional settings.

In this paper, we present analyses based on data collected during two multi-year collaborations (e.g. 7 years and 5 years) with middle-grades mathematics teachers in the Washington Park school district and Jefferson Heights school district[2]. Washington Park is a small district located in the southwest that serves a K-8 population. There are seven schools in the district, three of which house grades six through eight. Jefferson Heights is a county-wide district in the southeast and has eleven middle schools. Both were ranked in the top ten on the Princeton Review's high stakes accountability rankings.

Our primary goal for the collaborations was to support the teachers' reconceptualization of their instructional practice so that their students' ways of reasoning became resources for planning. However, the ongoing analysis of our work at the two sites differed significantly in spite of common goals. We found marked differences in the teachers' trajectories for learning that necessitated ongoing revisions to our conjectures for supporting the learning at each site. Activities, tasks and/or discussions that were useful at one site did not necessarily prove to be useful at the other site. This caused a dilemma within the research group[3] as our initial conjectures were based on the success of the mathematical instructional sequence that was the basis of our collaborations at both sites. Working to understand the differences across the two sites led us to focus on delineating the characteristics of the respective institutional settings in order to clarify the impact of institutional context on teachers' instructional practice. This understanding allowed us to situate our work at each site within the context of the teachers' work. For this reason, a primary product of our collaborations was a framework for analyzing the institutional setting of teachers' work (see Cobb, McClain, Lamberg, &Dean, 2003; Cobb & McClain, 2004).

Theoretical Framework

As a result of our understanding of the importance of institutional context, we treat instructional leadership and teaching as distributed activities. We also adhere to the stance that teaching is a social activity and draws on both human and material resources. In addition, as design researchers (cf. Brown, 1992; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2000), we treat our collaborations as design spaces in which we work to establish environments that support the emergence of student-centered teaching practices. This is a highly interventionist endeavor. Much like Brown (1992), we engineer learning environments to support our collaborative efforts.

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Although we began with a conjectured learning trajectory for supporting the learning of the teachers, we were constantly modifying and revising the initial conjectures in light of our ongoing analysis of our work with the teachers. Much like Simon's (2000) *Mathematics Teaching Cycle,* we informed our interactions by our ongoing work as shown in Figure 1.

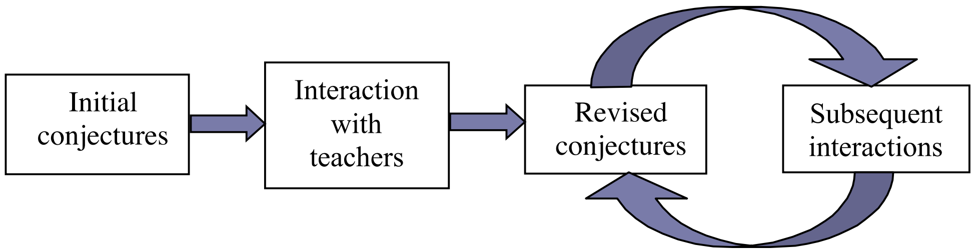

Figure 1. Cycles of intervention and revision.

This process allowed us to proceed in a logical manner while taking the teachers' current ways of reasoning as a basis for planning.

We also take the theoretical stance that teachers' learning is socially situated within the activities of a professional teaching community (cf. Franke & Kazemi, in press; Grossman, Wineburg, & Woolworth, 2000; Lehrer & Schauble, 1998; Rosebery & Warren, 1998; Stein, Silver, & Smith, 1998; Warren & Rosebery, 1995). This is consistent with Rogoff's (1994) claim that "learning and development occur as people participate in the sociocultural activities of their community" (p. 209). Visnovska (2007) notes "[t]his orientation conceptualizes learning as a process that is inherently related to the social and cultural contexts in which it occurs (p. 54). For this reason, we view teaching as lying in the intersection of networks of communities. Therefore, understanding these communities is critical to understanding what constitutes teaching in a particular school or school district.

Methods and Results

Initially, the research team could not account for the extreme differences in the development of both (1) the mathematical understandings of the teachers and (2) the development of professional teaching communities across the two sites. Given the use of the same research-based instructional sequence, the careful collaborative planning of the research team across the two sites and our perceived similarities in the groups of teachers, we were perplexed. Building from Simon et al. (2000), we took the stance that the teachers at both sites were acting rationally in the context of their setting. Therefore, in order to understand the differences across the two sites, we determined that we had to understand the institutional settings of the teachers.

As a result, we began gathering the data necessary to help us identify the network of communities within each district. In particular, we were able to identify *school leadership communities*, *mathematics leadership communities* and the two *professional teaching communities*.

In order to understand the network of communities within each school district, we used what Spillane (2000) refers to as a snowballing strategy and Talbert and McLaughlin (1999) term a bottom-up strategy to gather data in order to delineate the communities of practice within the two districts whose missions or enterprises were concerned with the teaching and learning of mathematics. To this end, we conducted audio-recorded semi-structured interviews with the collaborating teachers at each site to document the institutional settings of their work as they perceived and understood them. (Elsewhere we have described this as the teachers' *instructional reality*.) The issues addressed in these interviews included the professional development activities in which the teachers had participated, their understanding of the district's policies for mathematics instruction, the people to whom they were accountable, their informal professional networks, and the official sources of assistance on which they drew. We corroborated these interview data by including questions that address the same issues on a survey that we administered as part of our pilot work to all the mathematics teachers in the schools in which the collaborating teachers work. As a second step in the snowballing process, we interviewed the individuals identified by analyzing the teacher interviews and surveys to understand both their agendas as they relate to mathematics instruction and the means by which they attempt to achieve those agendas.

Our focus when analyzing these data was to identify the communities of practice within the districts whose missions or enterprises were concerned with the teaching and learning of mathematics. We also documented the evolving interconnection between these communities by identifying:

- *Boundary encounters* in which members of two or more communities jointly engaged in activities that focused on the learning and teaching of mathematics. This first type of interconnection arises when routine participation in the practices of one community involves boundary encounters in which teachers engage in activities with members of another community.
- People who were members of two or more communities and who could thus act as *brokers*. Brokers can bridge between the activities of different communities by facilitating the translation, coordination, and alignment of perspectives and meanings (Wenger, 1998). Their role can therefore be important in developing alignment between the enterprises of different communities of practice.
- *Boundary objects* that were used by members of different communities as they organized for mathematics teaching and learning, and made mathematical teaching and learning visible. This practice involves the use of a common boundary object by members of two or more communities as a routine part of their activities.

Following the first round of analysis, we used a modification of the constant comparative method to further analyze the data. In this process, we focused on both the leadership practices of school and district leaders, and on teachers' instructional practices. This process then delineated the affordances and constraints of the communities on each other. It is in this sense that we speak of the practices of each community being partially constituted by the institutional setting in which its members act and interact.

We have also documented changes in the collaborating teachers' instructional practices by collecting modified teaching sets (Simon & Tzur, 1999) twice each year for each teacher since the commencement of the project. A modified teaching set consists of an audio-recorded pre-interview that focuses on the teachers' instructional planning, a video-recorded classroom observation followed by an audio-recorded post-interview that focuses on the teachers' instructional planning and on reflections of lessons. The issues that we addressed in these semi-structured interviews included:

- What does the teacher view as the most important ideas or skills in the area of mathematics addressed in the lesson?
- How does the teacher take account of students' reasoning when planning for instruction?
- How does the teacher use and adapt textbooks and other resources?
- What aspects of student reasoning are visible to the teacher during the observed lesson and what artifacts does the teacher use to make them visible?
- How does the teacher adjust instruction based on observations of students' reasoning?

To ground the analysis of these teaching sets, a survey that addressed these same issues was administered to all the mathematics teachers in the schools in which the collaborating teachers work. This data in addition to the teaching sets was used to establish *regularities of practice* among the teachers at the two sites. Our focus when analyzing these data has therefore been to document shifts in the ways in which the teachers plan, conduct lessons, and analyze lessons.

Our analysis revealed differences in the regularities of practice across the two sites that could be directly related to our analysis of the institutional context. As an example, our analyses of the teaching sets (i.e., video-recorded classroom observations and follow-up interviews) revealed that the teachers at both sites revised their classroom instructional practices significantly in the course of our collaboration with them. In doing so, they increasingly came to view curriculum materials as resources that they could adapt to achieve their instructional agenda rather than as blueprints for instruction. As an example, the teachers in Washington Park originally held a *fidelity* approach to instruction. The district's professional development efforts had focused on training teachers to enact the NSF-funded curriculum with fidelity. Administrators assumed that this *fidelity to the curriculum* would lead to increased student achievement. This conception of professional development gives agency to text resources and places fidelity to the curriculum as the endpoint of professional development engagements. This approach contrasts sharply with the overall goal of professional development work with the teachers at both sites, namely that the teachers would place students' current ways of reasoning at the forefront of instructional planning and decision making. When the emphasis of instruction is on building on students' current understandings, teachers require professional development that supports them in developing the knowledge to adjust instruction to their students' needs and understandings. In the case of the teachers in Washington Park, the analysis of their teaching practices revealed that there was a significant shift in how they viewed text resources. The analyses documents the following sequence of changes:

1. Giving agency to text resources - a fidelity to the curriculum approach.
2. Teacher professional decision making with text resources as basis.
3. Instructional planning that places a premium on students' current understandings

This sequence of shifts is one indicator of the ways in which the teachers at this site have reorganized their instructional practices.

A significant difference between the sites was traced to the lack of brokers at Jefferson Park. The shift to using texts as a resource instead of a blueprint occurred within the Washington Park professional teaching community within the first year of the collaboration. The brokers were able to facilitate this process through their boundary encounters with administrators. The constant flow of information between the professional teaching community and the school leadership community supported the teachers' ability to shift from a fidelity approach to using the text as a resource.

This transition was much more labor intensive in Jefferson Heights since the professional teaching community and the school leadership community were disjoint entities with very little communication between the two. As a result, a goal for the Jefferson Heights professional teaching community became that of bridging the communication gap between the two communities. A series of activities was designed to allow the teachers to communicate with the school leadership community about their activities within the professional teaching community. This was a process that required its own conjectured trajectory.

Discussion

A number of scholars have documented that teachers' instructional practices are profoundly influenced by the institutional constraints that they attempt to satisfy, the formal and informal sources of assistance on which they draw, and the materials and resources that they use in their classroom practice (Ball, 1996; Brown, Stein, & Forman, 1996; Feiman-Nemser & Remillard, 1996; Nelson, 1999; Senger, 1999; Stein & Brown, 1997). The findings of these studies indicate the need to take account of the institutional setting in which the collaborating teachers develop and refine their instructional practices.

For this reason, we argue that up-scaling an innovation from a high-capacity district to other urban districts cannot be accomplished merely by attempting to develop more efficient ways to reify the innovation. As Carpenter and colleagues note (Carpenter, Blanton, Cobb, Kaput, & McClain, 2002), innovations cannot simply be codified and handed over to others. This is the case even if new information technologies are used (Brown & Duguid, 2000). To be successful, the dissemination process has to involve the restructuring of the target districts such that the communities of practice that comprise them might become as effective as the corresponding communities of the high-capacity district. This required restructuring process is both profound and daunting in that it penetrates the inconspicuous, recurrent, and taken-for-granted aspects of teaching and leadership. However, in the absence of such a restructuring, it is highly probable that even if objects that reify the innovation are seen as relevant, they will be used in very different ways and come to have different and quite possibly conflicting meanings when they are incorporated into the practices of communities in the target districts. These boundary objects are not carriers of meaning and will not be constituted in practice in the same manner. For this reason, unless the structure of the communities within a school district are such that they support innovation, the boundary objects will likely be used in a manner not intended by the designer. One might argue that we are claiming that professional development can only be effective in high-capacity settings. We are not. What we are arguing is that unless the institutional setting of teaches is taken into account during the course of collaboration, it is probable that the innovation will not sustain past the intervention. Given the high cost of human and material resources needed to conduct such professional development collaborations, it then becomes incumbent on those of us who work with teachers to take the time to analyze and understand the setting of teachers' practice so that collaborative efforts have the possibility of both transforming practice and sustaining new practice. It is for this reason that we focus on the importance of understanding the institutional setting of teachers' practice.

Endnotes

1. The first two authors were members of the research team. They worked at different sites. The third author was a member of the professional teaching community at one site.
2. The school district names are pseudonyms.
3. The research team was comprised of Paul Cobb, Kay McClain, Chrystal Dean, Teruni Lamberg, Jose Cortina, Qing Zhao, Jana Visnovska and Lori Tyler.

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RESEARCHING MATHEMATICS LEADER LEARNING: INVESTIGATING THE MEDIATION OF MATH KNOWLEDGE NEEDED FOR TEACHING ON LEADERS' COLLECTIVE WORK IN MATHEMATICS

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This research report examines the resources leaders access as they engage in collective mathematical activity. By examining resources we explore how leaders' participation in the collective work looks and sounds different across groups. Furthermore, through our mixed-method research design we investigate possible linkages between leaders' resources and a group's mathematical knowledge for teaching. This novel use of MKT is central in our exploratory study to raise questions and hypotheses about what leaders need to know to lead.

Background

The data for this paper were drawn from a larger study, Researching Mathematics Leader Learning (RMLL) (2) investigating how leaders learn to cultivate mathematically rich learning environments for teachers. Two key assumptions under gird our work. First, to improve children's mathematical learning, teachers need to develop deep understandings of mathematics (Ball & Bass, 2000; Hill et al., 2005; Ma, 1999). Second, leaders of mathematics PD need to learn to create and nurture PD climates where teachers have rich opportunities to grapple with and understand mathematics more deeply (Wilson & Bern, 1999).

There is widespread agreement that improving teaching and learning requires that teachers participate in high-quality PD (Darling-Hammond & McLaughlin, 1999; Loucks-Horsley et al., 2003). However, what leaders need to know in order to construct high quality PD is under-defined (Even, 2004; Stein et al., 1999). We have chosen, with sound theoretical support, to focus on developing leaders' understandings of norms for mathematical reasoning or sociomathematical (SM) norms. These are the norms that guide the ways people interact mathematically (Yackel & Cobb, 1996). Our work is informed by the classroom research on SM norms where norms are established and cultivated in students' mathematical activity (Kazemi & Stipek, 2001; Yackel & Cobb, 1996). We extend this work, focusing on how leaders may engage teachers in mathematically rich environments where productive SM norms are essential in supporting mathematical discussion and debate (3).

This paper examines the mathematical resources leaders bring to mathematical activity to understand what might be entailed in leaders developing an understanding of productive SM norms and putting these norms into practice in mathematics PD. We recognize that leaders' collective mathematical activity is not the same as leaders working with teachers on mathematics. We conjecture that if we know what resources leaders have access to, and how the resources are taken up by leaders when collectively engaged in productive mathematical activity, then we may better understand what resources are available to leaders and perhaps needed to cultivate mathematically rich PD environments.

Theoretical Framework

Recent research examining students' learning of mathematics and science has advanced our understandings of the social and cultural context in which learning takes place. In particular, researchers focus on the role of discourse in learners' development and ways of cultivating productive discipline specific engagement (Cornelius & Herrenkohl, 2004; Engle & Conant, 2002; Lampert, 1990; Wertsch & Toma, 1995). Discourse within much of this research is a cultural tool mediating development (Wertsch, 1991). Moreover, discourse is guided by patterns or regularities (Bahktin, 1986). These patterns or regularities are the normative ways of engaging and thus guide participation.

Researchers examining teachers' normative participation in mathematics PD suggest that discourse typically engages colleagues in social pleasantries and avoids dialogue that may prove uncomfortable for participants (Lord, 1994; Wilson & Berne, 1999). Recent research in math PD shows that efforts are being made to shift this norm, yet it is a difficult pursuit (Elliott, Mumme, & Carrol, 2006). Our research investigates how to cultivate productive mathematical discourse and understand what might be entailed in engaging in such discourse. To do so, we look to the science education literature where Engle and Conant (2002) advance four principles necessary for productive disciplinary engagement in classrooms.

The researchers' four principles for productive disciplinary engagement are, "(a) problematizing subject matter, (b) giving [learners] authority to address such problems, (c) holding [learners] accountable to others and to shared disciplinary norms, and (d) providing [learners] with relevant resources" (p. 399). These principles highlight the nature of disciplinary discussions in classrooms. Our work extends this research from learning opportunities with students to learning opportunities in math PD.

The principles, extended to PD, suggest that facilitators and learners must problematize the mathematical work in which they are engaged. Supporting taking on authority to engage in problems means that learners are contributors to learning with the potential of shaped what is learned. Holding learners accountable to others and the discipline requires that the facilitator and learners seek to understand contributions. In addition, learners are accountable to the discipline of mathematics meaning that they engage in making meaning, using evidence, pursuing mathematically important ideas, and constructing justifications for mathematical ideas (NCTM, 2000). The fourth principle is the provision of relevant resources. Provision of resources are not only necessary for productive disciplinary engagement, but may interact with the other principles. Resources may or may not be tangible objects such as time and text materials. They also are the resources that support: (i) the discourse practices of problematizing content, (ii) accessing discipline specific ways of engaging, and (iii) fostering learners' authority.

We found these principles insightful for framing what we saw unfold during the collective engagements of leaders solving mathematics tasks and discussing solution methods. Because the research on facilitation of math PD is under-theorized, we find ourselves extending the literature from classroom research to inform our work, in particular research investigating discourse and sociocultural views of development, to inform the project's research questions. Engle and Conant's (2002) principles guide us in making sense of what we observed and the variety of resources leaders used during their collective mathematical activity. This research report investigates these resources to better understand the factors that are at play as leaders engage in the work of learning about the facilitation of productive mathematics PD.

Leaders' capacity to engage in productive discussions of mathematical work draws on a number of fields of knowledge or resources. In the past, this knowledge was conceptualized as the professional knowledge needed for teaching including content knowledge (CK), pedagogical content knowledge (PCK), general pedagogical knowledge (PK) and knowledge of context (CN) (Grossman, 1990; Shulman, 1986). As leaders engage in collective mathematical activity it seem likely that they may try to make meaning of the mathematics by sharing different methods for solving tasks, connecting methods or examining the utility of various methods for different types of numbers. Leaders' meaning making accesses resources associated with the knowledge *of* mathematics and knowledge *about* how mathematics works (Ball, 1990). PK and PCK resources may take the form of sharing particular instructional strategies or insights on students' ways of solving particular mathematical problems. PCK also includes knowledge of curriculum and mathematical ideas that tend to be difficult for students.

The recent work by researchers in the Learning Mathematics for Teaching (LMT) group, have advanced that this knowledge is specialized mathematical knowledge needed for teaching (MKT) (Ball & Bass, 2000; Hill et al., 2005). MKT connects CK and PCK – drawing from across the domains. The LMT researchers claim that MKT is knowledge, not pedagogy, needed for the work of teaching (Hill et al., 2005). Our analysis explores how MKT is potentially activated and shared as leaders collectively solve mathematics problems and engage in productive disciplinary discourse.

Research Methods

Leaders for this study (n = 36) were volunteers from preexisting leader groups located in geographically distinct regions, NW and SW. These leader groups were charged with leading math PD in various contexts with teachers. The NW site was comprised of more leaders than the SW site (24 v. 12). The NW leaders on the whole had greater experience and worked with teachers spanning the k-12 continuum. The SW site had fewer leaders and these leaders were mostly elementary focused with less experience facilitating than the NW site. A few leaders were full time PD staff, however a majority of leaders, across both sites, were responsible for working with students in some capacity during the day.

The RMLL staff constructed a six-day seminar series focused on facilitation of math PD. For this report, we qualitatively examined the collective work of leaders in one whole group and the subsequent small group discussion from the first seminar, titled *Janice's Method*. Four small groups' engagements were captured on video and audio tape. In *Janice's Method,* leaders mentally solved the task 92-56, engaged in a whole group discussion where various solution methods were scribed onto a poster at the front of the room. Small groups were invited to discuss how the methods were similar and different, which methods worked for addition or subtraction, and which methods worked for any numbers.

Fieldnotes were constructed during the seminar and video data were transcribed after the seminar to accurately capture the leaders' discourse. The data examined for this paper were fieldnotes from the whole group and four small group discussions and the leaders' journals, where notes and mathematical work were recorded.

Before coding the fieldnotes, data were segmented into episodes based on what was being discussed and who was involved in speaking (Tharp & Gallimore, 1988). Episodes could include only one or many speakers. Episodes' boundaries were determined by shifts in topic when a speaker offered a different idea that took the discussion onto another topic. Some episodes consisted of one speaker because an idea would be shared, but no discussion was pursued. An episode potentially accessed a number of resources. Each leader's utterance was coded as it related to a resource. Coding associated with each episode was compiled into a table calculating the frequency of different codes, the number of speakers, and number of utterances.

The analytic framework guiding the examination of the qualitative data was developed from our literature review. Our coding scheme focused on four major resources. Resources aimed at developing mathematical meaning (MM), sharing pedagogical content knowledge (PCK), eliciting pedagogical knowledge (PK) and contributing contextual knowledge (CN). The MM code was used when leaders shared methods for solving the task and generalized the method to a mathematical model or for any numbers. Secondary codes were developed to document regularities in leaders mathematical discourse resulting in codes for exploring how a solution was developed, why a solution worked, categorizing the types of solutions, and generalizing if the method always works. Further analyses were completed on the mathematical meaning making of these groups. Illustrative episodes of the MM code were selected from each group by identifying the episode with the greatest number of different MM codes.

The Learning Mathematics for Teaching (LMT) survey was administered with leaders during the first seminar. RMLL staff constructed the survey, guided by LMT research faculty, drawing on the content domains of number, algebra, and geometry. These content areas were cross-referenced with items that focused on generalization and mathematical explanations as a part of the problem context. The survey consisted of 32 items. A one-parameter response theory (IRT) model (Hambleton, Swaminathan, & Rogers, 1991), using the software BILOG (Mislevy & Bock, 1997), was used to calculate leaders' scores and explore item and score properties. Using a Rasch-IRT model, the reliability index was (>.9). The same reliability was determined using Cronbach's alpha. The Rasch-IRT scores ranged from -1.7 to a + 1.9. Rasch score intervals were determined by identifying by finding natural breaks in the raw data and coordinating raw scores with Rasch scores. The low interval was -1.7 to -0.8, mid interval -0.9 to + 0.5, and high interval was + 0.6 to 1.9.

The use of LMT in this study was not meant to distinguish individual score gains, but serves as a proxy for estimating the potential mathematical knowledge for teaching resources available for leaders. Our primary analysis focuses on the resources that are accessed in collective mathematical discourse. After sharing the results of the analysis of these data we will coordinate group's average LMT scores to explore potential connections and implications. This novel use of LMT is explored here to raise questions and hypotheses about using MKT in this way.

Results

Each seminar began with solving and discussing a mathematical task. At the center of the seminar was a videocase of a mathematics PD facilitator engaging teachers in the same task. During the seminar, leaders' pursued *what* were the mathematical explanations shared in the videocase and *how* where these explanations shared. Leaders collective mathematical work and sharing insights on their mathematical methods set the stage for the videocase.

During the first seminar, the facilitator posed the mental math task of 92 - 56. Individually leaders solved the task. Afterward, the facilitator requested responses and posed questions asking leaders how they solved the task or probing why a particular method worked. The facilitator problematized the situation to launch small group engagement. In addition during whole group discussion the facilitator's questioning held leaders accountable to each other and norms for mathematical reasoning. The facilitator probed solution methods, asked others to explain a leader's thinking, and pressed to know why something worked rather than the steps or procedures comprising a solution. The facilitator positioned leaders in roles of authority acknowledging leaders' capacity to pursue questions and ideas. Engle and Conant (2002) suggest giving students authority " is a matter of [learners] having an active role … in defining …and resolving…problems" (p. 404). A number of times during a seminar, the facilitator or other leaders would raise a problem resulting in small group and whole group discussion.

After the task 92-56 was problematized, small groups were prompted to discuss how the methods compared and which method worked for any number. We examined how four groups collectively engaged in pursuing these questions and what resources they accessed during their discussion. From our analysis we saw that each of the four groups, in general, accessed the same types of mathematical meaning (MM) resources during discussion. Group's reviewed methods that were shared with the whole group and categorized the various methods by the mathematical strategy employed. For example, all groups suggested solutions were using 'benchmark" or "friendly" numbers when leaders rounded the minuend or subtrahend. Other leaders suggested solutions were constructed by moving the minuend and/or subtrahend to numbers that could easily be subtracted and the result would need to be "adjusted" to "compensate" for the change. A number of leaders connected this compensation method to a distance model for subtraction (Fosnot & Dolk, 2001). Leaders also connected across methods and three of four groups made statements suggesting they were considering general cases, or generalizing from arithmetic. Two groups related multiple representations of the task. Leaders not only accessed knowledge of the mathematics they accessed knowledge of important processes and big ideas within school mathematics. Leaders coordination of representations and connection across methods are important mathematical processes that can provide deeper insights (NCTM, 2000).

Groups' engagement appeared different when we looked beyond the resource of mathematical meaning making. All four groups accessed PK and PCK resources while they were engaged in discussions. However, the frequency of the PCK code for the two SW groups (15 and 24 utterances) was higher than the frequency found with the two NW groups (3 and 7 utterances). A third code, CN, was frequent in the SW groups. CN was used to signify leaders talking about a variety of contextual issues that the group was invested in discussing. For example, leaders in one SW group discussed how they had learned mathematics procedurally and its implications for their teaching. The other SW group shared their understandings of how individual's life experience played into the nature of solution methods. For the two SW groups, most of their discourse on CN-type topics was intertwined with PCK and PK resources.

Our analysis of illustrative episodes of discourse with the greatest variety of MM codes revealed further differences among the groups. Although every group was engaged in productive disciplinary activity, the degree to which they were able to consistently enact all of Engle and Conant's principles ranged from still developing to a more complete embodiment of the four principles. The two SW groups seemed to struggle staying focused on the mathematics to develop their own understandings. This suggests that leaders were still developing how to hold each other accountable and press for accountability to the norms of the discipline. This was illustrated when examining one of the SW group's discussion. During the small group discussion, leaders shared their concern that they only understood mathematics procedurally, yet their mathematical discussion primarily focused on providing a rationale for one member's method and not an in depth analysis of the task. When one member asked if the member's method would always work, another member simply answered, "yes, because they [addition and subtraction] are inverse operations." Leaders chuckled in response, saying that they did not remember this. The leader providing the information suggested that the leaders were not expected to know this since their curriculum did not cover that terminology. The leaders did not hold each other accountable to a deeper understanding of the mathematics after they had disclosed that they knew they needed to build their knowledge.

The four groups engaged in collective mathematical work were able to garner a number of resources as evidenced in their discussions. Our examination of resources showed that each group had access to a number of mathematically relevant resources. However, it seems that the nature of the resources varied across groups. One possible explanation for the varying discussions was that the collective work of leaders is highly situated and discussions accessed resources most intellectually pressing for these leaders. Certainly understanding students' mathematical reasoning is important and for a number of leaders this information was something that they were currently sharing with teachers during mathematics PD. In the SW, leaders were working with their staffs on implementing curriculum k-6. It seemed reasonable that leaders would share how different curricula approach subtraction as a resource that might provide insights for their collective work in the seminar. Further investigation is necessary to substantiate this explanation. Further analyses of subsequent seminar data will inform our conjectures.

Another possible way of understanding the differences we saw in the four groups focuses more on the differences in the nature of the mathematical resources they accessed. We examined leaders LMT scores after collecting our initial data from these small groups and noting that the discussions seemed to unfold differently across the sites. These data revealed that leaders in the groups who spent more time digging into the mathematics had a higher average Rasch-IRT score than the two SW groups who focused discussions on other resources and seemed to still be developing mathematical resources relevant to the mathematical work. In addition, the NW leaders' scores represented the full range of score bands, low, mid and high. The SW leaders' scores were from the low and mid range intervals. One possible explanation for why we saw the differences across groups was that the nature of the mathematics resources access by leaders was quantitatively different. LMT *may* serve as a means of capturing leaders' mathematical resources. Here too, further analysis is needed to uncover patterns in leaders engagement.

Conclusions

Our work is exploring factors that may be at play for leaders to understand how to cultivate mathematically rich PD environments. Questions arise, such as was the nature of these leaders' engagement typical of the remainder of their engagements in the seminars? Do different types of mathematical tasks press for different mathematical resources, thus leaders may engage in subsequent discussion much differently. Patterns will need to be uncovered across seminars to gain a better understand of the role that LMT might play in our analysis. However, this work uncovers some potentially interesting trends and conjectures about the nature of leaders learning about productive mathematics PD. The work advances the field by considering how we might coordinate qualitative data on resources available to leaders of mathematics PD with quantitative measures meant to capture the mathematical knowledge needed for teaching.

Endnotes

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2.RMLL (ESI-0554186). The opinions expressed in this paper are the authors and do not necessarily represent the views of the National Science Foundation.

3.See, Kazemi, E. Elliott, R. Hubbard, A. Carroll C. & Mumme J. (2007, October).Doing mathematics in professional development: Theorizing teacher learning with and through sociomathematical norms. *Proceedings of the annual meeting of the Psychology of Mathematics Education—North American Chapter, Lake Tahoe, NV.* (this volume)

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STUDENT NEEDS VS. DISTRICT MANDATES: TEACHERS' COMPROMISES IN THE ERA OF HIGH-STAKES TESTS

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This paper examines the various compromises teachers make between district mandates and meeting student needs. Using classroom observations analyzed with the QUASAR framework (Stein, Grover, & Henningsen, 1996) we found that, even in a high-accountability climate, teachers in the same school made strikingly different mathematics instructional decisions. We found none of the teachers using the mandated text to teach for understanding.

The work of teaching always involves balancing pressures and making compromises among competing demands. In the current post-NCLB climate, teachers in low-performing schools in particular are experiencing pressure from state and district policy-makers to raise student performance on state tests. In California, teachers are expected to teach numerous and specific content standards and use fairly traditional state-adopted textbooks. Many district officials have attempted to standardize instruction by dictating the parts of the text teachers are to use at a given point in time. Teachers must also interpret and attend to the needs of their students, and ideally will teach their students for understanding of mathematical concepts and processes, as opposed to narrow mastery of procedural skills. It may not always be possible to follow the district mandates and meet student needs, and teachers may have to make a compromise between these two objectives. We believe that the exploration of this set of tensions and the inevitable compromises teachers must make is both timely and important in the current environment so that we can support teachers in negotiating these tensions.

Theoretical framework

To frame our understanding of the topic, we draw on the work of Remillard on the importance of attending to the teacher/curriculum relationship (Remillard 1999, 2005). We take the position that teachers are active developers of their classroom curricula, and we operate under the assumption that individual teachers create a curriculum that they believe is adapted to their circumstances, drawing on and adapting their textbook as one source of guidance among other influences. Researchers have long documented differing styles of textbook use among teachers and related differences in the content to which students are exposed (Freeman et al, 1983; Freeman and Porter, 1989). These differences have been attributed to various teacher factors including their view of the textbook as a content authority, their convictions about what should be taught, and their knowledge of the subject matter. Although the effects of high-stakes testing environments on teaching have been examined the existing literature is not clear about how individual teachers who are pressured to both raise student test scores and use a particular textbook may respond with curricular choices. We find especially interesting and under-documented the tensions inherent in attempting to use a relatively traditional (non-reform-oriented) text to meet students needs in a high-stakes-testing environment, especially how different teachers in the same circumstance respond to this tension.

Methods

This report uses classroom observations of four second-grade teachers from a single high-poverty elementary school in California with low performance on state tests. The school's district had been designated as "in need of improvement" and district officials had responded by distributing a pacing guide mandating which chapter of the text teachers were to cover each month. The district also produced periodic benchmark tests, coordinated with the pacing guide, which all teachers districtwide were expected to administer.

The teachers in this study were all participants in a university-affiliated professional development project which focused on students' mathematical thinking. Each teacher was observed five times during the 2006-07 school year, once in early winter and four times in January. Observations focused on the materials used, the tasks for students, and the teacher's actions and utterances. The lessons were analyzed using the framework developed by the QUASAR project (Stein, Grover, & Henningsen,1996) to characterize features of the tasks and the levels of cognitive demand of lessons as they were enacted. Each teacher also participated in two one-hour interviews focusing on their classroom practice, their goals for their students, and their interpretation of district instructional mandates.

Context

The four participants in this study were all experienced second grade teachers. Mrs. M had taught for 23 years, and was in her seventh year at Maxwell school. Mrs. T., an 18-year teaching veteran, was in her fourth year at Maxwell. Mrs. P. had taught for fifteen years, ten of which were at Maxwell. Mrs. C, the newest teacher, had spent all six years of her career at Maxwell. The school provided some time during weekly grade-level meetings during which teachers could discuss curricular issues. The teachers also had the opportunity to discuss their curricular choices during some of the activities of the professional development program.

The district-mandated mathematics text, produced by a major publisher, is organized in a traditional way. Each of the 12 chapters covers a single broad topic such as Place Value or Addition with Regrouping, and most lessons are two-page spreads on a single concept, beginning with an introduction of the concept, followed by several guided practice questions and then a section of independent practice problems which usually focus on computation. Most lessons have a few word problems or review items at the end. Teachers reported during interviews that they were expected to use the text as their primary instructional materials, but felt that the district did not discourage them from supplementing the text with other materials. The pacing guide distributed by the district lists one or two chapters from the book per month, but does not specify which pages teachers are to cover each day or week. The pacing guide reorders some of the chapters in the book, but otherwise mainly follows its structure. The guide divides the school year into three benchmark periods, indicating testing windows after each. The district benchmark test observed during this study was a twenty-six-item, machine-scored multiple choice assessment that focused heavily on procedural skills.

Results

Our analysis of the observational and interview data revealed that, even in a high-accountability climate, four teachers in the same school made strikingly different instructional decisions about mathematics. We found wide variation in the topics the teachers addressed, the materials they drew on, and the expectations for student engagement in mathematics as the lessons were enacted. Overall, we found none of the teachers using the text to teach for understanding. We organized our findings into two time scales: the lesson-level decisions that teachers made about how and whether to use the text, and their broader long-term planning decisions.

Lesson-Scale Teaching Decisions

Figure 1 offers a look at the observation results.

Figure 1. **Lesson features by teacher.**

Teacher	Lesson Sources			Task Features			Level of Cognitive Demand			
	Text	Teacher Made	Other	Multiple Represent-ations	Multiple Strategies	Explanation Required	Procedures without connection	Procedures with connection	Doing Math	Other
Mrs. T.	4 = 80%	0	5 = 100%	2 = 40%	0	1 = 20%	5 = 100%	0	0	0
Mrs. C.	3 = 60%	1 = 20%	2 = 40%	2 = 40%	1 = 20%	0	3 = 60 %	1 = 20%	1 = 20%	0
Mrs. P.	2 = 40%	2 = 40%	4 = 80%	2 = 40%	0	0	5 = 100%	0	0	0
Mrs. M.	0	5 = 100%	2 = 40%	5 = 100%	5 = 100%	4 = 80%	0	3 = 60%	2 = 40%	1 = 20 %

Figure 1 summarizes the initial analysis of the twenty lessons observed (five per teacher). The first block of the table, Lesson Sources, indicates whether the teacher drew on the district-adopted text, an activity of the teacher's own invention, or some other source (old text series, a published book of worksheets problems from the professional development program or released state test items). The task features and levels of cognitive demand were drawn from the QUASAR analysis (Stein, Grover, & Henningsen, 1996). We needed to adapt the QUASAR framework to include the category of "other" to accommodate a lesson whose task was potentially rich in mathematics but was only partially completed during the time allotted, resulting in an awkward fit in any of the existing analysis categories.

Examination of Figure 1 reveals some striking differences among teachers. There is wide variation among the teachers in the materials they used. (Materials use figures add to more than 100% because teachers often drew on more than one material source during a single math lesson.) Mrs. T. used the text in 80% of the observed lessons, while Mrs. M. did not use the text at all. Mrs. T. and Mrs. P. both drew heavily on other commercial materials to supplement their text use, while Mrs. M. was much more likely to invent her own materials.

Task features varied from teacher to teacher as well. Mrs. M.'s instruction was much more likely than that of other teachers to require students to use multiple strategies, justify their answers, and make use of multiple representations. 60% of the remaining three teachers' lessons used only a single representation, usually symbolic, for the mathematical content being studied, and their lessons usually did not require students to use multiple strategies or explain their answers with mathematical justification. The levels of cognitive demand of the lessons tended strongly toward procedures without connections to underlying mathematical ideas, with the exception of Mrs. M, who was never observed teaching procedures without connections. Mrs. C. offered a wider range of demands in her instruction than did Mrs. T. or Mrs. P., both of whose lessons without exception either focused entirely on procedures without connection or devolved during the lesson to that level. The one lesson observed in which Mrs. C's students engaged in rich mathematical thinking used problem tasks from the professional development program and did not draw on the text.

Some brief description of the teachers' instruction around a particular topic, addition with regrouping, may help illuminate their different approaches. The textbook addresses addition with and without regrouping in separate chapters, and further places three-digit addition seven chapters after students' introduction to two-digit addition. Subtraction with regrouping is treated separately from addition, and place-value understanding is separated from operations work entirely. The district pacing guide unites two-and three-digit work within each operation, but otherwise adheres to the textbook's ordering of the topics. This organization is at odds with a perspective of teaching centered on students' thinking, such as that offered by Cognitively Guided Instruction (Carpenter, Fennema, Peterson, Chiang, and Loef, 1989). Adherents to this latter perspective would advocate linking instruction of addition and subtraction within a problem-solving context, while simultaneously developing and building on students' place-value understanding. Rather than teaching students to follow a standard algorithm, cognitively guided teachers would present students with problems to solve and encourage them to develop their own strategies to solve them, helping students to develop their mathematical understanding through discussion and other techniques which build on students' thinking. Teachers wishing to try this approach would have to depart from the district's pacing guide, and perhaps its mandated text, to do so.

All four teachers were observed teaching a lesson involving addition of two-digit numbers with regrouping. Mrs. C.'s students had had some previous introduction to the topic and she gave a brief procedural review, completing two examples with the students before they practiced independently on a text workbook page without the aid of base-ten blocks. She insisted that a student who was experimenting with adding the tens before the ones follow the standard algorithm and begin with the ones instead. Mrs. P. gave a similarly brief introduction to the topic, telling students that there was room in the ones column only for one digit and so if the sum "on the right" was more than 9, they had to put the 1 in the sum over above the "other side". After she demonstrated a few examples, her students practiced problems on the board that Mrs. P. copied from a commercial worksheet. In her lesson, Mrs. T. used a text page. Students were supposed to use base-10 blocks to model two-digit addition problems and then record their work symbolically. Students worked in pairs, but Mrs. T. directed students through each step of the problems, and the blocks were used more as a computational tool than as a model of the underlying structure of the number system. Mrs. M. did not dedicate a lesson explicitly to addition with regrouping, but over the course of several lessons had students make, use, and discuss various representations of two-digit numbers and work on and discuss word problems with two-digit numbers which could be solved using addition. The approaches the teachers used varied widely, both in their use of materials and their expectations for what students could and should do.

During the interviews, the teachers elaborated on their differing goals for their students in mathematics, based on their interpretations of their students' needs. These goals then affected the teachers' choices of lesson materials. Mrs. P. hoped to help her students perform well on the state testing, and believed they needed frequent, uncluttered practice on basic skills to do so. She found that the textbook often did not fill this need. "You know, it all revolves around the STAR test, and getting their good grades for the STAR test, so we're practicing…Usually the textbook is way over their head. And I want something that is basic, that doesn't have all this other stuff in it….because they're just second graders, you know, a lot of them can't even read the directions…I'll say, what's more friendly, the textbook or the papers I have? If the papers I have are lower and slower and come across easier, then I'll use that, if the textbook's too confusing." Mrs. M also often passed up the textbook, but for very different reasons. She explained that she felt the textbook focused too much on procedural skills and not enough on students' conceptual understanding and reasoning skills. With respect to addition with regrouping, she explained, "I think you can teach them the pattern of the algorithm, which is what I've been told to do by my boss, but I think that unless they start understanding what the numbers mean, that it's a crapshoot, they're going to forget it…. So I won't use any of those pages…because I'm convinced that that just confuses them… [T]hat is not a teaching tool, that is a conundrum for too many of my children, and I just can't, in good conscience, [laughs] do that to a child who doesn't get place value." Mrs. M often created her own lessons instead which she felt better addressed her students' needs to grapple with concepts and engage in problem solving at a deeper level than the textbook offered. Mrs. C and Mrs. T both expressed views of their students' needs which seemed to fall somewhere in between those of Mrs. M and Mrs. P. Their descriptions included seemingly competing goals: the desire to have students think and reason for themselves and the belief that students needed to be clearly shown how to complete procedures and then practice them frequently. Both tended to use the textbook regularly in deference to the district's mandates, but supplemented it sometimes to provide more problem solving or conceptual development, and at other times to provide additional practice in procedural skills. There thus was a spectrum of beliefs among the teachers about the text in relation to student needs, ranging from the perspective that the procedural focus of the text was appropriate but the presentation and amount of practice provided was inadequate, to the belief that the text's procedural focus was problematic and the conceptual and problem-solving materials so minimal that they were hardly worth using and should be replaced with other materials. The various ways in which the teachers supplemented the text reflected their differing goals.

Longer-Term Planning

The teachers all mentioned the district's pacing guide as having some effect on their long-range planning, but due to its loose structure and minimal enforcement, they did not find it particularly burdensome. The teachers all reported addressing the topics in the guide, but with varying materials, rather than relying only on the text. Mrs. M, who appeared to weigh her students' needs much more heavily than the district mandates in her instructional decisions, was aware of the topics required by the district, and was willing to address them with students, but not by using the textbook as the district expected. As she explained, "[W]e have a pacing guide and we are dictated that… I believe we are supposed to be covering the money unit and we're supposed to be introducing regrouping. So I am covering the money unit and introducing regrouping. In my own, interesting way [laughs]." The teachers did express concern over the fact that the pacing guide seemed to delay topics students typically found more difficult, such as multiplication and subtraction with regrouping, until late in the year. They reported feeling tempted to start these topics early so that they could continually revisit them over the year, something the pacing guide did not provide for. Mrs. C described her conflict: "I want to go into doing regrouping subtraction, because that's where they're going to need more help, and I don't think according to the pacing guide we're going to have as much time as we wanted. So I wonder if I can skip to that. I'm hesitant, though, because they told us that we have to give the benchmark test… and once the benchmark test is coming along, we say, OK, let's see what we have to do. Review what we have to, or go over what we have to. I know I'm not really following the pacing guide as they're suggesting, but it depends. By now my group doesn't need it, doesn't need the time they suggest." This quote reveals Mrs. C's ambivalence about following the district's plan to hold off on these topics versus her own professional instinct. She was aware of her students' needs, but was reluctant to gainsay the district in order to address them. Mrs. P, Mrs. C, and Mrs. T all reported adhering more or less to the pacing guide, although they did reorder topics when they felt it made sense to do so. This data suggests that the teachers viewed the pacing guide as a document which was open to interpretation. Everyone subscribed to it in some form, with some understanding it to mean that they should use the text to address the topics in the order and on the timeline suggested, while others felt that as long as they were addressing the suggested topics within the benchmark periods, they were paying the pacing guide its due respect.

Discussion

These findings support Remillard's (1999) characterization of teachers as curricular interpreters and shows that four teachers dealing with the compromise between following district mandates and meeting student needs did so in different ways. The teachers had different perceptions of what it means to meet student needs, necessitating this difference. These perceptions of needs spanned a range, from a focus on direct instruction and repetitive practice of computational skills to a focus on problem solving and conceptual development. No matter their beliefs about what their students needed, however, all the teachers found the need to ignore, adapt, or at least loosely interpret the district's mandated documents- the textbook and pacing guide- in order to adequately meet those needs. The teachers all reported being affected by these documents, but in differing ways and to different degrees.

Lesson-Level Decisions

Two teachers chose to mostly ignore the district's mandated textbook, for differing reasons. The two teachers who tried to use the text as their main instructional resource had trouble using it to address students' conceptual and problem solving needs, even when they used manipulatives alongside the text pages. The lesson observations showed that when these teachers used the text, the focus tended to be on procedures and the cognitive demands of the activities remained low. While lessons which did not involve the text could also have low cognitive demands, it is notable that only those lessons in which the teacher did not use the text incorporated high cognitive demand, multiple problem-solving strategies, and student explanations. The text as a document seemed to lend itself to a teacher interpretation that focused on procedures. If the instructional goal of the teacher or the district is to promote procedural competence and nothing else, the text may be an appropriate primary instructional material. But if goals for students include developing conceptual understanding, the text proved difficult for teachers to use in that way.

Longer-Term Planning Decisions

The teachers generally accepted the district's pacing guide, mainly because they found it open to interpretation. Some felt that the listing of chapters and dates meant "use your text to teach these topics in this order at these times", while others read it to mean "address these topics in your own way by these dates". The lack of specificity allowed them each to feel like they were adhering to the pacing guide, even without consistent use of the textbook. The data show that, the pacing guide notwithstanding, teachers continued to have their own priorities about what to cover and how much time to devote to it. For this group, subtraction with regrouping represented one topic they wanted to give plenty of time for, and they were largely willing to diverge from the pacing guide to ensure their students learned this topic. The teachers perceived that students needed to develop the concept/procedure slowly over a long period of time, and were uncomfortable with the pacing guide's short, encapsulated focus. The structures of the text and the pacing guide did not allow for the long period of time needed to master the algorithm and/or develop the concepts, and teachers thus had to choose between following these documents and employing a more developmental perspective of student learning. In this instance, these teachers chose their students over the district mandates.

Conclusion

It can be assumed that all interested parties, including the state, district officials, teachers, and advocates of mathematics education reform share some goals for students. Procedural competence backed by conceptual understanding and reasoning and problem solving ability would likely be agreed upon by all as worthy goals. However, ideas of the appropriate vehicles to use to achieve these goals differ. The state and district administrations appear to believe that a mandated text and pacing guide are levers to change teachers' practice to ensure that all students have access to certain skills and materials. However, this study indicates that at least in some cases, these efforts do not have the intended effect of standardized curricula. In concurrence with earlier research (Freeman and Porter 1989), we found that teachers continue to prioritize their students' needs as they interpret district-mandated documents, even in the current climate of pressure towards standardized instruction.

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THE ROLE OF HIGH SCHOOL TEACHERS' ANXIETY AND PROFESSIONAL IDENTITY IN LEARNING MATHEMATICS WITHIN PROFESSIONAL DEVELOPMENT CONTEXTS

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In this paper we share our professional development effort within one urban high school. Throughout intervention teachers' on-going needs and practical problems impeded efforts to facilitate change in mathematical understandings and practices. Results revealed that the complex relationship between teachers' mathematical anxiety and professional identity, including fear of public recognition of a lack of content knowledge, was crucial to learning.

Introduction and Theoretical Framework

In this paper we share our collaborative efforts within one urban secondary school. For this project we worked with three high school mathematics teachers to improve their understandings of mathematics and instructional design and assessment, while concurrently assisting them in becoming Highly Qualified (U.S. Dept. of Education, 2000). What is clear in the research on mathematics teacher education is that without on-going professional development that addresses teachers' understandings of mathematics and supports their efforts to improve practice within their own classrooms, no gains can be made in students' mathematics achievement (Ball, 2000). We thus created an on-site professional development program that involved us collaboratively teaching two courses (Algebra and Functions for Secondary Mathematics Teachers, Instructional Design and Assessment) that we integrated to meet the cognitive and professional development needs of these teachers. This effort was situated within a large (1,500) charter school located in a large, financially challenged city.

Early in this work we recognized that these teachers were faced with daily struggles that prevented them from arriving ready to participate in the goals of our professional development community. Throughout this experience we grappled with how to address their on-going non-academic needs and practical problems, while maintaining an effort that would facilitate change in mathematical understandings and practices. We became acutely conscious that it was essential to listen to these teachers and integrate their dilemmas into what was explored within the sessions. We therefore initiated the study out of a desire to better comprehend why this effort was not functioning as we had originally intended; namely, how do we bridge the learning to mathematics? Our original research goal for the project thus moved from, "How can we structure teacher learning within an authentic community that addresses the complex realities of inner-city practice?," to our new research questions: "Had teachers generated defensive facades to prevent others from realizing their lack of content knowledge?" and "Why were the project goals related to participants' mathematical content knowledge not being achieved?"

Professional Development Design

For this professional development experience we (researchers/instructors) worked with three practicing high school mathematics teachers to improve their understandings of mathematics,

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and instructional design and assessment, while concurrently assisting them in becoming Highly Qualified. Based on current research recommendations (e.g., Wilson & Berne, 1999), we created and co-taught an on-site professional development program that involved collaboratively teaching two courses (Algebra and Functions for Secondary Mathematics Teachers, Instructional Design and Assessment) that we integrated to meet the cognitive and professional development needs of these teachers, and allowed them to earn graduate course credit toward state-level certification. Specifically, we designed these courses based on participants' pre-assessments, classes they were currently teaching, district curriculum, and content embedded within state-mandated standardized assessments (e.g., functions). Courses met weekly over the course of one school year in a collaborative forum where the participants could share ideas about teaching in conversations that were grounded in the their actual practice.

We aligned the goals in these courses to reflect our common orientation toward teaching and teacher education. These goals include active knowledge construction, opportunities for on-going reflection, a focus on enduring mathematical understandings, alignment of course goals with authentic activities (e.g., Stein, Smith, Henningsen, & Silver, 2000), as well as modeling teaching practices that support these tenants. Recognizing that current reforms and federal mandates expect teachers to understand both content and innovative pedagogical practice, we concentrated on primary foundational structures that supported teachers' learning of both content and pedagogy.

Research Design and Methodology

We conducted case studies (Merriam, 1998), with final analysis looking at a comparison across cases. Participants were high school math teachers (3), who were dedicated to their profession and school and held undergraduate degrees in mathematics. They taught at a large, urban school (grades 6-12), that overall had concerned and dedicated faculty (based on observations from a previous grant project). The student body was 99.8% African-American. Although the student/teacher ratio was low (22:1), teacher turnover exceeded 60% each year and a high percentage (72%) of teachers did not yet possess state-level certification. Additionally, in 2005 over 75% of the students were unable to pass the standardized state mathematical assessments; scores had declined annually since 1999.

Data was collected throughout the 2004-2005 school year (August–June). Primary data sources included: transcripts of audio-taped instructional sessions (30) and informal community meetings (10), researcher fieldnotes of all sessions and meetings, transcribed interviews with participants, participant journal entries and course assignments (e.g., content exams, problem sets, narrative reflections, lesson design), initial surveys and end of program evaluations, observation field notes of participants' practice (5 per participant), e-mail correspondences between participants and researcher-instructors, and researcher journals. A large emphasis was place on analysis of teachers' written and verbal dialogue, classroom interactions, and researchers' systematic reflection through journals. Observing teacher dialogue and interactions has become an effective way to promote teacher learning in professional development contexts (e.g., Lloyd & Frykholm, 2004; Sherin & Han, 2004). Recognizing that current reforms and federal mandates expect teachers to understand both content and innovative pedagogical practice, we concentrated on primary foundational structures that support teachers' learning of both content and pedagogy.

We conducted formal, semi-structured interviews (30-45 minutes) with each participant at the beginning, middle, and end of the professional development experience. Initial interviews centered on participants' responses on open-ended initial surveys. Survey questions asked participants about their educational and teaching backgrounds, role within the school, beliefs about students, mathematics, and teaching, expectations for the professional development experience, and why they had chosen to currently teach secondary mathematics and in an inner-city setting. Middle of program interview questions asked participants to reflect on their learning and continued expectations of the pd experience. End of program interview questions were similar in nature to those in the middle of program interviews, however more questions asked participants to reflect on their experiences, as well as garnered information related to participants' perceptions of the impact of the professional development program on their learning and practice.

To understand participants' practices we observed each participant five times over the course of the year. For each participant, we chose to spread observations across different courses to determine if practice differed by content/students, with the final observation occurring in the same classroom as the initial visit to determine possible subtle change in practice. Observations also provided a shared understanding between participants and researchers that grounded conversations centered on participants' practice.

Analysis Approach (Three Phases)

Phase 1: Data were analyzed using *direct interpretation* (Stake, 1995) to garner emergent themes within individuals to understand the substantive changes in participants' content knowledge and practice, and the role that the experience and context played in those changes. Coding illustrated effectiveness of program through an analysis of focus and duration of course meetings, content of course assignments and participant dialogue, and researcher reflections on programmatic decisions.

Phase 2: Data were *aggregated* (Stake, 1995) across individuals to understand participants' perceptions of the role the professional development effort played in their growth.

Phase 3: We did a final analysis across the entire year of professional development to determine impact on long-term growth, as well as an articulation of how this program evolved over the course of the research project.

Validity issues were addressed by triangulating data, coding independently by two researchers (allowing for cross-validation of results), and externally validating coding by the long-term nature of the project. Reliability was enhanced by researchers keeping separate journals throughout the course of the project and its planning, in which they recorded personal reactions to the pd experience, emergent ideas, possible related literature, ethical considerations and dilemmas, and general perceptions of participants and the impact of program. These journals also served as an additional source of data, the comparison of which helped to triangulate and validate findings.

An Overview of Findings

Data revealed the following: (1) Teachers suffered from mathematical anxiety, (2) Depth of the anxiety went beyond what we expected based on documented gaps in content understandings, (3) Anxiety stemmed from two primary sources: public recognition of content deficits and self efficacy related to professional identity, (4) School context played a strong role in fostering these teachers' mathematical anxieties, and (5) Moving forward with the professional development effort was contingent upon immediately and openly addressing the teachers' content knowledge and anxieties.

For example, all three of these teachers diverted conversation more during mathematical lessons within the professional development program than in discussions that centered on how to best design a lesson or assessment. This finding caused us to originally suspect that the teachers had generated defensive facades to prevent others from realizing their lack of conceptual content knowledge (mathematics). Although we knew from pretests that the teachers had gaps in their understanding of secondary mathematics, it was only through systematic study that we truly recognized the depth of their anxiety over their lack of content understandings and fear of public recognition of this deficit. Research reveals that math anxiety exists among preservice and inservice teachers and influences practice; however, this finding is primarily documented at the elementary level (e.g., Cohen & Leung, 2004; Hembree, 1990). Arem (1993) defines math anxiety as "a clear-cut, negative, mental, emotional, and/or physical reaction to mathematical thought processes and problem solving" (p. 1). We found that the construct of *math anxiety*, although rarely noted in secondary mathematics education, best described the reactions we witnessed with these teachers. For example, we were taken aback when these practicing high school teachers exhibited what appeared to be symptoms of math anxiety (e.g., unwillingness to complete math homework assignments, avoidance of participating in collaborative problem solving) to the level that prevented engagement in mathematical lessons. Hence, our frustration over not immediately getting to the "mathematics" represented a real concern, particularly given that we had designed our effort to address all research recommendations for mathematics teacher education.

Further analysis revealed the strong role that their school context was playing in this process. Within this building these teachers were deemed as master teachers; in particular, one of the teachers, even without state certification, was placed in the role of Department Chair and another was put on the School Improvement Team. So, on the one hand, these teachers were publicly recognized by administration as being strong teachers. On the other hand, they were also encouraged to engage in our professional development program to achieve state-level certification and told daily that students' standardized mathematics assessment scores must improve. This dual message resulted in the teachers forming a professional identity that stemmed from competing forces of maintaining the perception of public mastery, while simultaneously trying to figure out how to engage and learn. How content knowledge manifested through the teachers' professional identities became imperative to moving forward with their professional development in mathematics.

Discussion

Findings from this research study suggest that teacher educators can better facilitate professional development by providing teachers a vehicle through which they can feel empowered to voice and make change. Teachers want to feel as though their opinions are heard and valued. In a study of professional development with secondary mathematics teachers, also in an urban context, Lachance and Confrey (2003) provided such a vehicle for teachers by developing an interactive community that encouraged collegiality and sharing. They found that grounding discussions in mathematical activities allowed for the generation of community, which could then provide teachers with a foundation for instructional change. While we agree that both components (content and community) are needed to facilitate change in practice, our work suggests that the establishment of a community based on a trusting relationship might be a necessary requirement to fully engage teachers in learning mathematics. As Lachance and Confrey (2003) note, "there is very little in the literature discussing the development or existence of teacher communities that addresses the notion of using mathematical content (or other subject content) as the "issue" around which teachers can interact and professional communities can develop," (p. 132). The experiences conveyed in this study will ideally move teacher educators forward in their thinking about development of teacher communities.

Within this professional development effort what we found genuinely surprising was the level of anxiety the teachers had over their own insufficient knowledge of mathematics, as well as how intricately it was bound with both their personal and professional identities. While these findings may be anticipated in an elementary context, it is not often discussed in high school situations and therefore must be further investigated. In future professional development situations we will not hold assumptions related to any teacher's content knowledge, beliefs about content knowledge, and how these aspects (in addition to context) define and shape the development of a teacher's view of good teaching.

This study aims to contribute to what is understood about the impact of mathematical anxiety and professional identity on teacher learning within urban, secondary contexts. Future work should explicitly explore the following questions: What assumptions do we hold about high school mathematics teachers' views about mathematics? What role does math anxiety play in a teacher's professional identity? How should professional development be structured to help teachers recognize and overcome their own math anxiety?

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THE ROLE OF THE FACILITATOR IN PROMOTING MEANINGFUL DISCOURSE AMONG PROFESSIONAL LEARNING COMMUNITIES OF SECONDARY MATHEMATICS AND SCIENCE TEACHERS

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This report describes the construct of decentering and its influence on the discourse of a professional learning community (PLC) of secondary mathematics teachers. We used decentering as a construct to describe a behavior in which one attempts to understand the mathematical thinking and/or perspective of someone else. Analysis of data from four PLCs over two semesters revealed that a PLC facilitator's ability to decenter influenced the quality of mathematical discourse in the PLC. As a facilitator's ability to decenter increased, the discourse among members of the PLC became more meaningful. In addition, the degree to which a facilitator decentered when interacting with other PLC members appeared to be influenced by the facilitator's understanding of the topic under discussion. This research is informing the instructional design for PLCs and PLC facilitator training. The findings also have the potential to contribute to the theoretical construct of decentering.

This study investigated the mathematical discourse among members of four different professional learning communities (PLCs) of secondary mathematics teachers. In this report we focus on PLC facilitators' roles in promoting meaningful mathematical discourse among the learning community's participants. By "meaningful mathematical discourse" we mean substantive conversations about understanding, learning, and teaching mathematics. We took a design research approach in which a strategy to support PLC facilitators was continually refined as we investigated its impact on the quality of the PLC's mathematical discourse. Our findings revealed that facilitators who made efforts to understand the thinking and perspectives of other PLC members (we call this decentering) were better able to engage the members of the community in meaningful discourse. This paper describes five manifestations of decentering and their effect on the mathematical discourse among the PLC's teachers. It also illustrates the ways in which four PLC facilitators, over time, decentered, some with increasing flexibility and increasing effect on the PLC's discourse. Findings also revealed that a facilitator's level of understanding of the concept that was central to the lesson influenced the quality of the PLC discourse. This was revealed in her/his questions, choice of tasks, and discussions with members of the PLC.

Theoretical Framing of the Study

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.

To facilitate teachers in a professional learning community requires that the facilitator place himself in the teachers' shoes. "Placing oneself in another's shoes" is a classic instance of what Piaget (1955) identified as *decentering*, or the attempt to adopt a perspective that is not one's own. Steffe and Thompson (2000) extended Piaget's idea of decentering to the case of interactions between teacher and student (or mentor and mentee) by distinguishing between ways in which one person attempts to systematically influence another. In their telling, decentering involves the ways a person adjusts his or her behavior in order to influence another in specific ways. In that process, each person acts as an observer of the other, creating models of the other's ways of thinking. Steffe and Thompson distinguish between two primary ways in which persons interact with each other and two kinds of models that people make of others in the course of interacting. As an interactor, a person can be a first- or second-order observer of the other. They can also create first- or second-order models of the other. A first-order observer is one who interacts with another in a non-decentered mode, taking non-reflectively the assumption that the other's thinking is identical with the observer's. As a second-order observer, the observer assumes that the other person's behavior entails a rationality of its own and attempts to discern that rationality. That is, a second-order observer creates models of the other's thinking and is aware that it is a model. It is necessary to clarify that a person can be a first-order observer if the observer realizes another's thinking is different than his or her own but does not attempt to build a model of this thinking. Also, if an interactor is speaking such that he or she believes the other person understands the utterances (of the interactor) just as intended, the interactor is acting in a non-decentered way.

Methods

The data for this study was collected from PLC members who were concurrently enrolled in a three-hour graduate mathematics education course and an accompanying one hour, school-based PLC for secondary mathematics and science teachers. The size of the PLCs ranged from three to seven members, with each PLC composed of teachers from the same school. The goal of the course was to deepen teachers' understanding of rate-of-change and function through a covariational approach to teaching function (Carlson, Jacobs, Coe, Larsen, Hsu, 2002; Thompson, 1994). The PLC model in this project drew from past research on "records of practice" in which communities of teachers study student work and classroom video (Ball & Bass, 2002). The PLC activities included a lesson study component, in which a lesson was developed, implemented and observed, while the researchers assisted the teachers in collecting and analyzing data (Ma, 1999, Shimizu, 2002). Following the lesson implementation and observation, the teachers use the data collected to make revisions to the lesson.

The content focus of the PLC followed that of the course. The PLC sessions and agendas were designed to improve teachers' ability to i) engage in conceptual conversations about knowing and learning central ideas of the course; ii) discuss and assess student thinking; iii) develop inquiry based, conceptually focused lessons; and v) engage in meaningful reflection on the effectiveness of their instruction.

Each PLC had one member, selected because of her/his potential to emerge as a teacher leader, who acted as facilitator. The facilitator received 18 hours of summer training and met weekly with university faculty to plan for each PLC session. PLC facilitators were responsible for managing the discourse during their PLC sessions. They were expected to follow a general agenda that was developed by project faculty, although they were encouraged to deviate from the agenda as needed to allow meaningful discussions to continue. Facilitators received coaching between meetings from a project leader who had reviewed video recordings of recent PLC meetings. The coach met for 30 minutes with all four facilitators once per week to offer general suggestions for improving their facilitation abilities. During the meeting the coach also asked specific questions aimed at promoting reflection about the facilitators' interactions with other PLC members in their PLCs (e.g., "Sharon, why don't you allow Mary to answer questions?" and "Dan, why are you doing most of the talking?"). These specific questions were gleaned from the data that was reviewed the previous week.

Each course and PLC meeting was videotaped. The videos were coded by two researchers who worked as a team to code all videos for a designated PLC between PLC meetings. Selected videos were discussed among the research team each week, with video excerpts related to decentering also shared with the PLC coach. During these meetings the research team identified emerging theoretical constructs in a manner consistent with a grounded theory approach.

Results

Analysis of the PLC video data revealed five observable manifestations of decentering. We have characterized these Facilitator Decentering Moves (FDM) as follows.

FDM1: The facilitator shows no interest in understanding the thinking or perspective of a PLC member with which he/she is interacting.

FDM2: The facilitator appears to build a partial model of a PLC member's thinking, but does not use that model in communication with the PLC member. The facilitator appears to listen and/or ask questions that suggest interest in the PLC member's thinking; however, the facilitator does not use this knowledge in communication.

FDM3: The facilitator builds a model of a PLC member's thinking and recognizes that it is different from her/his own. The facilitator then acts in ways to move the PLC member to her/his way of thinking.

FDM4: The facilitator builds a model of a PLC member's thinking and acts in ways that respect and build on the rationality of this member's thinking for the purpose of advancing the PLC member's thinking and/or understanding.

FDM 5: The facilitator builds a model of a PLC member's thinking and respects that it has a rationality of its own. Through interaction the facilitator also build's a model of how he/she is being interpreted by the PLC member. He/she then adjusts her/his actions (questions, drawings, statements) to take into account both the PLC member's thinking and how the facilitator might be interpreted by that PLC member.

Tracking the facilitation abilities of the four PLC facilitators over one year revealed that two of the four facilitators made observable shifts in their ability to decenter over the year. A third facilitator was effective in promoting meaningful exchanges between the other PLC members, although he did not show noticeable improvements in his ability to decenter when interacting with his colleagues. The fourth facilitator made no efforts to decenter, nor did she make moves to promote meaningful mathematical discourse in her PLC. In this section we describe and contrast ways in which the facilitators engaged with and used decentering in their interactions. We also discuss PLC facilitator shifts that we observed.

In Jason's early weeks as a facilitator he was observed asking general questions and made various moves (e.g., probed a PLC member to draw a picture, prompted a PLC member to explain her thinking) to promote discussions among members of his PLC. As the semester progressed his ability to promote meaningful discussions between the PLC members improved, although he was rarely observed decentering when interacting with his colleagues. Analysis of the video and other written data revealed that Jason's weak understanding of the concept under discussion was an obstacle to his decentering. At strategic moments during discourse he regularly missed opportunities to inquire about the thinking of a PLC member, although his awareness of his mathematical limitations combined with his desire to facilitate rich discussions resulted in him regularly involving other members of the PLC to facilitate meaningful exchanges, i.e., exchanges in which the interactors built and used models of each others mathematical understandings. This pattern of interaction is illustrated in the following exchange in which PLC members were discussing their solutions to a photo enlargement problem.

In this problem, a 6" by 8" photo was to be enlarged. The enlarged width dimension was given and the PLC members were asked to find the enlarged length dimension and provide a rationale for their approach. The PLC agenda also prompted the facilitator to manage the conversation so that the PLC members emerged with an understanding of the connections and usefulness of expressing a proportion as both a constant ratio and a functional relationship in which one quantity is expressed as a constant multiple of the other. (This had been the focus of the course activities just a few days earlier.) In the following except Jason posed general questions to Richard about his thinking after he had constructed a written response to the question.

Jason: What would you want your students to see if you know that there is a … there are two quantities that are in a constant ratio? What would you want your students to say or see?

Richard: That … what I think is maybe that even though the uh difference in numbers six to fifteen and eight to twenty and like the amount … the total amount that they want is different … the uh I was gonna say percentage wise they went up it's the same. Like they like … what I got was uh … six was increased or … yeah six is forty percent of fifteen and eight is forty percent of twenty. So they both … so the … so what I got was the photo enlarged forty percent like even though they are different numbers forty percent of a bigger is gonna be a bigger number and forty percent of a smaller number is gonna be a smaller number. So that even though the difference between the numbers … that the overall way they are changing I guess is staying the same. So if I were to go even bigger than … if I were to go from the twenty to the fifteen I still all I did … the twenty and the fifteen and enlarge it even more its still going to be just that forty percent different … difference.

Jason: Mm k. Now you two (points the other two members) he's your student and he just explained that to you and remember we're just talk… we're just trying to find out what he knows about constant ratio. We are given two quantities … k? What are ya … how do you want to interview him? What would you say?

First, we noticed that Jason showed interest in Richard's thinking but he did not act on the thinking expressed by Richard (FDM2). Instead, he involved other members of the PLC in questioning Richard. In doing so, Beth, the mathematically strongest member of the PLC, began questioning Richard. Her questions revealed that she believed Richard was unable to see how the fixed ratio between the two proportional quantities could be used to construct a functional relationship.

Beth: Could you also come up with a statement that would show the relationship between the height and the width that would work in any situation?

Richard: Betwe … a relationship between the height. … (looks at Beth)

Beth: (Interjecting) Between the length and width of this photo that would work in any situation? That you could always.

Richard: Ya. Ya …and actually when I was … just thinking about that … ya know the six is … if I were reducing it that the forty percent works what I would need to do is take uh … six and divide that into fifteen to find out what the … to find out what the real change in percentage was cause I was enlarging and not shrinking it what I … what I said was if I was shrinking if I were shrinking it going from fifteen to six then I just shrunk the fifteen forty percent but I didn't necessarily enlarge the six forty percent.

As this interchange unfolded, Beth continued to make requests for responses and constructions to advance Richard's thinking toward a more connected and flexible understanding of proportionality. In other excerpts she also appeared to understand and appreciate the value of expressing the relationship between the two quantities as both a constant ratio and a functional relationship with a constant multiple. Her subsequent questioning revealed that she was relying on her understanding of proportionality and the model she built of Richard's thinking to decenter in her interactions with Richard (FDM4).

Our third facilitator, Sharon, initially did not decenter when interacting with other members in the PLC. She did not show interest in understanding their thinking. Instead, she dominated the conversations by providing explanations that were based on her understanding of the content under discussion (FDM1). Her explanations did not initially build on the thinking of other PLC members, although she did make an attempt to provide correct explanations when she believed that a PLC member was having difficulty. After only a few PLC sessions, over which time she received specific suggestions from the PLC coach, did she began to decenter when interacting with members of her PLC. She was observed listening and questioning her colleagues and appeared to develop models of their thinking, although her questions and interactions were focused more on moving them to her way of thinking (FDM3). She was acting as a first-order observer of her colleagues.

In an early PLC session, Sharon led her PLC in a discussion of the covariation of height and volume of water in a cylinder. As seen in the following excerpt, she showed no interest in understanding the thinking of her colleagues and her focus was on explaining the "correct" way to think about the problem (FDM1).

Sharon: Well, we're not necessarily comparing volume to height, but we're looking at relationship between…

Lisa: wide and narrow cylinders

Sharon: … two variables, and in this case it sounds like wide and narrow cylinders is more appropriate than saying height and volume like we did in the box because, um, if I remember correctly, that liquids take the shape of their container, but a liter is a liter is a liter, no matter where it is. So if we take the same amount of water, aren't we taking the same volume of water and just giving it a different shape?

Lisa: Mmm Hmm. OK… So we're varying?

Sharon: So what are we varying? We're varying the height with the…

Lisa: with the size of the cylinder

Sharon: …based on the size of the cylinder. So the height versus probably the radius…

In a later session, the PLC members discussed the photo enlargement problem described above. The PLC members had previously discussed ideas of constant ratio, scale factor, and constant multiplier in the context of proportional relationships. Subsequently, we observed Sharon probing the PLC members to help them express a proportional relationship described in terms of a common ratio as a function that relates length and width by a constant multiple. In discussions with her colleagues, she stated, "we should be able to write base as a function of height, or height as a function of base. So Anne (gesturing to one PLC member) pick one quantity, and Sally (gesturing to a different PLC member) take the other, and try to write a function." It is noteworthy that her questioning did not appear to be based on the PLC members' current thinking. Rather, her questions appeared to be for the purpose of guiding the other PLC members to her way of thinking (FDM3).

In Dan's early weeks as a facilitator he did not decenter when interacting with other PLC members. Our data revealed that although he appeared to listen to other PLC members when they spoke, he did not make an effort to understand their thinking. During the weekly coaching sessions the PLC coach pointed this out to him, and soon thereafter we noticed shifts in Dan's interactions with members of his PLC. He shifted from doing most of the talking and explaining, and very little listening, to making regular attempts to understand the thinking of other members of the PLC. He was also observed drawing on his understanding of individual PLC members' thinking and understandings to guide his questions and actions (FDM4). As one example, Dan had asked Bill to provide an explanation of his reasoning when working through a proportional situation involving scaling. At one point in his questioning, Dan (the facilitator) turned to Larry (another member of the PLC) and asked, "Are you going to let Bill (another member of the PLC) get away with all that algebra stuff without explaining what he's doing?" Dan then got distracted by his attention to Bill's solution and began to ask Bill another question. In mid sentence he realized that he had not listened to Larry's response. It was at this point that he caught himself, turned back to Larry, and invited Larry to ask Bill for clarification of aspects of his solution process. As he listened to Larry's questions and discussions with Bill, Dan made utterances that suggested he was trying to understand Larry's thinking. Dan subsequently engaged in a conversation with Larry in which he asked numerous questions that were directed at understanding Larry's thinking. His questioning demonstrated that he was aware that Larry had his own way of thinking about this problem and that Larry's way of thinking was different from his own. The remaining utterances of this exchange revealed that Dan leveraged his model of Larry's thinking to engage in a meaningful and satisfying exchange that led to the advancement of Larry's understanding (FDM4). Dan was acting as a second-order observer in this exchange.

Karen, our fourth facilitator, exhibited extreme discomfort with the content and regularly made utterances that suggested she had not adopted the philosophies of the project (e.g., she did not value examining student thinking and expressed discomfort in exploring multiple approaches to solving a problem). She was not receptive to suggestions made by the PLC coach. The video data also revealed that she made no effort to listen to other PLC members, and on occasions when she did ask a question, it was not based on her understanding of how her colleagues were thinking. Karen was the only facilitator who showed no discernable shifts during her participation in the project. There were no instances when Karen decentered during her interactions with her colleagues. She was focused on obtaining answers to problems and expressed that she believed the goal was for each person in the PLC to get an answer. In fact, at one point during the second semester when the PLC coach encouraged her to explore the thinking of her colleagues, she questioned why one would be interested in understanding the thinking of others. As an example of Karen's approach to facilitating, we provide excerpts from an interchange that took place fairly late in the semester, in which Thomas, one of the PLC members, was discussing his understanding of an acceleration, rate, and time problem.

> Thomas: Kind of like the y equals "a" t-squared plus "b" t plus "c", and you multiply a times t squared, you have to end up with something in meters, so here you're multiplying these two things together, or dividing them, whatever, and you need to be ending up with meters per second.
>
> Karen: See, I didn't, I didn't read it as that at all. I did it another way. (Karen then moved the discussion to a different topic.)

In this excerpt, we see Thomas presenting a procedural description for working a problem. He revealed very little understanding about the quantities or relationships in the situation. Karen did not appear to be interested in Thomas' thinking. Instead, she made a comment that focused on her approach to the problem. As was typical of Karen's actions as a facilitator, she did not seem to be interested in understanding the thinking of the other PLC members (FDM1). As such, she had no basis on which to decenter when interacting with her colleagues.

Discussion

Sharon and Dan improved their ability to decenter over the course of the year. As their ability to decenter increased, the discourse among members of the PLC became more meaningful. In particular, efforts to model other PLC members' thinking resulted in more meaningful discussions about ideas of knowing, learning and teaching mathematics in their PLCs.

The mathematical discussions in Jason's PLC also became more meaningful over the year, although these improvements did not result from Jason decentering during interactions with his colleagues. His weak mathematical understandings appeared to limit his ability to decenter in the context of discussions about mathematical ideas. As a result we found the high level of discourse in Jason's PLC to be surprising. We had previously conjectured that meaningful mathematical discourse relied on the facilitator's understanding of the content being discussed as a coherent system of ideas to be learned. Although we continue to believe that Jason's weak knowledge of the content limited his effectiveness as a facilitator, we now realize that a facilitator's effectiveness can be enhanced by making strategic moves. In the case of Jason, he asked questions that promoted reflection and decentering by other members of the PLC, as they were encouraged to interact with each other. The quality of the exchanges that followed Jason's moves to facilitate discussions among the PLC members appeared to be affected by the decentering abilities and depth of mathematical understanding of other members of the PLC.

Implications

This research adds to the knowledge of the attributes of an effective PLC facilitator. The five manifestations of PLC Facilitator Decentering Moves (FDMs) should be useful for building a theory of PLC facilitator behaviors and discourse. They will also be useful in developing facilitator training tools and workshops. We anticipate that teachers' participation in PLCs that have a focus on decentering and meaningful communication will impact teachers' interactions with students. It is likely that as a PLC facilitator improves her/his ability to decenter when interacting with colleagues about mathematics teaching, learning and knowing, he/she will be better able to decenter when interacting with students. After participating in PLCs for a year, multiple teachers described how her/his efforts to understand other PLC members' thinking has positively impacted the nature and quality of their interactions with students. We will be video taping these teachers' classrooms during the upcoming year and will continue to investigate their decentering during PLC sessions and teaching.

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"TWEAKING" TASKS FROM A MIDDLE SCHOOL MATHEMATICS REFORM CURRICULUM: LONGITUDINAL DESIGN PROCESSES AND THE ROLE OF THE INSTRUCTIONAL CONTEXT

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This study explores how teachers revise tasks from a middle school mathematics reform curriculum over time, with consideration of how the instructional context influences the process of revision. The study takes place in three districts that are long-term implementers of the CMP curriculum. Thirty-one participants were interviewed regarding their district's implementation of CMP, with three of those participants selected for a collective case study that focused intensively on specific episodes of revision. The results suggest that the process of revision, and the resources teachers draw from in the process, changes over time. As teachers successively enact CMP tasks, they draw more from their experiences and from each other. The stability of the instructional context was important in teachers' feelings of competency in using CMP effectively. The results also implicate a form of teacher knowledge important to reform efforts.

One of the major outcomes of the current mathematics education reform is the development, release, and widespread implementation of a dozen comprehensive curricula based on the 1989 Standards document published by the National Council of Teachers of Mathematics (The K-12 Mathematics Curriculum Center, 2005). A number of school districts have adopted these curricula as a means to transform mathematics instruction (Remillard, 2005), a phenomenon that invites a focus on the ways that teachers use these materials to design instruction. Research that investigates teachers' use of NSF-funded curricula suggests that teachers' processes are, to follow Eisner's (1988) characterization, idiosyncratic, personalistic, and experiential (Herbel-Eisenmann, Lubienski, & Id-deen, 2006; Lloyd, 1999; Remillard & Bryans, 2004; Sherin & Drake, accepted).

The situated, idiosyncratic, and experiential nature of teachers' use of curricular materials connects to the notion of teaching as design. In the design perspective, a teacher takes the resources and tools that are present in a given context (*available designs*), uses them to design instruction (*designing*), the result of which is a unique set of instructional conditions and activities (*the redesigned*) (New London Group, 1996). The notion of design implies the involvement of "complex systems of people, environments, technology, beliefs, and texts" (New London Group, p. 73). Thus, design implies that planning and enacting instructional tasks are activities embedded in particular contexts.

Kennedy's (2005) study of nearly 500 teaching episodes describes the complexity that teachers encounter as they design and enact instructional tasks. Kennedy elaborates the myriad and often competing purposes teachers ascribe to their instruction, purposes that may undermine the intentions of reformers. An important implication of Kennedy's work is that researchers need to consider the ways in which the ideals of a reform vie with the demands, both intellectual and managerial, placed on teachers in U.S. schools. Consequently, researchers need to interpret

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teachers' use of curriculum materials not in terms of fidelity to the ideals of a reform, but rather in terms of a process that takes into account the affordances and constraints in a given context.

Much of the research on teachers' use of curriculum materials focuses on the interaction between teachers, curriculum, and context at a given point in time. An important extension of this work is to consider the ways in which the design process functions over time, a consideration that raises two related questions that form the focus of this study. First, how do teachers' iterative revisions of tasks reflect the affordances and constraints of their instructional context? Second, how do teachers' design processes function in relation to the goals set forth by the teacher, the district, and the curriculum designers? Looking at the design process over time provides insights into the long-term impact of reform efforts, in terms of both curriculum and instruction. This is especially relevant in the context of mathematics education, which has seen the development of comprehensive curricula and the expenditure of considerable resources to implement these curricula as a means of transforming practices related to teaching and learning.

Context for Study

The study is situated in three districts that are long-term implementers of the Connected Mathematics Project (CMP) curriculum (Lappan et al., 1998). The three districts have participated in several large federal grants at the local research university aimed at middle school mathematics reform and have further expended substantial material resources to implement CMP. The districts were selected because they had used CMP for more than five years, had committed substantial resources to supporting the implementation, and had consistent access to outside expertise related to the use of the curriculum.

CMP is an NSF-funded middle school curriculum. NSF-funded curricula are comprehensive and coherent, develop mathematical ideas in depth, promote sense-making, and engage students through the use of meaningful contexts and applications (Trafton, Reys, and Wasman, 2001); as such, they represent a substantive departure from commercially-developed curricula. CMP is the most widely adopted middle school NSF-funded curriculum and was rated as "exemplary" in terms of its content (US Department of Education, 1999). CMP is a 3-year curriculum for grades 6-8, designed to provide students with multiple opportunities to explore and formalize their understanding of key mathematical ideas within five major "strands" (numbers and operations, geometry, measurement, data analysis and probability, and algebra). The curriculum is organized into units, each comprising of 3-5 "investigations" where students explore a key mathematical concept or process. Each investigation begins with the presentation of a meaningful real-life problem/situation that embodies the mathematical concept/process under study ("launch"), followed by a combination of whole class and small group guided explorations, and concluding with a discussion in which the mathematical concept/process at the core of the investigation is explicitly identified and its understanding reinforced

Study Design

The study draws from two data sets. The first data set consists of 31 interviews from a study on the three districts' implementations of CMP. The scope of these interviews addresses the question about the affordances and constraints in the instructional context. Of the 31 interviewees, 22 were present or former middle school teachers, 6 were present or former district-level administrators, and three were current principals. Each district was represented by at least five teachers and two administrators. The second data set includes more extensive data collection in relation to the research questions identified above on three teachers. These three teachers constitute *a collective case study*, a set of instrumental cases intended to provide insight into an issue (Stake, 2000). I explore the ways in which these teachers adapt CMP over time and the resources they draw upon to do so.

Teachers and Teaching Context

Each of the three case study teachers had at least four years teaching experience with CMP. Two of the teachers taught in Denton, one of the highest scoring districts in the state, and one taught in Belleview-Burney, both inner ring suburbs of a moderately sized Northern city. Denton was one of the first districts to adopt CMP and several of its teachers have conducted CMP workshops under the auspices of large professional development grants based at the local research university. The implementation of CMP in Belleview-Burney encountered challenges related to shifting district-level administrators and policies in the initial years, though the ten teachers from this district interviewed for the implementation study report being well-supported in their use of the curriculum. Belleview-Burney is in the lower half of districts in terms of test scores in the county in which it and Denton are located, which has manifested itself in increased sensitivity and focus in relation to preparing students for the state test.

Data Collection

The primary sources of data consist of semi-structured interviews, videotaped classroom observations, and video-stimulated interviews. Each of the three focus teachers was interviewed from two to five times, depending on their participation in a professional development project. Additional data include semi-structured interviews of twenty teachers from the study on the districts' implementation of CMP.

Semi-Structured Interviews

For the implementation study, I interviewed the participants regarding: (1) their beliefs about the learning and teaching of mathematics; (2) their beliefs about CMP; (3) their practices around planning, pedagogy, and assessment; (4) contextual factors such as school demographics, district curriculum guidelines, and the impact of state testing; and (5) the resources they draw on to plan their mathematics lessons.

Observations of Class Sessions

For each of the three case study teachers, I videotaped five to eight consecutive classes from the beginning of a particular unit. These observations allowed me to investigate the nature and extent of adaptations of the tasks as they appeared in the CMP student materials. From these observations, I selected ten brief classroom episodes so that they: (1) spanned a variety of tasks; (2) included both launch and summary portions of lessons; and (3) represented key points in the development of the main mathematical concept for the unit. I then used these ten episodes as the basis of the video-stimulated interviews, discussed below.

Video-Simulated Interviews

I used video clips to elicit in specific instances of curriculum use (1) the purposes and processes of revision, including adaptations and revisions from the base curriculum and from how they designed the task in previous years, (2) their reflections on how the task design engaged students, (3) the experiences and resources they drew on to design the task, and (4) how they might revise the task in the future, based on how students reacted to the design.

Data Analysis

The semi-structured and video-stimulated interviews were analyzed using both typological and inductive coding (LeCompte & Preissle, 2003). The typological coding was used to develop general categories according to the structure of the questions in the semi-structured interview (e.g. beliefs about mathematics, pedagogical practices, general design of CMP), the categories of resources teachers use for planning (e.g. student text, teacher resource materials, interactions with colleagues), and comments specific to the use of the CMP student materials (e.g. purpose of task, adaptations, future revisions). Within each category, codes inductively emerged as I made passes through the data. For example, I noted differences in the ways teacher talked about the first time they taught a particular unit from subsequent enactments, so I created a code marking whether a comment referred to the first enactment of a task or unit.

Results

I initially present some themes from the results from the implementation study. These themes include resources teachers drew on as they designed tasks, how the process of design changed over the course of successive implementations of the same unit, and factors in the instructional context that affected the design process. I then focus on the three case study teachers, with an emphasis on how the teachers integrated reflections from prior implementations, goals they identified for the task, the tasks as they appeared in the student materials, and their revisions of CMP tasks. I explore (1) the extent to which the teachers appeared to be reaching stable designs that served the purposes of being efficient and effective and (2) the coherency and efficiency of the teachers design processes.

Implementation Study Results

The results from the 22 teachers, including the three case study teachers, suggest that teachers drew on a variety of resources to design instruction but that these resources changed over time as teachers successively implemented units. Furthermore, the stability of the instructional context played a significant role in teachers' developing knowledge of how particular task designs functioned within their instructional context.

Resources Teachers Drew on To Design Tasks

Teachers drew on various resources in order to understand the sequence of tasks, key mathematical points to emphasize, the ways that students would react to tasks as written in the student materials, and any challenges or obstacles they might face in enacting a particular design. The resources they drew on to develop this understanding changed over time as teachers successively taught the same units. Initially, the teachers relied on the CMP teacher resources and unit-specific training to provide the information, but gradually they referred to their own experiences and collaborative planning with colleagues as the most important resources.

Many of the teachers reported reading the teacher resources quite closely, especially the first time or two teaching a unit. Ms. Kantor noted:

> And, so I definitely always look at it, the first time, especially the first time I teach a unit I look at it, and throughout certain points in the unit I am always going back to the summary and seeing OK, what are the questions, or what are some ideas that kids might come up with that the new ones have a lot more examples of what kids might say and different strategies and explains that strategy. (Interview 120506)

Subsequently, teachers began to develop knowledge of how the curriculum functioned in their instructional context and they relied heavily on that knowledge. Teachers who were able to plan collaboratively with colleagues almost universally stated that this was valuable to remembering past enactments and using those reflections to design the task in a way that functioned more productively for their purposes.

The Design Process over Successive Implementations

Ten of the 22 teachers explicitly mentioned adhering very closely to the tasks as represented in the student texts the first time they taught a unit. Ms. Karp, one of the case study teachers, stated:

> So usually, when I go through CMP the first time I'm pretty... I stay with the way the book has its set up and then I find, once I've gone through a lesson, how I might have done it differently. (VSR Interview 010107)

After the initial enactment, however, teachers began to revise and supplement the tasks according to how CMP had functioned previously in the instructional context. Three of the primary purposes for revision were to emphasize basic skill development, which was a big concern in light of the state-mandated tests, to provide students with resources to study and complete homework problems, which teachers stated also helped to appease parent concerns about CMP, and to make the tasks more efficient in terms of achieving the stated learning goals. The first two purposes usually required some form of supplementing with teacher-created materials or worksheets from other sources. The third goal, to increase efficiency of the CMP task while maintaining much of its features, is central to my research questions.

Five of the teachers used the work 'tweaking' to describe the iterative process of fine-tuning the task design to make it more productive, which teachers mostly described as removing unintended distractions to student engagement with the key idea for that task. This result, which was echoed in most of the teachers' comments, is consistent with Kennedy's (2005) findings that one of teachers' main concerns in designing and enacting instruction is to reduce distractions, though this often means removing some of the most intellectually engaging task features. The case study teachers that I observed, all of whom discussed their efforts to fine tune tasks, did not do so to remove the intellectual difficult parts of tasks, but rather to focus students on particular concepts. The process of 'tweaking' was also indicative of teachers' expanding knowledge of how CMP functioned with their instructional context, knowledge that made them feel more confident that they were engaging students intellectually while developing the knowledge for which they were held accountable.

Stability in Instructional Context

A number of teachers stated that it was difficult for them to develop competence with a particular unit due to changes in the instructional context. There were multiple sources of instability in instructional context, which can be classified into changes in curriculum and changes in instructional priorities. The changes in curriculum were due to two main factors, one being changes in the content of the state standards and tests and the other being a focus on accelerating students in the middle school so that they would be ready to complete AP Calculus in high school. Consequently, teachers at a given grade level were often asked to teach a number of new units each year. One of the districts began an initiative to differentiate instruction that proved cumbersome and impacted how teachers were able to teach CMP.

The result of this instability was that teachers were seldom given the opportunity to teach the same units in the same instructional time with the same institutional policies from year to year. In addition, the opportunities to interact consistently with the same colleagues were rare, affecting the kinds of collaboration and support available to teachers. Given the challenges associated with learning to teach a curriculum that differs substantively from most teachers' own experiences learning mathematics, the lack of opportunity to gain familiarity and comfort with particular units made it seem to the teachers as if they were constantly on a steep learning trajectory.

Case Study Teachers

The case study teachers provide further insight into the processes by which the teachers revised the CMP tasks to suit their given purposes. There were noticeable differences between the three teachers, particularly in terms of how well each was able to connect their prior experiences in a given unit, their stated learning goals for a task, and their subsequent revisions. Ms. Kantor, for example, made explicit connections between past implementations, the mathematical goals of the task, and the subsequent revisions she made. The analysis of her interviews suggests that the revisions she made were appropriate given the challenges she identified in earlier enactments and the lesson observations indicated that the revisions were effective in getting students to engage with the goals that she identified. Ms. Cassini, on the other hand, made revisions that did not seem to directly address difficulties she cited from past enactments, especially in terms of student engagement with ideas. Rather, Ms. Cassini's revisions were intended to help students organize their thinking more generally and were more focused on logistical task features. As a result, in the observations of her lessons, it was unclear how her revisions served to focus student thinking on the mathematical goals she identified. Ms. Karp had the least coherent process. She relied on experiences using other curricula and was prone to making revisions on the fly, during a task enactment rather than in the planning process. In terms of using past enactments to inform subsequent enactments, as a measure of teacher learning of how to use CMP in a given context, Karp exhibited the least specific evidence of this. She employed general heuristics to describe her process of task design, while Cassini and Kantor spoke in very specific terms regarding the ways in which they revised CMP tasks.

Discussion and Implications

The results from this study have implications for efforts to use an NSF-funded curriculum as a catalyst and for reform. The first implication is that over time, teachers draw on different resources as they design instruction. Initially, they draw from the teacher resources materials and from curriculum-specific professional development, but over time they draw more from their own experiences and, when available, collaborative planning. Districts who are interested in the long-term effectiveness of the use of an NSF-funded curriculum should facilitate collaborative planning and initiate collegial discussion around the curriculum as much as possible.

A second implication is that the development of teachers' knowledge and competence is enhanced by stability in the instructional context. Given the widely documented challenges teachers encounter when enacting a curriculum for the first time, instability in the instructional context creates a perpetually steep learning curve that leaves teachers exhausted, a phenomenon noted by Tyack and Cuban (1995) and by Kennedy (2005). Teachers in the district with the most stable context were the most satisfied with CMP and with their district's efforts to implement it.

The third implication is that the idiosyncratic and personalistic nature of teachers' curriculum processes suggests that teachers will develop at different rates the knowledge of how to use CMP effectively in a given context. This form of knowledge, which I term *curriculum-context knowledge* (CCK), is vital to the effective use of a curriculum. The most effective of the three case-study teachers had an intricate knowledge of how CMP tasks functioned in terms of engaging her students with specific ideas, a knowledge she was able to articulate in detail and utilize in her revisions of tasks. Developing CCK is a form of teacher learning that is relevant to mathematics reform due to the emphasis placed on curriculum as a mechanism to reform instruction and needs to be an explicit focus of professional development. This study, with its focus on revisions over time, provides some preliminary insights into the nature and importance of CCK.

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AN EXAMINATION OF TEACHER-DESIGNED MATHEMATICAL TASKS FOR URBAN LEARNERS

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This investigation studied secondary mathematics teachers' understanding of performance-based tasks with respect to the contextual design of tasks that promote urban students engagement in and pursuit of learning mathematics. The findings showed after engagement in a one-week institute, teacher-designed tasks remained inconsistent with characteristics of tasks that are culturally relevant and require high cognitive demand for urban learners.

Consistently, literature addressing urban education states urban learners look for challenging situations that connects to real-world contexts. Urban learners want to see the "big picture" and understand how the curriculum will relate to their future endeavors. Teaching for conceptual understanding provides a means of increasing the learners' critical thinking skills, connects concepts to real-world contexts, and creates a learning environment in which the connections within mathematics are developed as opposed to the teaching of mathematics topics as segmented and discreet (Gay, 2002; Howard, 2003). With the move toward standards-based teaching, whereby students engage in performance-based mathematical tasks, students have access and more opportunities to approach learning mathematics conceptually and within real-world contexts.

The move toward standards-based teaching aligns with the expectations of a culturally relevant classroom in which mathematics lessons are developed and taught in ways to promote high levels of cognitive demand in student learning. In this learning environment students are engaged in performance-based mathematical tasks, have access and more opportunities to approach learning mathematics in ways to develop a deep understanding of mathematics and an appreciation of the beauty of mathematics.

While this movement toward standards-based instruction has been in existence for almost two decades (National Council of Teachers of Mathematics, 2000), the need for research addressing the challenges of and pathways to implementing standards-based instruction—especially in urban classrooms—is crucial. Thus this research study was designed to examine urban secondary mathematics teachers' understanding of performance-based mathematical tasks. The research question for this study is: how do teachers of urban learners define performance-based tasks and how do their definitions manifest into their design of tasks?

Methodology

The participants of this study were 30 secondary mathematics teachers who were selected by their respective principals to participate in a week-long, summer institute with sustained follow up during the academic year. During the weeklong intensive workshop, the participants were immersed in activities to strengthen their knowledge of performance-based mathematical instruction. Also the teachers were introduced to the Mathematical Task Framework (Stein, Smith, Henningsen, & Silver, 2000), as a process in for analyzing and creating performance-based tasks with respect to the elements of culturally relevant pedagogy. While engaged in the workshop, the participants reflected on their teaching practices, as well, as their personal meanings of performance-based tasks. Data collection included a survey and teacher-constructed tasks. The survey provided baseline data on the participants' knowledge of performance-based tasks. Constructed tasks refer to the process of tasks creation, analysis, and revision. An analysis of three participants provides a snapshot of the data results. Analysis Framework

To analyze the cognitive demand of each task the Mathematical Task Framework set forth by Stein et al. (2000) was utilized. The framework categorizes the cognitive effort employed to complete a task into four areas: doing mathematics, procedures with connections, procedures without connections, and memorization. If a task has the characteristics of the latter two categories a low level of cognitive effort is exhibited. The other two categories require a high level of cognitive demand. The higher domains require tasks to have multiple pathways towards a solution. The process of reaching a solution is valued more than the solution itself and the process of finding a solution most likely will not have an algorithm that can be readily applied. In this domain the task is also connected to real-world phenomenon. Within memorization and procedures without connections, tasks have an explicit and implicit pathway toward a solution and in many cases the task is neither connected to related mathematical topics nor real-world scenarios. A scale of one to four was used to determine if the task had a connection to real-world occurrences and clear pathways towards a solution.

To determine the level of the participants' understandings of an exemplary task, Suzuki and Harnisch's (1995), criteria for performance tasks was applied. This criteria include (a) replicating real-world events, (b) having various pathways to reach a solution, (c) demonstrating the continuity of mathematics instead of discrete segments, (d) providing a space for students to communicate understanding of the concepts, and (e) having a rubric for clear explanation of expectations. These guidelines were used to assist the teachers in creating tasks that are high in cognitive demand as well as challenging and meaningful for students.

According to Gay (2002) through the application of varied instructional techniques, culturally relevant pedagogy uses students' prior knowledge and cultural positioning to make connections between the students' community and home as well as validate their cultural stance. To determine the cultural relevance of the task, each task was reviewed for components that allowed for culturally relevant connections and use of prior knowledge. Questions were asked in the analysis process were: (a) How are the students' interests incorporated, (b) In what ways are students allowed to express their individuality, and (c) What are the connections to the students' community?

Discussion and Results

Each of the three participants was considered as a separate case for analysis. In Table 1 we display a sample of the participants' initial definition of a task and their constructed tasks. The developed tasks of these three participants are in alignment with their personal definition of a performance task. However, *Kyle's and Catherine's* tasks contained most of the criteria for a performance-based task that Suzuki and Harnisch (1995), set forth. Participants' ability to design tasks that meet the criteria of standards-based instruction that simultaneously serves as an approach to engage urban learners in mathematical tasks was minimal. Most of the tasks created did not have a connection to culturally relevant instruction. Culturally relevant instruction poses questions that stimulate critical reflections on the mathematics in connection with the students' environment. The tasks were connected to real-world phenomena, but not specially connected to occurrences that students could relate to in their situations with respect to their cultural identities and backgrounds. Thus this study shows teacher engagement in professional development designed to foster instructional practices for performance-based instruction will require an extended period of professional development beyond an intensive one-week institute. This project will continue over a two-year period as we further investigate the length of time required for secondary mathematics teachers of urban learners to develop culturally relevant classrooms that are grounded in standards-based practices.

Table 1. Participants Responses

Participant	Definition of a Task	Constructed Task
Kyle	A task is something to do, quite literally. It requires an active participation from those performing the task	You have been working in acting for several years and now your payday had come. You are about to be signed by a big production studio and are promised a third tier contract. The studio gives you three options from which to choose. The first gives you a handsome signing bonus of $1,000,000.00 and $150,000.00 per film. On the second plan, you would get $250,000 per film but a signing bonus of only $100,000. The first two options promise you 3 films a year for 4 years. The third plan offers neither a signing bonus nor per-film pay, but will pay you one-half of one percent of all the profits from each film. You will have to research the studio's track record of profits from their films over the last 5 years. Make your decision based on the data.
Catherine	Open ending, standards-based assignments. Usually hands-on, real-world application/ problem	Cameras, telescopes, and surveying equipment all have tripods as stands. A tripod has 3 legs. The length of the legs can be adjusted. Do you think three legged stands are better than four legged stands? Why or Why not?
Cevia	A detailed objective stating what you would like the students to learn/ obtain from the lesson/ unit	Solving a three variable equation – Choose two equations and eliminate a variable. Choose two more equations and eliminate the same variable. Using the two new equations from steps one and two, eliminate another variable and solve for the last variable. After solving for a variable, substitute the value back into equations one or two from steps one or two and solve for the remaining variable in that equation. By this step, you should know the values of two of the three variables. Substitute the values of the two variables into your original equations and solve for the last variable. Write the answer as an ordered triple.

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BARELY IN S.T.E.P: HOW PROFESSIONAL DEVELOPMENT AFFECTS TEACHERS' PERSPECTIVES AND ANALYSIS OF STUDENT WORK

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The purpose of this study is to determine how professional development affects teachers' perspectives on and analysis of student work. To achieve this purpose, this study examines, analyzes, and scores responses from four groups of teachers with varying levels of professional development experience. The results suggest that sustained professional development positively influence teachers' interpretation, understanding, and assessment of student work.

Recent research suggests that U.S. teachers lack the specific content knowledge necessary for teaching mathematics (Ball, 2005). This study examines if involvement in sustained professional development affects teachers' specific content knowledge required for teaching mathematics at the elementary school level. This study is a small part of the much larger project S.T.E.P., Studying Teacher's Evolving Perspectives, headed by Randy Philipp and Vicki Jacobs. Randy Philipp, Bonnie Schappelle, and the authors of this paper analyzed two tasks from the Content Assessment Test of this project. The teacher participants complete the Content Assessment Tests along with other tests, assessments, and interviews. The teacher participants included four groups: prospective teachers (PST), initial participants (IP), advancing participants (AP), and teacher leaders (TL). Providing the control group, the prospective teachers are college students enrolled in a teacher program at a large university located in southern California. The initial participants are elementary school teachers currently enrolled but not yet attending the professional development program, Cognitively Guided Instruction (CGI). The advancing participants are elementary school teachers with at least two years of CGI. The teacher leaders are elementary school teachers with at least four years of CGI. All current teacher participants teach in an elementary school located in southern California. The overall goal of S.T.E.P. is to improve professional development on a national level. S.T.E.P. attempts to reveal the gains from professional development by studying four groups with varied experience in professional development.

Therefore, stemming from the goals of S.T.E.P., the purpose of this study is to understand and analyze the mathematical views and specific content knowledge of teachers. More specifically, this study investigates how participation in sustained professional development influences teachers' perspectives on and analysis of student work. Recent research suggests that *good* teachers not only need mathematics content knowledge and pedagogical knowledge, but that teachers also need specific content knowledge for the teaching of mathematics. For example, teachers need these special skills to design lessons, evaluate student work, and create lesson plans (Ball, 2004). Hill, Rowan, and Ball (2005) even use assessment materials to examine the correlation between student achievement and teachers' specific mathematical content knowledge. These researchers report that mathematical content knowledge positively affects student achievement. Thus, in particular, this study examines the relationship between elementary school

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teachers' mathematical content knowledge and participation in professional development programs.

Mathematics education has always been a major concern in the United States, especially with the release of each new national report assessing students' mathematical achievement. This growing concern causes changes in policies at the national level, requiring more specialized classes, certificates, and degrees for teachers. For example, educational reform policies, like No Child Left Behind, require that teachers attain more advanced degrees or take more advanced mathematics classes in an effort to improve students' mathematical achievement and understanding (Ball, 2005). However, teachers may not need more advanced mathematics courses, especially with Begle's (1979) alarming research that suggests a negative or insubstantial correlation exists between teachers with post-calculus courses or credits and higher student achievement (as cited in Ball, 2005). Therefore, attention now turns towards professional development. Perhaps increased professional development will improve student achievement and understanding in mathematics and will change current teaching practices. Thus, this study is significant, even on a national level, because it attempts to address these concerns. This study examines how involvement in professional development affects teachers' perspectives on and analysis of student work.

Theoretical Framework

This study intends to find a correlation, if any, between the years of professional development a teacher has in cognitively-guided instruction (CGI) and the teacher's understanding of grade-appropriate mathematical concepts. In particular, the study chose to look at mathematical proficiency and the issue of transparency. The issue of transparency was brought up in the article *Divisibility and Transparency of Number Representation* by Rina Zazkis (in press) and was the initial impetus for this study. Zazkis defines a property of a representation to be *transparent* "if this property 'can be seen' considering the representation" (in press, p. 89). In other words, if a person attends to the property that a given representation was supposed to bring forth, then the representation would be considered transparent to that person in respect to the given property. In this case, the study hoped to find what properties the participants found transparent in the standard and non-standard addition and subtraction algorithms. Furthermore, the study wanted to probe if professional development had an effect on the transparency of different mathematical properties.

In order to study these transparency issues, however, one must first define what constitutes mathematical proficiency. To match with the S.T.E.P. program, this study chose to use the five strands of proficiency from *Adding It Up: Helping Children Learn Mathematics* by the National Research Council [NRC] (2001). In chapter four of the book, the five strands are listed as follows: adaptive reasoning, conceptual understanding, procedural fluency, productive disposition, and strategic competence. Conceptual understanding includes knowledge of mathematical concepts and ideas. Procedural fluency includes ability to use, manipulate, and apply mathematical algorithms. Strategic competence includes the ability to solve and reason through mathematical problems. Adaptive reasoning includes the ability to reason logically about mathematical relationships and concepts. Productive disposition includes one's attitude or perception of mathematics as a logical, rational, and worthwhile subject. It is important to consider all of these aspects when discussing mathematical proficiency. The authors stress that "the five strands are interwoven and interdependent in the development of proficiency in mathematics" (NRC, 2001, p. 116). In particular, when the rubrics for this study were built, these five issues were continually discussed to make sure that all five were being addressed.

Methods

The subjects for this study were practicing K-3 elementary school teachers from a single elementary school in the southern California. The teachers varied in their levels of experience, and were divided up into three categories: those with no professional development in cognitively-guided instruction (CGI), those with two to three years of professional development in CGI, and those with four or more years of CGI. These groups were called initial participants, advancing participants, and teacher leaders respectively. Since these teachers varied in the number of years of experience in the classroom, a fourth category was added as a baseline. This group consisted of prospective elementary school teachers who were just about to start a course focused on children's mathematical thinking. The participants were asked individually to look at written student work and to explain what mathematical issues the students may have been trying to address. This study looked at two of those tasks, namely Andrew's task (figure 1) and Terry's task (figure 2). Fifty responses were randomly chosen and blinded. The responses were fairly evenly distributed across the four categories.

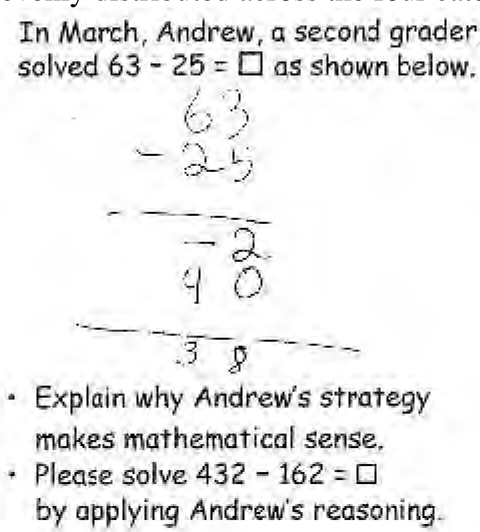

Figure 1

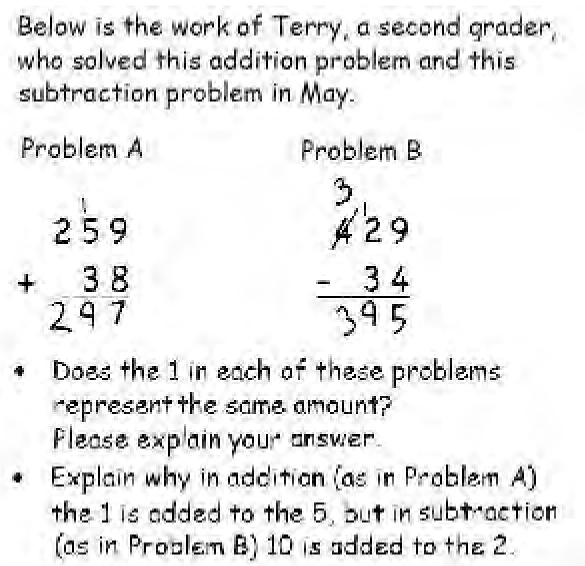

Figure 2

Data analysis took place in two phases. The first consisted of the building of a rubric to score the participants' responses to the task. This was accomplished by taking five of the participants' responses at random and organizing them from least to most sophisticated. Five more random responses were then analyzed and from these ten responses a rubric was formed. Another ten responses were then used to check the viability of the rubric. The rubric was then used to score the remaining thirty responses. These categories were used to see what effect, if any, professional development in CGI has on teachers' understandings. The second part of the data analysis consisted of matching up a respondent's score with their category.

The rubrics were finalized into the following set of scores:

Terry's Task

Score	Characteristics
0	No effort, or explanation verifies an incorrect view
1	Only talks about the algorithm, but knows the difference between the 1's
2	Views the 1's as "the same" but explanation deals with place value issues
3	Talks about issues of regrouping; talks about place value only in a rudimentary way
4	Understands issues of place value; implies understanding but does not state it clearly
5	Exhibits a complete understanding of place value issues

Figure 3

Andrew's Task

Score	Characteristics
0	Failure to explain Andrew's approach on top and failure to apply it on the bottom
1	(a) correct explanation of Andrew on top without attention to place value and incorrect application on bottom; or (b) generally following explanation on top with attention to place value but something missing and incorrect application of Andrew on bottom; or (c) correct application on bottom with no or incorrect explanation of Andrew on top
2	(a) correct explanation of Andrew on top but without attention to place value and correct application on bottom; or (b) correct explanation of Andrew on top and correct application on the bottom but something mathematically incorrect
3	Score of 4 but either (a) inconsistently talks about the negative 2 vs. subtracting 2, or (b) general sense that the respondent understands the algorithm with attention to place value but either something missing or unclear explanation
4	Clear explanation of Andrew's approach on top, correct application on the bottom, no reference to place value as place holders, and consistently approaching the 2 as subtracting 2, adding negative 2, or explicitly making connections between these two ways of thinking

Figure 4

Results

Terry's task was designed to probe the participants' understandings about the standard addition and subtraction algorithms. The goal was not to see if the participants knew how to use the standard algorithms (the assumption was made that they did), but rather to see what their understandings were about the number issues occurring in the algorithm. In particular, the purpose was to determine if the participants saw the difference between the carried and the borrowed one, which brings up issues of place value. In problem A, the 1 represents ten ones which have been regrouped into one group of ten. Thus, the one group of ten is added to the five groups of ten and the three groups of ten giving a total of nine groups of ten. The 9 in the sum of 297 represents this. In problem B, the 1 represents one group of a hundred that was borrowed and regrouped into ten groups of ten. This is why the two, symbolizing two groups of ten, is added to ten and not to one. In other words, four hundreds and two tens is the same as three hundreds and twelve tens. Thus, the three groups of ten are taken away from the twelve groups of ten giving nine groups of ten. The 9 in the resulting difference 395 represents this.

From the graph below (figure 5), it is clear that the majority of pre-service teachers (PST) scored a one, which means they knew that the ones were different, but were unable to articulate why they were different, apart from perhaps talking about the nature of the addition and subtraction algorithms in broad terms. Unlike the PST, the initial participants (IP) do not have one predominant peak. However, there was a higher percentage of IP than PST who said that the ones were the same. This difference may be due to the fact that the PST are still attending college and think about the algorithms differently than the practicing teachers. A clearer trend can be seen with the advancing participants (AP). In their case, only four people scored a two or lower. The vast majority scored a four or a five, which seems to indicate that professional development does have an impact on teachers' understandings. Lastly, the teacher leaders (TL) present an interesting story. The majority of them got a score of three, four, or five, which seems to indicate that professional development has a positive impact on the teachers. However, there is a small peak of TL who scored a zero. While this may seem to be contradictory to the previous conclusion, it may not be. Perhaps, instead, this shows that when TL believe that they have knowledge about a mathematical task, they do. On the other hand, if they do not feel they have this knowledge, they are willing to admit it. This would match the data, for an answer of "I don't know" would score a zero.

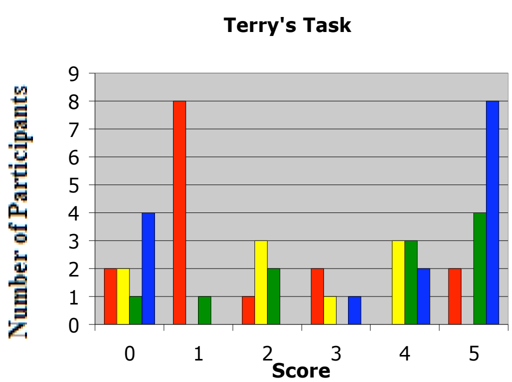

Figure 5

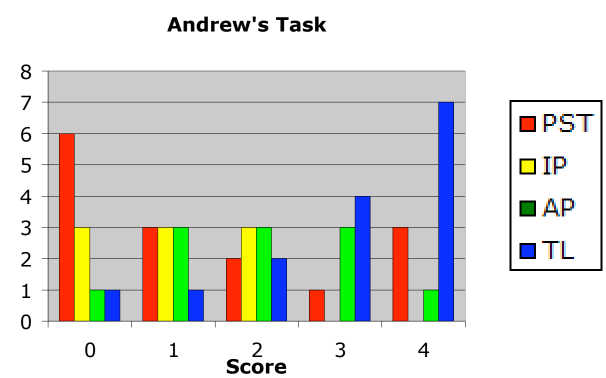

Figure 6

The results for Andrew's task (figure 6) are pretty evenly distributed among all five scores, illustrating that the scoring was fair and unbiased. Andrew's alternative algorithm, for the most part, causes the PST confusion, since there is a large peak at score 0. PST probably have procedural fluency with the traditional method, but these low-scoring PST lack the flexibility to apply the traditional methods and the justifications for the traditional method to Andrew's strategy. However, there is a surprising peak at score 4 for the PST. PST might be more familiar with the concept of negative numbers when compared to IP's or AP's knowledge of negative numbers. This familiarity with negative numbers from recently taking high school or college level math courses might account for the slight peak at score 4. IP either received a score of 0, 1, or 2. Moreover, out of 9 responses from IP, 3 received a score of 0, 3 received a score of 1, and 3 received a score of 2—a very even distribution. Most AP received a score of 1, 2, or 3. The results for the TL are in complete contrast with the results for the PST. The results suggest that most TL are able to make sense and successfully reason about Andrew's strategy, illustrating adaptive reasoning, conceptual understanding, and procedural fluency. Overall, there seems to be a correlation between the years of professional development and the higher scores for this task.

When the data from the two tasks were compared, several trends seemed to match up. On both tasks, the PST and IP scored relatively low, whereas the AP and TL scored relatively high. Also, the TL scored higher than the AP on average. The tasks were relatively similar in difficulty for the four groups with a similar distribution of scores.

Conclusion

The results of this study indicate that sustained professional development does affect teachers' analysis of and perspectives on student work. Exemplifying the need for specific content knowledge for mathematics, Ball, Hill, and Bass (2005) state, "Every day in mathematics classrooms across the country, students get answers mystifyingly wrong, obtain right answers using unconventional approaches, and ask questions…teachers are in the unique position of having to professionally scrutinize, interpret, correct, and extend this knowledge" (p. 17). This study examined how teachers interpret and respond to a mathematically correct answer obtained by using an unconventional algorithm as evidenced by Andrew's task. Furthermore, in Terry's task, this study examined how teachers interpret the meaning of the procedures used in standard algorithms. This study investigated whether sustained professional development plays a part in teachers' responses. The results from this study suggest that professional development can improve teachers' understanding of students' standard and alternative algorithms.

For both tasks, the results are evenly distributed among all the scores, illustrating that the tasks and the scoring were fair and unbiased. The extra years of professional development seem to allow the participants to explicitly and consistently discuss Andrew's strategy by making connections and using the correct mathematical language, which are characteristics of flexible and complete conceptual understanding. Similarly, for Terry's task the extra years of professional development cultivated a better understanding of place value issues, which otherwise may have been lost in the use of the standard algorithms. The results of this study indicate that sustained professional development positively affects teachers' analysis of and perspectives on student work. With recent national concern for improved mathematics education, this study, in a small way, illustrates how professional development can improve and contribute to student achievement.

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CONTENT COURSES FOR MIDDLE SCHOOL TEACHERS LEAD TO HIGHER SELF EFFICACY

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Introduction and Theoretical Framework

Licensure for middle school content teachers has varied across states as most states do not have a licensure or certification program specially designed for middle school teachers. Thus, many middle school teachers were found to be teaching out of field or did not have proper licensure to teach science or mathematics (NCLB; P.L. 107 – 110, 2002). In response to this finding, the No Child Left Behind legislation required middle school teachers to become highly qualified in terms of their content area. Highly qualified status requires each teacher to have at least 24 credit hours of college level content courses or successful completion of the Praxis exam (NCLB; P.L. 107 – 110, 2002) in the discipline. This has led to close examination of the effects of content courses on knowledge, pedagogical practices, disposition, and self-efficacy. This paper reports the impact of content courses on middle school teacher's self-efficacy.

Self-efficacy is defined as individuals' judgments of their own capabilities to accomplish particular levels of performance and is believed to impact individuals early in the context of new learning (Bandura, 1993). Most researchers believe that self-efficacy is two-dimensional in terms of personal teacher efficacy (PTE) and teaching outcome expectancy (TOE) (Swars, 2005). In other words, PTE is a teacher's belief in his or her skills and abilities to be an effective teacher. Whereas TOE is a teacher's belief that effective teaching brings about student learning regardless of external factors. Studies have shown that teachers with high self-efficacy tend to embrace student-centered teaching methods, use more innovative teaching strategies, and are more likely to use more difficult teaching approaches than teachers with lower self-efficacy (Swars, 2005).

The theoretical framework driving this research involves the knowledge that a teacher with high self-efficacy demonstrates the attributes of a highly qualified teacher. Therefore, this paper explores the following research questions: 1) what external factors influence teachers' self-efficacy scores as reported on the self-efficacy survey form? 2) Is there a significant difference between teachers' personal teacher efficacy and their teaching outcome expectancy?

Method

Context and Participants

The Rocky Mountain-Middle School Math and Science Partnership is a National Science Foundation-funded, 5-year project that targets middle school teachers and students in seven Denver-area school districts. The project links these school districts with faculty from University of Colorado at Denver and Health Sciences Center's College of Liberal Arts and Sciences and School of Education and Human Development as well as faculty from four other university partners to increase the subject-matter content and pedagogical content knowledge of middle school teachers. In No Child Left Behind language, the project is trying to increase the numbers of "highly qualified" middle school teachers. The project's primary component is to provide math and science content courses to middle level teachers. Since the project's inception in 2004, fifteen content-based math and science courses have been developed and co-taught by faculty from the College of Liberal Arts and Sciences, School of Education and Human Development, and our K-12 partners. Ninety teachers participating in the MSP grant courses responded to the self efficacy survey.

Survey Instrument

The survey instrument selected to measure teachers' self-efficacy was the STEBI-B. This instrument was developed by Riggs and Enochs (1990) and was based upon the Teacher Efficacy Scale (TES) developed by Gibson and Dembo (1984). The TES is a 30 item instrument that measures teacher efficacy on two scales – PTE (Personal teaching efficacy) and GTE (general teaching efficacy). The STEBI-B was developed to better analyze science teachers self-efficacy it consists of 23 items and is also broken down into two scales – Personal Science Teaching Efficacy Belief (PSTE) and Science Teaching Outcome Expectancy (STOE).

The actual survey instrument used was based upon the STEBI-B's 23-question survey, but was modified to include math and science. The research team chose not to administer the MTEBI (Enochs, Smith, & Huinker, 2000), the instrument that focuses on mathematics teachers self efficacy, as teachers participating in grant courses took both math and science courses. In addition, they felt this allowed consistency in data analysis. Six additional questions were added to the survey in order to measure whether low motivation in students can affect self-efficacy, and whether working with English as Second Language (ESL) students affects self-efficacy.

Analysis

Using the Statistical Package for the Social Sciences (SPSS), a principle factor analysis with varimax rotation was conducted on the data to assess the underlying structure for eighteen of the items on the original STEBI-B. The six added questions were eliminated from the analysis, as the reliability is still being examined. We also eliminated questions 3, 11, 13, 14, and 20 due to wording and multicollinearity issues. Two factors were requested, and after rotation, the first factor accounted for 21.8% of the variance, and the second factor accounted for 37.4% of the variance. The results compare favorably with the two previously conducted STEBI-B reliability studies (Riggs & Enochs, 1990; Bleicher, 2004). Reliability scales tests were also conducted to determine the Cronbach alpha for the two sub-scales. Paired samples t-tests were conducted to detect if statistically significant differences occurred between the two sub-scales and the total self-efficacy score. Multiple regression and bivariate correlations were conducted on the background variables (district employed, teaching area, type of endorsement, type of degree, and number of MSP courses taken) to determine if predictor variables exist and if there were statistically significant differences between group means.

Results

Factor analysis results indicated the presence of the two factors PSTE and STOE, suggesting the results are reliable for determining the teachers' efficacy scores in the two categories. To assess whether the items included in the two factors formed a reliable scale, Cronbach's alpha was computed. The alpha for the PSTE items was .84 and the alpha for the STOE items was .81, which indicates that the items form a scale that has good internal consistency reliability. Paired samples t-tests were conducted to determine if group mean scores on total self-efficacy, PSTE efficacy, and STOE efficacy had statistically significant differences. Assumptions were checked and met. The total self-efficacy mean was significantly lower ($t(89) = -7.998$, $p < .001$, $d = -.84$) than the PSTE mean, the total self-efficacy mean was significantly higher ($t(89) = 7.556$, $p < .001$, $d = .80$) than the STOE mean, and the PSTE mean was significantly higher ($t(89) = 8.133$, $p < .001$, $d = .86$) than the STOE mean. The effect sizes are considered large for the area of study. The assumptions for multiple regression were markedly violated, thus bivariate correlations were conducted and a statistically significant positive correlation ($r(89) = .213$, $p = .045$) did exist between the STOE average and the number of MSP courses taken.

Discussion

This study reveals that teachers' outcome efficacy is positively impacted by the number of content courses taken in science and math. Generally speaking, most teachers have a strong personal efficacy yet their outcome efficacy is significantly lower. However, when the number of content courses that a teacher participates in increases we have found a positive correlation with their level of outcome efficacy. Self efficacy is critical in teaching, and research has shown that student achievement is higher when teachers have higher self efficacy both personal and outcome based. However, research has shown that teacher self efficacy is most malleable during teachers' induction years.

To this end, an important finding that this study adds to our literature base on teacher education is that self-efficacy can be changed in experienced teachers. Further qualitative analysis of these data is being conducted to better understand how mathematics and science courses further enhanced teachers' self-efficacy.

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DOCUMENTING THE EMERGENCE OF "SPEAKING WITH MEANING" AS A SOCIOMATHEMATICAL NORM IN PROFESSIONAL LEARNING COMMUNITY DISCOURSE

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The purpose of this paper is to describe the sociomathematical norm of speaking with meaning. Speaking with meaning reflects the type of mathematical communication expected when a group of individuals are engaged in problem solving. We observed the emergence of this norm in professional learning communities comprised of mathematics and science teachers and use this data to illustrate its usefulness.

Introduction and Background

This paper describes the sociomathematical norm of *speaking with meaning* and its emergence in a Professional Learning Community (PLC). After studying the interaction patterns of four PLC's over one year, we observed that the quality of mathematical discourse was not very high. Based on these findings we introduced the term *speaking with meaning* as a way of making the nature of the discourse that we wanted to emerge among the members of the PLC more explicit.

The PLC's are defined as a collection of math and science teachers from the same school (ideally) with one teacher designated as a peer facilitator. The PLC design draws heavily from investigations of lesson study. The participants in the PLC's for this study were taking a graduate course that was focused on developing their understanding of the function concept. The purpose of the PLC's was to engage teachers in *meaningful discourse* about issues of learning and teaching mathematics content related to what they were learning in the course. For the purpose of this research we describe "meaningful discourse" as communication about knowing, learning and teaching that draws on coherent understanding of the content and the process of learning the content.

Each PLC has an assigned peer facilitator who manages the discourse for the PLC. The facilitator is initially trained and supported through weekly coaching sessions. Facilitators are also provided a PLC agenda that specifies points of discussion, questions and social norms for PLC interactions. Facilitators are encouraged to monitor PLC interactions so that the PLC members listen to and try to make sense of each other's solutions and offer justifications for their solutions. In this setting, the facilitator is trained to ask questions that promote *speaking with meaning*. They probe PLC members for clear articulation of their thinking and press PLC members to offer meaningful justifications for claims and statements. Within this environment, the PLC members' actions reveal what they believe are acceptable forms and patterns of communication.

Theoretical Perspective

Thompson, Philipp, and Thompson (1994) denote distinct differences between calculational and conceptual orientations. A calculational conception implies that a correct solution need only be justified using calculational sequences which are judged by criteria which may not be explicit to the whole audience. People who have a calculational conception have a tendency to give responses that consist primarily of numbers or numerical operations and procedures for arriving at an answer. In contrast, an individual who has a conceptual orientation is more concerned with the overall context within which a problem lies. He or she is also more focused on a broader system of ideas and ways of thinking and speaks about quantities and relationships when describing approaches or solutions. His or her explanations are typically grounded in the context and conceptions of the problem. The concept of *speaking with meaning* draws heavily upon the notion of a conceptual orientation. It is used to describe the type of "meaningful discourse" that is expected when individuals are involved in problem solving. *Speaking with meaning* implies that responses are conceptually based, conclusions are supported by a mathematical argument, and explanations are given using the quantities involved. Our viewing *speaking with meaning* as a norm reflects our observation that it has emerged as normative behavior within some PLC's.

Sociomathematical norms refer to normative behaviors that are specific to mathematics, such as understanding what constitutes an acceptable mathematical solution, and emerge from what counts as acceptable mathematical behavior in the classroom (Yackel & Cobb, 1996). The sociomathematical norm of *speaking with meaning* is used to illustrate the type of mathematical behavior expected of the teachers as they participate in their PLC's. In fostering the emergence of the sociomathematical norm for what constitutes a sufficient explanation or justification, the teachers established criteria for what it means to *speak with meaning*. These justifications need to be conceptual and embedded in the context of the problem. For example, explanations regarding rate would need to include language that describes how the amount of distance covered changes when considering changes in time. It is important to note that this lens for viewing our data did not emerge until after observing the discourse of the PLC's. It was in watching videos of the PLC's that we observed *speaking with meaning* emerging as normative behavior of the PLC.

Discussion and Implications

In order to establish PLC's that engage in meaningful discourse with regards to teaching and learning mathematics, it is important to be aware of some interaction patterns that are more typical in low functioning PLC's. During the first year of studying PLC interactions we observed teachers who spoke using partial phrases. Also, their solution explanations were often incoherent and did not connect to the context of the original problem. Based on these observations we created interventions aimed at improving the quality of discourse in the PLC'S. In the first class of the semester, the instructor managed a discussion with the class in which she negotiated productive patterns of communication for the class and PLC. From their negotiation emerged the term *speaking with meaning*. Teachers appeared to agree that they should attempt to speak meaningfully when discussing ideas and solutions.

Based on analysis of our data we found that it is difficult for inservice teachers to *speak with meaning*. It is important that the facilitator ensure that PLC members not only justify their own comments, but also probe each other when utterances are offered that are vague, incoherent or lacking meaning. We found that facilitators who were coached on specific actions to promote *speaking with meaning* were more effective in moving the PLC toward *speaking with meaning* as a norm that was spontaneously enacted in discourse within the PLC. The facilitators benefited by hearing specific suggestions that were based on observed shortcomings in the facilitators' actions during his or her PLC.

This research has applications to both theory and practice. This study intends to add to the theoretical constructs sociomathematical norms. Specifically, *speaking with meaning* refers to the kind of normative behavior we would like to observe in a PLC of secondary mathematics and science teachers. Within that scope we use the phrase *speaking with meaning* to encompass the ways in which PLC members should communicate with one another regarding mathematics.

What counts as *speaking with meaning* is negotiated in a PLC and can differ from PLC to PLC.

The term *speaking with meaning* emerged from negotiations during the first day of class. The term carries with it the ability to operationalize what is a sufficient explanation; it also describes the attributes of meaningful mathematical communication. It has brought clarity to how to make a sufficient justification and is now an intervention that the teachers can use in their classrooms.

Speaking with meaning has the dual nature of being both a theoretical construct and an intervention. Researchers will be able to use this construct to gauge the quality of mathematical discourse in the teachers' classrooms. It was also used to help train facilitators so they could better manage the discourse of their PLCs. Further, *speaking with meaning* provides both teachers and researchers a lens with which they can judge the effectiveness of their attempts to enact and promote *speaking with meaning*.

By observing the emergence of *speaking with meaning* within a PLC we are becoming more aware of interventions and actions that may lead to *speaking with meaning* becoming normative within a PLC. This can help inform training of facilitators so that they can better engage the other members of their PLC. As these facilitators are peers of the other group members, they do not have extensive formal training necessary to bring about these normative behaviors. Therefore, appropriate training methods and interventions are very important in preparing the facilitators to manage the discourse within their PLC's. Becoming more aware of actions that produce *speaking with meaning* as a spontaneous behavior among all members of the PLC will help improve interventions designed to support *speaking with meaning* as a norm. Research reported in this manuscript was support by National Science Foundation grant number HER-0412537

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FROM PROFESSIONAL-DEVELOPMENT PARTICIPANT TO DOCTORAL CANDIDATE: THE JOURNEY OF ONE MIDDLE-SCHOOL MATHEMATICS TEACHER

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This case study reports on changes in the practices of one sixth-grade mathematics teacher over four years of professional development and implementation of new curriculum materials. Analysis of data points to an unexpectedly dramatic change in her knowledge, perspectives, and classroom practices. This study attempts to identify factors critical in producing such a dramatic and lasting impact.

The journey of Celia, from professional development participant to doctoral candidate, is both dramatic and inspiring. With 7 years of teaching experience, Celia lacked confidence and depth of knowledge of mathematics. She embarked on what turned out to be a four-year odyssey of professional development activities that resulted in lasting changes in her knowledge, perspectives, and practices. The question arises, "What factors were critical in producing such a dramatic and lasting impact on this teacher?" This paper will attempt to answer this question.

Background

Analyzing and describing the impact of professional development activities has the potential to significantly improve the design of such activities (Becker & Pence, 2003; Farmer, Gerretson, & Lassak, 2003; Murata & Takahashi, 2002). However, a report by RAND (2003) suggests that professional development practices have not been studied extensively and there remains a great need for well-developed descriptions of successful professional development practices

In this study, we address three questions. First, what changes in Celia's classroom practices, perspectives, and knowledge emerged and remained over the four-year period? Second, what professional development activities were critical to effecting lasting change on her knowledge, perspectives, and practices? Third, what factors external to the professional development activities appear to have impacted their effectiveness?

Perspectives and Framework

Our perspectives on the nature of mathematics, learning, and teaching guide our design of professional development activities. We believe that understanding of mathematics involves (1) the ability to solve problems, (2) the ability to connect mathematical ideas to one another and to real-life contexts, and (3) the ability to communicate mathematics to peers. We believe that learning of mathematics is best achieved by active engagement in activities that call on and develop the abilities involved in understanding of mathematics. Thus, the teaching of mathematics should be designed to provide such opportunities to students and professional development should help teachers develop, implement, reflect on, and revise such learning opportunities.

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Research Methods

This research employed a qualitative, case-study design. To establish descriptors of this teacher's lasting changes in her knowledge, perspectives, and practices, we conducted an interactive interview at the end of year 2, solicited responses to structured journal prompts at the end of year 3, and took structured notes during observations of implementation of new curriculum materials in years 1-4. To investigate possible causes for lasting changes, we also retrospectively analyzed data collected at an earlier point in our study (Roddick & Bergthold, 2004). At that point in the study, we followed an "account of practice" strategy developed by Tzur, Simon, Heinz, & Kinzel (2001), gathering data from teachers' written responses to structured journal prompts, oral comments made during interviews, and actions exhibited during classroom observations.

Case Study Subject

The teacher in this case study, Celia, was purposefully selected. Over one year of content-based professional development institutes, taught by the authors, she demonstrated initiative to improve her teaching, a desire to learn new content, a willingness to try new practices, and engagement in ongoing reflections. Celia, with 7 years of teaching experience, had taken the required 6 hours of mathematics coursework for future elementary school teachers while in college. However, she lacked confidence in her mathematical knowledge and abilities and demonstrated little breadth or depth in her knowledge of mathematics.

Celia participated in an intensive 20-hour, one-week workshop in summer 2003, where she collaborated to develop and adapt mathematics learning activities for sixth-grade mathematics courses. These activities were designed to address more than one concept and sought to make connections among concepts. Throughout the following academic year these activities were pilot tested, reflected upon, and refined. In the two years after this, the refined activities were implemented again with additional reflection and new activities were designed and pilot tested.

Celia's Journey (Results)

Over the four years Celia exhibited dramatic, lasting changes in her knowledge, perspectives, and practices, relative to where she began. With 7 years of teaching experience, Celia lacked confidence in her mathematical knowledge and abilities and demonstrated little breadth or depth in her knowledge of mathematics. She had many of the same frustrations felt by other mathematics teachers, yet her interest in professional development was evident. Over the course of the study her motivation to change was fueled by the support from her principal, belief in her need to improve her teaching, and the enthusiasm of her students as changes were implemented in her classroom.

Celia's knowledge of mathematics shifted from relatively limited knowledge of basic algebra techniques to a much broader array of conceptual knowledge and procedural abilities, dramatically increasing her confidence. She came to view mathematics as a connected body of knowledge, inspiring a desire for her students to experience mathematics in the same way.

For Celia, the intensive summer workshop to develop new learning activities was a catalyst for complete revision of her perspective on teaching. She began to rethink how, what, and why she was teaching, and decided to make major changes. Instead of reviewing at the beginning, she believed that a unit on problem solving would point her students in the direction she wanted to go for the rest of the year. She began to view patterns as a central theme throughout the sixth grade mathematics curriculum. By the end of the project, she believed that learning activities were important elements in successful teaching.

Over the course of the project, Celia's teaching focused more and more on problem solving, exploration, and activity-based lessons. She sought ways to implement activity-based, inquiry-based lessons rather than traditional lecture-and-homework practices. Over the course of a school year, she incorporated approximately fifty activity-based lessons into her teaching. Ongoing development of new activities in the following two years formed the basis for further inquiry, leading to completion of a master's degree and enrollment in a doctoral program in education.

We also observed Celia become a voice for change in her school's mathematics department. As her students' enthusiasm grew, word began to spread throughout the school that her students were excited about mathematics. Administrators encouraged other teachers to observe Celia's class, thus promoting important discussions among the mathematics teachers in her school.

Discussion and Implications

Celia's newfound knowledge of mathematics opened up a whole new world for her. The more she learned, the more she wanted to learn, and the more she wanted to convey this kind of understanding to her students. The forced development, implementation, observation, discussion, and revision of learning activities during the formal professional development part of the project had a dramatic impact on her confidence and willingness to continue with this kind of teaching practice.

There appeared to be two critical external factors to Celia's success. First, with only 7 years of teaching experience, Celia did not have a career's worth of experience to overcome in developing new understandings, perspectives, and practices. Second, she was sufficiently unhappy with her own teaching that she was ready to adopt something completely different than what she had been doing all along.

Over the course of this project, we found there were both professional development activities and external factors that were critical in effecting significant change for Celia. Although the journey of this middle-school teacher could neither be considered commonplace, nor strictly the result of the professional development program, it does demonstrate what is possible for a dedicated and motivated teacher, given sufficient support through her school and appropriate professional development activities.

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INVESTIGATING THE SCALE-UP OF A TECHNOLOGY-RICH INNOVATION

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This report examines innovation scale-up within the 'train-the-trainer' model of professional development. We investigate how ideas central to the mathematics, technological environment, and pedagogical practices, are envisioned, interpreted and enacted as the professional development proceeds from researchers to teacher-educators to teachers. Preliminary analysis reveals variation in perceived benefits of SimCalc Mathworlds™*, which manifests itself in teachers 'privileging' certain aspects of the project.*

While there is sound evidence of technology's potential to help students deepen their mathematical understanding, few tools have crossed the gap from research prototype to wide-scale implementation (Roschelle & Jackiw, 2000). The SimCalc Scale-up project (1) was designed to focus specifically on issues of scale: Can innovative technology-based mathematics scale up to be useful for a wide variety of teachers in a wide variety of settings?

Our research, a sub-study of this larger project, examines innovation scale-up within the 'train-the-trainer' model of professional development. In essence, we investigate how ideas central to the mathematics, technological environment, and pedagogical practices, are envisioned, interpreted and enacted as the professional development proceeds from researcher to teacher-educators and ultimately to teachers.

Context

SimCalc Mathworlds™ is innovative software which supports the creation and modification of functions as well as providing the means to simulate motion based on these functions. The SimCalc Mathworlds™ environment links graphical, tabular and algebraic representations, through simulations, which students can algebraically or graphically edit, thus providing the student opportunity to investigate how changes in one representation impact other representations (Kaput & Schorr, in press).

Much of the current research on teacher professional development projects focuses on an individual program at a single site or at closely related sites. While it is beneficial to engage in comprehensive professional development over extended periods of time, many projects lack the resources to do so. Consequently, PD often involves focusing on several people who then share their knowledge with a wider audience (often referred to as a "train-the-trainer" model). In this study we investigate what Borko refers to as Phase 2 research: studying a single PD program enacted by more than one facilitator at more than one site (Borko, 2004). The central goal of such research is to determine if the program can be enacted with integrity especially as it becomes more and more removed from the original PD providers.

This research examines a portion of the SimCalc scale-up project in which the SimCalc Mathworlds™ software was used in combination with a 3-week replacement curriculum unit for 8th graders. For this intervention, responsibility for teacher training was passed from the research team to the conventional professional development model in place in Texas, in

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Stateline (Lake Tahoe), NV: University of Nevada, Reno.

which the Dana Center (housed at the University of Texas) trains teacher-educators through a Training-of-Trainers (TOT) workshop, who then train teachers in their region. In this case, teachers then implemented the SimCalc unit within their normal classroom practice.

This situation provided the opportunity to investigate all levels of the 3-stage 'train-the-trainer' (TOT) model of professional development, specifically within the context of a technology-rich innovation. In particular, our research question is: How did the participant researchers, teacher-educators, and teachers perceive:

1. The mathematical goals of the intervention, and
2. The role of SimCalc Mathworlds™ in accomplishing these goals.

In the course of videotape analysis, we highlight what Kendal and Stacey define as *privileging (Kendal & Stacey, 2001)*. For our purposes, *privileging* will refer to a facilitator or teacher highlighting or giving priority to certain aspects of the intervention over others.

Methods

Both TOT and teacher-training workshops were observed and videotaped, and all workshop facilitators were interviewed. At least two teachers from each ESC region were then selected, based upon availability, for classroom observation by the first author. Each classroom observation took place over the course of 2-3 days. In general, videotaping focused on the facilitator (during workshops) and the teacher (in the classroom). Teachers were interviewed after each session. Interviews were transcribed, noting in particular the teachers' (tacitly or explicitly) expressed mathematical and SimCalc-related goals. Videotapes were analyzed based on several criteria including emphasis made on certain types of representations, solution strategies, use of the technology and overall teaching strategies.

Results

Within the interview data, we have noted variation in the way participants perceive the benefits of the software. The TOT facilitators emphasized how the software allows an inversion from the usual order of student learning: instead of going from equations to graphs to motion, SimCalc allows the student to go from motion to graph to equation, leading to potentially deeper understanding "because they don't have to mediate it all through the symbols, they can use the graphs and the simulations to understand the symbols." The teacher-educators tended to focus on the benefit of linked multiple representations. Variation was somewhat more pronounced among the teachers. Some used terms such as "visual", "hands-on", and "manipulative" to describe the main affordances of the software, while others focused on the fact that students could change something and immediately see how this effected a change elsewhere.

Preliminary analysis appears to indicate that these are more than just superficial differences in language, and may be indicative of deeper differences in the ways in which the software and associated curriculum materials were used by the various teachers. Further, it appears that these differences exist within ESC regions, despite the fact that all teachers within a region took the same training course. Finally, the types of differences found appear to be similar to those found across other regions.

The following example describes differences in perceived benefits of the software and observed privileging of different mathematical goals. Teachers 01 and 03 attended the same training course, but expressed somewhat dissimilar views on the software. Teacher 03 states "It helps them [low achievers] most definitely because by the time they would take an equation, generate a table and then plot those points, they will have forgotten why we were doing all of that …. And they would go oh I don't know it's just a graph. Where as now, it's there for them. *They have to interpret that graph* [emphasis added].” Teacher 01 was positive in many ways, but also expressed concern that the students “lost the actual experience of collecting the data. And plotting the points … it’s already plotted for them, so they don’t have to actually have to use the x and y axis …”

In a lesson occurring midway through the curriculum unit, both classes were given the task: “Our goal for next year is to end the year with $110,000 in our bank account [current balance is $50,000]. We would like to put the same amount of money in the bank each month. Complete the graph below to show this.” In Teacher 01’s class, students sketched the straight-line graph as the teacher read the following question: “How much money do we need to put in the bank each month to reach our goal?” One student immediately responded “$5000”. His explanation was as follows: “We’re at $50,000, and I counted $5000 for each one.” This explanation, in combination with the video data, suggests that he may have remembered the dynamic simulation of a similar problem (*see figure 1)*, and was visualizing the growth over the time period—a positive result of the software.

Figure 1: Simulation and Graph in MathWorlds

In a subsequent interview, Teacher 01 reflected: “He didn’t necessarily know how to verbalize, well I take the sum of the increase that was put in the account, and divided it by 12 months, to get an equal deposit each month. So, some of them can do the mental math, but some of them don’t know how to verbalize it and show it. I think that’s something that we need to focus more on tomorrow.” We note that she appeared to “privilege” the calculation method of solution, and the curricular goal to have students verbalize their answers.

In contrast, when Teacher 03 reflected on her experiences with this problem, she stated: “When they drew it … some of them said, oh that’s 5,000 … well, they know to read this graph to get their answer, so.” In this case, it would appear the teacher felt comfortable with graphical representations even in the absence of calculation-based solution methods.

Continued analysis will allow us to further describe the perceptions held by participants, and how these may be similar or different across training sites. This research holds the possibility of contributing to the existing knowledge base regarding future scale-up projects, especially those with an emphasis on technology, using this professional development format (minimal research team influence beyond the “train-the-trainer” sessions).

Endnote

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PROFESSIONAL DEVELOPMENT COLLABORATIVE: SHIFTING PROSPECTIVE TEACHERS' GEOMETRY CONTENT EXPERIENCES TO FIRST GRADE CLASSROOMS

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This study focuses on professional development of first grade teachers by immersing prospective teachers, taking a geometry content course, into their classrooms to carry out mathematics instruction by way of a teaching experiment. A case study methodology was embraced and data collected to establish factors of importance in identifying needs for changing teachers' classroom instruction.

Although the *Professional Standards for School Mathematics* (NCTM, 1991) has been around for 15+ years, progress in embracing what this document supports has been slow in coming into classrooms. The standards identified four key areas for teachers to adhere to:

- Setting goals and selecting or creating mathematical tasks to help students achieve these goals;
- Stimulating and managing classroom discourse so that both the students and the teacher are clearer about what is being learned;
- Creating a classroom environment to support teacher and learning mathematics;
- Analyzing student learning, the mathematical tasks, and the environment in order to make ongoing instructional decisions (p. 5).

Mathematics educators certainly agree that these areas are critical, but how to help teachers transform their classrooms into such learning environments does not happen overnight and certainly does not happen in isolation.

Professional development has been a growing area of research in all areas of education and has continued to receive widespread attention with the NCLB Act implemented in 2002. One problem that educators continue to deal with is that professional development, as a whole, remains unfocused (Cwikla 2004) and at times misdirected. Thus, depending on local and state guidelines, professional development has no uniformity for teachers from one district to another. While this may pose some inconsistencies in how professional development is organized and carried out, one area that researchers are focusing greater attention on is use of classrooms as professional learning experiments. The professional community is collecting evidence that indicates the best environments to study mathematics learning and teaching are today's classrooms (Ball and Cohen 1999; Clark 2001).

One strategy of professional development that the mathematics education community really has not studied in any depth is the use of field-based experiences of prospective teachers to promote professional development for inservice teachers. By providing prospective teachers opportunities to plan, work with children in real elementary classrooms, and dialogue with teachers about the instruction, inservice teachers have the potential to gain new insights and interpretations of presenting material for young learners in their classrooms. While targeting both groups is important, the focal point of this study is on the inservice teacher since little attention has been given to researching this group in such a manner.

The purpose of this study is to highlight factors that support teacher change when conducting professional development programs or institutes. Changing or adjusting mathematics instructional practices to be "more in line with" documents like the *Standards*

(NCTM, 1991, 1995, 2000) is a complex and oftentimes slow process and the mathematics education community recognizes that teachers need help. This study also has the potential to impact not only inservice teachers' practices, but also prospective teachers' future practices.

The Setting

The study took place in an elementary school in a southeastern state with diverse demographics. More than _ of the school population received free or reduced lunch. Although this particular school district is considered to be in a rural to suburban geographic location, this particular school has a large number of minority students and is known to have low parental involvement and participation in school functions. While the school was considered one of the "lower-end" schools in this particular district, teachers and staff held high expectations of the students and their performance in the classroom.

The participants in the study were four first grade teachers – one male and three female. The number of years teaching elementary school ranged from one to 18 years, where one teacher had been in the school her entire teaching career and the other three were new to the school that year. One teacher, Ms. Jones, will be presented.

Methodology: Data Collection, Analysis, and Results

The research reported in this study employs a classroom teaching experiment design where prospective teachers' planned and carried out instruction, first grade students' were receivers of the instruction, and inservice teachers' were involved directly and indirectly in the teaching and learning experiment. A case study methodology was embraced and data from this experiment was collected by use of teacher observations and interviews.

Themes from the observations were generated and coded as well as data collected from teacher interviews. Glaser & Strauss's technique of constantly comparing the data to generate new categories as necessary was also implemented (1967). The themes or categories were used to identify factors or patterns of importance in adjusting teachers' own instruction in light of the mathematics reform movement. Lastly, the themes were used to make an attempt to generate a model for carrying out professional development that promotes teacher change.

McClain and Cobb's work (2001) was used as a framework to guide this particular study. Their research analyzed the development of sociomathematical norms in a first-grade classroom to determine what teachers might do to support the types of experiences advocated in the reform documents. While this particular study is not searching for answers to the same problems, it certainly lends itself to aspects of this particular study.

While there were numerous benefits for the prospective teachers in carrying out this teaching experiment, Ms. Jones gained a great deal from the collaborative. She was an experienced teacher with 18 years in the classroom at the same school. Her strength in teaching was in language arts – particularly in reading and creative writing. When it came to mathematics, Ms. Jones was quite traditional in her use of worksheets with very few activities involving "hands-on" materials or development of concepts. Although she was open to new ideas, she often expressed difficulty in keeping her students "on task" when she allowed for more open instructional experiences. She addressed concerns early on about the level of difficulty of some of the situations the university students organized to present to the children but was open to them "trying them out." What she came to find out was that her first graders could handle the explorations and were totally immersed in making sense of the material. After students had completed the activities, she shared a number of comments:

- I will be certain to use that story and tie it to the math lesson for the week. I love books and reading and hadn't thought much about connecting math and literature.

- I hope to find additional ideas to build off what your students did in my class.
- I could not believe the types of responses my students gave the teachers. I didn't think some of them could think that deeply.

After compiling data, four main professional development categories emerged:

- **Support** to develop new activities/situations that are in-line with State standards;
- **Collaboration** with others – in this case it was the prospective teachers in the mathematics content course;
- **Learning** content, how children think, and how different children are in their interpretations; and
- **Stimulation** of the learner to engage in something that is exciting to him/her.

While these factors have appeared in other research related to professional development and teacher change, there were some minor variations in what Ms. Jones professed. One of the major factors for her was changing her mindset to accept that prospective teachers had "worthy" ideas to implement into classroom instruction. She indicated that she often found student teachers lacking in solid instructional ideas – it ended up being more "glitz and glamour." This experience encouraged her to look at the novice a little bit differently.

Ms. Jones also expressed a positive reaction to the notion of never having to leave her own classroom to look at her own teaching practices and the impact they have on her students. She saw and heard things from learners that she had never seen or heard before and reflected on how she could incorporate more valuable mathematical experiences – geometry and other areas – into her classroom instruction. This was crucial for her in looking at this experiment as a professional development opportunity.

Discussion

While there is growing concern about the mathematical preparation of teachers (NCTM, AMS, MAA), educators continue to grapple with how to support teacher change for those who are presently in the classroom. This is especially true of elementary teachers who are responsible for teaching mathematics. Although this study did not tread on new territory, the design of it was somewhat unique in looking at how to offer teachers professional development opportunities without leaving the confines of their classroom. Implementing teaching experiments for professional development is one that many teachers may "buy into" since they will actually have some "payback" in their own classroom.

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STRENGTHENING MATHEMATICS TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE THROUGH COLLABORATIVE INVESTIGATIONS IN COMBINATORICS

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The NSF-funded Rice University Mathematics Leadership Institute immersed high school lead teachers in collaborative, combinatorics problem-solving experiences during an intensive four-week summer institute. The program challenged participants' pedagogical content knowledge and their views about collaborative problem solving as evidenced by statistically significant gains in test scores, their self-reported ratings on content and problem-solving abilities, and excerpts from journal writings.

According to Freudenthal (1991), only a small minority learns when mathematics is taught as a ready-made subject in which students are given definitions, rules and algorithms from which they are expected to proceed. Instead, learners should be given opportunities to reinvent mathematics in the manner that mathematicians create mathematics. Within professional development settings, in order for teachers to reinvent mathematics in this way and experience disequilibrium as their students do, the content should be in a domain which teachers typically do not know deeply.

Through a combinatorics domain problem-solving approach, the NSF-funded Rice University Mathematics Leadership Institute (MLI) (1) provided teachers opportunities to reinvent their mathematics pedagogy and knowledge. This allowed teachers to experience firsthand the instructional approaches they were expected to use with their own students (NSDC, 2002). We used challenging mathematics content with deliberately modeled learner-centered pedagogical approaches to develop teachers' pedagogical content knowledge (Shulman, 1986).

Program Methodology and Analysis

Thirty-two high school mathematics lead teachers, participating in their second of two intensive one-month summer institutes, were immersed in mathematical experiences for which they earned graduate credit. This experience served as the catalyst for reinventing teachers' views on the learning of mathematics. We adapted problem sets in the combinatorics domain from Park City Mathematics Institute materials (Kerins, Sinwell & Matsuura, 2004). We introduced teachers to this experience using the Simplex Lock problem, which they were invited to solve by the end of the program—once they had acquired a deeper understanding of the concepts through their experiences with collaborative problem solving. Each day, teachers solved problem sets consisting of Essential Problems (for everyone to solve), Neat Problems (similar to the essential problems but reaching toward the next topic) and Challenging Problems (for those who were ready for an intellectual stretch). Teachers were not expected to complete all problems and could solve them in any order. On the last day of the program, some teachers shared their solutions to the Simplex Lock problem. Those who had not solved it followed various solutions competently because they had acquired sufficient conceptual knowledge during the program. Table 1 shows representative problems from the first and third weeks of the program.

Table 1
Sample Problems from the 2006 MLI Summer Leadership Institute Combinatorics Strand

Essential Problems	**Neat Problems**	**Challenging Problems**
Week 1		
A "train of length 5" is a row of rods whose combined length is 5. 1. How many trains of length 4 are there? 3. Find a formula for the number of trains of length n. Come up with a convincing reason that your rule is correct.	7. If there are three flavors of ice cream, how many different three-scoop cones can you make using each flavor exactly once? 12. If you can make 220 different three-scoop bowls of ice cream, how many different three-scoop cones can you make?	17. What's the mean length of car used when you make all the trains of length 5? Is there a general rule at work here? Can you justify it? 18. How many three-scoop bowls could you make at Ben & Jerry's if you were allowed to duplicate flavors? Is there a general rule?
Week 3		
1. {{C}, {V}} means a bowl with one scoop of chocolate and a bowl with one scoop of vanilla. Here are some possible combinations: •{ } • {{C, V}} • {{C, V, S}} • {{S, V}, {C}} • {{S}} • {{V}, {S}}. How many different desserts are possible? 3. Without a calculator, expand $(h + t)^5$. 5. Spend at least 15 minutes thinking about the lock problem.	9. In a box of 12 batteries, it is known that 5 are dead. Four batteries are selected at random. Find the probability that (a) exactly one dead battery is selected. (b) all four of the selected batteries are dead. (c) at most two of the selected batteries are dead.	12. In row 7 of Pascal's Triangle, the numbers 7, 21, and 35 appear consecutively. Interestingly, these three numbers form an arithmetic sequence. Does this ever happen again? If so, find the next three times it happens. If not, prove it can't happen again.

We administered pre- and post-tests based on items similar to the Essential Problems. Test scores of teachers were used to conduct a paired samples t-test to measure the change in their combinatorics content knowledge as a result of participating in the summer institute. The highest score possible was 40. Teachers' mean score on the pre-test was 10.4 (median = 11). The post-test mean score on the same measure was 33.9 (median = 36.5). The change in score was statistically significant, t (df = 31) = –20.79, $p < .0001$ (2) and suggests that teachers' combinatorics content knowledge increased significantly as a result of participating in the summer institute.

Individuals can be transformed by critically reflecting on situations that are not consistent with their preconceived notions. Such situations or dilemmas often prompt new interpretations of experiences. Facilitating such transformative learning experiences are most effective when instructors assist learners by encouraging them to examine or question assumptions that underlie their beliefs, feelings, and actions; assess the consequences of their assumptions; identify and explore alternative assumptions; and test the validity of their assumptions through reflective dialogue (Mezirow, 2000). Providing opportunities for teachers to reflect on experiences that do not fit their preconceived ideas about or understanding of mathematics and how it is taught is a major component of the MLI's approach to teacher development.

Collaborative settings in which participants can engage in discourse and reflection to consolidate their knowledge assists learners adapt prior knowledge to new challenges (Freudenthal, 1991) particularly in. During the MLI's combinatorics instruction, answers or solutions to problems were not provided. Instructional staff deliberately refrained from affirming answers but prompted teachers to discuss their results with their peers. Teachers' journal entries revealed their early frustrations. For example,

- *What I hate is that I have people monitoring me and cannot answer if my solution is correct.*

Later, their acceptance and excitement about the learning experience emerged:

- *The combinatorics problems are finally starting to make sense. From day one, as a group we generated answers and answers only. I had an answer but could not explain my thought process. They are more challenging every day and it challenges my understanding to not just get an answer but explain my process and thought.*
- *I worked with a new group today. It is wonderful to see how different people work on a problem.*
- *I wish I could get across to my students the satisfaction of the challenge of an advanced math problem. I want them to enjoy the struggle the way I am.*

At the conclusion of the 2006 summer institute, based on a post-program survey, over ninety percent of teachers agreed that they had increased their knowledge and understanding of how to solve combinatorics problems as well as their ability to solve mathematics problems.

Endnotes

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2. Preprogram N = 31; Postprogram N = 32—the larger postprogram N was due to the delayed arrival of one teacher during the summer institute.

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TALKING MATHEMATICS: A CASE STUDY OF ONE KINDERGARTEN TEACHER'S PRACTICES TO SCAFFOLD MATHEMATICAL DISCOURSE

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This study documents one kindergarten teacher's journey as a reflective practioner as she implements research-based practices that facilitate students' voicing of mathematical thinking and reasoning in discussions. The process through which the teacher established the norms of verbalizing and justifying one's ideas was replete with challenges, successes, and pedagogical transformation.

Educational research promotes mathematical discussion as a necessary component of mathematics learning. However, facilitating student discussion of mathematical concepts and procedures and encouraging students to explain their thinking can be difficult (Smith, 1996). This study describes one kindergarten teacher's journey as a reflective practitioner as she implements research-based discourse practices that scaffold her students' ability to give voice to their mathematical thinking and reasoning.

Background and Theoretical Framework

Drawing from situated theories of learning (Lave & Wenger, 1991), we posit a dynamic view of mathematics wherein learning is a process of negotiating meaning within a community of practice and learning to participate in the larger discourse of mathematics (Goos, 2004). In order to participate in the mathematics community, it is essential students are "making conjectures, abstracting mathematical properties, explaining their reasoning, validating their assertions, and discussing and questioning their own thinking and the thinking of others" (Lampert, 1990, p.32). Scaffolding this kind of mathematical discourse presents unique challenges in lower elementary grades where students may be unfamiliar with collaborative learning, relatively inexperienced with listening to one another, and unaccustomed to verbalizing and justifying their mathematical ideas. In this study we seek to better understand the teacher's role in supporting students' emergent and varying abilities to "talk mathematics" and participate in the discourse of a mathematics community.

Methodology

This is an ethnographic case study examining how one kindergarten teacher (Ms. P) modified her teaching to support and elicit student explanation, reasoning and justification. We use methodologies informed by action research in which the classroom teacher plays a collaborative role in determining the goals and agendas for personal and classroom transformation. Changes in the classroom teacher's practices along with her challenges and insights were documented and described through an analysis of classroom transcripts, stimulated recall of video excerpts, student interviews, and the teacher's journal.

Analysis and Findings

The data revealed that Ms. P engaged in the following discourse practices: increasing wait time, following-up both incorrect and correct responses, allowing students to contribute to the flow of class discussion, asking students to revoice and comment on one another's ideas, and asking authentic, probing questions. Transcripts of classroom video revealed that Ms. P consistently used these discourse practices throughout the year to encourage her students to "talk mathematics." Her initial goal was to move beyond an emphasis on correct answers and help students uncover and articulate the reasoning behind the answers. As an example consider the following transcript which occurred during the last month of the school year.

Speaker	Text
Teacher	Can anyone else explain to me how did you know what numbers you were supposed to use? Veronica?
Veronica	Um I was counting in my mind and I know that it was five.
Teacher	Ok and so how did you get to five when you were counting?
Veronica	Um I was like counting in my head.
Teacher	Can you say out loud what you were doing in your head?
Veronica	1,2,3,4,5.
Teacher	Ok and why did you stop at 5?
Veronica	'Cause I think it was five.
Teacher	Ok can anyone else explain to me what you were thinking? It sounds like Ali and Veronica, both of them, they knew how to do this and they were able to get the answer in their head. But I want someone to really explain to me. Let's listen to the problem again … Martin what did you do to get your answer?
Martin	Um I did math. Like um I did three plus two, that equals five.
Teacher	Ok and how did you know, what made you know you should use the numbers three and two? Why didn't you use 7 and 8?
Martin	Um I heard it in the story like three boys were playing in the housekeeping and two girls joined it.
Teacher	And then how did you know that 3 and 2 made 5? How did you know that?
Martin	Because here's three right now (holds up 3 fingers), two join in (adds 2 more so all 5 are held up).

Table 1. Transcript from an Introduction to Addition Lesson

In this excerpt, Ms. P asked a series of authentic, probing questions designed to encourage both Veronica and Martin to explain their thinking. Despite Ms. P's repeated efforts to draw out Veronica's thinking, she was unable to give a clear and precise explanation. This is the reality of the classroom. Simply implementing new discourse practices did not always facilitate the students' abilities to verbalize their thinking. In fact, students often struggled, giving confusing or incomplete explanations. The process through which Ms. P established the norms of verbalizing and justifying one's ideas was replete with challenges like this. Other challenges included deciding which students to call on, handling unexpected and unrelated student ideas, knowing how hard to press a student, and balancing time constraints. Despite these challenges, Ms. P continued to value "really asking and really listening": in her journal she stated, "It surprised me how many right answers I might miss if I don't take time to listen and try to understand what a child is thinking. When I might jump in and redirect, it's better to wait first and see where the child is going. Their minds might surprise me" (9/2/06). As reflected in this journal entry, not only was she scaffolding her students' ability to verbalize their thinking, but Ms. P herself also gained a clearer picture of their mathematical understanding. Transformation of her practice led to better information about her student's capabilities. This, in turn, led her to reflect on and question the teacher's role in a mathematics classroom, what it means for students to understand math, how a teacher assesses this, and the role of curriculum and task selection.

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TEACHERS CONSIDERING IMPLICATIONS FOR MATHEMATICS LEARNING AND TEACHING IN THE CONTEXT OF THEIR OWN LEARNING DURING PROFESSIONAL DEVELOPMENT

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This study examined the impact on middle school teachers' conceptions of mathematics learning and teaching through reflections on their mathematical learning experiences within a content-based professional development course. In their reflections, the teachers noted learning and teaching processes aligned with standards-based recommendations. The teachers' reflections were more reform-oriented when they thought about their learning experiences in the role of a student before jumping to thoughts about teaching.

This study examined the impact on middle school teachers' conceptions of mathematics learning and teaching through reflections on their mathematical learning experiences within a content-based professional development course. Content-based professional development programs in mathematics focus on mathematics relevant to the K-12 curriculum but do so at a level appropriate for the teachers as adult learners (Basista & Mathews, 2002; Campbell & White, 1997; Hill, Rowan, & Ball, 2005; Saxe, Gearhart, & Suad Nasir, 2001; Schifter, 1998; Swafford, Jones, & Thornton, 1997). The facilitators of the professional development implement the mathematics in a student-centered or reform fashion, providing a model for the teachers of how such instruction might look in a classroom. In such programs, reflecting on mathematical content is often a prominent aspect, and various programs have reported on the impact of such reflections. For example, Burk and Littleton (1995) found that such reflections support teachers in increasing their knowledge of mathematics content. However, it appears to occur less often and is less often reported what happens when teachers are asked to go beyond the mathematics and to revisit and think about their *learning experiences* and implications of those experiences for how their students learn best and how to facilitate such experiences in their own classrooms. In the aforementioned professional development model, this additional component was incorporated. These reflections encouraged teachers to consider learning and teaching implications in the context of their own learning. Specifically, it allowed the teachers to personally experience and then explicitly consider the fruits of student-centered, reform-oriented instruction.

The intent of this study was to investigate what teachers were noticing in their reflections on their mathematical learning experiences. The associated research questions were:

- What do teachers express about how they best learned mathematics through these learning experiences? and
- How do teachers extend these ideas to considerations of how their students may learn best and how to teach accordingly?

The theoretical perspective was phenomenology due to the interest in the teachers' perspective of the mathematical learning experiences. A phenomenological study focuses on "descriptions of what people experience and how it is that they experience what they experience" (Patton, 1990, p. 71). Sixteen grade 4-9 teachers participated in the study; 8 of them were not highly qualified in mathematics. The professional development course consisted of a 2-week summer institute and the content focus was Number and Operations. The data included copies of the teachers' written reflections and field notes. The teachers

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Stateline (Lake Tahoe), NV: University of Nevada, Reno.

completed sixteen written reflections throughout the professional development responding to questions such as "What was it like to learn math this way? What was it like to be in the student role?" and "How might these learning experiences impact your future teaching?" The field notes were used to capture the context of the mathematical learning experiences so that the teachers' reflections could be analyzed within their respective occurrences in the professional development. Data analysis utilized the processes of Grounded Theory (open coding, axial coding, and selective coding) (Strauss & Corbin, 1998) to identify the impact of the mathematical learning experiences on the teachers' conceptions of learning and teaching mathematics.

Upon reflecting on their mathematical learning experiences, the teachers noted several supportive learning processes, which as expected aligned with standards-based recommendations: working collaboratively with peers, using visual and written representations, hearing multiple ways to approach a problem, developing mathematical models and testing them on successive examples, and writing about their mathematical ideas. They also found multiple instructor actions helpful: asking thought-provoking questions, capturing in writing mathematical ideas as they were expressed, providing feedback, allocating adequate time to explore ideas, allowing learners to make sense of the mathematical ideas, and creating a safe environment. Finally, they noticed beneficial aspects of the various tasks: open-ended and challenging (not too difficult or easy).

The teachers echoed many of these aspects when they considered how their students might learn best and how to provide better instruction. For example, many of the teachers reported that they hoped to use visuals, manipulatives, written representations, and models in their classrooms; to slow down instruction, revisit mathematical topics as needed, and provide multiple opportunities and examples to ensure that all students mastered the material; to ask questions that probed students' thinking about mathematics; to utilize group work; and to have students explore the mathematics themselves rather than telling the students how to complete mathematical tasks. With respect to selecting and using specific mathematical tasks, the teachers mentioned that they would attend to the mathematical concepts elicited by the activity, the cognitive challenges associated with the activity, and how to adapt or extend the activity as needed for their students. Finally, many of the teachers commented on their increased confidence in teaching mathematics as a result of the professional development.

These results mirror many of the results of other content-based professional development programs (Basista & Mathews, 2002; Campbell & White, 1997; Hill, Rowan, & Ball, 2005; Saxe, Gearhart, & Suad Nasir, 2001; Schifter, 1998; Swafford, Jones, & Thornton, 1997). For example, other programs have found increases in teachers' content knowledge, teachers recognizing mathematics as a sense-making domain, teachers viewing themselves as initiators of mathematical thought, and changes in classroom practice. In the classroom, teachers have been observed making students' thinking more central, using less drill and practice, actively engaging students in the mathematics, and demonstrating confidence and beliefs aligned with teaching in a reformed fashion. However, this study offers new ideas for teacher development by also sharing lessons learned about engaging teachers in more substantive reflections about learning and teaching mathematics.

As part of the analysis process, teachers' reflections were classified according to the level of insight about learning and teaching. Insights about processes that supported one's mathematical learning were classified as low if they expressed a view of mathematics as a collection of rules and procedures or if they appeared to rely on others (such as authority figures) to make sense of the mathematics. Other insights about learning were classified as high if they provided details and specific descriptions of processes that supported their learning. Insights were classified as medium if they provided general (more cliché like) descriptions. Insights about implications for teaching were classified as low if they expressed a transmission model of teaching or if they discussed removing the exploration or reducing the cognitive load for the students. Teaching insights were classified as high if they described specific or detailed pedagogical strategies that supported students' making sense of the material or of a reform-oriented approach to instruction. Teaching insights were classified as medium if they described in more general terms student-centered implications for teaching.

Classifying the teachers insights as such allowed the following theme to emerge: Teachers' reflections were more insightful when they thought about their learning experience in the role of a student before jumping to thoughts about teaching. When teachers took on at least a student hat (meaning considering their own learning experiences) in a reflection, their associated insights about teaching and learning were approximately 20% more likely to be classified as medium or high. This theme verifies the importance of teachers reflecting on their own learning experiences as a way to possibly impact their conceptions on learning and teaching. Teacher educators, upon asking teachers to reflect on their mathematical learning experiences, may want to use reflection prompts that ask teachers to take on a student hat prior to taking on a teacher hat. In conclusion, asking teachers to consider implications for learning and teaching in the context of their own professional development experiences appears to serve as a productive forum for fostering insights about learning and teaching mathematics.

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THE FIGURED WORLDS OF TEACHING: FINDING COHERENCE IN TEACHERS' IDENTITIES

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The purpose of this paper is to investigate how teachers' membership in different figured worlds shape the identities for teaching that are constructed both within and across these figured worlds. Specifically, we analyze how these different figured worlds shape the ways in which teachers envision what is involved in planning and orchestrating mathematics instruction. As we will discuss, this clarification gives us grounds on which to problematize the notion of implementation of new instructional materials to teachers' classrooms via a short-term professional development workshops organized for that purpose.

This paper considers the ways that teachers' *identities for teaching*—that is, their understanding and enactment of what it meant to be a teacher, shape the ways that they engage with and plan to implement new curricula in the context of sustained professional development. We consider the development of multiple identities for teaching as they are constituted through teachers' participation in diverse *figured worlds* (Holland, Skinner, Lachicotte, & Cain, 1998). Understanding the identities that teachers develop is an important aspect of supporting teachers to change their practice (Kazemi & Franke, 2004), as changing one's practice is not solely a matter of learning new content (Schifter, 2001), but also requires reconceptualizing what it means to *teach* new mathematics content (Cobb, McClain, Lamberg, & Dean, 2003; Franke, Carpenter, Levi, & Fennema, 2001). In so doing, teachers may come to see new demands as sensible and even valuable, resulting in a new vision of what it means to know and teach. We propose that teachers' understandings of what it means to teach are not individually-held ideas and beliefs that a teacher applies uniformly across contexts. Instead, findings from a five-year long intensive professional development suggest that teachers' identities are constructed through their engagement in particular figured worlds, and more importantly, that their experiences within these figured worlds come to shape the ways they think about specific aspects of their teaching. The contributions of this paper are twofold. Pragmatically, this paper contributes to our understanding of how to support teachers to change their practice through professional development by examining how teachers plan for their own teaching as they work with new curricula. Theoretically, this paper contributes to our understanding of the teacher identity by considering how teaching is constituted in the communities in which teachers are members, and how that membership shapes teachers' active decision-making.

Teacher Identities in Figured Worlds

According to Holland et al. (1998), a figured world is: "a socially and culturally constructed realm of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others" (p. 52). It is through our engagement in these different figured worlds, Holland et al. state, that our *positional identities* develop. Just as one can be a member of multiple communities, so too is one often participating in multiple figured worlds which shape and give meaning to both the creation and interpretation of activity. Although most often figured worlds are separate (for example, the figured world of the office is marked by many differences from the figured world of an aerobics class), it is also possible for multiple figured worlds to exist at the same time (for example, if you bring a friend from work to your aerobics class). Thus figured worlds are often overlapping and can at times be in conflict with each other.

For teachers who are engaged in sustained professional development, there are (at least) two different figured worlds that are being enacted at the same time. One involves how the activity of teaching is defined and made meaningful in professional development sessions, and the other involves the ways those same activities are made meaningful in teachers' classrooms. In both spaces, what it means to teach is constituted through the practices and engagement of the members of the figured world. For some teachers, these worlds might be so similar as to be seamless; for others, these two worlds can be markedly different. The teachers who are considered in this paper were in the latter position, with the figured world for teaching that was constructed in the professional development sessions being markedly different from the figured world for teaching that was constituted in schools (c.f. Cobb et al, 2003).

Methods of Analysis

The data considered for this paper come from the last year of a 5-year professional development effort that focused on supporting teachers' development of instructional practices that put students' reasoning at the center of their practice. In the final year of our collaboration, the sessions primarily focused on supporting teachers to adapt instructional materials for use in their own classrooms. The teachers drew on statistics units from two different curricula, both of which would be considered to be aligned with "reform" principles: 1) a NSF funded curriculum that was in its first year of use in the school district in which the teachers worked (NSF); and 2) a statistics instructional sequence that was previously developed during a NSF-funded classroom teaching experiments (cf. McClain & Cobb, 2001) (Minitools). The teachers were familiar with this instructional sequence as it was used in professional development sessions.

Data for this paper come from videotapes of six day-long sessions from the 2005-2006 school year, during which the teachers were reviewing two different curricula. As the teachers engaged in critiquing the two sets of curricula, it became apparent that they took strikingly different orientations when envisioning using them in their classrooms. This difference was intriguing and emerged as an important research question to be pursued. In order to examine these differences more closely, videotapes of the sessions were reviewed systematically by attending to the conversations that unfolded around the two curricula. This process involved three steps: (1) delineating times in the conversation when either curriculum was the explicit focus of conversation; (2) analyzing those sections broadly for differences in the ways that teachers spoke about content, implementation, and the nature of learning, and (3) closer analyses of turn-by-turn utterances to determine how conversations unfolded and how teachers were positioned (Harre & Langenhove, 1999) relative to each other and to the practice of teaching.

Results and Discussion

Findings from this analysis revealed that there were notable differences in the ways that teachers discussed the two curricula. Rather than conceptualizing the classroom in one consistent way, teachers focused on very different aspects of "implementation" when working with different curricula—despite the fact that both curricula focused on the same mathematical topics—statistics. When teachers critiqued the NSF curriculum, they focused on the feasibility of covering the content within a desired time frame. In other words, the teachers focused on *process,* with relatively little emphasis on the nature of the mathematical reasoning intended by particular activities. In contrast, when critiquing the minitools curriculum, the teachers focused primarily on the mathematical intent of the lessons, and the specific properties of the instructional activities that would support or hinder the realization of the mathematical intent.

The fact that two different visions of teaching emerged when discussing what, on the surface, appeared to be similar curricula raises interesting theoretical and pragmatic issues about the ways that knowledge is situated in practice. Although it was clear that the teachers had developed a vision of teaching that was aligned with the goals of the professional development sessions (i.e., to focus on student thinking with respect to a trajectory of learning statistical ideas), this vision was not enacted when a different curriculum was under discussion. This points to an important challenge in professional development design; how to systematically support teachers in connecting the values and meanings constituted by these different figured worlds. In addition, this finding relates to an important theoretical point that teachers' experience of what it means to be a teacher is not a static system of beliefs, but is rather constructed in particularly meaningful *figured worlds* which give meaning to the practice of being a teacher. In one case, when the teachers were reviewing the NSF curriculum, it appeared that they were positioned in the figured world of the school system, as evidenced by their reference to concerns that were shaped and reinforced through the institutional context of their schools (c.f. Cobb et al., 2003). In contrast, when the teachers were discussing the minitools curriculum, it appeared that their instructional decisions were situated in the figured world of the professional development sessions, in that they were making decisions and justifications that were normative for that group and were aligned with their histories of participation with those curricula.

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THE MARSHMALLOW PROBLEM: ENACTED TEACHER LUST IN A COLLEGIATE MATHEMATICS COURSE

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This article examines the interaction between a constructivist teaching approach and Mary Boole's construct of teacher lust within a content course for middle grades mathematics teachers. Examples of teacher lust and their antecedents are discussed.

The term teacher lust was first coined by Mary Boole. Her conception of the construct described a teacher's desire "to proselytize, convince, control, to arrest the spontaneous action of other minds"(Boole, 1931, p. 1412). I submit there are actually two facets to Boole's construct of teacher lust – an internal impulse and an observable action. *Experienced teacher lust* is the internal impulse or desire to act in the manner Boole describes. For example, when seeing a student struggle with a given problem, a teacher may feel inclined to interject and "help" the student to solve the problem by directing the student's learning. Should the teacher act upon this desire and attempt to impose their mathematics directly onto the child, they will have engaged in *enacted teacher lust*. Actions associated with enacted teacher lust may include defining a solution path, explaining the mathematics at hand, or demonstrating how to solve the given problem. This article describes one teaching episode which took place in a content course for preservice middle school teachers. I present examples of enacted teacher lust and provide an explanation for the impetus of these actions.

Theoretical Perspective

My research focused on the interplay between teacher lust and constructivist teaching approaches. My working hypothesis was that teachers who attempt to operate from a constructivist paradigm are more likely to experience and become troubled by feelings of teacher lust. As a result, I selected my participants based on their intent to work from a constructivist paradigm. My definition of constructivist teaching was based upon the work of Steffe & D'Ambrosio (1995) and of Simon (1995): teaching in which problem posing is the primary method of instruction and whose intent is active interaction and engagement with students' mathematical constructions. When the goal of the teacher is to investigate and scaffold students' mathematical understanding, engaging in enacted teacher lust has the consequence of impeding or removing students' opportunities and ability to make sense of mathematics.

Methods and Analysis

A data-collecting schedule was employed that allowed for observation for an extended period of time on multiple occasions. For the equivalent of two weeks each class meeting was videotaped. After the two weeks of observation, the classroom data was examined using Mason's (1998) six modes of interaction: *Expounding, Explaining, Exploring, Examining, Expressing, and Exercising*; in order to identify examples of teacher lust to analyze and discuss with the participants. From further analysis of these examples overarching themes emerged. Seminal examples of these themes were selected and compiled into video segments which were used as stimuli during the semi-structured interview conducted following the first two weeks of videotaping. The participants were given the opportunity to discuss and debate my observations and impressions of their practice, to raise and discuss episodes they selected as moments of teacher lust for themselves, and to discuss the potential antecedents for these incidents of teacher lust. This cycle was repeated twice more during the semester.

Results

Samantha, the instructor for the course, spent the majority of time in class each day posing problems to her students and allowing them time to work in small groups to solve, discuss, and present their solutions. She acted as a facilitator in class and often responded to student questions by redirecting or posing new questions for them to think about. The marshmallow problem was a task designed to encourage students to develop a closed formula for the sum of the first *n* square numbers. Students were shown an oblique pyramid, made of marshmallows with a 5x5 array on the bottom level, a 4x4 array on top of it, followed by layers of 3x3, 2x2 and 1x1 arrays of marshmallows. When asked to write an expression for the number of cubes, a student first suggested that the structure was composed of $1^2 + 2^2 + 3^2 + 4^2 + 5^2$ marshmallows. The students were asked if they could find another way of representing the number of marshmallows. After an incorrect expression was suggested, Samantha encouraged the students to think about the formula for the volume of a pyramid. When the students had difficulty recalling the formula, Samantha stated it. As a way to demonstrate that the volume of a pyramid with a given base area was 1/3 the volume of a rectangular prism with the same base area, Samantha produced three oblique paper pyramids, which could be placed together to form a cube. After one student tried to make a cube from the paper pyramids, the instructor told him to abandon his approach and to try and "put the tips together". When he could not do so successfully, Samantha took the objects from him and did it herself. Next, Samantha produced three oblique pyramid structures made of 1 + 4 + 9 blocks each, and demonstrated how to put the three structures together to form a rectangular prism, albeit with some cubes missing. The prism formed was a 4 x 4 x 3 prism with one gap on the second level, two gaps on the third level, and three gaps on the fourth level. Samantha asked the students to write an expression for the total number of cubes in the structure, which she described for them as "a 4 x 4 x 3 prism with a gap of one, a gap of two, and a gap of three". The students offered the expression 4 x 4 x 3 – (1 + 2 + 3). Samantha then asked them to write an equation for just one of the pieces. John offered 1 + 4 + 9, which Samantha wrote as $1^2 + 2^2 + 3^2$, "just to be more general". From this the students were able to write an equation that described the number of cubes in one structure in two ways.

$$1^2 + 2^2 + 3^3 = \frac{1}{3}[4 \bullet 4 \bullet 3 - (1 + 2 + 3)]$$

The next task posed was for the students to adapt this equation for the original marshmallow structure. The students were confused as to whether the fraction on the right hand side of the equation should be 1/3 or 1/5. In order to emphasize that it would always take three oblique pyramids to make the prism Samantha produced three cube structures made of 1 + 4 blocks, which she put together to make a 3 x 3 x 2 prism with a gap of one and a gap of two. Once this was explained, the students were able to write a formula for the original structure.

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \frac{1}{3}[6 \bullet 6 \bullet 5 - (1 + 2 + 3 + 3 + 4 + 5)]$$

The students were still confused and some were not convinced that this equation was correct. Samantha pressed on however and asked the students if they thought the formula would generalize. With little wait time given for responses, she told them that it would. Samantha's wrap of the task began with, "This isn't really a particularly important formula or anything, it's just kind of – I think it is a really neat connection because it is, you know, the volume of pyramids and cones related to this situation". When the students asked where they might teach this topic to their middle school students, Samantha explained that it was not a topic that they were necessarily going to teach, but it could be used as an enrichment activity for students to look for patterns and relationships.

Discussion

This task offers many opportunities for students to engage, explore, conjecture, and reason within the field of mathematics. The implementation of this task however, removed many of these opportunities. There were several moments of enacted teacher lust evident in this teaching episode, and here I will present the antecedents for these actions. In the beginning of the task, Samantha made a choice to show the key connection between the task and the pyramid formula for the students. She justified this choice by explaining that she didn't expect them to "develop that from scratch…If I didn't bring up that connection it would not have been made". The impetus for this decision was based upon time constraints, and this issue often impacted Samantha's experiences with teacher lust. Throughout this task she made pedagogical choices to move things along in the interest of time. Samantha directed the actions of the students in order to curtail what she called "unproductive activity". She wanted time spent on what she considered the important or interesting facets of the problem. Because of this she did not allow the students to fully interact with what she considered ancillary tasks. "This [putting together the cubes and pyramids] isn't a key idea I am trying to get them to build. I just want them to see a connection here. And so it's definitely a conscious choice to move things on and keep things going because I don't want to waste time on this". Beyond time constraints, another main antecedent for these examples of teacher lust was Samantha's desire for her students to see the beauty of mathematics. Prompted by her comment that the formula the developed wasn't particularly important, I asked why she chose to use this task. Her justification was that it was connected to the idea of finding sums of series, and also, "it is just really cool. I mean this stuff is neat".

The intent of this task was for her students to experience a "really cool" connection in mathematics between the volume of a pyramid and the sum of consecutive square numbers; two ideas that are not obviously related. But the students did not really experience these connections for themselves; they were simply in the audience as the connection was demonstrated. It was clear from our discussions that Samantha wanted students to see this connection, but also realized that due to the limited time she had in the semester, she could not devote the necessary time to have the students fully engage in this task. To reconcile these conflicting issues, she made a conscious choice to present the problem more like a demonstration. As a result the excitement Samantha wanted her students to experience through her task choice was lessened and the task was completed with little or no real understanding gained by her students. These actions, although justifiable for Samantha, removed opportunities for the students directly engaging with the mathematics and as a result, to make genuine mathematical connections.

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TOOLS TO SUPPORT LOW ACHIEVERS: A MIXED METHODS STUDY OF STUDENTS LEARNING FRACTIONS (1)

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This study examines effects of the implementation of content specific and student-need specific computer-based learning tasks with students struggling to learn about fractions. We employed a three-phase mixed methods approach with 89 teachers of Grades 7, 8 and 9 to test a model for increasing lower performing students' understanding, confidence, and achievement within the classroom context.

Concepts and skills related to fractions have typically been very challenging for Grade 7, 8 and 9 teachers to teach and for students to learn. We are interested in how to increase student understanding in this area. Our research questions were: 1.Does the implementation of content specific and student-need specific computer-based learning tasks (defined by Fullan, Hill & Crévola, 2006, as CLIPs) contribute to higher achievement, specifically of students with substantial learning needs related to fractions? 2. How are the CLIPs used by teachers and students? 3. Are the effects of CLIPs on achievement influenced by student grade, prior achievement, or motivation?

Theoretical Understandings and Orientation

Studies of low ability learners in classrooms implementing Standards-based mathematics teaching show mixed results. Some researchers have reported null or weak effects (Baxter, Woodward, & Olson, 2001; Kroesbergen Van Luit & Maas, 2004; Mayer, 1998) whereas others (Bottge & Hasselbring, 1993; Riordan & Noyce, 2001; Ross, Xu, & Ford, 2006) have found positive effects. The mixed results of these studies might explain why Mayer (1998) found that low ability students were much less likely than their higher ability peers to have access to Standard-based teaching. Maccini and Gagnon (2002) found that special education teachers were less confident about their ability to teach the Standards and expressed a strong preference for direct instruction over constructive techniques with lower achievers.

We attempted to improve achievement of lower ability students in a district that is committed to Standards-based mathematics teaching for all students in mainstreamed settings (i.e., with minimal use of withdrawal from regular classrooms). The key strategies we employed were: (1) Identification of need using diagnostic tests of PRIME (*Professional Resources and Instruction for Mathematics Educators)* to identify the Zone of Proximal Development for particular at risk learners; (2) Focused instruction for moving students through the developmental continuum of PRIME (Small 2005); and (3) Additional scaffolding provided by interactive software.

A Mixed Methods Approach

Our research team implemented a mixed methods approach because it provides empirical data on overall effectiveness of the intervention and context specific qualitative data on implementation issues. (Yanchar & Williams, 2006).

Phase 1: Identification of problem areas

In 2005-06 we collected data from underachieving grade 7 and 8 students (Ross, Ford, & Xu, 2006). We identified 21 problem types for Number Sense and Numeration in which student achievement was consistently low. All of these problems were related to the topic of fractions. Moss and Case (1999) found that the domain of rational numbers is difficult for middle school students to understand (conceptual knowledge) even though students may demonstrate reasonable competency with the related algorithms (procedures). In our study, a team of eight expert teachers generated two items for each of the 21 problem types. Then, teachers ranked the problem types in terms of learning difficulty and disciplinary importance. Further, students indicated how confident they were in being able to solve each type of problem. These data were used to generate the blueprints for six learning tasks representing fractions and equivalent fractions. The teacher team interacted with technical staff to produce interactive software (hereafter we refer to these as the CLIPs).

Phase 2: Qualitative Study of CLIPs Implementation

In order to examine how the CLIPs function to support or hinder math learning in practical contexts, we conducted case studies in three classrooms. Two classrooms were observed during implementation of the initially developed CLIPs. One classroom was reserved for observing implementation of revised CLIPs. Data collection methods included formal and informal teacher interviews as well as detailed observations of pairs of students in math class before, during and after the use of CLIPs. Our goals were to understand: (1) the context of the lesson, (2) how the student's mathematical thinking emerged through the lesson and, (3) which aspects of the lesson (especially the CLIPs components) hindered and/or facilitated student affect (e.g., self-efficacy, anxiety, effort, engagement, beliefs about math and math learning).

Phase 3: Quantitative Testing of CLIPs

We are using a randomized field trial with a delayed treatment design as our method of quantitative data collection. The sample consists of 89 grade 7-9 math teachers in one district who volunteered for an in-service program on the use of CLIPs in combination with PRIME. We are collecting achievement and motivational data on three occasions to examine the effects of CLIPS on student learning and to determine whether the effects of CLIPs on achievement is influenced by student grade, prior achievement, or motivation.

Results and Discussion

Phases 1 and 2 of the study (identification of learning needs and development of CLIPs & qualitative analysis of CLIPs implementation) are now completed. Phase 3, the quantitative data collection will be completed in the Fall. The full report of the study will be made available in December 2007 at: http://www.oise.utoronto.ca/field-centres/tvc.htm. The expected contribution of the study to academic knowledge is evidence of the effects of a Standards-based mathematics treatment adapted to the needs of lower achievers. This study is testing the claim that lower achievers can be successful if (1) the diagnosis enables teachers to provide appropriate instructional tasks (2) instruction is targeted to specific student learning needs, and (3) student mathematical talk is guided by high quality tasks delivered through interactive software.

Endnote

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USING STUDENT WORK TO SUPPORT TEACHERS' PROFESSIONAL DEVELOPMENT IN TWO CONTRASTING SCHOOL DISTRICTS

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In this paper, we document the contrasting ways in which professional development (PD) activities around student work became constituted in two differing PD settings, yielding different opportunities for supporting teachers' learning. Our analysis illustrates the importance of situating PD design in the institutional context in which teachers work, even when the designed PD activities are intended to focus on supporting teachers' mathematical and pedagogical reasoning.

Using classroom artifacts has increased in popularity in mathematics teacher professional development (PD) as a promising means of supporting teachers' learning (Ball & Cohen, 1999; Smith, 2003). It reflects efforts to promote practice-based PD in which researchers proactively seek to establish connections between classroom teaching and PD activities. However, as Lampert and Ball (1998) have cautioned us, classroom artifacts do not "make a curriculum for teachers' learning. [They are] merely records of practice, with a promising potential to be made into curriculum" (p. 109). Therefore, it is important that researchers and teacher educators be more articulate about *how* classroom artifacts can be built into a potential curriculum for supporting teachers' learning or *what* are some key factors that can deeply influence this process. Our goal is to contribute to the second question by illustrating that the ways that teachers examine records of their classroom practice in PD are profoundly influenced by their instructional practices in the classroom, which are in turn situated in their institutional context of teaching (Cobb, McClain, Lamberg, & Dean, 2003).

We draw on a five-year PD collaboration with middle-school teachers in two urban districts that focused on middle-school statistics. In each district, we conducted six one-day work sessions and three-day summer institute per school year. The broader research question concerned the process of supporting teachers' development of instructional practices in which they would come to focus on students' reasoning in their instructional decision-making. To this end, we used classroom artifacts (e.g., student work) as a means to help teachers understand the diversity of students' statistical reasoning and use it as a resource in instructional planning. It surprised the research team that the activity of examining student work was constituted differently at the two sites. In district one, the teachers were concerned about being portrayed as ineffective when bringing unsatisfactory student work to PD sessions. As a result, a number of them pre-taught their classes (Dean, 2006). When analyzing student work, the teachers were focused on the correctness of students' answers whereas the process in which students arrived at these different answers seemed untraceable and less important. Little connection did the teachers see between examining student work and instructional planning. In contrast, in district two, the teachers tried to make sense of various students' solutions and to categorize them according to their level of mathematical sophistication. In doing so, they explicitly talked about the PD activity as a valuable opportunity to deepen their own mathematical understanding.

The contrasting ways in which teachers approached student work at the two sites indicate the complexity that teacher educators may encounter when records of teachers' classroom practice are used to support learning in PD. To proceed productively in our work with the

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Stateline (Lake Tahoe), NV: University of Nevada, Reno.

teachers, it became imperative that we attend to teachers' classroom instructional practices around the use of student work. Additionally, it proved beneficial to view these practices as deeply situated in the broader institutional context of teaching. Although both school districts had high-stakes accountability programs, they differed in several significant ways (Cobb et al., 2003). In district one, school and district leaders viewed mathematics teaching as a routine and predictable activity and coped with accountability pressures by attempting to monitor and regulate teachers' instructional practices. When we first began to work with the teachers, their instructional practices were highly privatized, focusing primarily on covering the curriculum with little attention paid to students' mathematical understanding (as the teachers were not equipped with resources to do so). In contrast, school and district leaders in district two viewed mathematics teaching as a complex and demanding activity and coped with accountability pressures by providing teachers with access to new tools and new forms of knowledge. As a result, an active informal teacher network was in place before we entered the site, focusing on helping each other to plan and teach mathematics lessons by capitalizing on a variety of instructional tools.

It is through understanding the specific institutional contexts that the teachers' orientation towards student work in district one becomes understandable. The teachers were regularly assessed based on the extent to which their students could produce the correct answer. Teachers, therefore, used student work as an assessment tool of students' achievement, seeing it as an indicator of the conclusion of an instructional activity. Using it to plan for instruction was alien to these teachers' instructional experiences and therefore an unreasonable PD activity from their perspective. In contrast, the teachers in district two saw student work as records of students' diverse ways of reasoning and linked it to the type of classroom instruction that students received. The teachers valued meetings and conversations about how their students solved the problems differently. They saw a PD activity built around student work as aligned with this instructional orientation and thus useful not only in helping them interpret students' solutions but also in deepening their own statistical reasoning.

To summarize, we argue that when planning to use classroom artifacts (e.g., student work) as a means of supporting teacher learning, it is critical to understand the role of these artifacts in the context of teachers' classroom practice. As we illustrated, in order to understand that role, it is key to situate teachers' classroom instruction in the institutional context of their work.

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ACTIVITY STRUCTURES OF MIDDLE GRADES' MATHEMATICS CLASSROOMS

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The purpose of this poster is to describe common Activity Structures (Lemke, 1990) of middle grades' classrooms to provide a more nuanced understanding of mathematics classroom practices. These findings reflect a first step in investigating these Activity Structures to understand the complexities of the broader practices that occur in mathematics classrooms.

"Traditional" mathematics classroom practices have been described as consisting of a very specific sequence of activities: the teacher provides answers to the assignment from the previous day; the teacher or the students work out the solutions to the more difficult questions; the teacher provides a brief explanation of the new material; and students practice problems based on the new concept (Welch 1978). While this generic description of traditional mathematics classrooms provides a glimpse of what some mathematics instruction looks like, it does not provide a broader landscape of mathematics classroom practices. We do not seek to categorize classrooms within the "traditional" and "reform" dichotomy. Rather, the purpose of this poster is to describe common Activity Structures (i.e., how people interact with each other using a mutually constituted set of expectations about what can happen next (Lemke 1990)) of middle grades' classrooms to provide a more nuanced understanding of mathematics classroom practices.

We draw from the larger data set of an NSF-funded study (Beth Herbel-Eisenmann, PI) on mathematics classroom discourse in which eight classroom teachers (grades 6-10) from seven schools in the Midwest participated. Following Lemke's (1990) methodology developed for use in science classrooms, we investigated the variety of and time spent on various classroom Activity Structures in mathematics classrooms. The teachers were observed for one-week periods in September, November, January, and March. We found a total of 37 different Activity Structures in the mathematics classrooms. Across the 81 hours of classroom observations, we found that 20 percent of class time was spent Going over Homework. About 21 percent of time was spent doing either Seatwork, Groupwork, or Partnerwork. Another 7 percent of time was spent "Setting up" activities such as Groupwork or Homework. By drawing on the work of Lemke (1990) for this analysis, we were able to see that there are a range of Activity Structures taking place in mathematics classrooms. The findings reported in this poster reflect a first step in investigating these Activity Structures to understand the complexities of the broader practices that occur in mathematics classrooms.

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BUILDING A TEACHING COMMUNITY THROUGH BUILDING 3D GEOMETRY LESSONS

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Grossman, Wineburg and Woolworth's (2001) described the *essential tension* of professional development, noting the need for teachers to focus on the improvement of student learning, "direct applicability" and to improve their own knowledge of the content they are teaching, "intellectual renewal". We describe a teaching community in which designing 3D geometry lessons provided an opportunity for genuine mathematical inquiry as well as a forum for teachers to discuss pedagogical issues from both a micro and macro perspective. We speculate that characteristics of this particular domain have the potential to be a gateway to changing practice.

The teaching community consisted of the authors and a group of six elementary school teachers at 3 different elementary schools in Northern California, two of which are low performing. We met monthly to design and discuss geometry lessons. Field notes from meetings were used for this study.

We engaged in a design cycle where we began by sharing general observations about lessons which led the teachers to make specific observations about their students' mathematical thinking. Often these observations sparked a mathematical exploration in which one of the teachers usually got stuck and elicited help from the group. We then brainstormed how we should help a student in her position. We considered lesson possibilities and this often led to more mathematical exploration. In this way we vacillated between discussing mathematics, students' thinking about the mathematics, and pedagogy.

After conducting their first lesson one teacher reported that she was "surprised by what my kids can do." Another noted that one of her students who was typically disengaged from schoolwork showed interest and persistence she had not seen in him before. Because the lessons we designed required students to build and describe polyhedra of their own design, the teachers had an opportunity to see their students engaged in independent thinking. One of the teachers noted that this made her "want to explore different ways to teach math so my kids can build it and construct it for themselves." In essence she had an *existence proof* that student-centered instruction was effective and now she was interested in trying it in other domains.

This group achieved professional community status, Grossman et al (2001) in a relatively short time because of the nature of the topic we were exploring. 3D geometry is a topic that the teachers had little experience with. Since it is relegated to a few pages at the end of the text, they had not thought much about it or developed habits of teaching it, making them willing to try alternative forms of instruction that they might not try in a more familiar domain. They were motivated to collaboratively plan lessons in this domain because they don't have preconceived ideas about how it should be taught. They also did not expect themselves or their colleagues to have deep content knowledge in the domain so they readily engaged in mathematical explorations and freely admitted when they were having difficulty.

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DEEPENING PRACTICE: EMERGING CONCEPTIONS OF REPRESENTATION AMONG SECONDARY MATHEMATICS TEACHERS

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The latest version of National Council of Teachers of Mathematics [NCTM] standards (2000) elevated the status of representation to one of five process strands. Considered indispensable to communicating, organizing, and understanding mathematical concepts and relationships, representation also allows students to recognize interdisciplinary connections as well as to real-world contexts. For teachers, representations are crucial to providing insight into how their students think and solve problems. However, education research has suggested that students experience difficulties with such processes, and that teachers also struggle to make sense of these processes when doing and teaching mathematics (Stylianou et al, 2007).

As NCTM standards become absorbed into state and local curricula, it is critical that teachers gain the pedagogic content knowledge necessary to guide their students in utilizing representation in the classroom. However, this transformation cannot happen overnight without focused, practice-oriented professional development as well as a continued system to support teachers throughout their practice (Smith, 2001; Ball and Cohen, 1999).

The purpose of our work is to understand how teachers and students understand and enact the process of representation in their classrooms. In light of the need for professional development to first strengthen teachers' instructional strategies, we asked the following question: *To what extent do teachers' conceptions and understanding of representation develop through intensive, practice-oriented professional development?*

Theoretical Framework

In this study we are following a theoretical framework based on two underlying themes: a) the relationship between social and cognitive domains of learning and b) aspects of teacher growth through professional development. First, we are considering the symbiotic relationship between social and cognitive domains of learning and the interaction between the two (e.g., Cobb & Yackel, 1995). Using the two domains, we use a synthesis of aspects of representation identified in earlier studies as a lens to approach this topic. The four cognitive aspects of representation in individual problem solving we draw from are its functions as a means to a) understand the information provided; b) record ones problem solving activities; c) facilitate exploration; and d) monitor and evaluate progress. Along with the cognitive, we consider a sociocultural perspective where learning is situated in everyday practice. We regard representations as communication tools in the context of the classroom.

Next we consider Ball and Cohen's (1999) approach to practice-based professional development as a means to enhance teachers' instructional practice. In light of this research that suggests that professional development can be superficial, fragmented and disconnected to classroom practice, we developed a "pedagogy of investigation" (Lampert & Ball, 1998). This meant an emphasis on mathematical investigation and analysis of teacher's own thinking and representation, but also carefully chosen vignettes of K-12 classroom life. These "strategic documentations of practice" included samples of student work, and videotaped sessions of classroom lessons.

Methods

Participants

Seven middle-school teachers were participants in the study. The teachers were selected on the basis of their willingness to participate in the study and were chosen from two school sites, both with a heterogeneous urban student population. Like their students, the teachers represent a wide range in terms of their own formal mathematical education, confidence in their own ability to "do" math, and successful experiences as students.

Procedures

Earlier work (Stylianou et al, 2007) suggests that teachers have a narrow conception of the role of representation in their mathematical practice, and it therefore plays at best a peripheral role in their classrooms. In order to address the need to develop a more complete understanding of the role of representation, we developed an intensive, week-long, professional development institute in which participants were immersed in practice-oriented problem solving focusing on representation. This model situated the participants in the same types of practices that could be implemented in their own classrooms. In particular the participants were given opportunities to model situations mathematically, construct solutions and perhaps most importantly, justify their thinking to their peers. By reflecting on themselves as learners of mathematics, they began to consider new beliefs about their practice, sometimes even revising or even contradicting prior beliefs (Fosnot & Dolk, 2002).

Though it was clear that representation was the central theme of the week, the facilitators never provided a formal definition of representation. Instead, the community of learners constructed their own conceptions of representation throughout the course of the institute. These definitions were recorded publicly and deepened as the week unfolded.

Data and Analysis

Data from a variety of sources helped us to analyze the effects of the institute on teachers' conceptions: semi-structured interviews at the beginning and end of the week, videotape recordings of class discussions, posters of each problem solving investigation created by the teachers, a survey of mathematical beliefs, and a audio recording of a discussion of their hopes and fears in continuing the work throughout the year.

The primary source of the data was the semi-structured interview. The first stage focused on teachers' conceptions of representation in the discipline of mathematics (i.e., teachers' conceptions as individuals who are knowledgeable about mathematics), whereas the second stage focused primarily on their conceptions of representation in the context of secondary mathematics (i.e., teachers' conceptions as individuals who are teachers of mathematics).

Initial questions focused on teachers' conceptions about the nature and role of representation in mathematics. Additionally, during the interview, teachers were shown and asked to evaluate different sets of researcher-constructed solutions to problems – solutions that varied with respect to the types and extent of representation use. The solution sets provided a context for examining the nature of what teachers find acceptable representations; in particular whether they find a particular representation more appropriate, useful or generalizable than others, and if so why.

The data analysis was grounded in an analytical-inductive method in which teacher responses were coded using external and internal codes and then classified according to relevant themes.

Results

The growth of the participants' conceptions of representation can be best described along three dimensions:

Growth in defining representation and its role in problem solving: While teachers in their own problem solving initially showed evidence that they utilize representation for multiple purposes, they were not necessarily metacognitively aware of their actions. The majority of the participants at the beginning of the week struggled to define representation by simply suggesting types of representation such as graphs, tables, etc, rather than a coherent set of ideas. They had not yet developed the vocabulary to extend their definition. This suggested an initial limited understanding of the role of representation. However, by the end of the week, the participants had developed more cohesive definitions that took into account notions of concreteness, tools for problem solving, and as a form of communication, and had developed an understanding of its role both as process and product.

Growth in teachers' conceptions of representation as an instructional tool: The participants initially chose formal, algorithmic representations as efficient models for problem solving. When queried about the value of multiple representations, or of incorrect or incomplete representations, teachers were reticent to acknowledge their place during instructional time. By the end of the workshop, the participants had begun to understand the importance of representation as a form of assessment in order to identify how their students think and attack problems, as well as identifying where students may struggle with concepts. Finally, participants exhibited an emerging self-critique of the limited use of multiple representations in classroom instruction, especially with respect to algebra.

Development of a learning community: At the closing of the institute participants shared their "Hopes and Fears" for the project. The teachers noted that they had "formed a learning community," and that they were excited to continue the work throughout the academic year. The teachers were also eager to participate in another professional development institute in which they would become teacher-leaders who would "turn-key" the workshop with a new cohort of teachers the following summer.

Discussion

In this study, we examined whether purposeful professional development could enhance teachers' understanding of the role of representation in problem solving and instruction. Our findings suggest that this type of experience provides teachers with an experiential referent from which to evaluate their own teaching practice.

Additionally, we modeled strategies that could be implemented in teachers' own classrooms, and in several, critical moments were explicit about the modeling (allowing teachers to see our own thinking as facilitators of others' learning). This supported teachers in developing more fully realized conceptions of representation. This study suggests how a practice-driven model can effect positive change in pedagogic content knowledge, and ultimately, success for students.

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LOOKING OUTSIDE THE MATHEMATICS CLASSROOM: PROFESSIONAL DEVELOPMENT THAT INTEGRATES MATHEMATICS AND LIVED EXPERIENCES

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This study compares the features and results of three professional development seminars given in the same school district in a moderately sized Midwestern city. Participants in one of the seminars (S1: Critical Mathematics) were secondary school teachers, while in the other two (S2: Inform Mathematics Teaching Practice with Student's Lived Experiences); and S3: (Developing Conceptions of Everyday Practices for Use in School Mathematics), were elementary school teachers. In the S1 seminar, teachers worked to develop a mathematics lesson that incorporated both mathematical and social justice goals in an attempt to support students in "reading and writing the world with mathematics" (Gutstein, 2006). In the S2 seminar, teachers engaged in a close examination of the in and out-of-school experiences of a single student from each of their classrooms in an attempt to use knowledge of the student outside of the classroom to inform the planning of instruction for that child. In the S3 seminar, teachers surveyed students to determine their out-of-school practices and created lessons to incorporate those practices in the mathematics class.

In looking across these seminars, several notable commonalities support a comparative analysis of the results. In each seminar, (a) teachers were engaged in activities meant to support their development as multicultural mathematics teachers, that is to connect mathematics teaching to the greater socio-political context in which the students live; (b) the teachers all self-selected into these seminars and as such were willingly engaged in attempting to link issues of mathematics and issues of equity; (c) all the teachers in the studies were White and were teaching in a community, like many in the United States, which was experiencing an increase in ethnic and racial diversity among the student population .

The power of this comparative study is the similarity of the issues that emerge despite differences in seminar design and grade level. Various seminar activities were used (lesson study; meetings with parents; shadowing of children; home surveys to access home practices; development of lessons built on student experiences). In spite of engaging in a variety of seminar activities across the K-12 teaching spectrum, the results of comparing these studies points to the difficulty of keeping a focus on both mathematical development and issues of equity. Teachers chose to participate in these seminars, and yet, with the best of intentions for attending to issues of mathematics AND equity, it proved difficult in each of these instances to maintain this dual focus. On a positive note, in each study, the teachers began to grapple with the issue of the social distance between teachers and their students. This has implications for honoring and validating the experiences that students bring to school, so that they can be taught mathematics more effectively (Nieto, 2004). In that respect, these studies indicate that professional development that informs the lens with which teachers look at and understand children who are unlike themselves has the potential to change mathematics teaching practice.

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ADDRESSING EQUITY IN PRESERVICE MATHEMATICS TEACHER EDUCATION THROUGH MATHEMATICAL TASKS CONTEXTUALIZED IN SOCIAL ISSUES

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Our study's aim is to examine a preservice secondary mathematics teacher's conceptions of educational equity, expectations for diverse students, and reactions to engaging in mathematical tasks contextualized by social issues. Data reveal multiple dimensions of teachers' equity-related thinking and the tensions they may experience when confronting equity issues. Results suggest ways in which teachers' beliefs may inform their instruction.

A key concern of teacher education programs is preparing a relatively homogeneous population of teachers to meet the needs of an increasingly diverse student body (Sleeter, 2001), including poor and minority students who have been historically underserved in mathematics education (Lubienski & Crockett, 2007). Many teachers hold beliefs about schooling and student achievement that may interfere with educational equity (Sleeter, 2001). Furthermore, research in mathematics education suggests a link between teacher expectations and student outcomes, as mediated by pedagogical decisions (e.g. Silver & Stein, 1996; Zevenbergen, 2003). Teacher educators must better understand the beliefs and expectations of prospective teachers in order to provide experiences that challenge them. The aim of this case study (Patton, 2002) is to examine one preservice secondary mathematics teacher's (PSMT's) (a) conceptions of educational equity, (b) expectations for diverse students, and (c) reactions to engaging in mathematical tasks contextualized by social issues. Studies of teachers' equity-related beliefs are seldom situated in the context of mathematics education, as teacher education programs frequently separate constituents' content, methods, and multiculturalism coursework. The tasks designed for this research provide an example of helping PSMTs engage equity issues while maintaining a simultaneous focus on mathematics learning. They also provide the field of mathematics education with a unique research tool for revealing teachers' beliefs about issues of educational equity. Our research contributes to the field by exploring the complexity involved in examining teachers' equity-related beliefs. Results inform teacher education with the larger aim of improving mathematics instruction for all students.

Literature Review

We assert that mathematics teacher education must address (a) content knowledge and (b) issues of equity in response to common weaknesses among preservice and practicing teachers and in order to meet the goals of current reform efforts (NCTM, 2000). First, despite extensive coursework in mathematics and the study of advanced topics, many PSMTs lack conceptual understanding of content in the high school mathematics curriculum (Even & Tirosh, 1995). Teachers' deep understanding of mathematics content is crucial for ensuring high quality mathematics education, as large-scale studies demonstrate a positive relationship between teachers' mathematics knowledge and their students' achievement (Hill, Rowan, & Ball, 2005; Rowan, Chiang, & Miller, 1997).

Addressing equity in mathematics education necessitates an examination of students' access to, understanding of, and persistence in mathematics throughout schooling and beyond. It also involves considering the consequences of students' participation in various curricula (Secada, 1989). In the US, White, middle-class students and their poor and minority counterparts incur different rates of achievement (Lubienski & Crockett, 2007) and persistence (Hrabowski, 2002/03) in mathematics-oriented fields. Furthermore, preservice teachers—who are overwhelmingly White, female, and lower-to-middle class (Sleeter, 2001)—report being unprepared for and exhibit resistance to teaching diverse students (Dee & Henkin, 2002; Terrill & Mark, 2000).

Teachers' expectations for students and their conceptions of education are two promising venues for impacting the achievement gap. Preservice and practicing teachers frequently hold differentiated expectations for students of different race/ethnicities or class. They also tend to explain academic differences by employing deficit models of poor and minority students' home culture rather than considering the contribution of classroom, school, or societal factors to the achievement gap. This is of primary concern because research has established an association between high teacher expectations and high mathematics achievement among underserved students (Gutierrez, 1997; Silver & Stein, 1996). Teachers' conceptions of education may also interfere with goals for equity, as they commonly believe the primary purpose of schooling is to indoctrinate students into American culture (Cockrell, Placier, Cockrell, & Middleton, 1999; Davis, 1995). This runs counter to goals for educational equity, which require that decisions about education take into account peoples' socioeconomic conditions and their ability to participate fully in democratic society (D'Ambrosio, 1990).

Theoretical Framework

Secada (1989) has said that an evaluation of educational *equity* necessitates going beyond simple notions of *equality* of the inputs, processes, and outcomes of schooling. Instead, it requires questioning the appropriateness of educational decision-making from multiple perspectives and making "judgments about whether or not a given state of affairs is just" (p. 68). Mathematics education research overwhelmingly ignores these issues as they relate to teachers. We argue that teachers need to have a working definition of equity in order to gauge their own classroom practices and the practices of their larger grade, school, or district community.

A major barrier to educational equity is the low expectations that teachers often have for poor and minority students. Expectations often are manifested in teachers' notions of ability and effort, which in turn impact their pedagogy (Zevenbergen, 2003). Common conceptions of ability—as innate, fixed, correlated with speed, easily assessed, and normally distributed—are at odds with goals for equity in that they "are grounded in ideologies that maintain race and class privilege through the structure as well as the content of schooling" (Oakes, Wells, Jones, & Datnow, 1997, p. 484). Oakes and colleagues found that teachers often gave racial and cultural explanations for the different levels of perceived ability and used this ideology to rationalize schooling practices that disproportionately disadvantaged poor and minority students.

Methodology

This qualitative inquiry was conducted in a secondary mathematics methods course taught by the second author. All PSMTs completed three mathematical tasks assigned at near-equal intervals throughout the semester as part of the required coursework. The first two tasks were coupled with related research articles that the PSMTs read prior to their implementation. The tasks are designed to allow PSMTs to use mathematics concepts they will be expected to teach, and they are contextualized by equity related social issues and employ authentic data. Thus, they respond to the aforementioned concerns regarding PSMTs' weak content understanding and unsophisticated notions of equity. They also serve as a research context for investigating teachers' beliefs and expectations for students.

For the first task, PSMTs used statistics concepts to explore state and national achievement gap data. They considered the contribution of numerous factors to the achievement gap as they used Microsoft Excel® to aggregate standardized test scores, identify trends in the data, and represent findings in multiple ways. The second task involved the use of algebra to investigate the practice of payday lending—using a borrower's checking account information as collateral for a short-term high-interest loan. Students retrieved loan procedure data from an internet source and constructed linear graphs and equations to represent the costs of borrowing money for different lengths of time. Considering the relative cost of various scenarios and the factors that contribute to a lower-income person's decisions provided insight into the seemingly-unwise financial choices made by some borrowers. The third task required PSMTs to create their own mathematical activity contextualized by an equity-related social issue of their choosing and utilizing authentic data. This task provided an opportunity for PSMTs to create curricular materials designed around an issue of personal interest.

The participant, Josh, is a White male senior-level mathematics major enrolled in the course. He was selected for participation in this study as a result of his willingness to discuss equity issues and his interest in our project. Data sources included a focus survey, three semi-structured interviews, and Josh's solutions and reflections related to the mathematical tasks. The survey was administered at the beginning of the semester and primarily inquired into the participant's beliefs about teaching, schooling, student learning, equity, and the achievement gap. The first interview immediately followed the survey and allowed for clarifications and extensions of Josh's beliefs. The second interview occurred a week after Josh's completion of the first task and further clarified his stance as well as capturing his initial reactions to the task. The third and final interview occurred after all three tasks had been completed, and it probed Josh's reactions to all of the tasks. Our identification of Josh's conceptions of equitable mathematics education and expectations for students was informed by data collected in the survey and interviews. The interviews and written assignments provided evidence for examining his responses to the mathematical tasks.

Data were transcribed and coded in order to determine emergent themes and they underwent constant comparative analysis (Strauss, 1987). Initial codes for the thematic analysis were informed by the research literature, and the list of codes was revised throughout data collection and analysis as constant analysis dictates. In our discussion of the results, we analyze Josh's conceptions of equitable mathematics education in relation to Secada's definition of educational equity. Our goal was to determine the extent to which Josh possessed a sophisticated and nuanced conception of equity and the ways in which he believed various factors contributed to the pursuit of equitable mathematics education. Oakes and colleagues' (1997) discussion of different social constructions of ability informed our analysis of the participant's expectations for students' perceived ability and effort along lines of race and class. Beyond understanding *what* our participant expected for diverse students, we also explored his stated reasons for these expectations. To address Josh's reactions to the tasks, we coded and analyzed items referring to his response to the mathematical content, the equity issues, and his reasons for valuing or not valuing certain aspects of the task. We searched for themes in the data and report our findings below.

Results

In this section, we present Josh's conception of equitable mathematics education, his expectations for diverse students, and his reactions to the mathematical tasks. Josh is an "information-rich case" (Patton, 2002) because while it initially appeared that his beliefs aligned with goals for educational equity, subtleties in his views shed light on the complexity of preparing teachers to contribute to equitable mathematics education.

Conception of Equitable Mathematics Education

In the survey, Josh stated that the goals of schooling and of mathematics education should be to prepare students for a variety of employment opportunities, enhance everyday reasoning, and promote values and citizenship. Josh believed that equity in mathematics education would be realized if the percentage of students performing at different levels on standardized tests was identical for each racial/ethnic and class group, thus demonstrating his focus on a primarily outcome-oriented conception of equity. For instance, in response to an interview question asking how to identify an equitable high school mathematics department, Josh declared, "I'd say the best measure is just the statistics of like how students are doing." He saw specific inputs (e.g. the distribution of material resources) and processes (e.g. teachers' efforts to establish norms of respect and offer extra help to those in need) as contributing to achievement outcomes.

Josh frequently associated the achievement gap with funding, maintaining throughout the data collection phase that "insufficient funding for poor neighborhoods" and the results of this shortage were fundamental contributors. He expressed certainty that teachers should be held partially responsible for mitigating the achievement gap, but he was vague about the ways that they could address the problem. Josh described himself in the survey as wanting to be an "activist" in getting money to the schools, but again was unclear with respect to specific actions he could take. He was more confident in his recommendation that teachers work closely with students that have individual difficulties.

Expectations for Diverse Students

Analysis of survey and interview data provided information about Josh's expectations for diverse students. Josh proposed that all high school students should be required to take four years of mathematics, yet he believed that courses below the level of algebra should be readily available and allowed to count towards this total. He set Algebra I as the minimum requirement for high school graduation.

Josh believed achievement was measured by course grades, and saw it as a consequence of both effort and ability. Effort, to Josh, was the leading contributor to mathematics success and was reflected in students' homework completion score. Josh maintained that ability—the factor he listed as third-most influential in students' success—could best be measured through standardized tests and would be quite apparent to a classroom observer. Josh was convinced that a teacher could easily differentiate between students of different abilities as well as students exerting high or low effort, according to the speed at which they find correct answers in class and their homework completion. He asserted that ability has no association with race, but was unsure about its association with class, because of the possibility that ability may be hereditary. Josh attributed the lack of effort he perceived among poor and minority students to society's message that they are inferior.

Reactions to the Mathematical Tasks

Josh valued the open-endedness of the tasks as well as the requirements to use written communication and multiple representations, all which he saw as contributing to the mathematical complexity of the tasks. He felt the use of real data enhanced the tasks' authenticity and that this feature could increase students' motivation. Josh valued the topics of the tasks for his personal learning, reporting raised curiosity as a result of his study of the achievement gap and his newfound understanding of the dilemmas facing low-income families that choose to make use of payday lending. He seemed to be convinced by the tasks that mathematics could coexist with an examination of social issues without becoming "watered-down." Despite these positive reactions, he also expressed that it was important to maintain a focus on the mathematics and not delve too much into "social studies" issues. When asked what aspects of the tasks he would include in his future teaching, Josh talked at length primarily about the mathematical features. Pressed further about the equity content, he briefly entertained the notion of incorporating explorations of social issues into a high school mathematics course, but felt this would best be done in conjunction with a teacher in another subject. He also expressed concern that students might find such issues discouraging.

Discussion

Josh's conception of equity involved primarily the equality of achievement outcomes. He stated that particular inputs were unequally distributed among different groups of students, and asserted that these contribute to the disparity in achievement outcomes. He reached a restricted consideration of equity as defined by Secada (1989), however, because he did not question the appropriateness of educational activities and outcomes nor did he entertain qualitative notions of fairness in schooling. For instance, Josh did not consider outcomes beyond K-12 education when judging equity, such as persistence in mathematics-oriented fields or economic equality. He also did not question the current mathematics curriculum or mathematics teaching. Data suggest that Josh's willingness to consider equity issues in mathematics instruction may be influenced by his views of the nature of mathematics, as he exhibits traditional beliefs about the content that should be included in mathematics instruction. Other researchers have also proposed that teachers' beliefs about subject matter may impact their willingness to adapt curriculum and instruction to the needs of diverse learners (Stodolsky & Grossman, 2000). While the importance Josh placed on funding suggested an awareness of systemic problems in schooling and the impact of the achievement gap on groups of students, a great deal of Josh's notion of equity involved having teachers address what he perceived as the learning difficulties of individuals. In order to contend with the systemic factors contributing to achievement disparities, conceptions of equity must attend to educational trends among subgroups of the population as opposed to examining individual differences.

Josh's view of ability may counter goals of educational equity, but his view of effort is more closely aligned with these goals. He indicated that ability was innate, correlated with speed, and could be easily identified—all qualities that Oakes and colleagues (1997) found to be associated with resistance to educational reform. Although Josh sees effort as associated with race and class, he recognizes the powerful role of societal expectations on shaping students' efficacy and in turn, their effort. This demonstrates his awareness of systemic problems—most notably the relative privilege experienced by White upper- and middle-class students—and is aligned with goals for equity. This awareness suggests that Josh may be sensitive to some of the additional pressures faced by underrepresented students and may be likely to hold high expectations for student effort and try to instill confidence. His views on the role of teachers involved addressing funding issues, reaching students of different abilities, and creating a positive atmosphere for students, indicating that curricular content was not a significant feature of instruction to him. The strength of his recommendation that high school students take four years of mathematics is diminished by his belief in establishing Algebra I as a minimal graduation requirement. This idea is consistent with the low expectations frequently held for underserved students, as it may lead to lateral movement within mathematics course-taking rather than promote student access to advanced mathematics (Buckley, in press).

Josh's reactions to the tasks supply further evidence for our claim that his consideration of educational equity is restricted, but they suggest ways to mitigate teachers' resistance to confronting equity issues. While the mathematical features of the tasks were most salient to Josh, we speculate that the mathematical complexity may be crucial for helping establish the legitimacy of equity issues in the minds of PSMTs. Josh's seemingly contradictory reactions to the possibility and appropriateness of integrating equity issues into a mathematics course expose another critical tension that must be resolved before he can fully realize the goals of educational equity in his own classroom.

Significance

Our study of a single information-rich case (Patton, 2002) reveals multiple dimensions of teachers' equity-related thinking and the tensions they experience when confronting equity issues. It also provides insight into the manner in which teachers justify such beliefs and suggests implications this may have for instruction. While previous studies have relied heavily on straightforward measures and/or quantitative methodologies, our study highlights the necessity for a nuanced approach to understanding and challenging teachers' views through both research and practice. One such approach may involve situating teachers' explorations of equity issues in a subject-matter context. Another productive course of action may be to carefully address the foundation of teachers' beliefs in teacher education rather than simply the beliefs themselves. Understanding more about PSMTs' expectations of underserved students ultimately positions mathematics teacher educators to empower the teaching constituency to teach in ways that are more equitable and expands the research base for equity issues in mathematics education.

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ASSESSING COGNITIVE ACTIVITY IN THE CONTEXT OF LETTER WRITING

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We report on the results of a two-year study in which pre-service teachers (PSTs) used letter writing as a medium for posing tasks to algebra students. We measured PSTs' growth in terms of elicited cognitive activities, as indicated by students' responses. Results include demonstrated progress in PST's abilities to elicit and assess activities described by NCTM's Process Standards.

Recent reform efforts attempt to focus teachers' attention on the processes of mathematics, as well as the products. The challenge for teacher education programs is to engage preservice teachers (PSTs) in designing and posing tasks that might elicit high levels of mathematical activity from students. Silver et al (1996) conducted research on task posing in the context of PSTs generating problems from problem stems. The researchers found that PSTs tend to design tasks that are inadequately stated and unlikely to elicit high levels of cognitive activity. These findings highlight two key motivations for the research reported here: the need for PSTs to develop an ability to design better tasks and the need for methods courses to provide authentic contexts in which this can occur.

We have built on the work of Sandra Crespo (2003) by using letter writing between PSTs and students to provide opportunities for authentic, iterative exchanges of tasks and responses. We conducted a two-year study of PSTs' growth in eliciting high levels of cognitive activity from students through task posing. In this paper, we restrict our attention to those cognitive activities described by the *Process Standards* (NCTM, 2000).

In our first year of implementation, we paired PSTs with high school algebra students for twelve weeks of letter writing with a task and a response exchanged each week for each pair. We used the results of these exchanges to operationalize measures of cognitive activity, including the *Process Standards*, as we assessed students' responses. Our assessments were highly inferential, but we (the authors/researchers) conducted them individually with solid reliability on most measures (the exceptions were on measures of cognitive activity that were rarely assessed by either author). Having operationalized the measures, we decided to engage a *second* cohort of PSTs in performing similar assessments as part of their letter-writing activities during the following year.

Just as students benefit in shifting mathematical authority from math texts and teachers to themselves, we hoped that PSTs might benefit from an analogous shift in their methods courses. PSTs' engagement in authentic experiences of assessing students' cognitive activity might have the effect of shifting authority from instructors to PSTs and of increasing PSTs' sense of teacher efficacy. In implementing our approach with the second cohort, we envisioned the following potential outcomes for PSTs:

- appreciation for mathematical processes as well as products
- better understanding of students' cognitive activities

- increased ability to individualize task design and elicit higher levels of cognitive activity, based on thorough assessments of students' previous responses

In this paper, we address the ways in which PSTs demonstrated growth in these areas. In particular, how do PSTs understandings of students' cognitive activity evolve over the course of letter writing exchanges? In what ways do elicited responses from students indicate growth in PSTs' ability to assess such activity and design more effective tasks?

Theoretical Framework

A radical constructivist perspective (Glasersfeld, 1995) informs our assessment of cognitive activities and supports our most basic theoretical assumption: PSTs' letter-writing interactions can engender evolving models of students' mathematics, which will inform future tasks that PSTs pose. These models should become more powerful in the sense that the PSTs will be able to elicit higher levels of cognitive activity, and in the sense that PSTs will be able to abstract pedagogical content knowledge from these experiences and use them to inform task design with other students. Likewise, we view the ways in which students operate as becoming more powerful when they can operate in greater generality. Our operationalization of the NCTM *Process Standards* indicates generality in terms of students operating in new ways and in new contexts. For instance, we operationalized *Connections* to describe instances in which we could infer that a student had constructed new relationships between two or more previously constructed concepts.

In agreement with other recent work (Lester & Kehle, 2003), we argue that we can better judge tasks by making inferences from interactions between problem poser and problem solver than by considering student-independent variables, such as attributes of the tasks themselves. This argument is also supported by the *Principles and Standards* document, which states that "mathematical tasks have been enacted without sufficient attention to students' understanding of mathematics content" (NCTM, 2000, p. 5). Thus, the researchers and PSTs alike engaged in making inferences about students' cognitive activities in response to the tasks that PSTs posed.

Methods

The seventeen PSTs participating in this study were enrolled in the first of two methods blocks with a methods course taught by the first author. The methods course focused on mathematical learning and included discussions of the NCTM *Process Standards*. Although the block included 40 hours of field experience in secondary schools with valuable learning opportunities for the PSTs, these experiences varied greatly depending on the placements, and we could not count on them to provide opportunities for prolonged engagement with students. We felt that we needed to provide such opportunities for PSTs in order for them to learn from students by making inferences about the students' mathematics.

Starting with the previous year's cohort of PSTs (n=22), we decided to employ letter writing as a medium for posing tasks and assessing responses. We paired each PST with an algebra student from a local high school and transferred letters and responses back and forth via the students' classroom teacher. Data from the second cohort (the seventeen participants on whom we report here) consisted of PSTs' written tasks, students' written responses, and PSTs' assessments of tasks and responses. We focus on PSTs' assessments, as well as our own, which span ten weeks (iterations) of letter writing. Because the high school began a new term in the middle of the university's semester, PSTs wrote their first five letters to one class of Algebra II students and wrote their last five letters to a new class of Algebra II students. Both groups of students had the same teacher and continued with the Algebra II curriculum where the previous class had left off.

For each task, PSTs recorded which cognitive activities they expected their task to elicit from their student partners. Students used about 20 minutes of class time to respond. Once PSTs had received the students' responses, they had two days to assess them. PSTs used students' writing to infer the kinds of cognitive activity in which students had engaged. For each inference, PSTs provided supporting indicators from students' writings. For each discrepancy between expected and assessed activities, PSTs provided an explanation. Once PSTs had completed their assessments and designed their new tasks, each researcher assessed students' responses.

Results

Figures 1 through 5 illustrate summarized results for the ten weeks of letter writing, in terms of the percentages of weekly tasks that were expected or assessed to elicit the NCTM *Process Standards*. The blue and red bars (leftmost bars, labeled 'E' and 'A') represent the PSTs' expected and assessed cognitive activities; the yellow and green bars (rightmost bars, labeled 'R1' and 'R2') represent the two researchers' assessments. We used Cohen's Kappa to measure the reliability of the researchers' assessments; such measures are especially important given the inferential nature of our assessments. Sim and Wright (2005) cite Landis and Koch who suggest the following delineations for interpreting Kappa: <=0=poor, .01-.20=slight, .21-.40=fair, .41-.60=moderate, .61-.80=substantial, and .81-1=almost perfect. Our reliability results are as follows: *Communication* 0.48, *Connections* -0.02, *Representation* 0.56, *Reasoning & Proof* 0.51, *Problem Solving* 0.72. So we see moderate to substantial reliability for all of the researcher assessments except for *Connections*.

What can we learn from Figure 1 about the evolution of PSTs' understandings of students' mathematical communication and their abilities to elicit such activity from students? First, we note that the heights of the four bars appear to tighten over the ten weeks. Based on this observation, we hypothesize that PSTs became better and better at eliciting *Communication* when they made such an attempt. We also hypothesize that PSTs' understandings of *Communication* approached those of the researchers, who had achieved moderate reliability in their independent assessments. We can test our hypotheses by applying additional reliability statistics between PSTs and the researchers over the ten weeks.

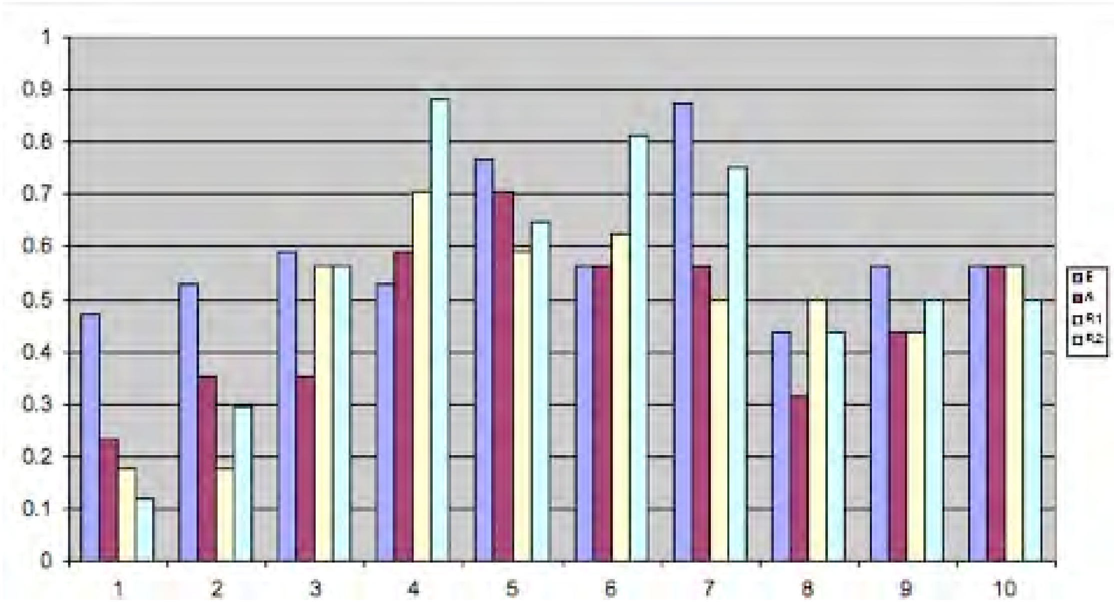


Figure 1. Communication

We also notice that, according to the researchers' assessments, PSTs elicited *Communication* more and more frequently over the first four weeks, and then the frequency dropped to a level about that of Week 3. We saw a similar pattern in last year's data (Norton & Rutledge, in review), and we will see that elicited activity, in general, followed such a pattern this year.

The most notable aspect of Figure 2 is the infrequency with which we assessed *Connections*. Two factors seemed to contribute to this. First, as defined by NCTM (2000), *Connections* can refer to connections between mathematical concepts and various situations in which students apply them, or *Connections* can refer to new connections that students make between existing concepts. Because we also measured responses using Bloom's taxonomy, which includes *Application*, we reserved *Connections* for the latter interpretation. However, and despite efforts in class to clarify the distinction we made in operationalizing *Connections* and *Application*, PSTs often assessed *Connections* when students' responses indicated *Application*. Second, we assessed responses conservatively, assessing a particular cognitive activity only when we found strong indications of it, and it is difficult to find indication from which to infer that a student had made a new connection between existing concepts. We needed indication that the connection was indeed new. Thus, we rarely assessed *Connections* with the first or second years' cohorts. This fact helps explain the unreliability of the researchers' assessments; we have found that cognitive activities that were least assessed were assessed least reliably (Norton & Rutledge, in review).

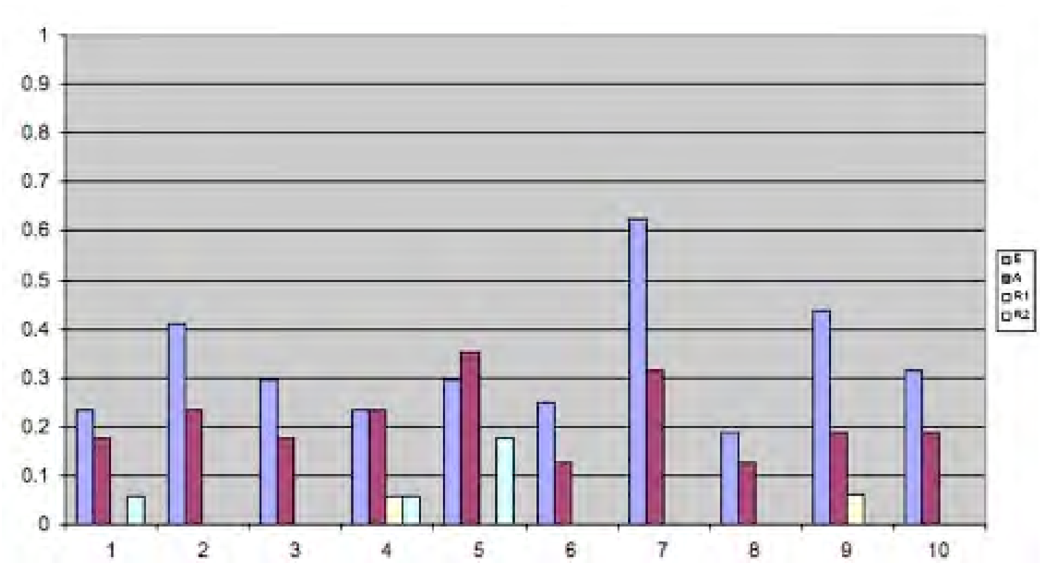


Figure 2. Connections

Figure 3 illustrates general trends toward tightening assessments (A, R1, and R2), and toward increased *Problem Solving*. We hypothesize that PSTs' understandings of *Problem Solving* became more and more compatible with our own. We further hypothesize that PSTs improved in eliciting *Problem Solving* from students. Notice that aside from Week 4, PSTs elicited *Problem Solving* most in the final four weeks and with increasing frequency across those weeks. The first year's data showed a similar trend.

As with *Connections*, PSTs rarely elicited *Reasoning & Proof* (Figure 4). Weeks 8 and 9 provide some indication that PSTs' improved in eliciting this activity, but we cannot identify a trend in Figure 3 to support this. Nor does it appear that PSTs' understandings of *Reasoning & Proof* approached our own.

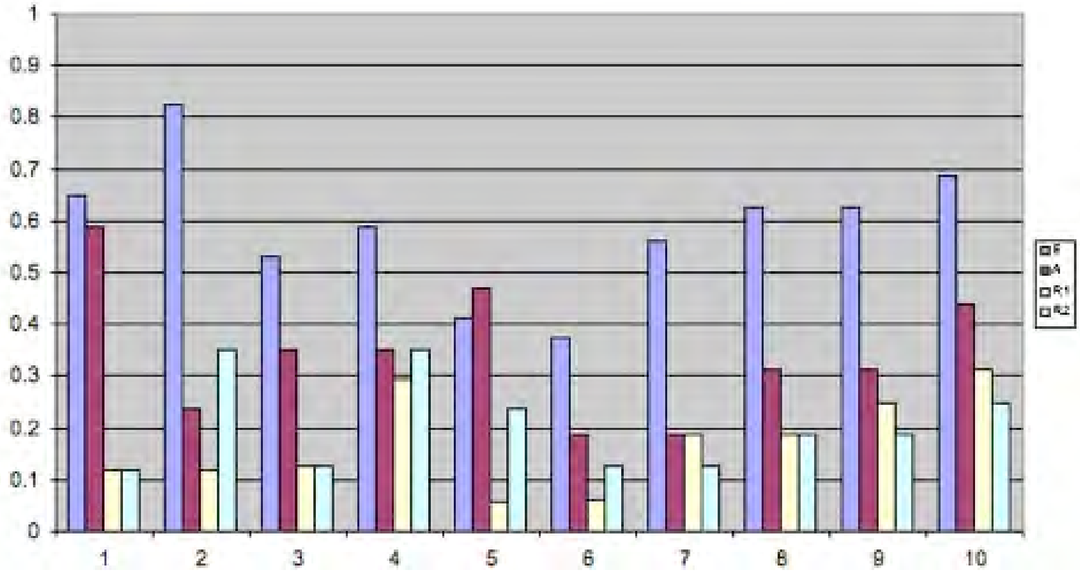


Figure 3. Problem Solving

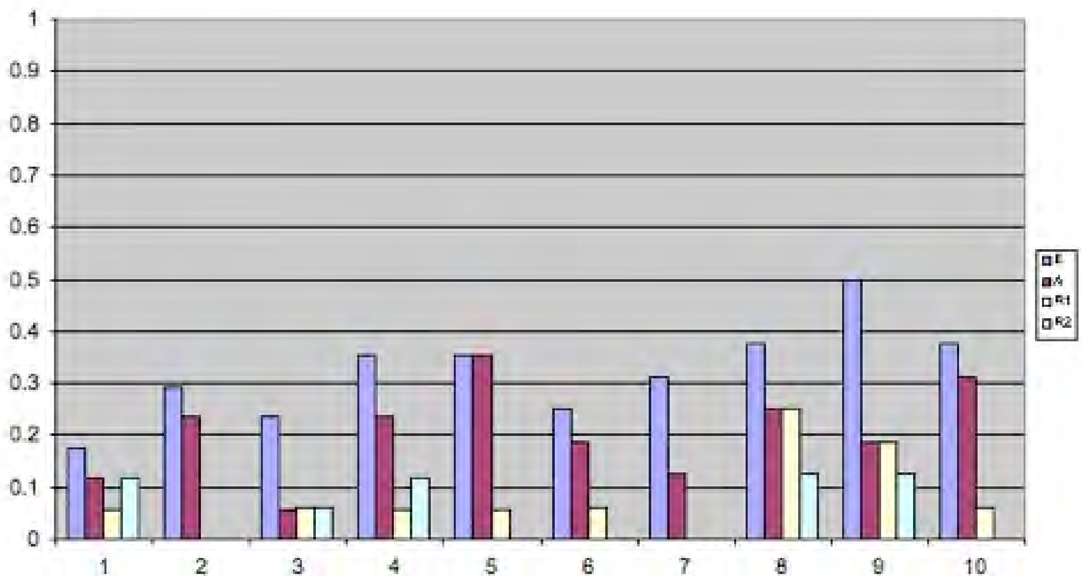


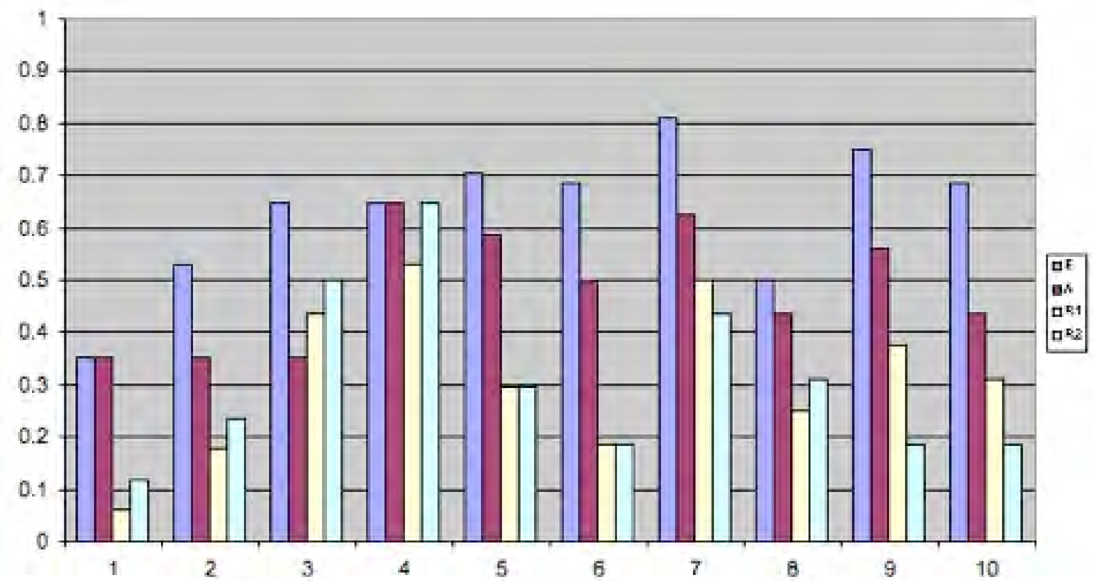
Figure 4. Reasoning & Proof

Notice that the trends for *Representation* (Figure 5) closely resemble those of *Communication*, except that PSTs' assessments did not seem to approach our own. Moreover, results of t-tests indicate that the second year's PSTs performed significantly better in eliciting both *Communication* and *Representation*: the mean for *Communication* was significantly higher (at the .01 level) for the second five weeks when compared to last year's mean; the means for *Representation* were significantly higher (at the .01 level) for the first and second five weeks when compared to the first year's mean.

Summary

The bars in Figures 1 and 3 illustrate trends that indicate PSTs' understandings of *Communication* and *Problem Solving* approached those of the researchers. We did not see such indications for *Connections*, *Reasoning & Proof*, or *Representation*. It is interesting to note that the researchers' own agreement (as measured by Kappa scores) was lowest for *Connections*, *Reasoning & Proof*, and *Representation* last year. Moreover, our Kappas improved the most for *Reasoning & Proof* and *Representation* (by about .20 each) from last year to this year. This indicates that those activities were also difficult for the researchers to assess. We need to focus more on negotiating the meanings of those particular processes with PSTs, and explicitly discuss student responses that might indicate such processes. For reasons noted above, we need to further reconceptualize our operationalization of *Connections*.

PSTs demonstrated significant improvement over last year in eliciting *Communication* and *Representation*. Although there are other differences to consider between last year's cohort and this year's cohort, we attribute most of the improvement to PSTs' assessments of students' cognitive activity. PSTs also demonstrated improvement in eliciting cognitive activities in general throughout the 10 weeks of letter-writing this year.



<u>Figure 5.</u> Representation

Figure 6 illustrates improvement by comparing weekly averages (per PST) of total elicited cognitive activities. Note that '*' indicates a significant improvement over Week 1 at the p=.05 level, and '**' indicates such an improvement at the p=.01 level. As with *Communication* and *Representation* (in fact, these measures carried most of the weight in Figure 6), and to a lesser degree with *Problem Solving*, we see a trend of significant improvement by the fourth week, with a slight drop and leveling off to follow. Still, elicited activities for Weeks 5, 7, 8, 9, and 10 remained significantly higher than Week 1 (the improvement of Week 6 over Week 1 showed significance of p=0.12).

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Recall that Week 5 was the final letter for the first class of Algebra II students; Week 6 was the first letter to the new class of Algebra II students. This partially explains the dip in Figure 6. But why didn't elicited activities in later weeks ever surpass those of Week 4? From last year's data (with corroboration from the classroom teacher's observations) we hypothesized that the novelty of letter writing had served as a motivator for students (and PSTs) in the first few weeks; although PSTs demonstrated proficiency at eliciting most activities, by the fifth week the novelty had worn off. Letter-writing pairs continued to work productively, but with no further significant improvement.

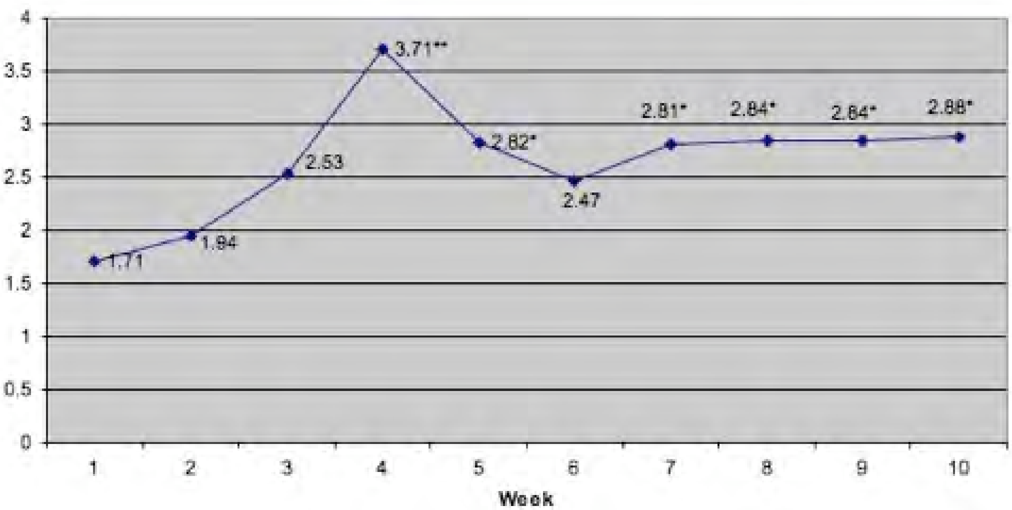

Figure 6. Weekly averages of elicited cognitive activities per PST.

Our results do indicate improvement, both in terms of PSTs' understandings of the NCTM *Process Standards* and in terms of their abilities to elicit such processes within the interactions of letter-writing pairs. Results also inform ways in which we could improve the letter-writing assignment for future cohorts of PSTs. In addition to those improvements already mentioned in this summary, we will also consider decreasing the number of letter-writing exchanges and increasing the amount of time PSTs and students have to respond. This could provide more time for classroom discussion and PSTs' reflection between exchanges, while maintaining student and PST engagement for the duration of the semester.

Discussions about *Reasoning & Proof* could support better understandings of that process. Although PSTs seemed to understand *Problem Solving* better than *Reasoning & Proof*, PSTs rarely elicited either process. Late signs of improvement in eliciting these processes indicate that further support could lead to further improvement. If letter-writing exchanges occurred every other week, PSTs could prepare drafts of tasks to share in class. They could also spend more time in cooperative groups assessing responses, which would provide further opportunity for the class to discuss task analysis in an authentic context.

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BEYOND METHODS: PRESERVICE TEACHERS' USE OF MATHEMATICS IN THEIR DAILY LIVES

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Do reform-based elementary mathematics methods courses change preservice teachers' perceptions about what it means to know and to do mathematics solely for the purpose of effective classroom teaching? Or, are we also influencing how preservice teachers approach, and solve mathematical problems that occur in their daily non-teacher lives? Analyses of preservice students' reports of their out-of-classroom mathematical activity combined with their reflections upon a scholarly essay depicting different learning paths from student to teacher show that our methods course did influence preservice teachers' attitudes toward and competence with their day-to-day mathematical activity. Suggested implications focus on empowering new teachers to understand, guide, and redirect students' invented and intuitive solution strategies for novel and routine mathematics problems.

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.

Objectives

To improve the efficacy of elementary mathematics teachers, methods courses have shifted away from telling preservice teachers how to "transmit" mathematical knowledge to students, to modeling and using constructivist pedagogies (e.g., Hiebert et al, 1997; Lampert, 2001; Schifter, 1996). This shift has allowed pre-service teachers to experience the conceptual objectives of reform-based mathematics teaching, while enriching their content knowledge through shared solution strategies and perspectives. Constructivism in general and research-based ideas such as cooperative learning and Cognitively Guided Instruction (e.g., Carpenter, Fennema & Franke, 1996) in particular, open many students' eyes to how much teaching and learning mathematics have changed since they were elementary students. Consequently, many new teachers enter the profession with the sort of pedagogical content knowledge (Shulman, 1986) that leads to rich, engaging, and successful mathematics classrooms.

Our objectives for this study, however, go beyond the reported pedagogical content successes of the modern methods classroom. We sought to determine if our methods course was changing preservice teachers' perceptions about what it means to know and to do mathematics *solely* for the purpose of effective classroom teaching. Or, were we also influencing how preservice teachers approached, and solved mathematical problems that occurred in their daily non-teacher lives? Specifically, we examined whether or not a reform-driven mathematics methods course that included weekly field experiences had any impact on how preservice teachers engaged with their everyday life tasks that required mathematical thinking. And, if change did occur, what were the personal contexts and motivations for those transformations?

Theoretical Framework

Instructors of methods courses are often faced with students who have deficits in their content knowledge (e.g., Ball, 1990; Bischoff, Hatch, & Watford, 1999; Merseth, 1993); negative associations with learning mathematics (Ball, 1990; Bischoff et al, 1990); and a resistance to teaching mathematics differently from how they were taught (Ball, 1990; Brown, Cooney & Jones, 1990; Ebby, 2000). Furthermore, many elementary-level preservice teachers consider themselves poor at mathematics because they were unsuccessful with the subject in school.

On the flip-side, some preservice teachers who were successful in school mathematics never consider that teaching mathematics could be anything other than passing along mathematical formulas and procedures for students to memorize and practice. These teacher candidates may resist the idea of being ongoing learners of mathematics because they believe they can already do what is on the elementary curriculum. These students never consider that many of their procedures are devoid of conceptual understanding and that there is more than one way to solve what seem like standard problems (Bere, 2006).

When looking to frame strategies for studying and accommodating the gamut of learners in our methods classes, an article by Caroline Ebby (2000) influenced our inquiry. She followed changes in preservice teachers' attitudes and understandings about teaching and learning elementary mathematics as they integrated reform-based methods coursework with complementary fieldwork. She identified three paths of learning and characterized these paths using three common student profiles labeled “Julia,” “Amy,” and “Michelle.”

First, “Julia” began her course- and fieldwork from a position of confidence. She judged her success with school mathematics based on “right answers” and completing context-appropriate formulas. By the end of the course, Julia recognized that *understanding* concepts and reasoning behind procedures was integral to learning both for herself and her students.

“Amy,” was proficient in standard algorithms and procedures. She was also, however, interested in her peers’ alternative strategies and explanations. This respect for different perspectives led Amy to integrate easily into the reform-minded environment of her field experience classroom and was excited by students’ enthusiasm for using and sharing their own methods for solving problems.

Finally, “Michelle” described herself as bad at math, and her passive participation in the methods course revealed her low self-esteem and unwillingness to take risks. Through Michelle’s experiences in both the methods classroom and her fieldwork, she came to understand that verbalizing her thinking about a problem did matter and could in fact facilitate a correct solution. This understanding opened her eyes to the learning processes of young children and turned her on to teaching mathematics in a way that would allow “children to discover on their own how it works” (p.89).

We found Ebby’s students very familiar and compelling. Her categorization of students resonated with us and thus, we wanted to preserve this established framework for our own inquiry. We also, however, wanted to move beyond Ebby’s framework to investigate whether or not students’ in vivo- and post-methods views of themselves influenced how they approached and solved routine mathematical tasks that they, as adults, encountered everyday. Because we did use Ebby’s profiles, we effectively adopted the assumption that methods students discussed in this paper were already integrating their course- and fieldwork. Thus, we focused on the impact of that integration on students’ out-of-classroom mathematics experiences.

Methods

Data was collected from 22 students enrolled in a 10-week graduate-level reform-based methods course, which included 15 hours of fieldwork. The data came from two sources. The first was a weekly assignment in which students were asked to submit a description of an everyday situation that required some sort of mathematical analysis. Students were to describe the problem situation and provide strategies for how they solved it at the time. Situations included shopping, cooking, traveling, baseball statistics, and buying a home. The second data source was a reflective essay about which preservice teacher in Ebby’s article students identified with and why. This reflective essay was assigned in the ninth week of the quarter.

Analyses began by identifying and coding specific characteristics of Julia, Michelle, and Amy. Specifically, each of Ebby’s prototype students demonstrated unique characteristics as a learner and teacher of mathematics (see Figure 1 for a complete list of codes). Julia’s coded characteristics included: a positive self-identity as a leaner of math (JPOS); “good” at math based on external definitions such as tests (JEXT); and an active participant in the methods class

(JPART). Amy's coded characteristics included: a positive self-identification as a learner of math (APOS); a strong background in math (ASTR); able to reason mathematically and without using memorized rules (AREAS); and interested in seeing how others use informal approaches to solve problems (AMULT). Michelle's coded characteristics included: negative self-image as a learner of mathematics (MNEG); an avoidance of mathematics whenever possible (MAVO); a passive learner in her methods course (MPAS); and difficulty explaining and reasoning her problem solving strategies (MREA).

Next, using the same codes, students' reflective essays were analyzed for identification with a specific profile and for examples of that identification. Then, the final stage of analysis was to examine if identification with Julie, Amy, or Michelle, transferred to respective students' day-to-day encounters with mathematics and if any changes occurred over the 10-week period. To do this, weekly math assignments were examined for instances of how Ebby's preservice teacher characteristics were manifest. For example, one student wrote a very simple math problem concerning train fare and spent most of the assignment reflecting on creative suggestions for using train system maps in the classroom. This example was interpreted and recorded as MAVO, "avoidance of math when possible."

Julia

JPOS- Positive self-view as learner of math
JEXT- "Good" at math based on external definitions of success-test, etc.
JPART- Participated in class and shared often (as an adult student)
JPEER- Listened to peers to learn from others
JMULT- Looked for multiple ways to solve a problem
JCOMM- The importance of communication
JHPASS- Historically passive in math class, as a K-12 student "used the rules"

Amy

APOS- Positive self view as a learner of math
ASTR- Strong background in math- i.e. math major
AREAS- Able to reason mathematical problems- did not have to memorize the rules
AINS- Developing and insider view- interested in how others learn rather than focusing on self-development
APART- Active participant in class as an adult/in teacher prep courses
AMULT- Interested to see how others use informal approaches to solve problems

Michelle

MNEG- No confidence as a learner of math, negative self-image
MAVO- Avoidance of math
MREFL- Reflective of own learning to come to a new understanding of what it means to teach/learn math
MPASS- Passive learner of math as adult/in teacher prep course
MCON- "Concealed" resistance to doing math.
MREA- Difficulty explaining and reasoning problem solving
MMULT- Discovery of multiple ways to solve problems

Figure 1. Coded characteristics of Ebby's profiles

Results

Results show that trajectories of problem solving strategies evidenced in students' weekly math assignments were in keeping with students' identification with one of Ebby's profiles. Below we describe this connection using an example of a "Julia", an "Amy", and a "Michelle."

In her reflective essay, Kate described herself as a "Julia," defining herself as "good at math" based on external factors like tests and homework. Ultimately, Kate progressed from demonstrating conventional mathematics skills to understanding and communicating *how* she came to an answer. Kate noted that by the end of the quarter, she was interested and impressed to hear how her peers solved math problems and remarked in her reflective essay, "I honestly had no idea there were so many ways to solve a math problem" and "[n]ow that I understand the value in seeing how my peers solve problems, I can see how important it will be for my future students to participate in this process."

Kate's growth as a mathematical thinker was reflected in her weekly assignments, which were clear, detailed, and mathematically challenging. Her first assignment was about decreasing a recipe for a dinner party. Kate set up her problem clearly and described the task as needing to alter the measurements in the recipe to reduce the number of servings from 10 to 4. She gave specific guidelines such as "[t]he new amounts need to be in amounts that can be measured using conventional kitchen measurements i.e. _ cup and some amounts need to be converted to another measurement, i.e. 1 tablespoon = 3 teaspoons". While she clearly defined and conventionally set up the problem, she did not solve it. In fact, for the first three weeks, Kate presented problems with no solutions or reflections. Not until week 4 did she begin to show process and to communicate her reasoning.

During weeks 4 and 5, she took her weekly math problems into her fieldwork classroom and began to ask students to find at least two different ways to solve each problem. In week 4, Kate once again composed a clear and challenging math problem. Her problem was for a 6th grade class and was about seismic and tsunami wave travel and time. Her math problems began to differ this week because she was not giving strict parameters within which students had to figure out the problems. Rather, her directions became less stringent, which allowed multiple interpretations of the problem. Her stated purpose for now approaching problems in this manner was to encourage students to discover multiple solutions. Kate herself, however, still did not solve the problems.

Another shift occurred in week 6 when Kate began to solve the math problems herself and to demonstrate her thinking. Kate's week 6 math problem was about figuring out the best cell phone plan based on schedule and minutes used. This is the first week in which Kate actually solved a problem and showed her work. This was the first step in an even more significant change, which occurred in weeks 7-9.

In weeks 7-9, Kate clearly communicated her problem solving through detailed answers. For example, in week 7 Kate's problem was a logic puzzle where students had to fill in a chart's missing information about the countries participating in the Winter Olympics, (which were going on at the time), and medals awarded using data provided and clues given. Kate both solved this problem and detailed how she solved the problem. This final step relates back to her reflection and self-identification as "Julia" and the significance of communication in mathematics. Kate appears to have shifted her *own* math solving abilities. That is, she progressed from setting up problems in standard algorithmic ways using conventional "rules" to actually processing her thinking and communicating and explaining her thinking.

Nora was a self-described "Amy." According to her reflection paper, she had a positive self-view as a learner of mathematics (APOS), a background in college level math courses, and was comfortable with the conceptual ideas of mathematics (ASTR and AREAS). Like "Amy", Nora was more interested in how others learn mathematics than in her own self-development (AINS and AMULT). Nora stated in her reflection essay: "Working with children, often on homework in an after-school program, I began having them explain to me what they were doing and how they understood the problem. I used this approach because I was interested in their thinking process and wanted to identify where they were confused and what they understood." Besides student development, Nora, like "Amy", was also interested in how children use multiple and informal approaches to solve problems, "Instead of correcting [the children] and presenting math the way I understood it, I would ask probing questions to help them discover their own solutions."

Nora's weekly math assignments reflected her confidence in math and also her progressive interest in how children learn math. In weeks 1-2, Nora presented standard math problems that were based in her everyday life. For example, one of her problems was based on whether or not to buy a new car. In keeping with Nora's comfort with math, she presented the problem and easily communicated her thinking processes using estimation. This problem illustrated how she uses mathematical reasoning instead of relying on memorized rules (AREAS).

Nora's weekly math assignments progressed steadily from weeks 2-6. Nora presented weekly math problems that she encountered in her real life such as how she divided lesson plans by topic and time in week 5 and used in-depth communication to discuss her reasoning. However, in week 7 a shift in her work began to emerge. This shift aligned with "Amy's" transformation. Nora began to focus on how the learner learns math and she was interested in how learners use informal approaches to solve problems. For example, in week 7, Nora detailed her work with children aged 3-5. Knowing children this age have yet to learn the "rules" of math, she began to probe in order to find what they understood conceptually. Her math problem was focused on asking the children how many children from the class were present one day and how many teachers there were. Nora then continued asking addition and subtraction questions such as: If there are 7 kids and 2 teachers how many people altogether? Nora "studied" them and their thinking processes as they solved these problems.

In weeks 8 and 9, Nora's weekly math problems were based on questions from a standardized test that the children in her fieldwork observations were solving. Nora presented the questions and then explained the different ways the children solved the problems. For example, in week 8 Nora wrote the problem: "A toy store sells bicycles and tricycles. Bicycles have two wheels and tricycles have three wheels. If there are nineteen wheels all together, how many bicycles are there and how many tricycles?" Nora demonstrated through drawings and explanations how children came to their solutions. Her emphasis in this weekly math assignment was not the problem and how *she* solved it, but in the processes and variations children used to solve the problems. This links Nora back to an "Amy" characteristic. That is, she was more interested in how others learn than in focusing on her own self-development (AINS).

Annie described herself as a "Michelle," with an initial negative self-image as a mathematician (MNEG). Annie began her reflective essay by describing her painful struggles with elementary school mathematics classes. She detailed how she was lost during math class and when she asked for explanations, teachers just repeated what was in the textbook. Her frustration as a math learner was summed up with her statement: "Math made me feel inadequate." Similar to "Michelle," throughout the methods course Annie began a process of reflection on what it means to be a learner and teacher of mathematics, including the discovery of multiple ways to solve a problem.

During week 1, Annie's problem was based on figuring out her train fare for the week. Her problem was as simple as how much more would taking the train cost per week if there were a fare increase. Annie, an accomplished artist, spent a majority of the problem explaining how she could connect this problem to other subjects doing creative art projects and using the internet to research more information on public transportation. Annie did not solve the problem or explain her reasoning. This avoidance of math (MAVO) is a clear characteristic of "Michelle."

Week 2's problem was very similar to week 1. Annie spent the majority of her weekly math problem, which was a simple addition and subtraction bank account problem, detailing how she would have a discussion about banking with children. The difference in week 2's math problem is that Annie did do the basic computation and solved the problem.

Problems for weeks 3-5 demonstrated a shift in that Annie moved from a basic problem that she would teach in a class to how she used numbers in her everyday life. From therein, her math problems began to increase in complexity. For example, in week 5 her math problem was about how long it would take her to drive from Florida to Chicago and at what time she would arrive. She incorporated information such as speed limits and time zones. Annie did not solve the problem, but she did begin to explain her reasoning. She detailed her thinking processes with respect to why she did what she did, and included the algorithms that she used. By week 5, Annie was clearly becoming more reflective of her own skills and confident in her number sense.

By weeks 6 and 7, a dramatic change occurred in how Annie solved her weekly math problems. Her computation skill level was basic, but the manner with which she solved the problems progressed. Annie began to offer multiple solutions to her weekly math problem. For example, in week 6 Annie figured out how much a smoking habit was costing her per week, month, and year. Annie solved the problem by multiplying the total cost by each period of time (i.e. $2.54 x 7 days = $17.78 per week) and then as a second solution, she multiplied the average price per pack by the number of packs in a period of time (i.e. .5 packs x 7 days = 3.5 packs per week; 3.5 packs x $5.07 = $17.78 week). Annie demonstrated her "Michelle-like" characteristics through her use of multiple ways to solve problems (MMULT). Overall, Annie's weekly math assignments were a clear documentation of her growth throughout the methods course.

Discussion

A recurrent and prominent concern voiced by our preservice teachers is that they will not understand their students' alternative or invented strategies for solving math problems. More precisely, they are concerned that they will not be able to see beyond their own reliance on standard algorithms or memorization to know if their students are indeed thinking in ways that will move them toward a correct solution (Ball, 1990). Furthermore, it is unrealistic to believe that any repertoire of how children or adults think about any given problem can ever be complete and recorded in a teacher's reference manual or textbook. Thus, in addition to sharing solution strategies to problems posed in methods classes and seeing children and teachers in action in constructivist settings, we believe it is important for preservice and practicing teachers to challenge themselves regularly outside the classroom in order to develop competence and flexibility with for instance mental calculations for determining a tip versus pulling out the cell phone calculator. If this outside-the-classroom dedication and practice does not develop, we believe that preservice and practicing teachers will not develop the confidence necessary for keeping up with, interpreting, guiding, and redirecting students' thinking about both novel and routine mathematical situations.

Just as classroom teachers are committed to educating students for success both in and out the classroom, methods instructors should be looking for every opportunity for preservice teachers to use the sort of flexible mathematical thinking skills and number sense outside of teaching that they will be expecting of their young students. By analyzing the learning trajectories of methods students both in and outside the classrooms, we were able to see the developmental changes that students of varying abilities experienced. Further, whether an "Amy," "Michelle," or "Julia," most students worked with their prior knowledge, experiences, and attitudes to become better mathematical thinkers as they learned about how important it is for students to connect cognitively to problems, and to use personal perspectives to solve them. Methods students, regardless of their identification, were inevitably surprised to learn of their classmates' alternative strategies and were motivated to explore others' thinking as a means to enrich their own. "Julia's" were surprised to learn that their solutions were not the only correct ones; and "Michelle's" were surprised to learn that they had viable mathematical thoughts and in fact could teach the "Julia's" new strategies. In general, bringing a mathematics "past" to the methods learning table and having opportunity to work with that past, leveled the "learning" field and led to many students reconsidering their mathematical capabilities.

Relationship to PME Goals

Ultimately, our goal as methods instructors is to help students become good if not great teachers. Sharing our successful strategies and hearing about others' accomplishments can only improve the overall quality of mathematics teachers everywhere. Furthermore, although in its nascent stage, the present line of inquiry can be broadened to include research that will add to the literature on teacher and preservice teacher beliefs; and on teacher change. Were the methods students discussed in this paper changing their *beliefs* about mathematics as well as their practices? Class discussions would suggest that yes, our methods students were changing their ideas about what the most efficient ways of solving routine problems might be. Further inquiry, however, into that question, and the impact that change has on elementary students' learning is necessary. We wonder: Can a teacher truly achieve optimum success using constructivist pedagogies in the classroom while continuing to rely on standard algorithms and rote methodologies in daily life? Authenticity is an important component of education if students are going to see the out-of-the-classroom value of what they are learning. Teachers can easily bring such authenticity to the classroom by discussing real problems solved in a variety of ways.

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CONTENT KNOWLEDGE ACHIEVEMENT: PLACING MATHEMATICS CONTENT FOR PROSPECTIVE K-8 SCHOOL TEACHERS IN CONTEXT THROUGH FIELD EXPERIENCES

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This study reports on changes in prospective teachers' knowledge of content, pedagogy, and school cultures developed during field experiences in their mathematics courses. Analysis of data points to positive outcomes and reactions. This paper will address the motivating factors, design issues, and results of implementing field experiences for prospective K-8 teachers in the context of their mathematics coursework abstract goes here, and should not exceed ten lines.

Factors Motivating Implementation of Field Experiences

The Content Knowledge Gap

A key characteristic of a highly qualified teacher is a strong base of subject matter knowledge. According to Darling-Hammond (2000a, 2000b) the subject matter knowledge of a teacher is a significant factor in student achievement. Ma (1999) suggests that this applies, in particular, to mathematics. Others have posited that the successful mathematics teachers need a strong base of pedagogical content knowledge (Wilson, Floden, & Ferrini-Mundy, 2002; Wilson, Shulman & Richert, 1987). Thus, it seems to be crucial to ensure that prospective K-8 teachers gain a deep and lasting understanding of mathematics. Yet, at our university, future K-8 teachers seldom seem to achieve the depth of content knowledge we feel is necessary for their future success as teachers of mathematics. This ongoing problem became the impetus behind our ultimate decision to implement field experiences in our mathematics content courses.

Theoretical Framework

Our theoretical framework began as a speculative construct to explain why students in our three-semester sequence of mathematics content courses for prospective K-8 teachers seem to have difficulty achieving a deep understanding of mathematics. Many would attribute such difficulties primarily to inadequate command of prerequisite material, which is a problem for our students, but it seemed to us that there may be other issues as well. Specifically, prospective teachers seem to persist in a belief that their mathematics coursework does not connect in any meaningful way to their future careers. Everything appears to be a philosophical exercise to them, simply a hurdle to cross, disconnected from reality. We speculated that achievement of mathematical content knowledge for prospective teachers depends on two factors, as illustrated in Figure 1.

Our working framework suggested that positive changes in students' beliefs about the meaningfulness of their mathematics coursework to their future careers might bring about positive changes in their understanding of the mathematical content. In practice, no amount of class discussion of the realities of teaching in today's public schools seemed to affect this belief. What appeared to be necessary to bring about change in this belief was exposure to real-world teachers, teaching mathematics in real-world classroom settings.

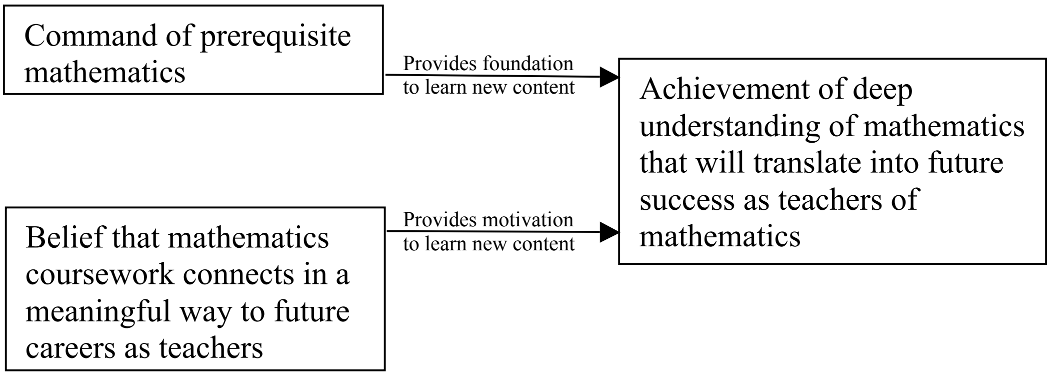

Figure 1. Theoretical framework

Field Experience

Our goals in implementing a field experience component in our mathematics content courses for future teachers were four-fold. We hoped to enhance students' (a) mastery of course content, (b) appreciation of the realities of contemporary classrooms and of effective teaching strategies, (c) sense of efficacy, as prospective teachers, and (d) desire to teach mathematics and/or confirm the level at which they would like to teach.

We have offered the field experience in all three mathematics content courses for undergraduate prospective teachers at various points since Spring 2004. In Spring 2004, the field experience was an optional extra credit assignment to allow for comparison between students who did and those who did not acquire this experience. Based on data collected in Spring 2004, the field experience was henceforth required of all students in each section taught by the authors. For this study, we restrict our attention to data collected during Spring 2004, Spring 2005, and Spring 2006 from the second course in our three-semester sequence of mathematics content courses for undergraduate prospective teachers. (Field experience data from the first course, Number Systems, and the third course, Intuitive Geometry, will appear in future reports.) The content in the second course includes proportional reasoning, algebra, probability, and statistics. The field experience assignment in this course required students to observe a middle school mathematics class once per week for ten weeks and write weekly reflections.

Research Questions

The present study sought to address four questions about the impact of infusing a field experience into mathematics content coursework.

- Overall, did the field experience help students develop mastery of the course content? In particular (for Spring 2004 students), did students completing the field experience perform better on course exams and the final exam than students not completing the field experience? In addition, did students (Sp04, Sp05, Sp06) themselves feel that the field experience helped them develop understanding of the course content?

- Overall, did the field experience help students to develop a greater sense of their own potential efficacy as future teachers? Here, we define a prospective teacher's *perceived potential efficacy as a future teacher* to mean her perception of the extent to which she will acquire confidence in her teaching ability and confidence about meeting her future students' needs.
- Overall, did the field experience alter students' perceptions about teaching in diverse middle school classrooms, and if so, how? That is, did they grow and mature in their perceptions about current school cultures, diversity of student needs, teacher work loads, expected mathematical content, and/or modern pedagogical strategies?
- Overall, did the field experience help students confirm their career goals? If so, to what extent? If not, did the experience expand their horizons about what career might be a better fit? What changes in career preparation, if any, did students adopt as a result of this experience?

Methods

Participants

Participants in this study (Sp04: $n = 24$, Sp05: $n = 57$, Sp06: $n = 55$) ranged from freshman to senior level undergraduates, most of whom had declared an intention to pursue careers as elementary school teachers. Many were enrolled in teacher-preparation tracks of one of six majors containing such tracks. Students in such a track are required to take a three-semester sequence of content courses in mathematics: Number Systems; Concepts in Mathematics, Probability, and Statistics; and Geometry. They can choose a concentration in mathematics, which involves three additional courses in mathematics, but most students in the second course of our required three-semester mathematics content sequence have not even considered this option, let alone whether to pursue it.

Data and Data Collection

Students completed pre- and post-surveys, consisting of Likert-type and open-ended items focusing on their goals and expectations. We developed the first survey, *Field Experience Expectations* (FEE), to compare students' expectations at the outset to perspectives they acquired by the end of the experience. A second survey, *Teacher Efficacy* (TE) was adapted from surveys originally developed by Kushner (2003) and Turner, Cruz, and Papakonstantinou (2004), which itself is based on work by Gibson and Dembo (1984). Gibson and Dembo's survey is regarded as an industry standard for research on teacher efficacy, although it does not focus on the teaching of mathematics *per se*, and it is intended for use with teachers or teacher candidates who are further along in their professional development than the students in our study.

Students provided demographic information and completed a variety of written reflections, including (a) a starting points narrative completed prior to the first classroom visit, in which they articulated their expectations and goals relating to their field experience;

(b) observation reflections, consisting of structured weekly log entries completed within two days of each classroom visit; and (c) a final field experience reflection, completed after their final classroom visit, where they wrote about their experience over the course of the semester.

Analysis

Likert-type items from the pre- and post-surveys were analyzed quantitatively while open-ended survey items and narrative reflections were analyzed qualitatively. To achieve a more complete picture of students' development toward each of the four goals, multiple sources of evidence were analyzed.

Content Knowledge

For students in the Spring 2004 data set, three hour exam scores and final exam scores were recorded, averaged within each group (field experience, no field experience), and compared. To evaluate students' (Sp04, Sp05, Sp06) perceptions of the impact of the field experience on their understanding of the course content, we sought out comments to this end on the FEE post-survey and final reflection.

Perceived Potential Efficacy as Future Teachers

We implemented our Teacher Efficacy (TE) survey beginning with the Spring 2006 students as a way of operationalizing our definition of *perceived potential efficacy as future teachers*. Students' responses to multiple Likert-type statements (six-point scale) were scored to obtain various measures of their confidence in their future teaching ability or about meeting the needs of their future students. Students were measured in this way prior to starting the field experience and upon completing it. Changes in students' mean scores from the pre-TE-survey to the post-TE-survey were analyzed for possible increases in each of these items.

Realities of Teaching Mathematics in Public Schools

To evaluate changes in students' (Sp04, Sp05, Sp06) perceptions of the realities of teaching mathematics in public schools, we evaluated their responses to the final reflection item, "What, if anything, surprised you about your field experience?" Responses were initially simply coded as Y (yes) or N (no) to indicate whether anything surprised that student. Then responses were inductively studied, categorized, and coded to enable descriptions of commonalities. Additional evidence of these commonalities was sought in other comments on the final reflections and students' stated expectations prior to starting the field experience.

Confirmation of Career Goals

To evaluate students' (Sp04, Sp05, Sp06) career goals, we qualitatively evaluated their responses to the final reflection open-response item, "How has participating in a middle school mathematics class affected your desire to teach in a K-8 school?" Responses were coded as CT (confirmed my desire to teach), CnotT (confirmed my desire not to teach), CE (confirmed my desire to teach elementary school), CM (confirmed my desire to teach middle school), CnotM (confirmed my desire not to teach middle school), ConsM (made me consider teaching middle school mathematics). Note that these categories are not mutually exclusive.

A response was coded as CT if it was implicit from a student's response. The remaining categorizations were only attributed if the student's language was quite explicit.

Results

Analysis of the data suggests that most students made positive gains toward all four our goals for the field experience assignment. Details of results on each research question follow.

Content Knowledge

Spring 2004 Results

In the first year of implementation we compared exam scores of participants (n = 24) versus non-participants (n = 31). In a comparison of grades from the first exam to the fourth exam the results were the following: participants in the field experience scored 7.4% lower on the first exam than non-participants, but their scores on the rest of the exams were nearly identical (1.6%, 0.49%, and 0.95% lower, respectively). The apparent advantage by the field experience students may have stemmed from their experience working with the middle school students. In evidence from the QMT post-survey all participants agreed or strongly agreed with the statement, "This field experience has helped me learn the material in this course." Full details of the Spring 2004 methods and results are given in Strage and Sliva (2006). Since we had such an overwhelmingly positive response from the students who participated in this initial semester, we made participation in the field experience mandatory in subsequent courses taught by the same instructor.

Self-Perceptions of Impact

In their final reflections many students commented on the mathematical content presented in the middle school classes they were observing, describing in various ways how it impacted their own understanding of mathematics.

- "This semester was wonderful. A lot of things make more sense to me because I actually see it. I learn to put in practice what I learn from the students to my class and vice versa."
- "From my field experience, I learned teaching strategies, discipline strategies, and more ways to think about mathematical concepts. My knowledge was built on various algebraic concepts during my visits to the classroom."
- "I learned the material itself, because by observing I either got acquainted with the concept before I encountered it in my class, or I had the concept reviewed and reinforced."
- "I really realized the importance of being able to present concepts from different perspectives, since children learn in many different ways."
- "I learned classroom management, and also was able to touch up on math skills I had forgotten. I learned how to get students' respect and how to keep them motivated."
- "[I learned] the importance of explaining things in more ways than one. I learned that math can be fun and how it relates to things we see and use in everyday life…. I learned that doing homework is crucial to understanding math and doing well on tests."

- "I learned that I have no desire to ever teach middle school math. Although I did enjoy working with the students, they gave me an insight into what I look forward to as a parent. I also was able to get another example of some of the material we went over in class although most the time they were behind us in the material we were doing. I really enjoyed the examples the teacher gave. They were very helpful in understanding the materials."

Perceived Potential Efficacy as Future Teachers

Descriptive analyses of changes in students' responses to items on the TE pre- and post-surveys suggest that they did, indeed, develop greater confidence in their potential as mathematics teachers. More specifically, well over three quarters of respondents reported equal or higher levels of confidence in their ability to teach math effectively on the post-survey than on the pre-survey (88%). Similar positive trends emerged in their beliefs in their ability to become an exemplary math teacher (79%), in their ability to improve their math teaching ability (78%), in their ability to adjust or explain an assignment so that a student could understand it (82%). Similar positive differences emerged in response to questions about how quickly, in comparison to other people, they felt they could pick up effective math teaching techniques (83%) and learn to teach math effectively to their students (76%).

Realities of Teaching Mathematics in Public Schools

Analysis of final reflections revealed that nearly all (94%) of the field experience participants were surprised by at least one element of their experience. Among those participants, the most common element of surprise (expressed by 31% of them) was the behavior of middle school students, citing surprise at good behavior and poor behavior. Many students indicated surprise at, how many different classroom management strategies teachers used (or didn't use) and how critical they are (15%), how unexpected the content and curriculum were (10%), how diverse the students are (7%), how different schools and mathematics classes are today than when they were middle school students (7%), how difficult and challenging teaching appears to be (6%), how many different pedagogical strategies teachers use every day (4%), and how technology was used (4 %). One student wrote of an experience that encompassed many of these elements.

> "The thing that surprised me the most about my field experience this semester is how much the classroom teaching styles have changed and how the subjects are taught. Any of the math classes that I observed were following the California standards and had them up in the classroom and in the textbooks that the students used. There also seemed to be more teachers showing the different steps of how to approach certain problems or equations. This I thought was really helpful not only for myself to refresh my memory about certain math concepts but also for the students to really try and grasp how to approach equations. I also was surprised as to how much emphasis is put on teaching how to approach equations. I also was surprised as to how much emphasis is put on 'teaching to the test'. In my child development classes it is discussed very much and how it might hinder some students as well as teachers from really getting to know the full potential of students."

Confirmation of Career Goals

When asked, "How has participating in a middle school mathematics class affected your desire to teach in a K-8 school?" 80% of the respondents indicated that the field experience helped them confirm a desire to teach. Out of those confirming a desire to teach, 3% confirmed a desire to teach middle school mathematics and 26% indicated that they were considering teaching middle school mathematics after completing the field experience at this level. On the other hand, 49% of the respondents indicated that the field experience helped them confirm that they did *not* want to teach at the middle school level and/or that they definitely wanted to teach at the elementary school level. In fact, students whose confidence in their ability to teach math effectively decreased over the course of the semester indicated that the experience had enabled them to clarify that they did, indeed, still want to teach, but that they wanted to teach at the elementary level. They cited characteristics of adolescents, and the behavior management challenges they observed, as reasons for preferring the younger grades.

To follow up on the apparent positive shift towards teaching mathematics at the middle school level, we evaluated enrollments in our Math Minor for K-8 Teachers and found that enrollments in this minor nearly doubled from about 13 per year prior to Fall 2004 to about 25 per year after that point. We don't attribute this entirely to the field experience, as we have taken great care to advertise the minor and strongly encourage our students to enroll. However, we do see this increase as a sign that at least some of those students who expressed in interest in teaching middle school mathematics decided to pursue the next step in making that a real possibility.

Discussion

The field experience component has been included in each of the three courses with positive reactions from students and progress towards each of the four goals. The experience seems to be especially dramatic for participants in the second course, given that it occurs in a middle school setting. This setting seems to include more extreme expectations and surprises for participants than the elementary school settings required for the field experiences in the other two courses. Written comments from students on weekly observations, the surveys, and final reflections make it very clear that they perceive the experience to have a dramatic impact on their thinking about mathematics as a topic and about teaching mathematics.

We would like to better understand those aspects of the experience that impact students' content knowledge. Why do students seem to learn the content while observing the content being taught to K-8 students? Our investigation thus far leads us to believe that our students are less stressed when they are asked to merely observe what is being taught. Perhaps, the lack of stress or anxiety enables learning to occur rather than being blocked. Conceivably, there is also an importance in watching actual students learn the content. By seeing what they will be expected to teach students in their future careers, it seems to be hitting home for them that the mathematics content knowledge needed by effective teachers is significantly greater than they expected, perhaps leading them to conclude that their college content courses are more relevant than they assumed.

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CREATING THIRD SPACES: INTEGRATING FAMILY AND COMMUNITY RESOURCES INTO ELEMENTARY MATHEMATICS METHODS

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In this paper, we examine theory and practice related to creating spaces for pre-service teachers to learn about incorporating family and community resources into elementary mathematics and literacy. Using interview data, as well as excerpts from pre-service teachers' reflections, we first define overlaps between mathematics and literacy, and between family and community funds of knowledge and school-based content. We then describe how pre-service teachers learned about, experienced, and enacted these overlaps in three distinct, but intersecting spaces – the elementary classroom, the university methods class, and the community.

In this paper, we examine theory and practice related to creating spaces for pre-service teachers to learn about incorporating family and community resources into elementary mathematics and literacy. Our framework for understanding pre-service teacher learning related to families and communities as resources for mathematics and literacy instruction locates this learning as happening concurrently in an elementary school, in the community in which the elementary school is located, and in university classrooms (Figure 1). Furthermore, we operationalize teacher learning as changes in:

- Personal narratives of self as a learner and teacher.
- Professional identities and practices as elementary teachers.
- Understandings of the mathematics and literacy practices and resources of children, families, and communities.

We are especially interested in understanding the "third spaces" and overlaps that are created and highlighted by this framework for investigating teacher learning. In particular, in this paper, we are focusing on the overlaps inherent in the last element of pre-service teacher learning – related to the literacy and mathematics practices and resources of children, families, and communities – and the tensions, challenges, and benefits of exploring and supporting pre-service teachers learning about these overlaps in three intersecting spaces – elementary school, community, and university.

Research Question

With this framework as our guide, we posed the following research question:

> What are the boundaries, key activities, tools, roles, and resources that define the overlaps and third spaces among the communities and funds of knowledge involved in an elementary school-university methods collaboration?

In this paper, we use pre-service teacher reflections and interviews to identify some of the boundaries, overlaps, spaces that were particularly salient for their learning during the first year of a school-university partnership. After a brief overview of our theoretical framework, we present the results of our analyses along with implications for future research and practice related to teacher learning in third spaces.

Theoretical Framework

In recent years, researchers have provided evidence that helping in-service teachers access and build on the mathematical "funds of knowledge" of students' families and communities can be a powerful mechanism for supporting the mathematical learning of students, parents, and teachers (e.g., Civil, Andrade, & Anhalt, 2000; Civil & Bernier, 2006; Gonzalez, Moll, & Amanti, 2005). Others have suggested that teachers can, and must, learn to view the increasing diversity of their student populations as a resource, rather than a deficit (Rodriguez and Kitchen, 2005), while still others have proposed "third space" as a theory for understanding the learning and exchange of capital that can occur with individuals from diverse communities and practices come together for the purpose of teaching and learning (e.g., Moje et al., 2004).

While all of these theories hold great promise and have led to significant (though typically small scale) successes with in-service teachers, very little research exists documenting or theorizing how teacher educators might create the kinds of spaces needed for pre-service teachers to learn to access and build on funds of knowledge, take advantage of diversity as a resource, and build productive third spaces in their own classrooms (though Gonzalez, Moll & Amanti, 2005, and Rodriguez & Kitchen, 2005, have each begun to describe this process). Here, we build on the theoretical constructs of funds of knowledge and third space to develop a framework for understanding the ways in which pre-service teachers learn about families and communities as resources for instruction for both literacy and mathematics.

Methods

Two cohorts (N=51) of elementary literacy and mathematics methods students have participated in a new partnership between the university and a local elementary school. This partnership is designed to help pre-service teachers learn about incorporating family and community resources into literacy and mathematics instruction. The local elementary school provides an interesting context for this work because its student body has undergone significant transitions in recent years, due to school closures, redrawn district boundaries, and increasing rates of immigration and poverty in the local community. For example, the percentage of students receiving free and reduced lunch has more than tripled in the past three years.

Data for the project include: 1) Interviews with pre-service teachers at the beginning and end of the semester; 2) Interviews with in-service teachers and school principal at the end of each semester; 3) Pre-service teachers' work, including lesson plans, reflective essays, and blogs; and 4) Field notes from Family Math and Literacy Nights (organized and run by pre-service teachers). These data were analyzed using open and iterative coding in which codes were initially assigned based on the research questions and then refined as findings emerged from the data. In this paper, we focus in particular on the pre-service teachers' interviews and reflections from the second semester of the project.

Results and Discussion

Our data analyses led us to focus on understanding the boundaries, roles, artifacts, and tensions related to two key overlaps: 1) between literacy and mathematics and 2) between children's families and communities funds of knowledge and school-based content knowledge. We found that pre-service teacher learning about and within these overlaps was strongly influenced by the spaces – elementary classroom, university classroom, and community – in which the learning occurred. Therefore, after sharing results related to both of the overlaps identified above, we then discuss some of the learning and tensions pre-service and in-service teachers experienced due to the spaces in which the overlaps were explored and enacted.

Mathematics and Literacy

As part of this project, literacy methods and mathematics methods course activities were coordinated and partially co-taught. One activity that was used to explicitly link literacy and mathematics focused on using children's literature to teach mathematics. Findings suggest that, while pre-service teachers identified the overlap between literacy and mathematics as one of their areas of greatest learning during the semester, tensions also arose over the mathematical and literacy value of either of these activities. For example, what is the mathematical value of mathematizing a piece of children's literature? Is the mathematics trivialized or inauthentic in these situations? Alternatively, does using children's literature allow teachers to connect mathematics to the lives, experiences, and interests of children and families? These questions, along with a series of additional activities and tools, helped to define the literacy-mathematics overlap for pre-service teachers.

At the end of the spring semester, pre-service teachers were asked, "What are important connections between literacy and mathematics at the elementary level?" They responded to this question in writing. Analysis of the students' responses revealed three trends related to the question. First, several pre-service teachers were clear that considering the connections between mathematics and literacy was something they had not considered before the semester. The statements below from two pre-service teachers are representative of the reflections of many of the pre-service teachers:

> I used to think that math and literacy weren't really connected…. To group the two together teaches students that it's hard to use one without the other.
>
> I loved that this class focused on the relationship between literacy and mathematics and made me think about how the subjects don't have to exist separately. In fact, I have learned that math and literacy work well when paired together in a classroom.

Second, many pre-service teachers discussed the relationship between children's literature and Cognitively Guided Instruction (CGI) (Carpenter et al., 1989) as an important connection that they observed, particularly in the lower grades. This points to an area in which the overlap between what pre-service teachers were learning in their methods classes and what K-2 in-service teachers were experiencing in professional development activities (related to CGI) came together to enhance pre-service teachers' learning. In the words of one pre-service teacher:

> Through this class I have learned that books can be a fun and interactive way to introduce a math concept. … CGI problems can be used to help students remember the main characters of a story.

Finally, pre-service teachers identified a wide range of connections between mathematics and literacy that, in several instances, went beyond using children's literature to introduce a mathematics topic. While the most common connections were between children's literature and CGI, or between children's books and mathematics more generally, other pre-service teachers described the connection as an important component of children's understanding in both content areas, as a more authentic reflection of the ways in which literacy and mathematics are used outside of school or, as one pre-service teacher wrote, "… the most important connection between literacy and mathematics is communication." Therefore, for these pre-service teachers, the boundaries between literacy and mathematics that had initially seemed quite solid, became increasingly permeable as the semester went on.

Learning about Families and Communities

While learning about families and communities as resources for instruction was the initial and primary focus of this project, it was also perhaps the most difficult overlap to create and maintain. In part, interviews with the in-service teachers suggest that this was due to the transitioning student population at the school with which the veteran teachers had yet to establish a close bond. Several times, we found evidence of pre-service teachers "re-voicing" in-service teachers' impressions about children's families and community, despite the fact that the pre-service teachers did not have the same experiences with the community that the in-service teachers did. In some ways, then, this overlap was dominated by competing, though not necessarily conflicting, discourses – one provided by the in-service teachers and one provided by the university faculty. At the same time, as each semester progressed, pre-service teachers found ways to define this overlap for themselves. For instance, one pre-service teacher volunteered at an evening family event at the school and observed:

> It was a great night and I really enjoyed being able to observe the kids interact with their family and friends. I saw several of the kids from my classroom and got a very new picture of them. In one case it seemed as though I was seeing a completely different student from the one I have conversed with on Wednesdays. I would highly recommend this "outside" experience to everyone.

Pre-service teachers reflected frequently on the value of the diversity of cultures and experiences represented by the elementary children. As one pre-teacher stated:

> The diversity at [School Name] is a wonderful source of knowledge for this school. They have students from all over the world…

Over the course of the semester, pre-service teachers described multiple instances of learning from and about the varied experiences of children and families. However, pre-service teachers were less successful in linking this diversity of experience and knowledge to academic content and had few opportunities to build on these funds of knowledge during instruction. For these pre-service teachers, the boundaries between funds of knowledge and academic content proved to be quite rigid and very difficult to cross in the context of a one-semester course. In the sections that follow, we describe how these boundaries were experienced differently in the spaces of community, university, and elementary classroom – even when these spaces were physically co-located in a single building.

Sharing Space

Our goal of helping pre-service teachers learn to incorporate family and community resources into their instruction required us to, literally and physically, be in a space in which pre-service teachers had access to children, families, and communities. In this project, that meant spending one or two days per week at the local elementary school. Pre-service teachers spent the morning in elementary classrooms and, in the afternoon, convened as a group in the school's library. Establishing and defining this shared, or overlapping, physical space was initially difficult, as the elementary school, like many schools, was constantly using all of its available spaces. However, defining everyone's roles within this overlapping space was even more challenging, though also very important to establishing the theoretical "third space." For instance, was this a space in which pre-service teachers could critique what they observed in in-service teachers' classrooms? Could university faculty participate in this critique? Did in-service teachers feel welcome to participate in the pre-service teachers' classes and vice versa? Posing and responding to these questions helped define the boundaries of this shared space.

Overlaps and Spaces

In addition to the challenges of sharing space, we also noted two ways in which these different spaces impacted how pre-service teachers experienced and enacted the overlaps between literacy and mathematics and between family and community funds of knowledge and academic content. The first example points to a particular challenge in implementing school-university partnerships. While pre-service teachers were cautious about describing what they noticed their elementary classrooms critically, they did repeatedly point to the lack of instructional practices related to the overlaps between literacy and mathematics or between community/family funds of knowledge and school content that we were emphasizing in the methods classes. The two quotes below, from two different pre-service teachers, illustrate this point:

> I did not have the opportunity to see the students in my practicum classes use literacy to learn/access mathematics much, or vice-versa, except for during the two lessons I taught.
>
> I have learned that schools often treat students' experiences in school as isolated experiences. In fact, I did not see either of my practicum classrooms take advantage or recognize students' experiences outside of the classroom. … From this class, I have learned that many things happen outside of the classroom that deserve recognition.

It is important to note that these two pre-service teachers offered these critiques in the context of overall very positive assessments of their experiences in the project and, in particular, of their relationships with and learning from their cooperating teachers. It is also very interesting to note that one of their cooperating teachers was the only in-service teacher (out of 12) who described content-based interactions with parents (as opposed to interactions with parents around student progress, field trips, volunteering, etc.) in her end of semester interview – she described a practice of having parents lead book groups with her students. Therefore, while these quotes certainly point to a tension that is inherent in this kind of project, they also represent an opportunity, or challenge, to make the boundaries that define the spaces of in-service and pre-service teacher learning more permeable, such that in-service teachers can learn about these new spaces, or overlaps, along with the pre-service teachers and pre-service teachers can gain a broader view into in-service teachers' practices.

Our second example was more positive in terms of pre-service teacher learning. While the goals of the project were explicitly to focus on linking mathematics and literacy *and* on helping pre-service teachers learn to access and build on community and family funds of knowledge, the connections between these two goals were less explicit during methods instruction. However, pre-service teachers were often struck by how clearly they saw the overlap between literacy and mathematics when they experienced it in a community space, suggesting that the boundaries between literacy and mathematics and between the university and community spaces were, in fact, quite transparent, or easy to cross. This finding is reflected in one pre-service teacher's description of her learning after an assignment to identify literacy and mathematics activities in a community setting:

> I was amazed how easy it was to connect both literacy and mathematics in a real setting and wondered why I never thought of it before… This is definitely something I want to do when I get my own class, even if it is just outside on the playground!

Implications

The theme for PME-NA 2007 is " Exploring Mathematics Education in Context." The findings described in this paper raise questions related to what it takes, both theoretically and practically, to contextualize mathematics education for elementary pre-service teachers in a local elementary school and community. In particular, as Figure 1 suggests, our framework for investigating pre-service teacher learning suggests the possibility of multiple overlaps being learned and experienced in multiple spaces. At each of the intersections and overlaps, there are boundaries that must be crossed in order to create the kinds of "third spaces" described in the literature. We have found that, while some of these boundaries are quite easy to cross and quickly become permeable for the pre-service teachers, others, such as the boundaries between funds of knowledge and school-based content, or even between university and elementary school spaces, are much more difficult to cross. As researchers and teacher educators, our challenge is to identify practices and activities that will support pre-service teachers in more easily crossing these boundaries.

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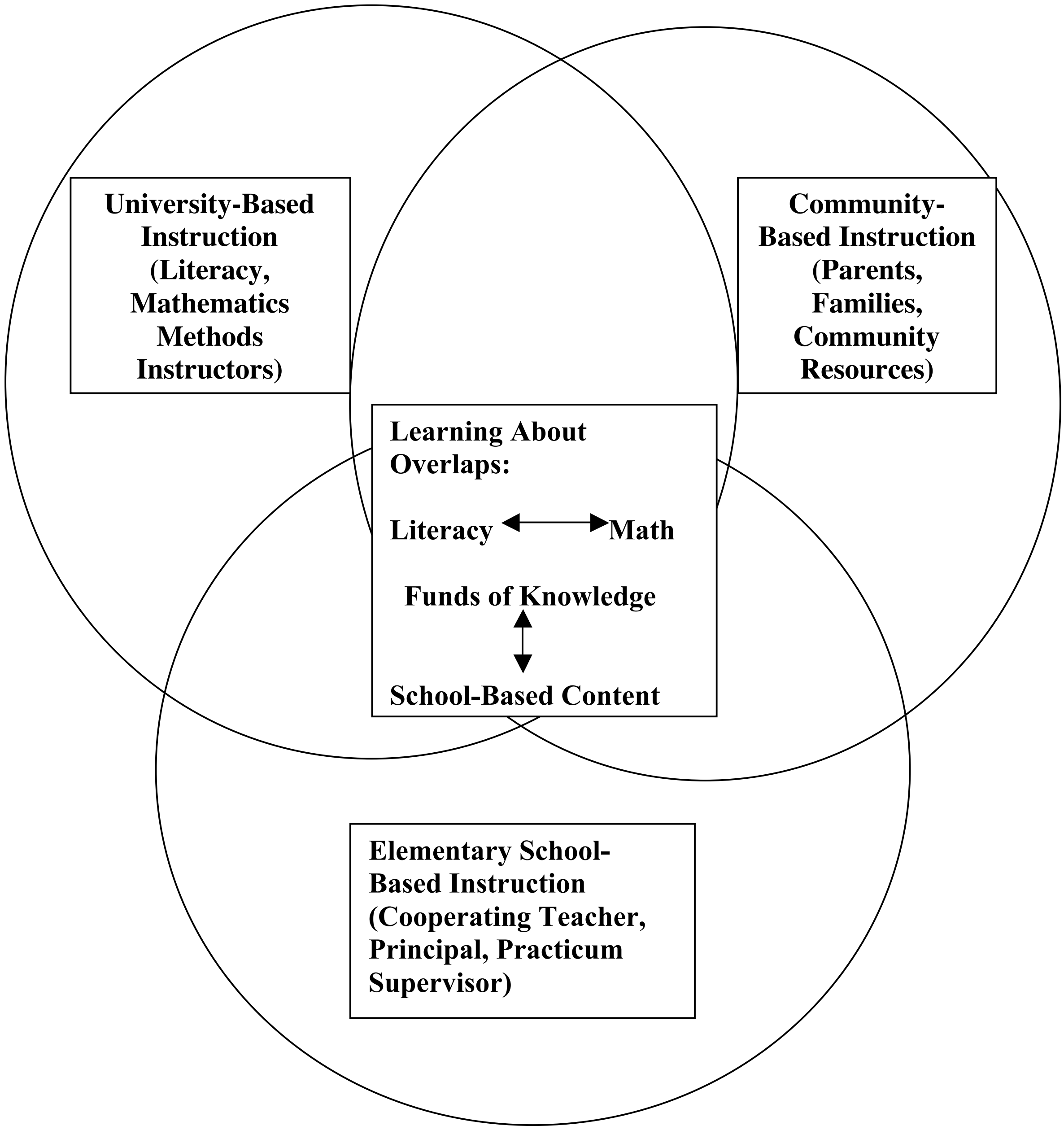

Figure 1. Framework for Investigating Pre-Service Teacher Learning

ELEMENTARY PRESERVICE TEACHERS' INFORMAL CONCEPTIONS OF DISTRIBUTION

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Relatively little research on the distributional thinking has been published, although there has been research on constituent aspects of distributions such as averages and variation. Using these constituent aspects as parts of an conceptual framework, this paper examines how elementary preservice teachers (EPSTs) reason distributionally as they consider graphs of two data sets having identical means but different spreads. Results show that the subjects who reasoned distributionally by considering both averages as well as variability in the data were likelier to see the data sets as fundamentally different despite the identical means used in the task.

The purpose of this paper is to report on research describing the conceptions of distribution held by elementary preservice teachers (EPSTs) in response to a task involving a comparison of two data sets. Two data sets, in the form of stacked dot plots, were presented to the EPSTs. Then two aspects of their statistical reasoning germane to the task were examined: Reasoning about the average of each distribution and reasoning about the variation of each distribution. While research has uncovered different ways that people think in regards to measures of central tendency (Mokros & Russell, 1995; Watson & Moritz, 2000), fewer studies has been done on how people coordinate averages and variation when comparing distributions. Of particular interest was the role that variation, or variability in data, played in the subjects' conceptions of distribution. This interest stems from the primacy that variation holds within the discipline of statistics (Wild & Pfannkuch; 1999).

Furthermore, although precollege students have been the focus for many researchers interested in statistical reasoning, relatively less attention has been paid to the statistical thinking of the teachers of those students. Even less prevalent has been published research on how preservice teachers reason statistically (Makar & Canada, 2005). Therefore, this study addresses the following research question: What are the informal conceptions of distribution held by EPSTs as they compare two data sets? After describing some related research concerning the aspects of distributional reasoning used in the analytic framework, the methodology for the study will be explicated. Then, results to the research question will be presented, followed by a discussion and implications for future research and teacher training programs.

Conceptual Framework

The key elements comprising the conceptual framework for looking at distributional reasoning are a consideration both aspects of center (average) and variation, which implies taking an aggregate view of data as opposed to considering individual data elements (Konold & Higgins, 2002). Coordinating these two aspects is what enables a richer picture of a distribution to emerge (Mellissinos, 1999; Shaughnessy, Ciancetta, & Canada, 2004; Makar & Canada, 2005).

For example, Shaughnessy and Pfannkuch (2002) found that using data sets for the Old Faithful geyser to predict wait-times between eruptions provided an excellent context for highlighting the complementary roles of centers and variation in statistical analysis. The question they posed to high school students was about how long one should expect to wait between eruptions of Old Faithful. At first, many students made an initial prediction that disregarded the variability in the distribution and were solely based on measures of central tendency (such as the mean or median). Shaughnessy and Pfannkuch (2002) point out that "students who attend to the variability in the data are much more likely to predict a range of outcomes or an interval for the wait time for Old Faithful... rather than a single value" (p. 257).

Similarly, Shaughnessy, Ciancetta, Best, and Canada (2004) investigated how middle and secondary school students compared distributions using a task very similar to the task used in the current study reported by this paper. Given two data sets with identical means and medians but with strikingly different variability, participants made comparisons of the distributions based on their reasoning about the averages and variation, with higher levels of distributional reasoning attributed to those responses that cohesively joined both components of centers and spread. The researchers (Shaughnessy et. al., 2004) found that subjects' conceptions included the notion of "variability as extremes or possible outliers; variability as spread; variability in the heights of the columns in the stacked dot plots; variability in the shape of the dispersion around center; and to a lesser extent, variability as distance or difference from expectation" (p. 29). Their findings and recommendations echoed that of Mellissinos (1999), who stressed that although many educators promote the mean as representative of a distribution "the concept of distribution relies heavily on the notion of variability, or spread" (p. 1). Thus, recognizing the importance of getting students to attend to the aspects of average as well as variation when investigating distributional reasoning helped inform not only the task creation for this research but also the lens for analysis of the EPSTs' responses.

Methodology

The task chosen to look at EPSTs' thinking about distribution when comparing data sets was called the Train Times task, and was motivated by a similar tasks initially used in previous research (e.g. Shaughnessy et. al., 2004; Canada, 2006). The task scenario describes two trains, the EastBound and WestBound, which run between the cities of Hillsboro and Gresham along parallel tracks. For 15 different days (and at different times of day), data is gathered for how long the trip takes on each of the trains. The times for each of these train trips are rounded to the nearest 5 seconds and are presented in Figure 1.

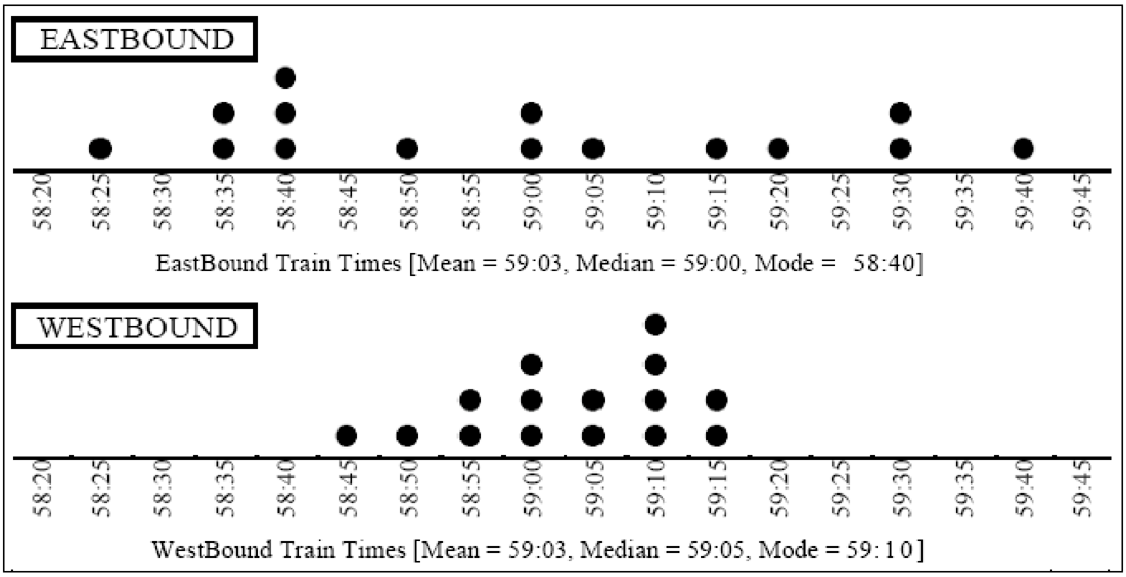

Figure 1: The graphs used in the Train Time task

The task was deliberately constructed so that the EastBound and WestBound train times have the same means, yet different amounts of variation are apparent in the graphs. As a part of the task scenario, subjects were told that the Transportation Department was deliberating whether or not one train was more reliable than the other. Subjects were asked whether or not they agreed with a hypothetical argument that there was “no real difference between the two trains because the data have the same means”, and to explain their reasoning. This methodology follows that of Watson (2000), where subjects are asked to react to a common line of reasoning. A similar technique has been used in other research on statistical thinking (e.g. Shaughnessy et. al., 2004; Canada, 2006). Would subjects be persuaded by the hypothetical argument of “no real difference” in times because of the identical means? Would they argue on the strength of the different modes, which are often a visual attractor for statistical novices reasoning about data presented in stacked dot plots? Or would they attend to the variability in the data, and if so, how would they articulate their arguments?

The subjects were EPSTs who took a ten-week course at a university in the northwestern United States designed to give prospective teachers a mathematics foundation in geometry and probability and statistics. Virtually none of the EPSTs expressed a direct recall of ever having had any prior formal instruction in probability and statistics, although their earlier education at a precollege level may well have included these topics. Early in the course, and prior to beginning instruction in probability and statistics, subjects were given the Train Times task as a written-response item for completion in class. The task was not given as part of a formal evaluation for the course, but rather as a way of having the subjects show their informal sense of how they were initially thinking. Two sections of the course, taught by the author, were used for gathering data, and a total of fifty-eight written responses were gathered from the EPSTs. The task was then discussed in class, and the discussions were videotaped so that further student comments could be recorded and transcribed.

The data, comprised of the written responses and transcriptions of the class discussions, was then coded according to the components of conceptual framework which related to the distributional aspects of centers and spread. Responses could be coded according to whether they included references to centers, or to informal notions of variation, or to both.

Results

Almost 35% (n = 20) of the EPSTs initially agreed with the hypothetical argument that there was "no real difference between the two trains because the data have the same means". While it might be expected that subjects who were predisposed to think of the mean as the sole or primary summary statistic for a set of data might support the hypothetical argument, a careful analysis of the responses showed different degrees to which subjects relied on centers and variation in their explanations. Thus, in addressing the primary research question ("What are the informal conceptions of distribution held by EPSTs as they compare two data sets?"), results are presented first according to responses that focused primarily on centers, then primarily on variation. Finally, examples of those responses that integrated centers with an informal notion of variation are presented as representing a form of distributional reasoning.

Centers

Out of all subjects, 24.1% (n = 14) had responses that included what were coded as *General* references to centers, and the exemplars that follow show the initials of the subject as well as an (A) or (D) to show whether they initially agreed or disagreed with the hypothetical argument of "no real differences" presented in the task scenario:

SE: (A) Because the average is the same for both of them

DW: (A) Each train had the same average time

Although it can be presumed that the subjects equate "average" with the mean, the *General* responses for center included no specific language.

In contrast, the 39.7% of subjects (n = 23) who had *Specific* references to centers were more explicit as far as what they were attending to:

SG: (A) The "mean" means the average, so both trains do travel for the same length of time

LT: (D) I would probably go with the mode, because it is the most common answer

RB: (D) I would go by the median on this one

Note how subjects LT and RB, in focusing on the mode or median, disagreed with the hypothetical argument of "no real difference." Indeed, although the means for the data sets in Train Times task were identical, the medians and modes differed, and some subjects with *Specific* center responses picked up on these differences:

CM: (D) The median and modes are not the same, meaning results varied

LN: (D) The median & mode are different. Because the data is very different in its variation

Here we see CM and LN tying their observations of differences in measures of centers to an informal notion of variation.

Variation

For the purposes of this research, variation need not be defined in formal terms such as a standard deviation (for which these subjects had no working knowledge), but tied to the informal descriptions such as those offered by Makar and Canada (2005). In particular, the essence of variation is that there are differences in the observations of the phenomena of interest. Of all subjects, 32.8% (n = 19) gave a more *General* reference to variation:

CG: (D) Because the data for both are different in variation of time

TS: (D) The trains could all have different times sporadically

LR: (D) Because the time patterns are different between the two trains

Note that in these examples, the theme of differences among data comes out in the natural language of the response. Clearly the sense of variation as differences is a naive and basic idea, but one that is fundamental and a potentially useful springboard for a deeper investigation as to how to describe those differences.

A slightly higher percentage of all subjects, 37.9% (n = 22) had more *Specific* references to variation, and the main motivation for looking at specific constituent characteristics of variation came from the related literature on thinking about variability in data (e.g. Shaughnessy et. al., 2004; Canada, 2006). These characteristics include relative spread, extreme values, and range. For example, consider these two exemplars of more *Specific* reasoning about variation:

EK: (D) Because looking at the charts, the data is more spread out going EastBound than it is going WestBound

AD: (D) Because the times for the EastBound trains are very spread out while the WestBound trains' times are clustered together.

Although EK and AD did not capture formal numerical descriptors of variation about a mean, they did use informal language to convey an intuitive sense of the relative spread of data. Other subjects included variability characteristics in their responses by paying attention to extreme values:

AU: (D) No, because the EastBound has more outliers and is more scattered

AN: (D) One EastBound train took 59:40 while the longest WestBound train took only 59:15, and that is almost a 30 minute difference

Note how AU shows sensitivity to the presence of outliers, while AN includes references to the maximal value in each data set. In addition to *Specific* characteristics of variation captured by responses suggesting a focus on relative spread or extreme values, some responses made explicit connection between both maximum and minimum values:

AU: (D) EastBound has a higher range, from 58"25 to 59:40, & WestBound's smaller range is from 58:45 to 59:15

DM: (A) The range is 1:15 seconds EastBound and 0:30 seconds WestBound.

It was interesting to note that even while acknowledging the different ranges, DM still chose to agree with the hypothetical argument in the task.

Distributional Reasoning

As noted in the previous exemplars, some responses focused more on centers and others on informal notions of variation. However, in line with the previous research (e.g. Shaughnessy, Ciancetta, Best, & Canada, 2004), responses coded as distributional needed to reflect an integration of both centers *and* variation. Of the total 58 subjects, 43.1% (n = 25) such responses that reflected distributional reasoning:

HH: (D) Because the mean is an average, and to get an average you will most likely use varying numbers. All the times for the most part on EastBound trains are different. Just because the mean is the same doesn't change that.

AJ: (D) The data is different, although the average is the same. We can see, for example, the difference in consistency of the WestBound train, where the times are closer together, and hold nearer to schedule

AB: (A) The mean is the average of times. BUT, there is a greater spread of times on EastBound vs. WestBound . And East mode is lower than West mode.

Note the richness in the exemplars provided above, as subjects integrate center and spread in their consideration of the two data sets. We see, for example, how HH understands about combining "varying numbers" to get an average. AJ actually lays the groundwork for making an informal inference, in the way that the WestBound may be to more reliable train because it holds "nearer to schedule". Meanwhile, AB agrees with the hypothetical argument of the task, despite apparently taking note of the means and commenting on the differences in mode and spread of the data sets. When asked further for an explanation of his stance in agreeing, he remarked to the effect that "Still, they are basically the same on average". Such is the power of the mean in many peoples' minds as a way of summarizing data.

Discussion

Out of all 58 subjects, 20 (34.5%) initially agreed and 38 (65.5%) disagreed and with the hypothetical argument of "no real differences" between the trains. But this research is about how EPSTs reason distributionally, and so it was crucial to dig into their explanations. The exemplars provided have been intended to show how subject responses could reflect a focus on centers, on variation, or on both. Since responses could be coded for multiple aspects, including some facets of statistical thinking not reported on in this paper, as a final note it is interesting to look at the breakdown of who agreed versus who disagreed with the hypothetical argument based on whose responses coded *only* for centers, or *only* for variation, or coded for distributional reasoning (both centers and variation). Looking strictly within those groups of responses, the percentage of those subjects who agreed or disagreed is presented in Table 1.

Table 1: Support for "no real differences" by type of reasoning

Type of Reasoning		(A)gree	(D)isagree
Only Centers	(n = 12)	75.0 %	25.0 %
Only Variation	(n = 16)	31.3 %	68.8 %
Distributional	(n = 25)	20.0 %	80.0 %

Again, the percentages in Table 1 are out of the respective numbers of responses falling within the given types of reasoning (the numbers do not total 58 because some students had explanations that went outside the themes of this paper). Comparing those percentages with the total pool of subjects (34.5% agreeing and 65.5% disagreeing), several interesting observations can be made. First, while more than half of all subjects disagreed, the majority of those subjects who only relied on reasoning about centers agreed with the hypothetical argument of "no real differences". Second, the percentage of those subjects who only used variation reasoning and disagreed is quite close to the percentage of all subjects who disagreed. Third and most important, the subjects whose responses reflected distributional reasoning had the highest percentage of disagreement with the hypothetical argument. This echoes the findings of Ciancetta (2007), who in working on similar tasks with more than 200 tertiary students also found that the majority (80%) of those subjects who only relied on reasoning about centers agreed with the hypothetical argument of "no real differences" and a majority of subjects who relied on reasoning either about centers or about distribution disagreed with the hypothetical argument. However, all of the tertiary students who reasoned about variation disagreed, and only about two-thirds of those who reasoned about distribution disagreed.

The first two groups of subjects, those who decided that there is "no real difference" because of equal centers or decided that there is a difference because of different variability, may have relied on the simplest and most obvious reasoning available to them to support their decisions. While these responses are not necessarily incorrect, they do lean toward the superficial as neither comparison captures a complete "picture" of each distribution as a whole entity. From the responses of the third group of subjects, it is clear that they may have considered various aspects of the distributions in more detail and consequently those aspects were integrated in their responses. The distributional reasoning evident in the responses of this group is essential for future and current teachers of statistics. The ability to reason about data sets distributionally may allow one to more easily discuss and assess reasoning based only on individual characteristics, such as center or variation.

Conclusion

This study was guided by the question "What are the informal conceptions of distribution held by EPSTs as they compare two data sets?" Although limited to a single task that was by nature contrived so as to invite attention to identical means yet differing amounts of spread in two data sets, the research suggests that EPSTs do reflect on aspects of distributional reasoning, and this paper gives a sense of how those aspects manifested themselves in the responses of the subjects. However, just as prior research has shown with middle and high school students (e.g. Shaughnessy et. al., 2004), we also see that EPSTs make more limited comparisons of distributions when they focus on centers while not attending to the critical component of variability in data. In contrast, the distributional reasoners who attended to both centers *as well as* variation made richer comparisons within the given task.

While further research is recommended to help discern the most effective ways of moving EPSTs toward a deeper understanding of distributions, certainly tasks such as the one profiled in this paper provide good first steps in helping universities offer opportunities to bolster the conceptions of the preservice teachers they aim to prepare. In turn, these novice teachers can then promote better statistical reasoning with their own students in the schools where they eventually serve.

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PROSPECTIVE TEACHERS' USES OF A VIDEOCASE TO EXAMINE STUDENTS' WORK WHEN SOLVING MATHEMATICAL TASKS USING TECHNOLOGY

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Introduction and Theoretical Framework

Whether the use of technology will enhance or hinder students' learning depends on teachers' decisions when using technology tools. These decisions are generally informed by teachers': 1) knowledge of mathematics, 2) knowledge of technology, and 3) knowledge of pedagogy. These three components are essential to a teacher's technological pedagogical content knowledge (TPCK) (Koehler & Mishra, 2005; Niess, 2005), a type of knowledge several authors have characterized as necessary for teachers to understand how to effectively use technology to teach specific subject matter. Consider the following example that illustrates these three components of teacher knowledge. A teacher who is teaching a lesson focusing on comparing two different distributions needs to know how to: compute measures of center and spread (knowledge of mathematics); use technology to create representations of a distribution (knowledge of technology); and design activities that align with approaches students may take when asked to compare two different distributions (knowledge of pedagogy). Where these three forms of knowledge intersect, Niess (2005) describes four different aspects that comprise teachers' TPCK: 1) an overarching conception of what it means to teach a particular subject integrating technology in the learning process; 2) knowledge of instructional strategies and representations for teaching particular topics with technology; 3) knowledge of students' understandings, thinking, and learning with technology; and 4) knowledge of curriculum and curriculum materials that integrate technology with learning. This study focuses on understanding more about prospective teachers' knowledge of students' understandings, thinking, and learning with technology.

Videocases have been used by several mathematics educators and researchers to focus teachers on decisions that are made during instruction and highlight mathematical thinking of both students and teachers (Bowers & Doerr, 2003; Horvath & Lehrer, 2000; Lampert & Ball, 1998). The videocase format provides a permanent record of classroom events (Roschelle, 2000), enables the observer to review a particular episode multiple times, and can be edited to highlight important classroom moments and remove extraneous noise. Video makes it possible to recast the teacher as an "observer" rather than "actor." Taking an observer perspective can allow teachers to focus on particular pedagogical activities rather than attend to all situations that arise in classrooms that require teachers' attention (Sherin & van Es, 2005; van Es & Sherin, 2002). To assist teachers in noticing and analyzing important classroom events, van Es and Sherin developed and used a multimedia tool in two different studies; one study involved inservice and preservice teachers while the other focused on new teachers pursuing an initial license to teach mathematics. In one study, the multimedia case divided the video into three segments which focused on discourse, the teacher's role, and student thinking. These researchers found the non-linear organization of the multimedia tool helpful in moving teachers from a chronological evaluative description of what happened during their class to a more in-depth analysis of important events supported by evidence identified in the video. Also, these researchers observed that teachers shifted their focus from actions performed and decisions made by the teacher to the mathematical thinking of students.

Building on prior research on the use of videocases with teachers and using TPCK as a framework, we created a videocase to assist prospective teachers develop their "knowledge of students' understandings, thinking, and learning with technology" (Niess, 2005). With a focus on student thinking, the research question we investigated was, "How do prospective teachers make sense of students' work and understandings when students are using technology to solve mathematical tasks?"

Context of Study and Description of Videocase

The study took place in an undergraduate mathematics education course for prospective middle and high school mathematics teachers (PSTs) focused on the learning and teaching of mathematics with technology. The course was offered once per week for three hours for 15 weeks. During five weeks of the course, the instructor and PSTs used one module from the *Preparing to Teach Mathematics with Technology: An Integrated Approach* curriculum (PTMT, Lee, Hollebrands & Wilson, under review). This module focuses on data analysis and probability topics while using tools such as *TinkerPlots (Key Curriculum Technologies, 2005), Fathom 2.1 (Key Curriculum Technologies, 2007), spreadsheets,* and graphing calculators and integrates pedagogical questions throughout the materials. The module contains six sections, the second of which is the videocase lesson and is the focus of our analysis for this study. The videocase lesson includes several components: 1) a video file and synchronized transcript of two middle school students working on a mathematical task using *TinkerPlots*, 2) scanned written work from the students, 3) the *TinkerPlots* data file and the mathematical task on which students worked, and 4) questions that engage prospective teachers in reflecting on their own mathematical work, anticipating students' work, and analyzing students' work shown in the video.

The design of the videocase was informed by suggestions and implications from work by Lampert and Ball (1998), Towers (1998), Sherin and van Es (2005), and Bowers, Kenehan, Sale, and Doerr (2000). Towers (1998) and Lampert and Ball (1998) discuss the challenges prospective teachers have in shifting their attention from classroom events to student thinking. They suggest including videoclips focused on students rather than the teacher, providing students' written work, and asking purposeful questions that focus prospective teachers on the work of students to assist this shift of attention. These suggestions were incorporated into the design of our videocase that only shows a teacher at the beginning when she is introducing the lesson and posing specific questions and otherwise mainly focuses on students' work. Since the students in the video are working with a computer, and we intend to develop prospective teachers' TPCK, we made a design decision to use a picture-in-picture format that coordinates and synchronizes the video of students' work with *TinkerPlots* with the video of the two students' working together. Furthermore, the larger of the two frames is the video of the computer work. This is intentional so that prospective teachers can focus on the particular actions that students perform within the software environment while solving the mathematical task.

In the design of their videocase materials, Lampert and Ball (1998) and Bowers et al (2000) specifically include and advocate for having prospective teachers engage in the mathematical tasks that are used in the videos. Bowers et al (2000) further recommend that prospective teachers engage in the mathematical task before viewing the video. In our videocase, we have prospective teachers use *TinkerPlots* to analyze the same data set and answer questions students were asked. The prospective teachers engage in this mathematical task before viewing the video, reflect on their own mathematical thinking and use of the software, then make predictions about how middle school students might approach the same task. We believe this process of reflection and anticipation of students' work can provide opportunities for surprise and possible perturbations when prospective teachers observe students doing work that is not as they expected.

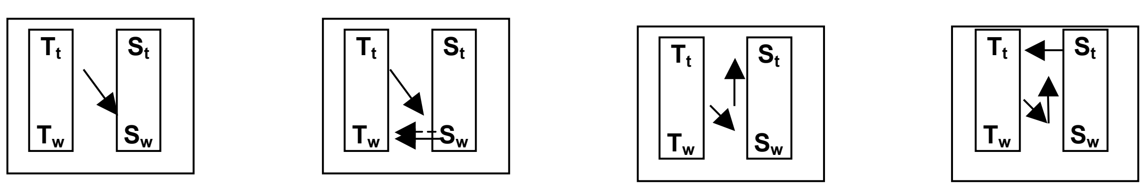
Methods and Data Sources

Data were collected from four PSTs' work in pairs on the three-hour videocase lesson, the second three-hour lesson on data analysis using *Tinkerplots*. Jay and Paula, and George and Britney were juniors in a mathematics education program seeking a license to teach high school mathematics and taking the course during the spring semester prior to student teaching. During the data analysis and probability unit, all sessions were videotaped to capture PSTs' work on paper and with technology. For this study, annotated transcripts of PSTs' work from the lesson that included the videocase were created and analyzed.

Members from the research team coded the transcripts according to whether the PSTs appeared to be using mathematical, technological, or pedagogical knowledge. These initial codes led to noticing distinct differences in the nature of a PST's focus in their analysis of students' work. To help characterize the differences we were noticing, we developed a mapping system to model the focus of PSTs' attention. An assumption embedded in the model is that it is impossible to have direct access to one's thinking, but we assert there is a relationship between one's thinking and one's actions and words.

Results and Discussion

Ways in which preservice teachers analyze students' work with technology can be characterized in terms their attention to student thinking as distinguished by evidence of PSTs' attention to students' work and their hypothesis about its relationship to students' thinking. This awareness is represented in the use of arrows in the diagrams in Figure 1. In the diagrams, S_T and S_w represent students' thinking and external work, respectively. The arrow represents where the teacher is focusing his or her attention. A solid arrow indicates that there is evidence of a PST's attention to students' work or thinking while a dotted arrow indicates lack of direct evidence but an inference can be made based on the PST's previous work. PSTs' attention may be explicitly focused on only the work of students or may be focused on the thinking of students as inferred from their work. These diagrams describe four categories of teachers' examination of students' thinking: description, comparison, analysis, and restructuring.



Description Comparison Analysis Restructuring

Figure 1. Models to illustrate five categories of preservice teachers' analyses of students' work.

Description

This category is characterized by PSTs' explicit attention to students' actions with the technology, words students have written or said, or mathematical terminology and symbols used by students. The only filter evident in their work can be viewed in terms of what is or is not reported, which may provide information to the researcher about what the PSTs' view as important. As an example of PST's work in the description category, consider George and Britney's response to the question, "Summarize the approach that this pair of students used to solve the task." They noted, "They [students] sorted the data based upon graduation rates and then stacked it vertically. They also color coded the data by admission, public or private. They then clicked on each individual point and compared them in their heads." The diagram that was used to characterize what these PSTs were doing is shown in Figure 2. Because there is evidence in their words of their focus on students' work, a solid is arrow is used to indicate this attention.

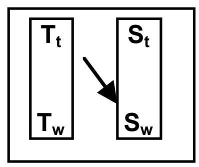


Figure 2: Diagram depicting George and Britney's focus on the actions performed by students with the technology.

In this example, PSTs' descriptions focus only on the actions the students performed with the technology as they solved the task. There is no evidence of PSTs' awareness of how those actions may be related to students' thinking or how the work of students is related to their own work or thinking.

Comparison

For PSTs to move beyond reporting what they see students doing and saying, they may need to relate students' work to their own work and experiences. These types of responses were categorized as comparisons.This category is characterized by PSTs' comparison of their work with the technology on the task and how it relates to the work of the student. In this category, teachers compare students' actions with their own actions, implicitly or explicitly. Thus, the comparison of the students' actions to their own actions does not necessarily lead to a consideration of students' thinking. For example, in summarizing the students' approach, George typed, "They sorted the data based upon graduation rates and then stacked it vertically. They also color coded the data by admission, public or private. They then clicked on each individual point and compared them in their heads. They did not take full advantage of the software." Because George had used parallel box plots on the same task, we infer that he is making an implicit comparison between his own work that took advantage of standard representations available in the tool for summarizing a distribution and students' work on the task that used a representation that focused their attention on individual cases and did not facilitate their view of the data as an aggregate.

Another student, Paula, summarized the students' approach as follows: "This was similar to our approach in that we separated them [icons representing graduation rates at universities in North Carolina] and stacked them also. They [the students] also recognized that there was an outside influence on graduation rate. They differed from us in that they did not graph the graduation rate versus anything. They also only investigate student-faculty ratio instead of looking at various variable[s] as we did." Paula's summary focuses on the actions of the students, without making inferences about what these actions may imply about their thinking, and explicitly relates these actions to her work.

Figure 3: Diagram on left with dotted arrow used to represent George's examination of the students' work and diagram on right with solid arrow used to represent Paula's examination of the students' work.

Whereas George makes an implied comparison, Paula clearly acknowledges the similarities and differences of her own actions with the technology and the actions of the students. These differences are shown in the diagrams included in Figure 3.

Analysis

While its natural for PSTs to focus on the actions of students, particularly when actions are necessary to perform tasks in a technological context, our goal is to assist PSTs' to consider the thinking of students that can be inferred from their actions. This focus on student thinking is central to the category we identified as Analysis. When teachers begin to use their technological, pedagogical, and/or mathematical knowledge to interpret students' work and make inferences about what students are thinking, then they are in a category described as analyzing. For example, in response to the question "describe how the students' attention to the attribute "student to faculty ratio" affected their analysis," George responded, "The students were perplexed by the fact that colleges with a high student to faculty ratio could have a higher graduation rate than colleges with lower ratios. They incorrectly assumed that a lower student to faculty ratio automatically meant that the school would have a higher graduation rate." George had observed the students clicking on individual data icons in the distribution and trying to make sense of a few data points in the middle of the distribution that did not follow the initial trend they had found. Thus, he was making an inference about what they may have been thinking based on the work he observed them do with the technology. Another student, Jay, critiqued the task by saying, "I would say it was an appropriate question, relatively easy to answer but the only limitation is that they do not have much knowledge of college." Jay is evaluating the task based on a coordination of his work, his thinking, and his observations of the students' work and inferring that the students might not have a complete understanding of some of the variables in the data set.

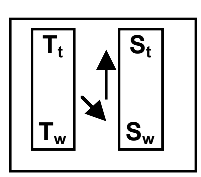


Figure 4: Diagram showing the PST's focus on students' work and inference about students' thinking.

In the examples, both George and Jay are focusing on students' work and using that work to make inferences about students' thinking. In the case of George, he states that students might be assuming that lower student to faculty ratio is related to a higher graduation rate. In Figure 4, this is depicted with an arrow showing George's focus on what students were doing (clicking on individual icons and examining the corresponding data card) and an upward arrow indicating how he uses this information to describe what students might be thinking. Jay uses his observations of the students (downward arrow) to infer that the students have limited knowledge of the context of the data set (upward arrow).

Restructuring

An important aspect of the Analysis category is a focus on student thinking. It is our goal for PSTs to use their analyses of students thinking to inform teaching. This is the feature characteristic of the category we identified as Restructuring. In this category, PSTs' analyses of students' thinking becomes integrated into their own ways of thinking about mathematics, technology, and pedagogy. Characterizing a PST's restructuring requires evidence over a series of episodes where the PST can be categorized as Analysis and then has opportunities to demonstrate how such analysis affects their mathematical, technological, and pedagogical knowledge. We believe George was Restructuring when he builds off of the last example to create a question for students to consider: "Does having a low student to faculty ratio mean that graduation rates are automatically lower?" Here, we see his observation that the students believe a direct relationship between student-faculty ratio and graduate rate exists that was mentioned above has informed the question that he would pose to students (see Figure 5).

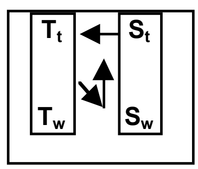


Figure 5: Diagram depicting the work of George as he makes inferences about students' thinking based on students' work and uses this to inform the development of a question he would pose to students.

Restructuring may provide evidence that PSTs are incorporating their analysis of student thinking into their TPCK and drawing upon this knowledge as they make pedagogical decisions related to the teaching of mathematics with technology.

Discussion

Teachers' decisions about how and when to use technology tools is informed by their knowledge of technology, pedagogy and mathematics (TPCK). To support prospective teachers' development of this knowledge, we focused on one of four aspects of TPCK, knowledge of students' understandings, thinking, and learning with technology, through the creation of a videocase of students working on a mathematical task while using technology. We found differences in what PSTs focused and reflected upon while examining student work. In particular, differences were noted in PSTs' attention to student work and their use of student work to make inferences about student thinking. These differences were highlighted in the four different categories that were created to describe PSTs' examination of students' work. These four categories may be useful in thinking about the ways in which prospective teachers analyze students' work on mathematical tasks using technology. We conjecture that the nature of a PST's examination of students' work seems influenced by the specific questions they are answering and that repeated opportunities are needed for PSTs to develop habits of attending to students' work in a way that fosters analysis and restructuring. For example, questions that asked PSTs to create a task for the students in the video often provided insights into the extent to which they drew upon their knowledge of students' understandings. Our categories are similar to the categories others have developed to described teacher reflection (Manouchehri, 2002) and noticing (Sherin & vanEs, 2005), but go one step further by describing the locus of a teacher's attention within a particular category and how it relates to students' work and thinking.

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STUDYING ELEMENTARY PRESERVICE TEACHERS' LEARNING OF MATHEMATICS TEACHING: PRELIMINARY INSIGHTS

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This report focuses on the first of a multi-year research project that is studying preservice teachers' learning of three essential mathematics teaching practices—(1) posing mathematics problems to students, (2) interpreting students' thinking, and (3) responding to students' mathematical activity. These are prominent practices in the work of mathematics teaching yet little is known about how they develop over time. The initial work reported here focuses on the challenges and insights we have encountered in the first year of work on this project. This work includes: (a) refining definitions of the practices we are studying, (b) constructing an Opportunities to Learn (OTL) map across courses in a particular teacher preparation program, and (c) exploring research instruments that could help us study what teacher candidates learn about these practices over time.

The practices of *posing*, *interpreting* and *responding* are central to the everyday work of teachers. To illustrate, consider the work of launching a lesson with a mathematical task, and then listening, and responding to what the students say and do with such task. These practices take place daily across all sorts of mathematics classrooms. In spite of the attention these three practices receive in Teacher Education (TE) and materials, little is known about what specific aspects of these practices are studied in teacher preparation and how to document beginning and novice teachers' development of these practices over time.

The larger research agenda for this project (1) is to study the practices of posing, interpreting, and responding using two studies. The *cross-sectional study* will characterize these practices at three stages of elementary teacher preparation (mathematics for teaching courses, mathematics methods courses, practice teaching internship) in a particular institution (Michigan State University). *The longitudinal study* will focus on a small number of cases (10-15) to explore in depth the development of the practices as prospective elementary teachers move through the program and into their first and second years of teaching.

The work undertaken in year one is foundational to the cross-sectional and longitudinal phases of the project. The questions driving this first year of work include:

- How can the practices of *posing, interpreting*, and *responding* be defined in ways that are useful to research and teaching?
- What *opportunities for learning* these practices do teacher preparation courses provide to teacher candidates?
- What kinds of tasks and observation *instruments* would best elicit and reveal these practices in prospective and practicing teachers?

Theoretical Framing

This project draws on and contributes to current investigations on the nature of mathematical knowledge for teaching by focusing on three specific mathematics teaching practices—*Posing, Interpreting and Responding* (PIR is our acronym)—that demand and

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.

require such specialized knowledge. Mathematical knowledge for teaching is related to what teacher educators call "pedagogical content knowledge," which is the capacity of a teacher to transform his or her content knowledge into forms that are pedagogically powerful (Shulman, 1986; Wilson, Shulman & Richert, 1987). Mathematical knowledge for teaching, in turn, has been defined as the mathematical work that arises in the context of teaching practice, and acknowledges that knowing mathematics in order to solve a problem and knowing how to represent mathematical concepts in order to make them accessible to students are two different types of knowledge for teaching (see Ball & Bass, 2000).

Investigating the three practices proposed in this project is important for multiple reasons. The practices of posing, interpreting, and responding are vital and visible in the everyday work of mathematics teaching. They are challenging in any form of mathematics teaching but are especially demanding in the kind of teaching envisioned in mathematics education reform documents (e.g., NCTM 2000, 1989). They have been singled out as essential to the improvement of mathematics teaching and learning (NCTM, 1991), and found to be hard to implement as well as to have direct impact on students' learning and experiences in mathematics classrooms (e.g., Boaler, 2002; Henningsen & Stein, 1997). These are practices that are underdeveloped in prospective teachers of mathematics (e.g., Borko et al. 1992; Moyer & Milewicz, 2002). In spite of the amount of attention these practices receive in materials designed to support teacher learning in a variety of contexts, little is known about how these practices develop or might be developed in beginning teacher preparation.

Method

In this report we focus on the three areas outlined earlier that comprise the work for year one of the project. Brief descriptions of each of these areas of work are provided next.

The first two areas—**refining definitions** and **constructing an OTL map** of these practices—focused on an analysis of existing documents (in particular, existing program standards, course syllabi, required readings and textbooks) and consultations with experienced mathematics and TE instructors at the institution where this research project is based. The revised PIR descriptions and OTL map are being used to aid the development of the PIR tasks and observation instruments.

The **development of PIR instruments** began with an examination of existing instruments that have been used in prior research studies (e.g., Borko et al., 1994; Kennedy, Ball, & McDiarmid, 1993; Ma, 1999; Tirosh, 2000). An initial analysis of the tasks from three large projects' released tasks (Ball, Hill, Rowan, & Schilling, 2002; Kennedy, Ball, & McDiarmid, 1993; and Phillip & Sowder, 2004) revealed that no task in this collection covered all three of the target practices so existing tasks would need re-writing to be used in this project. Because of the interrelatedness of these three practices within the context of actual mathematics teaching we are designing PIR tasks that will elicit these practices in ways similar to how these are performed in real classrooms. Each PIR task will describe a teaching scenario followed by a series of interrelated posing, interpreting, and responding prompts.

Currently the research team has begun the pilot testing of 5 PIR tasks with two groups of participants (novice and experienced) to determine the ability of these tasks to discriminate between these two groups. The experienced group consists of teachers with classroom experience (teachers associated with professional development projects at our institution and/or who are mentor teachers in the TE program); and the novice group includes freshmen undergraduates who have yet to take any Teacher Education courses. The next step is to then pilot test the PIR classroom observation instrument using collected observations and videos of two consecutive math lessons of the experienced teacher participants.

The final versions of PIR Tasks (6-8) and the PIR Observation Instrument will take into account how well these tools elicit and reveal the three practices. The *stability* (or internal consistency) of the coded practices across the two instruments (tasks and observations of actual teaching) is of concern to us. However, because we are still investigating the interplay between what we are calling *imagined* and *enacted* practice, and because we believe that discrepancies between hypothetical and actual practice may at times be productive, we suspect that PIR practices performed in written tasks may differ from the PIR practices performed in the act of teaching (for experienced teachers). These discrepancies will be examined, and may result in our choosing to discard or revise the task, or may lead to findings about the interplay between imagined and enacted PIR practice. Another important criterion in selecting and refining the tools will be their ability to *discriminate* between the two pilot test populations (novice and experienced teachers).

Insights

As our research team has worked on the activities planned for year one, there have been numerous challenges and insights. In our presentation we will share a more detailed account of the work we have undertaken and what we have been learning so far. In this paper we offer just a few examples from each of the areas of work we outlined for year one.

Work of refining PIR definitions. Consider the challenges associated with the work of naming, defining, and describing the PIR practices at the right level of detail for research and practical purposes. Table 1 shows the evolution of one of the original descriptors for the practice of responding to students' mathematical ideas. The initial descriptions were derived from the research literature and have undergone several iterations of revisions as the research team has engaged in analysis of teacher preparation texts and interviews with experienced mathematics and teacher education faculty that teach in the MSU program. Our research team is continually revising these definitions and is always questioning the choice of particular words (i.e., imagined, enacted, repertoire) and not others.

A noteworthy addition to the initial research-based PIR descriptions is the distinction between *Imagined* and *Enacted* forms of these practices. It became clear, for example, that teachers' work on their teaching practices both, outside and inside the classroom; in the imagined and hypothetical realm as well as in the physical space of their classroom. This distinction became clear as the University instructors we interviewed spoke about their work in their courses; and in particular seemed to aim more directly at developing *images* of mathematics teaching practices and less directly at the *enactment* of these practices. This distinction is necessarily represented in the revised PIR map.

Making the distinction between the imagined and enacted forms of teaching practice is important to us because research on teachers has generally described discrepancies between what teachers say about their practice and what they are observed doing in their classrooms as problematic. Instead of assuming that it is desirable to have teachers' talk about their practices exactly match their work in the classroom, we see these as two different windows into teaching, which allow us more complex ways of analyzing the work of teaching.

The revised descriptors also reflect our effort to develop a language for talking about posing, interpreting and responding in ways that do not align with either reform or traditional descriptions of mathematics teaching but rather represent a wider range of teaching "genres" which are indeed studied and experienced in mathematics teacher preparation courses.

Table 1: Evolution of one of the descriptors for the practice of Responding

Original Description (07/2005)

3.1. Teacher constructs responses to students' correct, incorrect, and novel work that probe their understanding and that challenge them to revise or extend their mathematical ideas.

Novice Performance

Responds in ways that dismiss, praise, or correct students' mathematical ideas.

Expert Performance

Responds in ways that deliberately engage students in examining the validity, connectedness, and efficiency of their ideas

Revised Description (05/2007)

Imagined Practice

3.1. Teacher has a collection of hypothetical responding strategies to students' correct, incorrect, and novel work. Can identify and construct a range of different types of responses when presented with hypothetical teaching scenarios.

Novice Performance

Exhibits a narrow collection of imagined responding strategies. Construct a limited set of hypothetical responses that primarily attend to the product of the students' mathematical thinking.

Expert Performance

Exhibits a broad collection of imagined responding strategies. Construct a varied set of hypothetical responses that give balanced attention to the product and process of the students' mathematical thinking.

Enacted Practice

3.1. Teacher has a repertoire (a set of well practiced and deliberate) responses (possibly a subset of their imagined response strategies) to students' correct, incorrect, and novel work. Some examples include: Revoicing (rebroadcasting) students' math ideas; Asking students to restate someone else's reasoning; Asking students to agree/disagree with someone else's reasoning; prompting students to add to others' ideas.

Novice Performance

Exhibits a narrow repertoire of responding strategies to students' correct, incorrect, and novel work. Most prominently responding strategies are teacher centered across genres of teaching (e.g., demonstration or discussion lesson).

Expert Performance

Exhibits a broad repertoire of responding strategies to students' correct, incorrect, and novel work. These patterns of responding strategies vary according to genre of teaching (e.g., demonstration or discussion lesson).

Work of constructing OTL map. To learn about the *Opportunities To Learn* that MSU offers to preservice elementary teachers we collected the mathematics education courses' syllabi and textbooks and interviewed faculty who teach and supervise these courses. One important finding as we map these opportunities to learn across the program has been identifying where in the MSU program these practices are studied.

We have learned that all three mathematics education courses in the program focus heavily on the practice of *posing* mathematics problems, but with a variety of definitions and views regarding this practice. For instance statements about what constitute good elementary school mathematics problems are quite different when we look in the required textbooks for the MATH and the TE courses. To illustrate, the required math textbook states that "word problems should be short, clear, succinct, interesting, realistic or whimsical, self-contained with a single answer." The TE textbook, in contrast, states that a good math problem is: "any task or activity for which the students have no prescribed or memorized rules or methods, nor is there a perception by students that there is a specific 'correct' solution method."

We have also learned that the practice of *responding* is not studied at all in the MATH course, and receives moderate attention in the senior year TE course and heavy attention in the internship year TE course. Knowing when and where prospective elementary teachers study (or not) the focal practices informs the generation of hypotheses for the data we'll get in response to the tasks will give during the cross-sectional study in year 2 of the project.

Work of designing PIR instruments. After much deliberation and review of hundreds of teacher learning tasks, we chose 4 tasks that were originally designed to study *knowledge of mathematics for teaching* and of *children's mathematics.* These tasks have been used in The TNE (Teachers for New Era) project at MSU for the past 2 years for program evaluation and reform work. We have adapted these tasks by including PIR prompts (while still using the original stem of the task). We have designed a fifth task that was expressly designed for this project. This fifth task asks the participants to generate a hypothetical class discussion around a particular task: "What goes in the box: 8+4=[ ]+5? (Carpenter, Franke, & Levi, 2003).

The task we share below targets the *imagined* form of PIR practices as is suggested by the statement: "What can you *imagine* saying and doing?" Part (a) is about reformulating a proposed task (posing), and part (b) is about drawing attention to important mathematics in students' work (interpreting) and about using students' work in the classroom (responding).

Figure 1: PIR Task (Adapted from MSU-TNE Task)

2. (a) Imagine you are teaching a lesson about two-digit subtraction and you ask the class to explore different ways to solve the following subtraction. The students look puzzled. What do you imagine <u>saying</u> and <u>doing</u> next?

$$\begin{array}{r} 37 \\ \underline{-19} \end{array}$$

Teaching Scenario Continued:

2. (b) After giving students some time to work on the task you call on their attention and ask for volunteers to share their strategies. Imagine that one of the students shows the following strategy. What can you imagine <u>saying</u> and <u>doing</u> to the student and to the class, and why?

$$\begin{array}{r} 37 \\ \underline{-19} \\ -2 \\ \underline{20} \\ 18 \end{array}$$

Our expectation as we began to pilot these items was that experienced teachers would provide a range of practices they could imagine using in the given (and possibly similar) situation and would provide a lot more detail, whereas participants with no (or little) teaching experience would have a narrower collection of responses and less detail of these imagined practices. However, as we pilot the tasks, we have developed a new hypothesis, which is that experienced teachers may actually have a narrower set of responses that are drawn from large collections of stories about past practice (and that they are therefore more attached to), while novices may respond with a longer list of possibilities that are less grounded, and they may be less committed to each of the things on this list of possibilities. For instance, in response to a task which asked teachers to generate multiple ways of introducing a 2-digit multiplication problem, one experienced teacher said she had trouble thinking of second and third ways, because she knew her initial idea – which emphasized place value – would be most effective. In contrast, a preservice teacher said the same question was easy because she had studied multiple ways of doing these problems in her methods course.

Work on how much or little attention students pay to the context of given mathematics problems has suggested three sorts of solver's interactions with the context of a given problem: 1) by ignoring the context in which the problem occurs, 2) by demanding more context for the problem before solving it, or 3) supplying the context themselves by assuming what the context is (e.g., See Cooper & Harries, 2002). So far, dealing with context has been important to how pilot study participants have responded to the PIR tasks, leading us to hypothesize that teachers' responses to our tasks may be similar. Teachers piloting our tasks have either claimed that they were very difficult because of the lack of context or have supplied the context themselves. For example, one experienced teacher claimed that the question was difficult or impossible to answer without knowing about the work the class had already done in math class this year, the difficulties the students have had, and the teacher's reasons for choosing the problem. This teacher suggested that we add such details to the tasks. Another teacher began each task by listing assumptions she was making in order to complete the task, such as the assumption that the scenario would be taking place after the class had been working on multiplication for a while, and that the teacher would have some mathematical goals for the students that included an understanding of place value.

Another consideration regarding context is that experienced and novice teachers may attend to different aspects of the context when they demand or supply it. For example, one novice teacher in our pilot study claimed that the tasks were difficult without knowing in which grade the scenario was taking place. In contrast, an experienced teacher who demanded more context claimed that it was necessary to know what students in the scenario had already discussed in class about the mathematics relevant to the problem. The experienced teachers, in these cases, were more interested in knowing the history of the particular class imagined in the scenario, while the novice teacher in this case was more concerned with the grade level at which the scenario was imagined to have occurred.

In Closing

In the foregoing report of our collective insights we have hinted at some of the challenges we have also faced during the first year of work on this project. To close we discuss more explicitly the sorts of challenges that have made this work hard but also intellectually rewarding. We alluded to the challenge of finding the 'right' words to name and describe the practices we are studying. We have constant deliberations and discussions among ourselves and with our advisory board and invited colleagues about the words we choose (and discard) to represent our understanding of the practices we are studying. The very notion of what a 'practice' means and whether we can, for example, even talk about 'imagined' practice when the very word practice suggests an action or application in real situations. One definition found in the Oxford English Dictionary's reads: "practice n.: the actual application or use of a plan or method, as opposed to the theories relating to it." And yet, we continue to believe that the formulation of plans, ideas, and images of teaching is a part of the work of teaching which requires the application of learned theories and dispositions, and so we have chosen to continue to use the word 'practice' to describe the inner work of a teacher, at least for now.

We have also encountered challenges as we set out to design PIR tasks that will help us study the development of these practices over time. As we seek to design PIR tasks we are faced with the question of what makes a written task a PIR task (rather than a regular math task) but also with the question of what is a 'good' PIR task. To this end, we have chosen to embed each task within a teaching scenario in order to elicit teachers' knowledge about mathematics specific to teaching practice, which is different than testing their mathematics knowledge separately. In asking teachers to respond to particular mathematical problems within specific problems of posing, interpreting, and responding in classroom teaching, we hope to gain more understanding about the interplay between teachers' mathematical understandings, mathematical knowledge for teaching understandings, and pedagogical understandings.

Endnote

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THE MATHEMATICS OF CHILDREN'S THINKING: AN EXAMINATION OF TEACHER EDUCATORS' USE OF INVENTED STRATEGIES IN A MATHEMATICS CONTENT COURSE FOR PROSPECTIVE ELEMENTARY TEACHERS

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This study examined how, in a mathematics content course for prospective elementary teachers, four instructors implemented lessons that used video clips of children's invented strategies. The lessons were designed to access mathematical knowledge within the context of teaching. Analysis of classroom discussions uncovered key issues and challenges that the instructors grappled with when trying to coordinate mathematical ideas with children's thinking. This study focused on those factors that contributed to maintaining or reducing the level of the mathematical activity. These results raise questions about how children's work can be implemented in a mathematics course, and have implications for the professional development of college faculty who are teaching these courses.

The idea of blending content and methods in teacher education is not a new one (Ball, 2000). Though more work is needed in this direction, results have shown some potential benefits of this practice (Ambrose, 2004; Dogon-Dunlap & Liang, 2006; Wiles, 2002). Little work has gone into the practice of implementing these methods from the perspective of the teacher educator (Doerr & Thompson, 2004). As we are starting to see the presence of children's thinking in mathematics courses for elementary teachers, a need arises for the development of models that help us understand the implementation of these alternative methods. This paper seeks to further illuminate the issues surrounding the implementation of lessons in a mathematics course for pre-service elementary teachers (PSTs) that make use of written and video examples of children's invented strategies. Two questions were central to the design of this study:

(1) In what ways do mathematics instructors implement lessons that make use of children's invented strategies?

(2) What are the mathematical issues that arise as instructors and PSTs engage with children's strategies?

Theoretical Framework

Much of the work involving the integration of children's work into the classroom has centered on case based methods. While cases are more than video clips or examples of student work (Shulman, 1992), the use of cases in education framed my orientation toward using video clips in mathematics. Cases provide a venue where a student can examine their developing theoretical ideas about instruction within the context of practice (Merseth, 1999). Similarly, video clips of children's strategies allow a teacher to examine mathematical ideas in the context of teaching. The presentation of a child's invented strategy presents a mathematically problematic situation for a preservice teacher, much as a case is problematic

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from a pedagogical perspective. These episodes are not meant to represent a case of teaching practice, but represent an authentic encounter with mathematics content as they might arise in teaching practice.

The idea of authentic mathematical encounters was central to the formation of the lessons that were used in this study. The process of unpacking the mathematics of a child's strategy exemplifies a particular kind of mathematical understanding that is unique to the field of teaching. Ball and Bass (2000) identified this as "pedagogically useful mathematical understandings" (p. 89). This mimics the uncertainties inherent to practice and illustrates the kind of real time mathematical analysis that is a significant part of a teacher's professional activity.

In order to examine the implementation of mathematics tasks, the Mathematics Task Framework (Stein, et al, 2000) provided a way to examine tasks and track how they unfolded during classroom instruction. Mathematics tasks can be categorized generally as either high level or low level based on the cognitive demands involved. Low level tasks were characterized by having no connections to mathematical concepts, had little ambiguity, and required no explanations. These tasks generally required either direct recall, for *memorization tasks*, or using procedures without examining the underlying mathematical concepts, categorized as *procedures without connections.* High level tasks directed the students to make connections to underlying concepts and required cognitive effort. Though certain high level tasks, called *procedures with connections,* might involve carrying out procedures, they usually involved multiple representations and required the student to actively explain their thinking. Those tasks classified as *doing mathematics* required the students to engage in non-algorithmic thinking, make use of prior knowledge, monitor their thinking while analyzing the task.

According to the framework, tasks progressed through three stages, first they were taken from instructional materials, then they were set up by the instructor, and finally were implemented by the students. I applied this framework to understand the factors that were key for maintaining high levels of student thinking through tasks that center on children's thinking.

Methods

Four instructors were observed for this study. Two instructors, George and Carlos, had Ph.D.s in mathematics and were Postdoctoral fellows in the department. The other two, Sarah and James, had Masters degrees in mathematics and were adjunct faculty in the department. Each was teaching the first semester mathematics course for preservice elementary teachers with approximately 25 students in each section. None of the instructors had experience teaching at the elementary level, though Sarah used to be a secondary mathematics teacher. In addition, George and Carlos had previously never taught the content course for prospective elementary teachers.

I visited and video taped three lessons in each instructor's class. The first focused on children's invented strategies for addition and subtraction. The second and third lessons focused on invented strategies for multiplication and division, respectively. These lessons would often span multiple class periods. During class I would take field notes and observe the discussion when the PSTs were engaged in small group interaction. All written work was collected and copied for later analysis. In addition, I was present at all training sessions for the mathematics instructors and kept a journal detailing my interactions with them.

In order to find patterns and themes, elements of grounded theory (Strauss & Corbin, 1998) were employed. I first developed categories of lesson implementation based on the mathematical and pedagogical themes that emerged. I also identified instances of interaction between the instructor and the preservice teachers and the extent that the instructor either raised or lowered the cognitive demand of the task (Stein, et al, 2000). I used these codes to examine the PSTs' written work and then identified additional ways that they engaged with children's strategies from a mathematical perspective.

Cognitive Demands of Unpacking Children's Strategies

The lessons that the instructors implemented all followed a similar structure. At the outset, PSTs generated alternative strategies for arithmetic problems without using remembered algorithms. The PSTs observed several clips of children solving arithmetic problems, representing varying levels of sophistication and efficiency, all producing correct solutions. For each strategy, the PST was instructed to describe what the child did, and then apply the strategy to two new arithmetic problems, one with similar numbers and the other involving larger numbers. The students were then directed to compare and contrast the various methods they observed and outline the key understandings that the children exhibited that allowed them to solve the problems in the ways that they did.

While the appearance of children's thinking made this appear to be a pedagogical task, the structure of the task was such that the students' attention could be on the mathematical ideas that were underlying the children's strategies. For example, a child who solves the problem 29 x 4 by saying, "20 times four is 80, nine times four is 36, and 80 plus 36 is 116," was exhibiting the use of the distributive property of multiplication over addition. The child is also making use of important ideas related to place value. These mathematical ideas are often overlooked by PSTs whose primary experience with arithmetic is through the systematic execution of remembered algorithms.

If we are to categorize the children's thinking lesson as a mathematics task, then it would be categorized as a high level task, specifically at the *doing mathematics* level. In order to explain the strategy, the PSTs must actively make connections to their knowledge of number and operations. The strategies were ambiguous enough that there were several reasonable ways that a PST could adapt it to solving new problems. There are also several ways that the strategies could be compared, highlighting connections to many different mathematical ideas.

Evolution of Cognitive Demands

The set-up and implementation of the tasks varied quite a bit in the four classes. In each of the classrooms, there was evidence of a lowering of cognitive demands. The following sections identify four factors that played an important role in either maintaining or lowering the cognitive demand of the task.

Clarity of Expectations

After viewing the video clips and having the PSTs work in small groups, all of the instructors then gathered the students to discuss their thinking. In the cases of James, Carlos and George, these sessions would start with directions that often lacked some clarity of purpose. For example, Carlos opened the first discussion on addition and subtraction strategies by saying, "So, what do you think about these methods... Do you like them... What else do you think?" The students' responses were often ambiguous and not focused on mathematical content. Sarah, on the other hand, had a more clear purpose in mind at the outset. The following took place in the first lesson centered on addition and subtraction

> Sarah: So what do the children understand that lets them use these strategies?
> PST: Place Value
> Sarah: Okay, what about place value did they seem to know? How did that help them solve it in the way that they did?
> …
> Sarah: Are there differences in how they are thinking about or using place value?

Rather than collect general sentiments about the methods, Sarah began with the mathematics at the fore. She also made it clear that she was looking for a discussion centered on more detailed mathematical ideas. George began to attend to the whole class discussions in similar ways during the later lessons:

> George: We saw before how children break apart the numbers to help them solve addition and subtraction problems. What allowed them to break up the numbers how they did with multiplication?
> PST: They used multiplication facts that was [sic] easier for them
> George: True, but how do we know that they can do this? Sometimes children will use a method that miraculously works in a particular situation, but not in general. Is this one of those cases?

Here, like with Sarah, the instructor was reacting to the PST's response in a way that made the expectations clear. This was much slower to occur in both James and Carlos's classes.

Building on Students' Prior Knowledge

The PSTs in all four classes were expected to have come to class with their own nonstandard strategies for the problems they would see the children work on. Only in Sarah's class did the PSTs discuss their own ideas. In these episodes, it was clear that there were a number of different ways that the PSTs were thinking about the operations, some of which complimented the strategies that the children on the tape would use. These ideas, however, were never returned to after the clips were viewed. In other classes, the instructor did not ask for students to share their thinking, but the PSTs did at times discuss their own experiences. For example, in James's class the following exchange took place, "I can't remember thinking like that. I never thought about it, maybe now that I have a better understanding of numbers. In the fourth grade there is no way I would have been able to do that," followed by a second student who commented, "For me it was way easier to use tens and fives. It was individual to me, though, it is student for student. If I tried to teach the whole class like this, I would confuse more people." In this episode, and in similar ones occurring in other classes, the discussion was acknowledged by the instructor, but not attended to.

Sustained Press for Explanations

When the discussions turned toward general mathematical ideas, such as the meaning behind the traditional algorithms, the instructors often required the PSTs to explain their thinking. When the PSTs discussed the children's strategies, however, there was often no teacher comment and no expectation for elaboration. For instance, when students would indicate that a method was inefficient, they were not expected to elaborate on what efficiency meant or what specific aspects of the child's they were referring to. This also extended to students' comments about the mathematical basis of a strategy. For example, one child's method for solving 159 divided by 13 involved dividing 100 by 13 and then 59 by 13. In James's class, a PST commented on this by stating, "I realized that it is kind of how we do long division. When you divide a two digit number into a three digit number, really we are splitting up the number." The instructor did not require to student to qualify this statement in any way, even though the statement was mathematically incorrect.

In each of the lessons, the PSTs were tasked to apply the children's methods to new problems. This had the potential of being a cognitively challenging task because there were a number of ways that a particular method, having seen it applied only once, might have been extrapolated to new problem. These cognitive demands were significantly lowered because the students were not required to defend their position of how a method might be expanded, even though there were often significant differences in the newly applied methods the PSTs generated. During the discussion, none of the instructors actively questioned the PSTs about their thoughts. Since PSTs were not pressed to defend their claims by appealing to the students' words or the mathematical basis of the strategy, any perspective about a child's thinking was accepted as possibly valid.

Focus on Methods Rather than Mathematical Concepts

To varying degrees, all four instructors approached the children's strategies as methods to be learned and categorized rather than as vehicles for making mathematical connections. For instance, in Carlos's implementation he walked the students through each of the children's methods, asking what they thought of them and what he/she did. Discussions of the strategies often focused on differences rather than on ways that they were connected to each other. His questions centered on particulars of strategies without connecting to underlying mathematical concepts, such as how the methods highlighted different models of the arithmetic operations and their properties. When concepts were brought out, they were done so in isolation from the children's thinking.

James brought out more mathematical concepts, but often pointed these connections out rather than making them a part of what the students would discover when examining the children's methods. As with Carlos, he held to the strategies as objects of study. In an episode, the PSTs were discussing a clip where a child used a remembered fact that 13 x 13 is 169 to help him solve 159 divided by 13. The PSTs were concerned with how efficient this strategy might be:

PST 1: [He] couldn't do it with larger numbers. If the square is far away from what he is looking for, it wouldn't be useful.
PST 2: For larger numbers, 500 divided by 12, he could do 12 x 12, and then take chunks of 144 away.
James: Is that exactly what he did?
PST 2: No, but in that case that's what we think he would do, he's familiar…
James: But would that be following his strategy?
PST 2: No.

Sarah and George were more consistent in tying the children's strategies to mathematical concepts. When the PSTs were examining multiplication strategies, both consistently pushed the students to place the children's methods in the context of the different models of multiplication that they had examined, and used their questioning to bring out the distributive property and how the ideas inherent to the children's methods underlie the traditional multiplication algorithm. This was continued in the division activities:

George: What does this method feel like?
PST: It is like when we did the multiplication, where we multiply each piece
George: What principle is behind the scenes there?
PST 2: Its like the distributive property
George: Good, now what does Shannon do?
PST 2: Splits 159 divided by 13 into 100 divided by 13 and 59 divided by 13
George: Okay, is it legal to use the distributive property like this with division?
PST 3: I think it works, but what about, like 105 divided by 12, the five is too small.

This discussion of how the distributive can be applied to division continued, until George again brought the PSTs back to the children's strategies

George: Are there other children who are also using the distributive property?
PST: Yeah, I guess Elaine did. She did 13 x 10 and then 13 x 2 to get the answer.
George: Are Elaine's and Shannon's use of the distributive property the same, or different?
PST 2: Yeah, they broke into place value
PST 3: Not really, well yeah they did, but Shannon broke up the 159. Elaine broke up the answer.

The students then examined the roles of the dividend, divisor, and quotient in a division problem, and then used these ideas to compare each of these methods to the long division algorithm.

Mathematics of Children's Thinking

These factors played an important role in how the PSTs participated in the four classes and what kind of opportunities for mathematical learning took place. In James and Carlos's classes, their focus on the children's strategies as methods had the effect that when the PSTs talked about the children's methods, their focus were almost exclusively on *what* the child did. The discussions of these methods always turned to issues of efficiency, but nothing else, so a number of opportunities were missed. Examining strategies that the PSTs themselves created might have helped the students to explicitly build off of their prior knowledge and experience. Also, by focusing on the particulars of the methods, they cut off a lines of reasoning that could have been helpful in parsing through different ways to conceptualize the mathematics.

This was contrasted to the ways that the PSTs in George and Sarah's class engaged with the children's methods. Though issues of efficiency and unpacking what the children were doing were examined, the comments of the students also turned to *why* the method worked and how they were connected together. In addition to efficiency, their discussion also brought out issues of generalizability and transparency. At the same time, they too tended to focus on the children's strategies as methods to be remembered, albeit conceptually based methods.

Discussion

Though the use of children's strategies has tremendous potential, it is clear from these results that their introduction does not guarantee deep mathematical activity. The students in Carlos and James's classes were engaged with the children's methods, but they did not draw out high level mathematical activity. In contrast, George and Sarah were able to maintain higher-level expectations and connect with mathematical concepts.

Instructors who wish to make use of children's thinking in a mathematical course must think carefully about their goals. This research tells us that if this is not done, mathematical payoff may not be enough to justify the amount of time that must be devoted to it. While the students in these classes may have benefited in other ways from the lessons, mathematical thinking often took place at a lower level. It is interesting to note that even instructors with extensive mathematics training did not end up focusing on the mathematical ideas, rather methods took a more central role. As the use of children's thinking becomes more prevalent in mathematics texts (see Bassarear, 2007), it becomes vital that instructors become aware of the factors that might contribute to mathematics learning. As such, these results speak to a need for examining the professional development of college mathematics instructors more carefully. There is also general need for teacher educators to turn their attention and research on themselves to better understand that complexities of their practice. A great deal of work has gone into understanding the process of becoming a K-12 mathematics teacher, a similar body of work needs to be developed for post-secondary education.

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THE MULTIDIMENSIONALITY OF LANGUAGE IN MATHEMATICS: THE CASE OF FIVE PROSPECTIVE LATINO/A TEACHERS

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This paper reports a study of prospective bilingual teachers and the intersectionality of language and mathematics in a non-traditional field experience. Qualitative data were analyzed in order to map Spanish and English language practices to teachers' assistance strategies. We define assistance strategies as those actions that are intended to help students understand content. Findings demonstrate the complexity of prospective teachers' use of language in mathematics teaching and suggest implications for teacher preparation programs.

Preparing teachers to become linguistically responsive educators has been a current issue in Teacher Education (Darling-Hammond, & Bransford, 2005). Most work relating to this issue has been done on monolingual prospective teachers, while there is very little work on bilingual prospective teachers, especially in the domain of mathematics. The purpose of this paper is to report a study designed with the intent to investigate how prospective teachers grapple with the intersectionality of mathematics learning and language through their participation in a non-traditional field experience. The participants were Latina/o prospective teachers who served as facilitators in *Los Rayos de CEMELA,* and after-school project. Specifically, this paper examines how shifts in language use (English – Spanish) relate to prospective teachers' interpretations of students' mathematical engagements and the subsequent assistance strategies they use with students. We define assistance strategies as those actions that are intended to help students understand content. The study describes how language choice mediates assistance in doing mathematics.

The study presented here draws on current work carried out by the Center for the Mathematics Education of Latinos/as (CEMELA)[i] which focuses on the research and practice of the teaching and learning of mathematics for Latino/as in the United States through the integration of socio-cultural theory, language, and culture. CEMELA has created after school projects at two of its sites, one of which (*Los Rayos de CEMELA*) is the source of the present study. The after-school projects are designed to investigate the linguistic and cultural resources that support bilingual Latino/a students' mathematics learning. and are a general adaptation of the work of the Fifth Dimension (Cole, 1996). Los Rayos is guided by other similar projects (e.g., Gutierrez, Baquedano-Lopez, & Alvarez, 2001) including *La Clase Mágica* (Vasquez, 2003). These works have utilized the after-school as a way of understanding literacy; CEMELA has extended the work to consider mathematics.

Theoretical framework

To understand the complex nature of the mathematical activities and the interactions that take place in *Los Rayos de CEMELA*, we use both a socio-cultural and an activity theoretical view of learning and development. Through these lens we focus our analysis on the prospective teachers' interactions with children and on their reflections on mathematics teaching and learning.

Cultural historical theory (Vygotsky, 1978) shifts the view of learning from an individual internal phenomenon to one based in social interaction. Higher mental processes derive from external and practical actions or activities between people. These actions are mediated by signs and tools, including language which is the most prevalent. Language also may be the most important semiotic tool because it is the most obvious and natural means that people use to make sense of phenomena, to transmit values and beliefs, and to socialize children into the practices of a culture (e.g., Halliday, 1993; Lave & Wenger, 1991; Vygotsky & Cole, 1978; Wells, 1999).

The after-school setting is a tool in itself in that it presents a different yet familiar setting from a traditionally structured teaching practicum or tutoring-oriented activity. *Los Rayos de CEMELA* embeds students' self-motivated actions and interactions with others that together form a "cultural practice". The way mathematics is done in the after-school is different from how it is done in classrooms, perhaps in small ways, but nevertheless different. This difference challenges students' assumptions about what it means to do mathematics and in this way forms a "practice" of doing mathematics. At the same time this difference challenges the facilitators' (in this case the prospective teachers') assumptions about what it means to assist students in doing mathematics while utilizing a variety of linguistic and cultural resources. Through their participation in *Los Rayos de CEMELA* prospective teachers are involved in hybrid practices which are "polycontextual, multivoiced, and multiscripted" and help create a *Third Space* which allows for assumptions and connected actions to emerge in a natural way (Gutierrez, Baquedano-Lopez, & Tejeda, 1999).

Methods

Five (four female and one male) Latino/a pre-service elementary teachers in their junior year were recruited during the second week of their mathematics methods course in Fall of 2006. All participants self identified as being fluent in both Spanish and English and four of them (all females) are seeking endorsements in bilingual education.

The participants met with fourth-grade students in *Los Rayos* once or twice a week for one and a half hours each time through an eight-week period during the Fall 2006 semester. Each participant worked with a group of two to four students during each session and was encouraged to serve as a sibling-like "more experienced other" rather than a mathematics tutor or teacher. All sessions in the after-school were videotaped. Additionally all participants were required to take detailed field-notes of their interactions with the students focusing specifically on how language (Spanish and/or English) was used, on students' mathematical strategies, and on their own assistance strategies. Furthermore, each participant played the role of *el Maga* (a bilingual math wizard) who communicated electronically with the students twice a week (via a message board) about mathematics. The students were not aware of who actually wrote as *el Maga*, and this situation encourages a free and natural exchange between correspondents. Finally, during this eight-week period, all participants met in a weekly a two hour debriefing seminar designed to engage participants in active discussions about students' linguistic and mathematical behaviors and interactions, and the participants' own reflections about the same items. All seminars also were audio-taped.

In order to map language use with assistance strategies data were analyzed using a constant comparative method (Glasier and Straus, 1967) in which relations amongst language choice and assistance strategies used, became apparent as the data were continuously examined.

Evidence

Data analysis reveals two patterns regarding language use and assistance strategies. Firstly, assisting students in mathematical activities in Spanish proved to be a challenge for most of the prospective teachers. They realized that fluency in conversational Spanish does not translate to fluency in academic (in this case mathematical) Spanish. Secondly, language was used in fluid and non-determined ways. There was no evident predictor for the language each prospective teacher chose to use and language choice was not always conscious or purposeful. We use Patton's (2002) intensity sampling strategy - whereby cases which represent categories of extreme interest are focused on in more detail - in the selection of cases to present patterns in language use and assistance strategies. For challenges involved in the use of Spanish when assisting students in mathematical activities we draw upon the experiences of Julie[ii], and to describe how language was used in fluid and non-determined ways when conversing with children about mathematics the experiences of Rex are discussed. Both cases highlight the role of language in assisting students.

Challenges in doing mathematics in Spanish

All study participants self identified as being fluent in Spanish and English. In fact, all participants' first language is Spanish. However, doing mathematics in Spanish, facilitating mathematical discussions in Spanish, and assisting students in mathematical activities in Spanish all proved to be a great challenge for three out of the five study participants, namely Julie, Deborah, and Rex.

During the first debriefing meeting, prospective teachers were asked to do the mathematical activities they were going to use with the children in *los Rayos* that week. All activities were provided in both Spanish and English, in the same forma that students would get them. Julie chose to use the Spanish version of the activities and informally teamed up with two other prospective teachers, Diana and Sonia, who also chose the Spanish version. The following describes Julie's experience with the Spanish materials.

While Julie was reading the activity she frowned and said "WHAT?" She then resorted to the English version of the activity. After reading it in English she said "Oh, now I get it!" and solved the problem right away. Julie kept resorting to the English version of the activities and although she discussed the context of the activities in Spanish she would resort to English when talking about mathematics. Later on, she reflected on her difficulty in understanding mathematical Spanish:

> **Julie**: …well like cause you if know let's say your L1 is Spanish but you like you know English, if they give you a word problem and it's in English you would be able to do it but then if they give it to you in Spanish you won't be able to do it. Does that make any sense?
>
> **Eugenia**: O.k. can you say that again? So what's your first language?
>
> **All**: Spanish.
>
> **Eugenia**: So English is your second
>
> **Julie**: But I learned English, I learned English math. So if they give me a word problem like they did like today and I read the Spanish one I had to reverse and I had - I - I know how to read Spanish and I understand it but I comprehend math a lot better if I read it in English…like I'll read it and I'll be "o.k…" but when I - when I flip it over I'll be like "Oh o.k.! O.k. umhum!" You know?

Julie was seeking bilingual endorsement at the time of the study and had read in their bilingual education course that conversational fluency in one cultural language does not translate to academic fluency in the same cultural language. However, it was not until she experienced trying to comprehend, discuss and solve a mathematical problem in Spanish that she realized this. Julie had a hard time comprehending the mathematics in the problem she was given and had to resort to the English version of the problem because she had learned mathematics in English. She started to realize that the reason she was not able to think about mathematics in Spanish was because she had never been taught mathematics in that language. In other words, Julie was not familiar with Spanish mathematical discourse and this resulted in her difficulty comprehending the Spanish mathematical text, even though the problem involved elementary mathematics in which she was competent.

During the after-school sessions, Julie would use both English and Spanish – mostly code-switched - with the students she worked with. She would always make a considerable effort in making sure that the mathematical conversations in her group were done in both languages as she realized the importance of children being exposed to mathematical talk in both Spanish and English. In a personal conversation with one of the researchers, she expressed that she felt saddened by the fact that she had a hard time thinking and talking about math in Spanish and she wanted to make sure that her students would not share the same experience as she did. In addition to this, Julie expressed many times during the debriefing meetings that it was necessary for her to read the English version of the activities and then solve the activities in English before attempting to assist students in Spanish.

Julie: I feel like if I could read an English worksheet then I could translate it in Spanish but if I get a Spanish worksheet and I read it then I'll take a lot longer to understand it…cause I could translate it and I could teach it to them in - well not teach - in Spanish.

Deborah: Me too. Exactly.

Rex: Yea the same way.

Even so, Julie found it difficult to assist students in mathematics using Spanish only. Instead she would blend Spanish and English in both mathematical and non-mathematical discussions. Code-switching came very natural to her and the students she worked with during non-mathematical discussions. However, during mathematics Julie consciously blended both vernacular and mathematical talk in Spanish and English as an effort to experiment with and expose her students to Spanish mathematical discourse. Julie later indicated in her field-notes that she had become so used to code-switching in Spanish and English during the mathematical activities that she preferred the activities that were written in a format that blended the two languages in the same sentences and paragraphs. In her field-notes, she said: "…one thing I did like about this problem was that it was both in Spanish and English because it was written in a way that many students speak, including myself…"

This example shows the difficulty that teachers like Julie, as well as Deborah and Rex, who are Spanish bilingual but not biliterate (e.g. , able to speak, read, and write mathematics) face in thinking and talking about mathematics in Spanish. However, the three prospective teachers dealt with it differently; one blended the two languages in all contexts, one only in non-mathematical contexts, and one stayed with English only.

Language choice

Unlike Julie who consistently code-switched when communicating with the students in *los Rayos*, Rex and Deborah used mostly English while Diana and Sonia used mostly Spanish. An interesting finding is that in numerous instances participants used English (instead of Spanish) when facilitating a mathematical discussion with a group that had a Spanish dominant student present but never used Spanish when an English dominant student was present. For example, Sonia and Diana would switch to English (and facilitate group mathematical discussions in English) when an English dominant student joined their groups, even though Spanish dominant students were present in the same group. On the other hand, Rex and Deborah would not switch to Spanish when a Spanish dominant student was present in their group. Instead they would continue carrying on the mathematical discussions in English. The only times Spanish were used in these cases was during non-mathematical conversations and/or to discipline the students.

Rex's language choice presents an interesting example. One of the students in the group he facilitated, Rafael, was Spanish dominant and would consistently choose the Spanish version of the activities, while the other two students (Rodrigo and Mario) would choose the English version most of the times. Rex would facilitate all mathematical discussions in English because (as indicated in his field-notes) he assumed that the students had been taught mathematics in English only and because they would always talk about mathematics in English. However, after working with the same children for four weeks, Rex noticed a switch to Spanish while playing a math game called "the grocery cart game." In his field-notes he wrote:

> "…Another interesting event happened today when the students started speaking in Spanish. I believe Rodrigo started it when he spoke about items on the menu in Spanish and Rafael responded in Spanish. They then went to add the items' price in Spanish. It seemed like Rafael added in Spanish by that he said the numbers, counted the numbers, and used terminology in Spanish.

Rex observed that Rafael used Spanish exclusively when doing the mathematics involved in the game. In later field-notes he indicated more instances during which Rafael resorted to Spanish when talking about the mathematics involved in the group activities. Yet he did not use any Spanish when conversing with Rafael about mathematics and continued talking to him in English. In his field-notes he frequently noted that Rafael (the Spanish dominant student) was not engaged with the mathematical activities and was often "off task."

However, when Rex conversed with Rafael via the electronic message board, the language used by both of them was always Spanish. Additionally, the entire dialogue between *el Maga* (Rex) and Rafael was about mathematics. Rafael kept challenging *el Maga* with math problems and *el Maga* responded by posing a new problem for Rafael. Rafael, in turn, who was "off task" during the group activities was extremely engaged in trying to solve the problems *el Maga* posed. In fact, there were a couple of instances where he would not leave *los Rayos,* the after-school, until he solved *el Maga's* problem. As an illustration of the *el Maga*/Rafael dialogue we have chosen the following:

> **Roberto**: Hola maga. Cuanto es 5076 menos 6987 si lo saves dimelo? Bye maga que diviertas hasta quando sea miercoles. [*Hi Maga. What is 5076 minus 6987? If you know tell me. Bye maga I hope you have fun on Wednesday.*]

El Maga (Rex): Hola Roberto. La respuesta es -1911. Es un numero negativa. Sabes la respuesta de 125 suma 875? Cuidae y portate bien. [*Hi Roberto. The answer is -1911. It's a negative number. Do you know the answer to 125 plus 875? Take care and behave.*]
Roberto: Hola maga. Como estas? El numero que me diste que eran 125 mas 875 yo lo tengo. Era 1000. Yo te quiero dar otra esta va ser muy dificil. Es 1,000,000,000 mas 2999,999,875,564? [*Hi Maga. How are you? The number that you told me that 125 plus 875 was I have. It was 1000. I want to give you another one. This is very hard. It is 1,000,000,000 plus 2999,999,875,564?*]

This is a very typical example of how *el Maga* and Rafael communicated via the electronic message board. Rex used the message board as a space for having a mathematical dialogue with Rafael in Spanish, and Rafael used it as a space to explore more challenging problems.

The forgoing examples of choices in language raise the following questions: Why did Rex continue carrying mathematical discussions in English with his group, even though he became aware of Roberto's dominance in both conversational and mathematical Spanish? Was this to accommodate the other two students who he had identified as being English dominant? Why did he not accommodate the Spanish dominant student? Was this because Rex, the prospective teacher, had a better written command of mathematical Spanish but felt insecure about his oral command of it? Was his language choice a conscious one?

Results and Conclusion

Findings from this study reveal the complexity of the interplay between language and mathematics. We examined bilingual prospective teachers and found that the use of two languages as a teaching and learning resource is not as straightforward as one would think. Being fluent in two languages does not necessarily mean that one can do or teach mathematics in both languages. Results suggest that mathematics educators need to rethink what is involved in teaching and learning mathematics in two languages and how complex this can be. However, if we wish for teachers to be equipped with the means to adequately address ELL and bilingual students' needs in mathematics, the Mathematics Teacher Education research community must investigate more (new) ways of preparing prospective teachers to meet these students' needs.

Conceptualizing the nature and role of language and culture in the teaching and learning of mathematics is not an easy task and research in this domain is in its infancy. According to Zeichner (2005) "the research on the preparation of teachers to teach underserved population should pay special attention to the preparation of teachers to teach English-Language Learners, because almost no research has been conducted on this aspect of diversity in teacher education." Even though there is a growing body of research on Latinas/os in mathematics, research that investigates the preparation of teachers to teach mathematics to Latinas/os who are English-Language Learners or bilinguals is almost non-existent.

Relationship of Paper to Goals of PME-NA

This research project was conducted with the intent to better understand the socio-psychological aspects of doing mathematics and assisting children with mathematical activities in two languages (Spanish and English.) In order to understand how to better prepare teachers to address the specific needs of students who are marginalized from the educational system in the United States, such as ELLs and bilingual Latinas/os, we must understand the complex factors that affect and shape prospective teachers' experiences in doing and facilitating mathematics in two languages. Through the examination of bilingual Latina/o prospective teachers' experiences as facilitators in a mathematics after-school setting, we highlight the need for providing prospective teachers with more opportunities to experiment and investigate the role and nature of language in mathematics.

Endnotes

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[ii] All names are pseudonyms, which participants chose for themselves.

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TRANSFORMING TEACHERS' KNOWLEDGE: THE ROLE OF LESSON STUDY IN PRESERVICE EDUCATION

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This study investigated how preservice elementary teachers experienced lesson study in their mathematics methods courses and how lesson study facilitated their knowledge development as teachers. Certain characteristics of the lesson study process were identified that supported the development and transformation of three types of teachers' knowledge: content knowledge, pedagogical knowledge, and pedagogical content knowledge.

Preservice teachers' knowledge of good math teaching and learning may be limited by their own classroom experiences. They might never have witnessed effective reform math teaching to understand the possibilities that it offers for sustained student learning. The apprenticeship of observation is indeed powerful in shaping beliefs (Lortie, 1975). To address this phenomenon, some teacher educators incorporate systematic observation of preservice teachers' practice to facilitate knowledge transformation. One such way to facilitate this transformation is through the implementation of lesson study in preservice programs.

Theoretical Framework

Lesson Study

Lesson study is a professional development tool that originated in Japan. Since the 1990s, it has spread rapidly in the United States (Fernandez, 2005; Lewis, Perry, & Murata, 2006; Lewis & Tsuchida, 1998). Several key features of lesson study serve to transform teachers' knowledge (Fernandez, 2005): lesson study enables teachers to work collaboratively in the lesson planning cycle (assess, plan, teach, reflect); and it provides contexts to demonstrate reform math teaching in the classroom by helping teachers see the lesson through the eyes of the students. Participation in this form of professional development has the potential to transform teacher beliefs which has an impact on teacher knowledge.

Teacher Knowledge

Three types of teacher knowledge were formalized by Shulman (1987) and others (Grossman, 1990; Hill, Schilling, and Ball, 2004) that need to be transformed in order to support teacher learning: (1) content knowledge, (2) pedagogical knowledge, and (3) pedagogical content knowledge. In other words, teacher educators need to provide opportunities for teachers to: (a) deepen their understanding of math content, (b) build a repertoire of effective teaching strategies, and (c) connect emerging knowledge of content and teaching strategies to reconceptualize how students can best think about and learn math.

In lesson study, teachers are supported to go beyond a superficial comprehension of the math content (e.g., procedural understanding) and gain a stronger conceptual grasp of the material they teach. Lesson study provides the impetus for teachers to examine current research findings of student learning, pre-assess their students based on these findings, and plan an effective lesson, broadening their existing ideas of effective teaching strategies. Through this process, lesson study continuously focuses teachers' attention toward students' thinking on certain math topics, highlighting effective mathematics teaching with a conceptual focus.

Using the three types of knowledge conceptualized by prior research as a framework, this research report presents a case study that describes how two cohorts of preservice elementary teachers experienced lesson study in their mathematics methods courses and how these experiences facilitated their knowledge development and transformation. This study aims to answer the following research questions:

How does lesson study facilitate the development and transformation of content knowledge, pedagogical knowledge, and pedagogical content knowledge of preservice teachers? What characteristics of lesson study are integral to this process?

Methods

Two cohorts of elementary preservice teachers in their respective math methods courses at a major research institution in the western United States participated in the study. All teacher education students engaged in the lesson study process which was central to one quarter's course(s). In order to identify if and how teachers' content knowledge, pedagogical knowledge, and pedagogical content knowledge were changed and how lesson study played a role in this change, a combination of both a quantitative and qualitative research design was used.

For quantitative data, both cohorts of elementary preservice teachers took a pre- and post-survey (Hill, et. al., 2004) at the start and end of the teacher education program. This survey was developed by the Study of Instructional Improvement (SII)/Learning Mathematics for Teaching/Consortium for Policy Research in Education (CPRE) at the University of Michigan and primarily assesses preservice teachers' pedagogical content knowledge and content knowledge. All questions from these pre- and post-tests were coded as corresponding to the types of teacher knowledge. This enabled analysis of each knowledge on a separate basis. The groups' overall scores were also used to determine the teachers' learning in the courses.

Qualitative methods, including unstructured interviews and classroom observation of the lesson study groups' planning processes and final lessons, were utilized to understand the complex learning shift. Teachers wrote open-ended reflections immediately after their lesson study experience as well as at the end of the program. Data were collected in the form of reflection papers on 20 teachers from Cohort 1 during the 2005-2006 academic year, and on 17 teachers from Cohort 2 during the 2006-2007 academic year. Teacher reflection data were first coded and analyzed for the types of knowledge that teachers formalized through the lesson study process as a whole. Percentages were calculated to determine which codes were referenced most often. Then, characteristics of lesson study were identified and linked to the teacher learning.

Results

The pre- and post-survey results show increases across the knowledge types and across both cohorts (see Table 1 for a summary of the results).

	Cohort 1 (2005-2006)			Cohort 2 (2005-2006)		
	Overall	Content Knowledge	Pedagogical Content Knowledge	Overall	Content Knowledge	Pedagogical Content Knowledge
Pre-Survey	72%	69%	73%	75%	68%	78%
Post-Survey	78%	78%	78%	85%	84%	85%

Table 1. Percentage of correct answers on pre- and post-surveys

While it is important to note the increase in overall scores for both cohorts (6% for Cohort 1 and 10% for cohort 2), it is also notable how the content knowledge and pedagogical content knowledge scores increased respectively. Cohort 1 increased 9% for content knowledge items and 5% for pedagogical content knowledge items; Cohort 2 increased 16% for content knowledge items and 7% for pedagogical content knowledge items.

Analysis of preservice teachers' reflections show that their beliefs were also challenged through engagement in the lesson study process. Teachers' content knowledge, pedagogical knowledge, and pedagogical content knowledge were all made central and transformed in some way. Appendix A presents excerpts of their reflections regarding the transformation of these three types of teacher knowledge.

Content Knowledge

While our data set is limited and we cannot claim definitive change in teachers' content knowledge of their respective lesson study math topics, the survey results show that their overall content knowledge was improved (see Table 1). Zooming in further, the qualitative data suggest that teacher beliefs on math as a subject were also transformed. Teachers became more aware of the way in which their own schooling limited attainment of a deeper understanding of various foundational math content topics. This realization of their lack of conceptual knowledge allowed teachers to admit that relearning elementary mathematics conceptually was necessary in order to better teach. Collaboration with peers helped them grapple with complicated mathematical concepts that are foundational to their familiar operational procedures. In addition to gaining a stronger grasp of the content itself, teachers realized that even “simple” mathematics concepts can be quite complex. One teacher mentioned:

> “...in exploring division of fractions I realized how little I understood conceptually about that math topic. For me, brainstorming real life situations that require division of fractions and figuring out what that operation is really 'doing' to the numbers was extremely challenging. It made me realize what a superficial understanding of the topic I have.”

Pedagogical Knowledge

Preservice teachers' pedagogical knowledge was also transformed as a result of their participation in the lesson study process. Teachers learned firsthand how both published materials and empirical research could inform pedagogical decisions. Teachers were encouraged to examine standards, curricula, and research data. This enabled teachers to realize that they need not be hampered by their limited experience with classroom planning. As a result of lesson study, many felt there was a breadth of resources available to facilitate future lesson planning.

Findings from empirical research also played a role in transforming pedagogical knowledge. Teachers conducted pre-assessments in the form of one-on-one student interviews. This allowed a fair number of teachers to begin to see the value of formative assessments, as opposed to the traditional summative assessments. There was also the opportunity to review post-assessment data during the debrief of the live lesson. Both types of assessment led teachers to a greater appreciation of the student voice and how it should always inform instruction. One teacher noted:

> "I learned so much from studying the pre-lesson student assessment and analyzing student work after the lesson. I had not realized how incredibly rich student work samples are for guiding instruction. In my mind I usually think of student assessments in a formal way, as tests or final projects. However, through the lesson study process I learned the value of really taking time to look at student work, even daily assignments, to gain an understanding of where my students are at and where we need to go next."

Pedagogical Content Knowledge

Finally, a transformation also occurred in terms of teachers' pedagogical content knowledge. Fernandez (2005) defines pedagogical content knowledge:

> Such knowledge entails understanding how students think about specific content, in particular the difficulties it presents to them, and being familiar with productive strategies that can be used in the classroom to further develop students' thinking and help them overcome their difficulties (Fernandez, 2005, p.2).

Lesson study focused teachers' attention on development of their pedagogical content knowledge. As expected, the preservice teachers left with a range of abilities in this area. Most teachers related their increased awareness of the importance of closely paying attention to student thinking. One teacher stated:

> "The pre-assessment was extremely useful in building my background knowledge about how students might approach and comprehend division of fraction problems. While I could anticipate different tactics that students might use and brainstorm about the concept of dividing fractions, it became more real to analyze the strategies they actually used. They thought of tactics like drawing and repeated addition that I had not previously considered. Reviewing their work truly informed our lesson by showing us how developed students were in their thinking about fractions and how we could guide their thinking. Because I am usually more lenient about assessing students, this activity truly emphasized the importance of this element of teaching and why I should do it in the future. Assessing helps teachers understand student thinking and captures the growth of their learning. This was extremely cognitive in that I had to think about students' background knowledge and why they might resort

> to certain ways of solving the problem. For the sake of making better informed lessons I will continue to assess student learning and interest."

This closer look into student thinking was the learning that was mentioned the most by both cohorts. The scores on the pre- and post-survey items on student thinking grew over time for both cohorts of teachers. Figure 1 (attached) shows that the mean score increased over 6 percentage points for Cohort 1 and nearly 10 percentage points for Cohort 2. The standard deviation decreased by 0.6 and 2.3 respectively, showing that the teachers at the lower end of the spectrum at the beginning of the courses were brought more in alignment with the expected curve at the end.

In addition to the teachers' increased ability to analyze student work, lesson study also provided teachers with the opportunity to wrestle with the big ideas behind reform math. Through lesson study, teachers were able to debate which knowledge, procedural or conceptual, is more important to student understanding. While teachers had read about and discussed this in their mathematics methods courses, the experience of planning a lesson brought its importance to the forefront. Teachers became much more aware of the need for balance between these two aspects of mathematics learning and the ability of all students to succeed in comprehending mathematical concepts.

Many preservice teachers left the lesson study experience with more skill in how to address gaps in student thinking. Certainly the lesson study experience has propelled preservice teachers in the right direction. With more classroom experience, they will become adept at dealing with all students.

Discussion

The study shows that lesson study has the capacity to influence preservice teachers' beliefs by transforming three types of knowledge that are crucial for effective teaching. At the very least, lesson study pushes preservice teachers to examine their existing math content knowledge, pedagogical knowledge, and pedagogical content knowledge. The lesson study experience forces teachers to examine critical elements of teaching in an authentic way – through the lesson planning-teaching-reflecting cycle.

What is specific to lesson study that allows this type of learning to occur? First, the opportunity to experience the coherent lesson planning cycle is critical. Preservice teachers seldom have the chance to engage in the entire cycle in their own classrooms. Second, the lesson debrief with experts enables teachers to gain assistance in interpreting student thinking. We cannot expect teachers to have mastered this skill upon entering the classroom. Rather, they need scaffolded learning opportunities such as those provided by lesson study. Third, because lesson study is based on collaboration with colleagues, multiple voices can contribute to the creation of the best learning environment for students. Finally, lesson study places the student at the center of the professional development, rather than the teacher. This allows for a judgment-free experience, encouraging teachers to speak freely and openly about questions and issues that are central to their advancement as professionals.

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Appendix A – Teacher Quotes

Content Knowledge

Confronting the Apprenticeship of Observation	"...the initial brainstorming was challenging, yet rewarding...It made me reflect on my own schooling and the background knowledge I will have to build in order to design rewarding math lessons."
Realizing One's Limited Understanding	"...I realized how little I understood ... about that math topic. For me, brainstorming real life situations that require division of fractions and figuring out what that operation is really 'doing'...was... challenging."
Deconstructing "Simple" Math	"...[lesson study]...really made us...deconstruct all of our assumptions, misconceptions, ideas of and issues with the basic operation of subtraction."

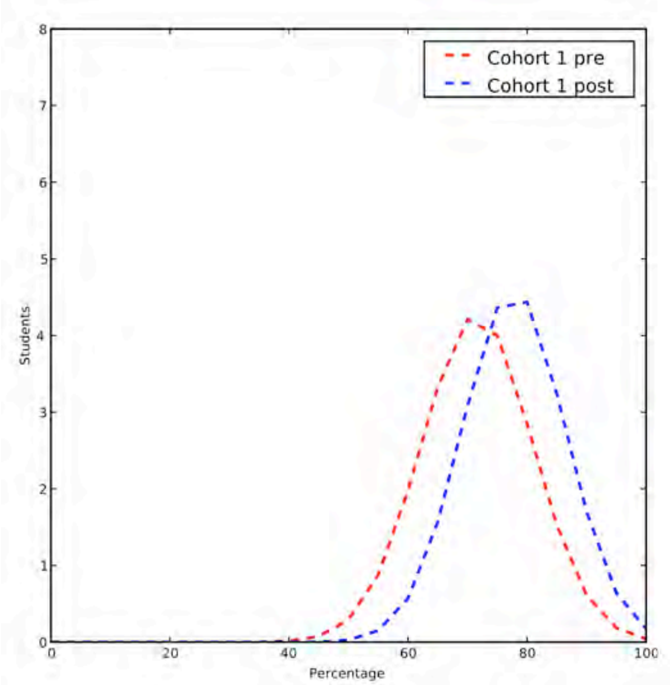
Pedagogical Knowledge

Basing Lesson Ideas on Research	"Nearly every decision...was informed either by...professional material (including standards, curriculum books, research findings) or by extensive discussion...The standards ...helped us to focus and continually re-center our thinking and our project plans."
Using Assessment Data	"I usually think of student assessments...as tests or final projects. However, through...lesson study...I learned the value of...taking time to look at student work, even daily assignments, to gain an understanding of where my students are at and where we need to go next."
Planning Lessons	"I learned how important it is for a teacher to really break down a subject into its smallest components before teaching it. I had previously thought that subtraction was so basic that there aren't really sub-topics that comprise it."

Pedagogical Content Knowledge

Focusing on Student Thinking	"Creating a lesson in which student thinking was scaffolded but still exploratory was...difficult and it was...rewarding...it was so difficult not to...ask students questions..., but by doing so I...got to...look at student thinking."
Reform Math/ Problem-Based Lesson	"...lesson study...opened my eyes to how difficult it is to teach Math through exploration and inquiry, but also how beneficial...it is to...students' learning."
Teaching Concepts Versus Procedures	"...we focus on [procedural fluency] with 'traditional' math...it has got to be our goal...to... find a balance between developing the children's conceptual understanding as well as their more automated skills..."

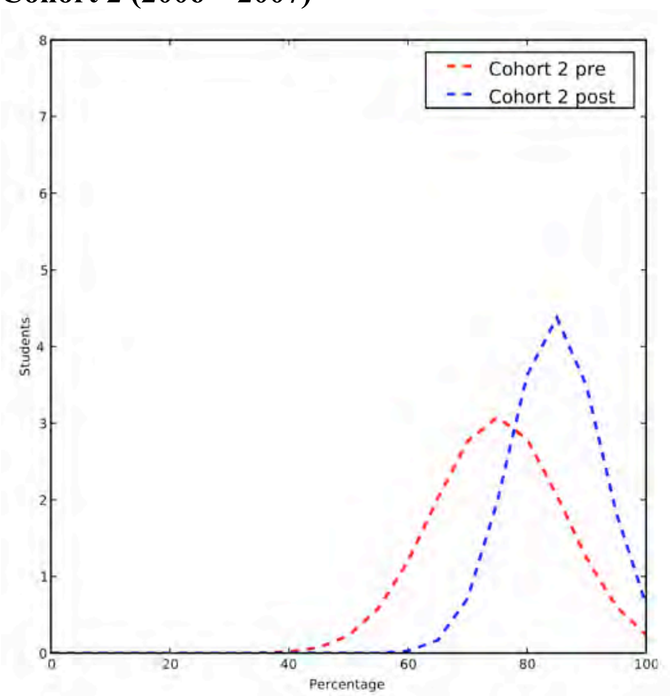
Cohort 1 (2005 – 2006)



Group	Mean	Std Dev
Cohort 1 Pre	71.6	9.3
Cohort 1 Post	77.8	8.7

t-test: 95.8% confidence

Cohort 2 (2006 – 2007)



Group	Mean	Std Dev
Cohort 2 Pre	75.1	11.0
Cohort 2 Post	84.8	7.7

t-test: 99.3% confidence

Figure 1. Distributions of overall pre- and post-survey results

A LONGITUDINAL STUDY OF ELEMENTARY PRESERVICE TEACHERS' MATHEMATICS PEDAGOGICAL AND TEACHING EFFICACY BELIEFS

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This study investigated the mathematics beliefs of 103 elementary preservice teachers in a developmental teacher preparation program that included a two-course mathematics methods sequence. Pedagogical beliefs became more cognitively oriented during the two methods courses; these beliefs then remained stable during student teaching. Personal efficacy for teaching also significantly increased throughout the program.

Background

Our teacher preparation program recently changed due to a mandate from our university system requiring an increase in the number of mathematics courses for elementary preservice teachers. Prior to this change, our program included two elementary mathematics content courses and two mathematics teaching methods courses. Our program was revised to provide a mathematics endorsement for completing four mathematics content courses and one methods course. In an effort to document the impact of these changes on our elementary preservice teachers, we began a longitudinal research effort we call the Mathematics Education Research Project (MERP). This specific study reports longitudinal changes in beliefs of the preservice teachers during the pre-endorsement program and compares their beliefs at the end of the program to the beliefs of inservice teachers.

Theoretical Perspectives

The relationship between teachers' beliefs and teaching is well-established. Beliefs influence teacher behavior and decision-making (Thompson, 1992) and change in beliefs is a crucial precursor to real change in teaching. We know these beliefs develop over time (Richardson, 1996); they are well-established by the time a student enters college (Pajares, 1992); they develop during what Lortie (1975) terms the *apprenticeship of observation* while a student; and teacher preparation programs have a limited amount of time to impact change in preservice teacher beliefs—usually two years or less. Many studies on changing preservice teachers' mathematics pedagogical beliefs focused on aligning these beliefs with a reform perspective. However, these studies often looked at change during only one course or semester; some

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.

reported achieving the desired effect while others did not. Most studies on preservice teachers' mathematics teaching efficacy beliefs also examined the effect of a single methods course, but these studies more uniformly reported significant increases in mathematics teaching efficacy. While these snapshots make important contributions to the body of knowledge on these beliefs, it is also important to examine programmatic effects on these beliefs over time.

Research Questions

1. How do elementary preservice teachers' mathematics pedagogical beliefs and teaching efficacy beliefs change during a teacher preparation program?
2. What is the relationship between these beliefs during a teacher preparation program?
3. Is there a difference between these beliefs of elementary preservice teachers and inservice teachers?

Methodology

The five cohorts of elementary preservice teachers in this study (n = 103) attended a large urban university in the southeastern United States. These cohorts were admitted concurrently and completed all education courses together. The pre-endorsement program consisted of four semesters of coursework, including a mathematics methods course during both the second and third semesters. Each of the first three semesters also included two-day-a-week field placements followed by a semester of student teaching. The field placements and coursework followed a developmental model, with pre-student teaching placements starting in pre-kindergarten and finishing in fifth grade. So, the first mathematics methods course focused on grades PreK-2 and the second focused on grades 3-5. The mathematics methods courses were taught by faculty in the elementary education department who share a common philosophical orientation toward the teaching and learning of mathematics. Important goals of the courses included developing (a) beliefs consistent with the perspective of the *Principles and Standards* (NCTM, 2000), (b) understanding of children's thinking about important mathematics concepts, (c) abilities to create problem-solving learning environments for children to facilitate discourse and understanding, and (d) abilities and confidence as a lifelong learner and doer of mathematics. In addition to university general education mathematics requirements, our program included two mathematics content courses for elementary teachers taught through the mathematics department.

The 66 inservice teachers in this study worked at a large, suburban elementary school located relatively close to the university. The elementary school and university have a Professional Development School (PDS) relationship, and the elementary school provided a number of field placements for our teacher preparation program. One of the researchers served as PDS university liaison to this school and was familiar with the mathematics curriculum used by the teachers. This curriculum took a traditional approach rather than a reform perspective.

The preservice teachers completed two instruments, four times each, as referred to in the following one-semester intervals: (a) week one of the first mathematics methods course, (b) week fourteen of the first mathematics methods course, (c) week fourteen of the second mathematics methods course, and (d) week fourteen during student teaching. The inservice teachers completed the two instruments toward the beginning of the academic school year. The two Likert-scale surveys were the Mathematics Beliefs Instrument (MBI) and the Mathematics Teaching Efficacy Beliefs Instrument (MTEBI). The MBI has 48-items designed to assess beliefs about the teaching and learning of mathematics and the degree to which these beliefs are cognitively-aligned (Peterson, Fennema, Carpenter, & Loef, 1989, as modified by the Cognitively Guided Instruction Project). The three MBI subscales include: (a) relationship between skills and understanding (CURRICULUM), (b) role of the learner (LEARNER), and (c) role of the teacher (TEACHER). The MTEBI consists of 21 items and includes the Personal Mathematics Teaching Efficacy (PMTE) subscale addressing teachers' beliefs in their capabilities to teach mathematics effectively and the Mathematics Teaching Outcome Expectancy (MTOE) subscale addressing teachers' beliefs that effective teaching of mathematics can bring about student learning regardless of external factors (Enochs, Smith, & Huinker, 2000).

Results

A repeated-measures analysis of variance indicated the preservice teachers had significant increases in overall MBI subscale scores. The preservice teachers' beliefs became more cognitively-aligned during the teacher preparation program with these significant changes occurring across the two methods courses with the exception of the CURRICULUM subscale, which increased significantly only during the first methods course. During student teaching the scores on the three MBI subscales remained largely unchanged.

Data from the PMTE subscale revealed the preservice teachers had significant increases in their overall personal efficacy for teaching mathematics. These increases occurred consistently across both methods courses and student teaching. MTOE subscale scores also showed significant overall increases in outcome expectancy beliefs, with this change largely occurring during the semester of the second methods course and remaining essentially constant during student teaching.

Correlations between teaching efficacy beliefs and pedagogical beliefs indicated that at the beginning of the first methods course there were no significant relationships between MTEBI and MBI subscales scores. However, in general, at the end of the first and second methods courses and again after student teaching the MTEBI and MBI subscales scores showed slight to moderate correlations.

An independent samples t-test comparing the preservice teachers' MBI and MTEBI subscales scores at the end of student teaching to the inservice teachers indicated significant differences on all three MBI subscales but not on the MTEBI subscales. The preservice teachers' pedagogical beliefs were more cognitively oriented than inservice teachers and these two groups had similar teaching efficacy beliefs.

Discussion

Although preservice teachers enter teacher preparation programs with relatively well-entrenched beliefs about mathematics teaching and learning (Pajares, 1992), our results suggest that programs can have an impact on those beliefs. We found that during their coursework, preservice teachers developed beliefs more cognitively oriented and hence consistent with a reform perspective and became more efficacious about their skills and abilities to teach mathematics effectively and to influence student learning. Even during student teaching, personal teaching efficacy continued to increase while teaching outcome expectancy and pedagogical beliefs remained stable. It is optimistic that this enculturation experience in the schools did not undermine previous changes. The stability of these beliefs during student teaching seems to suggest that the distinctive features of the teacher preparation program, including two semesters of mathematics methods and time-intensive, developmental field placements, helped in developing well-established beliefs.

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ALTERNATIVE CERTIFICATION IN URBAN SCHOOL DISTRICTS: THE CASE OF THE NYC MATHEMATICS TEACHING FELLOWS

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This research is designed to investigate a major alternative certification program for teachers of mathematics: the New York City Teaching Fellows. Our objectives are to study (i) how the Teaching Fellows program prepares its candidates to teach mathematics, (ii) how the Fellows teach once they are in the classroom, and (iii) the Teaching Fellows understanding of the urban students and families their school serves.

This research project, facilitated by MetroMath - The Center for Mathematics in America's Cities, is designed to investigate the largest alternative certification program for teachers of mathematics in the United States, namely the New York City Teaching Fellows (NYCTF) program. The specific objectives of the project are to study (i) how the Teaching Fellows program prepares its candidates to teach mathematics, (ii) how the Fellows teach once they are in the classroom and how this instruction compares with the NCTM's vision of teaching for understanding, and (iii) the Teaching Fellows understanding of the urban students and families their school serves.

Background and Framework

Evidence indicates that the level of teacher quality is a major factor affecting student achievement (Darling-Hammond, 2000; Wayne & Youngs, 2003). This linkage poses a serious problem for urban areas, since low SES urban schools are more likely to be staffed by less qualified teachers (Levin & Quinn, 2003). Furthermore, since the implementation of the *No Child Left Behind* law and its stipulation that every classroom must have a certified teacher, districts have scrambled, in the context of a nationwide shortage (Ingersoll, 2000), to find qualified mathematics teachers. States and cities have cooperated in recent years to institute a variety of policies to address the shortage of qualified mathematics teacher resulting, most prominently, in various alternative routes to teacher certification.

One particular alternative certification program is the NYCTF Program started in 2000. The NYCTF Program currently supplies over 60% of new mathematics teachers in NYC public schools – c. 300 new mathematics teachers in 2006 alone. Participants in the NYCTF Program currently attend summer courses at one of four colleges in NYC. The following fall they begin teaching full-time in NYC Middle Schools or High Schools while continuing to take classes for two to three academic years in order to complete a Master's degree in Mathematics Education.

Our research questions are:

1. What is the nature of teaching and learning in Math Fellows classrooms? Do the teaching practices of the Math Fellows facilitate the study of, or the engagement of students with, conceptually challenging mathematics?

2. How do the structures of the school environment afford or constrain the Math Fellows professional development and their ability to teach mathematics for understanding?
3. What are the teaching fellows' sense of their own understanding of their urban students' identities, cultures, beliefs, communities and practices? How are these understandings developed and how do they influence the Fellow's mathematics instruction?

By conceptually challenging mathematics we mean mathematical activity emphasizing reasoning, explanation and investigation in which making connections and understanding as outlined by the NCTM (2000). The "how or why" is an integral part of the mathematical activity. In this domain, mathematical meanings are as important as mathematical procedures.

Methods and Data Sources

We examine both the teachers' training under the auspices of the NYCTF program and their experiences working in NYC public schools, using macro and micro approaches to data collection: (i) a broad survey study which examines characteristics of all four university programs and characteristics of a particular Teaching Fellow cohort of approximately 300 pre-service candidates. The initial survey covers a wide range of issues such as the demographic background of the Fellows, their mathematical backgrounds, the influences leading them to teaching, and their expectations of the program. Two follow up surveys focus on the Fellows' classroom experiences, their experiences in their university classes, and how these experiences relate to their background and expectations; (ii) a set of case studies which look in detail at the mathematics instruction and experiences of eight Teaching Fellows as they simultaneously navigate their certification program and their beginning years of teaching mathematics in NYC Public Schools. The primary data sources are fortnightly video observations of their teaching and regular interviews with each Fellosw. This data is supplemented by field notes of the observed lessons, Fellow self-reflection journals of the observed lesson; student surveys; student focus groups; out-of-class observations; and interviews with administrators, mentors, and coaches. This multiplicity of data sources has allowed us to build a detailed picture of each of the eight fellows as they have progressed as novice alternatively-certified urban mathematics teachers.

Results and Discussion

Macro-Lens

The survey data show that somewhat less than half of Teaching Fellows come from, or attended college, in New York State. This is surprising given the recent finding that 85% of all new teachers take their first job within 40 miles of where they grew up (Boyd et al., 2006). The Teaching Fellows program is quite successful at recruiting broadly but, while originally billed as a career-changers program, for most Fellows in our study, this is a first real career after college.

85% of the Math Fellows were not mathematics majors and enter the classroom with limited understandings of post-secondary mathematics. This poses a significant problem to the NYCTF program, the partnering universities that train them, and perhaps even the schools that hire them.

A concern in NYC public schools is the attrition rate of new recruits. 62% of mathematics majors and 47% of non-majors stated that they plan on leaving NYC within five years. However the non-majors are also almost twice as likely to plan on teaching in NYC for 11 or more years.

Micro-Lens

Initial analysis of our case studies data suggests that mathematics instruction in the participants' classrooms is generally teacher-centered and, in most cases, heavily procedural. While teacher-centered, procedural mathematics instruction is typical of U.S. classrooms (Stigler & Hiebert, 1999), it runs contrary to the many progressively minded city and state educational policies (e.g., citywide adoption of reform textbooks, America's Choice whole school reform programs) and to the training the Math Fellows receive in their mathematics methods classes. In all but one of the classes we are observing, we find the surface-level features of reform mathematics instruction (e.g., students work in groups, use of reform-oriented mathematics texts). However, when examined beyond the surface level, the work students do in seven of the eight classrooms is mostly procedural (e.g., notetaking, learning to apply previously taught procedures to similar problems). Observations of classroom discourse and interviews with the Fellows suggest that reform-oriented training, policies, and texts, are being trumped by high-stakes testing and accountability in mathematics. This resonates with Lipman's (2004) finding that teachers who teach low-income urban students of color face acute pressure to teach to the test. With this in mind, it might be considered surprising that one of our eight participants is taking steps to facilitate conceptually challenging mathematics in her classroom.

A second, related, initial finding is that the Teaching Fellows place considerable, perhaps inordinate, emphasis on classroom management. In many of the classes, the teachers' own reflection on the observed lessons finds that they consider class a "success" if the behavior was orderly. An analysis of seventeen in-depth interviews suggests the Fellows feel adequately trained in classroom management but not in mathematics methods or the curriculum.

In terms of our third research question about the relationship between the NYCTF Fellows and their urban students, the interview and survey analysis make it clear that there is considerable social distance between the two groups. The majority of the Fellows are white and Asian, high SES, and from suburbs or small towns, whereas most of their students are African American or Latino, low SES, and urban. In interviews, many - not all - of the participants readily admit their lack of connection to the students and communities they serve. Fifteen of the seventeen interviewees were tracked into gifted or honor programs as children and the majority went to prestigious private universities. Once in the classroom, however, they often find themselves teaching students in lower track classes in low performing schools. The evidence suggests that there is a connection between the large social distance between Fellows and their students and control-oriented mathematics instruction.

Conclusion

The study is ongoing but has already shown that the Teaching Fellows are committed and bright young people and that the program is facilitating cooperation between schools and the University Partners. A clear concern arising from the data is that many of the Teaching Fellows have deficiencies in their mathematical backgrounds, their understandings of urban communities and students, and that those deficiencies seem to be impacting the quality of their teaching and their ability to engage students in conceptually challenging mathematics.

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DEVELOPING FRACTION UNDERSTANDING IN PROSPECTIVE ELEMENTARY TEACHERS

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This study sought to address the question, "how can teacher educators productively challenge, change, and extend what prospective teachers bring to methods courses?" specifically with respect to their understanding of fractions. The results indicated that prospective elementary teachers bring with them to their methods courses a limited understanding of fractions and that experiences in methods courses can change and extend those understandings.

The difficult and complex task of teaching mathematics for understanding is often challenged and complicated by the fact that many teachers' experiences with mathematics as learners result in their limited understandings of important concepts. Prior to beginning education or mathematics methods coursework, prospective teachers have spent years learning mathematics from teachers whose pedagogic practices primarily reflect a traditional orientation focused on procedural understanding. Thus, prospective teachers have difficulties envisioning mathematics teaching for understanding from a perspective that is more aligned with constructivist ideas about learning.

Deborah Ball posed the following question for mathematics educators in a paper presented at the eighth annual meeting of PMENA: How can teacher educators productively challenge, change, and extend what teacher education students bring? This study sought to address this question specifically with respect to prospective teachers' understanding of fractions and was two-fold in its efforts to accomplish this. First, this study examined the understanding of fractions prospective teachers bring with them to methods courses? Second, this study asked whether the experiences and learning opportunities created as part of an intermediate mathematics methods course challenged, changed, or extended what prospective teachers understand about fractions.

Theoretical Framework

In 1988 Kieren reported that middle level students in the U.S. rely heavily on rules and tricks and their rote memory of these to solve fraction problems. Since that time there have been calls for change in fraction instruction to move from procedural instruction to pedagogic practices aimed at developing conceptual understanding (Lamon, 2005; Van de Walle, 2007, NCTM, 2000). However, helping intermediate and middle level students develop a rich number sense about and understanding of fractions rather than relying on procedures and tricks is not easy. Similar to teaching problem solving teachers must foster opportunities and experiences for students to develop a certain way of thinking about fractions. This necessitates that teachers not only need to have a well developed and meaningful understanding of fractions themselves but also pedagogic practices that provide experiences for students to explore and construct ideas about fractions. This research is based on the premise that prospective teachers bring with them to their methods courses a somewhat

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limited understanding of fractions and that a methods course taught from a constructivist orientation may change and extend that understanding.

Methodology

The participants in this study, students enrolled in an intermediate mathematics methods course, included 42 females. Each participant was an elementary prospective teacher in her final year of study and had successfully completed 12 credit hours of mathematics including a course designed specifically for elementary education majors that addresses foundations of numbers (set theory, numeration, and the real number system), number theory algebraic systems, functions and applications, and probability. Additionally, each participant had successfully completed a mathematics methods course focused on grades pre-K through 3.

Participants were asked to respond to the following two short answer tasks:

Question 1: Which fraction is the largest?

Consider the following three fractions:

$$\frac{99}{100}, \quad \frac{6}{7}, \quad \frac{15}{16}$$

Which is the largest? Explain your reasoning.

Question 2: Description of a fraction – What is a fraction?

Suppose a third grader asked you "What is a fraction?" How would you respond?

Data were initially collected over two class periods the week prior to any instruction, experiences with, discussion, or readings on fractions. Post-data were collected at the end of the course following all fraction instruction. Following each data collection point data were analyzed by each researcher independently scoring data as either correct or incorrect. Each explanation provided was also scored as either correct or incorrect and were subsequently analyzed for emergent themes related to participant understandings.

The activities in the intermediate mathematics methods course involved the participants in problem solving with fractions, working with hands-on fraction models, creating and modeling of fraction addition, subtraction, multiplication, and division problems, and ordering fractions. Students also read a variety of articles focused on teaching fractions developmentally, establishing fraction benchmarks, and constructivist approaches for multiplication and division of fractions.

Results

The results of this study indicated that prospective elementary teachers brought a limited understanding of fractions with them to their mathematics methods courses. Analysis of the pre-assessment data collected from Question 1 (Which fraction is the largest?) indicated:

- 20 participants (48%) chose 99/100
- 12 participants (29%) chose 6/7
- 2 participants (5%) chose 15/16
- 8 participants (19%) indicated all three fractions were equal.

Analysis of the explanations revealed that of the 48% who chose 99/100 provided explanations that were often incorrect. For example, "I chose 99/100 because it had the largest numbers." Regardless of the fraction chosen as the largest the explanations revealed that participants relied heavily on part-whole understandings of fractions as they related to area models in order to make their determination. Many participants relying on an area model to make sense of the question and determine their response focused incorrectly on the "part missing from the fraction" and thus chose 6/7 as the largest fraction. Additionally, many who chose 6/7 did so because they were taught in elementary school that "the smaller the denominator the larger the piece." Likewise, of the 19% of participants who indicated the fractions were equal all but one explained that the fractions were equal because "they are all one away from the whole." The results of the pre-assessment data from Question 2 supported the findings of Questions 1 in that participants overwhelmingly understand fractions as "a part of a whole."

The analysis of the post-assessment results indicated that the participants' experiences with fractions in the mathematics methods course had an impact on their fraction understanding. The results of Question 1 indicate:

- 29 participants (69%) chose 99/100
- 9 participants (21%) chose 6/7
- 0 participants (0%) chose 15/16
- 4 participants (10%) indicated that all three fractions were equal.

Analysis of the explanations revealed that of the 69% who chose 99/100 none had faulty or incorrect explanations. Of those who chose 6/7 as the largest fraction most still maintained explanations related to a trick they had been taught in elementary school related to the size of the denominator (i.e. "the smaller the denominator the larger the piece). Those who indicated that the fractions were the same again pointed to the idea that the "fractions were all one away from the whole." The data analysis of the results from Question 2 indicated that while participants still largely relied on part-whole explanations they had expanded their explanations to include drawings, models, or scenarios to further explain what a fraction is.

Discussion

This study revealed that this group of prospective elementary teachers brought with them a limited understanding of fractions to their mathematics methods courses. Although these prospective teachers had many years of school mathematics including four college level mathematics courses and one primary level mathematics methods courses their reasoning about simple fraction concepts was often incorrect and based heavily on misconceptions they had previously developed or, at best, understandings of fractions as part of a whole. Lamon (2005) suggests that a part-whole understanding of fractions is not sufficient by stating "students whose fraction understanding has been limited to part-whole relationships have a very limited understanding of fractions; thus multiple meanings of fraction should be used in instruction." This has important implications for mathematics teacher educators with regard to the kinds of challenges we face in improving not only our prospective teachers' content knowledge but their pedagogic knowledge as well. The second focus of this study implies that the experiences and opportunities we provide for prospective teachers in our methods courses can change and extend their understandings of fractions. It is important however, that these experiences not mimic those that have helped create their misconceptions and limited understandings; the experience should be varied and provide prospective teachers opportunities to explore and develop their own meaningful understandings of fractions.

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EMPOWERING URBAN PRESERVICE MATHEMATICS TEACHERS THROUGH EXPLORATION OF NAEP DATA

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This study examines how exploring NAEP data informs elementary and middle preservice teachers' perspectives of mathematics teaching and learning as it relates to African American students. The preservice teachers' professional attributes are categorized and discussed with respect to the three broad propositions of Culturally Relevant Pedagogy: conceptions of self and others, social relations, and conceptions of knowledge.

The goal of this study was to examine elementary and middle school preservice teachers' perspective of mathematics teaching and learning as it relates to African American students. The study was inspired by the Learning from NAEP Professional Development materials that call for the understanding of the intricacies of assessment data and how such data relate to student learning in mathematics classrooms.

With the No Child Left Behind Act of 2001, there is the pressure for school districts to close the achievement gaps between racial and ethnic groups. According to the 2000 NAEP Mathematics Assessment, 40% of White students achieved proficiency. However, only 10% of Black students achieved mathematics proficiency, making the difference in achievement gap between Whites and Blacks substantially high. Further examination of the NAEP student performance trends from 1990, 1992, 1996, to 2000 reported by Braswell, Lutkus, Grigg, Santapau, Tay-Lim, and Johnson (2001) revealed that the achievement gap between Whites and Blacks has not shrunk; instead the gap remains wide.

In light of the disparities highlighted, and in light of understanding the factors attributed to the disparities, we sought to address the question: "How does exploring NAEP data inform preservice teachers' perspectives of mathematics teaching and learning as it relates to African American learners?"

Theoretical Framework

The Theory of Culturally Relevant Pedagogy (Ladson-Billings, 1995) served as the theoretical framework for examining how exploring NAEP data informs pre-service teachers' perspectives of mathematics teaching and learning as it relates to African American learners.

The pre-service teachers' professional attributes are categorized and discussed with respect to the three broad propositions of culturally relevant pedagogy: conceptions of self and others, social relations, and conceptions of knowledge. Ladson-Billings (1995) asserts that culturally relevant teachers exhibit the following broad qualities with respect to the underlying propositions: (a) Conceptions of self and others indicates that culturally relevant teachers hold high expectations for all students and believe all students are capable of achieving academic excellence; (b) social relations indicates that culturally relevant teachers establish and maintain

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positive teacher-student relationships and classroom learning community as well as are passionate about teaching and view it as a service to the community; and (c) conceptions of knowledge implies that culturally relevant teachers view knowledge as fluid and facilitate students' ability to construct their own understanding.

Methodology

Project participants consisted of fifteen elementary and fifteen middle grades pre-service teachers enrolled in an alternative teacher preparation program. The composition of the pre-service teachers in this study was eight Blacks and twenty-two Whites. Generally, the participants are monolingual, embrace conservative views of schooling, and tend to have limited understanding of cultures other than their own.

In light of the research question, we used a qualitative research design and interpretive approach to data collection and analysis. Observations and written responses of pre-service teachers were the primary data source for the study. Interpretive analysis was used to identify themes and generate descriptions of the pre-service teachers' perspectives of mathematics teaching and learning of African American students.

The thirty pre-service teachers participated in a Gallery Walk that displayed NAEP data and graphs of various demographic groups; they then examined NAEP State data with respect to race, gender, and socioeconomic status (SES). Through the activities, the pre-service teachers were asked to respond (orally and written) to a variety of questions related to equity issues as well as to reflect on the perspectives of their own mathematics teaching and learning as it relates to African American students. Data collection included (1) field notes of reactions to the Gallery Walk and (2) written narratives in response to the variety of questions related to equity issues.

Results and Discussion

Analyses of the data indicates that engaging in the exploration of NAEP data encouraged pre-service teachers to consider and articulate their professional ideological stance related to African American students and mathematics teaching and learning. The following narrative excerpts reflect key findings of the study.

Conceptions of Self and Others

Who Am I and What Do I Believe: Perceptions of themselves as teachers of mathematics.

- "I was shocked to see the level of disparity between the math achievement gap of minority and non-minority students. I am motivated to make a difference in my classroom."
- "Using this data will cause me to reflect on my own practices as they relate to these students and attempt to discover how I can best help them to be more successful not only on math tests, but in the classroom arena and ultimately in life."
- "Since my cultural background is different from that of students coming from an urban setting, additional work on my part will be necessary to become a successful math teacher of African Americans students."

Pygmalion in the 21st Century: Perceptions of African American students as learners of mathematics.

- "Exploring this data solidifies my understanding of the math achievement gap and the need for high expectations and culturally responsive teaching."
- "As I look at the NAEP data I search for reasons why African American children are falling farther and farther behind in mathematics.

Running a Tight Ship: The nature and structure of the mathematics learning environment.

- "I need to utilize various methods to assess my students and use those findings to inform and differentiate math instruction.

Social Relations

Can't We All Get Along: The nature of teacher-student and student-student interaction and relationships within the mathematics learning environment.

- "It is critically important that math teachers be aware of the stereotypes that insist and not knowingly or unknowingly perpetuate those stereotypes."

Conceptions of Knowledge

From Cookie Cutter to Constructivism: Notions of constructing and assessing mathematical knowledge.

- "Highlights the need for math teachers to utilize various instructional strategies and provide support to students as they try to make sense of mathematical concepts."
- "While I understand the intent, I question the ability for a standardized test to truly measure the mathematical abilities of the student."

The Death of the "Urban Legend": Conceptions of mathematical power and competence; building upon students' informal mathematics experiences, lives, and interests.

- "This data highlights something I already believed in – the importance of culturally responsive math teaching that draws upon what students bring to the table and build on it. Every child's experiences, regardless of race, are equally valuable and important."
- "Reinforces the importance of trying to make math lessons more relevant for students."
- "In the urban school environment I will need to employ teaching techniques that are culturally relevant and tap into the cultural capital that African American students bring to the classroom."
- "Getting to know your students is critical. The more you learn about your students, the more capable you are of developing culturally relevant math tasks that have meaning to them.

The data suggest that critically looking at NAEP data prompted the pre-service teachers to examine their views related to instructional planning, decision making, and implementation as it relates to diverse learners. Their reactions and narratives highlighted concern for creating an equitable learning environment in which ALL students - regardless of race, ethnicity, gender, socioeconomic status, language, learning disability, or other characteristics successfully realize mathematics achievement. The pre-service teachers asserted that creating a culture of mathematics achievement for ALL students requires equity-explicit knowledge, skills, and dispositions, and began to consider how they might develop and exhibit those attributes. Overall, exploring NAEP data functioned as a catalyst for positively informing pre-service teachers' perspectives of mathematics teaching and learning as it relates to African American learners.

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KNOWLEDGE OF CHILDREN'S LEARNING OF MATHEMATICS: A COMMON DENOMINATOR IN PRESERVICE, TEACHER, AND PARENT EDUCATION

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Preservice elementary teachers' learning of mathematical content, teachers' instruction in mathematics, and parents' work with their children on mathematics outside of school can all be enhanced through a focus on how children learn and understand mathematics. This paper will describe our efforts with all three audiences to improve the mathematics education of children by focusing on children's mathematical thinking.

Preservice elementary teachers' learning of mathematical content, teachers' instruction in mathematics, and parents' work with their children on mathematics outside of school can all be enhanced through a focus on how children learn and understand mathematics. This paper will describe our efforts in working with all three audiences to enhance the mathematics education of children by focusing how children learn and understand mathematics. For teachers and preservice teachers this understanding is part of the knowledge necessary for teaching mathematics (Hill & Ball, 2004). Parents can also benefit from specialized knowledge of mathematics when working with their children.

Theoretical Framework

Hill, Rowan, and Ball (2005) found that teachers' mathematical knowledge for teaching was positively correlated with student achievement. They hypothesized that "knowledgeable teachers may provide better mathematical explanations, construct better representations, better "hear" students' methods, and have a clearer understanding of the structures underlying elementary mathematics and how they connect" (p. 401). The mathematical knowledge necessary for teaching is more than mathematical content knowledge; it is also the knowledge of how children learn and think about mathematics. Teachers must know how to explain a mathematical concept to those that have not yet constructed an understanding of the concept, and in so doing, it is essential they understand how the student might be thinking mathematically, what explanations might make sense to the child, and in addition they must learn to make sense of students' explanations and solutions. This type of knowledge is predicated upon knowledge of how children learn and think mathematically. Our focus is to explicitly connect this knowledge with preservice teachers' learning of mathematics, teachers' instructional practice, and, in a similar way, parents' work with children.

Often teachers and preservice elementary teachers' mathematical knowledge is disconnected from what they are teaching. Yet true experts in any field do not view knowledge as isolated facts but rather as contexts of applicability (Bransford, Brown, & Cocking, 1999). The National Council of Teachers of Mathematics Research Companion states, "Children's thinking of mathematics needs to be the center of mathematics instruction" (2003, p. 49). Research has shown that one factor that influences children's learning of mathematics is how well teachers base instruction on children's ways of thinking (NCTM 2003). Our aim is to encourage both experienced and prospective teachers to organize mathematical knowledge around the context of how children learn mathematics and link that understanding to how they will be teaching mathematics to children.

Elementary teachers, and especially preservice teachers, are typically unaware of the research on how children learn and understand mathematics. Despite efforts to connect research and practice, teachers are often resistant to reform efforts that focus on translating research into practice. This resistance is due in part to the complexity of the classroom and external pressures such as those created by standardized testing. Our materials focus on children and how they think about mathematics and are specifically written for practicing teachers, preservice teachers, and parents.

Connecting mathematics to children's learning of mathematics will better enable parents to effectively help children learn mathematics outside of the school setting. Primary caregivers are often ill prepared to work with children in mathematics. They frequently lack the mathematical knowledge to help their children with their homework. Some parents are particularly frustrated in math because of their own limited understanding and inability to explain mathematics to their children. Further, they often do not understand how the children are thinking about mathematics. By knowing how children think mathematically, they can make appropriate interventions or provide learning opportunities. Mathematics requires a conceptual or relational understanding in order to teach and tutor children. Children whose parents are actively involved with their children in school achieve more than children of parents who are less involved.

Methods

To test our hypothesis that knowledge of children's mathematical thinking could be used with preservice teachers, teachers and parents both quantitative and quantitative methods were used. During the spring semester of 2006, data was collected in two phases: 1) initial pre and post efficacy data from 41 pre-service teachers enrolled in two sections of Mathematics for Elementary School Teachers at a small Midwestern University; 2) Data from pre-service teachers enrolled in treatment and comparison classes in seven sections of a *Mathematics for Elementary Teachers* course from four Midwestern Universities of varying sizes. Questionnaire data were collected using four subscales consisting of Likert-style items about: beliefs about mathematics and its teaching; efficacy in understanding and teaching mathematics; and knowledge of children's thinking about mathematics. The questionnaire subscales described below have been tested psychometrically using pilot data.

The qualitative methods consisted of journaling and interviews. Both teachers and parents were asked to keep ongoing journals of times when they considered or used information from the project materials in their teaching. They were asked to specifically describe how they used this information, when they did something differently or tried something new based on an idea from the text. They were also asked to describe any instance where a child's mathematical thinking was like or unlike that described.

Results

Preservice Teachers

Statistically significant results on all pre and post intervention differences indicate that participants felt more confident in all content and teaching areas after using materials that focused on children's mathematical thinking in their content course than before. Results indicated that prospective teachers feel significantly more confident about their own mathematics ability and their ability to teach elementary mathematics to children after taking a course that focused on the children's thinking approach. Before the course, participants were more confident about their own ability than their ability to teach it to children. After the course, this confidence gap narrows. Interview data was used to triangulate these results.

- *Beliefs about Mathematics and Its Teaching* (13 items, Cronbach's Alpha =.73). *Knowledge of Children's Mathematical Thinking* (18 items, Cronbach's Alpha = .81)
- *Efficacy to Understand Subject Matter* (6 items pre/post course, Cronbach's Alpha =.90)
- *Efficacy to teach Subject Matter* (6 items pre/post course, Cronbach's Alpha = .88)

Teachers

In our qualitative analysis of the interviews and journaling, teachers learned more about how children think mathematically. In an interview, a middle school teacher, who also taught math content courses for elementary teachers, said:

> One specific thing that I learned had to do with the concept of ten. Until reading about this in your materials, I had assumed that the concept of ten as a unit was pretty easy for students. I'd seen many suggestions for using manipulatives (base-ten blocks or an equivalent), but I'd never seen anything suggesting that even with the manipulatives it's a big step for children to think of ten as a unit.

Parents

A parent of a sixth grader noted that the materials changed how she helped her daughter with her mathematics homework. She found herself more willing to let her daughter experience mathematics rather than just telling her how to do it, and perhaps more significantly she began to realize that her way of helping her daughter may not be the best approach. In her journaling, she bulleted the following changes:

- restraining myself to not "jump in" and allow her to experience problem solving for herself
- becoming more open to using other resources/tools to assist her, as "my way" of thinking/teaching, may be more harmful than helpful to her

Discussion

Our findings to date indicate that focusing on knowledge of children's mathematical thinking raises prospective teachers' efficacy to understand and teach mathematics as well as having an impact on their beliefs about mathematics and its teaching. There is initial qualitative evidence that using this approach also influences teachers and parents in their teaching and work with children.

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PEDAGOGICAL CONTENT KNOWLEGDE "IN-ACTION"

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This study builds on past research related to pedagogical content knowledge (PCK) to create a working list of mathematically-oriented pedagogical practices (MPPs) for researchers to use to study PCK in action. This list will be of use to both researchers and practitioners in the advancement of secondary mathematics teaching.

Secondary mathematics teachers are expected to be competent in a variety of practices in the classroom; however, their success as teachers has often been measured by the knowledge of teaching they hold in their minds. A disconnect exists between what is being measured in research studies and what organizations such as the National Council of Teachers of Mathematics (1991, 2000) expect them to perform in their classrooms.

The need to learn more about how teachers learn these practices as well as their success with these practices necessitates an articulation of these particular instructional approaches. This study builds on past research related to the cognitive construct pedagogical content knowledge (PCK) to create a working list of mathematically-oriented pedagogical practices (MPPs) for researchers to use to study PCK in action. This list will be of use to both researchers and practitioners in the advancement of secondary mathematics teaching.

Framework

The cognitively defined construct of PCK has been characterized as a particular type of subject matter knowledge specific to the field of education. It is this knowledge that is thought to help teachers communicate their subject matter knowledge in a manner that makes sense to their students (Shulman, 1986, 1987; Wilson, Shulman, & Richert, 1987). The backbone of PCK stems from a teacher's procedural, conceptual, and overall understanding about the field of mathematics and how to convey the important ideas of the subject to others. This includes the recognition of misconceptions of certain concepts and methods for alleviating these misconceptions so learning can progress beyond them.

There appears to be consistency in the way PCK is characterized in the mathematics education community (Ball, Lubienski, & Mewborn, 2001). The components of PCK have been drawn from Shulman's (1986, 1987) characterization, but significant progress has not been made on the expansion of these notions. Shulman's framing of PCK has been used in research with few if any modifications. The lack of forward movement with the construct of PCK leads to a broad characterization of PCK and a lack of research on the specific entailments of this construct and the practices related to this construct as applied in the classroom. Furthermore, since

Shulman's identification of PCK occurred before the reform movement, the characterization is more teacher-centered rather than student-centered. Hence, when MPPs were delineated in relation to this study they were designed to fit a more reform-oriented classroom.

Method

An initial list of MPPs was developed from literature on PCK to examine PCK in action. These practices were examined and expanded upon in a Teaching Algebra Seminar through intensive observations and analysis. Observations were made approximately three times per week for one 15 week semester. Field notes and video recordings were taken during each observation. Data were analyzed using standard methods of qualitative research (Patton, 2002).

The Teaching Algebra Seminar consisted of 11 pre-service secondary mathematics teachers (PSMTs) and a mathematics faculty member. Each PSMT taught his/her own section of college algebra at a large, Midwestern university. The Teaching Algebra Seminar met each day the PSMTs taught College Algebra (typically three times per week). In this seminar, PSMTs discussed any concerns from their teaching that day. They also discussed the teaching of the upcoming lesson. This venue was used to study and create a working list of MPPs which could be studied by researchers and reflected upon by practitioners.

Results

The delineation of MPPs provided a list of practices that align with current reform movements (e.g., NCTM 1989, 1991, 2000; NRC, 2001). These practices are not fixed, but rather serve as a starting point for categorizing teaching practices that demonstrate a firm understanding of mathematics coupled with the ability to teach in a student-centered manner.

Results of the study determined eight practices fit the criteria to be considered MPPs: (a) facilitating mathematical discourse, (b) helping students create powerful examples and representations, (c) responding to students' questions, (d) analyzing curricular materials, (e) analyzing teaching and planning, (f) understanding students' thinking, (g) addressing the needs of all learners, and (h) creating mathematical assessments. Each of these practices is important for reform-oriented teaching of secondary mathematics. The Teaching Algebra Seminar served as a place to observe discussions surrounding these practices. By observing and analyzing PSMTs discuss their teaching practices, a list of teaching practices related to PCK in action was formed. These MPPs will be discussed in detail during the short oral presentation.

Discussion

Intertwining mathematics and pedagogy, these practices serve as a way to observe what teachers know about mathematics instruction from a situative perspective of learning. Researchers, mathematicians, and teachers can all benefit from a working list of reform-oriented teaching practices. This list of MPPs will provide a common ground for the research of MPPs. The observation of teachers with an eye to certain practices will help refine, modify, and better understand MPPs. Ultimately the findings from studying MPPs can be taken into consideration when designing teacher education programs improving teaching and teacher learning.

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PREPARING PRE-SERVICE TEACHERS TO MODEL MATHEMATICS WITH TECH-KNOWLEDGY

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This paper discusses the learning experiences of pre-service teachers in a project called Tech-knowledgy for Building Mathematical Knowledge, where they explored mathematics concepts using various models and representations via virtual manipulatives. As teachers learned how to select and evaluate effective virtual manipulatives and incorporate them meaningfully in teaching mathematics, they gained confidence in their ability to model mathematics using technology.

Integrating technology effectively in the content areas can give students access to advanced concepts, higher ordered thinking and sophisticated learning. In mathematics, that means using technology as “tech-knowledgy”, as a tool to construct mathematical knowledge. Technology in mathematics allows teachers and students to represent abstract mathematics concepts that may be difficult to illustrate and gives access to opportunities for rich learning experiences and meaningful class discourse.

Theoretical Framework

Today, our children are growing up in a technology advanced society where working flexibly and thinking critically with technology while solving problems is an increasingly important skill. Jonassen (1996) defined computers as mind tools that should be used for knowledge construction while engaging learners in critical thinking about the content they are studying. The NCTM *Principles and Standards* stated, “Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances student learning” (p. 24). According to this document, technology supports effective mathematics teaching when teachers create appropriate mathematics tasks that capitalize on the strengths of technology, which are the ability to graph, visualize, simulate and compute. The worldwide web offers teachers a wealth of mathematics technology resources and websites with a variety of virtual manipulatives. With the abundance of technology resources, teachers’ new challenge is learning how to be selective in choosing the best instructional tools and creating a learning environment to effectively integrate technology for learning. Teachers need to rethinking and shifting their limited views of technology as a product yielding tool to a more powerful “mind tool” to construct knowledge. This implies that teachers must learn and experience for themselves how to select, evaluate, design, teach and learn using these innovative tools. Providing experiences for pre-service teachers to re-learn fundamental mathematics concepts using models and modeling can help them be more confident and competent to teach elementary mathematics.

Methods

Research Question

This research was conducted to explore how preservice teachers' experience with Tech-knowledgy for Building Mathematical Knowledge impact teachers' confidence and attitude about teaching mathematics using technology.

Participants

The participants in this study were twenty-one pre-service teachers in the Elementary Mathematics Methods classroom. These preservice teachers enrolled in the mathematics methods course met each week for three hours and were assigned to an elementary classroom in local schools during the semester to observe and teach lessons.

Data Sources

The data sources used for this study included qualitative and quantitative data. The descriptive data included surveys called Prior Use of Manipulatives and Technology with likert scales, pre and post survey on Confidence in Teaching Mathematics and Integrating Technology in Mathematics, and a final survey on attitudes towards Mathematical Models and Technology Integration. The qualitative data included open-ended survey questions, teachers' reflection on technology integrated mathematics lessons, class discussion and teacher interviews.

Procedure

The Tech-knowledgy for Building Mathematics Knowledge project involved three processes. In the first process, pre-service teachers explored mathematical models on the computer through virtual manipulatives and discussed the mathematics involved in the task. In the next process, teachers had individual assignments where they were to select and evaluate at least four mathematical models available online and discussed the instructional value and effectiveness of the models with their peers. Most importantly, teachers engaged in a rich discussion about how to structure the lessons to use the mathematical applet as a mind-tool for students to construct important mathematical knowledge. In the final process, pre-service teachers incorporated a technology mathematics applet to plan and teach a lesson in an elementary classroom. Finally, they reflected on the lesson and shared their learning with their peers.

Results

Results from the Prior Use of Mathematics Manipulatives and Technology in mathematics survey showed that teachers had some exposure to mathematical tools used for modeling in their learning and/or teaching experiences. The manipulatives that they were most familiar with were the calculators, play money and measurement tools used for length, volume, weight and temperature. They reported having some familiarity with base ten models, color chips, geometric solids, pattern blocks and computer mathematics games. Not surprisingly, these pre-service teachers had no prior knowledge of virtual manipulatives. Teachers commented that when they were in school, manipulatives were not often used and most of their learning was procedural with drills on basic skills. By the end of the course teachers' confidence in teaching mathematics and using technology in teaching mathematics showed a marked increase with majority of the teachers saying that they were feeling more confident teaching mathematics and using technology to teach mathematics in the post-survey than in the pre-survey.

The results from the Mathematical Models and Technology survey showed that all teachers agreed that students can learn more mathematics deeply with the appropriate use of technology and that technology allows effective ways for conceptualizing and modeling mathematical ideas (See Table 1).

Mathematical Models and Technology

	Strongly agree	Agree	Unsure	Disagree	Strongly Disagree
Statement 1.Students can learn more mathematics more deeply with the appropriate use of technology.	43%	57%			
Statement 2. Technology tools can provide visual models that many students are unable to generate independently.	71%	29%			
Statement 3. Technology in the mathematics classroom is best used for remediation, or reinforcement of skills.			29%	57%	14%
Statement 4. Technology in the mathematics classroom is best used to promote students' analytical, creative, and other higher ordered thinking skills.	14%	29%	29%	29%	
Statement 5. Technology allows more time for conceptualizing and modeling mathematical ideas.	29%	57%	14%		
Statement 6. Technology offers teachers options for adapting instruction for individual needs of students.	71%	29%			
Statement 7. Teachers can learn more ways to model mathematics concepts by using virtual manipulative applets.	57%	43%			

Table 1. Results of Mathematical Models and Technology survey

The themes that emerged from the analysis of teacher lesson reflections and in class discussions were that virtual manipulatives were effective in that they offered

- Visualization that provides a link between concrete and abstract
- Strategy for students to use for collaborative knowledge construction
- New (nontraditional) ways to model math ideas
- Ease of differentiation and tiered learning
- User-friendly tasks with visuals for English language learners and visual learners

- Interactive tools to entice special needs learners who could not focus on work with manipulatives.

Conclusions

Modeling mathematics concepts can be easily facilitated by the use of technology and teachers and students benefit from the many affordances unique to this learning environment. For example, students can have access to visual and dynamic mathematical models to learn mathematics. Teachers can use technology to represent multiple models and illustrate abstract mathematics concept with ease while offering differentiation and scaffolding for diverse learners. Preparing teachers to consider unique design features for planning effective integration of technology in mathematics is critical. One of these considerations is to allow students to work collaboratively when exploring mathematics with virtual manipulatives to give opportunities for communication of mathematical ideas. In addition, teachers need to pose critical thinking questions that allow students to conjecture, test and prove their thinking and explore relationships and patterns in mathematics while using these tools. Engaging in productive mathematical discourse is critical to revealing the important mathematical learning that occurs with technology. Teaching with technology has promising outcomes for our students' learning and can support meaningful knowledge construction.

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PRESERVICE MATHEMATICS AND SCIENCE TEACHERS' EVIDENCE-BASED INQUIRY INTO INTERNET LITERACY PRACTICES

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Reform efforts in the teaching and learning of mathematics have included an emphasis on both verbal and written communication (National Council of Teachers of Mathematics, 2000). An important component of communication for mathematical sense-making is literacy. Literacy—even when narrowly considered as "reading and writing"—cannot be separated from broader perspectives and contexts of language (Gee, 2001). For example, within inquiry-oriented mathematics instruction, reading has been recognized as providing a "mode of learning" rather than merely "a set of skills for extracting information from text" (Siegel & Fonzi, 1995, p. 635). However central literacy might be to teaching and learning, secondary preservice teachers (PSTs) often tend to be dismissive of efforts to incorporate practices that focus explicitly on literacy (O'Brien, Stewart, & Moje, 1995). Many PSTs perceive content area literacy strategies to be merely a collection of technical terms and tools that are essentially "an additional burden on an already overloaded instructional agenda" (p. 448), rather than as a vehicle for supporting content area learning. This paper describes a small-scale intervention study of secondary mathematics and science PSTs and their learning about both print and *new literacies of the Internet* (Leu, Kinzer, Coiro, & Cammack, 2004) within the context of subject area methods classes, rather than in a traditional "reading in the content area" course. The research is part of a larger project funded by the *Carnegie Corporation's Adolescent Literacy Preservice Initiative* that joins the perspectives of mathematics and science educators with literacy experts, focusing on improving the capacity of teachers to enact research-based pedagogy in foundational print literacy as well as the *new literacies of the Internet.*

Background

Drawing upon work from linguistic and cultural traditions, Moje, Young, Readance, and Moore (2000) have called for an expanded definition of literacy practices. This expanded definition includes attention to *new literacies*, rooted in the Internet and other digital technologies (Leu, et al., 2004). These new literacies have rapidly commanded an increasing part of our daily lives—both in and outside of schools. Researchers have begun to study the practices and pedagogy integral to new literacies—that is, high-level forms of comprehension and composing skills that are required when using the enormous information potential of the Internet for learning. Reading comprehension on the Internet, for instance, requires richer and more complex skills than simply reading a book. Reading search engine results, critically evaluating the veracity of online information, tracking information through a sequence of inferences made at various hyperlinks, communicating clearly and efficiently via email, and many other new reading and writing skills are required online (Leu, et al., 2004).

New literacies are clearly relevant to mathematics and science teachers who use the Internet both for their own learning of content, as well as for incorporating online learning opportunities for their pupils. As instructors of mathematics and science methods courses, we were particularly interested in these issues as they pertain to our own efforts to prepare PSTs to make use of literacy skills as part of a broader sense-making context for learning and teaching of mathematics and science.

Methods and Procedures

As part of their content-specific methods course, 11 mathematics and 13 science PSTs conducted a scaffolded investigation into the literacy practices of middle and high school pupils in their practicum placements. The investigation was designed to elicit elements of both the traditional and new literacy practices of their pupils, as well as to provide the opportunity for the PSTs to conduct analyses of these practices and reflect on their learning.

The task was composed of three parts: (a) traditional literacy prompt, (b) new literacy prompt and (c) written analysis of the data collected in parts (a) & (b). The traditional literacy prompt (printed on a sheet of paper) included a paragraph of informational text and an associated line graph that were selected for content that engaged both mathematics and science subject matter. The pupil was requested to "think aloud" (Ericsson, 1993) as he/she read the paragraph and examined the graph. In the second part, the new literacies prompt, the PST asked the pupil to use the Internet to find additional information related to the text in part (a), while also "thinking aloud." The PSTs recorded the pupils' responses using data record sheets provided with the task. Each PST then wrote an analysis paper of his/her investigation, which addressed questions such as: What new skills and strategies are required while reading from the Internet? How are these different from print reading? What skills did you observe? What can you conclude about whether "new literacies of the Internet" really exist? (The prompts, protocols, and data record sheets are available for download at http://www.newliteracies.uconn.edu/carnegie/index.html.)

The PSTs' analysis papers and data record sheets were coded using the qualitative data analysis software NVivo 7. Initial coding schemes were derived in part from the literature related to print and new literacies and also through open coding of emerging themes. The researchers employed axial coding to make connections among the categories and to refine the coding schemes (Strauss & Corbin, 1990). Individual cases were set up with key identifiers related to both the PST and his/her participating pupil. Coded passages were organized and analyzed through use of queries. These data were organized in tables to facilitate analysis within and across cases. While our initial interest was to examine the text for evidence of new literacy skills and practices, the analysis raised a number of interesting questions about the ways in which PSTs view literacy practices more broadly, as well as the ways in which they construct evidence-based arguments from practice.

Results and Discussion

Examining the PSTs' responses to the tasks, we focused first on the nature of the impressions of particular literacy strategies (i.e., what did the PSTs notice and value about their pupils' performance?). The analysis of the investigations indicated that the PSTs did indeed find substantive differences in the literacy practices of pupils for traditional (print) literacies and new literacies of the Internet. For example, they noted qualitative differences in how their pupils attended to and negotiated print literacies and Internet literacies (e.g., print text tended to be read in a more linear fashion; whereas, Internet text appeared to be constructed by the reader from multiple sources, including greater evidence of skimming, considering the source of the information, etc.). The PSTs noted qualitative differences in how their pupils critically evaluated print literacies and Internet literacies (e.g., print text was typically perceived as having more "authority" than Internet text; pupils tended to be more skeptical of Internet sources).

The implementation of the scaffolded investigation task required the PSTs to collect data of student thinking and make evidence-based arguments. Because typical mathematics and science PSTs tend to be dismissive of efforts to incorporate practices that focus explicitly on literacy (O'Brien, et al., 1995), this work is significant because it not only helped to establish a tangible connection between literacy practices and learning in the content areas for these PSTs, it also helped build their capacities to enact evidence-based inquiries into their own teaching practices. Additionally, the evidence-based arguments provoked the PSTs to express opinions, thoughts, and conjectures about literacy that went beyond a "skill level" perception of literacy that might be expected from typical mathematics and science PSTs (O'Brien et al., 1995). Indeed, there appeared to be some awareness of literacy as a "mode of learning" (Siegel & Fonzi, 1995) that may enhance understanding of mathematics and science content.

This study provides teacher educators with both insights and tools for better understanding potential connections between literacy practices and content-learning. In particular, by pairing traditional literacy approaches with investigations in new literacies, implications for literacy instruction for secondary PSTs were uncovered. For example, we recognized that, because of the strong textual authority inherent in most mathematics and science texts, PSTs tend to look *through the text* as if it is transparent—focusing solely on the mathematics or science content. As a result, PSTs may not notice textual processing that is necessary for their pupils' understanding of the content itself. However, when the PSTs focused on the new literacies, they reported complexities in how their pupils processed the text. For example, they noted that the text was processed recursively with a kind of dialogue between the reader and the online sources. As complexities in negotiating and making sense of the online text were noted, the print text and its necessary processing became more visible to the PSTs. This provoked a recognition that a dialogue of sorts must exist between the reader and *any text* for meaning to be made (Bakhtin, Holquist & Emerson, 1986). This recognition is an important step for PSTs as they learn to help their students make meaning of texts and associated content. In sum, evidence-based inquiry with explicit attention to both print and new literacies of the Internet suggests promising avenues for influencing knowledge and dispositions of mathematics and science PSTs. As we continue to learn about the demands and opportunities that the Internet and other digital technologies create for our students, we seek to engage the rich set of questions that this work generated for us as mathematics and science teacher educators.

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PRE-SERVICE SECONDARY MATHEMATICS TEACHERS' EXPERIENCES RELATED TO REFORM-ORIENTED PRACTICES OF TEACHING MATHEMATICS

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Researchers and mathematics educators are trying to understand which learning experiences will help pre-service teachers develop visions of teaching and learning, as well as practices that align with reform-oriented movements (Shulman & Shulman, 2004). The purpose of this research is to contribute to the understanding of educational experiences that will afford PSMTs the learning of practices for teaching secondary mathematics.

Upon completion of their undergraduate degrees, pre-service secondary mathematics teachers (PSMTs) are being held to new expectations, which in turn brings into question the experiences provided to them as part of their undergraduate education (Conference Board of the Mathematical Sciences [CBMS], 2001; Darling-Hammond, 2000; Floden & Meniketti, 2005; National Research Council [NRC], 2001). Current reform efforts expect teachers to demonstrate practices that extend beyond directly telling their students how solve mathematical problems. Teachers are encouraged to instruct in a non-traditional fashion, with a focus on student-centered teaching (National Council of Teachers of Mathematics [NCTM], 1989, 1991, 2000).

Educating secondary mathematics teachers to learn these reform-oriented practices is not an easy task. The majority of PSMTs enter the field of education with at least 13 years of their own formal educational experiences, most of which conform to traditional teacher-centered ways of learning. Researchers and mathematics educators are trying to understand which learning experiences will help pre-service teachers develop new visions of teaching and learning, as well as practices that align with reform-oriented movements, to bring into their future classrooms (Shulman & Shulman, 2004). The purpose of this research is to contribute to understanding the educational experiences that will afford PSMTs the learning of such practices of teaching secondary mathematics.

Theoretical Framework

It has been recognized that there are competencies that can be learned for the practice of teaching mathematics (CBMS, 2001; NRC, 2001). Past research has focused on measuring knowledge as a construct existing only in the mind rather than evaluating a teacher's ability to apply practices to the classroom setting. For teaching, this knowledge has been termed pedagogical content knowledge (PCK) by Shulman (1986). Shulman claimed that for teaching mathematics PCK is at the intersection of pedagogy and mathematics and is the knowledge specific to the teaching of mathematics. Although the term, PCK, has allowed researchers a way to explore what teachers know about mathematics in order to teach the subject, examining what a teacher knows as something that exists in the mind can be problematic.

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.

This method for measuring what teachers know, does not take into account specific roles of teachers in the classroom interacting with students. The practices teachers are being held accountable in their classrooms are also placed in the background when examining what a teacher knows in such a manner. Measuring knowledge as a construct of the mind takes away the context of the classroom and focuses on knowledge devoid of context. A focus on *knowing in action* rather than knowledge as something that exists in a teacher's mind will allow researchers to gain a better understanding of what teachers are prepared to do in the classroom (Putnam & Borko, 2000). If indeed these practices are expected of teachers upon completion of their undergraduate coursework, the next logical step is to understand how PSMTs learn these practices.

This study examines the learning experiences afforded to a cohort of PSMTs and how these experiences contribute to developing practices related to the cognitive construct of PCK. The practices examined are termed *mathematically-oriented pedagogical practices* and are examined using the situative (Greeno, 2003; Lave & Wenger, 1991; Wenger, 1998) perspective of learning. Changes in PSMTs' participation in relation to mathematically-oriented pedagogical practices are examined.

Methods

A cohort of PSMTs (n=11) was purposively selected to participate in this study, as their experiences in a Teaching Algebra Seminar may shed light on educational experiences that help PSMTs learn mathematically-oriented pedagogical practices.

The Teaching Algebra Seminar was designed specifically to provide PSMTs with the experience of teaching their own semester-long college algebra course. PSMTs' teaching is accompanied by a tri-weekly seminar to reflect on their experiences with peers who are also engaging in similar experiences. Shifts in PSMTs' participation in relation to mathematically-oriented pedagogical practices as they develop as teachers in this Teaching Algebra Seminar are examined in this study. Data collection included multiple modes of inquiry: (a) pre- and post-course questionnaires, (b) interviews with the focal PSMTs, and (c) video-tapes and field notes from the observed Teaching Algebra Seminar sessions. Data were analyzed using processes of coding (Miles & Huberman, 1994) to reveal emergent themes of the aspects of the course that supported PSMTs' learning of mathematically-oriented pedagogical practices. As the learning of mathematically-oriented pedagogical practices emerged, I examined these practices to determine the changes in PSMTs' participation over the course of the semester.

Results

Over the course of the semester, the questions posed by PSMTs in the Teaching Algebra Seminar changed from focusing on course managerial concerns to those related to the practices of teaching mathematics. Aspects of the seminar that shed light on ways to strengthen the education of secondary mathematics teachers will be discussed.

This change appeared to be supported by the collaborative learning experience of the Teaching Algebra Seminar. PSMTs had the opportunity to practice and discuss the teaching of mathematical topics before they were taught, as well as reflect upon their actual teaching experiences in the classroom. Each PSMT and the professor brought his/her own experiences of teaching the course to the forefront to try to resolve pedagogical, mathematical, and pedagogical content issues ranging from choosing appropriate examples to making decisions regarding when a detailed mathematical explanation/proof is appropriate in teaching the algebraic concepts.

Discussion

The results of this study are informative to the mathematics education community in recognizing educational experiences that help PSMTs learn reform-oriented practices for teaching. The shift in PSMTs' discussions in the Teaching Algebra Seminar over time brings to the forefront the issue that their concerns related to classroom management issues may need to be addressed before they are able to focus on practices related to teaching mathematics. Teacher education programs can be strengthened by incorporating particular aspects of the Teaching Algebra Seminar that appear to help PSMTs improve upon practices which are deemed necessary for teaching secondary mathematics.

For instance, activities that occur in the seminar such as "learning through collaborative reflection" have been proposed in the *Principles and Standards of School Mathematics* (NCTM, 2000) document as a method for practicing teachers to improve their practice. An understanding of how the PSMTs' discussions related to their teaching practices evolve throughout the course could help researchers learn more about the value of these types of learning experiences.

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PRESERVICE SECONDARY TEACHERS' TECHNOLOGY EXPERIENCE IN A CARIBBEAN CONTEXT

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This study examines the experiences and perceptions of preservice mathematics teachers in a Caribbean context as they explored with technology in their practices. Despite facing similar challenges of developed countries when new qualified teachers apply new technologies, they found the benefits for students and teachers were phenomenal in stimulating interest, motivation and improvement in their students' mathematics performance.

In considering the economical concerns in the Caribbean region, it is critical that attention be given to the lessons learned from prior research in developed and developing countries. In particular, teacher attitudes, concerns about not having up-to-date equipment or faculty with technological expertise, the demands from school districts, parents and students to use CT in their classrooms, teachers' motivation, classroom management, teachers' role, support systems, and computer coping strategies (Ropp, 1999) will assist in the strategic and practical plan for the move forward. The lack of detail in the literature regarding computer technology (CT) integration in mathematics instruction in the Caribbean and the continued move toward mathematics reform are of concern and instrumental in the investigation of: "*What are preservice secondary school mathematics teachers' experiences within the junior high school classrooms as they explore the use of CT in their instructional practices in an English-speaking Caribbean country?*" This study is significant in providing contextual evidence of challenges and successes with the use of CT as PSSM teachers move forward in mathematics reforms. With the cost of the graphing calculator and Internet becoming less expensive and easier accessible in developing countries, the choice of mathematics computer software, Math Trek; the graphing calculator, TI-83Plus; and the Internet were utilized in this study.

Theoretical Framework

Blume's (1991) three-phase model provides mathematics teachers with opportunities: (1) to learn with the use of technology, (2) to reflect on their learning in the technological environment, and (3) to translate their own encounters that were facilitated by their instructor in a technology-rich environment, into similar encounters for their students. On the other hand, Senge's (1990) model consists of five learning disciplines: personal mastery, mental models, shared vision, team learning, and system thinking. Wetzel (1997) noted that when learning opportunities are provided for teachers to use CT in their instruction, they used the opportunity to effectively implement the technology in their instructional practices. He also found that providing a learner–centered approach (Huerta-Macias, 1993) and a focus on learning rather than teaching

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.

(Blume, 1991) had addressed some of the central issues of implementation and effectiveness on teachers' minds during their training.

Methods

The five PSSM teachers: Abiola, Levoli, Sean, Wayne and Yvon were among other PSSM teachers who attended a technology application course, taken in the third and final year of their program in a learner-centered environment. As part of the data collection to gain an in-depth understanding of their views, each teacher kept a journal of activities and reflections of each lesson (Bogdan & Biklen, 1998). Observations, journals, lesson plans and field notes provided the lens to triangulate and confirm their views with their actions on the integration of CT in the mathematics classroom. Towards the end of student teaching, 90-minutes interviews were scheduled with each participant to discuss factors that the preservice mathematics teachers considered to be important in expanding the data on their subjective views (Seidman, 1998).

Results and Discussion

Some common themes found were "Perceptions of CT Experience, Support of CT in the Mathematics Classroom, Availability and Accessibility, Concept of Time, Classroom Management, and Suggestions Based on their CT Experience. This study being one of few that explored a Caribbean perspective has provided evidence that this Caribbean setting has faced many similar challenges of developed and developing countries when new qualified teachers attempted to apply new technologies. The motivation developed in teacher education programs, and the risks new teachers are willing to take in exploring new ideas, teaching and learning strategies, and new technologies are expressed as in the cases of Abiola, Levoli, Sean, Wayne and Yvon (Junor, 2003). The "chalk and talk" approach that was evident in the way the PSSM teachers were taught was seen as "torture" in Levoli's views. However, the lack of resources, supporting principals, and other faculty, technical support, and environments that are conducive for teaching and learning during teacher preparation are seen to be prohibitive to the appropriate integration of CT (Senge, 1990). To encourage our new teachers to continue the path to an effective teaching career in the reform of mathematics education in the Caribbean, policy makers need to review the availability, accessibility and pedagogy of computer technology in the mathematics curriculum. It is crucial that principals, and faculty work together as a team to advocate for resources, share the vision of CT integration in mathematics, and develop personal mastery in the use of technology for their classroom. Taking a bottom-up top-down approach may be the way to go strategically. In this way the stakeholders will evidence effective efforts for further investment and development.

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PRESERVICE TEACHER EFFICACY THAT RESULTS FROM THE CONTEXT OF MATHEMATICS METHODS COURSEWORK

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Mixed-methods research with quantitative and qualitative methods complementing each other to examine changing preservice teachers' sense of efficacy. Results of the Teachers' Sense of Efficacy scale and written course work indicates an expected increase in efficacy and enhanced discourse, and unexpected size of efficacy score increase and sophistication of discourse as a complex sense of teaching practice development and orientation to teaching.

Teachers' sense of efficacy (Bandura, 1997), the teachers' "… judgment of [their] capabilities to bring about desired outcomes of student engagement and learning…" (Tschannen-Moran & Woolfolk Hoy, 2001, p. 783) implies a relationship of teacher efficacy to their instructional behaviour. Considering instructional behaviour formally develops in preservice methods courses, then from the experience of the preservice mathematics methods course context, what is the preservice teachers' sense of teacher efficacy and how is it expressed?

Theoretical Framework

Steele & Widman (1997) found mathematics methods courses increase preservice teachers' levels of confidence to teach and change their conceptions about mathematics and mathematics teaching and learning. The results of research by Steele & Widman (1997), Frykholm & Glasson (2005) and others indicate it is the nature of the activities within methods courses that has an impact on the pedagogical-content knowledge or knowledge of mathematics for teaching, and hence increase preservice teachers' sense of efficacy.

Feiman-Nemser (1989) describe five teacher orientations to classroom practice as possible attitudes and perspectives that influence teachers' behaviour. An integration of orientations may represent a more holistic perspective and a greater sense of professional development. Borich and Fuller (1994) (in Staton, 1992, p. 160), describe three phases to the developing sense of teacher practice as i) concern for self e.g., mastery of content, class control, ii) concern for task, e.g., teaching methods, and iii) concern for impact, e.g., student learning, affective needs of students. An integrated sense of these 'concerns' in preservice teachers may represent greater professional maturity, an increasingly positive attitude, and possibly greater teacher efficacy.

Therefore, as preservice teachers' sense of efficacy changes over the duration of a methods course, as they experience the mathematics, the content, and the pedagogy, what do they think is 'good mathematics for teaching and learning'? How does the preservice teachers' efficacy complement their expression of their understanding of the teaching and learning of mathematics?

Method

There are two concurrent research threads weaving their way through this study. Teacher efficacy is collected quantitatively with the short form of the Teachers' Sense of Efficacy Scale developed by Tschannen-Moran and Woolfolk Hoy (2001). A sense of what preservice teachers understand about the teaching and learning of mathematics is collected through written responses to the question, "What is good mathematics for teaching and learning?"

This study follows from a pragmatic (Greene & Caracelli, 2003) paradigmatic position to mixed-methods research. While data is collected at the start and the end of the methods course, the two parallel and discrete quantitative and qualitative methods to collect data neither interact with the other, nor inform the other. A greater sense of mixed-methods methodology comes into play during the analysis, and reporting of inferences and conclusions phases as the quantitative and qualitative methods act in a complementary manner to each other.

Results

Quantitatively, the unweighted means and total scores from the Teacher Efficacy Scale are calculated. Of the forty-one students in the course, thirty-two experience an increase of five to thirty percent in efficacy scores, six experience minimal or zero change, and three experience a decrease of five to eight percent in efficacy scores. This paper will focus on the 38 out of 41, or 93%, who experience minimal to great efficacy score increases.

Qualitatively, the preservice teachers' written responses provide a rich set of data to complement the understanding of teacher efficacy and preservice teachers' understanding of teaching and learning mathematics. In September five themes emerge, the affective sense of learning mathematics, the content of the mathematics they would teach, the mathematical processes (i.e., thinking, problem solving), the purpose of secondary school mathematics, and student abilities. In March, within the same themes, the writing articulates pedagogical-content knowledge issues and illuminates the preservice teachers' increased sense of the complexity of teaching and learning, and the inter-related nature of teacher, student, mathematics, affect, cognition, and purpose. For example, with respect to mathematics 'content', in September preservice teachers use words such *as skills, knowledge, theory, logic, steps*. In March preservice teachers use words such as *understanding, concepts, meaningful, connected, motivating, relevant*.

As well, in March the affective domain and mathematical processes are mentioned more, 'relevance' is emphasized three times as much, student abilities are emphasized almost twice as much, and two new themes dominate preservice teacher writing, classroom management and the image of self as teacher. For example, with classroom management, it is not just the case that preservice teachers pick up the vocabulary of classroom management from the course work and identify classroom management simply, as discipline and consequences; preservice teachers are able to articulate it as a seamless sense of classroom practice involving student, teacher, instruction, assessment, learning, teaching, and management of behaviour, curriculum, and resources. "*Good mathematics arise from a structured caring safe environment that is structured yet it also changes to incorporate the needs of the student by assessing how and what they understand. Good mathematics brings into it methodologies and learning styles that demonstrate the relationships and patterns in systems of greater complexity*" [Preservice Teacher, +30 increase in efficacy score].

Discussion

The expected results of the efficacy scale are an increase of the preservice teachers' sense of teacher efficacy over the duration of the one-year mathematics methods course. This result is hoped for and expected in a preservice teacher program. Also expected, as preservice teachers' efficacy increases, is the change in understanding of teaching practice, the enhanced clarity and elaboration with examples and connections to other themes. For example, mathematical processes change from skills and knowledge and that math should be "easy" to connection and meaningfulness through thinking, problem solving, and knowledge construction. The efficacy level of a teacher would necessarily be greater to consider implementing such challenging pedagogical positions in a classroom.

What is unexpected are the sizes of teacher efficacy increase, from 0% to 30%, and how the preservice teachers articulate their perceptions of teaching and learning mathematics, that is, the theoretical constructs that emerge from their words and the nature of their discourse. In September, the preservice teachers' writing shows the concern for content (Staton). For example, as HD [Preservice teacher, efficacy score increase +14] initially states, "*Good math is more about how you learn or teach math as opposed to the math process itself.*" However, in March preservice teacher writing shows concern for task and impact. For example, there is an increase in the discourse of affective issues and student abilities, and the addition of two new issues, classroom management and image of self as teacher in relation to learning.

In September preservice teachers perceive teaching and learning as skills and knowledge of mathematics, simple problem solving in real world applications – illustrating Feimen-Nemser's (1989) academic, practical, and technological orientations. After the methods course, preservice teachers perceptions turn to student abilities, relevancy of content to students, and connectedness of mathematics to mathematics – the personal and critical/social orientations. As GF [Preservice teacher, efficacy score increase +8] states, "*Good mathematics is when students get together after class and talk to their friends about what happened in math class, stories from math class, overall remembered things from math class*."

This is a shift from a content and procedural orientation, and concern for self and task to a more complex mix of the five orientations and concern for impact. The increased level of efficacy complements this increased preservice teacher ability to consider, accept, and apply a less mechanical approach to teaching, and employ a more inter-personal and inclusive approach to teaching. "*Math in general is not what we know. We know information, math is what we try to figure out using that information. ... All math is good math for someone*." KL [Preservice teacher, efficacy score increase +14].

There are likely other factors and influences, such as the experiences from the other preservice course work, but the focus of the subject-related mathematics methods course, and the clear focus of mathematics education in the efficacy scale provide the grounds to consider the results of the levels of efficacy and the results of the preservice teachers' writing as true representations of preservice teachers' perceptions and understandings of their level of efficacy for teaching and learning mathematics.

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PRESERVICE TEACHERS' MODELLING OF FRACTIONS: CONNECTING MEANING AND SYMBOLS

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We teach mathematics to preservice and inservice elementary school teachers. We have found that these students often have great difficulty in making sense of the procedures for working with fractions. We have had some success in helping students develop an understanding of operations on fractions by using manipulatives such as Cuisenaire rods. However, for many students, grasping the meaning of procedural algorithms is a challenge. Although many students are successful in modelling the problems, some continue to have difficulties in associating meaning with operations on fractions. In this research, we examine two representative problems that we use to attempt to help students associate meaning with basic operations.

Theoretical Framework

Our framework is based on the work of Reynolds (2005), Glass and Maher (2002), and Maher and Alston (1989). Reynolds reported that the use of manipulatives can help elementary school students develop understanding of fractions and operations on fractions. Glass and Maher showed that college students who are encouraged to make connections between models and mathematical operations can come to understand the operations. Maher and Alston documented an instance of a middle school student who had mastered procedural algorithms and who was able to solve problems with models, but who – like some of our preservice teachers – was unable to make the connection between the symbols and the models. They noted that, to the student, the traditional algorithms for working with fractions represented "meaningless ritual" (p. 248). We have found this to be the case with many of our college students, as well.

Methods

Data for this study are taken from five classes: two freshman-level mathematics courses for elementary school preservice teachers at Felician College with 33 students and three junior-level math methods courses for elementary school preservice teachers at Kean University with 57 students. Student work, classroom observations, and instructor notes provide the data for the analysis.

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Here we examine the students' work on two problems, the candy bar problem and division problems. For both problems, our goal was that the students should think about the meaning of the mathematical operations.

The candy bar problem

I had a candy bar. I gave 1/2 of the bar to Jose and 1/3 of the bar to Tara. How much of a candy bar do I have left?

Our objective for this problem was for students to explore different length rods and to find that only certain length rods can be used to model the problem. Would the students see a connection between this length and common denominators?

Division problems

What is 6 divided by 2? What is 6 divided by 1/2? Build models to justify your answers.

The traditional rule is: When dividing by a fraction, take the reciprocal of the divisor and multiply. Would the process of constructing a model help students make sense of the traditional rule?

Our objective for this problem was for students to explore the two different models for division (repeated subtraction model and partitive model) and to appreciate the reasonableness of the answers. We also wanted them to develop an understanding of why the traditional algorithm works.

Results

Students used a wide variety of approaches in their explorations. Some of the outcomes included:

- Candy bar problem: Although students were usually able to see that the candy bar problem works only with certain length rods, they often did not associate the lengths of those rods with the idea of a common denominator.
- Division problem: Students could usually see from the physical operations that it makes sense for the answer to be larger than the original number. However, they often had trouble connecting this to the "multiply by the reciprocal" rule.
- Some students had difficulty recognizing the structural similarity between problems with whole numbers and problems with fractions. This problem has been observed in elementary school students and is documented by Greer and Harel (1998), among others.
- Some students had difficulty in writing story problems that would be appropriate for a given calculation; they also had problems in converting story problems to appropriate calculations.

Discussion

We suggest that possible reasons for these issues are related to students' early training in the use of symbols without associated meaning. This might be an obstacle to later efforts at sense-making. Specifically:

- Students' previous experience with mathematics often gives them the impression that learning the procedure is important but that sense-making is not. Students who learn procedures without meaning often remain uninterested in addressing issues of meaning.
- Students' difficulties in translating words to mathematical symbols might be due to lack of previous experience. Some students' only experience of operations on fractions was in isolated numerical problems without context.

We suggest that some ways of dealing with these issues might include:

- Showing students that they need conceptual understanding in order to succeed as teachers. We suggest that those who supervise student teachers make a point of telling student teachers that part of their evaluation will include assessing the conceptual understanding of the children they instruct as student teachers.
- Making more explicit the connection between symbols and meaning. In some cases guiding students towards finding the connections might be a feasible alternative to waiting for them to make the connection on their own. For example, after students note that only certain length rods work for the candy bar problem, we might ask, "What does what you just did have to do with a common denominator?"
- Spending more time in teaching fractions. We suggest that professional development courses for teachers should include courses in the intensive study of fractions. It has been shown that students who have difficulties in understanding fractions also have problems with algebra and proportional reasoning (Lamon, 1999). Since algebra is often taught in middle school and algebraic thinking is a part of the curriculum in every grade, it is crucial that teachers at all levels be competent at and comfortable with algebraic concepts. The ability to manage proportional reasoning is similarly critical for progressing past elementary mathematics.

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PROPORTIONAL REASONING: (MIS)CONCEPTIONS OF ELEMENTARY PRESERVICE TEACHERS

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How might preservice elementary teachers' conceptual understanding be affected by being asked to solve and discuss three different versions of the same problem? What mathematical approaches they might use, and what connections could they make? In this paper, after describing the three different task versions and how we used them in class, we'll profile some of the diverse types of mathematical thinking that the preservice teachers demonstrated.

The purpose of this paper is to report on current research aimed at categorizing and interpreting the conceptions held by Elementary Preservice Teachers (EPSTs) in a proportional reasoning context. Just as proportional reasoning is of paramount importance in the K-12 school curriculum (Lesh, Post, & Behr, 1988; Behr, Harel, Post, & Lesh, 1992), so too is it critical that EPST's in university teacher training programs have a deep understanding of this concept.

The following central research question framed our research: What conceptions of proportional reasoning do students exhibit when asked to explore and discuss a task that is posed in three different ways? We wanted to investigate the range of mathematical approaches that students might use, what connections they might make between the parallel problems, and how their conceptual understanding might be affected. We called the set of all three versions of the task *Ratio Triplets.* The essential feature of the activity involved determining which of two packages of ice cream was the better buy: A 64-ounce container selling for $6.79 or a 48-ounce container selling for $4.69 (1).

While *Ratio Triplets* centers on a fairly common type of proportional reasoning task, we found that using different versions of the task afforded a good opportunity for our students to consider the meaning of ratios in multiple ways. In the presented paper, after considering the theoretical background relevant to the current research, we describe the three versions of the task and how we used them in class. Then, we profile some results of the different mathematical thinking that students demonstrated by using examples from each of the three versions, and finally offer a brief discussion of these results.

Theoretical Background

While relatively few studies have been aimed specifically at the proportional understanding of preservice teachers, there is a substantial body of prior research that exists on children's' understanding of ratio and proportion (see Behr, et. al., 1992, for a good overview). Several themes emerge from this corpus of research that are germane to the current study, since the *Ratio Triplets* task fundamentally involves a comparison of two rates. One theme is the importance of unit recognition, which is included with partitioning and equivalence as part of the "basic thinking tools for understanding rational numbers" (Behr,

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Lesh, Post, & Silver, 1983, p. 109). As Lamon (1993) notes, of particular importance to reasoning proportionally is "the ability to construct a reference unit or a unit whole, and then to reinterpret a situation in terms of that unit" (p. 133). Lamon also describes how "the process of norming can achieve yet another level of sophistication" (p. 137), whereby an independent unit may be selected as a basis for comparison. Finally, the basic theme of recognizing the mathematics underlying additive and multiplicative structures is of key importance in discerning proportional reasoners. This is an important piece of content knowledge for teaching with EPSTs, since, as Resnick and Singer (1993) note, "The early preference for additive solutions to proportional problems is a robust finding, replicated in several studies" (p. 123). We found a similar reliance on additive strategies even with the EPSTs in our study.

Methodology

Knowing how the NCTM Standards (2000) reflect calls from the literature to stress not only the importance of using multiple representations in problem solving, but also the importance of communication, we designed *Ratio Triplets* to incorporate three versions of the same basic problem. This design choice allowed us to examine EPSTs' understanding of ratio and proportion in a way that also promoted discourse about different ways of thinking mathematically.

In Version A, a dollars-per-ounce strategy for Mark is shown as well as an ounces-per-dollar strategy for Alisha. The correct quotient for each calculation is also provided. We wondered if students would recognize the meanings behind those calculations, and further, would they understand the interpretations of the results? Version B was identical to Version A in all but one respect: The outcome of each calculation was not provided. That is, the decimals were missing as well as the equality and approximation symbols. Since calculators were available, we wondered if students would just do a unit-rate conversion (effectively mirroring the computation results in Version A), or would they try something different? Version C had the same initial situation description, but omitted any reference to what strategies Mark or Alisha might have used. Instead, it invited any strategy for Mark and prompted a *different* strategy for Alisha. The focus on all versions was in getting the students to provide explanations and justify their thinking.

Results

Across all versions, we found a surprisingly rich diversity of explanations given by the EPSTs, some which were *reasonable* and many which were *questionable.* To illustrate the types of thinking offered, sample results will first be presented in terms of what the subjects had to say about Mark (the first strategy in all versions), and then about Alisha (the second strategy). We first distinguished responses about Mark's strategies according to whether they were *reasonable* (recognizing that the lower price per ounce is the better buy) or *questionable* (reasoning that the larger package is only $2.10 more, so Mark should buy the larger package). All of our claims are supported by examples. Our analysis used the lens of the types of themes that emerge from the literature on children's proportional reasoning.

Whereas the *reasonable* responses for Mark's strategies showed some conventional ways of thinking, we were surprised at the *questionable* responses. We wondered, for example, in Versions A & B where the ratios were already set up, where did those EPSTs get confused? In contrast to results for Marks' strategy, however, we found that the majority of EPSTs actually had *questionable* responses for Alisha. Of the 75 total subjects, only 37.3% gave reasonable responses for Alisha, which was surprising to us. There were many responses that said, in effect, that Alisha was not correct in attempting a different approach and that the only way to solve the problem and find a better buy was to consider cost per amount (as in Mark's strategy in Versions A and B).

Discussion

As we sought to engage our EPSTs in a discussion about mathematics for teaching, we used the Ratio Triplets task as a way to examine what they knew about proportional reasoning and also to promote conversations about the mathematical approaches in the different ways of thinking. We knew *a priori* that proportional reasoning is often difficult for students, "especially for those who do not understand what is actually meant by a particular proportional situation or why a given solution strategy works" (Weinberg, 2002, p. 138). There was an impressive array of differing strategies that prompted fruitful discussion in class about representing mathematical situations in a variety of ways, and especially about the importance of communicating one's own thinking and understanding that of others.

However, it was surprising to us that 18.7% of our EPSTs either gave *questionable* responses for Mark's strategies (on Versions A and B) or could not come up with a proportional strategy for Mark (in Version C). Particularly on Versions A and B, it was disturbing to find some these young adults unable to interpret the initial ratio set-ups that were offered. Moreover, the 62.7% of our EPSTs who gave *questionable* responses for Alisha's strategies often showed a very limited understanding of proportional reasoning. As our exploratory research suggests, even those preparing to be teachers may be entering their university training without a sufficiently robust conceptual understanding of proportional reasoning.

While more research is necessary to further unpack the dimensions of thinking exhibited by EPSTs in a proportional reasoning context, this research takes important first steps toward that process. Of particular importance was the structure of our *Ratio Triplets* task, since the lively class discussions that ensued when debating each others' explanations across the three versions helped foster a better conceptual sense of the actual mathematics while also modeling the kinds of pedagogical practices we'd like to see carried into the classrooms where these preservice teachers eventually will serve.

Endnote

1. Adapted from a task first proposed by Jane Lane, Eastern Washington University (2003).

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REPRESENTATIONAL UNIT COORDINATION: PRESERVICE TEACHERS' REPRESENTATION OF SPECIAL NUMBERS USING SUMS AND PRODUCTS

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I examined preservice teachers' (PST) coordination of representational units arising from the use of magnetic color cubes and tiles for representing various special numbers. The coordination and identification of these representational units of different - not necessarily hierarchical - types (multiplicative, additive, pseudo-multiplicative) appear to be important mathematical practices in dealing with problems involving representational quantities.

Theoretical Framework

In its true nature, coordination is about making various different things work effectively as a whole. In the context of my study, it refers to the conception of unit structures in relation to smaller embedded units within these unit structures, or, bigger units formed via iteration of these unit structures. Unit coordination (UC) has been previously studied by various researchers in the mathematics education field. The coordination of different levels of units in whole number multiplication problems is reminiscent of a key concept in multiplication, i.e., the notion of composite units (Steffe, 1988). The essence of multiplication lies in fact in distributive rather than repeated additive aspect (Confrey & Lachance, 2000; Steffe, 1992). The multiplication of 5 by 7, for instance, can be thought as the injection of units of 7 (each being units of 1) into the 5 slots of 5, each slot representing a 1. In this example, the conceptualization of each singleton unit describing a unity, i.e., 1, stands for a first level of UC. Moreover, 5 and 7 can be conceptualized (as composite units of 1) as 5_1 and 7_1, respectively, as a second level of UC. The product 5_7 which denotes 5 (composite) units of 7 (composite) units of 1, can be conceptualized as a third level of UC. Some other researchers also studied unit coordination in a fractional situation (e.g. Lamon, 1994; Olive, 1999; Olive & Steffe, 2002). Additionally, work on intensive and extensive quantities reflects unit coordination as well (Schwartz, 1988; Kaput, Schwartz, & Poholsky, 1985). Representational Unit Coordination (RUC) can be defined as the different ways of categorizing units arising from the modeling of identities on representational quantities as the area as a product and area as a sum of the corresponding special rectangles made of color cubes or tiles. In its most basic sense, area of, e.g., a rectangle, is defined as the product of its two dimensions. The identities on special numbers PST analyzed via color cubes and tiles were always about a rectangle – prime rectangle, composite rectangle, summation of counting numbers, odd and even integers generated as a growing rectangle, and polynomial rectangle. Coordination of a particular rectangle's two dimensions, i.e., the arrangement of these two linear units in a particular order as an ordered pair such as (a, b) or (b, a), defines the first part of my construct RUC which is of multiplicative nature. The analysis of the other important concept, area as a sum (of a special number rectangle), is prone to many more, not necessarily hierarchical, additive type RUC for which the addends, namely the areal units, are expressed as n-tuples in square brackets. RUC has more of a relational aspect, that way.

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Context and Methodology

I conducted my study with PST enrolled in the Mathematics Education Program in a university in the southeastern United States. I interviewed 5 PST individually twice during January & February 2007. Duration of each session was about 60-75 minutes and each interview session was videotaped. The focus was on problems on identities for prime and composite numbers along with summation of counting numbers, odd and even integers as well as products and factors of polynomials modeled with magnetic color cubes and tiles. I used thematic analysis supported by constant comparison of the interviews and retrospective analysis. I also simplified and extended *the generalized notation for mathematics of a quantity* (Behr et al., 1994) in such a way as to cover identities that equate summation and product expressions of special numbers.

Results and Conclusions

Multiplicative type RUC arose from various usages by the PST such as "It [areal 6] is [linear] 6 and [linear] 1", "This [linear x] and this [linear 1] to find this [areal x]", "When you put this length [linear 1] and that length [linear 1] together", "This edge [linear 1] right here and this edge [linear 1] right here", "This edge [linear x] by this [linear x] edge". For all such usages, I used a relational notation of the form (a,b) where a and b stand for the corresponding linear quantities represented by the dimension tiles. Moreover, a mapping structure of multiplication operation was spelled out by these PST. For instance, "And this area right here is x times y... to find that specific spot", can be represented using a functional notation such as f:(x,y)_xy. Here, f denotes the multiplication operation which maps the linear units, x and y, into the corresponding "spot", namely the areal xy. I observed more than one additive type RUCs which can be described using a functional notation ∑f(i)=g(n) where areal quantities f(i)'s are being summed from 1 to n (number of addends) and i is the stage number (ordering number for the addends):

- Equal Addends: These are the addends describing a composite number rectangle. With the functional notation, f(i)=c, for all i. All PST produced this type.
- Irreducible Addends (Type I): PST used these addends mostly when dealing with prime number rectangles. This is a special case for equal addends, f(i)=c, for all i, with c=1.
- Symmetric Addends: This type came from Ben's work on the summation of odd integers activity. Ben used these addends to describe the odd integers as symmetric L-shapes. For each symmetric L-shape, there are three addends only, i.e., n=3. One of these addends is equal to 1, and the remaining two addends are equal to each other. In other words, with the functional notation, one can write, f(2)=1, f(1)=f(3). For example, for the case of areal 9, which denotes an odd integer, f(2)=1, f(1)=f(3)=4. Note that f(1)+f(2)+f(3)=4+1+4=9, i.e., the odd integer itself.
- N+(N-1) type Addends: These are the addends describing a symmetric L-shape odd integer. In this case, n=2. With the functional notation, f(1)=N, and f(2)=N-1, i.e., the addends differ only by 1. John, Stacy, and Ben explained their ideas using this type when working on the summation of odd integers activity.
- (N+1)+(N-1) type Addends: These are the addends describing a nonsymmetric L-shape even integer. Once again n=2. Only John referred to this type when working on

the summation of even integers activity. A nonsymmetric L-shape even integer can be described using the functional notation f(1)=N+1, f(2)=N-1.

- Recursive Addends: f(i+1) is being added to the previous summation (Nicole). With the functional notation, this can be written as g(n+1)=g(n)+f(i+1).
- Summed Addends: The addends of the growing rectangle are areal units with different shapes made of color cubes representing the "area as a sum" part of the summation formula. For example, f(i)=i, f(i)=2i-1, f(i)=i+(i-1), f(i)=2i, for all i, for the addends corresponding to summation of counting numbers, odd numbers, odd numbers, and even numbers, respectively. 3 out of 5 PST came up with this usage. Ron and Ben, on the other hand, did not care about the color shapes generating the growing rectangle. Instead, they used Equal Addends in expressing the area of the growing rectangle as a sum: Namely they treated the growing rectangle as a composite number rectangle.
- Random Addends: n can be anything (Stacy). There are many different ways of writing the sum. With the functional notation, f(i)=anything, for all i. And f(i) is not necessarily equal to f(j) for any i≠j, where i, j denote the ordering number for the addends (areal units).
- Irreducible Addends (Type II): The area of the polynomial rectangle is written as the sum of irreducible areal units. 4 PST came up with this usage. For instance, for the 2x+y by x+2y+1 rectangle, the irreducible addends are [x^2, x^2, yx, xy, xy, y^2, xy, xy, y^2, x, x, y].
- "Boxes of the Same Color" type Addends: The area of the polynomial rectangle is written as the sum of the boxes of the same color. As an example, only 1 PST used the areal units [$2x^2$, 4xy, 2x, yx, $2y^2$, y] to generate the 2x+y by x+2y+1 polynomial rectangle.

There is one more type RUC, in between additive and multiplicative which I named Pseudo Multiplicative type RUC. This occurred for the "Area of the Boxes of the Same Color as a Product" in dealing with polynomial rectangles made of color tiles. For instance, the x+1 by 2y+3 rectangle has 4 boxes (x by 2y, x by 3, 1 by 2y, 1 by 3) of the same color. Nicole, Stacy, and John's products were x_2y, x_3, 1_2y, 1_3; i.e., of multiplicative nature. With the relational notation, these linear units, namely the length and the width of each "box" can be written as (x,2y), (x,3), (1,2y), (1,3). However, Ben and Ron's areas as a "product" for the same boxes were 2_xy, 3_x, 2_y, 3_1. In other words, the first term of each "pseudo-product" is a coefficient serving as a counting number indicating how many there are of each irreducible areal unit: Though written as a "product", Ben and Ron's expressions are of additive nature.

The identification and the coordination of representational units of different types (multiplicative, additive, pseudo-multiplicative) associated with color cubes and tiles are important aspects of quantitative reasoning and need to be emphasized during the teaching and learning process. Moreover, mapping structures serve as a bridge in between "area as a sum" and "area as a product" concepts in helping students and teachers make sense of identities on integers and polynomials: Why does the LHS have to be equal to the RHS?

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TEACHING ALL CHILDREN MATHEMATICS: PRESERVICE TEACHERS' PERCEPTIONS OF TEACHING DIVERSE LEARNERS

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Teachers must be prepared to teach mathematics to all students. In our experience, and that of other mathematics teacher educators, we have found that there are barriers to making this goal a reality, despite good intentions of faculty and preservice teachers. Some of these barriers, which have been well-documented by research, include:

- the homogeneity of the preservice and in-service teacher population which tends to reinforce stereotypical expectations;
- preservice teachers belief systems that include stereotypical expectations for children's achievement in mathematics, i.e., Asian and White students have more innate abilities in mathematics while African-American and Latino students have to "work harder" to achieve, or that boys demonstrate stronger mathematical abilities than girls;
- lack of understanding of how issues of gender, ethnicity, and class can interact to affect students' achievement in mathematics and how to address this; and
- lack of knowledge concerning how non-White, non-European cultures have contributed to the knowledge base in mathematics (Garmon, 2004).

Our purpose for this project was therefore to determine the readiness of CI 3030-*Investigating Mathematics and Learning* students to teach mathematics to diverse populations and to improve our students' readiness to teach mathematics to diverse populations.

Procedures

We decided to begin by adding to and modifying the diversity-related assignments in our CI 3030-*Investigating Math and Learning Class*. Based on our readings and review of the literature, we determined that we needed to get a measure of our students' current understanding of diversity, as it related to mathematics education, to develop an educational intervention, and then measure the effects of this intervention. We undertook the several tasks to achieve these goals. We adapted a survey from the following sources:

- Indiana Mathematics Belief Survey (Kloosterman & Stage, 1992)
- NAEP National Center for Education Statistics (2001). National assessment of educational progress question tool. Available from the NCES web site, http://nces.ed.gov/nationsreportcard/itmrls/
- Additional questions from the survey used in the study, *The Beliefs And Conceptions Of Elementary Preservice Teachers* (McCormick, Kapusuz, & Al-Salouli, 2004)
- Diversity questions developed specifically for the current project.

The survey includes five scales: *1) Confidence in Doing Mathematics; 2) Conceptual Understanding; 3) Effort; 4) Usefulness; and 5) Diversity.*

We gave this survey to our students at the beginning and end of the semester. We developed a reading list for our students and placed the articles in on-line reserve with the library. Students were directed to choose three articles, from three different categories, from the reading list. These articles are used for the article critique assignment in place in our courses. Each critique is required to be at least three pages in length and focus on a major issue raised in the article. At the end of the course, following the practicum, our students were asked to provide an answer to the following question, as an in-class essay: *What have you learned about diversity and mathematics teaching based on your readings and field experience for this course?*

Results

We completed the survey with approximately 80 students. We also asked them to complete the in-class essay described above. Preliminary results are interesting, if somewhat predictable. First, our survey results indicate that students are still undecided or are fixed in their opinions regarding diversity issues. For example, when asked to respond to the statement: *Instruction of at-risk students should focus on remediation.* (SA A U D SD) we saw the following response rate:

Mean before: 3.03
Mean after: 3.27

This indicates that the students came into our course undecided about this issue and appear to be leaving unchanged, despite the fact that we discuss the fallacy of this approach and the damage that can occur due to it. This results is also disturbing since, during Block I, while students are taking our course, they are simultaneously taking the course, CI 3000-*Learner Diversity*, where such issues are also addressed.

We also found it interesting that while students tend to provide somewhat neutral (or possibly, politically-correct) responses to the survey questions, their responses to the in-class essay question sometimes contradicted their responses to the survey. Here are some example responses to the essay question:

What have you learned about diversity and mathematics teaching based on your readings and field experience for this course?

- I personally felt like my diverse classroom was more alike in how the students learned.
- I didn't realize boys were smarter than girls in mathematics.
- I learned that all children learn differently.
- I learned that a student's culture and home life completely eafects their success in math and other subjects.
- I have learned that Hmong children love math and usually excel at math as well.
- I did not witness a lot of diff. b/t boys and girls in mathematics education.
- I learned that students learn differently.

- Just because someone is from another culture does not necessarily make them diverse in math.
- I did not learn much about diversity.
- I learned that sometimes diversity can throw a wrench into teaching math, because of existing language barriers.

We noticed in these responses several stereotypes, for example, "Asians do better in math." Of further interest was our students' completion of the practicum assignments. While some of the students acknowledged that they should address learner diversity in their classroom, not many actually did so in the actual lessons that they prepared. Their analysis of student interviews and their preparation of small-group lessons did not provide much evidence of applying their "knowledge" regarding diversity. Some of the students seemed to interpret SES as an indicator that a student could not be expected to perform better in mathematics because, "a student's culture and home life completely effects [sic] their success in math and other subjects." On a more positive note, our students did respond that they enjoyed the readings concerning diversity and found them to be interesting and useful.

Discussion

While preservice teachers do need experiences working with diverse populations, one must take care that their experiences do not serve to validate their stereotypes. If preservice teachers work only African-Americans in a remedial tutoring situations or with Asian students in a gifted program, then these experiences can be counterproductive to their understandings. Our preliminary findings appear to confirm this, in that, while we did not place our students in exclusive tutoring or gifted situations, the activities and interactions in the whole class setting allowed them to "cherry-pick" what happened so that their experiences mirrored their expectations and beliefs. To our surprise, we found that our students did this with the readings we assigned as well. While we assigned articles regarding mathematics education and gender that argued strongly that gender did not predetermine mathematical ability, some of our students interpreted the whole discussion as evidence that boys must be better in math. After teaching mathematics and mathematics education for many years, we are accustomed to dealing with students' misconceptions and the resilience of these misconceptions. However, we were still surprised at the complexities associated with preparing our students to teach diverse learners and how naïve, although well-intentioned, our students can be. This experience has led us to realize that this must be an area of greater focus in our courses and that we must continue to inform ourselves, craft better assignments and discussions, and continue to evaluate the preparedness of our students.

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THE COMPLEXITY OF THE BASE-TEN NUMBER SYSTEM: USING CHILDREN'S MATHEMATICAL THINKING TO HELP PRESERVICE TEACHERS CONNECT SYMBOLS TO THEIR VALUES

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I develop a framework for preservice teachers' (PSTs') conceptions of multidigit whole numbers and use it to (a) describe their conceptions and their difficulties explaining the mathematics underlying the algorithms, (b) show why a good understanding of tens and ones is insufficient to explain regrouping in numbers with more than 2 digits, and (c) discuss implications for education.

Algorithms are powerful tools in mathematics. With algorithms, one is equipped to solve a variety of problems quickly and accurately. However, without understanding the reasoning behind the steps of an algorithm, one may struggle in deciding when using the algorithm is appropriate and how the algorithm should be modified according to different contexts. Preservice teachers are able to execute algorithms for operations on multidigit whole numbers; however, they struggle when asked to explain why the algorithms make sense mathematically (Ball, 1988; Ma, 1999; Thanheiser, 2005).

To develop effective activities to address PSTs' struggles, we educators must understand the roots of their difficulties. Once those roots have been identified, one way to focus the PSTs' attention on the mathematics is to focus first on the PSTs' central concern: children. By carefully selecting artifacts of children's mathematical thinking, we can expand the PSTs' interest from their initial caring about the child to include caring about the child's mathematical thinking. PSTs have motivation to learn the mathematics when they realize that they themselves need to understand the mathematics to help the children understand (R. A. Philipp, Thanheiser, & Clement, 2002).

To understand the mathematics underlying the algorithms, PSTs need to first understand how numbers are composed. Research on children's understanding of numbers has shown that to understand two-digit numbers, one must see a ten simultaneously as 10 *ones* and 1 *ten*; however, even this relationship is not obvious (Fuson et al., 1997; Kamii, 1986). And although adjacent digits in any multidigit number can be treated as if they were *ones* and *tens*, whether understanding the relationship between *tens* and *ones* is sufficient to explain regrouping in larger numbers had not been studied. I set out to (a) describe PSTs' conceptions of multidigit whole number in the context of the standard algorithms, (b) explain the difficulties PSTs have when explaining regrouping in 3-digit numbers, and (c) discuss educational implications.

Mode of Inquiry

The data analyzed are drawn from two 75-minute semistructured interviews with 15 PSTs, each at a large, urban, state university. Questions for the interviews were developed by a team of researchers, piloted, and then modified. Tasks were planned in advance, but follow-

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up questions were posed on the basis of the PSTs' responses so that the PSTs' conceptions could emerge in their answers. PSTs were asked to answer questions in the context of the standard algorithms as well as in other contexts. For example, PSTs were asked to explain whether the value of a number stays the same when it is regrouped in the context of the subtraction algorithm. Each interview was transcribed and analyzed; Interview 1 served as a basis for Interview 2. Using a grounded theoretical approach with open coding, I categorized the PSTs' conceptions of multidigit whole numbers.

Results and Discussion

PSTs' conceptions of multidigit whole numbers are categorized into four major groups—two correct conceptions and two incorrect: thinking in terms of (1) *reference units*, (2) *groups of ones*, (3) *concatenated digits plus*, and (4) *concatenated digits only*. Only 3 of the 15 PSTs held a reference-units conception. They reliably conceived of the reference units for each digit in the number and could relate the reference units to one another; in 389, they could see the 3 as 3 *hundreds* or 30 *tens* or 300 ones and see the 8 as 8 tens or 80 ones. Two PSTs held a *groups-of-ones* conception. They reliably conceived of all digits in terms of groups of ones (389 as 300 *ones,* 80 *ones*, and 9 *ones*) but not in terms of reference units (i.e., the 3 in 389 was NOT seen as 3 hundreds or 30 tens). Seven PSTs held a *concatenated-digits-plus* conception. They conceived of *at least one* digit in the number in terms of an incorrect unit type (e.g., 389 as 300 *ones,* 8 *ones,* and 9 *ones*). Three PSTs held a *concatenated-digits-only* conception, conceiving of *all* the digits only in terms of *ones* (e.g., 548 as 5 *ones,* 4 *ones*, and 8 *ones*).

When asked to subtract 527 – 135, all 15 PSTs in the study knew that the value of regrouped minuend $4^{1}27$ was unchanged, but not all could justify that conclusion. Using two examples from this context, I describe PSTs who were able to explain regrouping between tens and ones but not regrouping between hundreds and tens. Vanessa, who held a *groups-of-ones* conception, struggled in relating the regrouped 12 in the ten's place to its value (120):

> When I look at it, it's like, just out of reflex it's 12, but if you, like, look at it and you think about it, it's actually 120 and 30 [in the ten's column of the subtraction problem] … but like out of just like reflex and human nature, it's a 12, and you are just subtracting 3 from it.

She did not use the reference unit *tens* to relate the 12 as 12 *tens* to its value 120. Vanessa adequately explained regrouping between tens and ones for 577 – 159 (577 is regrouped into $56^{1}7$): "The 7 [ones] wasn't big enough to subtract 9 from it, so I had to take from the 70. I just took 10 from the 70, and I added to the 7, so that made it a 17. So that left me with 6 [in the ten's place], as in 60."

Jamie, who held a *concatenated-digits-plus* conception, was confused about the value of the regrouped digit in $4^{1}27$. She said, "I don't know why that [4 in the hundred's place] would be 400 [rather than 490], if you're only borrowing 10," thinking of the regrouped digit as 10 *ones* rather than 1 *hundred* or 10 *tens* or even 100 *ones.* Jamie knew that 10 tens form a hundred and tried to use that fact: "I am taking one hundred; I am taking 10 tens"; but she still saw the regrouped digit as 10 rather than 10 *tens* or 1 *hundred*, saying that it is "just one 10 [not 10 tens]." Jamie was, however, able to explain regrouping between tens and ones conceptually for 17 – 9: "So instead of 7 ones and 1 ten, I am making it 17 ones," explaining each digit in terms of its reference unit (7 *ones* and 1 *ten*) and relating the reference units to each other (1 *ten* is 10 *ones*). Jamie knew from her experience with algorithms that adjacent digits are related by a 1-to-10 relationship but exhibited confusion about the relationship: "This [relationship between tens and ones] is obvious. It is 10. This [relationship between tens and hundreds] is not so obvious. But it is still 10 for some reason." Jamie was unable to use her knowledge of the value of each place and the relationship between the digits to make sense of regrouping beyond two-digit numbers.

Why were Jamie and Vanessa able to explain regrouping between ones and tens but not between hundreds and tens? Consider the regrouped two-digit symbols in the ten's ($4^{1}27$) and one's ($51^{1}7$) places. In the one's place, the new symbol (17) is read as 17 (*ones*), with *ones* being the appropriate reference unit. In the ten's place the new symbol (12) in the ten's place must be read as 12 *tens* rather than 12 *ones*. Thus an *explicit* shift in reference units is required to explain this regrouping. The relationship between hundreds and tens is not a simple extension of the relationship between tens and ones.

Implications and Example

Procedurally, standard algorithms do not differ for regrouping between tens and ones and regrouping between hundreds and tens. Conceptually, however, regrouping with three-digit numbers is more complex than regrouping with two-digit numbers. Instruction on regrouping often focuses in depth on regrouping between tens and ones, but, for many students, explicit attention is needed to regrouping between unit types that are not equal to *ones*—to explore all aspects of regrouping.

To engage both Vanessa and Jamie in considering the important relationships, we could ask them to view some carefully selected video clips of children while playing the role of the children's teacher. This engagement requires them to (a) decide whether a child's thinking is valid, (b) identify critical aspects of the child's mathematical thinking, and (c) decide how to proceed with the child. To help Vanessa and Jamie see numbers in terms of reference units, for example, we could confront them with a video clip (R. Philipp, Cabral, & Schappelle, 2005) in which a child states that there are 12 tens in 120 and 3 tens in 32. Can Vanessa and Jamie explain the child's thinking? They may be motivated to view numbers more flexibly and to relate them to the regrouping in algorithms when they realize that this child and others they may teach have understanding they have not yet developed.

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THE LIVED EXPERIENCES OF PRESERVICE ELEMENTARY TEACHERS IN MATHEMATICS CONTENT COURSES

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For this phenomenological study, we interviewed six elementary preservice teachers to understand their experiences in upper-division mathematics content courses taken during their teacher preparation program. The preservice teachers described their affective reactions to the experience, their attempt to situate the experience in their professional program and their images of the pedagogy.

Background

The elementary teacher preparation program at our institution recently changed in response to a mandate from the regents of the university system. The directive required elementary preservice teachers to complete 9 hours of upper-division (3000-level or above) *mathematics* coursework. Previously, preservice teachers had no required mathematics content coursework. New courses were developed and taught in the department of mathematics. Effect on student matriculation and progression through the elementary education program was immediate. Many students received *D*s, failed, or withdrew from the new courses. As mathematics educators, we were deeply concerned. Why were so many preservice elementary teachers not successful in these mathematics courses? We needed and wanted to understand this problem.

Related Research

For the past several years the question of adequate and appropriate mathematical content knowledge for elementary teachers has been at the forefront of discussions among the mathematics education community (Ball, 1990; Ball, 1991; Mewborn, 2001). Given the importance of a knowledgeable teacher in creating mathematical experiences in a classroom, e.g., using curriculum materials, choosing and using representations and tools, interpreting and responding to students' work, and designing assignments (Ball, Lubienski & Mewborn, 2001), the value and need for appropriate mathematics content knowledge and coursework is evident. However, previous research suggests that the shear number of higher-level mathematics courses does not guarantee an understanding of the mathematical concepts that translate into effective teaching (Begle, 1972; Eisenberg, 1977). Yet even in the face of these findings and current recommendations (Kilpatrick, Swafford & Findell, 2001), some professional organizations in the United States continue to suggest the solution to the problem of well-prepared elementary mathematics teachers is more, higher-level mathematics.

Research on post-secondary mathematics teaching and learning, outside the work on the calculus reform effort, is limited. Two recent studies were found that somewhat related to our question of why many of our students were not successful in there mathematics content courses. Powell-Mikle (2003) examined the perceptions of six African-American college students toward instructional practices in secondary and post-secondary mathematics courses. Her findings revealed that the students felt adequate instructor availability, prevalence of classroom discourse, clear instructor explanations, and a caring ethic were associated with exemplary practices and their success in mathematics. Weinstein (2004) looked at students' perspectives on the question, "How do college remedial math students define success, and what are they striving for in their math classes?" Based on his findings, he recommended that instructors decrease the amount of time spent on lecture and help students develop more confidence to do mathematics.

The Research Purpose

The purpose of our research was to understand the experiences of these elementary preservice teachers in their required, upper-level mathematics content courses. We felt we first needed to hear *their* voice, *their* perceptions. We wanted to hear how the experience situates itself in the students' professional development as an elementary teacher; and, we did not want to assume we knew the answers. We wanted to let the students speak for themselves.

Methodology

Given the purpose of our research, a phenomenological methodology seemed appropriate. This meant taking hold of the experience to be studied while "bracketing" our experiences outside the phenomenon being studied (Van Manen, 1990). As researchers, we needed to make explicit our assumptions, beliefs, and biases; not forgetting them, but holding them at bay as we explored and made sense of the experience. As we reflected, we realized we assumed the use of traditional teaching methods. We questioned if the specialized content knowledge (SCK) needed for teaching *elementary* mathematics (Ball, Hill & Bass, 2005) was being addressed. We believe SCK is gained by studying elementary mathematics. We also believe that our issues and concerns are not unique and will be of interest to others. It is with these biases and assumptions we began our study.

After identifying our research focus and making public our beliefs, we developed the interview protocol. The questions were framed to open up possibilities for the students and the researchers (Moustakas, 1994). They needed to hold the promise of helping us see the experience as the students described it and allow us to tell their story. We agreed that a question was only a starting point for the discussion and that the interviewer was free to pursue interesting comments or remarks in-depth.

The participants were six randomly selected elementary preservice teachers who were completing their student teaching and hence had completed all required mathematics coursework (Number and Operations, Geometry, Statistics). All participants were female. Two were African-American and four were Caucasian. At the convenience of the preservice teachers, we conducted interviews in faculty offices or at the student teaching school We made every effort in the interview to engage in the *Epoche process* (Moustakas, 1994), i.e., to display an unbiased, receptive presence. Our intention with each of the six students was to obtain a direct description of the experience as it was, without offering causal explanations or interpretative generalizations of the experience. A graduate student transcribed each of the tapes. Both researchers independently analyzed each of the six transcripts using a constant comparison method looking for language that would describe *what is it like to be an elementary preservice teacher in these upper-level mathematics content courses?* Significant comments were highlighted and common threads were identified. Individual analyses were compared and discussed. Final categories were identified and provide the results from which we developed the story of their lived experience.

Synthesis of the Story

Analysis of the interviews elicited three common ways the students experienced the courses. The first was a cognitive experience in which the student attempted to situate the mathematics coursework in their larger professional program. The second was an affective reaction to the experience of the courses. The last was objective observations about the nature of the pedagogy. Each will be described briefly here.

All six students described an inability to position the mathematics coursework within their professional development as elementary teachers. They indicated a disconnect between the two. The participants stated that *the courses are disconnected* and *there is no link to real schools.* They said the instructors *don't know what goes on in schools, they don't talk about students* [children], and *they don't know what goes on in our department* [elementary education]. One preservice teacher said the instructors are *blind to what we are doing with our lives*. The affective reaction to the experience was portrayed as a feeling of being devalued. The students used language such as *there was a sense of invisibility, I do not feel seen,* and *they* [the instructors] *do what they do and leave.* This was connected to a feeling of diminished self-efficacy: *I felt discouraged, I feel less confident* and *It makes you feel dumber.* Lastly, the participants' description of the pedagogy in the mathematics courses revealed a predominately traditional approach to instruction. They mentioned a preponderance of *lecture, note-taking,* and *power-point presentations.* They observed that the classes were *not hands-on*.

Discussion

How does the story the preservice teachers tell contribute to our understanding of the professional development of teachers in our program? Clearly, the disconnect between their professional program and mathematics content coursework suggests the need for better integration between the two experiences. We must find ways to connect pedagogy and content and ways to make the content knowledge relevant. All faculty participating in the professional development of teachers need to help students relate what they are learning to becoming elementary teachers.

The findings also indicate the importance of increasing the confidence of elementary teachers as doers and learners of mathematics. This is consistent with the work of both Powell-Mikle (2003) and

Weinstein (2004) who found that a caring ethic and developing confidence provides a foundation for learning. If college coursework is further undermining self-confidence in a group that is already identified as math-anxious (Hembree, 1990), we are not improving an already difficult situation.

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THE RELATIONSHIP BETWEEN PRE-SERVICE ELEMENTARY TEACHERS' IDENTITIES AND ENGAGEMENT WITH STUDENTS' MATHEMATICAL THINKING

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Recent reforms in mathematics teaching mark a departure from the traditional, transmission-based model, which requires that teachers tell, demonstrate, or show how to do something. Instead, they require that teachers organize their pedagogy in such a way to facilitate growth in *students'* ideas (National Council of Teachers of Mathematics, 2000). While research has long viewed engagement with students' thinking as a desirable outcome of teacher learning, recent studies have established it as a powerful *source* of teacher learning as well (Steinberg, Empson, & Carpenter, 2004). The potential of engagement with students' thinking to meet goals for mathematics education reform (NCTM, 2000) as well as result in teachers' generative change (Franke, Carpenter, Levi, & Fennema, 2001) makes it a timely topic of study.

The results presented here are part of a dissertation study currently in progress on the evolution of preservice elementary teachers' (PSETs') learning and identity development. This report focuses on the following question: How are PSETs' identities as teachers of mathematics related to their engagement with students' thinking during their first teaching experience? While the larger study explores the ways that teacher learning and identity evolve over time, this study adds to the research base on the reciprocal relationship between teaching practice and identity.

From an acquisition perspective of learning, knowledge must undergo a mysterious process of application or "transformation" (Shulman, 1987) in order to be used in teaching, which might depend upon an exhaustive list of other variables. This perspective also does not reflect the situated nature of teachers' knowledge within the context of the classroom. My research addresses the aforementioned limitations by viewing PSET knowledge through a situative lens. The situative perspective (also known as situated learning theory), views knowing as *doing* rather than *having* (Sfard, 1998). Thus, I am directly concerned with PSETs' participation with respect to a practice that relates to reform: engagement with students' mathematical thinking.

The treatment of identity in my research is guided by work that equates identity with a collection of reifying, endorsable, and significant stories about an individual (Sfard & Prusak, 2005). From a situative perspective, participation and identity are inextricably linked and develop concurrently (Sfard & Prusak, 2005). An individual's participation in a community of practice influences the stories that she and others tell about herself. Conversely, identity influences subsequent participation by creating goals for learning (Drake, 2006). The construct of *identity* shows promise for explaining participation because it allows researchers to study teachers' knowledge, beliefs, experiences, and emotions as an integrated and contextualized whole. Previous research suggests that identity may be valuable for helping researchers understand the ways in which a teacher engages with students' thinking and participates in opportunities to learn about teaching (2006). Although from the situative perspective knowing and identity are inextricable, few studies explicitly investigate the relationship between these constructs. Even fewer studies have examined this relationship in mathematics teacher education.

Data collection takes place in one section of an elementary mathematics methods course. As part of the course requirements, PSETs have four opportunities to work in pairs to plan and teach problem-based lessons to a class of elementary school students. The participants are four PSET volunteers who will be assigned to the same grade level for their teaching. Data include videotapes of PSETs' first teaching episodes as well as individual questionnaires and interviews. I will transcribe videotapes of the teaching episodes and draw on constant comparative methodology (Strauss & Corbin, 1998) to systematically code and analyze them. Initial codes will be informed by Franke and colleagues' *Levels of Engagement with Children's Mathematical Thinking* (2001), though I anticipate modifying these codes as ongoing analysis dictates.

Individual questionnaires and interviews provide information about participants' identities as teachers of mathematics and are administered prior to their first teaching episode. The questionnaire asks PSETs to reflect on their own experiences as mathematics learners and teachers as well as their visions of their future mathematics instruction; the follow-up interviews allow for clarification of their responses. Consistent with Sfard & Prusak's operationalization of identity as narrative (2005), to uncover each participant's identity as a teacher of mathematics requires an understanding of how they describe themselves as mathematics teachers and their vision of their future teaching. I will code the surveys and corresponding interview transcripts for each participant, noting emergent themes in the analysis. I present a case study for each participant, describing his or her identity in narrative form and engagement with students' thinking during instruction. I also look across the portraits of all three participants to determine whether there are general trends in the ways that identity and practice are related.

This study identifies challenges that teacher educators may face when attempting to promote reform-minded mathematics pedagogy among PSETs, both in terms of identity and practice. Understanding the ways that PSETs' identities relate to their goals for instruction and their practice may help us understand the knowledge and skills they seek to gain from learning opportunities in methods courses. It also informs the design of educational experiences that help PSETs move towards pedagogy aligned with visions for reform. This study adds to the research base by exploring the usefulness of "identity" as a construct for accounting for teaching practice. My larger research program, of which this study is a part, ultimately aims to understand the concurrent evolution of teacher learning and identity in preservice education.

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TWO PRESERVICE ELEMENTARY TEACHERS' PERCEPTIONS ABOUT THEIR LEARNING AND UNDERSTANDING OF FRACTIONS

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This study investigated two preservice teachers' emergent perceptions about learning and understanding fractions while they engaged with and reflected on fraction-based children's thinking activities in a content course. Our findings show varying perceptions in regards to mathematical content and their own understanding of mathematics. These findings are due, in part, to the level of confidence the participants brought with them.

This study investigated two preservice elementary teachers' perceptions about their learning and understanding of fractions, which emerged as they engaged with and reflected on fraction-based children's thinking activities in a content course. Due to the vast number of perceptions revealed, for this session we only focus on what perceptions emerged in regards to the interrelated themes of mathematical content and the preservice teachers' own understanding of mathematics. For our purposes, these perceptions were conceptualized as the result of applying individual prior knowledge, experiences, and beliefs to specific settings. These settings were comprised of individual assessments and group discussions within a content course and one-on-one interviews. We contend that a better insight into preservice teachers' perceptions is a critical tool for teacher educators to develop rich activities that promote continued mathematical growth.

Sociocultural theory (Vyotsky, 1978) framed our study. Learning was viewed as a function of environmental social interaction where knowledge was continually developed, shared, and transmitted. Focused discussions allowed the preservice teachers to form communities of practice (Lave & Wenger, 1991) in which they could extract meaning from children's explanations and share their developed knowledge with fellow classmates. By interacting with children's thinking in this way, we provided the preservice teachers an opportunity to re-think their mathematical identities (Franke & Kazemi, 2001) and to make mathematical discoveries of their own (Crespo, 2000).

Motivated by the literature on preservice teachers' knowledge of fractions (e.g. Ball, 1990; Tirosh, Fischbein, Graeber, & Wilson, 1998), three graduate students (the two researchers and an applied mathematician) developed a fraction-based module. This module was created as a part of the Center for Mathematics Education of Latinos/as (CEMELA)[1] and served as the children's thinking activities for this study since it centers on the written work and follow-up explanations of four 5th-graders. To document what perceptions emerged in this setting, two females in their junior year of coursework, Nora and Alisa[2], were chosen to be cases. They were selected based on their expressed interest and their responses on a background profile and mathematical autobiography, specifically because of their differences in their perceived ability level and educational backgrounds. We focused on two case studies because we wanted to document not only what perceptions emerged but also to understand the source of these perceptions, since in our view perceptions are based on an individual's prior knowledge, experiences, and beliefs.

Data came from a variety of sources. Both a pre- and a post-assessment on the children's thinking activities were completed, with similar content questions asked on both and a self-evaluation portion added to the post-assessment. A background profile and mathematical autobiography were also collected, and all in-class discussions were video- and audio-taped.

All written work done during the activities was collected, and two individual interviews were conducted. The first interview took place prior to the start of the children's thinking activities, with the primary purpose of understanding the case study participants' backgrounds and their responses from the pre-assessment. The focus of the second interview, which took place after the children's thinking activities were completed, was on the case study participants' reflection on their experiences, as well as on their responses from the post-assessment.

Findings indicate that while both case study participants attended to their own mathematical understanding, they did so in very different ways. Alisa perceived the children's thinking activities as an opportunity in which she could take a step back and begin to understand the concepts more deeply. This perception was seen various times throughout the data. Alisa stated that prior to the children's thinking activities she was aware that one could interpret a fraction in different ways based on the context in which it was embedded. However, she perceived herself as not completely knowledgeable about these different fractional interpretations. As a result of her experiences within this setting, Alisa was able to become more cognizant of this lack of knowledge. Additionally, Alisa commented on how her understanding of equivalence of fractions had been strengthened. For example, she had never really considered different-sized objects when making fractional comparisons. She stated that, "… half of one jelly bean and half of a hundred jelly beans, you obviously have more, but it doesn't mean that the half is not equivalent." In other words, she was commenting on the power of understanding that $\frac{1/2}{1} = \frac{50}{100}$, although the size of the wholes is different.

On the other hand, Nora generally avoided discussing her understanding of mathematics and instead perceived this lack of content understanding as a result of incomplete explanations from her instructor: "…cause we asked him multiple times, like 'Is this what you mean?' and he never really completely explained exactly what he meant." The only time Nora directly commented on her mathematical understanding was when she addressed her difficulties with the children's thinking activities stating, "The videos just confused me and made me work very hard to keep the understanding that I had about fractions."

In regards to mathematical content, Alisa was very open about how the children's thinking activities altered her view of fractions. Specifically, Alisa commented on the meaning of fractions and the relationship she saw between whole numbers and fractions. She noted that it was important to see a fraction as not just separate whole numbers but as part of a larger class of numbers in which there is a connection between pieces and wholes. She perceived whole numbers and fractions to be similar in certain aspects, but that fractions "are a whole set of new numbers on the number line … they have their own unique way of doing things." In contrast, Nora never openly attended to the mathematics content present in the children's thinking activities, even though there were numerous opportunities to do so.

While both case study participants saw themselves as mathematically capable, Alisa was more willing to share her view of mathematics than Nora. While there may be many reasons for this openness, it is our contention that the level of mathematical confidence the case study participants brought with them was a major factor. Alisa's perception was that she was very strong mathematically, which she associated to her family background and her prior academic achievements. This perceived strength is supported by the leadership role Alisa took within her group discussions and the deep level of self-reflection she provided on both her self-evaluation and in the interviews. Nora, on the other hand, noted her lack of confidence in understanding fractions, attributing it to her non-use of them in her everyday life. In the classroom, she remained very quiet and assumed a docile role within the group environment. Her reflections were mainly focused on her perceptions of the students and their thinking, and how mathematics, in her view, should be taught. It appears that her insecurity with fractions limited her vocalization of her mathematical experience, both in terms of her own understandings and her conception of mathematics.

With this study we are contributing to the research on the psychological factors that are associated with how preservice teachers learn and understand fractions. Specifically, our findings inform teacher educators of different factors to consider when developing teacher education content coursework, in particular the preservice teachers' level of mathematical confidence. Additionally, while prior research has shown the benefit of using children's thinking as a method to develop both preservice teachers' content and pedagogical content knowledge (e.g. Crespo, 2000; Tirosh, 2000), this study demonstrates that it can also be used as a setting for comprehending preservice teachers' perceptions. This knowledge aids teacher educators in determining the view of mathematical content and understanding of mathematics that preservice teachers possess. Understanding perceptions about mathematical content is vital because it can illustrate what misconceptions preservice teachers may have. Knowledge of the perceptions that preservice teachers have in regards to their own understanding of mathematics is essential so that teacher educators can develop instructional strategies and settings that support preservice teachers' continued mathematical growth.

Endnotes

1. CEMELA is a Center for Learning and Teaching and is funded by the National Science Foundation under grant ESI-0424983. The views expressed here are those of the authors and do not necessarily reflect the views of the funding agency.

2. Both preservice teachers' names are pseudonyms.

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USING STUDENTS' MATHEMATICAL THINKING TO ORCHESTRATE A CLASS DISCUSSION

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Learning to orchestrate class discussions that are based on students' mathematical thinking is one of the most difficult aspects of learning to teach in a reform-oriented way. Based on a research project in which student teaching was restructured so as to focus on using student thinking, we describe three factors that inhibited student teachers' abilities to lead productive classroom discussions: listening, understanding, and recognizing.

Mathematics classrooms wherein the teacher promotes mathematical discussion based on students' mathematical thinking, and then orchestrates that discussion in ways that facilitate yet deeper mathematics thinking, is a hallmark of reform visions for mathematics classrooms (National Council of Teachers of Mathematics, 1991). Orchestrating such discussions, however, seems to be one of the most difficult aspects of this approach to teaching (Sherin, 2002), particularly for novice teachers. Many factors likely contribute to this difficulty, some of which have been discussed in the literature. For example, Wood (1998) draws the distinction between two patterns of classroom discourse: funneling and focusing. In the funneling pattern, the teacher directs classroom discussion toward a preconceived "best solution strategy" and away from alternative and wrong strategies. By contrast, in the focusing pattern, the teacher seeks to capitalize on alternative and wrong strategies as a means of elevating students' mathematical thinking toward important mathematical ideas. When placed in a situation where students' mathematical thinking is being made public in the classroom, novice teachers often follow the funneling pattern, turning the sharing of alternative solution strategies into a lesson on the value of multiple strategies ("there are lots of ways to solve these problems") rather than a lesson on the important mathematical ideas that are motivated by these alternative solutions. Our research set out to put novice teachers (student teachers) in situations where they were eliciting students' mathematical thinking, and then to study how they navigated the road to using that thinking in their teaching.

One of the drawbacks of the traditional student teaching model is that student teachers spend much of their energy trying to survive instead of focusing on the act of teaching. This is because even a portion of a practicing teachers' job is a great deal of work and takes a great deal of time—the entire job is often overwhelming to student teachers. This focus on survival often results in student teachers and their cooperating teachers focusing their attention on developing skills in student and classroom management (Peterson & Williams, 2001). Little time is left or taken to discuss lesson planning and revision, using student thinking and core mathematical concepts.

Methodology

Six student teachers and three cooperating teachers were purposefully selected to participate in the study. The student teachers were chosen based on feedback from their past teachers, who were asked to recommend students who they felt were primed to excel during

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student teaching. Four of the student teachers in this study were placed in middle school classrooms with teachers who were approaching their instruction from an NCTM *Standards* perspective and were using a reform-based curriculum. The other two student teachers were placed in a high school setting with a new teacher who taught more traditionally but was open to learning new ideas and who supported the student teachers in implementing such ideas.

The student teachers spent significant time during the first five weeks planning together and teaching individually just one lesson per week. All six student teachers, the cooperating teacher and the university supervisor observed each of these lessons and participated in a reflection meeting at the end of the day. Student teachers also conducted directed observations and student interviews, and wrote extensively as a means of processing and synthesizing what they were learning. In this way, we set out to alter the structure of student teaching in ways that we believed (based on the literature and on personal experience) would refocus the purpose of student teaching away from classroom management and on perceiving, developing and using students' mathematical thinking in the classroom.

Analysis of the overall structure of this student teaching experience indicated that the refocus was achieved. Student teachers and cooperating teachers alike universally perceived the primary purpose of the student teaching experience to be that of learning how to facilitate and use students' thinking in the everyday acts of teaching. Although the structure focused student teachers and cooperating teachers on their students' mathematical thinking, merely eliciting student thinking is clearly not sufficient to produce mathematics lessons that are actually influenced by that thinking. We thus asked the following research question: What factors influence student teachers' abilities to use students' mathematical thinking to orchestrate a class mathematical discussion?

Results

Regardless of whether the student teachers were using reform curricula in a middle school or traditional curricula in a high school, they ran into similar issues when trying to conduct a whole class discussion that would assist all students to come to a deeper understanding of the underlying mathematics. We describe here three factors we identified that inhibited their abilities to lead productive classroom discussions: *listening, understanding,* and *recognizing*.

The student teachers believed that their lessons would be more productive if their students were given opportunities to make comments or to share their solutions to problems. Sometimes this "student sharing" became a goal in and of itself. Thus, student teachers sometimes felt that their goal had been met when students shared their work. They then failed to *listen* to the content of the explanation. In such situations, the substance of the student explanation was not viewed as useful, merely the fact that the student was explaining.

Once student teachers began to listen to the substance of the student thinking, they struggled to *understand* what was being said. When leading a class discussion where students are encouraged to share their thinking and methods for solving a problem, a student teacher's knowledge or experience may not allow them to understand a students thinking. Because the student strategy is unique or different, the student teacher may not understand the point the student is trying to make even though they are listening. Student thinking that is not understood cannot be used to enrich the class discussion for the benefit of all students.

Beyond listening to and understanding a student's mathematical thinking, the student teachers often lacked the ability to *recognize* the pedagogical or mathematical value of what was being said. At least two types of roadblocks inhibited the student teachers from recognizing the value of students' thinking. The first roadblock to recognition occurred when the student teachers filled in the blanks rather than asking the students to do so. Students often use imprecise language when answering questions or sharing their work. The student teachers often forgave this imprecision, assuming that the student understood what they were inadequately describing. They failed to recognize such moments as important opportunities to push the student to clarify their statements and thinking. A second roadblock to recognition happened when the student teacher had in her mind an idea that she wanted to come out of the student discussion (i.e., the funnel pattern). As a result of funneling toward that particular idea, she failed to recognize other mathematical ideas that could have been accessed by way of the students' thinking.

Once a student teacher has listened to student thinking with the intent of using that thinking, understood the mathematics presented, and recognized its pedagogical value, they are positioned to use that thinking for the benefit of all students. Even with this proper positioning, however, limited knowledge of how to use student thinking may inhibit their ability to do so. An example of this occurred when a student teacher was launching a lesson wherein students were going to solve problems using a calculator. They were to input equations and look at the tabular outputs to make decisions about the situation. In anticipation of this, the student teacher asked the students to identify the independent and dependent variables in the equation $C = 2\pi r$. This was to be a quick review so the students could properly input equations into a calculator. The student teacher wanted to hear that r was the independent variable and C was the dependent variable. However, one student said that if she put a piece of string around the circle, she could use that length to find the radius; this would make the circumference the independent variable and the radius the dependent variable. The student teacher listened to the student and understood the mathematical correctness of what was said. She also recognized that the student had uncovered a complexity of the concept of independent and dependent variables and that this would be a good time to explore that complexity. This exploration did not take place, however, because she did not know how to use the student thinking to initiate a class discussion. Because independent and dependent variables were not the goal of the day's lesson, the student teacher had not anticipated such a discussion and was unprepared to pursue it. The student teacher had the skills to listen and understand what the student was saying and even seemed to recognize the importance of what was said, but she did not have the knowledge to integrate the thinking into the lesson.

Conclusion

Although much has been said about the importance of using significant mathematical tasks in order to elicit students' mathematical thinking, relatively little is known about the factors involved in using that mathematical thinking effectively. These results illustrate the complexity of this issue. More work needs to be done on designing and researching the effectiveness of "learning to teach" activities that can help novice teachers learn how to listen to, understand and recognize the value of their students' thinking, and then be able to use that thinking in order to orchestrate meaningful mathematical discussions.

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"HEY PAT, I'VE GOT A SPECIAL PATTERN FOR YOU": LEARNING AND TEACHING WITH AL

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This study looks at elementary preservice teachers' thinking when taught by an instructor with specialized mathematics abilities. As they struggle to understand Al's multiplication algorithms and his use of square numbers as basic facts, they gain an appreciation for his divergent thinking and realize the depth of mathematical knowledge they need as teachers for all children in inclusive settings.

In 2001, Al and I met at a birthday party for a faculty member at the Center for Human Development. Al was "wowing" people with this ability to multiply large numbers very quickly. When Al sat down next to me, I asked him to show me another problem. He did. I then asked him if he was thinking about the two problems in the same way. He laughed and said, "Of course, this pattern works for all numbers but I have better ones for some numbers." Thus Al and I began to have weekly math meetings with Al calculating large products and me asking questions about his thinking. Al shared that he grew up institutionalized. He entered at age two and did not leave until such institutions were closed and he was placed in an adult group home. His education was limited. As Al and I worked, we began to develop a routine comfortable for both of us and I began to recognize Al's math vocabulary. Al's use of "pattern" is referring to a procedure he has developed for a particular set of numbers. Al's basic algorithm for multiplication is based on a visual image of a "barn-door." Other procedures for working with squaring numbers with zero as the center digit are called Mac Donald problems after the symmetry Al sees both in the Mac Donald M and in this set of numbers. While Al also has a love for calculating roots of numbers, I asked him to share his thinking about multiplication and teach a lesson to my preservice students. Al has been teaching with me every semester since.

This study documents our experience of teaching together and the mathematical understandings and teaching insights gained by me, by Al, and by the preservice student teachers in the elementary mathematics methods class. One student spoke for many when she wrote, "As a teacher in inclusive settings we interact with children with special needs but we don't always see that the student has special abilities. I would not have probed a student's thinking the way we did with Al today. I would have missed the child's strengths - my math is not good enough to recognize what Al was doing without his being able to talk me through it."

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BEYOND POSITIVE DISPOSITIONS: EXPLORING PRESERVICE TEACHERS' ORIENTATIONS ABOUT MATHEMATICS ACHIEVEMENT IN HIGH-POVERTY CONTEXTS

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With the changing demographics in the United States it is critical that preservice teachers be adequately prepared to provide effective, high-quality mathematics instruction to all students regardless of their race, ethnicity, gender, socioeconomic status, culture, or other characteristics. Recent demographic data indicates that approximately 48.3 million children, prekindergarten to grade twelve, are enrolled in United States public schools. Of these students, 43% are members of a racial or ethnic minority group, 19% speak a language other than English at home, and 14% receive special education services (Rooney, 2006). Further, the U.S. Census Bureau data reveals that about 18% of children under eighteen years old are living in poverty.

Now, more than ever equity in mathematics education is critically salient. With the passage of the No Child Left Behind Act, schools must address traditionally underserved populations (i.e., students of color, students who live in poverty, females, students who are not native speakers of English, students with disabilities) that have long been ignored. Teachers must function as agents of change and challenge the pervasive societal belief that only some students are capable of learning mathematics. They must hold high expectations for all students and these expectations must be reflected in all aspects of the mathematics teaching and learning process – from instructional planning and decision making to implementation and assessment. Teachers must also challenge the implicit, often unspoken, notion that only the experiences of some students are valuable and reflect mathematical knowledge. The lived experiences, prior knowledge, intellectual strengths, and personal interests of all students should be valued and utilized as a springboard for learning. Therefore, mathematics teachers must reconceptualize the nature of students' mathematical knowledge, bridging informal and formal mathematical experiences, and providing students opportunities to demonstrate their understanding of mathematics in a multitude of ways.

Given the critical importance of reaching all mathematics learners, this study sought to examine elementary preservice teachers' perspectives of mathematics teaching and learning in high-poverty contexts. Specifically, (a) How does teachers' orientations toward mathematics achievement differ from dispositions? and (b) What attributes reflect teachers' orientations toward mathematics achievement?

Theoretical Framework

The theory of culturally relevant pedagogy (Ladson-Billings, 1995) served as the theoretical framework for examining elementary preservice teachers' perspectives of mathematics teaching and learning in high-poverty contexts.

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The pre-service teachers' orientations toward mathematics achievement were categorized and analyzed with respect to the three broad propositions of culturally relevant pedagogy: conceptions of self and others, social relations, and conceptions of knowledge. Ladson-Billings (1995) asserts that culturally relevant teachers exhibit the following broad qualities with respect to the underlying propositions: (a) Conceptions of self and others suggests that culturally relevant teachers hold high expectations for all students and believe all students are capable of achieving academic excellence; (b) social relations infers that culturally relevant teachers establish and maintain positive teacher-student relationships and classroom learning community as well as are passionate about teaching and view it as a service to the community; and (c) conceptions of knowledge suggests that culturally relevant teachers view knowledge as fluid and facilitate students' ability to construct their own understanding.

Methodology

Participants in this study are elementary preservice teachers enrolled in an alternative certification program. The full-time, accelerated program focuses on and is specifically designed for individuals committed to teaching in urban, high-poverty contexts.

Participants engaged in professional learning communities throughout the study. The professional learning communities, referred to as mathematics circles, consisted of three preservice teachers. Together they explore and tackle (a) personal ideologies, (b) myths and stereotypes, and (c) pedagogical issues related to mathematics teaching and learning in high-poverty contexts. Circles met bi-weekly over the course of a semester. At the initial session preservice teachers completed a questionnaire related to their dispositions, ideologies, and orientation toward mathematics achievement in high-poverty contexts. At the subsequent sessions preservice teachers shared pedagogical artifacts (lesson plans and student work), and oral and written testimonials about field-based teaching episodes. Testimonials were stimulated utilizing prompts, but participants were also strongly encouraged to share any information that they wanted to with the group.

This study employed a qualitative research design. Data sources for the study were a questionnaire, pedagogical artifacts, written testimonials, and researcher field notes from oral testimonials. Interpretive analysis was used to identify themes and patterns among the data sets. Themes and patterns were coded and organized into three categories reflecting the broad propositions of culturally relevant pedagogy: conceptions of self and others, social relations, and conceptions of knowledge.

Results and Discussion

Preliminary findings indicate that orientations toward mathematics achievement do differ from dispositions. The National Council of Accreditation for Teacher Education (NCATE) defines professional dispositions as "The professional behaviors educators are expected to demonstrate in their interactions with students, families, colleagues and communities. Such behaviors support student learning and development and are consistent with ideas of fairness and the belief that all students can learn." This study found that orientations tended to be more action-driven (i.e., what teachers actually do) while dispositions reflected more of an affective stance (i.e., what teachers believe and feel).

This study also found that attributes of orientations toward mathematics achievement are closely aligned with the theory of culturally relevant pedagogy in that they express notions of self and others, social relations, and knowledge.

The preliminary findings offer insight into elementary preservice teachers' perspectives of mathematics teaching and learning in high-poverty contexts. Given the changing student demographics of 21st century schools, this study is germane and contributes to the knowledge base of mathematics teacher preparation, and more broadly to issues of equity and achievement in mathematics education.

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BREAKING NEW GROUND: UTILIZING AN ON-SITE UNIVERSITY FIELD EXPERIENCE TO FACILITATE TEACHER DEVELOPMENT

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This study investigated requiring an additional field experience for prospective secondary mathematics teachers prior to beginning an internship. This work extends this underdeveloped area of research by: (1) having prospective teachers be instructors during this field experience, and (2) providing a new context (remedial math courses at a university) through which to prepare teachers and study models of teacher education.

The on-going teacher shortage in California has prompted the Board of Education to offer an alternative option to the traditional semester of student teaching for students seeking an initial credential at the secondary level. This option (internship) allows prospective teachers to be hired for one school year, during which they assume full responsibility for classes and are mentored by the district. While this approach allows interns to be paid and provides districts with individuals to teach classes, many prospective teachers are not prepared for such extensive teaching responsibilities and struggle to successfully complete their internships. The minimal support interns are provided is not sufficient to compensate for the weak knowledge base of most interns.

This study examined one approach to supporting success within the internship option by exploring the outcome of requiring an additional field experience prior to beginning an internship. For this new field experience each participant taught a section of intermediate algebra at our university with mentoring and rigorous supervision provided by a faculty member in the mathematics department. The new field therefore provided prospective teachers with the opportunity to learn about how to teach algebra in a controlled situation (supervised university course), yet without the added dimension of attending to the needs of adolescent age students. This approach is modeled after *Lab Schools*, which were K-12 schools created on university campuses to provide a context for student teaching, and have been shown to be effective in teacher development (Cardellichio, 1997). The following research questions guided data collection: (1) What do prospective teachers appear to learn during the new field experience? (*Phase One*), and (2) What factors appear to support and/or hinder the effectiveness of the new required field in preparing prospective teachers for a successful internship? (*Phase Two*).

Preliminary Results

Preliminary results suggest: (1) participants felt more prepared and confident to teach algebra at the secondary level at the conclusion of the field experience, (2) participants expressed less fear about pursuing an internship than they did prior to beginning the experience, (3) participants gained knowledge in multiple aspects of teaching practice (e.g., pedagogical content knowledge), (4) participants perceived the collaborative structure of the meetings and observation feedback to be most useful aspects of their experience, and (5) the university supervisor believed that both participants grew in their ability to both teach and reflect on their practice.

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COLLABORATING *WITH* MIDDLE GRADES MATHEMATICS TEACHERS TO IMPROVE CLASSROOM DISCOURSE

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This poster reports findings from a five-year NSF-funded research collaboration (Herbel-Eisenmann, PI) involving the authors and eight middle-grades mathematics teachers. The goal of this collaboration is to understand how having mathematics teachers do action research on their classroom discourse can impact their discourse practices and beliefs over time.

The action research model adopted for this study is based on an apprenticeship model (CITE) in which: 1) the university-researchers (URs) collected and analyzed data to report back to the teacher-researchers (TRs) (AY 05-06), and 2) the TRs conduct their own action research projects within the community of this collaboration (currently). During Stage 1, each TR was observed for a week in September, November, January, and March. All observations were video-taped and transcribed using Transana (Fassnacht & Woods, 2005). The observations were coded based on Lemke's (CITE) Activity Structures (AS), allowing us to identify the number of minutes spent on each AS (e.g., Going over Homework, Seatwork). This information then was used to create pie charts of the percentage of time spent on each AS each week and across the three weeks (1). We also created mappings that showed the flow and organization of AS for each lesson. Finally, a turn-length word count was done to determine the number of words spoken by each teacher versus their students in each week as well as across the first three weeks. After reporting this descriptive quantitative information to teachers, they were asked to choose AS to be analyzed qualitatively by the URs, using systemic functional grammar (Halliday CITE).

Based on an analysis of project meeting transcripts, this poster will focus on the following findings about the teachers' discussions of the data and analyses that we reported to them:

- The teachers were surprised by the number of shifts in AS in their classrooms and thought that many of the shifts were transparent to students.
- The teachers were disappointed (but not surprised) by the number of words they spoke in comparison to their students.
- Most teachers asked us to analyze AS in which they were interacting with small groups or individual students.
- All teachers identified language patterns that they thought might be undermining their professed beliefs.
- Many teachers are interested in using revoicing more strategically, not "interrupting" students as often, being more careful about vague referencing, and trying to use "focusing" patterns rather than "funneling" (Wood CITE) ones.

Although there is a growing body of literature that analyzes mathematics classroom discourse, little of this work has been done *with* teachers to better understand what and how they find this information to be useful. The work reported in this poster can help teacher educators better understand teachers' perspectives on their classroom discourse.

Endnote

The fourth week of data was reserved for validity checks, e.g., following up on observed patterns, searching for discrepant events.

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ELEMENTARY PRESERVICE TEACHERS' EXPERIENCES WITH TANGRAMS

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Use of concrete manipulatives promotes students to develop long-term skills that are practical for helping children to move from the concrete to the abstract level. However, establishing this bridge between these levels requires careful construction of manipulatives by a teacher (Hartshorn & Boren, 1990).

In this research study, our goal was to examine how preservice teachers were challenged and succeed in tangrams activity, and how they would use tangrams in their classrooms and the reasons behind their choices. In addition, we examined how they would use a learning activity involving tangrams to teach other concepts of mathematics to students.

Twenty-nine junior elementary preservice teachers who were enrolled in Mathematics Teaching course participated in the tangram activity. The tangram activity consists of two phases that took three weeks for preservice teachers to complete. In the first phase, preservice teachers created 7 geometric figures of tangrams (i.e., two small size triangles, one medium size triangle, two large triangles, square, and parallelogram) following the instruction of Russell and Bologna (1983). In second phase, 5 different geometric figures (i.e., triangle, square, rectangle, trapezoid, and parallelogram) were created using 3 (i.e., two small triangles and one medium size triangle), 5 (i.e., two small size triangles, one medium size triangle, square and parallelogram) and 7 pieces of tangrams. Then, they were asked to calculate perimeter and area of each geometric figure, and compared their findings.

Although preservice teachers had no experience with use of manipulatives including tangrams, they were easily engaged and involved in the activity. However, they were challenged when they were creating geometric figures due to their lack of experience with the use of manipulatives. They calculated length of the sides of the geometric shapes (e.g., small triangle, parallelogram) using side of a square as 1 unit. As they calculated perimeter and area of each geometric figure, they compared values of perimeter and area across 5 geometric figures (a triangle, a square, a rectangle, a parallelogram, and a trapezoid). They noticed that perimeter for each geometric figure changes area is always constant.

Preservice teachers indicated using tangrams would be helpful for elementary grade students to better understand the characteristics of geometric shapes by rotating, turning over, or sliding the pieces. They also suggested that tangrams could be used for teaching fractions, ratio and proportions. However, they were unable to see that this activity leads to development of general formulas for finding the area of a particular shape. It is also helpful for students to better understand the formulas for finding perimeter and area of geometric figures and review them after they have been taught. Effective teacher preparation requires that preservice teachers involve in the learning activities that they will utilize in their classrooms. By reflecting on their participation in challenging and meaningful activities, they can begin to think more effective classroom practices (Thatcher, 2001).

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HOW PRE-SERVICE ELEMENTARY TEACHERS DEFINE BEING GOOD AT MATHEMATICS

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There is some evidence that pre-service teachers' (PSTs') prior experiences and beliefs affect their teaching confidence in mathematics (e.g. Brady & Bowd, 2005) and some evidence that pre-service teachers' prior experiences and beliefs affect their eventual teaching (e.g. Artzt & Curcio, 2003). One type of belief that may affect PSTs' teaching confidence and future teaching is PSTs' beliefs about what it means to be successful in mathematics. There has been some previous work in how students define success in mathematics (e.g. Nicholls et. al, 1990). However, there appears to be little or no work in how pre-service teachers define mathematics success. The goal of this study is to examine these beliefs. In particular, I pay attention to ideas of mastery and performance orientation and to ideas of natural ability and effort in mathematics.

The six participants were randomly selected from students who were enrolled in the first content course for elementary teachers at a mid-Atlantic university; all were female and five were first year students for whom the content course was their first college math class. The PSTs participated in an interview at the beginning and end of the semester. Both interviews were designed to examine the pre-service teachers' definitions of what it means to be good at mathematics by focusing on their past and present experiences in math courses.

I analyzed the data using a qualitative coding method that focused on the varying definitions that the pre-service teachers gave and ranked these definitions according to prominence. I found a fair amount of variation in how these six pre-service teachers defined being good at mathematics. For example, one PST seemed to define being good at math predominantly as mastering the material; yet another pre-service teacher seemed to define it as performing well on exams and getting good grades. Several PSTs seemed to define being good at math as having a natural ability for it and several defined being good at math as the amount of effort a student exerted toward mathematics. Overall, I found that the definitions provided by the PSTs were more robust than I had expected based on the available literature.

This study provides support for the variety in definitions of success in mathematics. Further, the study has implications for research on preparing teachers. For example, a PST who believes being good at mathematics is a matter of natural ability may be less inclined to learn the material in their content course well and may end up feeling unconfident in their ability to teach mathematics. More research is needed to understand these implications.

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MATH CONTENT COURSES FOCUSED ON CHILDREN'S THINKING: ADDRESSING PRESERVICE TEACHERS' EFFICACY AND BELIEFS ABOUT MATHEMATICS

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This study presents results from an ongoing evaluation that examines the outcomes of an NSF grant project that provided prospective teachers with a supplemental course text to help them develop an understanding of how children learn mathematical concepts. Results from earlier studies indicated that prospective teachers reported significantly higher efficacy regarding their ability to understand math and their ability to teach mathematics to children and attributed those differences to using the textbook. Results from the current study verify through control and experimental groups that efficacy beliefs can be changed in math content courses that focus on children's thinking. The present study also determined that prospective teachers' efficacy for understanding and teaching math is correlated to distinct beliefs about mathematics such as the belief that children solve and think about mathematics differently than adults and that children may be guided to mathematical understandings through inquiry methods rather than by direct instruction. Teaching math content courses focused on children's thinking may be a valuable way to address and change beliefs and efficacy about mathematics and may lead to changes in future teaching practices.

The results of this study are derived from the evaluation results of an NSF project, Connecting Mathematics for Elementary Teachers (CMET). The primary goal of the CMET project is to connect the mathematics that prospective elementary teachers are learning in their content courses with how children learn and think about mathematics. To reach this end, a supplemental textbook was designed to align with current math for teachers textbooks. This supplement provides college students with an insight into how children think about a variety of math concepts through sample problems and research into how children think.

This textbook has been aligned to supplement a variety of mathematics content textbooks for prospective elementary teachers and extend the traditional textbook by providing information about how children think, misunderstand, and come to understand mathematics. These descriptions are based on current research and include: how children come to know number, how they understand addition as a counting activity, how manipulatives may embody mathematical activity, and how concept image impacts children's understanding of geometry. It is important to note that CMET is not a methods textbook, but it is designed as a supplement for traditional mathematical content courses for elementary teachers.

USING IDENTIFYING NARRATIVES TO EXAMINE INTERNS' RESPONSES IN AN ELEMENTARY MATHEMATICS TEACHER EDUCATION COURSE

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The impact of Teacher Education (TE) courses on the practice of future teachers has been a long-standing question. Research often reports on the weak impact of such courses. This study tackles this issue from a somewhat atypical perspective--by investigating interplay between prospective teachers' identifying narratives and their interactions with the goals of a TE course focused on mathematics reform and equity.

Drake and Sherin (2006) looked at teachers' differences in adaptations of reform curricula alongside their mathematics narratives. Sfard and Prusak (2005) looked at differences in students' mathematics achievement in relation to their "identifying narratives", naming two categories: *actual identity* narratives (who one is in the present) and *designated identity* narratives (who one hopes or plans to become). Building on these, two teacher interns' identifying narratives were examined for insights into their responses in a mathematics TE course exploring issues of equity and reform. The author (also the instructor) chose two interns that seemed to have different responses to the course: Meredith cautiously accepted some new ideas, and Janelle resisted the ideas of the course. Data include pre- and post- course assessments, videotapes of course discussions, and assignments.

Analysis revealed different patterns of identifying narratives between the interns. Meredith's changed during the semester. She told 14 designated identity narratives in the data and 13 actual identity narratives. She told 10 identifying narratives in the first two weeks, 8 of which were designated, In the last two weeks, she told 3 identifying narratives, one of which was designated. The character of her identifying narratives changed; those in the first weeks mention a variety of wishes, and narratives in later weeks focus narrowly on a sense of agency and a student-centered pedagogical stance. In contrast, Janelle did not demonstrate qualitative changes in her identifying narratives, and after the pre-assessment, her number of them remained stable. Her lack of designated identity statements overall is notable. She told 36 actual identity narratives and 6 designated identity narratives in the data collected. They were about a variety of topics, such as communicating with parents and learning about curriculum. Five of her 6 designated identity narratives occurred during the first week of class, with the remaining one on the last week. Janelle did not discuss mathematics in her designated identity narratives. Her identifying narratives were characterized by a lack of agency, as she talked about her inability to influence students and society.

The analysis gleaned insights into the relationship between these interns' identifying narratives and responses to reform and equity in a math TE course. It is possible that the ways in which teachers construct identifying narratives impacts their responses to reform and equity in mathematics education. The study has implications for teacher educators and researchers seeking to understand and attend to teachers' visions of mathematics teaching and learning.

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PROMPTING TEACHER KNOWLEDGE DEVELOPMENT USING DYNAMIC GEOMETRY SOFTWARE

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Teachers' mathematical knowledge for teaching (MKT) has been found to impact teacher change (Hill, Rowan, & Ball, 2004). This study examines teachers' development of MKT as they implement dynamic geometry software (DGS) with a coach. We relied on Laborde's (2001) account of expertise in teaching using DGS as an analytical tool to categorize teachers' ways of using DGS to teach geometric reasoning. We found a relation between teachers' expressions of MKT and their ways of using DGS, based on qualitative evidence.

Introduction and Theoretical Framework

Teacher knowledge has taken center stage as researchers seek to improve mathematics education in the United States. However, content-specific professional development has not brought about significant teacher change. Alternatively, several studies have surfaced that point to specialized teacher knowledge as a key to effective reform. The seminal study relating teacher knowledge to student achievement found that *mathematical knowledge for teaching* (MKT) was a key to predicting student gains in first and third grade (Hill, Rowan, & Ball, 2004). We explore ways of improving one aspect of MKT, *geometric reasoning*, as teachers participate in collaborative professional development emphasizing the use of dynamic geometry software (DGS) in their classrooms.

Components of Mathematical Knowledge for Teaching (MKT)

Subject Matter Knowledge	**Pedagogical Content Knowledge**
Common Content Knowledge (CCK)	Knowledge of Content and Students (KCS)
Specialized Content Knowledge (SCK)	Knowledge of Content and Teaching (KCT)
Knowledge at the Mathematical Horizon	Knowledge of Curriculum

Table 1. Components of MKT

Ball, Bass, Goffney, and Sleep (2006) defined mathematical knowledge for teaching as the "mathematical knowledge, skills, [and] habits of mind that are entailed by the work of teaching" (Slide 5). More specifically, Ball and her colleagues categorize mathematical knowledge for teaching MKT in six parts. Common content knowledge is basic, lay-person knowledge of the mathematical content. Specialized content knowledge (SCK) is the way the mathematics arises in classrooms, such as for building representations. Knowledge of content and students (KCS), is knowing how students think about mathematics. Knowledge of content and teaching (KCT) involves knowing the most effective examples or teaching sequences. We understand Shulman's (1987) definition of pedagogical content knowledge to be a marriage of knowledge of content and students (KCS) with knowledge of content and teaching (KCT). Knowledge of curriculum and knowledge at the mathematical horizon are the final components of mathematical knowledge for teaching. In this study, we focused on specialized content knowledge (SCK), knowledge of content and students (KCS), and knowledge of content and teaching (KCT). We believe teachers engage students in substantive mathematics as they use DGS to relate and contrast variant and invariant elements in constructed objects and attend to special cases and boundary cases (Arshavsky & Goldenberg, 2005; Laborde, 2001; Presmeg, Barrett. & McCrone, 2007).

Because of the demonstrated link between mathematical knowledge for teaching and student achievement, this study sought, through microgenetic analysis, to unpack the psychological processes of how teachers develop MKT. The teachers in our study were engaged in professional development through an NSF-funded Graduate Teaching Fellows in K-12 Education (GK-12) project (Moore, 2003). GK-12 places graduate mathematics and science fellows in classrooms as *collaborative coaches* who design and deliver standards-based instruction with classroom teachers. Thus, the GK-12 model of professional development allows the transfer from theory to practice, a task which Bazzini and Morselli (2006) have identified as worthwhile, yet complex.

Research Questions

Because appropriate use of dynamic geometry software affords teachers new opportunities for discourse with their students we posed the following research question and sub-questions:

In what ways do teachers develop mathematical knowledge for teaching as they engage with a collaborative coach to prepare and implement inquiry-based lessons within a dynamic geometry computer environment? In particular in the DGS environment:

1. In what ways do teachers develop common content knowledge (CCK) and specialized content knowledge (SCK)?
2. In what ways do teachers develop knowledge of content and students (KCS) and knowledge of content and teaching (KCT)?

Methods and Analysis

Four 5^{th}-8^{th} grade teachers and three GK-12 fellows participated as case studies in this multi-tiered teacher development experiment (Presmeg & Barrett, 2003; Lesh & Kelly, 2000). However, for purposes of this report, we narrowed our focus to the mathematical knowledge for teaching development of one case study teacher. Mrs. Gerber collaborated with a fellow to teach 22 Sketchpad-based lessons in her 6-8^{th} grade classes over the course of two years. To collect data, we videotaped all but three of the 22 classes. In addition to videotapes, the teacher participated in interviews before and after each lesson and before and after the unit of Sketchpad-based lessons.

To triangulate the data, the fellows wrote pre and post reflections of the lessons. Moreover, to establish trustworthiness, the fellow from the second year (the third author of this report) engaged in respondent validation. The first author of this report acted as a participant-observer. Data was analyzed for development in mathematical knowledge for teaching with regard to common content knowledge, specialized content knowledge, knowledge of content and students, and knowledge of content and teaching. Inter-rater reliability of 81% was established among codes for identical portions of at least three transcripts per fellow, after which the first author coded the remaining transcripts.

To identify development of specialized content knowledge, data was further analyzed with respect to the levels of DGS use ascribed to novice versus expert teachers by Laborde (2001). Level one involved creating and measuring figures. Level 2 added dragging to amplify and explore geometric properties. Level three tasks allowed for dynamic solution strategies, although they continued to have paper and pencil counterparts. Level four tasks derived their meaning from and depended on dynamic geometry. The first and second author coded lessons for Laborde levels independently. We then reconciled differences by considering further details of the lessons and by reading and discussing observer field notes and fellow's written reflections.

Results and Discussion

To illustrate Mrs. Gerber's development of MKT, we discuss a classroom episode from each year. In the first episode, the students had constructed tessellations based on translating modified parallelograms by a marked vector. During the post reflection, the teacher and GK-12 fellow (first author) discussed how students had produced a tessellation with a hole in it. As they talked, it became apparent that Mrs. Gerber believed that the concept of vector depended on *position,* a misconception similar to that of a student described by Hollebrands (2007). In response, the fellow conveyed to her that vectors involve direction and distance, but not position. Thus, reflection, along with the capacity of Sketchpad to amplify what was variant (position) and what was not (vector), revealed the teacher's misconception.

Mrs. Gerber developed three components of mathematical knowledge for teaching in this episode. First, she developed specialized content knowledge **(SCK)** as she observed the fellow teaching students about vectors on Sketchpad. She commented, "I wasn't thinking of the vector showing the direction. And you [the fellow] made the comment, you said that that vector's going to show direction and distance. I thought, 'Yeah, that is what it's doing. It's moving it over.'" Her observation of the fellow in a practice-based setting enhanced her learning above the common content knowledge that she may have learned previously in a lecture course. Secondly, she developed specialized content knowledge (**SCK**) as she reflected with the fellow about a student production using DGS. The fellow identified her misconception and corrected her using the context of student work. Thirdly, she developed knowledge of content and students (**KCS**) as she and the fellow considered what the students had done wrong. Finally, the teacher developed knowledge of content and teaching (**KCT**) as she gained confidence to teach geometry using Sketchpad by watching the fellow teach and by answering student questions with the fellow present for support. During the post reflection, the fellow asked, "Do you feel comfortable enough to [use the DGS] with 6th grade [without my assistance]?" She responded, "I think I do if I can finish with 8th grade and yeah, because by 7th grade I felt better. I knew what I was doing." She had observed and co-taught the same tessellation lesson to first 8th and then 7th grade. Later in the year, she taught the lesson alone to her 6th grade class, providing evidence of growth.

In the second year a series of lessons using DGS was implemented to teach quadrilateral properties. To conclude the unit, the teacher and fellow collaboratively developed a lesson to assess whether students had learned how to classify quadrilaterals according to their properties. They decided to provide students with an unbreakable kite on Sketchpad, and assign them the task of identifying its properties and placing it within an existing taxonomy of quadrilaterals (See Figure 1). During the pre-lesson planning meeting, the fellow explained his idea for the task, which was beyond what the textbook presented. They discussed including *always, sometimes*, and *never* relationships within the arrows of the taxonomy which moved from general to specific figures (Battista, 1998). Together, Mrs. Gerber and the fellow decided that the model held true for the *always* propositions as one might travel upwards through the taxonomy, and the *sometimes* propositions held true as one moves down through the taxonomy. For example, a square is *always* a rhombus because it is located below the rhombus and connected by a down arrow in the taxonomy. As one moves upwards, from square to rhombus, one finds that a given square is always a rhombus. Alternatively, a rhombus is *sometimes* a square (as one would find by moving downwards along the arrow). They realized that propositions labeled *never* would not fit this taxonomy.

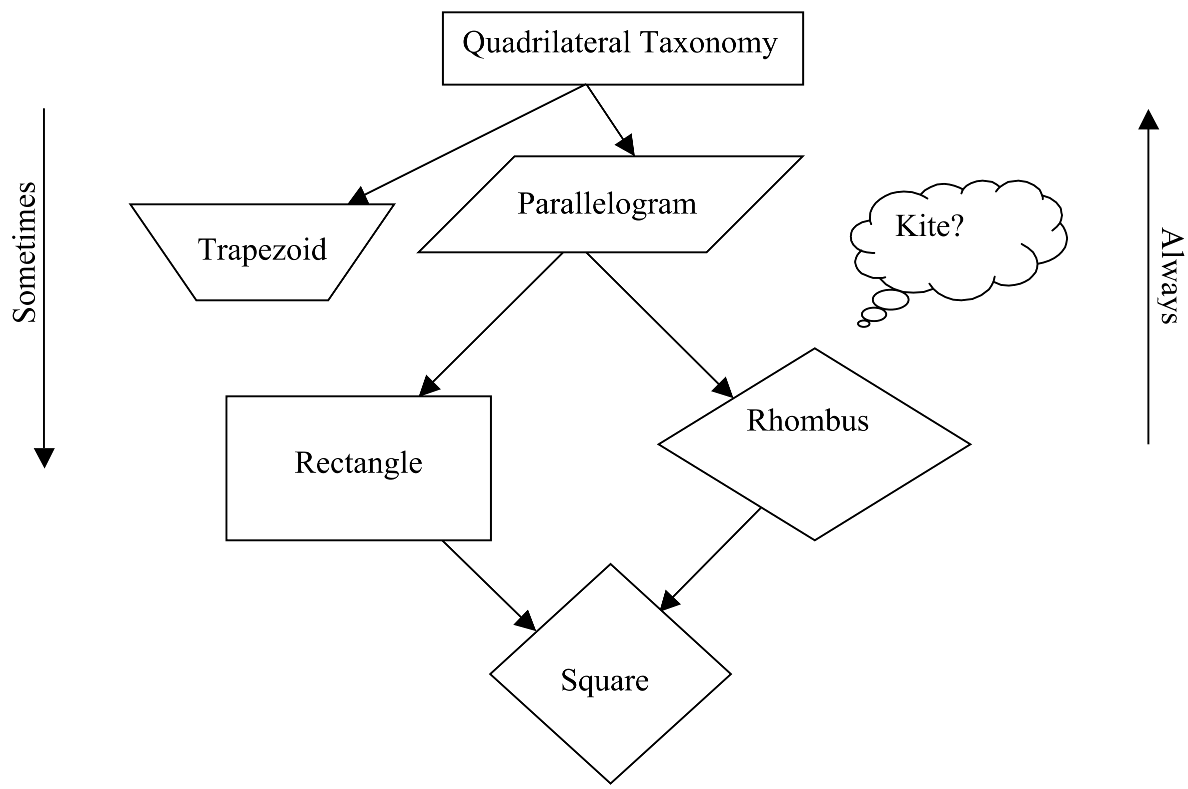

Figure 1. Quadrilateral Taxonomy

During the actual lesson, the teacher and a pair of students dragged the kite into the shape of a rhombus. Together they decided that a kite should have an arrow from it, downwards to a rhombus, signifying that a kite is sometimes a rhombus. This would match their experience in moving the kite around and forming it into a rhombus. Five months later in her final interview, Mrs. Gerber was not immediately certain that there should be an arrow from the kite to the rhombus. After reasoning through the properties of the figures in the taxonomy, she deduced that there should be an arrow showing that a kite is sometimes a rhombus. However, she indicated that she would like to check it in her notes to be sure, as can be seen in the following quote from her final interview.

Mrs. Gerber: Well and he [the fellow] did them, we did them in school too, but then we were trying to make sure that when we made this diagram that the kids could see where it was coming from because we were going with sometimes and always.

AK: OK

Mrs. Gerber: And so from the quadrilateral, uh, a quadrilateral is sometimes a parallelogram. Then, a parallelogram is sometimes a rhombus, or it's sometimes a rectangle.

AK: OK

Mrs. Gerber: OK. [Pause.] And then a rectangle or a rhombus is sometimes a square.

AK: So you had arrows going to the square.

Mrs. Gerber: Mhm, to the square. And the square was down at the end. Now, the trapezoid was up here. A quadrilateral is sometimes a trapezoid. But we put it over to the side because a trapezoid does not fit with the parallelogram. The properties don't. Or the rectangle or rhombus or square. And then we also had a kite. And kite came over here, and a quadrilateral is sometimes a kite. Now, you know what? I think we had an arrow from the kite to the rhombus. [long pause] I'd have to go back and look at the papers. But I think we did relate it down here that if the kite had four congruent sides, it could be a rhombus because your opposite angles, not opposite angles, well the ones opposite each other yeah, they would be could be congruent. I mean you could make a kite that way, and it would be OK. And so I think we had, I'm going to do this. [She drew a dashed arrow from the kite to the rhombus.]

Mrs. Gerber was finding that she did not have to recall every fact of geometry, nor look it up in a reference book. Instead, she was comfortable recalling this reasoning process on her own, even five months later. Another issue from the lesson arose when the fellow asked the class if a square is a kite. Students struggled with the question. Later, during a post reflection, the teacher and fellow discussed the reason for the students' difficulty and related it back to the property that a kite has *at least one* bisected diagonal. Students may have been thinking of a kite as having *exactly one* bisecting diagonal, which would have excluded the square (cf. Scher, 2005).

Mrs. Gerber again developed three elements of mathematical knowledge for teaching during this second episode. First, she and the fellow both developed specialized content knowledge (**SCK**) as they discussed the labeling of *always*, *sometimes* and *never* within the taxonomy. Mrs. Gerber developed knowledge of content and teaching (**KCT**) as she and the fellow raised the Laborde (2001) level of the lesson to level two from how she typically taught at level one or from a traditional text book without any use of technological tools. Using DGS *to study variation* contributed to the teacher's development of specialized content knowledge (**SCK)** as she and her students dragged the kite into a rhombus and discussed the relation of rhombus and kite. Her procedural knowledge of *properties* had developed into knowledge of *relationships* among properties as she engaged in pseudo-deductive reasoning in her final interview (Hollebrands, 2007). Moreover, her confidence to conjecture with students on Sketchpad indicates she was developing profound understanding of fundamental mathematics (Ma, 1999) and **SCK**. DGS further allowed Mrs. Gerber to hear her students and see their actions on DGS simultaneously, which led to knowledge of content and students (**KCS**). For example, as a student investigated the properties of a kite on sketchpad, the following discourse ensued:

Mrs. Gerber: Now you said opposite angles are congruent. So are you saying this one and this one?

Student: Yeah.

Mrs. Gerber: What about these two?

Student: No! [student modifies the property that he had typed.]

By questioning the student in relation to his sketch, Mrs. Gerber was able to figure out that a student had discovered one set of congruent opposite angles in a kite, but had neglected to investigate the other set. She learned that it took students *time* to reason through properties of figures, and she was surprised that some of her students who did not typically excel on paper and pencil tasks flourished in an open-ended DGS environment. This KCS result builds on Hollebrands' (2007) account of such an environment "by discussing or referring to a common shared screen, the teacher and student can communicate mathematical ideas through the use of visual images" (p. 168). Finally, as with the first episode, the post reflection allowed for the fellow and teacher to discuss a perturbation from the lesson. It was decided that a quadrilateral having *at least one* perpendicularly bisecting diagonal best defined the kite. Thus, the teacher developed specialized content knowledge (**SCK)** during the post reflection.

Student understanding in the second year was assessed through a 23-item pre/post test written instrument. The 29 7th-grade students were asked to respond with *always, sometimes*, and *never* to statements relating various subclasses of quadrilaterals. For example, one question stated, "A parallelogram is a rhombus." Students performed significantly better on the post test than on the pre test (p=.000). The mean score for the pretest was 9.4, whereas the mean for the post test was 14.7, with an average 5.3 point increase in scores. Using DGS facilitated student understanding and teacher knowledge.

Conclusions and Implications

The two episodes described here typify the coaching arrangement of this teacher with fellows and their use of a DGS. Lasting change in Mrs. Gerber's mathematical knowledge for teaching, however, can best be seen across the two-year span. In the first year, she focused largely on "teaching Sketchpad," whereas in the second year her focus was better characterized as "teaching geometry using Sketchpad as a tool." For example, learning to use DGS as a tool for showing geometric ideas facilitated her mathematical definitions of those ideas or objects since she found herself explaining to students what particular dragging or constructing actions within the software meant in terms of the geometric objects and relations among them. She learned to attend to student screens to find what aspect of her explanations or definitions they did not yet understand. She could then decide how to clarify her definition, thus developing specialized content knowledge (**SCK**).

Rather than relying solely on DGS lessons from the GK-12 program, she began in the second year to teach and envision her own DGS lessons. Lessons taught with a fellow were on average neatly one Laborde (2001) level above her own teaching, allowing her to develop knowledge of content and students (**KCS**) as she observed her students learning through dynamic processes. As Laborde stated, "a deep and precise knowledge of students' behavior and strategies" in DGS is an essential part of a teacher's knowledge (p. 309). Furthermore, Mrs. Gerber's lessons in the second year were designed to introduce concepts using DGS rather than merely working to reinforce them, a shift indicative of movement from a novice user of DGS toward expertise.

Thus, the practice-based, collaborative professional development environment established by the GK-12 program, along with the visualization afforded by dynamic geometry facilitated her development of geometric reasoning and mathematical knowledge for teaching. With respect to the first research question, Mrs. Gerber developed specialized content knowledge (SCK) to a much greater degree than common content knowledge (CCK). First, the invariant features of dynamic geometry facilitated richer mathematical discourse between the teacher and her students. Secondly, she developed SCK through exploration with her students in a DGS environment. Third, she gained mathematical knowledge by observing the fellow teach and by reflecting with the fellow about mathematical topics arising from her practice.

For the second research question, Mrs. Gerber developed knowledge of content and students (KCS) as she discussed student work and misconceptions with the fellow. Furthermore, she developed KCS as she observed and listened to students as they dragged figures on DGS. Finally, Mrs. Gerber developed knowledge of content and teaching (KCT) by watching the fellow teach at a Laborde (2001) level higher than her own. By helping students with a fellow present for support she gained confidence to implement "goal free" instruction, which Manizade (2006) identified as an element of pedagogical content knowledge of geometry and measurement for middle school. Thus, Mrs. Gerber demonstrated development and growth in mathematical knowledge for teaching throughout her two years of implementing DGS with the assistance of a collaborative coach. This in turn translated to deeper geometric reasoning for her students, substantiating Hill, Rowan, and Ball's (2004) finding that increased MKT of teachers supports student learning. Further study is needed to elaborate on coaching strategies and to link such DGS coaching to student achievement. This study was supported by National Science Foundation Grant # DGE-0338188.

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PROSPECTIVE SECONDARY TEACHERS' SUBJECT MATTER KNOWLEDGE AND PEDAGOGICAL CONTENT KNOWLEDGE: THE RAMIFICATIONS FOR TEACHING

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An exploratory study examined prospective secondary teachers' subject matter knowledge (SMK) and pedagogical content knowledge (PCK) of the concept of function when they taught an episode on exponential functions to a group of high school students. During their analyses of the lesson plans and teaching episodes, due to their weak and incomplete SMK, they failed to select an appropriate activity and examples of exponential functions. In addition, due to their weak PCK, the participants failed to ask appropriate questions to make the concept easier for students and frequently seemed unaware of students' thinking. The teaching episode also provided a relevant context to enhance their SMK and PCK of the concept of function. However, their instructional strategy for teaching changed insignificantly due to their incomplete SMK and PCK.

Challenge for preparing future mathematics teachers is recognized and what types of experiences prospective mathematics teachers need should be the central focus for the improvement of mathematics teaching in order to raise effective teachers (Cooney, 1994). It is essential to engage prospective teachers in sequences of planning, practicing, and reflecting on lessons. As they engage in preparation of a lesson plan, prospective teachers improve their knowledge of effective instruction beyond the particular lesson and subject matter (Fernandez, 2005). Over the past decade, many researchers have turned their attention to examining prospective teachers' subject matter knowledge (SMK) and pedagogical content knowledge (PCK). Since Shulman (1986) introduced these two concepts, prospective teachers' SMK and PCK have been investigated using questionnaires, and preparation and analysis of lesson plans, and hypothetical teaching situations. Although research studies (e.g., Even, 1989) have investigated prospective secondary teachers' SMK and PCK of the concept of function using questionnaires, it is surprising to see that the focus of these research studies do not include examining prospective secondary teachers' classroom instruction. The current study went beyond using questionnaires and conducting interviews by including an examination of a teaching episode.

Theoretical Framework

In this study, Even's (1989) framework consisted of seven aspects of SMK for teaching functions; these are essential features, different representations, the strength of the concept, basic repertoire, alternative ways of approaching, different kinds of knowledge and understanding of function and mathematics, and analysis of students' mistakes. These seven aspects were used to examine the participants' SMK and PCK of the concept of function and to analyze their responses to a function questionnaire and a card sorting activity. In addition, Even's framework provides a general overview of prospective teachers' SMK and PCK of the concept of function that is necessary for the present study. We used Wilson, Shulman and Richert's (1987) model to investigate the participants' growth in SMK and PCK as well as improvement in their understanding of the concept for teaching as a result of these tasks. In particular, we focused on preparation and analysis of lesson plans on exponential functions and video teaching episodes. We also organized the tasks under six aspects suggested by Wilson, Shulman and Richert (1987): comprehension, transformation, instruction, evaluation, reflection, and new comprehension.

Methods

An exploratory study was designed to examine prospective secondary mathematics teachers' subject matter knowledge (SMK) and pedagogical content knowledge (PCK) of the concept of function as well as the nature of relationships between their SMK and PCK for teaching the concept of function as they participated in tasks during the six weeks of data collection from a university in the southeastern United States. We also analyzed prospective secondary mathematics teachers' SMK and PCK of the concept of function when they taught a lesson on exponential functions to a group of high school students. The data were collected through three stages: (1) the function questionnaire, (2) the card sorting activity, (3) preparation and analysis of lesson plans on exponential functions and video teaching episodes.

The function questionnaire adapted from the studies of Even (1989) and Wilson (1992) included fourteen items addressing different aspects of SMK (e.g., examples of functions and non-functions, different representations of functions) and five items focusing on analysis of students' incorrect solutions. In the Fall 2005 semester, the function questionnaire consisting of nineteen mathematics problems was administered to thirty-three prospective teachers. The participants were asked to show their method of solution and explain their answers so that we could examine the nature of their SMK and PCK of the concept of function. The function questionnaire provided information about the prospective teachers' general knowledge of the concept of function and possible approaches for teaching the topic.

The card sorting activity (Cooney, 1996), with a different function or representation for a function on each card, included twenty-eight different examples of seven types of functions. These types of functions were linear, quadratic, polynomial, exponential, logarithmic, trigonometric, and rational functions, given in four different representations; tables, graphs, equations, and verbal descriptions. In the card sorting activity, the participants were asked seven questions and organized their knowledge about seven different categories of functions using different representations. The card sorting activity helped us determine how the participants utilized different representations and how they translated a function from one representation to another.

Jack and Sara, who presented robust mathematical understanding in two of the components (i.e., the function questionnaire and the card sorting activity), were selected for the study. After analysis of each task, we conducted interviews with participants. We asked them to elaborate on their responses. Jack was a senior teacher candidate and had a 3.58 overall college G.P.A, and Sara was a senior teacher candidate and had a 3.8 overall college G.P.A. Both of them were completing their education to receive middle and high school certification. Jack and Sara took advanced mathematics courses such as Calculus I and II, but only Jack had completed mathematics education courses at the time of the study.

Prior to preparation and analysis of lesson plans on exponential functions and the video teaching episodes, we wanted participants to analyze two lesson plans and teaching videos of these lessons because of their lack of experience in designing a lesson. We wanted the participants to gain some experience in preparing lesson plans as well as analyzing video teaching episodes of those lessons before they wrote a lesson plan and taught it. Therefore, we asked Mr. A at Southern High School to participate in this study. Mr. A, who had a masters degree in mathematics education and was pursuing his Ph.D. in mathematics education agreed to participate in this study. Mr. A agreed that he would prepare two lesson plans. The first lesson was on *Theoretical and Experimental Probability*. Instead of preparing the lesson plan before implementing it, Mr. A taught the lesson unprepared, and then wrote the lesson plan. In addition, teaching of the lesson was teacher-centered by design. The second plan was on *Fundamental Counting Principles* and was written before he taught the lesson.

We first gave the participants the lesson plans, then the videos of the teaching of those lesson plans. The prospective teachers analyzed the lesson plans, and then watched the videos of the teaching of those lesson plans. They analyzed the lesson plans as well as teaching of the lessons. Later, we met with Mr. A to talk about the classes he taught for the study. We video-recorded the interview with Mr. A so that we could have the participants watch this video as well as describe his teaching from our point of view. After their analysis, one sixty minute interview was conducted with each of the participants to discuss what they thought about the lesson plans as well as which lesson plan they found better and why. The participants were asked questions about what they thought about Mr. A implementation of these two lessons.

In the task of preparation and analysis of lesson plans on exponential functions and the video teaching episodes, the participants were given a lesson plan guideline and objectives, and asked to write a lesson plan on exponential functions. Objectives of this lesson were to have students graph exponential functions and identify data that displays exponential behavior. They tried to choose the most appropriate examples and activities in their lesson plan and tried to implement examples and activities that would be appropriate for the students' needs. We analyzed their lesson plans and then conducted one sixty-minute interview with each participant. Our goal was to examine how the participants prepared and selected the questions for their lessons. We also looked at how the participants used different forms of representations, ideas, examples, activities and explanations in their lesson plans. After analyzing their lesson plans, we videotaped the participants' classes while they were teaching their lessons to a group of high school seniors. Their teaching of the lesson took 40 minutes. When they taught their lesson on exponential functions, we looked at how they made it comprehensible to the group of high school seniors. We examined whether or not the participants recognized what made the learning of exponential functions easy or difficult for the students. Then, we conducted one sixty-minute interview with each participant. We asked each participant to watch the video of his/her teaching, and evaluate and reflect on his/her teaching of the lesson. We asked them what they thought about their teaching and whether any change occurred in their thinking about how they should teach exponential functions. We also examined whether or not the participants considered the group of high school seniors' current understanding to make instructional decisions. We looked at how the participants differentiated what the students comprehended and what they could perform (Graber, 1999), as well as how they provided the basis for their choices and actions, and how they used their subject matter knowledge of the concept of function for teaching. Analyses of their lesson plans as well as their teaching enabled us to examine how the participants transformed their subject matter knowledge for teaching.

Results

A student said that there are 2 different inverse functions for the function $f(x)=10^x$. One is the root function and the other is the log function. Is the student right? Explain.

Figure 1. Question

In the function questionnaire, the participants mentioned the nature of the univalence property and used the vertical line test exclusively to determine whether a relation was a function or not. The participants were trying to translate the equation (or a table) into a graphical form so that they could use the vertical line test. However, they did not explain why the vertical line test worked or what it meant to fail the test. They were unable to determine whether or not logarithm function and root function were inverses of $f(x)=10^x$. Thus, their responses revealed that the participants' SMK was weak and incomplete. For example, in the function questionnaire, the participants were asked to answer the question in Figure 1. The procedure in figure 2 shows how to find the inverse function of $f(x)=10^x$.

Interchanges variables of $y=10^x$:
Solves $x=10^y$ for y:
$\log x = \log 10^y$
$\log x = y \log 10$
$y = f^{-1}(x) = \log x$

Figure 2. Solution of the question

Jack was asked to determine whether or not log function and root function were inverses of $f(x)=10^x$ (see Figure 1). Jack interchanged the variables in the equation, $y=10^x$, then took the logarithm of both sides and wrote $\log(x)=y$. To find the second inverse function, he wrote $x=10^y$ and took the yth root of both sides but realized that the equation was not the same as $\log x = y$. At this point, he was still thinking that he must have made a mistake somewhere because taking the logarithm and yth root of both sides should have given the same function. Jack thought finding yth root of x equals to ten (i.e., $x^{(1/y)}=10$) would give him x value. He ended up with the value of 10. Even though Jack could not find the x value by taking yth square root of both sides, he still taught that this algorithm should give the inverse of $f(x)=10^x$. Jack thought that log and root functions were two different ways of writing inverse of $f(x)=10^x$ (see Figure 3).

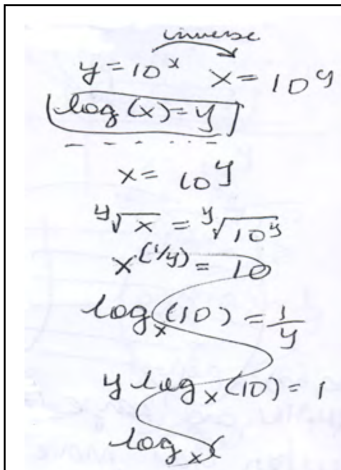

Figure 3. Jack's solution

Similarly, Sara too was unable to determine whether or not a logarithmic function and a root function were inverses of $f(x)=10^x$. Sara did not remember what inverse function was or how it should be presented. Sara knew that she needed to take the logarithm of both sides, but she did not know how to use logarithms to find the inverse of $f(x)=10^x$ (see Figure 4). When she was asked to explain whether or not the function, $f(x)=10^x$, had two inverses, she said, "I don't know that's what I am trying to do.

$\log f(x) = \log 10^x = x$
$\log f(x) = x$

Figure 4. Sara's solution

I don't remember what inverse function is and how to find it. I don't really know what I am looking for." For the same question, Sara provided the incorrect logarithm shown in figure 4.

In the card-sorting activity, on which the functions were written on the cards, the participants were able to sort functions according to their representations; it was easier for them to recognize functions presented graphically or algebraically, and overall, they had difficulty translating functions presented with verbal and table representations. At this point, it became clear that their understanding of the relationship between logarithmic and exponential functions was weak and incomplete. For example, in the card sorting activity, the participants were asked to sort the cards 1, 6, 7, 16, 21, 26 and 28 into two piles and describe the criterion they used to sort the cards in figures 5, 6, 7, 8, 9, 10, 11, 12 and 13.

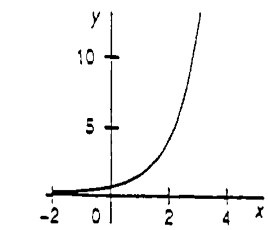

Figure 5. Card 1

In his response, Jack grouped the function cards into three piles according to their representations. He grouped the function on cards 1, 6 and 26 because they were given in graphical representations. Jack put the functions on cards 21, 16 and 28 together because they were given as tables. He indicated that the functions on cards 7, 18 and 20 were in the same group because together they were equations. Jack's hesitation about grouping these function cards continued as he talked about the other functions on the cards. He first said that function cards 7 (quadratic) and 21 (quadratic) were quadratic, and then card 7 (quadratic) could be exponential. For the function on card 20 (logarithmic), he was not sure whether it was exponential or logarithmic. Jack thought the function card 16 (logarithmic) was exponential. He also said, the function on cards 20 (logarithmic), 26 (logarithmic), 28 (exponential) were similar and could be logarithmic. He plotted the points and tried to match the points to the graph of a familiar function (i.e., logarithmic, exponential, or quadratic) instead of trying to find the pattern in the tables. As a result, he said he needed more data (or points) to determine the shape of the graph.

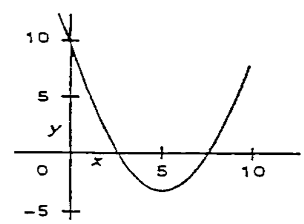

Figure 6. Card 6

$y = \frac{1}{3}x^2 - x + 1$

Figure 7. Card 7

x	1/100	1/10	1	10	100
y	-2	-1	0	1	2

Figure 8. Card 16

Jack correctly determined that the function on card 16 was logarithmic, however, he denoted the logarithmic function using an exponential equation $y = 3^x$ instead of writing $y = \log x$.

$y = (\frac{1}{2})(4^x)$

Figure 9. Card 18

$y = \log_4(x + 4)$

Figure 10. Card 20

x	-2	-1	0	1	2
y	1	0	1	4	9

Figure 11. Card 21

Then Jack put the function on card 16 under the category of logarithmic function. He said that logarithmic and exponential functions were inverses of each other but did not remember how to write the equation of the logarithmic function whose graph he drew. As can be seen in his work, the graph intersected the y-axis and did not cross the x-axis at $x = 1$ even though the logarithmic function $y = \log x$ did not cross or touch the y-axis and crossed the x-axis at $x = 1$. Jack knew that the function should be one-to-one to have an inverse function. However, he did not remember how to find the inverse of a logarithmic function and claimed that he would interchange x and y to find the inverse function.

Jack first thought that the equation of the function on the card 28 was $y = 2^{x}$, and then he changed the equation and wrote $y = 2^{-x}$. Toward the end of the interview, he was asked to put the functions into three groups. He grouped the cards 1, 18, 28 as exponential, 6, 7, 21 as quadratic, and 16, 26, 20 as logarithmic. Even though Jack did not have a good understanding of exponential and logarithmic functions, he determined the types of the functions by plotting the points given in the tables and correctly sorted them into three groups according to their types. During the interview, it became clear that Jack's strategy was mainly point-wise for the functions represented by a table. When he drew the graph of a logarithmic function card 16 ($y = \log x$), he failed to write the correct equation for the graph. In addition, he made some mistakes on the graph of $y = \log x$ which showed deficiencies in his content knowledge about logarithmic functions.

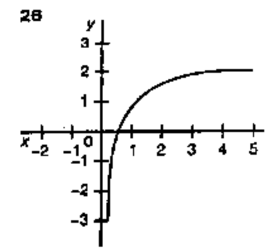

Figure 12. Card 26

x	-2	-1	0	1	2
y	4	2	1	1/2	1/4

Figure 13. Card 28

Similarly, Sara grouped the function cards using two different ways. First way, she grouped the cards according to their representations. She grouped the cards 1, 6, 26 as graphical representations, 7, 18, 20 as formulas, and 16, 21, 28 as tables. Second way, she grouped the function cards according to their types. She put the cards 16, 26, 20 as logarithmic, 1, 18, 28 as exponential), and 6, 7, 21 as parabolic. When Sara was asked whether or not the function cards could be sorted differently, she said, "I don't see other way. I don't see anything." As can be seen from Sara's solution, she had a good understanding of three categories of functions and the use of their representations. Sara sorted the functions cards into two piles using two different ways. In her response, she grouped the function cards using their types (i.e., exponential, quadratic, logarithmic) and different representations (graph, equation, verbal).

The participants were unable to choose appropriate questions or organize a lesson plan because of weak SMK. When they were describing their instructional strategies to facilitate students' learning during their analyses of the lesson plans, they failed to select an appropriate activity and examples of exponential functions (e.g., $y = (-5)^{x}$) due to weak and incomplete SMK. During the teaching episode, the participants failed to ask appropriate questions to make the concept easier for students and frequently seemed unaware of students' thinking. They failed to challenge students and ask critical questions due to their weak PCK. They also could not determine the fundamental components in exponential functions that students needed to know how to draw and understand the graph of exponential functions. Furthermore, they had difficulty explaining and discussing the definition and properties of exponential functions such as increasing, decreasing, domain and range that would help students, because their PCK for teaching was not comprehensive. When they were evaluating and reflecting on their teaching, they noticed students' difficulties and lack of understanding of the asymptote and graphing exponential functions but they did not think of using any instructional strategy to help them during their teaching. After their teaching, they suggested using instructional strategies, such as technology tools, or alternative approaches that may help students understand the concept or that initiate classroom discussion. The model of Wilson, Shulman and Richert (1987) was effective in helping the participants gain experience in preparing and teaching a lesson; however, their instructional strategy for teaching exponential functions changed insignificantly due to their limited SMK and PCK.

Conclusion

The teaching episode experience was beneficial for the participants. The teaching episode also provided a relevant context to enhance their SMK and PCK of exponential function. This experience provided an opportunity for prospective secondary teachers to begin linking theory and practice to teach and to analyze their teaching. By analyzing, evaluating, and reflecting on their teaching, they identified weaknesses and strengths of their teaching and their lesson plans. We suggest prospective teachers prepare lesson plans, videotape their lessons, and evaluate and reflect on their teaching in method courses, in stages of increasing awareness, under the supervision of their professors because "knowledge is developed through cycles of planning, implementing, and reflecting on lessons (Fernandez, 2005, p. 38)." As a result of their weak SMK, teachers were unable to choose appropriate examples and activities, to challenge their students, to ask critical questions to engage students in discussion or participation, to recognize students' difficulties, and to interact with students. Ball (1990) stated that SMK should be a central focus of teacher education in order to teach mathematics effectively. Accordingly, prospective teachers develop their PCK as they plan to teach as well as during actual teaching (Wilson, Shulman & Richert, 1987).

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TEACHER KNOWLEDGE AND CURRICULUM USE

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This paper describes an investigation on teacher knowledge and curriculum use. The results of the investigation reveal strong and weak aspects of teacher knowledge when using an elementary mathematics curriculum. The results also illuminate where the curriculum provides sufficient support and where it needs more clarification for teachers. In addition, this study tested a research tool (an observation coding protocol) for its usefulness in research on teacher knowledge and curriculum use.

This paper describes an investigation on teacher knowledge and curriculum use. Research questions are: How does teacher knowledge influence ways in which teachers use mathematics curriculum materials? How do the curriculum materials support teacher enactment of lessons? For this investigation, both teacher knowledge in enacted lessons and components of the written curriculum were examined to see the interactions among teacher knowledge, curriculum use, and the curriculum. The data for this study were drawn from a larger study that investigated teacher implementation of a new curriculum. This paper focuses on one particular teacher who lacked confidence in her knowledge of mathematics as well as mathematics teaching. This teacher was particularly interesting because even though she was not confident in her mathematics teaching, her enacted lessons revealed that she had some strengths to teach mathematics effectively. As such, her case was analyzed in depth by using the observation coding protocol by the *Mathematics Learning to Teach* Project (2006). A detailed rubric of the protocol, mapping how mathematical knowledge for teaching might appear in practice as she worked with the curriculum and students, helped identify strong and weak aspects of her knowledge when using the curriculum. It also helped elicit where the curriculum provided sufficient support and where it needed more clarification for teachers.

Theoretical Perspectives

Teachers' use of curriculum materials is a relatively new area of research. Curriculum is an important factor that determines what and how students learn mathematics, and many curriculum development projects in the 1990's (e.g., TERC Investigations project, Connected Mathematics Project, and Core-Plus Mathematics Project) intended to improve student learning by not only providing the content but also embedding appropriate pedagogy in their curriculum materials. However, depending on the teacher, the curriculum materials could be used in various ways in the classroom. Therefore, how teachers use curriculum materials greatly influences student learning experiences (Hiebert & Grouws, 2007; Stein, Remillard, & Smith, 2007). Here, teacher curriculum use means how teachers read and interpret the curriculum, plan and enact a lesson, and reflect on teaching along with the materials.

One key component of this study is the role teacher knowledge plays in curriculum use. Earlier work (e.g., Crumbaugh, Grant, Kim, Kline, & Cengiz, 2005; Grant, Kline, Crumbaugh, Kim, & Cengiz, in press; Manouchehri & Goodman, 2000) has established that teacher knowledge is an important variable in curriculum use, but little is known about ways in which it affects teacher curriculum use. Another key component is curriculum support – ways in which curriculum provides support for the curriculum enactment and curriculum features that influence how teachers implement the curriculum. Various features of curriculum influence how teachers read and interpret the curriculum and what instructional decisions teachers make, and various curriculum features can support teachers. Very little, however, is known about what features actually meet teachers' needs and how they should be provided. This paper highlights such a research effort and methods of the investigation.

Because of its importance in teaching, teacher knowledge has been a research focus among many teacher educators (Shulman, 1986). Teachers who have insufficient knowledge of the subject that they teach or who know procedures without understanding the rationale behind the procedures are not likely to teach mathematics effectively. But, what specific knowledge is required to teach mathematics? Simply knowing more mathematics does not guarantee the quality of teaching, let alone better learning outcomes (Ball, Lubienski, & Mewborn, 2003). Recently, researchers have begun to conceptualize kinds of teacher knowledge for mathematics teaching (e.g., Ball & Bass, 2003; Chinnappan & Lawson, 2005; Davis & Simmt, 2006; Hill, Sleep, Lewis, & Ball, 2007). In particular, Ball and her colleagues not only attended to the kinds of teacher knowledge for teaching, but also began to address where and how mathematical knowledge is used in mathematics teaching by examining videotaped lessons (Ball & Bass, 2003; Learning Mathematics for Teaching, 2006). With such efforts, they developed and tested coding schemes to determine the mathematical quality of what teachers say and do during instruction. Ball and her colleagues' work was incorporated in this study, because their attention was particularly given to teacher knowledge unveiled during instruction and they identified elements of such teacher knowledge.

Modes of Inquiry and Data Sources

In order to investigate the research questions, teacher knowledge in 13 enacted lessons as well as curriculum materials were examined. Kellie, the teacher in this study, taught for 20 years and was on the mathematics committee in her school. She, however, was not confident in her ability in mathematics, let alone her ways of teaching mathematics. Her lack of confidence corresponded to the results of an interview using mathematical tasks conducted at the beginning of the study. Her knowledge revealed during the interview was at the low level.

The lessons analyzed were taken from a larger study of a new curriculum implementation for three years (one year for pilot and two subsequent years of full implementation). The lengths of those lessons ranged from one hour to two hours, mostly one hour and a half. The topics of the lessons were number and operations, data analysis, and geometry. Besides the videotaped lessons and the curriculum materials, six interviews with Kellie during the three-year period were analyzed in order to triangulate the inferences from the other data. Her explanations about specific elements of teaching within a lesson or a unit helped support or dispute potential inferences.

Each videotaped lesson and the corresponding written lesson from the curriculum were analyzed by using the coding protocol mentioned earlier. The coding protocol was composed of five sections, including *knowledge of mathematical terrain of enacted lesson, use of mathematics with students,* and *mathematical features of the curriculum and the teacher's guide*, and each section contained 8 to 24 elements to code. Even though the coding protocol was useful to identify aspects of teacher knowledge and curriculum support, the protocol was not designed to examine the relationship between teacher knowledge and curriculum. Therefore, additional coding elements were needed for this study as the analysis proceeded. For instance, when a teacher alters a task, it can change the nature of the task as well as the levels of complexity of the task and thus influence kinds of strategies and reasoning students can engage in. This can indicate whether or not the teacher recognizes big ideas of the task. In such a case, it is also important to check whether the curriculum provides enough description about the task so that teachers can understand the purpose of the task and implement the task as intended. As such, additional elements were included for this research, some of which were change of task, use of suggested questions, and use of suggested models or representations.

For each coding element it was required to indicate whether evidence was "present" or "not present" and, when present, whether it was "appropriate" or "inappropriate." Coding was conducted for every five-minute segment with each of those coding elements (student work time was not coded), and frequencies of present-appropriate, present-inappropriate, and not-present-inappropriate were counted. While employing the coding protocol to identify aspects of teacher knowledge in the enacted lessons and aspects of the curriculum, using a holistic approach I also revisited each lesson in terms of teacher knowledge in use, the written curriculum materials, and how they interacted within the lesson. Finally, common themes were crosschecked across lessons.

Results and Discussion

The analysis of the data shows strengths and weaknesses of Kellie's knowledge for mathematics teaching. Also, the analysis illuminates aspects of the curriculum materials that supported Kellie to teach lessons. Such identifications helped infer the interplay between Kellie's knowledge and curriculum features for enacted lessons.

Aspects of Teacher Knowledge

Strengths of Kellie's Knowledge for Teaching

Coding every five-minute segment of the videotaped lessons revealed that Kellie certainly focused on the mathematics that students should learn; time spent for administrative purpose was minimal and class work was related to mathematical procedures and ideas. One of the strong aspects of her teaching was her explicit effort to elicit students' thinking and reasoning during whole group discussions. For example, a coding element, *elicit student explanation*, was present-appropriate in 40 out of the 74 five-minute segments (54%), while only three (4%) were coded as not-present-inappropriate. This means that whenever possible, she encouraged students to explain their thinking on the strategies and procedures they used. It was observed that often she searched for appropriate questions to ask in a way that could help elicit the essence of the thinking that students could not verbalize explicitly without prompt. Kellie indicated in an interview that she was committed to teaching mathematics with understanding and this was evident in her enacted lessons. Another strength of her teaching was explicit talk about ways of reasoning and mathematical practices. During the observed lessons, almost in every lesson Kellie mentioned the importance of explaining thinking, not just getting the answer. She also encouraged students to verbalize their thinking in public and allowed students time to think on their own: "Let him think." "I know it's hard. Go on." "Honey, I am not saying that you are wrong. I just want to know what you did. I want you to explain how you did." "This is where you guys gotta think." She was very patient with students when they provided their explanations, and probed students' explanation deliberately to capture the reasoning behind what they showed with manipulatives or on the board.

Even though one might argue that the strengths described above are not directly related to the teacher's knowledge, the coding protocol includes such elements as important aspects of teacher knowledge that should be utilized in enacted lessons – whether the teacher created opportunities for students to provide mathematical explanations; whether students' efforts to explain were adequately scaffolded; and how the teacher responded to students' comments, questions, ideas, or errors. In addition, teachers' explicitness about the work the students are supposed to be doing, about the meaning and use of mathematical language, about ways of reasoning, and about mathematical practices, affects kinds of learning opportunities for students, and doing so requires specific teacher knowledge. By utilizing such knowledge, Kellie could create the classroom environment where students were expected to think and explain their reasoning, instead of merely providing answers and showing procedures. Kellie expressed to students how hard it was to explain one's thinking, allowed students to take time, and asked them to be patient with each other. She also demonstrated how to respect and probe each other's thinking, and how to listen to understand others' reasoning.

Weaknesses of Kellie's Knowledge for Teaching

While the analysis revealed the above strengths in her teaching, it was evident that her knowledge in the enacted lessons had several weak areas. One of the challenges that Kellie had was that she introduced a task and students had difficulty working on the task. When this happened or when she felt the main task was difficult for her students, she altered the task into smaller activities. In fact, the task alteration led to the change in the focus of activities, but it was not clear whether she realized the effect of the task change, let alone whether she knew that the task she gave to students was different from the one in the curriculum. For example, when the main task was to figure out "Which is more, the number of school days or the number of non-school days? How many more?" she segmented the activity into smaller steps (i.e., finding the number of non-school days, finding the number of school days, and then finding the difference). The original activity could have led to multiple strategies and thinking, while the altered activity was straightforward and limited student learning opportunities.

Another weak area was related to using models in teaching. She rarely used multiple models, let alone made connections among them. In addition, she did not utilize certain models that the curriculum suggested. For example, she did not use a number line model shown to represent the difference in quantities and calculators as a means of counting backwards from a large number, such as 300. It seems that one possible reason was Kellie's unfamiliarity with the models.

Her use of recording was limited. While she carefully listened to students and valued thinking and explanation shared in public, she rarely made records of student strategies or thinking in public. It is not clear whether that was because she was just not used to make a public record of student thinking, because she did not know how to record effectively, or because she did not recognize the potential of the record for student learning opportunities. Shared strategies on the board or chart paper could have been beneficial to students when they attempted to examine or incorporate others' ideas. In a couple of lessons, she intentionally recorded what students offered during discussion. It was noticeable that such recording was influenced by the curriculum, even though she did not take all the recording suggestions by the curriculum.

It was apparent that Kellie was struggling to search for appropriate questions during whole group discussion. Often she communicated with the researcher about how hard it was to ask appropriate questions and asked for suggestions in particular classroom situations on site or after class. On the other hand, it was interesting to observe Kellie not utilizing some important questions that the curriculum materials suggested. She did not ask suggested questions that were conceptually important, such as "If we had made this graph at the beginning of the year, what might be different? How might the shape of our graph look?" Knowing that Kellie was a thorough reader of the curriculum and careful planner of lessons, it is not clear if that was because she made a note of those questions and forgot during the course of the lesson, or if she did not see the value of those questions.

Kellie did not discuss some important concepts, while she spent much time on something less important. For example, in a lesson where students figured out the number of children in their family and the difference between that number and the world's record number of children, Kellie spent 35 minutes introducing this task, taking most of the time getting a consensus on whom they should count as children in their families. It was important to establish a definition and a basis for comparison, and yet because of such extensive amount of time for getting the consensus, there was no time left for the next activity (finding pairs of two-digit numbers which make a total that is as close to 100 as possible), which seems to be the more important task in the lesson. [The "children in a family" task was not mathematically very challenging because the third graders were asked to find the difference between a one-digit number (the number of children in a family would be typically one digit) and 69, the world's record.] On the other hand, in this lesson she did not explicitly discuss the word *difference,* even though the curriculum suggested asking students if they understood the meaning of this word in the problem context and helping them see this as a particular mathematical meaning – the difference between two quantities.

Kellie sometimes misinterpreted suggestions by the curriculum. In some cases, the suggestions in the curriculum were not explicitly clear. In the "children in a family" lesson, the curriculum suggested, "Have students work in pairs, comparing their individual family data with the record. They should have cubes and 100 charts to work with. … Encourage them to use materials and *representations* to show their strategies" (p. 19). It was not explained in the curriculum why the two "materials" were suggested, even though there were examples of how students might use the materials to show their strategies. When enacting this lesson, Kellie provided students with cubes and 100 charts, and told them, "Make a representation to show how many more you would need to tie the world's record." Materials can be used as a tool to solve problems and to reason with, and it is not necessary to use them when students do not need them. Students could have had options to use the suggested materials or other representations when they needed. Also, the focus should have been given to solving the problem and student strategies to do so, rather than creating representations.

Support of the Curriculum Materials

As one of the Standards-based curricula, the curriculum that Kellie used provided rich problem contexts and useful information for teachers. Students were required to explore mathematical ideas in those problem contexts; reasoning and discussion were a default, not an option. In order to help enact the intended lessons, the curriculum provided detailed descriptions of the lessons, including important teacher questions, possible student responses, models to represent procedures and thinking, things to look at for informal assessment during lessons, and sample dialogues in the classroom discussion.

It is noticeable that when Kellie's lesson was closely aligned with the written curriculum, the enacted lesson went well overall. When she utilized the curriculum materials well (staying with the focus of the lesson, posing the task as suggested, asking suggested questions, using models as intended, etc.), students were exposed to good learning opportunities. On the contrary, in the lessons where Kellie did not ask important suggested questions, posed a task in a way that was different from what was suggested, missed some critical aspects of tasks, or ignored certain important elements of a lesson, the overall lesson did not seem to go well. This implies that the curriculum certainly provided features that supported Kellie to enact lessons.

The analysis of the curriculum, however, revealed that the curriculum could have provided more clarification in several areas. Multiple models were introduced with examples of how students might use them, but clear explanations of effective use of the models were often not provided. Perhaps explicit explanations on how the provided model could convey mathematical concepts and ideas, rather than just showing sample usage of the model, could have been helpful for Kellie. For example, the number line model was introduced in a lesson as a way to show how a student found the difference between two quantities, but there were no sufficient explanations about the usefulness of the model. Kellie, who was not familiar with the model, did not utilize the model in her lessons.

In addition, problem contexts could have been explicitly explained so that Kellie could understand big ideas in those contexts. Recall "the number of school days" lesson. Even though the task itself was open-ended and could provide opportunities for various strategies and thinking, Kellie altered it to a straightforward task. Sufficient explanations about the big ideas and the essence of the task could have led Kellie not to alter the task. Also, the "children in a family" lesson could have stated the meaning of subtraction as quantity difference explicitly and clearly in the lesson introduction and objectives. The purpose of activities was sometimes not clearly stated. Kellie did not seem to know the purpose of the "Quick Check" activity (showing boxes of clips on overhead for a second and having students share how many they saw), and the curriculum did not explain why and how the activities should be provided. This might have been provided elsewhere in the curriculum, but certainly not in the unit in which the activity was presented. Assuming that some teachers often choose units and activities to use, at least providing a proper reference on the purpose of activities, if it is described elsewhere, would be beneficial to teachers.

This curriculum did a good job in providing students' multiple methods and what students might say, but potential difficulties students might have were rarely included in the curriculum. Often no scale-up or scale-down problems were suggested, which was left to the teachers' decision. For example, Kellie found students having difficulty using "realistic" quantities when figuring out "what the heights of six third graders might be if the sum of their heights is 318 inches." Therefore, she had to spend quite some time helping students see a possible range of realistic heights of third graders. But, there was no explanation of this foreseen difficulty in the curriculum, and Kellie, a first-time user of the curriculum, had a hard time helping students complete the task.

Moreover, the curriculum included many examples and descriptions of computational steps and procedures, but reasoning of these steps was often not described. Questions that could help elicit students' reasoning were even rare in the curriculum. Because of this, not knowing what to expect from students in terms of reasoning, Kellie had difficulty figuring out kinds of questions to ask to elicit the reasoning behind the steps students took to solve problems.

Conclusion

In this study, the teacher's knowledge in enacted lessons revealed several aspects that prompted or hindered the implementation of the intended curriculum. While providing useful information for teachers, the curriculum had areas needing improvement in order to support better enactment of the curriculum. The study results show several cases where the teacher needed more information or clarification – the focus of a lesson, big ideas of a task, how a model could convey mathematical ideas, potential reasoning by students, and questions to elicit such reasoning. It seems that such ambiguity confused the teacher when she interpreted suggestions by the curriculum, and lack of information caused her difficulty in figuring out on her own what to do in certain classroom situations.

This study also tested the research tool to assess its usefulness in research on teacher knowledge and curriculum use. The results showed that the coding protocol helped identify certain aspects of teacher knowledge in use and specific aspects of the curriculum that supported the lesson enactment; the tool also had limitations when examining the relationship between teacher knowledge and curriculum use. Therefore, it was necessary to construct additional elements to consider, such as whether the teacher alters the main task, and if so, whether that substantially changes the activity. In fact, the revised coding protocol was very useful in attending to specific aspects of teacher knowledge during instruction rather than assessing an overall level of teacher knowledge in general, separated from teaching context.

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FACTORS THAT INFLUENCE NOVICE MIDDLE SCHOOL MATHEMATICS TEACHERS' ANALYSES OF THEIR INSTRUCTION AND OPPORTUNITIES TO LEARN FROM THEIR OWN TEACHING

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Novice middle school mathematics teachers face many demands as they adjust to their new professional role. How do these demands influence their efforts to improve their teaching? The purpose of this project is to examine the criteria novice mathematics teachers use to determine the effectiveness of their instruction and to examine the factors that influence novice middle school mathematics teachers' analyses of the effectiveness of their instruction.

Theoretical Perspective

An assumption guiding this study is that analysis of teaching is an important part of a teacher's work. Analyzing teaching involves specifying the learning goal, gathering evidence during instruction to determine whether students learned, designing cause-and-effect hypotheses to link teaching to students' learning, and revising instruction to help more students learn (Hiebert, Morris, Berk, & Jansen, 2007). This process enables teachers to learn from their practice. However, new teachers may not engage in analyses of their teaching in such a systematic manner. Also, they focus on factors related to their own performance as a teacher over students' understanding of mathematics content when they analyze their teaching.

The demands of their classroom, department, school, district, or local community may help or hinder teachers' efforts to attend to students' thinking when evaluating the effectiveness of their instruction. Novice teachers typically exhibit a rather high sense of efficacy that usually drops during the first few years of teaching due to the gap between their expectations (even if unrealistic) of what they can accomplish in the classroom and their professional performance and circumstances (Hoy & Spero, 2005). Doubts in their efficacy that arise from this gap could facilitate opportunities to learn.

This study contributes to the field of mathematics education in that we need to document criteria that contribute to teachers' sense of efficacy in the context of mathematics reform (Smith, 1996). In mathematics teacher education programs, novice teachers have been prepared to expect to teach according to principles such as providing students with opportunities to develop conceptual understanding of mathematics as well as procedural fluency and attempting for a balance between exhibiting authority as a teacher and providing students with opportunities to develop autonomy as mathematical thinkers. Given these expectations, novice teachers are likely to evaluate their teaching in light of agent-means efficacy concerns, or whether they were able to enact particular instructional practices. Alternatively, they could also consider agent-ends or means-ends efficacy criteria, which means the effectiveness of their lesson would be based upon whether the design of their lesson or curriculum (means) or their implementation of the lesson (agent) helped students learn or behave in particular ways (ends) (Wheatley, 2005).

Certain conditions may shape teachers' analyses of the effectiveness of their instruction. Morris (2006) found that pre-service teachers' analyses of a transcript of a mathematics lesson was influenced by whether or not they were aware ahead of time that students struggled to understand the content during the lesson prior to analyzing the transcript. In addition to encountering students who struggle to learn mathematics, novice teachers face a range of external demands and expectations that could shape what teachers attend to when analyzing their instruction. In this study, I addressed the following questions: What criteria do novice teachers use to determine the effectiveness of their instruction? What factors influence the teachers' attention to these criteria?

Method

Eight first- and second-year middle school mathematics teachers, graduates from the same teacher education program, participated in this study. Seven participants were female and one was male. All of the participants were White, and they were all between the ages of 22 and 25 at the time of the study. Teachers in this study worked in three different states, and they did not all use the same textbook series; only half of the teachers taught with NSF-funded curricula.

These teachers were observed twice and participated in one-on-one post-observation debriefing interviews with the author, one early in the fall semester and one near the end of the fall semester in 2006. Sample interview questions included: How do you think the lesson went? What were the best parts / worst parts about the lesson? Do you think the students learned what you hoped they would learn? How do you know?

Codes for criteria used to determine strengths and weaknesses of the lessons were developed through a constant comparative process (Glaser & Straus, 1967). The codes were developed according to general themes of a focus on student outcomes (ends), features of the lesson or curricula (means), or the teacher's efforts to implement the lesson or curriculum and manage students (agent), and external factors (such as district or school policies, parent expectations, school or departmental structures, curriculum features, etc.). For trustworthiness, code development and interpretation of data took place in collaboration with a graduate student who was a recent graduate of the same teacher education program as the participants.

Results

Results in this paper focus on two teachers in the same school who used the *Connected Mathematics* textbook series. One of the teachers, Ms. Alpha, was a female first-year teacher who taught seventh grade. The other teacher, Mr. Beta, was a male second-year teacher who taught sixth grade. External factors led to a focus on agent-means efficacy, or whether they were implementing instruction as expected from the school district, when evaluating their instruction. They also focused on student outcomes when evaluating their instruction, but emphasized students' engagement over students' learning when evaluating their instruction, potentially linked to attempting to effectively implement their curriculum.

As an example of the teachers' agent-means efficacy, content coverage and pacing were frequently mentioned among their criteria for evaluating their instruction, which was influenced by the district-level expectation that teachers were expected follow lesson plans designed by a committee of teachers in the school district and district curriculum guides that required the teachers to test by a certain date. Their lesson plans "called to do more than we could actually get to" (Mr. Beta, second interview); both teachers noted in their interviews that the lessons were designed for 90 minute class periods, but their class periods were only 65 minutes, so they consistently felt that either they were not able to teach effectively because they could not complete the lessons or that the lessons themselves were less effective because they were too long. As a result, these teachers learned, or tried to learn, what to keep and what to cut from their district-provided lessons. “We have closure and follow-up activities that they would like us to do, and unfortunately once we get through the lesson they want an extra activity after summarize to just reinforce and we never have time to do that so it’s something I always have to cut out. So, in the lesson I kind of have to teach what is absolutely necessary.” (Mr. Beta, first interview). This meant cutting some activities, which was a challenge, particularly for Ms. Alpha during her first year of teaching. “We’re always told to over plan, and to have plans after plans after plans, but I never have a plan for what to do if you don’t get through all of it, like how to close it up. Like at what point is a good point to stop, and just, you know, jump right to the closure. Or is there a particular part that you really need to get to or you should cut out if you don’t have enough time?” (Ms. Alpha, first interview). Making these choices involves pedagogical content knowledge about which activities support students’ achievement of particular learning goals.

These two teachers also focused on whether they achieved particular ends, or student outcomes, when evaluating their teaching. They were able to cite more specific evidence when describing engagement outcomes, such as whether students participated in explaining their thinking and whether groups worked well together, which are important components involved with implementing the *Connected Mathematics* textbook series. A student outcome Mr. Beta noted in both interviews was that his lessons were not as effective as they could have been because students resisted using representations, such as diagramming for multiplying fractions using poly strips to explore relationships among side lengths for triangles and quadrilaterals. His attention to student resistance as an outcome was partially due to being expected to use these representations as a part of implementing this curriculum, but perhaps it was also due to competing goals of developing procedural fluency as well as conceptual understanding in a short amount of time. When the teachers described outcomes related to students’ learning of content, they were often described generally, without much supporting evidence, such as saying that students “got it at the end” (Mr. Beta, first interview) or that students “seem to be understanding,” (Ms. Alpha, first interview).

Discussion

While novice teachers may have been encouraged in their teacher education programs to attend to students’ understanding when evaluating the effectiveness of their instruction, this is a challenge for new teachers as they cope with factors in their professional context. Novice teachers may be more likely to base their efficacy on criteria related to implementation of a curriculum or district expectations. This is understandable given a desire of a new professional to be evaluated positively in a context where these novice teachers perceive to have limited autonomy for making changes to their timeline for content coverage or structure of their lesson plans. Novice teachers would benefit from factors supporting attention to students’ conceptual understanding of mathematics in complement with their attention to students’ engagement in learning practices associated with their curriculum.

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FEATURES OF CONVINCING ARGUMENTS FOR MIDDLE GRADES PRESERVICE TEACHERS

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This study explores preservice middle grades teachers' preferences for certain kinds of arguments / proofs that convince them that $0.\overline{9}=1$. *Attempts are made to describe important features of arguments that are most convincing to future teachers at different stages in their intellectual development.*

The role of proof in the mathematics classroom has recently received attention beginning with the new NCTM standards document (National Council of Teachers of Mathematics, 2000) that urges curriculum writers to integrate activities involving proof throughout the mathematics content and to make it available and accessible to *all* students. Because proof is useful in fulfilling many roles such as verification, explanation, communication, discovery, and systemization (deVilliers, 1999), it is important that our future teachers feel confident when formulating sound mathematical arguments. At Miami University, the preservice middle grades program instructors attempt to create interactive learning environments in our classrooms so that students become comfortable with posing conjectures, formulating arguments, and, in general, learning how to behave more like mathematicians as they explore new ideas. We hope that our future teachers will leave with multiple representations for many concepts that will better prepare them for adjusting to their roles as middle grades teachers. If teachers lack flexibility with their own conceptual representations, they may lack the necessary understandings to help their struggling students.

The mathematical content of this study is solely the mathematical fact that $0.\overline{9}=1$. We decided to choose a fact that is not very obvious or intuitive. Specifically, this study attempts to use this context to answer the following overarching questions:

- When introduced to an unfamiliar, unintuitive mathematical fact, what kinds of arguments are most convincing?
- When comparing different subjects in this study, are there similarities in what finally tips the scale from unbelief to belief for freshman/sophomores as compared to seniors? In other words, does the feature or structure of an argument that convinces them change with mathematical maturity?

Comments from a prior study that addressed similar questions caused me to wonder if students hold the same beliefs about their own ability to comprehend a mathematical argument as they do for their future students. So a few more research questions were added and addressed to the students who were just about to graduate:

- Which argument(s) will you use to convince your future students that $0.\overline{9}=1$?
- If the argument that convinced you and the one that you would share with your students are not the same, why aren't they? If they are the same, why?

Review of Literature

Much has been learned about what preservice and in-service teachers value highly as verifications for certain arguments. Martin and Harel (1989) found that many preservice teachers accept inductive arguments as proofs regardless of their familiarity with the context. However, they also found that the same students accepted deductive arguments as proof and this led them to theorize that these two kinds of arguments represent two proof frames that are constructed in the minds of their students as the result of two different experiences; their everyday life experiences and their experience with mathematics in the classroom. They also found, like Fischbein & Kedem (1982), that even when some students seemed "convinced" by a deductive argument, they sought out empirical evidence to support the proof. Eric Knuth's work (2002) with inservice teachers who had much more mathematical maturity than those in previous studies revealed that *concrete features, familiarity, sufficient level of detail*, proofs that *show why*, and those that *prove the general case* were the characteristics preferred by teachers when asked what convinces them the most. Overall, he concluded that "the characteristics of arguments that the teachers found to be most convincing seemed, in large part, to relate more to form than to substance (p. 402)."

A framework useful in classifying the different kinds of arguments preferred by students is provided by Sowder and Harel (1998). They suggested that "Proof, or justification, schemes can be organized into three categories: *Externally based* proof schemes, *empirical* proof schemes, and *analytic* proof schemes." The externally based schemes rely more on an authority outside of the student's own understanding. Empirical schemes can either be when a student is convinced by a single, or sometimes several drawings or when they have found several examples that form a pattern that they extrapolate to the general case. Finally, there are two kinds of analytic proof schemes. The transformational analytic proof scheme is used when a student is able to make the general case of the argument with *reasoning* for why the generalization makes sense and an understanding of the connection between the examples and the general case. This kind of justification then leads to the ability to understand axiomatic analytic proof schemes which most mathematicians use when formalizing proofs.

Method

Three classes of middle childhood (gr. 4-9) preservice teachers were studied. Two classes (13 and 15 students) were in the second semester of their freshman (or sophomore) year and one class (18 students) was composed of seniors, most who were in the last semester of their university experience. All students in each class were presented with a questionnaire that asked them to state the argument that was most convincing to them and to explain why it was most convincing. The seniors' questionnaire also had two extra questions asking which arguments they will share with future middle grades students and why. The questionnaires were given to the freshman/sophomore students the day after they had finished their discussions in class concerning $0.\bar{9}=1$. Therefore, the variety of arguments was still fresh in their minds, and some of the students still were not convinced of the truth of the statement. The seniors had taken the same class three years prior and most likely did not have any other exposure to the concept other than perhaps through other classes or in their student teaching.

Data Analysis

The questionnaires were analyzed for common themes and similarities in comments. Efforts were made to classify the arguments into the three categories and subcategories provided by Sowder and Harel as well. However, the data available did not contain enough information to assign responses to a category without significant speculation on the part of the researcher. Therefore, the researcher resorted to a more grounded theoretical approach (Glaser & Strauss, 1967) and the responses were classified by the different argument(s) preferred, while taking several notes when a subject gave an external, empirical or analytical justification. If students provided more than one convincing argument, each was counted in the results. A few of the freshman/sophomore students (n=6) still did not believe that the two numbers were equal, so they provided arguments for why the statement was false.

Results

The sorting of the responses yielded several interesting findings, with the main results summarized below:

- Valid arguments weren't convincing to students whose prior understandings of limit or rational numbers weren't well-supported. (21.4% of the freshman and sophomore students still believed the statement to be false.)
- An algebraic argument was heavily favored by both groups (46.4% and 44.4%), but many reasons given for this preference placed too much emphasis on the authority of the form over the substance (an externally based proof scheme).
- An argument that works well for one student can be a stumbling block for another.
- On average, seniors shared 2.2 arguments (compared to 1.2 from the younger group) and 38.9% specifically said that the use of a variety of approaches to convince their future students would be better than using just one.
- The top two arguments preferred by the seniors are also arguments that would be most accessible to their future students since they involved fractions and decimals and used inductive reasoning by looking for patterns.

Conclusions and Implications for Teaching

Overall, Knuth's finding of form being valued over substance was confirmed by my data. Students preferred an argument for its structure and its familiarity for them, but I became doubtful that they really understood all of the underlying assumptions they accepted blindly when they accepted each argument. Perhaps the teacher should be better at pointing out the hidden assumptions at some point in the instruction. It also was a stumbling block to many students that the whole number 1 would have another representation, though these same students had no problem believing that a fraction can also be represented by a repeating decimal. This interference may result from having been told for so long that we have a whole number system with a "unique" representation for each number. Finally, the most preferred arguments remembered by the seniors were actually the most basic and least abstract of the approaches, which is probably good given the non-intuitive nature of this concept.

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KNOWN KNOWNS AND UNKNOWN UNKNOWNS: COMPARING PRE-SERVICE AND IN-SERVICE TEACHERS' ASSESSMENT OF MATHEMATICS CONTENT PREPAREDNESS

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A comparison of pre-service and in-service teachers' assessment of their preparation to teach mathematics lessons from six content areas is reported. In five content areas, pre-service teachers felt less well-prepared to teach than experienced teachers. In the area of algebraic reasoning, pre-service teachers felt more confident than experienced teachers in their preparation to teach algebra topics in elementary school.

The National Council of Teachers of Mathematics states "teachers must know and understand deeply the mathematics they are teaching and be able to draw on that knowledge" (NCTM, 2000, p. 17). A recent focus on mathematical knowledge for teaching (Ball & Bass, 2000) underscores the importance of teachers' mathematical content understanding in relation to student achievement (Ball & Hill, 2004; Mewborn, 2003). Teacher preparation programs have been called upon to improve preparation of teachers and to cultivate teachers' mathematical proficiency as stated by the National Research Council in *Adding It Up: Helping Children Learn Mathematics* (2001). What does a comparison of experienced and pre-service teachers tell us about improving teacher preparation programs for future elementary mathematics teachers? What changes in emphases for specific mathematics content in mathematics methods courses are suggested by such a comparison?

Using a database of 51,000 in-service teacher questionnaires, I compare how pre-service teachers' views of their own mathematics content preparedness differ from the views of teachers with experience in the classroom. As experienced teachers undertook intensive professional development in mathematics, it was found that a self-assessment of their preparedness to teach certain mathematics topics decreased slightly when they began professional development and then climbed as the number of professional development hours increased (Weiss, Pasley, Smith, Banilower, & Heck, 2003). In other words, as the difficulty of the task before them became evident, the experienced classroom teachers knew more about the content they did not know in teaching mathematics.

The study was conducted in two phases. In the first phase questionnaire data was collected from pre-service teachers (n = 152) at the end of a one-semester mathematics methods course. These results were compared to the results of a large national database of experienced teachers. Following an interesting lead in the data, the second phase of the study examined pre-service teachers' understanding of algebra, patterns and functions and compared these results to experienced teachers' knowledge of algebra topics.

The NSF-funded Local Systemic Change through Teacher Enhancement Initiative (LSC) collected teacher questionnaire data from over 51,000 teachers from 1995 – 2006. In one section of the survey, teachers were asked, "How well prepared do you feel to teach each of the following topics at the grade level you teach?" Teachers chose from a four-point scale ranging from "not adequately prepared" to "very well prepared" about their preparation to teach mathematics (Banilower, Rosenberg, & Weiss, 2006; Heck, Rosenberg, & Crawford, 2006). A sample of pre-service teacher candidates (N = 152) was asked a similar question on an anonymous survey at the conclusion of an elementary mathematics methods course. The question was altered to ask about "… topics at the grade level *you wish* to teach." Frequency of “fairly well prepared” and “very well prepared” responses for each topic were tabulated. Survey results exploring how well prepared teacher candidates feel about each of six mathematics topics are shown in Table 1.

Topic	Experienced Teachers	Pre-Service Teachers N = 152
Computation	96	93
Number and Number Theory	92	92
Measurement	91	81
Geometry and Spatial Sense	78	69
Data Analysis and Probability	78	70
Algebraic Reasoning	64	81

Table 1. A Comparison of Experienced Teachers and Pre-service Teachers Views of Content Preparedness in Mathematics (Percent of Teachers Who Feel Fairly Well or Well Prepared to Teach the Topic)

At the conclusion of mathematics methods courses for elementary school teachers, I ask teacher candidates to fill out surveys on content preparedness in mathematics. There are those teacher candidates who are aware of their shortcomings and who tend to self assess at a lower level of preparation to teach certain mathematics topics, and there are those candidates who are unaware of what they do not know—the *unknown unknowns*. These students may believe they are well prepared to teach and may not realize their lack of content knowledge until they are in the classroom. There are validity concerns with self-reported data on content preparedness (Berends, 2006; Levy & Lemeshow, 1999). Nevertheless, it does appear from Table 1 that in the traditionally emphasized content areas of computation and number sense, the pre-service teachers, on the whole, feel adequately prepared to teach number and operations. In the content areas of measurement, geometry, and data analysis, a smaller percentage of pre-service teachers feel well prepared to teach in comparison to experienced teachers. Interestingly, when considering the topic of algebraic reasoning, these pre-service teachers seem more confident than their experienced peers; a higher percentage of pre-service teachers report that they feel well prepared or fairly well prepared to teach algebra than experienced teachers. The result is surprising and may be a reflection of a new group of prospective teachers who have learned algebra throughout their schooling and feel more comfortable and confident with algebra.

To test this idea, the pre-service teachers (N = 152) were administered the Content Knowledge for Teaching Mathematics Measures (CKTM) (Ball, Hill, Rowan, & Schilling, 2002). These measures give a picture of the mathematical knowledge for teaching that teachers use in classrooms everyday. One section of the CKTM measures the knowledge of pattern, functions, and algebra content. The CKTM measures are normative and validated with a group of experienced classroom teachers (Hill & Ball, 2004) and reported in scaled IRT scores. Thus, a zero IRT score is the mean score for the experienced teacher group. The mean score for the group of pre-service teachers is 0.92, close to one standard deviation above the norm. These results suggest that the self reported data from this sample of pre-service teachers may, indeed, show that the pre-service teachers have some basis for believing that they feel better prepared to teach algebra topics than experienced teachers.

Until these teacher candidates have gained experience in teaching mathematics to children, their preparedness to teach algebra topics will be a *known unknown.*

In the first phase of the study, pre-service teachers reported that they felt least prepared to teach the content areas of geometry and data analysis, somewhat more prepared to teach measurement and algebra, and well prepared to teach number and computation. This study suggests four specific content areas be given more emphases in mathematics methods courses. As teacher preparation institutions work to improve the quality of their preparation programs, they may begin to examine and collect evidence of needed improvements in their programs by using a large database of responses from experienced teachers as benchmarks. Further studies on content preparedness and student achievement, as well as the relationship of self-assessment of content preparation to teacher performance are needed to fully implement changes in teacher preparation programs.

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MATHEMATICIANS ARE LAZY BUMS: AN INVESTIGATION OF TEACHERS' FRAMINGS OF MATHEMATICIANS

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Although popular media often provides negative images of mathematicians, we contend that mathematics classroom practices can also contribute to students' images of mathematicians. In this paper, we analyze classroom observations to explore how often, when, and how teachers frame the work of mathematicians to their students. The findings suggest that there may be a relationship between one's mathematics background and his/her references to mathematicians.

Introduction and Relevant Literature

When reading popular comic strips, one often finds manifested in the humor a portrayal of mathematically gifted students or mathematicians as geeks, nerds, social misfits, and even fools. Negative images of mathematicians can also be found in literature and in the movies (Furinghetti, 1993). While we believe that the representations found in popular culture influence students' beliefs about mathematicians, we contend that experiences in mathematics classrooms can also contribute to students' images of mathematicians.

The sparse literature on students' images of mathematicians has drawn on research methodology used in science education (e.g., Huber & Burton, 1995). For example, Picker and Berry (2000) asked 473 students in five countries to "draw a mathematician at work," to explain their drawings, and to identify a situation where they might need to hire a mathematician. Picker and Berry identified the following seven themes in the students' drawings: *mathematics as coercion*; *the foolish mathematician; the overwrought mathematician; the mathematician who can't teach*; *disparagement of mathematicians*; *the Einstein effect;* and *the mathematician with special powers* (pp. 74-75). Providing no empirical evidence, Picker and Berry claimed: "it is very rare to be in a mathematics class and hear the word mathematician used during the lesson.…[and] it is very unusual to hear pupils addressed in a mathematics class as mathematicians" (p. 90). The purpose of this paper is three-fold: We examine when and how teachers refer to mathematicians in their classrooms; we characterize the teachers' framings of mathematicians; and we discuss how these descriptions position teachers and students as mathematical participants with respect to the work of mathematicians.

Methodology

We draw from the data set of a larger NSF-funded study on mathematics classroom discourse (Herbel-Eisenmann, PI, Grant No. 0347906) in which eight classroom teachers (grades 6-10) from seven schools in the Midwest participated. Five of the teachers were certified to teach secondary mathematics, and the remaining three teachers were elementary certified.

Because we were interested in describing how classroom teachers framed mathematicians during their lessons, our primary data source was classroom observation data. During the first year ('05-'06), data were collected during a class period chosen by each teacher for four full weeks across the school year. All 148 observations were videotaped and transcribed. A search engine was used to find all instances of the word "mathematician" in the transcripts. During the

'06-'07 school year, we continued to observe the class of the teacher who is the focus of the case study for this paper because we found that most of the examples came from his classroom.

Results

In the first year, we found that 14 of the lessons contained the word "mathematician," and only teachers (not students) referenced mathematicians. All but one of the instances was found in classrooms taught by teachers who were certified to teach secondary rather than elementary. Furthermore, two of the elementary certified teachers who, on numerous occasions expressed concerns about their mathematical content knowledge, never mentioned mathematicians.

We identified three themes related to teachers' use of the word mathematician. First, and most often, teachers referenced mathematicians *in relation to time*. Within the references to time, we identified the two sub-themes of mathematicians' concern with *saving time* and time with respect to *the historical development of mathematics*.

Related to the first theme is the second theme of *mathematical convention*. In some cases, students were told to use particular notation because that was how mathematicians would do it. All examples of the third theme of *mathematicians in conflict* with others came from one particular teacher who we call Matt. Because Matt was the teacher who spoke about mathematicians 6 of the 14 instances the first year, we considered him as a separate case.

The case of Matt

Matt, a second year teacher, assigned to teach geometry, had more mathematics credits than the other project teachers. We observed 29 of Matt's lessons across six weeks. Matt spoke of mathematicians during all six weeks for a total of 19 times. When looking at how Matt discussed mathematicians, we identified two themes. First, Matt frequently mentioned time, both in a historic sense and in reference to wanting to do something quickly. Second, Matt often positioned mathematicians in conflict with others such as mathematicians versus logicians, mathematicians versus textbook authors, or mathematicians versus "real people."

Discussion

Our findings provide empirical evidence that there are instances of teachers framing the work of mathematicians and framing students as mathematicians. We find it particularly significant that none of these instances came from the two classrooms of the teachers who took the least mathematics credits. Additionally, Matt, the teacher who had more mathematics credits than the other participating teachers, referenced mathematicians most often.

These results also indicate that more work needs to be done to promote positive images of mathematicians. Additionally, consideration should be given to the possibility that classroom practices might also contribute to mathematics' unfortunate "image problem" (Malkevitch, 1989). As noted by Picker and Berry (2000), teachers seem largely unaware of their students' lack of knowledge about mathematicians and the role that they play in shaping and changing their views. While we cannot control popular culture portrayals, we speculate that teachers and teacher educators could be best positioned to counter the messages sent by these other sources if they more consciously attended to their references of mathematicians.

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MATHEMATICS TEACHERS' KNOWLEDGE THAT SUPPORTS INQUIRY TEACHING

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This paper deals with secondary mathematics teachers' knowledge that supports the teachers' use of inquiry-teaching approaches. It discusses the nature of, and relationships among, four domains of knowledge that are central in explaining when and how they use these approaches. It suggests the importance of learning tasks in teacher education that allow teachers to integrate these knowledge domains.

Current perspectives to reform mathematics education suggest that inquiry teaching [IT] is important for students to develop deep mathematics understanding and mathematical thinking. In inquiry classrooms, students are expected to construct mathematical meaning through reasoning, exploring, communicating, and collaborating with peers and the teacher while working on tasks that are inquiry-oriented (NCTM, 1991). However, my experience suggests that such classrooms are less common at the secondary level. Even after learning about IT approaches, high school teachers, in particular, may not implement them in the classroom. This paper reports on a study that investigated the relationship between the knowledge of teachers exposed to IT theory/approaches in preservice or inservice education courses and their teaching. Of particular interest is the nature and role of the knowledge of secondary mathematics teachers that support their use of IT approaches.

Related Literature and Theoretical Perspective

Quality teaching is directly related to teachers' knowledge (Shulman, 1986). Teachers' beliefs about mathematics have been shown to play an important role in how it is taught (Ernest, 1989; Thompson, 1992). Research also has provided evidence to support concerns about the adequacy of teachers' knowledge as a basis to teach mathematics in an inquiry or reform-oriented way (Ponte & Chapman, 2006). However, for the study reported in this paper, the focus is not on highlighting deficiencies in the teachers' knowledge, but to examine how their knowledge forms a basis of sense-making in their teaching. The study is framed in a cognitive perspective of interpreting teacher knowledge (Fenstermacher, 1994), in particular, the pedagogical knowledge they construct through theory, practice, and other experiences. This knowledge constitutes the teachers' perspectives of what they know, that is, what makes sense to them. The study also adopts the view that three major domains of mathematics teacher knowledge are necessary for proficient mathematics teaching: knowledge of mathematics, students, and instructional practices (Kilpatrick, Swafford & Findell, 2001). It considers these domains of knowledge from the teachers' perspectives.

Research Process

Three preservice and one inservice secondary mathematics teachers were studied. The preservice teachers were in the second year of their two-year B.Ed. program, which had a focus on IT. However, they were in predominantly traditional classrooms for their practicum. The inservice teacher was taking a graduate course related to inquiry pedagogy, but not specific to mathematics and at the beginning of the study had started working on implementing IT approaches in her practice.

The main sources of data were interviews, classroom observations, discussions, and teaching documents. The open-ended interviews included a focus on the participants' thinking about mathematics, IT and learning; actual experiences with IT; and thinking about what supported or inhibited their use of IT. Classroom observations were conducted for lessons including and not including IT approaches. Discussions included the teachers' knowledge of the mathematics concepts in the IT lessons. Documents included lesson plans.

Data analysis involved focusing on identifying the participants' knowledge and thinking about mathematics, students' learning and IT; situations of when, how and why they used IT approaches; and apparent relationships between their knowledge/thinking and IT. Themes were determined by identifying conceptual factors that characterized each participant's thinking and practice based on the information from the scrutiny of the data. "Patterns" emerged as the most dominant theme in relation to IT.

Results

Two of the preservice teachers (Reba and Sara) and the inservice teacher (Brea) used IT approaches. (Hereafter, they will be referred to as IT teachers.) The third preservice teacher (Cara) did not use any IT approach. (Hereafter, she will be referred to as non-IT teacher.) The IT teachers differed in how they planned and conducted their IT lessons, but exhibited prominent similarities that seemed to characterize their sense making in using IT. Only these similarities are discussed here. They involve four domains of knowledge and the relationships among them, which seemed to be central to explaining the teachers' use of IT approaches. The non-IT teacher differed from the IT teachers in these domains of knowledge. The following highlights the nature of these knowledge and the relationships among them.

Beliefs about Mathematics

The IT teachers held a similar core belief about mathematics that provided the foundation for when and how they used IT approaches. For Sara, this belief was "patterns, making connections between patterns and the world," for Reba "a lot of patterns … can be found everywhere," and for Brea, "patterns and connections and interconnectedness … [having] connections to the world." The non-IT teacher had a different focus, for example, "problem solving, a way of analyzing the situation and exercising your brain in a certain way."

Beliefs about Learning

The IT teachers held a core belief about students' learning that was compatible with IT and directly related to the belief about mathematics. This belief focused on having students "make the connections themselves" (Sara), or "see patterns for themselves" (Reba), or "make connections and see things as interconnected" (Brea). The non-IT teacher had a different focus, for example, "There are tons of different learners that learn at different rates. … Some people get it in a snap; some people take a lot longer to get it."

Mathematics Knowledge

The way the IT teachers held their knowledge of mathematics played a significant role in terms of when and how they were able to transform their beliefs about mathematics and learning into practice. For example, mathematics concepts that they readily understood, or already held, in terms of patterns were taught through IT approaches. The IT teachers were able to associate the concepts they held as patterns with specific IT approaches that focused on the mathematical structure of the concepts, for example, approaches involving a "compare/contrast" technique.

Instructional Knowledge

Finally, the IT teachers held instructional knowledge for engaging students in the learning tasks that was directly related to their belief about learning. Their thinking and practice indicated that this knowledge included: use of groups, open questioning/prompts, and flexible listening in an IT context. For example, regarding listening, Sara explained: "I always tried to make them tell me what they were doing. … I always tried to listen to their process … how are they thinking and why." Reba noted: "I would ask them, 'Well, what are you thinking of?' because then that could trigger them without me having to say how you solve this." Brea, "I am questioning what is being called forth in that moment. I am deeply trying to listen to their inquiries now." The non-IT teacher had a different view, for example, "I try and understand … if they're asking a question and they're like on the page before …that tells me that they haven't really listened to anything that I've done since that."

Classroom observations of the IT preservice teachers' practice revealed that their inquiry-oriented lessons had a similar structure, consisting of: an introduction stage, an exploration stage, a sharing and discussion stage, and a conclusion stage. The IT experienced teacher's lessons had a more complex, cyclic structure that included initial responses, inquiry-based discourse, investigations/research, collaboration, and extension/application.

Conclusion

The nature of, and relationship among, the teachers' knowledge in the four domains of knowledge are central in accounting for, and understanding, their use or non-use of IT approaches. The teachers did not only have to hold knowledge relevant to IT, but, of particular importance, they had to hold the four domains of knowledge in an interconnected way. For the IT teachers, the theme of the interconnectedness was pattern. When the inter connectedness was lacking, for example, the mathematics concept was not viewed as a pattern, the preservice teachers resorted to traditional teaching and justified it in terms of the contextual classroom constraints they perceived. For the non-IT teacher, there was no centrally held theme to connect any of the theoretical knowledge she held about IT. For example, she considered IT to be about "getting the students to discover concepts on their own as opposed to telling them what it is, so giving them an opportunity to explore a problem or a situation." But this was not evident in her teaching and she believed it was the traditional classroom context she was in that prevented her use of IT.

The findings suggest the importance for teachers to construct an integrated image of IT to be able to implement it. For the IT teachers in this study, this involved understanding mathematics as patterns, learner as inquirer of patterns and teacher as facilitator of student as inquirer of patterns. This validates the importance of providing teachers with learning experiences that treat these key domains of teacher knowledge in an integrated way.

Endnote

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TELLING TALES FROM OUTSIDE OF THE CLASSROOM: NARRATIVE AS A LENS ON HOW ONE TEACHER MADE CHANGES IN HER MATHEMATICS TEACHING PRACTICE

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This paper examines narratives one teacher told during participation in a Professional Study Group. Through various retellings, one sees her shift the responsibility for underachievement in mathematics from the child to herself. Specific plans the teacher made to adapt her teaching practice are noted. Through her retelling of events we gain access to what supported her in forming these plans.

There is a recognized concern over the lack of cultural congruence between the mainly female, white, middle-class teaching force in elementary school and the growing diversity of the students in schools nation wide (Howard, 1999; Nieto, 2004). The study adds to a growing body of information on methodologies that support teachers in becoming better teachers of "other people's children." This paper looks at narratives that one teacher, whom I call Ellie, told during participation in a Professional Study Group, and how through various retellings, one can see her shift the responsibility for underachievement in mathematics from the child to herself. One goal of the Study Group was to support the members in changing their teaching practice in mathematics so that it better addressed the needs of a student in their classrooms that struggled in mathematics. This paper chronicles the specific changes that Ellie made or planned to make in adapting her teaching practice in mathematics.

Theoretical Framework

Mathematically substantive pedagogy may not be sufficient to support teachers in teaching diverse populations of children. Pedagogy that is culturally relevant may also be necessary (Ladson-Billings, 1994, 1995). In order to develop a pedagogy that is both mathematically substantive and culturally relevant it may be helpful for teachers to examine children's competencies outside of the confines of the classroom and in this way build more authentically on the child's lived experiences.

Professional development that engages teachers in a close examination of a single child has been shown to support teachers in arriving at deeper understandings of that child's particular educational needs (Brooker, 2003; Gallas, 1992; Himley & Carini, 2000).This study built on these ideas by engaging teachers in a deep examination of a single child from their class who came from a background unlike their own.

Through an examination of the narratives of one participant, we gain access to the ways in which it is possible to use information gained in out-of-classroom settings to support changes in practice. The power of narrative is another thread that frames this study. In this case, Ellie provides us with a local case (Clandinin & Connelly, 2000) that can be pointed to as standing in contrast to our national narrative of the underachievement of non-White students due to deficits they bring to school. The analysis of the narrative allows for tracing changes in the teacher's point of view as she successively retells events (McVee, 2004).

Methods

This paper draws on data collected as part of a larger study (Author, 2006, in submission). In that study, six teachers participated in a Professional Study Group in which they each conducted a case study of a particular child from their respective classrooms who came from a background unlike their own. The participants in the study were six White in-service teachers who at that time worked in the same elementary school in a moderately sized Midwestern school district. This paper analyzes the narratives told by one of these six teachers. This teacher made two presentations to the Study Group about her target child. In addition the researcher/faciltiator met with the teacher before each presentation to support her in preparing the presentation. In this way, the teacher had four separate opportunities over the course of a semester to share information in narrative form about her target child.

Between the two presentations, the teacher spent one day shadowing her target child in the school setting. This provided the opportunity for the teacher to observe the child in situations to which a teacher does not usually have access such as lunch and recess, pull-out programs, and special classes such as art and gym. During this time between the two presentations, the teacher also met with the mother of her target student. The purpose of the meeting was for the parent to share with the teacher information about the child's out-of-school interests and experiences, using photographs she had taken as an artifact to support the discussion. One intention was to position the parent as the one from whom the teacher would learn about the child, reversing the traditional power dynamic between teacher and parent.

Study Group sessions, the meetings to prepare for the presentations, and the parent-teacher meetings were audio-taped. Field notes were taken at each meeting. All audio-tapes were transcribed to facilitate analysis. I examined the narratives this teacher told about her target child, and herself as well as about her interactions with the child's mother. This perspective provided a view of the degree to which this teacher was able, not only to learn significant new information about her target child that could support the teaching of mathematics to that child, but also the degree to which this teacher was able to act upon the information, by positioning herself as the party responsible for the underachievement.

Results

Through an analysis of the narratives told by Ellie, one sees that she not only identified specific mathematical abilities that her target child, whom I call Evan, exhibited outside of the classroom, she also brought that learning into the classroom. She discussed competencies such as the Evan's ability with counting that she had accessed while consulting with his mother. She presented problem solving and language competencies that the child possessed that she had accessed while observing him on the playground. She pondered ways to bring those out-of-classroom competencies into the mathematics classroom. She not only learned things about the target student, she learned things about herself as well. She moved from dismissing Evan's mother's estimation of his counting abilities, to considering more carefully that her own experience was one that should be open to examination.

Discussion

The telling of incidents and episodes about her target student as part of her participation in the Professional Study Group, provide an opportunity to see how Ellie examined the preconceptions she was bringing to assessing Evan's abilities, changing her point of view while the events she related stayed the same (Wolf, 1992). Ellie identified that Evan worked best in the workshop rather than the centers format. This analysis of the characteristics that supported his learning may have allowed her to think about how to restructure other time in the classroom. This ability to analyze existing practice, and analyze it not merely in a general way, but with regard specifically to her target student, may have positioned her to interrogate her practice in a way that supported her in either implementing or making plans to implement changes in practice.

The narratives show how she was able to change the initial stance she had toward Evan as a child who was continually out of step with the flow of classroom activities, rarely connecting to the classroom program. They demonstrate that Ellie accessed evidence of problem solving in unexpected spaces around the school, including the playground and is able through these experiences to identify herself and her teaching practice in mathematics as in need of change. We are also able to see through the narratives she tells, that Ellie has specific ideas about how to adapt her practice to better serve Evan's needs, as well as the needs of other struggling mathematics learners in her class.

This case confirms Villegas's (1993 as quoted in ; Zeichner & Hoeft, 1996) findings that conferring with parents and observing children in school are ways that give teachers access to children's lived experiences. The kind of specific knowledge that Ellie had about her target child has been documented as providing a basis from which teachers can potentially continue to grow and change (Franke, Carpenter, Levi, & Fennema, 2001). In the area of student-teacher relationships, my findings are also consistent with those of Civil (1998) and other funds-of-knowledge researchers, that the relationships forged between teacher and student as a result of participating in professional development that focuses on specific children are of great importance to the teachers. It may support the teacher in looking toward herself instead of toward the child for changes that can better support learning.

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TOWARDS EFFECTIVE TEACHING OF LOGARITHMS: THE CASE FOR PRE-SERVICE TEACHERS

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This report is a part of ongoing research on pre-service secondary mathematics teachers' content knowledge. The study draws data from two sources: the Math Play and the peer-job-interview. The analysis based on practice-based theory of mathematical knowledge for teaching (Ball et al, 2006), and the system of frameworks for interpreting understanding of logarithms and logarithmic functions (Berezovski & Zazkis, 2006). The results of this study indicate participants' disposition towards a procedural approach and reliance on rules, rather than on meaning of concepts.

The mathematical concepts of a logarithm and logarithmic functions play an important role in advanced mathematics courses. In recent years, several research studies reported on students' understanding and misunderstanding of logarithms at the high school (Berezovski, 2006; Berezovski & Zazkis, 2006) and undergraduate mathematics level (Kenney, 2005; Weber, 2002). It is also common knowledge in the field of mathematics education that teachers' mathematical knowledge for teaching has a strong impact on students' understanding. However, there is no significant body of research that focuses on teachers' mathematical knowledge of logarithms and logarithmic functions.

This paper focuses on pre-service secondary mathematics teachers' content knowledge. To fulfill a dual commitment as an instructor of the methods course, and as a researcher, the author utilizes two tasks: the Math Play and the peer-job-interview. They were used as research methodology and also to create and orchestrate experiences for the pre-service teachers to reflect, and grow in their understanding of how to teach logarithms and logarithmic functions.

Theoretical Frameworks

Ball et al, (2006) introduced the mathematical knowledge for teaching model. The model has four domains, each tied to the distinctive work teachers do.

a) *common content knowledge* (CCK) — the mathematical knowledge of the school curriculum;

b) *specialized content knowledge* (SCK) — the mathematical knowledge that teachers use in teaching that goes beyond the mathematics of the curriculum itself;

c) *knowledge of students and content* (KSC) lies at the intersection of knowledge about students and knowledge about mathematics. KSC includes knowledge about what students are likely to do with specific mathematics tasks;

d) *knowledge of teaching and content* (KTC) – the knowledge about instructional sequencing of particular content, and provides useful examples for highlighting salient mathematical issues.

Each domain is used to measure the particular mathematical knowledge. For example, domain a) manifests participants' knowledge of the definition of a logarithm, and their ability to apply it, presenting logarithms in the exponential form. Within b) domain I can evaluate teachers' understanding of mathematical content. The system of interpretive frameworks (Berezovski & Zazkis, 2006) for investigating high school students' understanding of logarithms was adapted to measure teachers' understanding. This model differentiates teachers' mathematical knowledge upon meaning of logarithms as numbers; operational meaning of logarithms; and logarithmic functions. As such it allowed the researcher to gain deeper insight into pre-service teachers' understanding of logarithms.

Methodology

Participants of this study were 47 pre-service teachers enrolled in the Design for Learning Secondary Mathematics course. Data collection relied on two major sources: the lesson plan with the Math Play and the peer-job-interview. Both tasks were created to simulate possible situations of the beginner teachers' experiences. Constructing such situations, Brousseau (1997) points out the necessity of organizing a milieu in which students can work assuming a maximum responsibility about mathematical knowledge. It is essential to employ such situations during the training time. They enable pre-service teachers to evolve their own mathematical knowledge, due to the interactions with situations; and, to experience the mathematical interactions with the students. They enable the mathematics educator to observe such developments with respect to teachers' mathematical knowledge.

The Math Play

Within the sequencing of lessons that student-teachers designed as a part of the course project, they were asked a) to summarize the lessons, carefully sequencing and connecting the lessons to provide optimal learning experiences; b) write a detailed lesson plan that incorporates a Math play. Participants were provided with an episode of a fictional mathematical interaction between a student and a teacher that presented a problematic situation in which student has developed a misunderstanding about logarithms.

There is a conversation between a teacher and a student (there are 30 students in a class):

T: Why do you say that $\log_3 7$ *is less than* $\log_5 7$*?*

S: Because 3 is less than 5.

Teachers were to diagnose the misunderstanding, formulate a plan for the remediation of the misunderstanding, and write out the balance of the interaction in the form of a play.

The data collection was based on the participants' math plays; the evidence of the level of teachers' content knowledge was established upon choices participants made in the remediation part of the play, and the diagnosis of the fictional student misconception.

The Peer-Job-Interview

This task was designed in the way that participants have to interview each other in the capacities of a head of math department in a secondary school and a substitute teacher applying for a 6-month contract covering for a teacher on a maternity leave. The conditions of this task were the following: the interviewer will do his/her best to verify and evaluate the potential candidate's knowledge and understanding of the mathematical content required to cover (for the purpose of this paper it is limited to logarithms). The potential candidate should do his/her best to answer the questions to demonstrate his/her competence. The data collected from this interview was the original recording of the interview, the transcript of the interview, and the written explanation of the participants' choices of the interview questions. Then, the fictional interviewer had to submit his/her written evaluation of the interviewee highlighting the candidate's qualification and suggestions to strengthen his/her knowledge.

Results, Conclusions

Majority of results show that mathematical knowledge possessed by the pre-service teachers participated in this study is limited. Such results draw upon the following aspects of investigation. Focusing on the SCK domain, I investigated to what degree logarithms are understood as numbers and whether the value of a logarithm influenced this understanding. In the traditional curriculum, the concept of logarithm is presented as an inverse of the exponent. A novice operating in this framework may correctly interpret, for example, the value of log39, by using the definition ($3^2=9$ _ $\log_3 9=2$). The main result obtained by exploration in the second interpretive framework is that participants' understanding of the operational character of logarithms is weak. While focusing on operations with logarithmic expressions, I explored the participants' awareness of the isomorphism between multiplicative and additive structures that determine the "rules" by which logarithmic expressions are manipulated. In investigating teachers' understanding of logarithmic functions the main focus is on the considerations teachers' conceptions of logarithms related to logarithmic function, and how they employ logarithmic properties and different representations in constructing successful problem solving situations.

The contribution of this study is in providing two educational tasks employed for training pre-service teachers. These tasks were constructed and adapted in order to create rich learning environment for the perspective teachers, and proved themselves worthy in this regard. Furthermore, this research provides a better understanding of the participants' difficulties involved in acquiring important mathematical concepts of a logarithm and a logarithmic function and also suggest possible explanations of the sources of these difficulties. As such, this study lays a foundation for future research on this topic. It paves the path to future development of pedagogy.

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IMPORTANT MATHEMATICS AND POWERFUL PEDAGOGY (IMAPP)

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The foci of IMAPP (a 3-year research and Professional Development (PD) project) are to deepen secondary school mathematics teachers' content knowledge and enhance their use of research-based powerful pedagogies. Teachers attend a summer mathematics content PD Institute (PDI) on ***legacy*** *&* ***future*** *mathematics content for secondary school mathematics curriculum and a pedagogy PDI, analyze their Iowa Test of Educational Development (ITED) data to identity two hard to learn concepts/ideas that will serve as fodder for designing and implementing plans for student learning through a Lesson Study Approach (LSA).*

Research Questions

[I] Content Intervention Questions: [Overarching] What opportunities to learn (OTL) mathematics content are afforded participants in a summer mathematics content professional development institute? [II] Pedagogy Intervention Questions: [Overarching] What levels of intervention, collaboration, and supervision of LSA can be effective in improving pedagogy to enhance students' understanding? [III] Students' Outcomes Question: [Overarching] What are the trend-shifts between ITED scores for students of participants/AEA before the content PDI, pedagogy PDI and the LSA interventions and after the interventions?

Methods

IMAPP will serve 20 participants in Year 1 (2007-2008 AY), 40 in Year 2 (of which 20 are repeating from Year 1), and 60 in the third year of the project (40 of which are repeating from Year 2). Participants will examine/explore the following ideas content PDI [1] Algebra; [2] Quantitative Literacy*;* [3] Statistics and Probability*;* and [4] Geometry [*Legacy – Yr1 (Coordinate), Yr2 (Trigonometry), Yr3 (Geometric Properties); Future – Yr1 (Vertex-Edge Graphs), Yr2 (Vertex-Edge Graphs), Yr3 (Transformations)*]. The participants will explore powerful pedagogies and related understandings around *How People Learn*, LSA theory and practice, teaching through problem solving, Problem-Based Instructional Tasks (PBITs) selection, and task implementation. Data saturation include, pre-post assessment of teachers' mathematics content knowledge of secondary school mathematics (concept maps, free-response and MC tests); observational data (field-notes, video and audio records) on PDIs, and teachers' classroom enactments; teacher interviews, Curriculum Enactment Suite, and Conception of Mathematics Inventory. Both qualitative (content analysis, constant comparison, structural and relational analysis, thematic/categorizations) and quantitative (comparison of means) data analysis will be deployed.

Ongoing Analysis and Preliminary Results

The poster presentation will share preliminary results from the analysis of data from the summer content and pedagogy PDIs, and initial observations of teacher enactments pre-PDIs, and beginning enactments post PDIs. The participants' report of awareness of *future mathematics content* agrees with our initial assumption about teachers' opportunity to learn the said content areas, with 92% of participants starting that they had either *no knowledge* or *small amount of knowledge* of vertex-edge graphs; 98% answered similarly on preferential voting; and

61% answering similarly on recursion. Analyses on IMAPP generated post-test of content and the Knowledge for Algebra Teaching (KAT) assessment (pre and post) are underway to parse out what new mathematics content the participants learned during the content PDI. Ongoing analysis of the data on lesson planning, teaching, and reflection on teaching based on enactment and student work samples are yielding images of complexities of effectiveness such as *knowledge of how people learn*, *content trajectory and content fidelity/coherence*, *mile-wide and inch-deep* lesson plans, *flexible pedagogical support for learning*, and *student engagement*. For example, it appears that there is a struggle, among participants, between *teaching through multiple representations* that addresses the principles as stated in the Principles and Standards for School Mathematics (NCTM, 2000) and *teaching to learning styles.* This is a struggle that IMAPP intends to unpack and repack in ongoing interventions and interactions with the participants. *Trying to do too much* (*mile-wide and inch-deep lesson plans*) has surfaced as an area that IMAPP needs to confront as we review lesson plans and observe enactments of hypothesized learning trajectories. Within the area of *content trajectory and content fidelity/coherence*, data analysis is surfacing a complexity of *repacking the unpacked mathematics for instruction*. Repacking the mathematics for instruction is an often neglected process in effecting learning. So what we find, for example, is enactments of lesson plans that provide a near surface treatment of the nuanced complexities of the subject matter and the intended *mathematical residues*.

Based on local analysis of ITED data, the participants chose the following *legacy content*: Percents, interpreting graphs, linearity, and linear programming. During the Pedagogy PDI in the summer of 2007, the demonstration lesson for the LSA was on linear programming. Twenty four high school students served as a focus-group to help the lesson team, teacher-peers/colleagues serving as observers, and Lesson Study experts make sense of the mathematics, teaching and learning treated in the lesson. The data analysis from that LSA cycle is still underway. Two school-based groups of participants have completed an initial trial of the lessons on *interpreting graphs* and *linearity* in the current 2007 AY. Data digitization and transcription are underway as we prepare the data for analysis. More LSA enactments are planned for the Fall of 2007and Spring of 2008. What we learn from the first iteration of IMAPP in the summer of 2007 and the 2007-2008 AY activities will inform the re-design of IMAPP for the summer of 2008 content and pedagogy PDIs.

The overall evaluations of the 2007 content PDI were quite positive, with 17 of the 23 evaluations items receiving an average score of four or better on a five-point likert-scale. Another two evaluation items received scores between 3 and 4 on the five-point likert-scale. The participants were equally positive about the 2007 pedagogy PDI, with 17 of the 27 evaluation items receiving an average score of four or better on a five-point likert-scale. Another nine evaluation items received average scores between 3 and 4 on the five-point likert-scale. More analyses are underway, but the little we are able to discern so far encourages us that we have embarked on a productive trajectory of engagement with teachers that will facilitate their identification with the core practices of the mathematics education community.

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PROSPECTIVE AND PRACTICING 4-12 MATHEMATICS TEACHERS' THINKING WITHIN A UNIVERSITY COURSE ABOUT PROPORTIONAL REASONING

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Instructors addressed three questions: What themes emerged regarding student thinking? How did these themes direct instructional decisions? How did the instructional decisions impact student thinking? Data included assessments, tapings, and other course artifacts. Regarding content and pedagogy, students progressed in thinking related to multiplicative reasoning and ratio sense yet continued to struggle with explaining thinking related to invariance and covariance.

Mathematics teacher preparation and professional development programs must attend more explicitly to teachers' development of proportional reasoning and related concepts (Sowder, Armstrong, Lamon, Simon, Sowder, & Thompson, 1998). Within a one-semester university course about proportional reasoning, the instructors carefully documented student thinking in order to address the following questions: (1) What themes emerged regarding student thinking during the course? (2) How did these themes direct instructional decisions related to this course? and (3) How did the instructional decisions impact student thinking?

There were 27 students in the class. Twelve students were undergraduate interdisciplinary majors seeking 4-8 mathematics certification and 15 students were post baccalaureate, M.Ed., or Ed.D. students. Nine students were middle, high school or community college mathematics teachers.

Within the course, students engaged in several activities related to proportional reasoning (e.g., Thompson & Bush, 2003), read professional and research articles related to proportional reasoning, and planned and evaluated lessons based on proportional reasoning concepts. These experiences were designed in such a way as to engage students in mathematical and pedagogical problem solving and to provide opportunities for the creation of meaningful representations of mathematical and pedagogical ideas.

To assess patterns and themes regarding student thinking, and to assess the effects of instructional decisions on student thinking, the instructors used pre-, mid-, and post-assessments, video of class meetings, audiotapes of small group discussions within the mathematical activities, student reflections, field notes taken during the class meetings, and other course artifacts. The resulting data were studied for the student use of multiplicative reasoning, evidence of ratio sense, and the application of invariance and covariance. We also attended to students' abilities to explain one's thinking related to these three components of proportional reasoning. The data were also examined for pedagogical insight needed to develop thinking and skills related to multiplicative reasoning, ratio sense, and invariance and covariance with students in grades 4-12.

The students progressed in their thinking related to multiplicative reasoning and ratio sense yet continued to struggle with explaining thinking related to invariance and covariance. This was true for thinking related to content and pedagogy. The poster session will include a discussion about the instructional decisions that may or may not have contributed to these findings.

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TEACHING DATA ANALYSIS INSIDE OF AN ALGEBRA CURRICULUM: THE ROLE OF CONTENT KNOWLEDGE

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Teaching is a complex activity. Teacher content knowledge is one of many factors that influence teaching, though we are not certain in what ways and to what extent (Wilson, Floden, & Ferrini-Mundi, 2001). While some studies have addressed what knowledge teachers should have (Fennema, et al, 1996; Hill & Ball, 2004), few studies have looked directly at how that knowledge plays a role in the classroom. Teaching frameworks claim that several factors, including teacher knowledge, might influence the implementation of curricular materials and instructional tasks (Stein & Lane, 1996). This presentation specifically focuses on the way that teacher content knowledge affects the design and implementation of tasks in an Algebra I/Data Analysis curriculum. This relates to the goals of PME-NA by seeking to extend understanding of how teaching Data Analysis is affected by the mental constructs of content knowledge.

This poster describes the ways in which two teachers' content knowledge affected their creation and implementation of instructional tasks within an Algebra I/Data Analysis curriculum. They are both veteran teachers at the same high school located in an urban setting in the Eastern United States. The Algebra I/Data Analysis class is high stakes in that all students are required to pass a state mandated exam pertaining to this content in order to graduate.

This study comes out of a larger, multiple case study project and for the purposes of this presentation I focus on two cases. Each of the analytic methods that follows draws upon the constant comparative method used in the development of grounded theory (Glaser & Strauss, 1967) in which data are constantly compared as they are analyzed against current conjectures. Eight observations of classroom teaching, pre and post teaching conversations, and two interviews (one focusing on testing content knowledge) were conducted with each teacher throughout the school year.

Observations and interviews of the two teachers indicated that one had a robust, deep understanding of Data Analysis and its relationship to Algebra (T4) while the other had gaps and misunderstandings regarding some concepts in Data Analysis (T6). According to the instructional task framework proposed by Stein & Lane (1996), T4 created and employed Data Analysis tasks with a "high level" of "cognitive demand." Interviews, conversations, and observations suggest that her knowledge about Data Analysis and its relation to Algebra played a direct role in the creation and implementation of the tasks. The Data Analysis tasks designed by T6, however, are categorized as "low level" activities in Stein & Lane's framework. Interviews, conversations, and observations indicate that her discomfort with and misinterpretations about Data Analysis contributed to the types of tasks she employed in her class. These findings suggest that teacher content knowledge can play a role in the types of instructional tasks used in the classroom.

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COLLEGE GEOMETRY STUDENTS' USES OF TECHNOLOGY IN THE PROCESS OF CONSTRUCTING ARGUMENTS

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Prior research on students' uses of technology in the context of Euclidean geometry has suggested it can be used to support students' development of formal justifications and proofs. This study examined the ways in which students used a dynamic geometry tool, NonEuclid, as they constructed arguments about geometrical objects and relationships in hyperbolic geometry. Five students enrolled in a college geometry course participated in an interview about properties of quadrilaterals in the Poincaré disk model. Toulmin's argumentation model and the MATCI framework were used to analyze students' uses of technology in the process of constructing arguments as they were working on various tasks.

Introduction and Theoretical Framework

Traditionally, College Geometry is a difficult course for students because it requires them to reason strictly from axioms and postulates rather than informal experiences and intuitive understandings. In order for students to appreciate the importance of the rigorous axiomatic approach, most college geometry courses introduce students to a less intuitive world of non-Euclidean Geometry. Students generally enter a college geometry course with twelve or more years of experience working within the Euclidean system of axioms, and their understandings of figures and relationships within this system are challenged when the axioms are modified. While geometry, in general, is a very visual subject, there are several limitations to students' uses of paper-and-pencil diagrams, especially when it comes to non-Euclidean geometries. A student may create inaccurate misleading diagrams and arrive to incorrect conjectures, or a student may create a correct diagram that is too specific and this may inhibit their ability to derive general conclusions and proofs that go beyond the drawing they have created (Schoenfeld, 1986).

Many mathematics education researchers and professional organizations have suggested the use of dynamic software programs to teach geometry (NCTM, 2000). These software programs enable students to construct accurate diagrams and interact with the diagrams to abstract general properties and relationships, because the ways in which the programs respond to the students' actions is determined by geometrical theorems. Research related to secondary students' uses of such programs has been shown to improve their understandings of geometrical concepts and support their development of formal proofs (Laborde, Kynigos, Hollebrands & Straesser, 2006). Such programs show promise for working with models of hyperbolic geometry, where interpretations of planes, lines, and angles are unconventional. Various technological tools have been developed to assist students in reasoning within different non-Euclidean systems (e.g., *NonEuclid*, Castellanos, 2007), but little research has examined how students' uses of such tools affects their mathematical thinking or influences the mathematical arguments they develop.

While proof and formal logic are two characteristics of mathematics that separates it from other sciences, students learning mathematics often engage in mathematical reasoning and sense-making activities prior to constructing a formal proof. It is in this "territory before proof" (Edwards, 1997) where students make the most use of technology tools, but few researchers have examined how students' use of such tools affects less formal mathematical arguments they develop. The purpose of this study is to investigate the ways in which students use dynamic geometry tools (*NonEuclid*) as they construct arguments about geometrical objects and relationships in hyperbolic geometry. The purposes of this study are closely tied to the goals of PME-NA in that it seeks to build upon and extend our knowledge of the psychological processes involved in students' construction of mathematical arguments when solving mathematical tasks with the use of a technology tool.

One model that has been used by several researchers (e.g., Stephan & Rasmussen, 2002; Lavy, 2006) in mathematics education to examine students' mathematical arguments is Toulmin's model of argumentation (Toulmin, 1958). This model views argumentation from a practical perspective rather than a pure logico-mathematical viewpoint. This model decomposes an argument into three main components: claim, data, and warrant. When making an argument a claim is made. The claim is often the purpose of the argument and it is generally based on some form of evidence, facts or general information called data. If the argument is challenged, then a warrant may be provided, which is the logical connection between the data and the claim. The warrant explains the relationship between the claim and the given data. To aide the audience in understanding the reasoning used in the warrant, additional information, called backing, may be provided to support the warrant. This model of argumentation was used to focus the researchers' attention on different aspects of a student's argument and examine the ways in which technology was used in response to a specific mathematical task.

Methods and Data Sources

For this study, five participants were selected to participate in a series of three interviews conducted by the first two authors of this paper. The third and fourth authors were involved in the teaching of the college geometry course and the development of the interview tasks and protocols.. The five participants included two students pursuing bachelors degrees in mathematics: Teresa and Calvin, and three students pursuing degrees in mathematics education: Gail, Ed, and Will. The one-hour interviews were conducted outside of class during the beginning, middle and near the end of the semester during which time they were taking the college geometry course which included five technology-based lab assignments. A video-camera captured students' written work and a video-recording device directly captured students' work on the computer. Interview transcripts were created from the videotapes. Data taken from the second of these three interviews were analyzed for this study

To analyze college students' mathematical arguments, Toulmin's model of argumentation was used to diagram claims a student made, isolate data the student used to support their claims, identify warrants the student provided to explain how the data were related to the claims and describe the backing if such backing was provided by the student. For each claim that was made by the student, the way in which the technology was used in various aspects of the argument were noted and the type of task on which the student was working was determined.

Because an argument that a student may provide in an interview setting is likely influenced by the task posed, once arguments were identified, the task on which students were working was also coded using the Mathematical Task Coding Instrument (MATCI, Heid, Blume, Hollebrands & Piez, 2002; Hollebrands & Heid, 2005). Tasks that were identified were of three different types: 1) the original task in the interview protocol, 2) questions posed by the interviewer, and 3) tasks taken on by the student that may have deviated from the original task presented to the student or a question posed by the interviewer. Task codes that were used included: Identify, Describe, Elaborate, Produce, Corroborate, Predict, Justify, Generalize, and Generate. These task codes are associated with three different categories: Concept, Product and Reasoning. Codes within the Concept category included Identify, Describe, Elaborate, each of these task has a goal of characterizing a mathematical concept. Codes within the Product category included Produce, Generate, Predict, and Generalize. All four of these tasks require students to create a specific mathematical object. The third category, Reasoning included the Justify and Corroborate tasks and these tasks involve students in developing a rationale for a particular claim. Task codes were placed on each Toulmin diagram to identify the task on which students were working as they were constructing an argument. A spreadsheet was created for each student to coordinate the different codes and this enabled the researchers to look for patterns in the codes.

Results

In the process of analyzing mathematical arguments, themes related to students' uses of technology and their relationships to different components of the argument and the tasks on which they worked were identified. One theme that became evident is that when students used the dragging feature of the software, the students were involved in tasks that were coded as Produce and Justify or Generalize. A second theme that was identified was students' uses of the appearances of diagrams on the screen as warrants and measures used as backing in conjunction with their work on a sequence of tasks coded as Predict, Produce, Describe. A third theme that was noted was that when students were involved in tasks coded as Justify, the students rarely made use of the technology in their warrants and backing. In conjunction with this theme, when measurements were used as backing in the construction of an argument, the tasks in which the students were involved were rarely within the Reasoning category. Elaboration and examples of these three themes will be provided in the following paragraphs.

Dragging in Response to a Produce Task Leading to a Justify or Generalize Task

In the process of constructing arguments, students made use of the dragging feature. This feature not only allowed students to view the change in the appearance of the diagram, but also allowed students to focus their attention on the ways in which the measures that were previously taken changed or remained invariant as they dragged. The dragging of a point in the diagram and the links between the diagram and measurements became data on which students based their arguments. In general, students were engaged in two types of tasks while employing the dragging feature. The first type of task was a production task students responded to by producing variations of a particular geometrical figure or by producing a completely different geometrical figure. During these production tasks, or upon its completion, students would then begin to make generalizations or justifications based on what they noticed.

To illustrate this theme, an example of how one student employed the drag feature to produce data for an argument and how this led to a generalization is provided. During the interview, Teresa was exploring the properties of a Lambert quadrilateral, a quadrilateral with exactly three right angles (see Figure 1). She was trying to determine whether there was a relationship between the lengths of the opposite sides of a Lambert quadrilateral. She noticed that a base of the quadrilateral ($\overline{AB}$), defined as a side contained between the two right angles, and its opposite side ($\overline{EG}$) were both longer than the other two sides ($\overline{AG}$ and $\overline{EB}$). Teresa dragged point A towards point B resulting in a Lambert quadrilateral such that $\overline{EB}$ and $\overline{AG}$ were longer than $\overline{AB}$ and $\overline{EG}$.

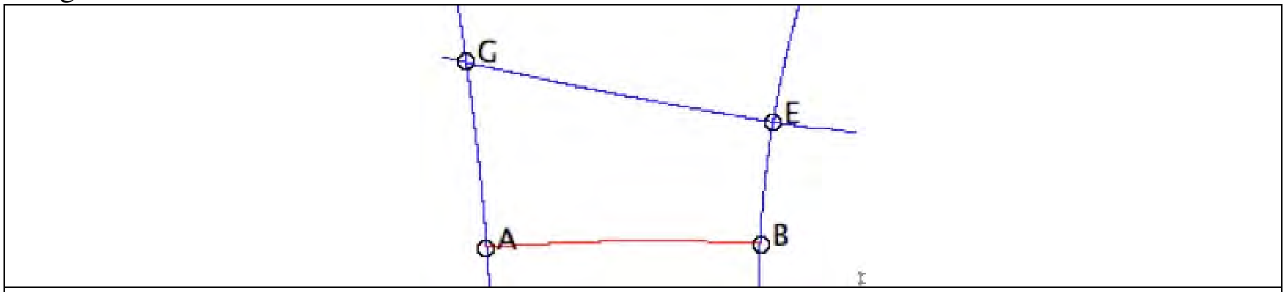

Figure 1. A Lambert quadrilateral, similar to the one used by Teresa and Ed

Teresa then claimed that $\overline{AB}$ and $\overline{EG}$ may not always be longer than $\overline{EB}$ and $\overline{AG}$ (See Figure 2a). In the process of constructing this argument, Teresa was engaged in two types of tasks. The first being a Produce task because she constructed a new example of a Lambert quadrilateral. The second task was a Generalize task because she stated that for all Lambert quadrilaterals that the base of the quadrilateral and its opposite side may not always be longer than the other pair of opposite sides.

	Data	Claim	Warrant/Backing
Figure 2a: Teresa's argument	*Drag to create a different Lambert quadrilateral**	"AB and EG may not always be longer than EB and GA"	*Measures and/or the appearance of the figures*
Figure 2b: Ed's argument	*Drag a point and view the appearance of the figure and the linked measurements*	"I think it's always going to be acute"	Warrant: "*The measurement is always less than 90"* Backing: "*As long as it is on that perpendicular it's a defined angle whenever it's not its all of the angles except for the first right angle becomes undefined"*

**italics denotes the use of technology*

Similar to Teresa's use of the drag feature to produce data for an argument, Ed employed this feature as well to construct an argument regarding the measure of the non-right angle of a Lambert quadrilateral. He dragged point *E* towards point *B* and noticed that measure of angle *G* remained acute. The data for this argument are the appearance and measurements of the changing Lambert quadrilateral afforded by the drag feature (See Figure 2b). He claimed that angle *G* will always be acute and he based this on the fact that the angle measure was always less than 90 degrees, which he used as his warrant. He backed up this warrant by demonstrating that when the point *G* was not on the perpendicular (it was actually on point *A)* only one right angle remained on the screen and the Lambert quadrilateral was destroyed. Similar to Teresa, Ed was involved in two types of tasks, the first being a Produce task, the second a Justify task.

Appearance as Warrants and Measures as Backing

In the process of constructing arguments, there were many instances when students used the diagram on the screen as data, the appearance of the diagram as a warrant and the measurements as backing. During these types of arguments the students appeared to be engaged in three distinct tasks. First, the students Predicted what they believe is true based on the appearance of the diagram. The students would then Produce diagrams and/or measurements that would either confirm or discount their predictions. Lastly, the students would Describe the relationship between their measurements and their prediction.

The following examples illustrate how students used appearances as warrants and measures as backing while engaged in a Predict, Produce, Describe task sequence. During the interview Will was asked to determine whether the diagonals of a parallelogram in the Poincaré disk model always bisect each other. In a previous task, he had constructed a parallelogram and he used this diagram to Predict that the diagonals probably do not bisect each other based on the appearance of the figure on the screen. He then used the Constructions menu to Produce the diagonals of the parallelogram, and the point where the diagonals intersect. He measured the diagonals and stated that the diagonals do not bisect each other because the measures are not equal (See Figure 3a).

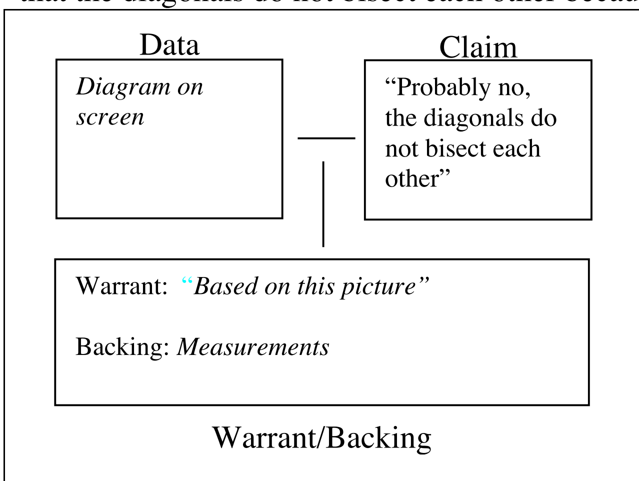

Figure 3a: Will's argument

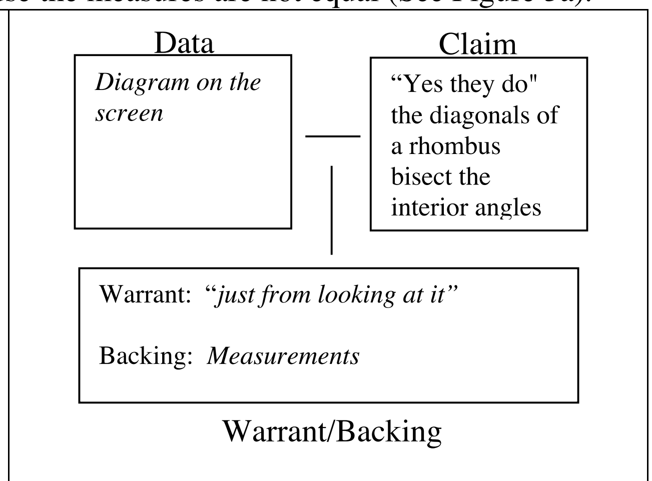

Figure 3b: Calvin's argument

In this example, the appearance of the diagram created using technology was used as a warrant that was backed up with measures while the student was responding to a sequence of tasks that involved predicting, producing, and describing.

As a second example, Calvin was determining whether the diagonals bisect the interior angles of a rhombus in the Poincaré disk model. In a previous task, Calvin had constructed a rhombus and its diagonals. With a focus on the appearance of the diagonals and rhombus, Calvin first predicted that the diagonals would bisect the interior angles. He stated that to make sure he was correct, he would produce some additional measurements. He found the measurements of these angles of interest and described their relationship by stating that the measurements were the same. Similar to Will, Calvin used the technology to produce a diagram and measurements. The appearance was used as a warrant, measurements were used as backing, and this took place in the sequence of tasks that involved Predicting, Producing and Describing.

The Absence of Technology Use as Part of the Warrant and Backing for Justification tasks

In the process of constructing arguments in which the students were engaged in Justification tasks, in general, the warrants and backing of their arguments did not involve the use of technology. However, at times, the students did make use of the constructed figure on the screen and previous measurements as their data. When the students did make use of the technology in their backing, specifically measurements, the students were involved in tasks that fall under the Product and Concept categories of the MATCI task framework. There was only one instance when a student used measures as backing when the student was engaged in a Reasoning task.

During the interview, Gail was provided a task to determine and justify whether a Saccheri quadrilateral, a quadrilateral with exactly two consecutive right angles and a pair of congruent opposite sides that share a ray of a right angle (See Figure 4), could also be a Lambert quadrilateral in Hyperbolic space.

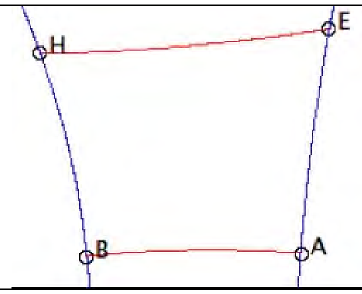

Figure 4. An example of a Saccheri quadrilateral.

In a previous task, she had constructed a Lambert quadrilateral on the computer screen and its appearance was used along with the definitions and properties of these two quadrilaterals, as data for her claim that a Saccheri quadrilateral cannot be a Lambert quadrilateral. She based this argument on the fact that if she moved the Lambert quadrilateral so that it had congruent sides, then the summit angles would both be 90, which would result in a quadrilateral with four 90 degree angles. She backed up her warrant using a property of quadrilaterals that sum of the angles of a quadrilateral in Hyperbolic Geometry must be less than 360 degrees, which she had learned previously in class (See Figure 5a). Gail imagined what would happen if opposite sides of a Lambert quadrilateral were congruent and used a property learned previously as backing.

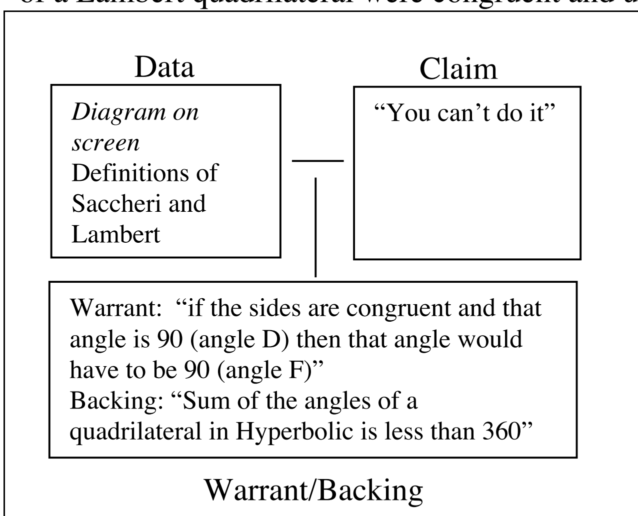

Figure 5a: Gail's argument

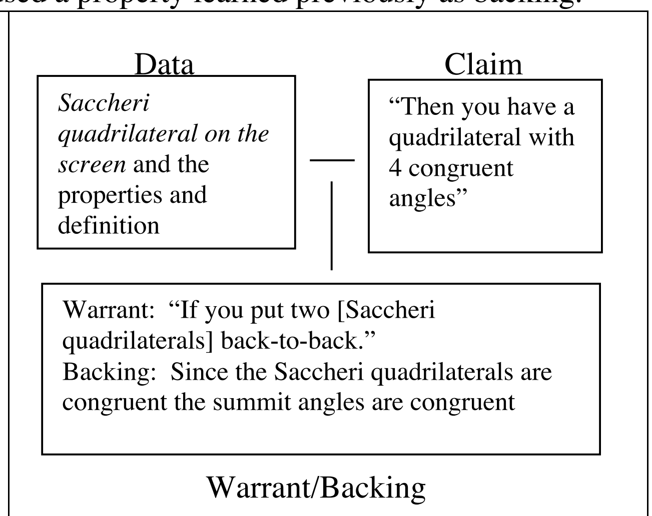

Figure 5b: Teresa's argument

In response to a task about whether an equiangular quadrilateral exists in Hyperbolic space, Teresa claimed that it was possible to construct. The data for her argument was a diagram of a Saccheri quadrilateral and its definition and associated properties, which she had noticed in a previous task. Her claim was based on the construction of a second congruent Saccheri quadrilateral that was produced by reflecting the original quadrilateral about its base, the side between the two right angles (See Figure 5b). Teresa had generated a procedure for constructing an equiangular quadrilateral using reflections. She justified her procedure by explaining that the summit angles of congruent Saccheri quadrilaterals will be congruent. In this example, Teresa uses the diagram on the computer screen as data for making a claim about a procedure that could be used to construct an equiangular quadrilateral.

Discussion

Several researchers have described ways in which students view technological evidence as proof (e.g., Chazan, 1993). None of the students in this study held that belief. However, when asked to solve problems that required arguments, students did use the technology in a variety of ways to create evidence that served as data for claims. Often students would be posed with a task from the Reasoning category and they would create subtasks within the Product or Concept category and respond to them by creating arguments where the technology was used as part of a warrant and/or backing. These subtasks were then used to return to the original Reasoning task, where the technology was used in the generation of data and definitions and properties were used as warrants and/or backings. This “breaking down” and “building up” of tasks may have been facilitated by the technology that makes it easy to quickly generate diagrams, and measures linked to those diagrams. By dragging students can generate data from which they can observe what changes and what remains the same, that often led to Generalizations and Justifications. However, when dragging for the purpose of generalizing or justifying, students need to know what to attend to and what to ignore and how to relate what they are observing to what they already know about the particular figure with which they are working.

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INCORPORATING DIGITAL TECHNOLOGIES TO THE MATHEMATICS CLASSROOM: IN-SERVICE TEACHERS REFLECT ON THE CHANGES IN THEIR PRACTICE

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In this paper, we report data from six primary and middle school in-service teachers in Mexico, who have been reflecting on the changes in their practice, and that of their colleagues, when incorporating digital technologies (DT) to their mathematics classrooms. The changes documented take into account different aspects: the teacher, didactical materials, classroom interactions, students' learning and motivation, technical aspects and the broader school community context.

Digital Technologies in Mexican Schools

The inclusion of new digital technologies in classrooms is inevitable, and teachers need to adapt to the changes brought about by these technologies, as well as harness their potential; but this is not straightforward. In our paper we will present data from case-studies of six Mexican teachers who have been reflecting on their changing practice, and the changes in the classroom and school culture, when incorporating digital technologies (DT).

Since 1997, the Mexican Ministry of Education has been making intense continuing efforts for incorporating digital technologies into Mathematics classrooms of the basic education system (primary and middle school levels). The largest projects in this effort are known as the Teaching Mathematics with Technology (EMAT) Program for middle schools, and Enciclomedia for primary schools. The EMAT program provides activities and a pedagogical model for incorporating the use of technological tools in mathematics classrooms, in a constructionist way, aimed to enrich the teaching and improve learning (Ursini & Rojano, 2000). The pedagogical model emphasizes changes in the classroom structure such as the requirement of a different teaching approach and the way the classroom needs to be set up. In particular, the pedagogical model emphasizes a collaborative model of learning, and a role of the teacher's as guide, mediator and promoter of the exchange of ideas and collective discussion. The main tools currently used in EMAT are Spreadsheets, Dynamic Geometry, Logo and CAS activities with the TI-92, and. On the other hand, Enciclomedia, which has been massively implemented in all primary schools in Mexico, aims to help teachers by providing resources, computer interactive activities and strategies (mainly designed to be used on electronic whiteboards), through links in an enhanced electronic version of the mandatory textbooks (Lozano et al., 2006).

Theoretical Framework

We consider digital technologies (DT) catalysts for change, since they have a great potential for revolutionizing school practices. In fact, there have been conflicts reported between the autonomy that students gain from the use of DT, versus the restrictions of conservative curricula (Facer *et al.*, 2000). This is one of the "forced" changes that educators need to face.

Mexican schools tend to be extremely traditional; one of the aims of EMAT was to use the DT as a means to transform the school culture. But, even though a decade has passed since the initial EMAT efforts, the benefits of the inclusion of DT in schools are not yet perceived and the difficulties in changing a school culture (including teachers' difficulties in adapting to the proposed pedagogical model) are more evident than ever (Sacristán et al., 2006); it is clear that the role of the teacher is critical for a successful implementation of DT in the classroom and for fostering meaningful learning in students.

Research results (e.g. Ertmer, 1999; Goldenberg, 2000) indicate that DT can help students learn in a more significant way, only through an *adequate use*; they also indicate that teachers with little experience in the use of DT, have difficulties in harnessing the power of these technologies as *tools for learning,* having as consequence a lack of significant influence of the DT in the school culture (McFarlane, 2001). Teachers not only need to be trained in the use of the new technologies but also need to understand how to use these tools, and change their practice in order to promote significant learning in their students. However, changing teaching practices is not straightforward: teachers can face struggles when attempting to modify their practices (e.g. Wilson & Goldenberg, 1998).

In order to provide strategies to help in the educational transformation and to overcome what Ertmer (1999) calls "second-order barriers", more understanding is needed of all the variables, difficulties and changes that teachers' face when attempting to incorporate new technologies into their practice. And what we believe can provide further insights is to document those changes from the perspective of the teachers themselves (as opposed to exclusively that of the researchers). We therefore want teachers to reflect on their own practice. That is the purpose of our study.

A Three-Year Development and Research Project with In-Service Teachers

We are involved in a development and research project that is linked to a three-year master's degree program in education. The project seeks to: a) train teachers in the use of DT tools (specifically those linked with the EMAT and Enciclomedia projects) for mathematics teaching and b) research and understand how changes in school practice and culture are brought about by the implementation of those tools into the classroom. More specifically, the project seeks to document the methods, techniques, resources and strategies that teachers use and develop when incorporating new technologies into their practice, and understand the influence and impact of, and on, the broader school community (since we consider teachers part of a symbiotic system that is the school community).

Methodology

In this project we have been working (to date, for almost two years) with six in-service mathematics teachers, all of them students of the aforementioned master's program. Their ages are between 35 and 45 yrs. old, with a minimum of eight years of in-service practice. Two of these teachers (Sarah and Jane), both female, are primary school teachers in fifth- and sixth-grade (children 10-12 yrs. old); the other four (Mary, Michael, Adrian, and Raymond) –one female, the other three male— are teachers of the three grades of the Mexican middle school (children 12-15). Four of these participants (Sarah, Jane, Adrian and Raymond) are also involved in training programs for their colleagues. Although all of them had taken some workshops on the use, for mathematics teaching, of some technological tools (such as the Enciclomedia or EMAT tools), the use they had carried out of these tools in their practice, before being part of the project, was very limited (if any).

During our development and research project, the teachers have been involved in three, almost simultaneous, activities: a) in the training, and development of abilities, for the use of digital technologies (DT) in the classroom; b) the design and planning of teaching strategies and activities that integrate DT; and c) engaging in observation and reflection-on-action (Bjuland, 2004) of the changes in their own teaching practice with the new tools. In addition, the four teachers who are involved in training programs for other teachers, have also engaged in observation, or training-and-observation, of some of their colleagues (not involved in the development project) when incorporating DTs. As part of the master's degree courses, the participants have also analyzed and discussed research papers and results.

As a theoretical and methodological framework for integrating these attempts of educational innovation with teacher learning research we follow the work of García et al. (2006), as well as that of Artzt & Armour-Thomas (1999) which provides a model for enabling teachers to reflect on their instructional practice.

The participants have been analyzing and reflecting upon the potentials, limitations and changes brought forth by the incorporation of DT into their practice, and that of their colleagues, from various perspectives:

(a) *The perspective of the teacher and the didactical use of DT:*
- changes in the role of the teacher (e.g. changes of the teacher as lecturer, to that of mediator) and the difficulties in those changes;
- changes in teaching methodologies;
- changes in their beliefs and conceptions;
- use and design of activities with DT;
- articulation of the DT activities with the curricular requirements;
- design of assessment techniques for DT activities;
- complementarity's of different DT tools among themselves, and with non-DT activities (such as those with paper and pencil) ;
- new mathematical knowledge and perspectives through DT.

(b) *The perspective of the classroom interactions:*
- changes in classroom structure;
- changes in teacher-student relationships;
- changes in student-student relationships (collaborative work);
- changes in the physical setup of the classroom, etc.

(c) *The possible impact on students:*
- in their learning;
- in their motivation (affect),
- beliefs, and
- classroom participation.

(d) *The technical perspective:*
- technical knowledge for the use of the DT tools and equipment;
- technical difficulties.

(e) *The social context:*
- changes in the school community;
- the role of school authorities;
- impact and collaboration with colleagues;
- the interaction with parents.

The participants have been collecting data by using video recording of their –and their colleagues'–classrooms; taking field notes when possible; designing questionnaires for colleagues, authorities and/or students; collecting their activities and assessment designs and sometimes students' work, and most importantly, writing weekly reports. In addition to that, we designed an initial questionnaire for evaluating the participants' conceptions on the use of DT and we have held bi-weekly meetings with the participants where they present oral reports, are informally interviewed, and engage in reflections and discussions with the other participants. Finally, a year after the beginning of the project, "independent" researchers (not involved in the development project) have carried out some observations of the participants during their practice. In this way we are able to carry out case studies of each of the participants, combining data from both their own reflections and reports, and from researchers observations and interviews. Below, we present some of the findings.

Some Interesting Results and Discussion

Initial Conceptions of the Participants on the Use of Digital Technologies

(Data from the initial questionnaire and from the participants' work during the first trimester).

The main beliefs that the participants had on the use of the DT was that they are a useful tool for teaching because DT facilitate the construction of graphical representations ("*it is easier to create graphs*"); and that they can save time for some activities in comparison those with paper and pencil. In the questionnaires, none of them mentioned any disadvantages nor seemed aware of the different knowledge or practices that DT can bring. However, many expressed a lack of confidence and concern in the pedagogical and technical knowledge that DT demand from the teacher, and some of them felt uncomfortable in using some of the tools with students; and one of the participants did not believe that the DT tools could help create significant learning in students.

Results of the Participants Self-Observations and from Observations of Their Colleagues

Since the beginning of the project, the participant teachers' beliefs, attitudes and practices have been in constant and profound transformation, and enriched by their long-term exposure and use of the DT in their classrooms, by their sharing of experiences and by their observations of other colleagues.

General Changes in Teachers' Practices and Beliefs

In terms of *confidence with the use of technology*, despite their initial reluctance, in retrospective most participants have realized that the only way they could build confidence was by gaining experience through the direct use of the DT in their practice. One of the participants expressed that although adequate training and continuous support helped her to change her practice, ultimately it is up to the teacher to make the changes.

In terms of *how their practices, and the classroom dynamics, have changed*, most of them recognize that with DT, they need to change the way they teach. Most participants have expressed how they have now taken a back-seat role in their classrooms, becoming more of observers and guides than lecturers. Mary and Michael, in particular, continue to emphasize this at every meeting. As Michael expressed it recently, they have realized more and more, and come to accept, that students can learn on their own, or from collaborative work, with the support of DT, and that they request less and less assistance from the teacher; Michael adds that he has learned to be patient and "open to what he can learn from his students". On the other hand, Janet, Sarah and Adrian have noted, that although their colleagues –which they observe— make attempts to allow students to collaborate and to have group discussions, these attempts are short-lived and they quickly revert to traditional forms of teaching as they feel uncomfortable letting students talk or hearing them laugh. The participants feel that it takes time for teachers to get used to this change in the classroom structure.

In fact, as the participants reflect more on their practice, they have become increasingly aware of the resistance to change and difficulties of their colleagues (as partly illustrated above). Janet and Sarah remark that although many of their colleagues use DT in their personal lives, most do not consider these tools, including calculators and spreadsheets, as adequate for didactic use. In fact, they have noted that many fellow primary school teachers only use the Enciclomedia tools for projecting the digitalized textbook, and not for exploratory mathematical activities. Also, in general, Sarah, Janet and Adrian have observed that many of their colleagues attempt to teach with DT in the same way as they did before (i.e. without DT) such as resorting to teaching algorithmic memorization, and are unable to recognize their own mistakes or the value of making mistakes. These participants have even expressed their frustrations in being unable to help their colleagues change or realize the potentials of DT. Interestingly, Raymond, who was initially unenthusiastic about the use of DT, has now made it an aim, to motivate fellow teachers for the use of DT in their mathematics teaching.

Most of the project's participants have become aware that the use of technology "is a tool and not an end" (Mary's words). Janet feels that some of the teachers she has observed, are not successful in the implementation of DT into their practices, because they are not clear why they are using it (i.e. they do not have a broader educational goal or plan) and just give students some DT activity for the sake of using the technology.

Awareness of Teacher's own Mathematical Knowledge Limitations and Changes in the Conception of the Type of Mathematical Learning Developed

Another aspect is that the DT use have made the participants, and their colleagues, aware of their own limitations in their mathematical knowledge. Sarah and Janet have noticed that often the teachers they have observed, refrain from certain DT activities, because they lack mathematical confidence and feel their deficiencies can be exposed by the DT activities.

But, in general, all the participants have realized that the knowledge developed with the DT can be of a more conceptual nature. Mary explains that she has learned to look more at what abilities her students are developing rather than looking for correct answers to problems.

DT Tool Use and Design of DT Activities

At the beginning of the project, all of the participants began with the use of pre-designed DT activities (either from EMAT or from Enciclomedia). Only recently, Mary and Michael began designing their own activities. In one of his first activities (word problem explorations with spreadsheets), Michael made some mistakes in the values of a problem: this led to unforeseen investigations on the part of the students; although these were interesting, the experience also made Michael realize the design difficulties and the importance of being more careful in trying out the activities himself. before giving them to students.

Another aspect of the use of the DT tools, has been the *complementarity between tools.* Most participants began considering that using a single tool was enough to provide students with a rich opportunity for explorations with technology. Only Mary thought otherwise, and early on she began analyzing which tools were suitable for teaching which subjects, and how different tools could complement each other. Only recently, other participants, such as Adrian, have begun to realize that a diversity of approaches and tools can be useful.

Development of Assessment Techniques for the DT Activities

A concern that has become more and more prominent as we end the second year of the project, has to do with the *assessment* of the DT activities and of students' work with DT. This has been one of Mary's main interests: she has realized that traditional assessment doesn't adequately evaluate the learning that takes place when using DT; she has asked herself how to assess students' work with DT and has realized that she is more observant now of students work with DT than at the beginning of the project; she is also investigating how to design problems in a test, which, in order to solve them, would require the use of DT tools.

Other participants are beginning to be concerned about assessment as well, not only of students' work but also assessment of the activities and strategies used. Adrian recently became concerned, and wrote, that there is a need to change the broader school assessment culture in authorities and parents (both of which should be aware that assessment with DT is different to traditional evaluations), teachers (who have to develop assessment strategies) and students (in particular, he feels that students need to see DT activities as more than just play).

Michael has also changed the way he assesses his students and his own work when using DT; in the beginning he looked for correct responses to problems; later he said: "after learning [the use of] the computer, now I assess not only the answers but also the abilities, the questions that my students pose; [I also evaluate] my new teaching strategies”. We are not altogether clear exactly how he can assess students’ abilities, but he does take into account the participation and collaborative work of his students.

Implementation Difficulties.

The difficulties are of two types: (i) pedagogical and of classroom management, and (ii) technical and administrative. In the first category, some participants, such as Raymond, found it initially difficult to cope with their students progressing at different paces and the need for the teacher to do more individual guidance work, which can be difficult due to time limitations. Another common concern is the need to properly prepare their classes in order to harness the potential of the DT tools; but also in this case, time limitations can make it difficult. With respect to this point, both Michael and Mary have now developed a work methodology that gives much more importance to class preparation involving DT activities.

In the second category, all the participants have expressed frustration with a multitude of technical and administrative issues. One difficulty is often the lack of school support. The teachers frequently have difficulties in gaining access to the technology-equipped rooms, and there are many technical problems with the equipment such as too few computers, lack of maintenance or restricted access that prevents children from saving their work. All the participants also express frustration with the amount of administrative cancellations of many of their programmed technology-based sessions.

Another difficulty relates to training. The participants observe that there is too little training available in schools, and often the trainers themselves are not proficient enough in what they teach, due in part to a cascading model of training. They also comment that other teachers, particularly the primary school ones, want more training in the use of the equipment (computers, beamers and electronic whiteboards) because they are afraid to damage it.

Observed Changes in Students’ Behavior and Learning

In terms of the impact on students’ behavior, attitudes and learning, all of the participants note an *increase in motivation and interest* in their students when using DT (Mary wrote that her students consider the work with DT, a game). The exception to this has been when they have observed other teachers “presenting” DT activities without allowing students to actively engage, collaborate, explore and/or discuss amongst themselves. In one of Michael’s first experiences, he also failed to engage students and felt that the classroom atmosphere was one of total apathy; but now he has learned to let students collaborate and explore and he feels that sometimes the classes run almost by themselves; he has also been surprised recently that his students are beginning to use the DT tools (such as spreadsheets) for solving problems in other school subjects (such as in carpentry workshops).

In terms of *mathematical learning*, both Mary (middle school) and Sarah (primary school) comment that their students have improved in traditional departmental mathematics tests since using the DT tools. However, although Mary has noticed a general improvement in conceptual understanding, she had also noted the emergence of new difficulties in the formal math performance of some of her students. This has made her realize the epistemological difference of DT-based learning versus traditional learning and now views them as complementary. Adrian also noted that although students can solve the work (e.g. using spreadsheets) they have difficulties explaining their results. Michael also noticed this and it led him to ask his students to explain all their work with DT in written form; since then, he has noted that students increasingly write and describe their work better.

Concluding Remarks

What we have observed, is that through the training and "forced" involvement (in the masters program) with the DT tools, these teachers' perceptions of their use has been enriched. Second, the classroom experiences of implementing DT has led them to reflect on the potentialities and limitations of the tools. More importantly, the opportunity to reflect upon, share their personal experiences with the other participants, and observe fellow teachers, has led them to develop a critical and reflective attitude, as well as enabled them to construct didactic strategies for the use of DT that are in accordance with the specific needs of their students. However, even these teachers have experienced difficulties in changing their practices; we keep in mind the words of Goldenberg (2000, p.8): "Provide instruction and *time* for teachers to become creative users of the technology they have." We will continue the follow-up of these teachers and the analysis of their changes which may provide insights for strategies for more successful integration of DT in the classroom and in the curriculum.

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A PRINCIPAL COMPONENTS ANALYSIS OF RATE AND PROPORTIONALITY USING SIMCALC MATHWORLDS

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Using data from an experimental research project on the effects of dynamic software on student outcomes in 7th-grade classrooms, we examine changes in student knowledge through a pretest/posttest assessment. With principal components analysis, we found that students who used the software cluster concepts differently than students who did not, grouping together test questions that combine multiple representations of rate and proportionality.

Over 12 years ago, the SimCalc MathWorlds software program was conceived to serve a need for innovative curricula that would prepare students to understand the concepts of change, a necessary precursor for calculus understanding (Roschelle, Tatar, & Kaput, in press). The data used in our study comes from a larger, 4-year research project on scaling up the use of MathWorlds to teach rate and proportionality in 7th-grade classroom involving across a large, southern state. Teachers in the treatment group taught a replacement unit, measuring student learning through a 30-item pre/posttest. Meanwhile, teachers in the control group taught rate and proportionality using their usual curriculum materials, with a total of 117 teachers in all. The data in our analysis comes from the period just after the teachers taught the 3-week rate and proportionality unit for the first time to their 1621 students.

The replacement unit creates multiple opportunities for students to engage in learning concepts of rate and proportionality, democratizing access for all learners (Roschelle, Tatar, & Kaput, in press). Specifically designed for students to interact highly with the MathWorlds software, thereby creating personal dynamic representations of rate and change, the replacement unit utilizes the principle that dynamic representation through technology leads to large gains in conceptual understanding (Kaput, 1992). The core of the replacement unit involves high interactions with the MathWorlds software, in which students learn to analyze, augment, build, and interact with various rate graphs. Students used a workbook to guide them through activities involving connecting graphs with real-world situations and creating piece-wise linear graphs based upon varying rates of speed.

This main focus of this study explores what student learning looks like within a framework of democratized access to technology; showing how opportunities to learn cluster in ways that align with amount of dynamic interactions. We use Principal Components Analysis because it is a statistical analysis technique that clumps test questions together into component groups based on student response (Meyers, Gamst, & Guarino, 2006). Using this analysis on our pre/posttest results, we found significant components in which students using the MathWorlds unit clustered conceptual understanding differently than students who had not been involved with the replacement unit. To analyze the results, the treatment group components were analyzed for changes in student learning that occurred during the MathWorlds unit, and those components juxtaposed with the components of the control group.

For their primary component group (Table 1), the control students grouped together items that dealt with Piece-Wise Linear Graphs, before and after the treatment, even if the underlying concepts behind the questions differed. This is very different than the primary component group for the students who used SimCalc MathWorlds (Table 2). They stopped grouping all Piece-Wise Linear Graphs questions together, instead combining questions with such varied surface features as Slope and Speed, Linear Functions, Graphs, Tables, and Piece-Wise Linear Graphs into one conceptual strand. These students seemed to understand that these separate representations of a linear function were related to each other on a deeper level. The modeling and interactive aspects of MathWorlds help students fuse together multiple representations of the same concept. These results support the work of Bodemer, Ploetzener, Bruchmuller, and Hacker, who showed that interactive media supports integration of multiple representations (2005).

The Scaling Up research project creates notable differences in the way students cluster concepts together, helping link conceptual knowledge beyond surface features. Students participating in the SimCalc MathWorlds replacement unit show a statistically significant difference in their understanding of rate and proportionality compared to the students who did not. The results shown here only add to the numerous data that can be gleaned from this project, proving that Jim Kaput's vision of democratizing access to the mathematics of change and variation learning is creating a powerful and scalable revolution in learning.

Table 1. Control Group Differences

Pretest Component	*Posttest Component*
Slope/Speed	Piece-wise graphs
Piece-wise graphs	Piece-wise graphs
Piece-wise graphs	Piece-wise graphs
Piece-wise graphs	Piece-wise graphs
Piece-wise graphs	Piece-wise graphs
Piece-wise graphs	Piece-wise graphs
Piece-wise graphs	

Table 2. Treatment Group Differences

Pretest Component	*Posttest Component*
Piece-wise graphs	Slope/Speed
Piece-wise graphs	Slope/Speed
Piece-wise graphs	Linear Functions
Piece-wise graphs	Tables
Piece-wise graphs	Linear Functions
Piece-wise graphs	Graphs
Piece-wise graphs	Graphs
	Piece-wise graphs

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COMPUTERS AS A MEAN TO IMPROVE TEACHERS' BELIEFS AND THEIR MODE OF INSTRUCTION IN THE CLASSROOM

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Research has the objective to find out the changes that are propitiated in the teaching of math by the use of a computer and a projector inside the classrooms of elementary schools. The study showed that, with the support of the computer program, the teachers developed their modes of instruction and their interaction skills, although this improvement was very variable.

In this paper we describe a combined research and educational project that has as its main purpose to find out the effect of the use of computers on the mode of instruction and interaction of teachers. For this, a didactical approach was proposed, using a single computer and a projector inside the math classrooms of elementary schools in Mexico.

We designed 120 interactive activities with the programming language Java. Their classification according to topics is: 1) Elements, 2) Additive problems, 3) Decimal system, 4) Mental calculation and estimation, 5) Multiplicative problems, 6) Geometry, 7) Fractions and 8) Probability.

Two problems with teaching with computers are that it is often done as individualized learning and that the students and teachers have difficulties getting acquainted with the software. In our project, the work was done with the whole group of students to enrich the learning process and, by contrast, the activities developed for this project contain each, a very specific content and situation, so teachers and students can use them almost intuitively.

Additionally, the teaching practices in elementary schools tend to give a lot of emphasis on the procedural, mechanical aspect of math and the teacher's practice consists mainly of explanations on a blackboard. Our proposal tries to change this also, centering the learning process on the students, their conceptual development and their thinking.

Theoretical Framework

To study the role of the teacher and his changes, we based our analysis on two similar frameworks. In an article about cognitively guided instruction, Carpenter et al (2000) stressed the importance of the teacher's knowledge about the mathematical thinking of children. The authors identified four levels of teachers' beliefs that correlate with their mode of instruction. A brief explanation of these is: I. They believe that math has to be taught explicitly and therefore they show procedures and ask the students to practice them. II. They start to question this explicit mode and therefore they give to the students some opportunity to solve problems by themselves. III. They believe that students can have their own strategies so they provide problems and the students report their solutions. IV. The previous mode of working becomes more flexible, in which the teacher learns from his students' productions and adapts his instruction to this knowledge.

In another study by Jacobs and Ambrose (2003) about how interviews applied by teachers to their students can improve their instruction by developing their questioning skills, the authors proposed a classification of the different modes of teachers' interaction during the interview. These define a profile of the teacher. The four proposed categories are: i) *Directive*. Intervention is active but there is too much control and assistance. Questions are leading, with the intention of teaching. ii) *Observational*. Behavior is passive, restricted to observe and to give verbal checkmarks like "fine", "good", with no follow-up questions. iii) *Explorative*. Behavior is active but questions are non-specific (without getting to the essence) and mostly restricted to incorrect responses. iv) *Responsive*. The teacher discovers the students' thinking and uses it to inquire competently and to create extensions of their ideas. Another similar study in this line of research was done by Moyer and Milewicz (2002). In fact, the above descriptions of the categories are a blend of Jacobs and Ambrose and Moyer and Milewicz frameworks.

Methodology

Three research assistants (teachers studying a master's degree in education) and three in-service teachers of a middle class elementary school participated in the didactical experiment (the three research assistants talked with the teachers and explained to them the project, the software package and showed them the activities). By pairs (one research assistant and one teacher), they selected a grade level to work in, one of the eight topics contained in the software package and a set of ten activities corresponding to that topic (the first pair worked on decimal system in second grade; the second on geometry in third grade; and the third on fractions in fourth grade). There were two stages in the research. First, each of the three research assistants worked in each of the teachers' classrooms with the ten activities and the corresponding teacher observed. Then, in the following school year, they exchange places, the teachers used the activities in their classrooms and the research assistants observed them. A fourth research assistant applied questionnaires, performed interviews and observed some of the lessons of each of the six participants to determine their mode of instruction and their interaction level with the students.

In each of the six experimental groups, the study followed the following steps: An initial questionnaire and interview with each of the six participant teachers to find out their beliefs about using computers in the classroom. A didactical experiment within the classrooms, working with the ten activities selected (with observations of some of these lessons). A final interview with each of the six teachers to find out the changes in their beliefs. All these interviews and classroom sessions were audio taped.

Results

The initial questionnaire revealed that only one teacher (will call her T1) didn't have any experience with computers (she expressed "I am afraid of computers") and that they all believe that computers are useful in teaching and had a good disposition of using computers for this purpose (although they had very little experience with this). In the final interview, they were more explicit and specific about this use: "This package showed me that computers facilitate the work of the teacher… promote mental calculation… and improve comprehension." "I believe I improved my form of teaching geometry and the programs are easy to use." "Now I can say that computers improved my teaching style because before, I was just assuming… I discover that the students think more." On another question about the factors that are more important in teaching with computers, in the initial questionnaire all claimed that "the activities are the most important". However, after the working sessions they all admitted that the teacher is an equally important element.

During the didactical experiments each teacher developed differently. The next table shows the advancement of each one as observed, according to the four levels of Carpenter and the four modes of interaction of Jacobs and Ambrose described before.

It is important to point out that all the teachers worked only eight sessions with the students. We can observe in this table that all the teachers were of the directive type (i) of interaction mode when they started the experiment and that all of them showed changes. Four of them (RA1, RA2, T2 and T3) moved from level I to level II in Carpenter's classification, but two stayed being directive (i) and the other two move into more observational (ii) or explorative (iii) mode. The other teachers (T1 and RA3) started already at level II of Carpenter, and had a much more noticeable improvement during the teaching sessions in both scales. It is worth noticing that the teacher (T1) was the one that didn't have any knowledge of computers at the beginning.

Teacher:	Carpenter's level (start)	Interaction mode (start)	Carpenter's level (end)	Interaction mode (end)
RA1	I	i	II	ii
T1	II	i	III	iii
RA2	I	i	II	ii and iii
T2	I	i	II	i
RA3	II	i	IV	iv
T3	I	i	II	i and iii

Conclusions

The results of this study suggest that the teachers, due to the software and its positive features, got motivated to change their usual way of teaching into a more appropriate method for this visual-dynamic situation. The software provides a favorable environment where the students can manifest their strategies to give sense to the notions in play, provided that the teacher opens up this possibility. According to the opinions of the students, the "screen" helped them to experiment, reach a solution and verify the answer.

In general, in the directive mode (i), the computer-software was used as a practice tool or to verify answers. The lesson had a repetitive structure of "Teacher's specific question (TSQ), student's answer (SA), TSQ, SA…" with the teacher always looking for the right answers. The students showed a lot of interest at the beginning of the sessions and wanted to participate, but because of this restrictive way of working of the teacher, the students eventually showed some restlessness and lost interest. The "Explorative" teacher used the activities to introduce and develop concepts, which corresponds with the two higher levels of Carpenter's framework.

We believe that this interactive software can support teachers to improve their mode of instruction and develop communication and expression skills of both, students and teachers.

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HIGH SCHOOL MATHEMATICS TEACHERS' USE OF MULTIPLE REPRESENTATIONS WHEN TEACHING FUNCTIONS IN GRAPHING CALCULATOR ENVIRONMENTS

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Purpose of the Study

The concept of function has been widely recognized as being foundational to school mathematics and mathematics in general (Romberg, Carpenter & Fennema, 1993). Research has shown that graphing calculators can improve students' conceptual understanding of functions by allowing the students to explore various representations of a function (Penglase & Arnold, 1996). The National Council of Teachers of Mathematics (NCTM, 1989, 2000) advocate a curriculum based on multiple representations, arguing that by encouraging students to incorporate different types of representations into their sense-making, the students will become more capable of solving mathematical problems and understanding underlying concepts. In this paper, I explore how high school mathematics teachers use multiple representations when teaching functions in graphing calculator environments. I pay special interest to how teachers plan their lessons to accommodate multiple representations of functions when teaching with graphing calculators and how the calculators in turn influence the teachers' approaches to teaching functions. I also seek to explore the effect of the teaching strategies and instructional tasks on the ways in which graphing calculators are used.

Perspectives and Guiding Frameworks

This study draws on a theoretical framework developed by Salomon, Perkins, and Globerson (1991) for studying the interaction between technology and the user. In this framework, Salomon et al. distinguish between two sets of *principal effects* that arise when one works in partnership with a technology tool, namely (1) *principal effects with* the technology and (2) *principal effects of* the technology. For purposes of clarity, I will refer to the first set as *planned effects,* and the second set as *emergent effects.* The work of Goos, Galbraith, Renshaw, and Geiger (2003), which provides metaphors for studying the interaction between calculator and user, is closely related to this partnership framework and so I will draw parallels to the metaphors when discussing some of the principal effects.

Characteristics of *planned effects* include elaborate planning (laying out the specifics concerning how the calculator will be used), executing the plan (using the calculator in the desired ways), and interpreting the results. The user (teacher) here predetermines exactly when it will be appropriate to turn to the calculator in the course of a lesson and in what ways this should be done. *Emergent principal effects* on the other hand are characterized by spontaneity, that is, effects that the user (teacher) does not intentionally plan for. These effects are then retained and may be applied to other related but not calculator dependent mathematical activities (Jones, 1993). Using the metaphor of technology as partner, Goos et al. (2003) describe "cognitive re-organization effects" (p. 79) as those characterized by using technology to explore new tasks or new approaches to existing tasks and to mediate mathematical discussion in the classroom between students and teacher or between small groups of students. I contend that for meaningful principal effects of technology to arise in a classroom, the teacher must be willing to allow his or her students to explore new situations with the calculators and guide the students into discussions that will help them make sense of their findings. In this study I investigated how principal effects that are planned for and those that emerge are manifested in secondary mathematics classrooms where graphing calculators are used.

Methods, Data Sources, and Analysis

Participants in this study were four high school mathematics teachers drawn from high schools in a medium-sized city school district in northeastern United States. Data were collected through semi-structured task-based interviews (Goldin, 1999), and classroom observations. The interview questions were divided into four major categories, namely (1) planning (what are the key things that teachers consider as they prepare to teach lessons on functions especially when they intend to use graphing calculators?), (2) sources of teaching tasks (where do teachers get their teaching activities/tasks and how do they use these tasks, i. e. do they modify them or not and what are the reasons for this?), (3) function representations (teachers presented with various tasks and asked to respond to the tasks as well as speculate on how their students might respond to those tasks), and (4) issues related to calculator usage.

Categories (1) and (2) helped me develop some insights into how teachers envision a lesson on functions in which graphing calculators are used and what outcomes they might expect, thus shedding some light on the planned principal effects. Categories (3) and (4) helped shed some light on the teachers' choices of representation in various situations, the kind of partnerships these teachers had developed with graphing calculators, and the kind of expectations the teachers held for their students when using graphing calculators. This was important to this study since the tasks provided a common ground for all the four teachers given that no two teachers taught the same lesson.

During classroom observations I took note of both the teachers' and the students' interactions with graphing calculators, paying special attention to how the teachers facilitated the interaction between students and calculators. In this regard, I examined the kind of instructions the teachers gave to their students, the actions the students took and the questions they asked their teachers as well as their peers, and how the teachers responded to the students' questions. All these helped provide data that would later be analyzed for emergent effects of technology.

Data were analyzed in two phases. In phase I, I carried out a microanalysis of the interview data for all the teachers, identifying broad theme statements from the interviews based on dominant phrases in the teachers' responses to items under the categories of planning and sources of tasks and also on the actions they took while attending to items under the categories of function representations and issues related to calculator usage. In phase II, I analyzed the data from classroom observations against the statements generated above. I tried to identify situations from the classes that could support these statements (or sometimes challenge them). I then refined the statements into three major themes, namely (a) teaching strategies, (b) types of instructional tasks, and (c) representational forms that emerge.

Results

While classroom organization varied from teacher to teacher, all teachers seemed to value involving students in decision making regarding calculator use. This would range from asking students to suggest what to do in order to get started with the calculator with respect to given information, to asking students to suggest how to modify various calculator menus in order to achieve various desired results. Often times the teachers encouraged students to share their work with the whole class using the calculator overhead projection unit. It was also common for teachers to ask probing as well as clarification questions. Occasionally the teachers would ask questions requiring students to compare solutions obtained using different representations and explain the differences if any. The teachers would also challenge students to interpret calculator results in the context of the problem situation and communicate their understanding of the calculator results to their peers.

Although in most cases the teachers seemed to balance among the various representations, equations and graphs seemed to dominate more than tables. Most instructional tasks made specific reference to either an equation for which a graph would be drawn and various explorations done on it, or a graph on which various explorations would be done. Only a few tasks specified use of tables. In cases involving word problems, it was common to see equations being generated then graphs drawn.

Analysis indicated that the choices for instructional tasks and teaching strategies are not unique to particular teachers; what seems to be unique however, is the pattern of representation forms that the teachers use. While some teachers will prefer to move from equation to graph and possibly to tables, others prefer going from equation to tables then graphs.

Discussion

The first step towards developing intelligent partnerships with technologies is for the user to be able to plan on how to use the tool, execute the plan, and interpret the results. Results of this study indicate that teachers can help their students towards this end by guiding them to actively participate in the process of working with calculators either in small groups or as individuals. The teachers in this study had plans on how they wanted their students to use the calculators in the classroom, but they often times gave the students a chance to suggest their own approaches first.

The second step towards forming intelligent partnerships with technologies is for the user to gain new insights that can be transferred to other situations where the technology tool is not necessarily used. The teachers in this study tried to help their students towards this end by requiring them to interpret their solutions to real life situations and also to explain their answers to their peers. This would ensure that the student develop some kind of ownership to the knowledge they were acquiring and hence be in a better position to retain it beyond the classroom.

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MIDDLE GRADERS' EMERGENT STRATEGIES USING ELECTRONIC MATHEMATICAL BOARD GAMES

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Here we present the results of an exploratory study into the functionality of a digital board game called Domino, in the learning of mathematics at middle school. In particular, we have some results from possible mathematical connections that students establish with other notions or processes of Math curriculum arising from playful experience. Although the intended mathematical structure of the game is symmetry, the actual structure defines a potential organization that the children concretize in different ways once they are engaged in the task for winning strategies.

To accomplish the exploratory study presented here, the digital board game *Domino* was designed (Raggi, 2006). Symmetry is the underlying mathematical structure for this game. When playing, a winning strategy is to place your dominos symmetrically opposite the opponent's placements. This computer game was introduced into the classroom as an exploratory material to advance knowledge of how use of this type of concrete materials mediates the development of mathematical activity among middle students. In this case, each of the students had to initially play against the computer, called here *Robi*. The task asked of them was to find a way to beat *Robi* or, if *Robi* won, to try to explain why *Robi* was able to beat them.

Theoretical Underpinnings and Methodology

Although the potential of digital games as rich learning tools is widely recognized (Sanford, 2006), the great improvements at schools have not yet materialized (Wijekumar et al., 2005). According to Wijekumar et al. (2005), it is still necessary to work moving students from a game affordance of a computer to a learning mode.

However among the initiatives of using games to try to encourage students to learn specific topics, are those that try to foster visual reasoning and self-engaging tasks designing a game construction kit (Kahn et al. 2006), or to explore the affordances of the utilization of mathematical electronic board games (Rodriguez, 2007), even to promote general action patterns for solving math or science problems.

Applying Saxe and Bermudez's (1996) theoretical constructs, Rodríguez (2007) analyzed children's mathematical activity in relation with the affordances of the *Dominó* computer game. Saxe and Bermudez (1996) pointed out that, even though there is an intended structure in a board game (in their investigation they used the specially designed *treasure hunt*,) another actual structure emerges as children play. In fact, artifacts and conventions used, reformulation of rules, and also children's prior understanding will give form to an alternate structure, which could partially respect the previously intended, to establish new emergent mathematical environments.

According to Saxe and Bermudez (1996), a mathematical environment is built by the constructions the child performs in interaction with the tools or materials available. These constructions must be understood or analyzed in that same environment, without overlooking the child's previous knowledge, so that we may understand how that child comprehends the new knowledge or how a new conception of a mathematical object occurs.

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Stateline (Lake Tahoe), NV: University of Nevada, Reno.

Concerning the *Domino* computer game, it was introduced in two middle school Math classes, each with approximately 25 students. In fact the game proved to serve as a tool to diagnose the state of the knowledge of symmetry among seventh and eight grade students, previous to the instrumentation of other teaching cycles with other computational material, (a dynamic geometry software). Figure 1 shows the screen the game displays:

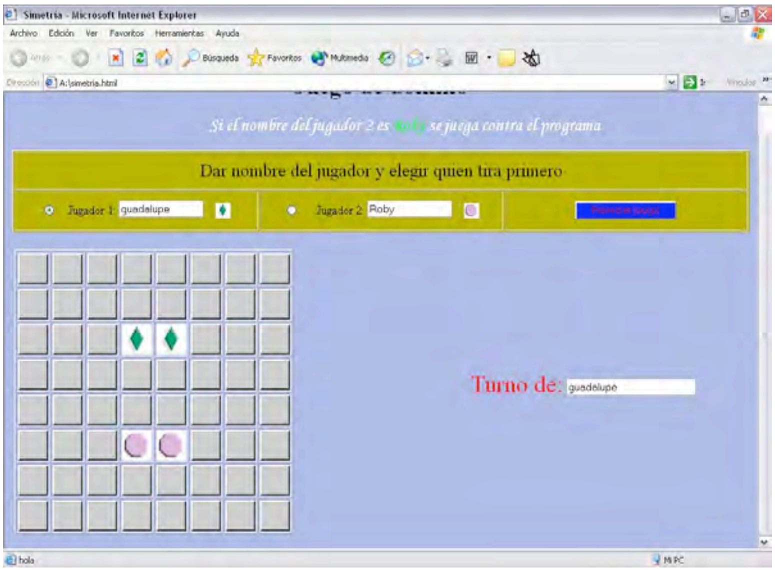

Figure 1: Screenshot of the *Domino* computer board game

The work sessions with the game were video-recorded, as were some of the instructor-student interviews when the games ended. These interviews aimed to record student hypotheses, conjectures, and explanations of the possible winning strategies; and important data was obtained after transcribing the protocols extracted from both video sources.

The fact should be highlighted that when the game was first shown to them the theme of symmetry had not yet been presented to the seventh graders. They did not recognize symmetry in *Robi*'s way of playing or winning. Rather, they developed another winning strategy, one based on counting and building inaccessible spaces on the board. Such was not the case with eighth graders (1), who had had symmetry presented to them the year before. In general, student performance of the eighth school year reveal that, as long as they experiment with the game and during execution of the task, they were able to activate mental processes that allowed them to demonstrate a pattern of behavior as such that characteristic to the mathematical activity related with problem solving (see Polya, 1954).

From a Verillon and Rabardel (1995) point of view, an artifact's structure and function will foster certain knowledge or effect in cognitive development. It is to say that the introduction and use of instruments, whether material or psychological (language, computational means, symbols, diagrams, maps, etc.) leads to consummating many structural and functional changes in the learner's cognition. Vygotsky confirms this in these affirmations:

> A complete series of new functions tied to the use and control of the chosen instrument are activated; the labor performed by the instrument makes an entire series of natural process worthless and replaces them; [the instrument] transforms the development and particular aspects (intensity, duration, continuity, etc.) of all the processes involved in the composition of the instrumental act. [In this way the instrument constitutes itself as] a new interceding element situated between the

object and the psychological operation directed toward it. (Cited by Verillon and Rabardel, 1995: 81-82, from Vygotsky, 1930: 42.)

Principal Results and Conclusions

A result of seventh graders playing against *Robi* was a rapid turn toward a different winning strategy, one which consisted of trying to leave blank spaces, and counting out how many were necessary according to which turn they had. They were able then to display their arithmetical abilities to find a new and not expected winning strategy.

Concerning eighth grade students, they recognized that symmetry was the intended structure of the game, in accordance with they had studied this topic the previous year. It is likely that the structure and function of the artifact employed (*Domino*) in this context have fostered cognitive development (Verillon and Rabardel, 1995), since it was observed that when students used a strategy that they believed to be a winner, they continued to use it and perfected it as long as it was functional, or discovered an alternative one.

Moreover, an opponent strategy, which began to be a winning one, was a cause for reflection and reformulation or construction of the new winning strategy. This heuristic is a characteristic for problem solving (Polya, 1954) promoted by the competition situation of the game. Nonetheless, the potential of this type of psychological instrument is still to be determined for solving specific math problems.

Endnote

1. The topic of symmetry is part of the curriculum for seventh graders; thus eighth grade students should have encountered this topic the previous year. In effect, this situation might be confirmed by applying the game in class with eighth students, since many of them quickly noticed that one way to win was to mirror the opponent's moves symmetrically.

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STUDENTS' USES AND INTERPRETIONS OF SYMBOLS WHEN SOLVING PROBLEMS WITH AND WITHOUT A GRAPHING CALCULATOR

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As students interact with graphing calculators to help solve math problems, one aspect that they need to attend to is the use of mathematical symbols, which are the components of mathematics that enable communication of solutions and ideas. However, the symbolic language in mathematics is often very confusing for students (Rubenstein & Thompson, 2001). This language barrier may be one reason that students turn to graphing calculators for assistance, but, most non-computer algebra system (CAS) graphing calculators cannot algebraically manipulate symbolic equations to produce useful results.

Studies have shown that students can perform as well in mathematics using graphing calculators as they can without them, but many instructors remain reluctant to teach or assess with this tool (Ellington, 2003). It is reasonable to assume that students who have learned to do mathematics using graphing calculators in high school will continue to use them on homework in college; however, a test that does not permit calculator use may require students to redefine their goal to include communicating a correct solution method for a problem using mathematical symbols. The activity engaged in during homework will no longer be helpful if students have not abstracted a relationship between their activities with the calculator and the symbolic representation of the mathematical concepts involved.

The intent of this research is to conduct a case study of precalculus college students with a focus on their mathematical thinking, particularly about symbols, as they solve problems with and without the assistance of graphing calculators. The question to be addressed is: What is the nature of the goals, activities, and symbolic communication that students engage in when doing work in a practice environment with access to a non-CAS graphing calculator and in an assessment environment without such access?

Framework

In order to address the research questions in this study, a theoretical framework is needed that will provide a lens for looking both at students' anticipation and reflection on goals and activities and the symbol sense that they exhibit as they work with the graphing calculator. The chosen framework is a combination of Simon, Tzur, Heinz, and Kinzel's (2004) reflection on activity-effect relationship framework (AER), and Pierce and Stacey's (2001) framework for algebraic insight. Simon et al.'s (2004) AER framework is built on Piaget's notion of reflective abstraction, and is designed as a way for explaining the development of new mathematical conceptions beyond those already available. The authors describe it as a theory to guide the teaching of mathematical concepts and the design of instructional interventions to address problems in learning mathematics (Simon et al., 2004; Tzur & Simon, 2004). It begins with a goal-directed mental activity, where the learner continually monitors the effects and results of the activity. The learner creates mental records of the relationships between each execution of the activity and the effect produced. By reflecting on these records and looking for patterns between the activities and their effects, the learner abstracts a new activity-effect relationship, which is the basis for a more advanced conception. As a tool for looking at learning, the AER framework can help in the identification of stages of learners' concept development, particularly whether students are at a *participatory* or *anticipatory* stage (Tzur & Simon, 2004).

Learners' goals and reflections in the AER mechanism are not always conscious to the learner (Simon et al, 2004) and will be difficult to observe. Thus, the algebraic insight framework will be incorporated to evaluate students' understanding when working with technology to solve mathematical problems (Pierce & Stacey, 2001). Algebraic insight is the subset of symbol sense that enables a learner to interact effectively with a CAS when solving problems, and is comprised of two components: algebraic expectation and the ability to link representations. For this study, algebraic insight will be applied to students use of non-CAS graphing calculators to try to solve and manipulate symbolic problems.

Methods

This is a qualitative, multi-case study. The case is defined as an undergraduate precalculus student who frequently uses a graphing calculator to assist in problem solving, thus each of the six different participants constitutes a separate case. The data collection took place during a summer semester at a university in the southern United States, and occurred in the following manner: The research began with individual task-based interviews with each of the students to discuss their experiences with using graphing calculators and solving mathematical problems. During the interview, students were asked to work on three to five algebraic tasks and were prompted to "think out loud" as they solved the problem and/or made use of the graphing calculator. This initial interview was followed by two group sessions where two students worked together on web-based homework assignments for their class. Students had access to textbooks, notes, graphing calculators, and each other in these sessions. A final piece of data came from students' work on their course tests. Students took the test in the classroom where the teacher does not allow calculators, but met soon after the test with the researcher to discuss the goals and activities in which they engaged on the test. The researcher then showed clips of the students' work in the homework sessions and the researcher and participant discussed similarities and differences between the approaches that students took in the practice and assessment environments.

All work in the initial interview, homework sessions, and final interviews was videotaped, and calculator keystrokes were recorded using computer software. These recording techniques where particularly useful in the homework sessions because they allowed the researcher to create a natural work setting for observing students' work with minimal interference (Berry, Graham, & Smith, 2005). The focus of the homework sessions and tests was on quadratic, general polynomial, and rational functions.

Results

At the time that this paper was written, the data collection was still in progress and the analysis was incomplete. Detailed results are to be presented at the PMENA conference, but some initial ideas and observations are shared here. For example, during the initial interviews, the researcher was surprised to find that none of the six participants were able to solve the algebra problem: Solve for *x,* if $x^3+2x-4=8$, by using the graphing features of the calculator, although most of the students had indicated on a survey given the first day of class that they could use the calculator for such a purpose. However, the students were fairly dependent on the graphing calculator for most basic calculations and for checking over any work they did by hand.

The students were all attentive to the fact that the calculators were not going to be permitted on their course tests, and so in the initial interview and homework sessions, most of them insisted that it was best for them to work the problem by hand so that they understood it in the way that would be expected of them on the test. However, when they became stumped, many of them turned to the calculator for help, although not always with a clear purpose in mind of how the calculator was going to be able to assist them. They all had different levels of comfort and trust when using the calculator, and different ideas about how it could be useful to them. For example, one student insisted that the calculator could not help her with any problem that involved a variable, unless it was to allow her to test a value in the expression, while another student went so far as to type an expression like $(x\text{-}16)/(x^2+3x\text{-}12)$ into the regular calculator screen (i.e., not the graphing screen) in order to “see what it says.”

Initial observations about the participants symbol sense leads the researcher to think that they were all fairly weak in this area. For example, one question on the homework asked to students to come up with the quadratic function whose graph was given. Two students followed a similar example in the textbook that used the alternate form of a quadratic function, $f(x) = a(x-h)^2+k$, where (h,k), the vertex, was given on the graph. The students had trouble distinguishing between letters as parameters and variables, and one student did not believe that the final answer could stay in this form, and struggled to get it in the form $f(x) = ax^2+bx+c$ before submitting the problem. All of the students used sloppy notation in their notes, and several made up their own notations to keep track of information in a problem.

These and other observations will be analyzed under the lens of the theoretical framework for this study and results will be shared at the PMENA conference.

Significance

With this study, the researcher hopes to contribute to the understandings of ways that students are intending to use graphing calculators to practice mathematical problems. This may, in turn, identify reasons why students who work successfully with a calculator on homework problems cannot perform well on tests without a calculator. The researcher is not necessarily looking for improvement in test scores, as the intent of this research is not to show that graphing calculator use is superior to pencil and paper methods alone, but will instead be looking for examples of algebraic insight and instances of reflections on the activity-effect relationships, and how the relationships developed with calculator use may have affected the ways that students solved similar problems without a graphing calculator. Attention will focus on how students choose to write up the solution to a problem that they solve with the graphing calculator, and whether they are attentive to the need to clearly explain their work using mathematical symbols.

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TEACHERS' USES OF VIRTUAL MANIPULATIVES IN K-8 MATHEMATICS LESSONS

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This study examined teachers' uses of virtual manipulatives in mathematics lessons across Grades K-8. Ninety-five lessons using virtual manipulatives were analyzed. The content in a majority of the lessons focused on two NCTM (2000) Standards: Number & Operations and Geometry. Virtual geoboards, pattern blocks, base-ten blocks, and tangrams were the applets used most often by teachers. The ways teachers used the virtual manipulatives most frequently focused on students understanding and investigating concepts. It was common for teachers to use the virtual manipulatives alone, or to use physical manipulatives first followed by virtual manipulatives. These results represent the first general exploration of the current use of virtual manipulatives in K-8 classrooms.

The release of the most recent NCTM Standards (2000) gave prominence to representation as a significant area of mathematics education research. Although this was the first appearance of representation as a Standard, teachers have used a variety of representations during mathematics instruction for many years. Representations commonly used in school mathematics include physical or concrete representations, visual or pictorial representations, symbolic or abstract representations, and dynamic electronic representations that combine characteristics of physical and pictorial models (e.g., *virtual manipulatives*, Moyer, Bolyard, & Spikell, 2002). This study focuses on the examination of teachers' uses of *virtual manipulatives* in mathematics lessons across Grades K-8.

Virtual Manipulatives, Representation, and Dual Coding Theory

Goldin (2003) defines representation as a configuration of signs, characters, icons, or objects that stand for, or "represent" something else. Students' capacity to translate among multiple representational systems influences their abilities to model and understand mathematical constructs (Goldin & Shteingold, 2001). Cognitive science has influenced educational research by proposing theoretical models that explain the encoding of information among representational systems. For example, *Dual Coding Theory* (DCT), proposed by researchers in the field of educational psychology, is the assumption that information for memory is processed and stored by two interconnected systems and sets of codes (Clark & Paivio, 1991). These sets of codes include visual and verbal codes, which can represent letters, numbers or words. According to DCT, presenting learners with both visual and verbal codes, which are functionally independent, has additive effects on their recall.

A common design structure for virtual manipulative applets is to include verbal codes (i.e., letters, numbers, and words) and visual codes (i.e., pictures) presented simultaneously (Mayer & Anderson, 1992). Applying DCT to instruction when virtual manipulatives are used infers that mathematics environments that activate multiple systems of codes have a greater potential for improving learning because two mental representations are available for use by the learner, rather than just one. Rieber's (1994) research shows that it is easier for students to recall information from visual processing codes than from verbal codes because visual information is accessed using synchronous processing, rather than sequential processing. Virtual manipulatives (or *VMs*), which are primarily visually-based tools, can facilitate greater access to memory when students are using this form of visual media.

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A recent review of research indicates that students using *VMs* either alone or in combination with physical manipulatives demonstrate gains in mathematics achievement and understanding (Bolyard, 2006; Moyer, Niezgoda, & Stanley, 2005; Reimer & Moyer, 2005; Suh, 2005; Suh & Moyer, 2007). To focus on teachers' uses of *VMs*, we analyzed teachers' reports of mathematics instruction where these tools were used using the following research question: What *VMs* are used by teachers in mathematics lessons, and how are they used?

Methods

A total of 116 teachers participated in the study, with two sections at each of four grade bands (K-2, 3-4, 5-6, 7-8), for a total of eight groups. Participants were kindergarten through eighth-grade teachers in eight different teacher professional development institutes taught by four instructors. The eight groups of teachers attended 40-hour summer institutes followed by four formal meetings during the fall and spring of the following academic year (8 hours). The purpose of the professional development was to improve mathematics instruction through readings, discussions, and hands-on experiences. Manipulatives and technology were two major resources used throughout all of the activities and were used daily with each group.

The primary source of data was teacher-developed lesson plans. During the school year, teachers developed and taught mathematics lessons. Researchers collected the written lesson plan reports at the end of the academic year. Teachers' written plans reported what was actually taught during the lessons in the classrooms. This provided evidence on how the *VMs* were used. Each of the 116 teachers designed five lessons, for a total of 580 lessons. Of these 580 lessons, 95 lessons used a virtual manipulative (28% Grades K-2, 16% Grades 3-4, 32% Grades 5-6, and 24% Grades 7-8), and these 95 lessons were the focus of this analysis.

Separate analyses examined: (1) the content of the lesson plans, comparing the content with the NCTM Standards (2000); (2) types of *VMs* used in the lessons within grade-specific groups; (3) categories describing *how* the *VMs* were used in the lessons, including (a) *investigate* (open-ended investigations/problem-solving activities), (b) *understand* (students developed understandings of specific mathematical concepts through teacher guidance and practice), (c) *introduce* (teachers introduced new concepts), (d) *game* (students played games), (e) *aide* (*VMs* used for remediation), (f) *model* (teachers demonstrated concepts, but students did not use *VMs*), and (g) *extend* (*VMs* used to extend concepts for students above grade level); and (4) connections in lessons between virtual and physical manipulatives.

Results

The majority of the virtual manipulative lessons were in the *Number & Operations* Standard (35%), followed closely by *Geometry* (32%). A lesser portion of the lessons focused on *Algebra* (13%), *Measurement* (13%), and *Data Analysis & Probability* (7%). Most frequently used of all *VMs* across the grade levels were virtual geoboards (11% of all lessons), pattern blocks (11%), tangrams (9%) and base-ten blocks (8%). The most frequent use of *VMs* within grade-specific groups included virtual pattern blocks in 22% and virtual tangrams in 19% of K-2 lessons, virtual base-ten blocks in 20% of 3-4 lessons, and virtual geoboards in 17% of 5-6 lessons and 17% of 7-8 lessons. No virtual manipulative was used by all four grade-specific groups, however eight *VMs* were used by three of the four grade-specific groups: virtual base-ten blocks, fraction circles, fraction squares, geoboards, geometric solids, number lines, pattern blocks, and tangrams.

As Table 1 shows, *how VMs* were used focused on *investigating* mathematical ideas or *understanding* mathematical concepts (45% and 37%, respectively). There were a greater number of lessons that included open-ended *investigations* at Grades K-2 (52%) and 5-6 (47%). A greater number of lessons at Grades 3-4 were designed to develop *understandings*

(47%), while an equal number of lessons at Grades 7-8 asked students to *investigate* and develop *understandings* (43%).

	Grade-Specific Groups				
	K-2 (N = 27)	3-4 (N = 15)	5-6 (N = 30)	7-8 (N = 23)	All (N = 95)
Investigate	14 (52%)	5 (33%)	14 (47%)	10 (43%)	43 (45%)
Understand	8 (29)	7 (47)	10 (33)	10 (43)	35 (37)
Intro	3 (11)	3 (20)	4 (13)	3 (13)	13 (14)
Game	1 (4)	0 (0)	1 (3)	0 (0)	2 (2)
Other (Aide, Model, Extend)	1 (4)	0 (0)	1 (3)	0 (0)	2 (2)

Note: Because groups contain different *N*s, data are presented in numeric and percent formats for comparison purposes.

Table 1. How virtual manipulatives were used in the lessons.

Table 2 shows the relationship of *VMs* with other mathematical tools used in the lessons (e.g., physical manipulatives). An approximate equal number of lessons used *VMs* alone (49 lessons) as used *VMs* together with physical manipulatives (46 lessons). A larger portion of lessons at Grades K-2, 3-4, and 7-8 (59%, 53%, and 52%, respectively) used both *VMs* and physical manipulatives, using the physical materials first, and then using the *VMs* during the second part of the lesson or as an extension. The majority of Grades 5-6 lessons used the *VMs* alone (77%).

	Grade-Specific Groups				
	K-2 (N = 27)	3-4 (N = 15)	5-6 (N = 30)	7-8 (N = 23)	All (N = 95)
Used Only VM	10 (37%)	6 (40%)	23 (77%)	10 (44%)	49 (52%)
Used PM, Then VM	16 (59)	8 (53)	5 (17)	12 (52)	41 (43)
Used VM, Then PM	0 (0)	0 (0)	2 (7)	0 (0)	2 (2)
Simultaneous Use (VM & PM)	1 (4)	1 (7)	0 (0)	1 (4)	3 (3)

Note: Because groups contain different *N*s, data are presented in numeric and percent formats for comparison purposes.

Table 2. The relationship of virtual manipulatives with other mathematical tools used in the lessons.

Concluding Remarks

As the results show, there were a variety of *VMs* in use by teachers for mathematics instruction across Grades K through 8. We were not surprised to find that geoboards, pattern blocks, base-ten blocks, and tangrams were the most frequently used *VMs* across the grade levels, as these are commonly used physical manipulatives with which many teachers are familiar. Research examining the benefits of *VMs* at specific grade levels and for specific mathematics content would further inform instructional decision making on the use of these tools in mathematics teaching. (References available upon request.)

GRAPHING IN GROUPS: COLLABORATIVE ACTIVITIES FOR CLASSROOM NETWORK

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We present an activity design that uses networked graphing calculators to help students explore mathematical concepts in a collaborative setting. In the design, each student in a small group controls an individual point on a graph such that together they can jointly manipulate curves defined by their collective points to perform a variety of tasks and investigations.

Stroup, Ares & Hurford (2005) provide a framework for studying the generative interaction between mathematics and network-supported design. They describe a dialectic in which mathematical structure offers possibilities for dynamic classroom interaction characterized by space-creating play, while social structure promotes agency and participation among the students. Our designs reflect efforts to capitalize on this dialectical interplay between mathematical and social structure in networked collaborative activities.

These designs use the NetLogo modeling environment and the TI-Navigator 3.0™ graphing calculator network (Wilensky & Stroup, 1999). In each activity, students work in server-defined pairs such that their individually-controlled points collectively define a line. As each student moves her individual point, the algebraic equation and the associated linear function is continually transformed on the collective display. The pair of students can then move their points and see the changing relationship between the positions of their points and the equation of the line they determine. A computer screen projection at the front of the classroom simultaneously displays several graphing windows, so that each pair or small group can occupy a distinct graphical space, yet all groups' activity can also be examined collectively.

Pairs of students can perform a variety of tasks involving these jointly controlled lines. The teacher might ask each pair of students to move their points in order to obtain the graph of a specified equation, such as $y = (2/3)x + 3$. Students might next be asked to find another line that maintains the same y-intercept, or that maintains the slope while adjusting the intercept. Each of these tasks places constraints on the ways students can position their points in relation to one another, and yet allows for an endless array of possible configurations—each, in other words, supports the kind of open-ended yet mathematically structured "play" that allows a class to collectively chart the space of a linear relationship.

The collaborative possibilities of these tasks can be further opened up by asking two pairs of students to find parallel lines or perpendicular lines on the same grid. The students must then manage not only the relationships between their respective pairs of points, but also the relationship between the pair of lines those points determine. All of these activities for both pairs and small groups invite students to consider the links between mathematical objects such as points or lines in relation to their interactions with one another. Our hope is that these bridges between social interactions and mathematical relationships will provide powerful resources for supporting students' understanding of the latter.

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ONLINE HOMEWORK IN UNIVERSITY ALGEBRA COURSES: PERCEPTIONS OF STUDENT LEARNING AND MOTIVATION

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Automated online homework is an increasing trend in college mathematics courses over traditional paper and pencil. One university's college algebra courses have moved the majority of homework assignments into the online environment, allowing for immediate feedback. We examined the degree to which online homework, specifically for algebra courses, affects perceptions of learning and motivation to learn.

Online homework is an increasing trend in college mathematics courses over traditional paper and pencil. One such instance of this is at a large Midwestern University, where the college algebra courses have moved to comprising almost all homework as online assignments. Studies have been conducted on the use of online homework in physics courses (Bonham, Beichner, & Deardorff, 2001) and chemistry courses (Cole & Todd, 2003), but parallel research was not found in the field of mathematics. The aforementioned studies investigated exam scores and found no significant difference in online versus traditional methods of completing homework. Due to the considerable use of online homework in courses such as the college algebra course, it is necessary to investigate its effectiveness in a mathematics class. Furthermore, the investigation needs to go beyond test scores and examine its effects on student learning and motivation to learn mathematics.

We chose to examine the degree to which online homework, specifically for algebra courses, affects perceptions of learning and motivation to learn for this group of college students. To aid the investigation, surveys were administered to all of the college algebra students enrolled in the course for one semester (n ~ 380). The Motivated Strategies for Learning Questionnaire (MSLQ) was adapted using both Likert-scale and open-ended response items. Data were compiled and analyzed by the research team in order to determine the online homework's effects on perceptions of learning and motivation to learn college algebra.

The data indicate a range of perceptions of the online homework. Some students appreciated the immediate feedback from the online homework and felt more accountable for completing the problems. Hence, they sought help to correctly answer online homework problems. Other students, however, stated they felt frustrated with the online homework. Their reasons for frustration were mainly due to difficulties with entering their solutions or technical concerns. Our poster session will include more detailed results from our study.

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TEACHERS' USE OF TECHNOLOGY IN MEXICAN JUNIOR SECONDARY SCHOOLS

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We present results from a study that tries to answer important questions which have emerged after many years of use of computational technologies in the mathematics classes of public junior secondary schools in Mexico.

In 1997, a still ongoing national program, the Teaching Mathematics with Technology – *EMAT* – project was introduced throughout Mexico by the Ministry of Education. Its aim is to incorporate computational tools (TI Calculators, Excel, Cabri-Géomètre, and Logo) to junior secondary mathematics classrooms using a research-based pedagogical approach, specifically designed to foster students' exploration, problem solving and whole class discussion. Teachers all over the country were trained for the implementation in classrooms of the tools, the approach and student-centered activities.

Given the importance and impact of this project, as part of the evaluation team, we were concerned about how teachers have assimilated the main ideas of the project and how they use them in their classes. For this, we follow the enactivism theory of knowing, which considers learning as an effective or adequate action (Maturana & Varela, 1992); from this enactivist perspective, the use of computer tools is part of human living experience and they are used to represent and negotiate cultural experience (Davis et al., 2000).

Using a combination of methodologies (including observations of technology-based classes in a sample of schools across the country, and interviews) after three years of study, we have found that teachers have many difficulties when trying to use technology to teach mathematics. Some of these problems are related to teachers' difficulties regarding the philosophy and pedagogical approach of the project. Other difficulties are related to teachers' knowledge of mathematics and to their fears on the use of technology.

On the other hand, teachers have found motivation for using the technological tools and materials mainly in response to how their students work with proposed activities, but they still have a lot of work to do for gaining experience and developing their practice in order to use the technology effectively.

In the poster we will present the details of the project, its framework, the methodology of the study and the results obtained.

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THE COLLABORATIVE SPEEDWAY: MOTION AND INTERACTION IN TWO DIMENSIONS

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This poster presents a scenario in which student pairs use networked graphing calculators to collaboratively control the motion of a single car by manipulating velocity vectors as they navigate a racing track displayed on a computer screen. We expect this activity to serve as an important example of the ways networked devices can support collaborative classroom problem solving.

Collaborative activities can be excellent ways to engage students, promote interaction, and advance learning, but it is not necessarily a given that all students will equally benefit from the process. What if one student does most or all of the work? What happens if one student becomes uninvolved in the activity? Research suggests that the quality of interaction among participants in a group has implications for learning (Barron, 2003). To promote equitable learning opportunities, it is important that these tasks include interdependence, parallel responsibilities, and meaningful exercises for all participants (Cohen, 1994).

This poster presents a collaborative scenario using the NetLogo (Wilensky, 1999) modeling environment and Hubnet (Wilensky & Stroup, 1999) networking tools in combination with a TI-Navigator 3.0™ graphing calculator network. In this scenario, student pairs take shared responsibility for controlling the motion of a single car to navigate a variety of racing tracks displayed on a computer screen. Driving the car in this environment involves adjusting the horizontal and vertical components of a velocity vector; these components are separately controlled through inputs to two different students' graphing calculators. The car begins at rest, and in each turn during the race, the two students controlling the motion each have the option of incrementing or decrementing their respective components by one unit, or leaving the component value unchanged. During the same turn, other teams of students controlling other cars in a collective display likewise have the same options for adjusting the velocity vector, and the teams race to see which can navigate the track in the fewest turns. Efficiently completing the race without leaving the track requires coordinated decision-making at each turn.

As teams compete in the race, the server creates a log of the team's velocity vector and Cartesian coordinates in each turn, as well as the distance traveled and the triangular area formed by the velocity vector and its components during each turn. At the end of a race, these four different sets of information are distributed as lists to the respective personal devices of various members in a group. Groups are then prompted to engage in a variety of collaborative mathematical analyses of these different artifacts in order to reflect on and revise their racing strategies, and to discuss ways of minimizing the number of turns required to traverse a given racing track prior to engaging in another iteration of the race.

This collaborative activity is intended to foster interaction among group members. Students have separate roles that contribute equally to the outcome. These same roles also limit the potential for one student to take over the workload. Additionally, we anticipate that the racecar context and group competition will support students' interest in the activity and their engagement with rich mathematical concepts. We expect this activity to serve as an important example of the ways networked devices can support collaborative classroom problem solving.

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